MATHEMATICS 1st YEAR

UNIT # 03

MATRICES & DETERMINANTS

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Contents

Sherazi Mathematics

اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Matrix:- A rectangular array of numbers enclosed by a pair of bracket is called a matrix.

e.g., $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix}$ or $\begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 4 \\ 3 & 2 & 6 \end{bmatrix}$

Note:- i) Matrices are denoted by capital letters such as A, B, C,... ..., X, Y, Z

ii) The elements or enteries of a matrix as denoted by small letters such as a, b, c, ..., x, y, z.

iii) The horizontal lines of elements are called rows of a matrix.

iv) The vertical lines of elements are called columns of a matrix.

Order of a matrix:-

The number of rows and columns of a matrix is called order of a matrix. i-e., If a matrix has m rows and n columns then its order is $m \times n$ (read as m-by-n). e.g.,

$A = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix}$ its order is 2×3

$B = \begin{bmatrix} 1 & 4 & 6 \end{bmatrix}$ its order is 1×3

* The matrix A is called real if all of its elements are real.

Types of Matrices

Row matrix:- A matrix, which has only one row, i-e., a matrix of order $1 \times n$ is called row matix. e.g., $[a_{11} \; a_{12} \; a_{13}]$,

$[1 \quad 2 \quad 3 \quad 4]$ etc

Column Matrix:- A matrix, which has only one column i-e., a matrix of order $m \times 1$ is called column matrix. e.g.,

$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix}$ etc

Rectangular matrix:- A matrix whose number of rows and columns are not equal is called rectangular matrix. e.g.,

$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{bmatrix}$, $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

Square matrix:- A matrix whose number of rows and columns are equal is called square matrix.

e.g., $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Null matrix:- A matrix whose all elements are zero is called null matrix. e.g., $[0 \quad 0 \quad 0]$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

* Row matrix and Column matrix are also called row vector and column vector.

Principal diagonal:- The diagonal from upper left corner to the lower right corner of a square matrix is called principal diagonal or main diagonal or leading diagonal. e.g.,

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ The entries a_{11}, a_{12} and a_{13} form the principal diagonal.

Secondary diagonal:- The diagonal from the lower left corner to the upper right corner of a square matrix is called secondary diagonal. e.g.,

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ The entries a_{13}, a_{22} and a_{31} form the secondary diagonal.

Diagonal matrix:- A square matrix in which all elements except the leading diagonal are zero is called a diagonal matrix. e.g., $\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$, $\begin{bmatrix}3 & 0 & 0\\0 & 3 & 0\\0 & 0 & 3\end{bmatrix}$

Identity matrix:- A diagonal matrix whose all elements of the main diagonal are 1 is called identity matrix denoted by I_n. $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}_{2\times2}$, $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}_{3\times3}$

Scalar matrix:- A diagonal matrix whose all elements of main diagonal are same is called scalar matrix e.g.,

$$\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}, \begin{bmatrix}K & 0 & 0\\0 & K & 0\\0 & 0 & K\end{bmatrix}$$

* Needs to remember *

Square matrix:- A matrix having m rows and n columns with $m = n$ is called square matrix

Rectangular matrix:- A matrix having m rows and n columns with $m \neq n$ is called rectangular matrix.

Diagonal matrix:- Let $A = [a_{ij}]$ be a square matrix of order n, if $a_{ij} = 0 \ \forall \ i \neq j$ and at least $a_{ij} \neq 0$ for $i = j$, some elements of the principal diagonal of A may be zero but not all, then matrix A is called a diagonal matrix.

Identity matrix:- Let $A = [a_{ij}]$ be a square matrix of order n. for all $i \neq j$ and $a_{ij} = 1$ for all $i = j$ then A is called unit matrix denoted by I_n.

Null matrix:- A matrix of order $m \times n$ with all elements zero is called null matrix.

Scalar matrix:- Let $A = [a_{ij}]$ be a square matrix of order n. if $a_{ij} = 0 \ \forall \ i \neq j$ and $a_{ij} = K$ (some non-zero scalar) $\forall \ i = j$ then the matrix is called scalar matrix.

Equal matrices:- Two matrices of the same order are said to be equal if corresponding entries are equal.

Addition of Matrices:- Two matrices can be added if both have same order.
* addition is done by adding corresponding entries of the matrices.
* Matrices of different orders cannot be added.

Transpose of a matrix:-
Transpose of a matrix is denoted by A^t and can be obtained by interchanging rows into columns or columns into rows.

Example 1. If $A = \begin{bmatrix}1 & 0 & -1 & 2\\3 & 1 & 2 & 5\\0 & -2 & 1 & 6\end{bmatrix}$ and $B = \begin{bmatrix}2 & -1 & 3 & 1\\1 & 3 & -1 & 4\\3 & 1 & 2 & -1\end{bmatrix}$, then show that $(A+B)^t = A^t + B^t$

Solution:-

$$A + B = \begin{bmatrix}1 & 0 & -1 & 2\\3 & 1 & 2 & 5\\0 & -2 & 1 & 6\end{bmatrix} + \begin{bmatrix}2 & -1 & 3 & 1\\1 & 3 & -1 & 4\\3 & 1 & 2 & -1\end{bmatrix}$$

$$= \begin{bmatrix}1+2 & 0-1 & -1+3 & 2+1\\3+1 & 1+3 & 2-1 & 5+4\\0+3 & -2+1 & 1+2 & 6-1\end{bmatrix}$$

$$A+B = \begin{bmatrix}3 & -1 & 2 & 3\\4 & 4 & 1 & 9\\3 & -1 & 3 & 5\end{bmatrix}$$

$$\Rightarrow (A+B)^t = \begin{bmatrix}3 & 4 & 3\\-1 & 4 & -1\\2 & 1 & 3\\3 & 9 & 5\end{bmatrix} \rightarrow (I)$$

Now

$$A^t = \begin{bmatrix}1 & 3 & 0\\0 & 1 & -2\\-1 & 2 & 1\\2 & 5 & 6\end{bmatrix}, \quad B^t = \begin{bmatrix}2 & 1 & 3\\-1 & 3 & 1\\3 & -1 & 2\\1 & 4 & -1\end{bmatrix}$$

$$A^t+B^t=\begin{bmatrix}1&3&0\\0&1&-2\\-1&2&1\\2&5&6\end{bmatrix}+\begin{bmatrix}2&1&3\\-1&3&1\\3&-1&2\\1&4&-1\end{bmatrix}$$

$$=\begin{bmatrix}1+2&3+1&0+3\\0-1&1+3&-2+1\\-1+3&2-1&1+2\\2+1&5+4&6-1\end{bmatrix}$$

$$A^t+B^t=\begin{bmatrix}3&4&3\\-1&4&-1\\2&1&3\\3&9&5\end{bmatrix}\longrightarrow(\text{II})$$

from (I) and (II)

$(A+B)^t = A^t + B^t$ Hence proved

Scalar Multiplication:-

Let $A=[a_{ij}]$ be an $m\times n$ matrix and K be a scalar. then the product of K and A can be obtained by multiplying each entry of A by K.

i-e., $KA=[Ka_{ij}]$

order of KA is $m\times n$

* If n is a +ive integer, then $A+A+A+\dots$ to n terms $= nA$

Subtraction of Matrices

If $A=[a_{ij}]$ and $B=[b_{ij}]$ are matrices of order $m\times n$, then we define subtraction of B from A as:

$A-B = A+(-B) = [a_{ij}]+[-b_{ij}]$

$A-B = [a_{ij}+(-b_{ij})] = [a_{ij}-b_{ij}]$

$i=1,2,3,\dots m$, $j=1,2,3,\dots n$

$A-B$ is formed by subtracting each entry of B from the corresponding entry of A.

Multiplication of two Matrix

Two matrices A and B are said said to be conformable for product AB if the number of columns of A is equal to the number of rows of B.

* Matrix multiplication is not commutative i-e.,

$AB \neq BA$

* If the product AB is defined, then the order of the product can be illustrated as given below:

Order of A	$m\times n$
Order of B	$n\times p$
Order of AB	$m\times p$

Example 2. If $A=\begin{bmatrix}2&-1&0\\1&2&-3\\1&2&-2\end{bmatrix}$ and $B=\begin{bmatrix}2&-2&3\\-1&-4&6\\0&-5&5\end{bmatrix}$, then compute A^2B.

Solution:-

$$A^2=AA=\begin{bmatrix}2&-1&0\\1&2&-3\\1&2&-2\end{bmatrix}\begin{bmatrix}2&-1&0\\1&2&-3\\1&2&-2\end{bmatrix}$$

$$=\begin{bmatrix}4-1+0&-2-2+0&0+3+0\\2+2-3&-1+4-6&0-6+6\\2+2-2&-1+4-4&0-6+4\end{bmatrix}$$

$$A^2=\begin{bmatrix}3&-4&3\\1&-3&0\\2&-1&-2\end{bmatrix}$$

$$\therefore A^2B=\begin{bmatrix}3&-4&3\\1&-3&0\\2&-1&-2\end{bmatrix}\begin{bmatrix}2&-2&3\\-1&-4&6\\0&-5&5\end{bmatrix}$$

$$=\begin{bmatrix}6+4+0&-6+16-15&9-24+15\\2+3+0&-2+12+0&3-18+0\\4+1+0&-4+4+10&6-6-10\end{bmatrix}$$

$$A^2B=\begin{bmatrix}10&-5&0\\5&10&-15\\5&10&-10\end{bmatrix}$$

Note:- Powers of square matrices are defined as: $A^2=A\times A$
$A^3=A\times A\times A$, $A^n=A\times A\times A\dots$ to n factors

Determinant of 2x2 Matrix

The determinant of a matrix is denoted by enclosing its square array between vertical bars instead of brackets. e.g.,

If $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ then $|A|=\begin{vmatrix}a&b\\c&d\end{vmatrix}$

$\rightarrow |A| = ad-bc$ For example

if $A=\begin{bmatrix}2&-1\\4&3\end{bmatrix}$ then $|A|=\begin{vmatrix}2&-1\\4&3\end{vmatrix}$

$\rightarrow |A|=(2)(3)-(4)(-1) = 6+4 = 10$

Hence the determinant of a matriz is the difference of the products of the entries in the two diagonals.

Singular Matrix:- A square matrix A is said to be singular if $|A| = 0$ e.g.,

$A = \begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}$, $|A| = \begin{vmatrix} 8 & 4 \\ 2 & 1 \end{vmatrix}$

$\rightarrow |A| = 8 - 8 = 0$ so

A is singular.

Non-Singular Matrix:- A square matrix A is said to be non-singular if $|A| \neq 0$ e.g.,

$A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$, $|A| = \begin{vmatrix} 1 & 2 \\ 4 & 9 \end{vmatrix}$

$\rightarrow |A| = 9 - 8 = 1 \neq 0$ so

A is non-singular.

Adjoint of 2x2 Matrix:-

The adjoint of a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is deonoted by adjA and defined as; $\text{adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Inverse of 2x2 Matrix:-

Let A be a non-singular square matrix of order 2. If there exists of matrix B such that

$AB = BA = I_2$, where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

then B is called multiplicative inverse of A and is usually denoted by A^{-1} i.e., $B = A^{-1}$. Thus

$AA^{-1} = A^{-1}A = I_2$

Example 3. For a non-singular matrix A, prove that $A = \frac{1}{|A|} \text{adj}A$

Solution:- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \longrightarrow (I)$

$\text{adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \longrightarrow (II)$

Let $A^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ then

$AA^{-1} = I_2$

$\rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

By def. of equal matrices,

$ap + br = 1 \longrightarrow (i)$

$cp + dr = 0 \longrightarrow (ii)$

$aq + bs = 0 \longrightarrow (iii)$

$cq + ds = 1 \longrightarrow (iv)$

Solving (i) and (ii)

By $d(i) - b(ii) \rightarrow$

$$\begin{aligned} adp + bdr &= d \\ -bcp \pm bdr &= 0 \\ \hline (ad-bc)p &= d \end{aligned}$$

$\rightarrow p = \frac{d}{ad-bc}$

By $c(i) - a(ii) \rightarrow$

$$\begin{aligned} acp + bcr &= c \\ -acp \pm adr &= 0 \\ \hline (bc-ad)r &= c \end{aligned}$$

$\rightarrow r = \frac{c}{bc-ad} = \frac{-c}{ad-bc}$

Solving (iii) and (iv)

By $d(iii) - b(iv) \rightarrow$

$$\begin{aligned} adq + bds &= 0 \\ -bcq \pm bds &= -b \\ \hline (ad-bc)q &= -b \end{aligned}$$

$\rightarrow q = \frac{-b}{ad-bc}$

By $c(iii) - a(iv) \rightarrow$

$$\begin{aligned} acq + bcs &= 0 \\ -acq \pm ads &= -a \\ \hline (bc-ad)s &= -a \end{aligned}$$

$\rightarrow s = \frac{-a}{bc-ad} = \frac{a}{ad-bc}$

Thus $A^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$

$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

By (I) and (II)

$A^{-1} = \frac{1}{|A|} \text{adj}A$ Hence proved

Example 4. Find A^{-1} if $A=\begin{pmatrix}5 & 3\\ 1 & 1\end{pmatrix}$ and verify that $AA^{-1}=A^{-1}A$

Solution:- $\because A=\begin{pmatrix}5 & 3\\ 1 & 1\end{pmatrix}$

$\rightarrow |A|=\begin{vmatrix}5 & 3\\ 1 & 1\end{vmatrix}=5-3=2\neq 0$

so A^{-1} exists.

$$\because A^{-1}=\frac{adjA}{|A|}=\frac{\begin{bmatrix}1 & -3\\ -1 & 5\end{bmatrix}}{2}$$

$$A^{-1}=\frac{1}{2}\begin{bmatrix}1 & -3\\ -1 & 5\end{bmatrix}$$

Now $AA^{-1}=\begin{bmatrix}5 & 3\\ 1 & 1\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}1 & -3\\ -1 & 5\end{bmatrix}$

$$=\frac{1}{2}\begin{bmatrix}5 & 3\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & -3\\ -1 & 5\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}5-3 & -15+15\\ 1-1 & -3+5\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}=\frac{2}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$AA^{-1}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\longrightarrow(1)$$

Also, $A^{-1}A=\frac{1}{2}\begin{bmatrix}1 & -3\\ -1 & 5\end{bmatrix}\begin{bmatrix}5 & 3\\ 1 & 1\end{bmatrix}$

$$=\frac{1}{2}\begin{bmatrix}5-3 & 3-3\\ -5+5 & -3+5\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}=\frac{2}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$A^{-1}A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\longrightarrow(2)$$

By (1) and (2) $AA^{-1}=A^{-1}A$

Hence proved

Solution of simultaneous linear equations by using matrices

Let the system of linear eqs. be

$$a_{11}x_1+a_{12}x_2=b_1$$
$$a_{21}x_1+a_{22}x_2=b_2$$

where $a_{11},a_{12},a_{21},a_{22},b_1,b_2\in R$

given system in matrix form

$$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\end{bmatrix}$$

Let $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$, $X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$, $B=\begin{bmatrix}b_1\\ b_2\end{bmatrix}$

If $|A|\neq 0$, then A^{-1} exists so

$$AX=B$$

Pre multiplying by A^{-1}

$\rightarrow A^{-1}(AX)=A^{-1}B$

$(A^{-1}A)X=A^{-1}B$ (associative law)

$I_2X=A^{-1}B$ $(\because A^{-1}A=I_2)$

$\rightarrow X=A^{-1}B$

or $\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\frac{1}{|A|}\begin{bmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix}$

$$=\frac{1}{|A|}\begin{bmatrix}a_{22}b_1-a_{12}b_2\\ -a_{21}b_1+a_{11}b_2\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}\frac{a_{22}b_1-a_{12}b_2}{|A|}\\ \frac{-a_{21}b_1+a_{11}b_2}{|A|}\end{bmatrix}$$

$\rightarrow x_1=\frac{a_{22}b_1-a_{12}b_2}{|A|}$, and

$x_2=\frac{-a_{21}b_1+a_{11}b_2}{|A|}$

Thus $x_1=\frac{\begin{vmatrix}b_1 & a_{12}\\ b_2 & a_{22}\end{vmatrix}}{|A|}$ and

$x_2=\frac{\begin{vmatrix}a_{11} & b_1\\ a_{12} & b_2\end{vmatrix}}{|A|}$

Example 5. Solve the following systems of linear equations.

i) $3x_1-x_2=1$
$x_1+x_2=3$

ii) $x_1+2x_2=4$
$2x_1+4x_2=12$

Solution:- i) $3x_1-x_2=1$
$x_1+x_2=3$

In matrix form,

$$\begin{bmatrix}3 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1\\ 3\end{bmatrix}$$

$$AX=B$$

$\rightarrow X=A^{-1}B\longrightarrow(i)$

where $A=\begin{bmatrix}3 & -1\\ 1 & 1\end{bmatrix}$, $X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

$B=\begin{bmatrix}1\\ 3\end{bmatrix}$

$|A|=\begin{vmatrix}3 & -1\\ 1 & 1\end{vmatrix} = 3+1=4\neq 0$

so, A^{-1} exists

$adj A=\begin{bmatrix}1 & 1\\ -1 & 3\end{bmatrix}$

$\because A^{-1} = \frac{adj A}{|A|} = \frac{1}{4}\begin{bmatrix}1 & 1\\ -1 & 3\end{bmatrix}$

so by (i) $X=\frac{1}{4}\begin{bmatrix}1 & 1\\ -1 & 3\end{bmatrix}\begin{bmatrix}1\\ 3\end{bmatrix}$

$\Rightarrow X=\frac{1}{4}\begin{bmatrix}1+3\\ -1+9\end{bmatrix} = \frac{1}{4}\begin{bmatrix}4\\ 8\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix} \Rightarrow x_1=1,\ x_2=2$

ii) $x_1+2x_2=4$
$2x_1+4x_2=12$

Solution:- In matrix form,

$\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}4\\ 12\end{bmatrix}$

$A\ X = B$

where $A=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$, $X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$, $B=\begin{bmatrix}4\\ 12\end{bmatrix}$

Now $|A|=\begin{vmatrix}1 & 2\\ 2 & 4\end{vmatrix}=4-4=0$

$\Rightarrow A^{-1}$ does not exist.

Exercise 3.1

Q1. If $A=\begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix}$ and $B=\begin{bmatrix}1 & 7\\ 6 & 4\end{bmatrix}$, then show that

i) $4A-3A=A$ ii) $3B-3A=3(B-A)$

Solution:- i) $4A-3A=A$

L.H.S $=4A-3A$

$=4\begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix}-3\begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix}$

$=\begin{bmatrix}8 & 12\\ 4 & 20\end{bmatrix}-\begin{bmatrix}6 & 9\\ 3 & 15\end{bmatrix}$

$=\begin{bmatrix}8-6 & 12-9\\ 4-3 & 20-15\end{bmatrix}=\begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix}$

$= A =$ R.H.S

Hence proved.

ii) $3B-3A=3(B-A)$

Solution:- L.H.S $=3B-3A$

$=3\begin{bmatrix}1 & 7\\ 6 & 4\end{bmatrix}-3\begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix}$

$=\begin{bmatrix}3 & 21\\ 18 & 12\end{bmatrix}-\begin{bmatrix}6 & 9\\ 3 & 15\end{bmatrix}$

L.H.S $=\begin{bmatrix}3-6 & 21-9\\ 18-3 & 12-15\end{bmatrix}=\begin{bmatrix}-3 & 12\\ 15 & -3\end{bmatrix} \rightarrow$ (I)

R.H.S $=3(B-A)$

$=3\left(\begin{bmatrix}1 & 7\\ 6 & 4\end{bmatrix}-\begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix}\right)$

$=3\begin{bmatrix}1-2 & 7-3\\ 6-1 & 4-5\end{bmatrix}$

R.H.S $=3\begin{bmatrix}-1 & 4\\ 5 & -1\end{bmatrix}=\begin{bmatrix}-3 & 12\\ 15 & -3\end{bmatrix} \rightarrow$ (II)

Hence L.H.S = R.H.S

Q2. If $A=\begin{bmatrix}i & 0\\ 1 & -i\end{bmatrix}$, show that $A^4=I_2$

Solution:-

$\boxed{\textbf{Note:-}\ i=\sqrt{-1} \Rightarrow i^2=-1}$

$\because A^2 = A\times A=\begin{bmatrix}i & 0\\ 1 & -i\end{bmatrix}\times\begin{bmatrix}i & 0\\ 1 & -i\end{bmatrix}$

$A^2=\begin{bmatrix}i^2+0 & 0-0\\ i-i & 0+i^2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$

$\Rightarrow A^4=A^2\times A^2=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\times\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$

$A^4=\begin{bmatrix}1+0 & 0+0\\ 0+0 & 0+1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I_2$

Hence proved

Q3. Find x and y if

i) $\begin{bmatrix}x+3 & 1\\ -3 & 3y-4\end{bmatrix}=\begin{bmatrix}2 & 1\\ -3 & 2\end{bmatrix}$

Solution:- $\begin{bmatrix}x+3 & 1\\ -3 & 3y-4\end{bmatrix}=\begin{bmatrix}2 & 1\\ -3 & 2\end{bmatrix}$

$\Rightarrow x+3=2$, $3y-4=2$

$\Rightarrow x=2-3$, $\Rightarrow 3y=2+4$

$x=-1$, $y=\frac{6}{3}=2$

so $x=-1$ and $y=2$

ii) $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} = \begin{bmatrix} y & 1 \\ -3 & 2x \end{bmatrix}$

Solution:- $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} = \begin{bmatrix} y & 1 \\ -3 & 2x \end{bmatrix}$

$\Rightarrow x+3 = y \longrightarrow (i)$

$3y-4 = 2x$

$\Rightarrow 2x-3y+4=0$

$\Rightarrow 2x-3(x+3)+4=0$ by (i)

$2x-3x-9+4=0$

$-x-5=0 \Rightarrow -x=5$

$\Rightarrow x=-5$ put in (i)

(i) $\Rightarrow -5+3=y \Rightarrow y=-2$

so $x=-5$ and $y=-2$

Q4. If $A=\begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix}$, find the following matrices;

i) $4A-3B$ ii) $A+3(B-A)$

Solution:- i) $4A-3B$

$4A-3B = 4\begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} - 3\begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} -4 & 8 & 12 \\ 4 & 0 & 8 \end{bmatrix} - \begin{bmatrix} 0 & 9 & 6 \\ 3 & -3 & 6 \end{bmatrix}$

$= \begin{bmatrix} -4-0 & 8-9 & 12-6 \\ 4-3 & 0+3 & 8-6 \end{bmatrix}$

$4A-3B = \begin{bmatrix} -4 & -1 & 6 \\ 1 & 3 & 2 \end{bmatrix}$

ii) $A+3(B-A)$

$A+3(B-A) = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + 3\left(\begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}\right)$

$= \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + 3\begin{bmatrix} 0+1 & 3-2 & 2-3 \\ 1-1 & -1-0 & 2-2 \end{bmatrix}$

$= \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 3 & -3 \\ 0 & -3 & 0 \end{bmatrix}$

$= \begin{bmatrix} -1+3 & 2+3 & 3-3 \\ 1+0 & 0-3 & 2+0 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 0 \\ 1 & -3 & 2 \end{bmatrix}$

Q5. Find x and y if

$\begin{bmatrix} 2 & 0 & x \\ 1 & y & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & x & y \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$

Solution:-

$\begin{bmatrix} 2 & 0 & x \\ 1 & y & 3 \end{bmatrix} + \begin{bmatrix} 2 & 2x & 2y \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$

$\begin{bmatrix} 2+2 & 0+2x & x+2y \\ 1+0 & y+4 & 3-2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$

$\begin{bmatrix} 4 & 2x & x+2y \\ 1 & y+4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 6 & 1 \end{bmatrix}$

$\Rightarrow 2x=-2 \Rightarrow x=-1$

and $y+4=6 \Rightarrow y=6-4=2$

so $x=-1$ and $y=2$

Q6. If $A=[a_{ij}]_{3\times3}$, show that

i) $\lambda(\mu A) = (\lambda\mu)A$

Solution:- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

L.H.S $= \lambda(\mu A)$,

$= \lambda\left(\mu\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$

$= \lambda\begin{bmatrix} \mu a_{11} & \mu a_{12} & \mu a_{13} \\ \mu a_{21} & \mu a_{22} & \mu a_{23} \\ \mu a_{31} & \mu a_{32} & \mu a_{33} \end{bmatrix} = \begin{bmatrix} \lambda\mu a_{11} & \lambda\mu a_{12} & \lambda\mu a_{13} \\ \lambda\mu a_{21} & \lambda\mu a_{22} & \lambda\mu a_{23} \\ \lambda\mu a_{31} & \lambda\mu a_{32} & \lambda\mu a_{33} \end{bmatrix}$

$= \lambda\mu\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = (\lambda\mu)A = R.H.S$

Hence proved

ii) $(\lambda+\mu)A = \lambda A + \mu A$

Solution:-

L.H.S $= (\lambda+\mu)A$

$= (\lambda+\mu)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$= \begin{bmatrix} (\lambda+\mu)a_{11} & (\lambda+\mu)a_{12} & (\lambda+\mu)a_{13} \\ (\lambda+\mu)a_{21} & (\lambda+\mu)a_{22} & (\lambda+\mu)a_{23} \\ (\lambda+\mu)a_{31} & (\lambda+\mu)a_{32} & (\lambda+\mu)a_{33} \end{bmatrix}$

$= \begin{bmatrix} \lambda a_{11}+\mu a_{11} & \lambda a_{12}+\mu a_{12} & \lambda a_{13}+\mu a_{13} \\ \lambda a_{21}+\mu a_{21} & \lambda a_{22}+\mu a_{22} & \lambda a_{23}+\mu a_{23} \\ \lambda a_{31}+\mu a_{31} & \lambda a_{32}+\mu a_{32} & \lambda a_{33}+\mu a_{33} \end{bmatrix}$

$= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix} + \begin{bmatrix} \mu a_{11} & \mu a_{12} & \mu a_{13} \\ \mu a_{21} & \mu a_{22} & \mu a_{23} \\ \mu a_{31} & \mu a_{32} & \mu a_{33} \end{bmatrix}$

$= \lambda\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \mu\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$= \lambda A + \mu A = R.H.S$

Hence proved

(iii) $\lambda A - A = (\lambda - 1)A$ see at page #10

Q7. If $A = [a_{ij}]_{2\times 3}$ and $B = [b_{ij}]_{2\times 3}$ show that $\lambda(A+B) = \lambda A + \lambda B$.

Solution:-

$\because A = [a_{ij}]_{2\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

$B = [b_{ij}]_{2\times 3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

$L.H.S = \lambda(A+B)$

$= \lambda\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}\right)$

$= \lambda \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$

$= \begin{bmatrix} \lambda a_{11}+\lambda b_{11} & \lambda a_{12}+\lambda b_{12} & \lambda a_{13}+\lambda b_{13} \\ \lambda a_{21}+\lambda b_{21} & \lambda a_{22}+\lambda b_{22} & \lambda a_{23}+\lambda b_{23} \end{bmatrix}$

$= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \end{bmatrix} + \begin{bmatrix} \lambda b_{11} & \lambda b_{12} & \lambda b_{13} \\ \lambda b_{21} & \lambda b_{22} & \lambda b_{23} \end{bmatrix}$

$= \lambda \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \lambda \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

$= \lambda A + \lambda B = R.H.S$

Hence proved

Q8. If $A = \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$ and $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. find the values of a and b.

Solution:-

$\because A^2 = A \times A = \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$

$A^2 = \begin{bmatrix} 1+2a & 2+2b \\ a+ab & 2a+b^2 \end{bmatrix}$

$\because A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so

$\begin{bmatrix} 1+2a & 2+2b \\ a+ab & 2a+b^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\rightarrow 1+2a = 0$ and $2+2b = 0$

$\rightarrow 2a = -1$ and $2b = -2$

$\rightarrow a = -\frac{1}{2}$ and $b = -1$

Q9. If $A = \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}$ and $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find the values of a and b.

Solution:-

$\because A^2 = A \times A = \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}$

$A^2 = \begin{bmatrix} 1-a & -1-b \\ a+ab & -a+b^2 \end{bmatrix}$

$\because A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so

$\begin{bmatrix} 1-a & -1-b \\ a+ab & -a+b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\rightarrow 1-a = 1$ and $-1-b = 0$

$1-1 = a$ and $-b = 1$

$\rightarrow a = 0$ and $b = -1$

Q10. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$, then show that $(A+B)^t = A^t + B^t$

Solution:-

$L.H.S = (A+B)^t$

$= \left(\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}\right)^t$

$= \begin{bmatrix} 1+2 & -1+3 & 2+0 \\ 0+1 & 3+2 & 1-1 \end{bmatrix}^t$

$= \begin{bmatrix} 3 & 2 & 2 \\ 1 & 5 & 0 \end{bmatrix}^t$

$(A+B)^t = \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 2 & 0 \end{bmatrix}$

$R.H.S = A^t + B^t$

$= \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}^t + \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}^t$

$= \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1+2 & 0+1 \\ -1+3 & 3+2 \\ 2+0 & 1-1 \end{bmatrix}$

$R.H.S = \begin{bmatrix} 3 & 1 \\ 2 & 5 \\ 2 & 0 \end{bmatrix}$

Hence $(A+B)^t = A^t + B^t$

Q11. Find A^3 if $A=\begin{bmatrix}1&1&3\\5&2&6\\-2&-1&-3\end{bmatrix}$

Solution:- $A=\begin{bmatrix}1&1&3\\5&2&6\\-2&-1&-3\end{bmatrix}$

$$A^2 = A\times A=\begin{bmatrix}1&1&3\\5&2&6\\-2&-1&-3\end{bmatrix}\begin{bmatrix}1&1&3\\5&2&6\\-2&-1&-3\end{bmatrix}$$

$$=\begin{bmatrix}1+5-6&1+2-3&3+6-9\\5+10-12&5+4-6&15+12-18\\-2-5+6&-2-2+3&-6-6+9\end{bmatrix}$$

$$A^2=\begin{bmatrix}0&0&0\\3&3&9\\-1&-1&-3\end{bmatrix}$$

$$\therefore A^3 = A^2\times A$$

$$=\begin{bmatrix}0&0&0\\3&3&9\\-1&-1&-3\end{bmatrix}\begin{bmatrix}1&1&3\\5&2&6\\-2&-1&-3\end{bmatrix}$$

$$=\begin{bmatrix}0+0+0&0+0+0&0+0+0\\3+15-18&3+6-9&9+18-27\\-1+5+6&-1-2+3&-3-6+9\end{bmatrix}$$

$$A^3=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O_3$$

Q12. Find the matrix X if;

i) $X\begin{bmatrix}5&2\\-2&1\end{bmatrix}=\begin{bmatrix}-1&5\\12&3\end{bmatrix}$

Solution:-

$$X\begin{bmatrix}5&2\\-2&1\end{bmatrix}=\begin{bmatrix}-1&5\\12&3\end{bmatrix}$$

$$X\ A = B$$

$$\rightarrow X = BA^{-1} \longrightarrow (i)$$

Now $|A|=\begin{vmatrix}5&2\\-2&1\end{vmatrix}=5+4=9\neq 0$

$$\text{adj } A=\begin{bmatrix}1&-2\\2&5\end{bmatrix}$$

$$\because A^{-1}=\frac{\text{adj } A}{|A|}=\frac{1}{9}\begin{bmatrix}1&-2\\2&5\end{bmatrix}$$

so (i) $X=\begin{bmatrix}-1&5\\12&3\end{bmatrix}\cdot\frac{1}{9}\begin{bmatrix}1&-2\\2&5\end{bmatrix}$

$$=\frac{1}{9}\begin{bmatrix}-1&5\\12&3\end{bmatrix}\begin{bmatrix}1&-2\\2&5\end{bmatrix}$$

$$=\frac{1}{9}\begin{bmatrix}-1+10&2+25\\12+6&-24+15\end{bmatrix}$$

$$\rightarrow X=\frac{1}{9}\begin{bmatrix}9&27\\18&-9\end{bmatrix}=\begin{bmatrix}1&3\\2&-1\end{bmatrix}$$

so $X=\begin{bmatrix}1&3\\2&-1\end{bmatrix}$

ii) $\begin{bmatrix}5&2\\-2&1\end{bmatrix}X=\begin{bmatrix}2&1\\5&10\end{bmatrix}$

Solution:-

$$\begin{bmatrix}5&2\\-2&1\end{bmatrix}X=\begin{bmatrix}2&1\\5&10\end{bmatrix}$$

$$A\ X = B$$

$$\rightarrow X = A^{-1}B \longrightarrow (i)$$

$$|A|=\begin{vmatrix}5&2\\-2&1\end{vmatrix}=5+4=9\neq 0$$

$$\text{adj } A=\begin{bmatrix}1&-2\\2&5\end{bmatrix}$$

$$\because A^{-1}=\frac{\text{adj}A}{|A|}=\frac{1}{9}\begin{bmatrix}1&-2\\2&5\end{bmatrix}$$

so (i) $X=\frac{1}{9}\begin{bmatrix}1&-2\\2&5\end{bmatrix}\begin{bmatrix}2&1\\5&10\end{bmatrix}$

$$=\frac{1}{9}\begin{bmatrix}2-10&1-20\\4+25&2+50\end{bmatrix}=\frac{1}{9}\begin{bmatrix}-8&-19\\29&52\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}-8/9&-19/9\\29/9&52/9\end{bmatrix}$$

Q13. Find the matrix A if,

i) $\begin{bmatrix}5&-1\\0&0\\3&1\end{bmatrix}A=\begin{bmatrix}3&-7\\0&0\\7&2\end{bmatrix}$

Solution:-

$$\begin{bmatrix}5&-1\\0&0\\3&1\end{bmatrix}A=\begin{bmatrix}3&-7\\0&0\\7&2\end{bmatrix}$$

suppose $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ then

$$\begin{bmatrix}5&-1\\0&0\\3&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}3&-7\\0&0\\7&2\end{bmatrix}$$

$$\begin{bmatrix}5a-c&5b-d\\0+0&0+0\\3a+c&3b+d\end{bmatrix}=\begin{bmatrix}3&-7\\0&0\\7&2\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}5a-c&5b-d\\0&0\\3a+c&3b+d\end{bmatrix}=\begin{bmatrix}3&-7\\0&0\\7&2\end{bmatrix}$$

$\rightarrow 5a-c=3\rightarrow(i)$ $\quad 5b-d=-7\rightarrow(iii)$

$3a+c=7\rightarrow(ii)$ $\quad 3b+d=2\rightarrow(iv)$

(i)+(ii) $8a=10$ $\qquad$ (iii)+(iv) $8b=-5$

$\rightarrow a=10/8 \therefore \rightarrow \boxed{a=5/4}$

so (i) $5\left(\frac{5}{4}\right) - c = 3$, $\boxed{b = -\frac{5}{8}}$

$c = \frac{25}{4} - 3 = \frac{25-12}{4}$ so (iv) $3\left(-\frac{5}{8}\right) + d = 2$

$\rightarrow \boxed{c = \frac{13}{4}}$

$\rightarrow -\frac{15}{8} + d = 2 \rightarrow d = 2 + \frac{15}{8}$

$\rightarrow d = \frac{16+15}{8} = \frac{31}{8}$

Hence $A = \begin{bmatrix} 5/4 & -5/8 \\ 13/4 & 31/8 \end{bmatrix}$

ii) $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} A = \begin{bmatrix} 0 & -3 & 8 \\ 3 & 3 & -7 \end{bmatrix}$

Solution:-

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} A = \begin{bmatrix} 0 & -3 & 8 \\ 3 & 3 & -7 \end{bmatrix}$$

$$B\,A = C$$

$$\rightarrow A = B^{-1}C \longrightarrow (i)$$

$$|B| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 \neq 0$$

$$\text{adj } B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\because B^{-1} = \frac{\text{adj } B}{|B|} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

so (i) $\rightarrow$ $A = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & -3 & 8 \\ 3 & 3 & -7 \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} 0+3 & -6+3 & 16-7 \\ 0+6 & -3+6 & 8-14 \end{bmatrix}$$

$$A = \frac{1}{3}\begin{bmatrix} 3 & -3 & 9 \\ 6 & 3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix}$$

Q6. iii) $\lambda A - A = (\lambda - 1)A$

* Remaining part of Q6 at page # 7 *

Solution:-

L.H.S $= \lambda A - A$

$$= \lambda\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda a_{11} - a_{11} & \lambda a_{12} - a_{12} & \lambda a_{13} - a_{13} \\ \lambda a_{21} - a_{21} & \lambda a_{22} - a_{22} & \lambda a_{23} - a_{23} \\ \lambda a_{31} - a_{31} & \lambda a_{32} - a_{32} & \lambda a_{33} - a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda-1)a_{11} & (\lambda-1)a_{12} & (\lambda-1)a_{13} \\ (\lambda-1)a_{21} & (\lambda-1)a_{22} & (\lambda-1)a_{23} \\ (\lambda-1)a_{31} & (\lambda-1)a_{32} & (\lambda-1)a_{33} \end{bmatrix}$$

$$= (\lambda-1)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= (\lambda - 1)A = \text{R.H.S}$$

Hence proved

Q14. Show that

$$\begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix} = rI_3$$

Solution:-

L.H.S =

$$\begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix}\begin{bmatrix} r\cos\phi & 0 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} r\cos^2\phi + 0 + r\sin^2\phi & 0+0+0 & r\cos\phi\sin\phi + 0 - r\cos\phi\sin\phi \\ 0+0+0 & 0+r+0 & 0+0+0 \\ r\sin\phi\cos\phi + 0 - r\sin\phi\cos\phi & 0+0+0 & r\sin^2\phi + 0 + r\cos^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} r(\cos^2\phi + \sin^2\phi) & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r(\sin^2\phi + \cos^2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} r(1) & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r(1) \end{bmatrix}$$

$$= \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}$$

$$= r\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = rI_3 = \text{R.H.S}$$

Hence proved

Properties of Matrix Addition, Scalar Multiplication and Matrix Multiplication

If A, B and C are m×n matrices and c and d are scalars, then following properties are true:

1. Commutative property w.r.t addition:-
$$A+B=B+A$$
2. Associative property w.r.t addition:-
$$(A+B)+C=A+(B+C)$$
3. Associative property of scalar Multiplication:-
$$(cd)A=c(dA)$$
4. Existance of additive
$$A+O=O+A=A \quad (O \text{ is null matrix})$$
5. Existance of Multiplicative identity:-
$$IA=AI=A \quad (I \text{ is unit matrix})$$
6. Distributive property w.r.t scalar Multiplication

(a) $c(A+B)=cA+cB$

(b) $(c+d)A=cA+dA$

7. Associative property w.r.t Multiplication:-
$$A(BC)=(AB)C$$
8. Left distributive property:-
$$A(B+C)=AB+AC$$
9. Right distributive property:-
$$(A+B)C=AC+BC$$
10. $c(AB)=(cA)B=A(cB)$

Example 1. Find AB and BA if

$$A=\begin{bmatrix}2&0&1\\1&4&2\\3&0&6\end{bmatrix} \text{ and } B=\begin{bmatrix}1&-1&0\\2&3&-1\\1&-2&3\end{bmatrix}$$

Solution:-

$$AB=\begin{bmatrix}2&0&1\\1&4&2\\3&0&6\end{bmatrix}\begin{bmatrix}1&-1&0\\2&3&-1\\1&-2&3\end{bmatrix}$$

$$=\begin{bmatrix}2+0+1&-2+0-2&0+0+3\\1+8+2&-1+12-4&0-4+6\\3+0+6&-3+0-12&0+0+18\end{bmatrix}$$

$$AB=\begin{bmatrix}3&-4&3\\11&7&2\\9&-15&18\end{bmatrix} \longrightarrow (I)$$

$$BA=\begin{bmatrix}1&-1&0\\2&3&-1\\1&-2&3\end{bmatrix}\begin{bmatrix}2&0&1\\1&4&2\\3&0&6\end{bmatrix}$$

$$=\begin{bmatrix}2-1+0&0-4+0&1-2+0\\4+3-3&0+12+0&2+6-6\\2-2+9&0-8+0&1-4+18\end{bmatrix}$$

$$BA=\begin{bmatrix}1&-4&-1\\4&12&2\\9&-8&15\end{bmatrix} \longrightarrow (II)$$

Also from (I) and (II), $AB \neq BA$

Example 2. If $A=\begin{bmatrix}2&-1&3&0\\1&0&4&-2\\-3&5&2&-1\end{bmatrix}$, then find AA^t and $(A^t)^t$.

Solution:- $A=\begin{bmatrix}2&-1&3&0\\1&0&4&-2\\-3&5&2&-1\end{bmatrix}$

$$A^t=\begin{bmatrix}2&1&-3\\-1&0&5\\3&4&2\\0&-2&-1\end{bmatrix} \quad \text{so}$$

$$AA^t=\begin{bmatrix}2&-1&3&0\\1&0&4&-2\\-3&5&2&-1\end{bmatrix}\begin{bmatrix}2&1&-3\\-1&0&5\\3&4&2\\0&-2&-1\end{bmatrix}$$

$$=\begin{bmatrix}4+1+9+0&2+0+12+0&-6-5+6+0\\2+0+12+0&1+0+16+4&-3+0+8+2\\-6-5+6+0&-3+0+8+2&9+25+4+1\end{bmatrix}$$

$$AA^t=\begin{bmatrix}14&14&-5\\14&21&7\\-5&7&39\end{bmatrix}$$

$$\because A^t=\begin{bmatrix}2&1&-3\\-1&0&5\\3&4&2\\0&-2&-1\end{bmatrix}$$

$$\rightarrow (A^t)^t=\begin{bmatrix}2&-1&3&0\\1&0&4&-2\\-3&5&2&-1\end{bmatrix} \text{ which is } A$$

so $(A^t)^t=A$

Exercise 3.2

Q1. If $A = [a_{ij}]_{3\times 4}$, then show that
i) $I_3 A = A$ ii) $AI_4 = A$

Solution:- i) $I_3 A = A$

L.H.S $= I_3 A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+0+0 & a_{12}+0+0 & a_{13}+0+0 & a_{14}+0+0 \\ 0+a_{21}+0 & 0+a_{22}+0 & 0+a_{23}+0 & 0+a_{24}+0 \\ 0+0+a_{31} & 0+0+a_{32} & 0+0+a_{33} & 0+0+a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = A$$

Hence $I_3 A = A$

ii) $AI_4 = A$

L.H.S $= AI_4$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+0+0+0 & 0+a_{12}+0+0 & 0+0+a_{13}+0 & 0+0+0+a_{14} \\ a_{21}+0+0+0 & 0+a_{22}+0+0 & 0+0+a_{23}+0 & 0+0+0+a_{24} \\ a_{31}+0+0+0 & 0+a_{32}+0+0 & 0+0+a_{33}+0 & 0+0+0+a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = A$$

Hence $AI_4 = A$

Q2. Find the inverses of the following matrices.

i) $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$

Solution:- Let $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$

$A^{-1} = ?$ $\because A^{-1} = \dfrac{\text{adj}A}{|A|} \longrightarrow (i)$

$|A| = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 3+2 = 5$

$\text{adj}\,A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$ so (i) becomes

$A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1/5 & 1/5 \\ -2/5 & 3/5 \end{bmatrix}$

ii) $\begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}$

Solution:- Let $A = \begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}$

$A^{-1} = ?$ $\because A^{-1} = \dfrac{\text{adj}A}{|A|} \longrightarrow (i)$

$|A| = \begin{vmatrix} -2 & 3 \\ -4 & 5 \end{vmatrix} = -10+12 = 2$

$\text{adj}\,A = \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}$ so (i) becomes

$A^{-1} = \frac{1}{2}\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 5/2 & -3/2 \\ 2 & -1 \end{bmatrix}$

iii) $\begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$

Solution:- Let: $A = \begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$

$A^{-1} = ?$ $\because A^{-1} = \dfrac{\text{adj}A}{|A|} \longrightarrow (i)$

$|A| = \begin{vmatrix} 2i & i \\ i & -i \end{vmatrix} = -2i^2 - i^2$ $(\because i^2 = -1)$

$|A| = -2(-1) - (-1) = 2+1 = 3$

$\text{adj}\,A = \begin{bmatrix} -i & -i \\ -i & 2i \end{bmatrix}$ so (i) becomes

$A^{-1} = \frac{1}{3}\begin{bmatrix} -i & -i \\ -i & 2i \end{bmatrix} = \begin{bmatrix} -i/3 & -i/3 \\ -i/3 & 2i/3 \end{bmatrix}$

iv) $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

Solution:- Let $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

$A^{-1} = ?$ $\because A^{-1} = \dfrac{\text{adj}A}{|A|} \longrightarrow (i)$

$|A| = \begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} = 6-6 = 0$

$\because |A| = 0$ so A^{-1} does not exists.

Q3. Solve the following system of linear equations.

i) $2x_1 - 3x_2 = 5$
$5x_1 + x_2 = 4$

Solution:-

In matrix form

$$\begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$AX = B$

$X = A^{-1}B \longrightarrow (i)$

$A = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}$, $|A| = \begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix}$

$|A| = 2+15 = 17 \neq 0$

so A^{-1} exists.

$$\text{adj}\,A=\begin{bmatrix}1 & 3\\ -5 & 2\end{bmatrix}$$

$$\because A^{-1}=\frac{\text{adj}A}{|A|}=\frac{1}{17}\begin{bmatrix}1 & 3\\ -5 & 2\end{bmatrix}$$

so (i) becomes

$$X=\frac{1}{17}\begin{bmatrix}1 & 3\\ -5 & 2\end{bmatrix}\begin{bmatrix}5\\ 4\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\frac{1}{17}\begin{bmatrix}5+12\\ -25+8\end{bmatrix}=\begin{bmatrix}17/17\\ -17/17\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}1\\ -1\end{bmatrix} \text{ so } x_1=1,\ x_2=-1$$

(ii) $4x_1+3x_2=5$
$3x_1-x_2=7$

Solution:-
In matrix form

$$\begin{bmatrix}4 & 3\\ 3 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}5\\ 7\end{bmatrix}$$

$$A\ X = B$$

$$\Rightarrow X=A^{-1}B \longrightarrow (i)$$

$$\because A=\begin{bmatrix}4 & 3\\ 3 & -1\end{bmatrix},\ |A|=\begin{vmatrix}4 & 3\\ 3 & -1\end{vmatrix}$$

$$|A|=-4-9=-13\neq 0$$

so A^{-1} exists.

$$\text{adj}\,A=\begin{bmatrix}-1 & -3\\ -3 & 4\end{bmatrix}$$

$$\because A^{-1}=\frac{\text{adj}A}{|A|}=\frac{1}{-13}\begin{bmatrix}-1 & -3\\ -3 & 4\end{bmatrix}$$

so (i) becomes

$$X=-\frac{1}{13}\begin{bmatrix}-1 & -3\\ -3 & 4\end{bmatrix}\begin{bmatrix}5\\ 7\end{bmatrix}$$

$$=-\frac{1}{13}\begin{bmatrix}-5-21\\ -15+28\end{bmatrix}=-\frac{1}{13}\begin{bmatrix}-26\\ 13\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}-26/-13\\ -13/13\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}2\\ -1\end{bmatrix}$$

so $x_1=2,\ x_2=-1$

(iii) $3x-5y=1$
$-2x+y=-3$

Solution:-
In matrix form

$$\begin{bmatrix}3 & -5\\ -2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1\\ -3\end{bmatrix}$$

$$A\ X = B$$

$$\Rightarrow X=A^{-1}B \longrightarrow (i)$$

$$\because A=\begin{bmatrix}3 & -5\\ -2 & 1\end{bmatrix},\ |A|=\begin{vmatrix}3 & -5\\ -2 & 1\end{vmatrix}$$

$$|A|=3-10=-7\neq 0$$

so A^{-1} exists.

$$\text{adj}\,A=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}$$

$$\because A^{-1}=\frac{\text{adj}A}{|A|}=-\frac{1}{7}\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}$$

so (i) becomes

$$X=-\frac{1}{7}\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\begin{bmatrix}1\\ -3\end{bmatrix}$$

$$=-\frac{1}{7}\begin{bmatrix}1-15\\ 2-9\end{bmatrix}=-\frac{1}{7}\begin{bmatrix}-14\\ -7\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-14/-7\\ -7/-7\end{bmatrix}=\begin{bmatrix}2\\ 1\end{bmatrix}$$

$$\Rightarrow x=2,\ y=1$$

Q4. If $A=\begin{bmatrix}1 & -1 & 2\\ 3 & 2 & 5\\ -1 & 0 & 4\end{bmatrix}$, $B=\begin{bmatrix}2 & 1 & -1\\ 1 & 3 & 4\\ -1 & 2 & 1\end{bmatrix}$ and $C=\begin{bmatrix}1 & 3 & -2\\ -1 & 2 & 0\\ 3 & 4 & -1\end{bmatrix}$, then find

i) $A-B$ ii) $B-A$ iii) $(A-B)-C$
iv) $A-(B-C)$

Solution:- i) $A-B$

$$A-B=\begin{bmatrix}1 & -1 & 2\\ 3 & 2 & 5\\ -1 & 0 & 4\end{bmatrix}-\begin{bmatrix}2 & 1 & -1\\ 1 & 3 & 4\\ -1 & 2 & 1\end{bmatrix}$$

$$A-B=\begin{bmatrix}1-2 & -1-1 & 2+1\\ 3-1 & 2-3 & 5-4\\ -1+1 & 0-2 & 4-1\end{bmatrix}=\begin{bmatrix}-1 & -2 & 3\\ 2 & -1 & 1\\ 0 & -2 & 3\end{bmatrix}$$

ii) $B-A$

$$B-A=\begin{bmatrix}2 & 1 & -1\\ 1 & 3 & 4\\ -1 & 2 & 1\end{bmatrix}-\begin{bmatrix}1 & -1 & 2\\ 3 & 2 & 5\\ -1 & 0 & 4\end{bmatrix}$$

$$B-A=\begin{bmatrix}2-1 & 1+1 & -1-2\\ 1-3 & 3-2 & 4-5\\ -1+1 & 2-0 & 1-4\end{bmatrix}=\begin{bmatrix}1 & 2 & -3\\ -2 & 1 & -1\\ 0 & 2 & -3\end{bmatrix}$$

iii) $(A-B)-C$

$$(A-B)-C=\left(\begin{bmatrix}1&-1&2\\3&2&5\\-1&0&4\end{bmatrix}-\begin{bmatrix}2&1&-1\\1&3&4\\-1&2&1\end{bmatrix}\right)-C$$

$$=\begin{bmatrix}1-2&-1-1&2+1\\3-1&2-3&5-4\\-1+1&0-2&4-1\end{bmatrix}-C$$

$$=\begin{bmatrix}-1&-2&3\\2&-1&1\\0&-2&3\end{bmatrix}-\begin{bmatrix}1&3&-2\\-1&2&0\\3&4&-1\end{bmatrix}$$

$$=\begin{bmatrix}-1-1&-2-3&3+2\\2+1&-1-2&1-0\\0-3&-2-4&3+1\end{bmatrix}$$

$$=\begin{bmatrix}-2&-5&5\\3&-3&1\\-3&-6&4\end{bmatrix}$$

iv) $A-(B-C)$

$$A-(B-C)=A-\left(\begin{bmatrix}2&1&-1\\1&3&4\\-1&2&1\end{bmatrix}-\begin{bmatrix}1&3&-2\\-1&2&0\\3&4&-1\end{bmatrix}\right)$$

$$=A-\begin{bmatrix}2-1&1-3&-1+2\\1+1&3-2&4-0\\-1-3&2-4&1+1\end{bmatrix}$$

$$=A-\begin{bmatrix}1&-2&1\\2&1&4\\-4&-2&2\end{bmatrix}$$

$$=\begin{bmatrix}1&-1&2\\3&2&5\\-1&0&4\end{bmatrix}-\begin{bmatrix}1&-2&1\\2&1&4\\-4&-2&2\end{bmatrix}$$

$$=\begin{bmatrix}1-1&-1+2&2-1\\3-2&2-1&5-4\\-1+4&0+2&4-2\end{bmatrix}$$

$$=\begin{bmatrix}0&1&1\\1&1&1\\3&&2\end{bmatrix}$$

Q5.

If $A=\begin{bmatrix}i&2i\\1&-i\end{bmatrix}$, $B=\begin{bmatrix}-i&1\\2i&i\end{bmatrix}$ and $C=\begin{bmatrix}2i&-1\\-i&i\end{bmatrix}$, then show that

i) $(AB)C=A(BC)$

ii) $(A+B)C=AC+BC$

Solution:- i) $(AB)C=A(BC)$

$L.H.S=(AB)C$

$$=\left(\begin{bmatrix}i&2i\\1&-i\end{bmatrix}\begin{bmatrix}-i&1\\2i&i\end{bmatrix}\right)C$$

$$=\begin{bmatrix}-i^2+4i^2&i+2i^2\\-i-2i^2&1-i^2\end{bmatrix}C$$

$$=\begin{bmatrix}3i^2&i+2(-1)\\-i-2i^2&1-i^2\end{bmatrix}C$$

$$=\begin{bmatrix}3(-1)&i-2\\-i-2(-1)&1-(-1)\end{bmatrix}C$$

$$=\begin{bmatrix}-3&i-2\\-i+2&2\end{bmatrix}\begin{bmatrix}2i&-1\\-i&i\end{bmatrix}$$

$$=\begin{bmatrix}-6i-i^2+2i&3+i^2-2i\\4i-2i^2-2i&-2+i+2i\end{bmatrix}$$

$$=\begin{bmatrix}-4i-i^2&3+(-1)-2i\\2i-2i^2&-2+3i\end{bmatrix}$$

$$=\begin{bmatrix}-4i-(-1)&3-1-2i\\2i-2(-1)&-2+3i\end{bmatrix}$$

$$\Rightarrow L.H.S=\begin{bmatrix}-4i+1&2-2i\\2i+2&-2+3i\end{bmatrix}\longrightarrow(I)$$

$R.H.S=A(BC)$

$$=A\left(\begin{bmatrix}-i&1\\2i&i\end{bmatrix}\begin{bmatrix}2i&-1\\-i&i\end{bmatrix}\right)$$

$$=A\begin{bmatrix}-2i^2-i&i+i\\4i^2-i^2&-2i+i^2\end{bmatrix}$$

$$=A\begin{bmatrix}-2(-1)-i&2i\\3i^2&-2i+(-1)\end{bmatrix}$$

$$=A\begin{bmatrix}2-i&2i\\-3&-1-2i\end{bmatrix}$$

$$=\begin{bmatrix}i&2i\\1&-i\end{bmatrix}\begin{bmatrix}2-i&2i\\-3&-1-2i\end{bmatrix}$$

$$=\begin{bmatrix}2i-i^2-6i&2i^2-2i-4i^2\\2-i+3i&2i+2i^2+i\end{bmatrix}$$

$$=\begin{bmatrix}-4i-(-1)&-2i^2-2i\\2+2i&3i+2i^2\end{bmatrix}$$

$$=\begin{bmatrix}-4i+1&-2(-1)-2i\\2+2i&3i+2(-1)\end{bmatrix}$$

$$R.H.S = \begin{bmatrix} 1-4i & 2-2i \\ 2+2i & -2+3i \end{bmatrix} \longrightarrow (II)$$

By (I) and (II)

L.H.S = R.H.S

ii) $(A+B)C = AC + BC$

$$L.H.S = (A+B)C$$

$$= \left(\begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} + \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix} \right) C$$

$$= \begin{bmatrix} i-i & 2i+1 \\ 1+2i & -i+i \end{bmatrix} C$$

$$= \begin{bmatrix} 0 & 1+2i \\ 1+2i & 0 \end{bmatrix} \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} 0-i-2i^2 & 0+i+2i^2 \\ 2i+4i^2+0 & -1-2i+0 \end{bmatrix}$$

$$L.H.S = \begin{bmatrix} -i+2 & i-2 \\ 2i-4 & -1-2i \end{bmatrix} \longrightarrow (I)$$

$$R.H.S = AC + BC$$

$$= \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix} + \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix} \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} 2i^2-2i^2 & -i+2i^2 \\ 2i+i^2 & -1-i^2 \end{bmatrix} + \begin{bmatrix} -2i^2-i & i+i \\ 4i^2-i^2 & -2i+i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -i-2 \\ 2i-1 & 0 \end{bmatrix} + \begin{bmatrix} 2-i & 2i \\ -3 & -2i-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+2-i & -i-2+2i \\ 2i-1-3 & 0-2i-1 \end{bmatrix}$$

$$R.H.S = \begin{bmatrix} -i+2 & i-2 \\ 2i-4 & -1-2i \end{bmatrix} \longrightarrow (II)$$

From (I) and II

L.H.S = R.H.S

Q6. If A and B are square matrices of same order, then explain why in general;

i) $(A+B)^2 \neq A^2 + 2AB + B^2$

Solution:-

$$L.H.S = (A+B)^2$$

$$= (A+B)(A+B)$$

$$= A^2 + AB + BA + B^2$$

$\because AB \neq BA$ in general so

$$AB + BA \neq 2AB$$

Now

$$L.H.S \neq A^2 + 2AB + B^2 = R.H.S$$

Hence $(A+B)^2 \neq A^2 + 2AB + B^2$

ii) $(A-B)^2 \neq A^2 - 2AB + B^2$

Solution:-

$$L.H.S = (A-B)^2$$

$$= (A-B)(A-B)$$

$$= A^2 - AB - BA + B^2$$

$\because AB \neq BA$ in general so

$$-AB - BA \neq -2AB$$

Now

$$L.H.S \neq A^2 - 2AB + B^2 = R.H.S$$

Hence $(A-B)^2 \neq A^2 - 2AB + B^2$

iii) $(A+B)(A-B) \neq A^2 - B^2$

Solution:-

$$L.H.S = (A+B)(A-B)$$

$$= A^2 - AB + BA - B^2$$

$\because AB \neq BA$ in general so

$$-AB + BA \neq 0$$

Now

$$L.H.S \neq A^2 - B^2 = R.H.S$$

Hence $(A+B)(A-B) = A^2 - B^2$

Q7. If $A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$, then find AA^t and A^tA.

Solution:-

$$AA^t = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}^t$$

$$= \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+9+0 & 2-0+12+0 & -6-5+6-0 \\ 2-0+12+0 & 1+0+16+4 & -3+0+8+2 \\ -6-5+6-0 & -3+0+8+2 & 9+25+4+1 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 14 & 14 & -5 \\ 14 & 21 & 7 \\ -5 & 7 & 39 \end{bmatrix}$$

Also

$$A^tA = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}^t\begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+9 & -2+0-15 & 6+4-6 & 0-2+3 \\ -2+0-15 & 1+0+25 & 1+0+25 & 0-0-5 \\ 6+4-6 & -3+0+10 & -3+0+10 & 0-8-2 \\ 0-2+3 & 0-0-5 & 0-8-2 & 0+4+1 \end{bmatrix}$$

$$A^tA = \begin{bmatrix} 14 & -17 & 4 & 1 \\ -17 & 26 & 7 & -5 \\ 4 & 7 & 29 & -10 \\ 1 & -5 & -10 & 5 \end{bmatrix}$$

Q8. Solve the following matrix equations for X:

i) $3X - 2A = B$ if $A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$

Solution:-

$$3X - 2A = B$$

$$\rightarrow 3X - 2\begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$$

$$\rightarrow 3X - \begin{bmatrix} 4 & 6 & -4 \\ -2 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$$

$$\rightarrow 3X = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 6 & -4 \\ -2 & 2 & 10 \end{bmatrix}$$

$$3X = \begin{bmatrix} 2+4 & -3+6 & 1-4 \\ 5-2 & 4+2 & -1+10 \end{bmatrix}$$

$$\rightarrow 3X = \begin{bmatrix} 6 & 3 & -3 \\ 3 & 6 & 9 \end{bmatrix}$$

$$X = \frac{1}{3}\begin{bmatrix} 6 & 3 & -3 \\ 3 & 6 & 9 \end{bmatrix}$$

$$X = \begin{bmatrix} 6/3 & 3/3 & -3/3 \\ 3/3 & 6/3 & 9/3 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

ii) $2X - 3A = B$ if $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$

Solution:-

$$2X - 3A = B$$

$$2X - 3\begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

$$2X - \begin{bmatrix} 3 & -3 & 6 \\ -6 & 12 & 15 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

$$\rightarrow 2X = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -3 & 6 \\ -6 & 12 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 3+3 & -1-3 & 0+6 \\ 4-6 & 2+12 & 1+15 \end{bmatrix}$$

$$2X = \begin{bmatrix} 6 & -4 & 6 \\ -2 & 14 & 16 \end{bmatrix}$$

$$\rightarrow X = \frac{1}{2}\begin{bmatrix} 6 & -4 & 6 \\ -2 & 14 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 6/2 & -4/2 & 6/2 \\ -2/2 & 14/2 & 16/2 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & -2 & 3 \\ -1 & 7 & 8 \end{bmatrix}$$

Q9. Solve the following matrix equations for A:

i) $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}A - \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix}$

ii) $A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$

Solution:-

$$\begin{bmatrix}4&3\\2&2\end{bmatrix}A-\begin{bmatrix}2&3\\-1&-2\end{bmatrix}=\begin{bmatrix}-1&-4\\3&6\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}4&3\\2&2\end{bmatrix}A=\begin{bmatrix}-1&-4\\3&6\end{bmatrix}+\begin{bmatrix}2&3\\-1&-2\end{bmatrix}$$

$$\begin{bmatrix}4&3\\2&2\end{bmatrix}A=\begin{bmatrix}-1+2&-4+3\\3-1&6-2\end{bmatrix}$$

$$\begin{bmatrix}4&3\\2&2\end{bmatrix}A=\begin{bmatrix}1&-1\\2&4\end{bmatrix}$$

$$\rightarrow BA = C$$

$$\rightarrow A = B^{-1}C \longrightarrow (i)$$

Here $B=\begin{bmatrix}4&3\\2&2\end{bmatrix}$, $C=\begin{bmatrix}1&-1\\2&4\end{bmatrix}$

$$|B|=\begin{vmatrix}4&3\\2&2\end{vmatrix}$$

$|B| = 8-6 = 2 \neq 0$ so B^{-1} exists

$$\text{adj } B=\begin{bmatrix}2&-3\\-2&4\end{bmatrix}$$

$$\because B^{-1}=\frac{\text{adj } B}{|B|}=\frac{1}{2}\begin{bmatrix}2&-3\\-2&4\end{bmatrix}$$

so (i) becomes

$$A=\frac{1}{2}\begin{bmatrix}2&-3\\-2&4\end{bmatrix}\begin{bmatrix}1&-1\\2&4\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}2-6&-2-12\\-2+8&2+16\end{bmatrix}$$

$$A=\frac{1}{2}\begin{bmatrix}-4&-14\\6&18\end{bmatrix}=\begin{bmatrix}-4/2&\frac{-14}{2}\\6/2&18/2\end{bmatrix}$$

$$\rightarrow A=\begin{bmatrix}-2&-7\\3&9\end{bmatrix}$$

ii) $A\begin{bmatrix}3&1\\4&2\end{bmatrix}-\begin{bmatrix}-1&2\\3&1\end{bmatrix}=\begin{bmatrix}2&0\\-1&5\end{bmatrix}$

$$\rightarrow A\begin{bmatrix}3&1\\4&2\end{bmatrix}=\begin{bmatrix}2&0\\-1&5\end{bmatrix}+\begin{bmatrix}-1&2\\3&1\end{bmatrix}$$

$$=\begin{bmatrix}2-1&0+2\\-1+3&5+1\end{bmatrix}$$

$$A\begin{bmatrix}3&1\\4&2\end{bmatrix}=\begin{bmatrix}1&2\\2&6\end{bmatrix}$$

$$\rightarrow AB = C$$

$$\rightarrow A = CB^{-1} \longrightarrow (i)$$

Here $B=\begin{bmatrix}3&1\\4&2\end{bmatrix}$, $C=\begin{bmatrix}1&2\\2&6\end{bmatrix}$

$$|B|=\begin{vmatrix}3&1\\4&2\end{vmatrix}=6-4=2\neq 0$$

so B^{-1} exists.

$$\text{adj } B=\begin{bmatrix}2&-1\\-4&3\end{bmatrix}$$

$$\because B^{-1}=\frac{\text{adj } B}{|B|}=\frac{1}{2}\begin{bmatrix}2&-1\\-4&3\end{bmatrix}$$

so (i) becomes

$$A=\begin{bmatrix}1&2\\2&6\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}2&-1\\-4&3\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}1&2\\2&6\end{bmatrix}\begin{bmatrix}2&-1\\-4&3\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}2-8&-1+6\\4-24&-2+18\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}-6&5\\-20&16\end{bmatrix}$$

$$\rightarrow A=\begin{bmatrix}-6/2&5/2\\-20/2&16/2\end{bmatrix}=\begin{bmatrix}-3&5/2\\-10&8\end{bmatrix}$$

Determinants

Minor of an element:-

Let $A=[a_{ij}]$ be a matrix of order $n\times n$. If we delete the ith row and jth column of A, then we get a $(n-1)\times(n-1)$ matrix. The determinant of the $(n-1)\times(n-1)$ matrix is called minor of the element a_{ij} denoted by M_{ij}. For example

$$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$$

$$\text{Minor of } a_{11}=M_{11}=\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}$$

$$\text{Minor of } a_{22}=M_{22}=\begin{vmatrix}a_{11}&a_{13}\\a_{31}&a_{33}\end{vmatrix} \text{ etc}$$

Let $A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$

$$\text{Minor of } 2=\begin{vmatrix}4&6\\7&9\end{vmatrix} \quad \because \begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix}$$

$$\text{Minor of } 8=\begin{vmatrix}1&3\\4&6\end{vmatrix} \quad \because \begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix}$$

etc

Cofactor of an element:-

Let $A = [a_{ij}]$ be a square matrix. then cofactor of a_{ij} is denoted by A_{ij} and defined as

$$A_{ij} = (-1)^{i+j} M_{ij}$$

For example,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A_{11} = \text{cofactor of } a_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$A_{12} = \text{cofactor of } a_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

etc

Determinant of a square matrix of order $n \geq 3$:

The determinant of a square matrix of order n is the sum of products of each element of a row or column by its cofactor.

For example, If A is matrix of order 3x3:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \text{ then}$$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

(by expanding Row first)

or

$$|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$$

(by expanding Column first)

If A is a matrix of order n i-e.,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1j} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2j} \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \\ a_{i1} & a_{i2} & a_{i3} \cdots & a_{ij} \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \\ a_{n1} & a_{n2} & a_{n3} & a_{nj} \cdots & a_{nn} \end{bmatrix}$$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + \cdots + a_{1j}A_{1j} + \cdots + a_{1n}A_{1n}$$

(by expanding row first)

or

$$|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} + \cdots + a_{i1}A_{i1} + \cdots a_{n1}A_{n1}$$

(by expanding column first)

Example 1. Evaluate the determinant of $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$

Solution:-

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{vmatrix}$$

Expanding by R_1

$$|A| = 1\begin{vmatrix} 3 & 1 \\ -3 & 2 \end{vmatrix} - (-2)\begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} -2 & 3 \\ 4 & -3 \end{vmatrix}$$

$$= 1(6+3) + 2(-4-4) + 3(6-12)$$

$$= 1(9) + 2(-8) + 3(-6)$$

$$|A| = 9 - 16 - 18 = -25$$

Example 2. Find the cofactors A_{12}, A_{22} and A_{32} if $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$ also find $|A|$.

Solution:-

$$\because A_{12} = (-1)^{1+2} M_{12} = -M_{12}$$

$$\rightarrow A_{12} = -\begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} = -(-4-4) = 8$$

$$A_{22} = (-1)^{2+2} M_{22} = M_{22}$$

$$\rightarrow A_{22} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 2 - 12 = -10$$

$$A_{32} = (-1)^{3+2} M_{32} = -M_{32}$$

$$= -\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = -(1+6) = -7$$

Now we find $|A|$.

Since $|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$

$$= (-2)(8) + 3(-10) + (-3)(-7)$$

$$\rightarrow |A| = -16 - 30 + 21 = -25$$

Properties of Determinants which Help their Evaluation

1. For a square matrix

$$|A| = |A^t|$$

2. If in a square matrix A, two rows or two columns are interchanged, the determinant of resulting matrix is $-|A|$.

3. If a square matrix A has two identical rows (or columns) then $|A| = 0$

4. If all the entries of a row or column of a square matrix A are zero, then $|A| = 0$

5. If the entries of row or a column in a square matrix A are multiplied by a number $K \in \mathbb{R}$, then the determinant of the resulting matrix is $K|A|$.

6. If each entry of a row or a column of a square matrix consists of two terms then its determinant can be written as the sum of two determinants.

7. If to each entry of a row (or a column) of a square matrix A is added a non-zero multiple of the corresponding entry of another row (or column), then the determinant of the resulting matrix is $|A|$.

8. If a matrix is in triangular form, then the value of its determinant is the product of the entries on its main diagonal.

Examples of above mentioned properties of determinants

1. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $A^t = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

$|A^t| = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

$\Rightarrow |A| = |A^t|$

2. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Now interchanging R_1 and R_2

$\begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix} = a_{21}a_{12} - a_{11}a_{22}$

$= -(a_{11}a_{22} - a_{21}a_{12})$

$= -|A|$

3. Let $A = \begin{vmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$|A| = 0\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - 0\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + 0\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$|A| = 0$

4. Let $A = \begin{bmatrix} a & b & c \\ a & b & c \\ x & y & z \end{bmatrix}$

$|A| = a\begin{vmatrix} b & c \\ y & z \end{vmatrix} - b\begin{vmatrix} a & c \\ x & z \end{vmatrix} + c\begin{vmatrix} a & b \\ x & y \end{vmatrix}$

$= a(bz - cy) - b(az - cx) + c(ay - bx)$

$= abz - acy - baz + bcx + acy - bcx$

$\Rightarrow |A| = 0$

5. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$|A| = a_{11}a_{22} - a_{12}a_{21}$

Now 'x' R_1 by K

$\begin{vmatrix} Ka_{11} & Ka_{12} \\ a_{21} & a_{22} \end{vmatrix} = Ka_{11}a_{22} - Ka_{21}a_{12}$

$= K(a_{11}a_{22} - a_{21}a_{12})$

$= K|A|$

6. $\begin{vmatrix} a_{11}+b_{11} & a_{12} \\ a_{21}+b_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix}$

L.H.S $= \begin{vmatrix} a_{11}+b_{11} & a_{12} \\ a_{21}+b_{21} & a_{22} \end{vmatrix}$

$= a_{11}a_{22} + b_{11}a_{22} - a_{12}a_{21} - a_{12}b_{21}$

$= a_{11}a_{22} - a_{12}a_{21} + b_{11}a_{22} - a_{12}b_{21}$

$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix}$

$=$ R.H.S

7. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ ka+c & kb+d \end{vmatrix}$

$L.H.S = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$= ad - bc$

$R.H.S = \begin{vmatrix} a & b \\ ka+c & kb+d \end{vmatrix}$

$= a\cancel{k}b + ad - b\cancel{k}a - bc$

$= ad - bc$

$\rightarrow L.H.S = R.H.S$

8. Let $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

$\rightarrow |A| = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix}$

$|A| = (a_{11})(a_{22})(a_{33}) = a_{11}a_{22}a_{33}$

Example 3. If $A = \begin{bmatrix} 2 & -2 & 3 & 4 \\ 3 & 1 & 5 & -1 \\ -5 & -3 & 1 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix}$, evaluate $|A|$

Solution:-

$|A| = \begin{vmatrix} 2 & -2 & 3 & 4 \\ 3 & 1 & 5 & -1 \\ -5 & -3 & 1 & 0 \\ 1 & -1 & 0 & 2 \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 & 3 & 0 \\ 0 & 4 & 5 & -7 \\ 0 & -8 & 1 & 10 \\ 1 & -1 & 0 & 2 \end{vmatrix}$ By $R_1 - 2R_4$, $R_2 - 3R_4$, $R_3 + 5R_4$

Expanding by C_1

$= 0 - 0 + 0 - 1\begin{vmatrix} 0 & 3 & 0 \\ 4 & 5 & -7 \\ -8 & 1 & 10 \end{vmatrix}$

$= -\begin{vmatrix} 0 & 3 & 0 \\ 4 & 5 & -7 \\ -8 & 1 & 10 \end{vmatrix}$

$= -\left\{0 - 3\begin{vmatrix} 4 & -7 \\ -8 & 10 \end{vmatrix} + 0\right\}$ By expanding R_1

$= -\{-3(40 - 56)\}$

$|A| = 3(-16) = -48$

Example 4. Without expansion, show that $\begin{vmatrix} x & a+x & b+c \\ x & b+x & c+a \\ x & c+x & a+b \end{vmatrix} = 0$

Solution:-

$L.H.S = \begin{vmatrix} x & a+x & b+c \\ x & b+x & c+a \\ x & c+x & a+b \end{vmatrix}$

$= \begin{vmatrix} x & a+x-x & b+c \\ x & b+x-x & c+a \\ x & c+x-x & a+b \end{vmatrix}$ $C_2 - C_1$

$= \begin{vmatrix} x & a & b+c \\ x & b & c+a \\ x & c & a+b \end{vmatrix}$

$= \begin{vmatrix} x & a+b+c & b+c \\ x & a+b+c & c+a \\ x & a+b+c & a+b \end{vmatrix}$ $C_2 + C_3$

Take x common from C_1 and $(a+b+c)$ as common from C_2

$= x(a+b+c)\begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}$

$= x(a+b+c)(0)$ ($\because C_1$ and C_2 are identical)

$= 0 = R.H.S$

Hence proved

Example 5. Solve the equation $\begin{vmatrix} x & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & -2 & 3 & 4 \\ -2 & x & 1 & -1 \end{vmatrix} = 0$

Solution:-

$\begin{vmatrix} x & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & -2 & 3 & 4 \\ -2 & x & 1 & -1 \end{vmatrix} = 0$

$C_3 + C_2$, $C_4 + C_2$

$\begin{vmatrix} x & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 2 \\ -2 & x & 1+x & -1+x \end{vmatrix} = 0$

Expanding by R_2

$-0 + 1\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 2 \\ -2 & 1+x & -1+x \end{vmatrix} - 0 + 0 = 0$

$C_2 - C_1$, $C_3 - 2C_1$

$$\begin{vmatrix} x & 1-x & 1-2x \\ 1 & 0 & 0 \\ -2 & 3+x & 3+x \end{vmatrix} = 0$$

Expanding by R_2

$$-1\begin{vmatrix} 1-x & 1-2x \\ 3+x & 3+x \end{vmatrix} - 0 - 0 = 0$$

$$\begin{vmatrix} 1-x & 1-2x \\ 3+x & 3+x \end{vmatrix} = 0$$

$\Rightarrow (3+x)(1-x) - (3+x)(1-2x) = 0$

$(3+x)\{(1-x) - (1-2x)\} = 0$

$\Rightarrow (3+x)(1-x-1+2x) = 0$

$(3+x)(x) = 0$

$\Rightarrow x+3 = 0,\quad x = 0$

or $x = -3,\quad x = 0$

$S.S = \{-3, 0\}$

Adjoint of a Square Matrix of order $n \geq 3$

If $A = [a_{ij}]$ be a square matrix of order n, then $[A_{ij}]$ is matrix of cofactors, adjoint of A is denoted by adj A and defined as

$$\text{adj} A = [A_{ij}]^t$$

For example,

if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\text{adj} A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

$$\Rightarrow \text{adj} A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Inverse of a Square matrix of order $n \geq 3$

If A is non-singular matrix of order n then its inverse is denoted by A^{-1} and defined as,

$$A^{-1} = \frac{1}{|A|}\text{adj} A$$

Example 6. Find A^{-1} if

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Solution:-

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = 1(2+1) = 3$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1(0-1) = 1$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} = 1(0-2) = -2$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = -1(0+2) = -2$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1(1-2) = -1$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = -1(-1-0) = 1$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = 1(0-4) = -4$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1-0) = -1$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 1(2-0) = 2$

so matrix of cofactor $= \begin{bmatrix} 3 & 1 & 2 \\ -2 & -1 & 1 \\ -4 & -1 & 2 \end{bmatrix}$

Now $|A| = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix}$ $R_3 - R_1$

$= 1\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} - 0 + 0$ Expanding by C_1

$|A| = 1(-2+1) = -1$

$$\text{adj } A = \begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$\because A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$A^{-1} = \frac{1}{-1} \begin{vmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{vmatrix}$$

$$\rightarrow A^{-1} = \begin{bmatrix} -3 & 2 & 4 \\ -1 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix}$$

Example 7. If $A = \begin{bmatrix} -1 & 2 \\ 1 & 4 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$, then verify that

$$(AB)^t = B^t A^t$$

Solution:-

$$\text{L.H.S} = (AB)^t$$

$$= \left(\begin{bmatrix} -1 & 2 \\ 1 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \right)^t$$

$$= \begin{bmatrix} -1-4 & -3+2 \\ 1-8 & 3+4 \\ 2+2 & 6-1 \end{bmatrix}^t$$

$$= \begin{bmatrix} -5 & -1 \\ -7 & 7 \\ 4 & 5 \end{bmatrix}^t$$

$$\text{L.H.S} = \begin{bmatrix} -5 & -7 & 4 \\ -1 & 7 & 5 \end{bmatrix}$$

$$\text{R.H.S} = B^t A^t$$

$$= \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}^t \begin{bmatrix} -1 & 2 \\ 1 & 4 \\ 2 & -1 \end{bmatrix}^t$$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 2 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1-4 & 1-8 & 2+2 \\ -3+2 & 3+4 & 6-1 \end{bmatrix}$$

$$\text{R.H.S} = \begin{bmatrix} -5 & -7 & 4 \\ -1 & 7 & 5 \end{bmatrix}$$

Hence L.H.S = R.H.S

Exercise 3.3

Q1. Evaluate the following determinants.

(i) $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$

Solution:- $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$ Expanding by R_1

$$= 5\begin{vmatrix} -1 & -3 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 3 & -3 \\ -2 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$$

$$= 5(-2+3) + 2(6-6) - 4(3-2)$$

$$= 5(1) + 2(0) - 4(1) = 5 + 0 - 4 = 1$$

(ii) $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$

Solution:- $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$ Expanding by R_1

$$= 5\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -2 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$$

$$= 5(2-1) - 2(-6+2) - 3(3-2)$$

$$= 5(1) - 2(-4) - 3(1) = 5 + 8 - 3 = 10$$

(iii) $\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$

Solution:- $\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$ Expanding by R_1

$$= 1\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix} + (-3)\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix}$$

$$= 1(18-20) - 2(-6+8) - 3(-5+6)$$

$$= -2 - 2(2) - 3(1) = -2 - 4 - 3 = -9$$

(iv) $\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$

Solution:- $\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$

Expanding by R_1

$$= (a+l)\begin{vmatrix} a+l & a-l \\ a & a+l \end{vmatrix} - (a-l)\begin{vmatrix} a & a-l \\ a-l & a+l \end{vmatrix} + a\begin{vmatrix} a & a-l \\ a-l & a \end{vmatrix}$$

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$$= (a+l)\left[(a+l)^2 - a(a-l)\right] - (a-l)\left[a(a+l) - (a-l)^2\right] + a\left[a^2 - (a-l)(a+l)\right]$$

$$= (a+l)\left[a^2+l^2+2al-a^2+al\right] - (a-l)\left[a^2+al-a^2-l^2+2al\right] + a\left[a^2-a^2+l^2\right]$$

$$= (a+l)(l^2+3al) - (a-l)(3al-l^2) + al^2$$

$$= al^2+3a^2l+l^3+3al^2-(3a^2l-al^2-3al^2+l^3)+al^2$$

$$= al^2+\cancel{3a^2l}+\cancel{l^3}+3al^2-\cancel{3a^2l}+al^2+3al^2-\cancel{l^3}+al^2$$

$$= 9al^2$$

(v) $\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}$

Solution:- $\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}$ Expanding by R_1

$$= 1\begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix} - 2\begin{vmatrix} -1 & -3 \\ 2 & -1 \end{vmatrix} + (-2)\begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix}$$

$$= 1(-1+12) - 2(1+6) - 2(-4-2)$$

$$= 1(11) - 2(7) - 2(-6)$$

$$= 11 - 14 + 12 = 9$$

(vi) $\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$

Solution:- $\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$

Expanding by R_1

$$= 2a\begin{vmatrix} 2b & b \\ c & 2c \end{vmatrix} - a\begin{vmatrix} b & b \\ c & 2c \end{vmatrix} + a\begin{vmatrix} b & 2b \\ c & c \end{vmatrix}$$

$$= 2a(4bc-bc) - a(2bc-bc) + a(bc-2bc)$$

$$= 2a(3bc) - a(bc) + a(-2bc)$$

$$= 6abc - abc - 2abc = 4abc$$

Q2. Without expansion show that

(i) $\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$

Solution:-

$$L.H.S = \begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 7-6 & 8-7 \\ 3 & 4-3 & 5-4 \\ 2 & 3-2 & 4-3 \end{vmatrix} \quad \begin{matrix} C_2 - C_1 \\ C_3 - C_2 \end{matrix}$$

$$= \begin{vmatrix} 6 & 1 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} \quad (\because C_2 \text{ and } C_3 \text{ are identical})$$

$$= 0 = R.H.S$$

Hence proved

(ii) $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} = 0$

Solution:-

$$L.H.S = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 & -1+3 \\ 1 & 1 & 0+1 \\ 2 & -3 & 5-3 \end{vmatrix} \quad C_3 + C_2$$

$$= \begin{vmatrix} 2 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & -3 & 2 \end{vmatrix} \quad (\because C_1 \text{ and } C_3 \text{ are identical})$$

$$= 0 = R.H.S$$

Hence proved

(iii) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

Solution:-

$$L.H.S = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 1 \\ 7 & 1 & 1 \end{vmatrix} \quad \begin{matrix} C_2 - C_1 \\ C_3 - C_2 \end{matrix}$$

$$= 0 = R.H.S \quad (\because C_2 \text{ and } C_3 \text{ are identical})$$

Q3. Show that

i) $\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$

Solution:-

$$L.H.S = \begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix}$$

Opening from C_3

$$=(a_{13}+\alpha_{13})\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix} - (a_{23}+\alpha_{23})\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32}\end{vmatrix}$$

$$+(a_{33}+\alpha_{33})\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}$$

$$= a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32}\end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}$$

$$+\alpha_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix} - \alpha_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32}\end{vmatrix} + \alpha_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{11} \\ a_{21} & a_{22} & \alpha_{22} \\ a_{31} & a_{32} & \alpha_{33}\end{vmatrix}$$

$= R.H.S$

Hence proved

ii) $\begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1\end{vmatrix} = 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1\end{vmatrix}$

Solution:-

$L.H.S = \begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1\end{vmatrix}$

$= 3\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 2 \\ 2 & 15 & 1\end{vmatrix}$ Take 3 common from R_2

$= 3\cdot 3\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1\end{vmatrix}$ Take 3 common from C_2

$= 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1\end{vmatrix} = R.H.S$

Hence proved

iii) $\begin{vmatrix} a+\ell & a & a \\ a & a+\ell & a \\ a & a & a+\ell\end{vmatrix} = \ell^2(3a+\ell)$

Solution:-

$L.H.S = \begin{vmatrix} a+\ell & a & a \\ a & a+\ell & a \\ a & a & a+\ell\end{vmatrix}$

$= \begin{vmatrix} 3a+\ell & 3a+\ell & 3a+\ell \\ a & a+\ell & a \\ a & a & a+\ell\end{vmatrix}$ $R_1+(R_2+R_3)$

$=(3a+\ell)\begin{vmatrix} 1 & 1 & 1 \\ a & a+\ell & a \\ a & a & a+\ell\end{vmatrix}$ Take common $(3a+\ell)$ from R_1

$= (3a+\ell)\begin{vmatrix} 1 & 0 & 0 \\ a & \ell & 0 \\ a & 0 & \ell\end{vmatrix}$ $C_2 - C_1$, $C_3 - C_1$

$=(3a+\ell)\left[1\begin{vmatrix} \ell & 0 \\ 0 & \ell\end{vmatrix} - 0 + 0\right]$ Expanding by R_1

$= \ell^2(3a+\ell) = R.H.S$

Hence proved

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy\end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2\end{vmatrix}$

Solution:-

$L.H.S = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy\end{vmatrix}$

$= \dfrac{1}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz\end{vmatrix}$ xC_1, yC_2, zC_3

$= \dfrac{\cancel{xyz}}{\cancel{xyz}}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1\end{vmatrix}$ Take common xyz from R_3

$= -\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2\end{vmatrix}$ Interchange R_2 and R_3

$= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2\end{vmatrix}$ Again interchange R_2 and R_1

$= R.H.S$

Hence proved

vi) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b\end{vmatrix} = 4abc$

Solution:-

$L.H.S = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b\end{vmatrix}$

Expand by R_1

$=(b+c)\begin{vmatrix} c+a & b \\ c & a+b\end{vmatrix} - a\begin{vmatrix} b & b \\ c & a+b\end{vmatrix} + a\begin{vmatrix} b & c+a \\ c & c\end{vmatrix}$

$=(b+c)[(c+a)(a+b)-bc] - a[b(a+b)-bc]$
$+a[bc-c(c+a)]$

$=(b+c)[ac+\cancel{bc}+a^2+ab-\cancel{bc}] - a[ab+b^2-bc]$
$+a[bc-c^2-ac]$

$=abc+\cancel{a^2b}+\cancel{ab^2}+\cancel{ac^2}+\cancel{a^2c}+abc-\cancel{a^2b}-\cancel{ab^2}$
$+abc+abc-\cancel{ac^2}-\cancel{a^2c}$

$= 4abc = R.H.S$

Hence proved

vi) $\begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix} = a^3+b^3$

Solution:-

$$L.H.S = \begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix}$$

Expand by R_1

$$= b\begin{vmatrix} b & 0 \\ a & b \end{vmatrix} - (-1)\begin{vmatrix} a & 0 \\ 1 & b \end{vmatrix} + a\begin{vmatrix} a & b \\ 1 & a \end{vmatrix}$$

$$= b(b^2-0) + 1(ab-0) + a(a^2-b)$$

$$= b^3 + ab + a^3 - ab = a^3+b^3 = R.H.S$$

Hence proved

vii) $\begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix} = r$

Solution:-

$$L.H.S = \begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix}$$

Expand by R_2

$$= -0 + 1\begin{vmatrix} r\cos\phi & -\sin\phi \\ r\sin\phi & \cos\phi \end{vmatrix}$$

$$= r\cos^2\phi + r\sin^2\phi$$

$$= r(\cos^2\phi + \sin^2\phi) = r(1)$$

$$= r = R.H.S$$

Hence proved

viii) $\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix} = a^3+b^3+c^3-3abc$

Solution:-

$$L.H.S = \begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & b+c & a+b \\ a+b+c & c+a & b+c \\ a+b+c & a+b & c+a \end{vmatrix} \quad C_1 + C_2$$

$$= (a+b+c)\begin{vmatrix} 1 & b+c & a+b \\ 1 & c+a & b+c \\ 1 & a+b & c+a \end{vmatrix} \quad \text{Take } (a+b+c) \text{ common from } C_1$$

$$= (a+b+c)\begin{vmatrix} 1 & b+c & a+b \\ 0 & a-b & c-a \\ 0 & a-c & c-b \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

Expanding by C_1

$$= (a+b+c)\left[1\begin{vmatrix} a-b & c-a \\ a-c & c-b \end{vmatrix} - 0 + 0\right]$$

$$= (a+b+c)[(a-b)(c-b) - (c-a)(a-c)]$$

$$= (a+b+c)(ac-ab-bc+b^2-(ac-c^2-a^2+ac))$$

$$= (a+b+c)(\cancel{ac}-ab-bc+b^2-\cancel{ac}+c^2+a^2-ac$$

$$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= a^3+b^3+c^3-3abc = R.H.S$$

Hence proved

ix) $\begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix} = \lambda^2(a+b+c+\lambda)$

Solution:-

$$L.H.S = \begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c+\lambda & b & c \\ a+b+c+\lambda & b+\lambda & c \\ a+b+c+\lambda & b & c+\lambda \end{vmatrix} \quad C_1 + (C_2+C_3)$$

$$= (a+b+c+\lambda)\begin{vmatrix} 1 & b & c \\ 1 & b+\lambda & c \\ 1 & b & c+\lambda \end{vmatrix} \quad \text{Taking } (a+b+c+\lambda) \text{ common from } C_1$$

$$= (a+b+c+\lambda)\begin{vmatrix} 1 & b & c \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

Expanding by C_1

$$= (a+b+c+\lambda)\left\{1\begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} - 0 + 0\right\}$$

$$= (a+b+c+\lambda)(\lambda^2 - 0)$$

$$= \lambda^2(a+b+c+\lambda) = R.H.S$$

Hence proved

x) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution:-

$$L.H.S = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \quad \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \end{vmatrix}$$

(Take common $(b-a)$ from C_2, $(c-a)$ from C_3)

$$= (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$ Expanding by R_1

$$= (b-a)(c-a)\left\{1\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} - 0 + 0\right\}$$

$$= (b-a)(c-a)\{(c+a)-(b+a)\}$$

$$= (b-a)(c-a)(c+a-b-a)$$

$$= (b-a)(c-a)(c-b)$$

$$= [-(a-b)](c-a)[-(b-c)]$$

$$= (a-b)(b-c)(c-a) = R.H.S$$

Hence proved

xi) $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

Solution:-

$$L.H.S = \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a & a^2 \\ a+b+c & b & b^2 \\ a+b+c & c & c^2 \end{vmatrix} \quad C_1 + C_2$$

Take $(a+b+c)$ common from C_1

$$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

Expand by C_1

$$= (a+b+c)\left\{1\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} - 0 + 0\right\}$$

$$= (a+b+c)\begin{vmatrix} b-a & (b-a)(b+a) \\ c-a & (c-a)(c+a) \end{vmatrix}$$

Take common $(b-a)$ from R_1 and $(c-a)$ from R_2

$$= (a+b+c)(b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$$= (a+b+c)(b-a)(c-a)\{c+a-b-a\}$$

$$= (a+b+c)(b-a)(c-a)(c-b)$$

$$= (a+b+c)[-(a-b)](c-a)[-(b-c)]$$

$$= (a+b+c)(a-b)(b-c)(c-a)$$

$$= R.H.S$$

Hence proved

Q4. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 & 5 \\ 3 & 1 & 4 \\ -2 & 1 & -2 \end{bmatrix}$, then find: i) A_{12}, A_{22}, A_{32} and $|A|$

ii) B_{21}, B_{22}, B_{23} and $|B|$

Solution:- i) $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$

$$|A| = 1\begin{vmatrix} -2 & 0 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ -2 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix}$$

$$= 1(-2+0) - 2(0+0) - 3(0-4)$$

$$|A| = -2 - 0 + 12 = 10$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ -2 & 1 \end{vmatrix} = -(0+0) = 0$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} = (1-6) = -5$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix} = (-1)(0-0) = 0$$

ii) $B = \begin{bmatrix} 5 & -2 & 5 \\ 3 & 1 & 4 \\ -2 & 1 & -2 \end{bmatrix}$

$$|B| = \begin{vmatrix} 5 & -2 & 5 \\ 3 & 1 & 4 \\ -2 & 1 & -2 \end{vmatrix}$$

$$= 5\begin{vmatrix} 1 & 4 \\ 1 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 3 & 4 \\ -2 & -2 \end{vmatrix} + 5\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix}$$

$$= 5(-2-4) + 2(-6+8) + 5(3+2)$$

$$|B| = -10 + 4 + 5 = -1$$

$$B_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 5 \\ 1 & -2 \end{vmatrix} = -(4-5) = 1$$

$$B_{22} = (-1)^{2+2}\begin{vmatrix} 5 & 5 \\ -2 & -2 \end{vmatrix} = (-10+10) = 0$$

$$B_{23} = (-1)^{2+3}\begin{vmatrix} 5 & -2 \\ -2 & 1 \end{vmatrix} = -(5-4) = -1$$

Q5. Without expansion verify that

i) $\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix} = 0$

Solution:-

$$L.H.S = \begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha+\beta+\gamma & \beta+\gamma & 1 \\ \alpha+\beta+\gamma & \gamma+\alpha & 1 \\ \alpha+\beta+\gamma & \alpha+\beta & 1 \end{vmatrix} \quad C_1 + C_2$$

Taking $(\alpha+\beta+\gamma)$ common from C_1

$$= (\alpha+\beta+\gamma) \begin{vmatrix} 1 & \beta+\gamma & 1 \\ 1 & \gamma+\alpha & 1 \\ 1 & \alpha+\beta & 1 \end{vmatrix}$$

$$= (\alpha+\beta+\gamma)(0) = 0 \quad (\because C_1 \text{ and } C_3 \text{ are identical})$$

Hence proved

ii) $\begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix} = 0$

Solution:-

$$L.H.S = \begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix}$$

Take $3x$ common from C_3

$$= 3x \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 5 & 3 \end{vmatrix}$$

$$= 3x(0) = 0 = R.H.S$$

Hence proved

iii) $\begin{vmatrix} 1 & a^2 & a/bc \\ 1 & b^2 & b/ca \\ 1 & c^2 & c/ab \end{vmatrix} = 0$

Solution:-

$$L.H.S = \begin{vmatrix} 1 & a^2 & a/bc \\ 1 & b^2 & b/ca \\ 1 & c^2 & c/ab \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} 1 & a^2 & \frac{a(abc)}{bc} \\ 1 & b^2 & \frac{b(abc)}{ca} \\ 1 & c^2 & \frac{c(abc)}{ab} \end{vmatrix} \quad \text{'x' } C_3 \text{ by } abc \text{ and '÷' outside}$$

$$= \frac{1}{abc} \begin{vmatrix} 1 & a^2 & a^2 \\ 1 & b^2 & b^2 \\ 1 & c^2 & c^2 \end{vmatrix}$$

$$= \frac{1}{abc}(0) = R.H.S$$

Hence proved

iv) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Solution:-

$$L.H.S = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

$C_1 + (C_2+C_3)$

$$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0 = R.H.S$$

Hence proved $(\because C_1 \text{ is zero})$

v) $\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix} = 0$

Solution:-

$$L.H.S = \begin{vmatrix} bc & ca & ab \\ 1/a & 1/b & 1/c \\ a & b & c \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} bc & ca & ab \\ \frac{abc}{a} & \frac{abc}{b} & \frac{abc}{c} \\ a & b & c \end{vmatrix} \quad \text{'x' } R_2 \text{ by } abc \text{ and '÷' outside}$$

$$= \frac{1}{abc} \begin{vmatrix} bc & ca & ab \\ bc & ca & ab \\ a & b & c \end{vmatrix} \quad (\because R_1 \text{ and } R_2 \text{ are identical})$$

$$= \frac{1}{abc}(0) = 0 = R.H.S$$

vi) $\begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix} = \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$

Solution:-

$$L.H.S = \begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix}$$

$$= \frac{1}{lmn}\begin{vmatrix} lmn & l^2 & l^3 \\ lmn & m^2 & m^3 \\ lmn & n^2 & n^3 \end{vmatrix} \begin{matrix} lR_1 \\ mR_2 \\ nR_3 \end{matrix}$$

$$= \frac{lmn}{\cancel{lmn}}\begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$$ Taking lmn common from C_1

$$= \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix} = R.H.S$$

Hence proved

vii) $\begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix} = 0$

Solution:-

$$L.H.S = \begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$$ Take 2 common from R_1

$$= 2\begin{vmatrix} a & b & c \\ b & b & b \\ c & c & c \end{vmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= 2bc\begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$ Take common b from R_2 and c from R_3

$= 2bc\,(0) = R.H.S$ ($\because R_2$ and R_3 are identical)

Hence proved

viii) $\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$

Solution:-

$$R.H.S = \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 7 & 2 & 7-1 \\ 6 & 3 & 5-3 \\ -3 & 5 & -3+4 \end{vmatrix}$$ Add C_3 of both

$$= \begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = L.H.S$$

Hence proved

ix) $\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix} = 0$

Solution:-

$$L.H.S = \begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} -ab & 0 & bc \\ 0 & ac & -bc \\ ab & -ac & 0 \end{vmatrix} \begin{matrix} bR_1 \\ cR_2 \\ aR_3 \end{matrix}$$

$$= \frac{1}{abc}\begin{vmatrix} -ab+ab & ac-ac & bc-bc \\ 0 & ac & -bc \\ ab & -ac & 0 \end{vmatrix} \quad R_1 + (R_2 + R_3)$$

$$= \frac{1}{abc}\begin{vmatrix} 0 & 0 & 0 \\ 0 & ac & -bc \\ ab & -ac & 0 \end{vmatrix}$$

$= \frac{1}{abc}(0) = 0$ ($\because R_1$ is zero)

$= R.H.S$

Hence proved

Q6. Find values of x if

i) $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$

Solution:-

$$\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$$

$$\Rightarrow 3\begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 4 \\ x & 0 \end{vmatrix} + x\begin{vmatrix} -1 & 3 \\ x & 1 \end{vmatrix} = -30$$

$\Rightarrow 3(0-4) - 1(0-4x) + x(-1-3x) = -30$

$-12 + 4x - x - 3x^2 = -30$

$-3x^2 + 3x - 12 + 30 = 0$

$-3x^2 + 3x + 18 = 0$

$\Rightarrow x^2 - x - 6 = 0$

$x^2 - 3x + 2x - 6 = 0$

$\Rightarrow x(x-3) + 2(x-3) = 0$

$\Rightarrow (x-3)(x+2) = 0$

$x - 3 = 0, \quad x + 2 = 0$

$\Rightarrow x = 3, \quad x = -2$

ii) $\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -2 & x \end{vmatrix} = 0$

Solution:-

$$\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -2 & x \end{vmatrix} = 0$$

$$\Rightarrow 1\begin{vmatrix} x+1 & 2 \\ -2 & x \end{vmatrix} - (x-1)\begin{vmatrix} -1 & 2 \\ 2 & x \end{vmatrix} + 3\begin{vmatrix} -1 & x+1 \\ 2 & -2 \end{vmatrix} = 0$$

$$\Rightarrow 1(x^2+x+4) - (x-1)(-x-4) + 3(2-2x-2) = 0$$

$$x^2+x+4-(-x^2-4x+x+4)+\cancel{6}-6x-\cancel{6}=0$$

$$x^2+\cancel{x}+\cancel{4}+x^2+4x-\cancel{x}-\cancel{4}-6x=0$$

$$2x^2-2x=0$$

$$2x(x-1)=0$$

$$2x=0, \quad x-1=0$$

$$x=0, \quad x=1$$

iii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

Solution:-

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$$

$$1\begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & x \end{vmatrix} + 1\begin{vmatrix} 2 & x \\ 3 & 6 \end{vmatrix} = 0$$

$$(x^2-12)-2(2x-6)+(12-3x)=0$$

$$x^2-\cancel{12}-4x+\cancel{12}+12-3x=0$$

$$x^2-7x+12=0$$

$$\Rightarrow x^2-4x-3x+12=0$$

$$x(x-4)-3(x-4)=0$$

$$(x-4)(x-3)=0$$

$$x-4=0, \quad x-3=0$$

$$x=4, \quad x=3$$

Q7. Evaluate the following determinants.

i) $\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix}$

Solution:-

$$\begin{vmatrix} 1 & -1 & 2 & 4 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix} \; R_1-R_2$$

$$= \begin{vmatrix} 1 & -1 & 2 & 4 \\ 0 & 7 & -4 & -5 \\ 0 & 3 & -5 & 1 \\ 0 & 5 & -10 & -10 \end{vmatrix} \; \begin{matrix} R_2-2R_1 \\ R_3-R_1 \\ R_4-4R_1 \end{matrix}$$

$$= 1\begin{vmatrix} 7 & -4 & -5 \\ 3 & -5 & 1 \\ 5 & -10 & -10 \end{vmatrix} -0+0-0 \quad \text{Expand by } C_1$$

$$= 7\begin{vmatrix} -5 & 1 \\ -10 & -10 \end{vmatrix} - (-4)\begin{vmatrix} 3 & 1 \\ 5 & -10 \end{vmatrix} + (-5)\begin{vmatrix} 3 & -5 \\ 5 & -10 \end{vmatrix}$$

$$= 7(50+10)+4(-30-5)-5(-30+25)$$

$$= 420-140+25 = 305$$

ii) $\begin{vmatrix} 2 & 3 & 1 & -1 \\ 4 & 0 & 2 & 1 \\ 5 & 2 & -1 & 6 \\ 3 & -7 & 2 & -2 \end{vmatrix}$

Solution:-

$$\begin{vmatrix} 2 & 3 & 1 & -1 \\ 6 & 3 & 3 & 0 \\ 17 & 20 & 5 & 0 \\ -1 & -13 & 0 & 0 \end{vmatrix} \; \begin{matrix} R_2+R_1 \\ R_3+6R_1 \\ R_4-2R_1 \end{matrix}$$

$$= -(-1)\begin{vmatrix} 6 & 3 & 3 \\ 17 & 20 & 5 \\ -1 & -13 & 0 \end{vmatrix} +0-0+0 \quad \text{Expand from } C_4$$

$$= \begin{vmatrix} 6 & 3 & 3 \\ 17 & 20 & 5 \\ -1 & -13 & 0 \end{vmatrix}$$

$$= 3\begin{vmatrix} 17 & 20 \\ -1 & -13 \end{vmatrix} - 5\begin{vmatrix} 6 & 3 \\ -1 & -13 \end{vmatrix} + 0 \quad \text{Expand by } C_3$$

$$= 3(-221+20)-5(-78+3)$$

$$= 3(-201)-5(-75) = -603+375$$

$$= -228$$

iii) $\begin{vmatrix} -3 & 9 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 9 & 7 & -1 & 1 \\ -2 & 0 & 1 & -1 \end{vmatrix}$

Solution:-

$$\begin{vmatrix} -3 & 9 & 1 & 1 \\ -3 & 12 & 0 & 3 \\ 6 & 16 & 0 & 2 \\ 1 & -9 & 0 & -2 \end{vmatrix} \begin{matrix} R_2+R_1 \\ R_3+R_1 \\ R_4-R_1 \end{matrix}$$

$$= 1\begin{vmatrix} -3 & 12 & 3 \\ 6 & 16 & 2 \\ 1 & 9 & -2 \end{vmatrix} - 0 + 0 - 0 \quad \text{Expand from } C_3$$

$$= -3\begin{vmatrix} 16 & 2 \\ -9 & -2 \end{vmatrix} - 12\begin{vmatrix} 6 & 2 \\ 1 & -2 \end{vmatrix} + 3\begin{vmatrix} 6 & 16 \\ 1 & -9 \end{vmatrix}$$

$$= -3(-32+18) - 12(-12-2) + 3(-54-16)$$

$$= -3(-14) - 12(-14) + 3(-70)$$

$$= 42 + 168 - 210 = 0$$

Q8. Show that $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = (x+3)(x-1)^3$

Solution:-

$$L.H.S = \begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$

$$= \begin{vmatrix} x+1+1+1 & 1 & 1 & 1 \\ 1+x+1+1 & x & 1 & 1 \\ 1+1+x+1 & 1 & x & 1 \\ 1+1+1+x & 1 & 1 & x \end{vmatrix} C_1+(C_2+C_3+C_4)$$

$$= \begin{vmatrix} x+3 & 1 & 1 & 1 \\ x+3 & x & 1 & 1 \\ x+3 & 1 & 1 & 1 \\ x+3 & 1 & x & x \end{vmatrix}$$

Take common $(x+3)$ from C_1

$$= (x+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$

$$= (x+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x-1 \end{vmatrix} \begin{matrix} R_2-R_1 \\ R_3-R_1 \\ R_4-R_1 \end{matrix}$$

Expanding by C_1

$$= (x+3)\begin{vmatrix} x-1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{vmatrix} - 0 + 0 - 0$$

$$= (x+3)(x-1)^3 \quad \text{(by determinant property)}$$

$$= R.H.S$$

Q9. Find $|AA^t|$ and $|A^tA|$ if

i) $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

Solution:-

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}, \quad A^t = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 9+4+1 & 6+2-3 \\ 6+2-3 & 4+1+9 \end{bmatrix} = \begin{bmatrix} 14 & 5 \\ 5 & 14 \end{bmatrix}$$

Now $|AA^t| = \begin{vmatrix} 14 & 5 \\ 5 & 14 \end{vmatrix} = 196 - 25 = 171$

and $A^tA = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 9+4 & 6+2 & -3+6 \\ 6+2 & 4+1 & -2+3 \\ -3+6 & -2+3 & 1+9 \end{bmatrix}$$

$$A^tA = \begin{bmatrix} 13 & 8 & 3 \\ 8 & 5 & 1 \\ 3 & 1 & 10 \end{bmatrix}$$

Now $|A^tA| = \begin{vmatrix} 13 & 8 & 3 \\ 8 & 5 & 1 \\ 3 & 1 & 10 \end{vmatrix}$

$$= 13(50-1) - 8(80-3) + 3(8-15)$$

$$= 637 - 616 - 21 = 0$$

ii) $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$

Solution:-

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}, \quad A^t = \begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+16 & 6+4 & 3+4 \\ 6+4 & 4+1 & 2+1 \\ 3+4 & 2+1 & 1+1 \\ 6+12 & 4+3 & 2+3 \end{bmatrix} = \begin{bmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 18 & -7 & 5 & 13 \end{bmatrix}$$

$$|AA^t| = \begin{vmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 18 & 7 & 5 & 13 \end{vmatrix}$$

$$= \begin{vmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 25 & 10 & 7 & 18 \end{vmatrix} \quad R_4 + R_3$$

$$= 0 \quad (\because R_1 \text{ and } R_4 \text{ are identical})$$

$$A^tA = \begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+4+1+4 & 12+2+1+6 \\ 12+2+1+6 & 16+1+1+9 \end{bmatrix}$$

$$A^tA = \begin{bmatrix} 18 & 21 \\ 21 & 27 \end{bmatrix}$$

$$|A^tA| = \begin{vmatrix} 18 & 21 \\ 21 & 27 \end{vmatrix} = 486 - 441 = 45$$

Q10. If A is a square matrix of order 3, then show that $|KA| = K^3|A|$.

Solution:-

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$KA = K \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$KA = \begin{bmatrix} Ka_{11} & Ka_{12} & Ka_{13} \\ Ka_{21} & Ka_{22} & Ka_{23} \\ Ka_{31} & Ka_{32} & Ka_{33} \end{bmatrix}$$

$$\Rightarrow |KA| = \begin{vmatrix} Ka_{11} & Ka_{12} & Ka_{13} \\ Ka_{21} & Ka_{22} & Ka_{23} \\ Ka_{31} & Ka_{32} & Ka_{33} \end{vmatrix}$$

$$= K \cdot K \cdot K \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{Taking common } K \text{ from } R_1, R_2 \text{ and } R_3$$

$$= K^3|A| = R.H.S$$

Hence proved

Q11. Find the values of λ if A and B are singular.

$$A = \begin{bmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{bmatrix}$$

Solution:- i)

Given matrix is singular so

$$|A| = \begin{vmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 4\begin{vmatrix} 3 & 6 \\ 3 & 1 \end{vmatrix} - \lambda\begin{vmatrix} 7 & 6 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 7 & 3 \\ 2 & 3 \end{vmatrix}$$

$$4(3-18) - \lambda(7-12) + 3(21-6) = 0$$

$$-60 + 5\lambda + 45 = 0$$

$$5\lambda - 15 = 0 \Rightarrow \lambda = 3$$

ii)

Given matrix is singular so

$$|B| = \begin{vmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{vmatrix} = 0$$

$R_3 - R_2$, $R_4 - 3R_2$

$$\Rightarrow \begin{vmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ -5 & 0 & -5 & 0 \\ -22 & \lambda-6 & -16 & 0 \end{vmatrix} = 0$$

Expand by C_4

$$-0 + 1\begin{vmatrix} 5 & 1 & 2 \\ -5 & 0 & -5 \\ -22 & \lambda-6 & -16 \end{vmatrix} - 0 + 0 = 0$$

Taking -5 common from R_2

$$-5\begin{vmatrix} 5 & 1 & 2 \\ 1 & 0 & 1 \\ -22 & \lambda-6 & -16 \end{vmatrix} = 0$$

Expand by R_2

$$-5\left\{-1\begin{vmatrix} 1 & -3 \\ \lambda-6 & 6 \end{vmatrix} + 0 - 0\right\} = 0$$

$$5(6 + 3(\lambda-6)) = 0$$

$$5(6 + 3\lambda - 18) = 0$$

$$5(3\lambda - 12) = 0$$

$$3\lambda - 12 = 0$$

$$3\lambda = 12 \Rightarrow \lambda = 4$$

Q12. Which of the following matrices are singular and which of them are non-singular?

i) $\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$

Solution:-

Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} - 0 + 3\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$$

$$= 1(4+2) + 3(6-0)$$

$$|A| = 6 + 18 = 24 \neq 0$$

$\Rightarrow A$ is non-singular

ii) $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$

Solution:-

Let $B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$

$$|B| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$$

$$= 2(5-0) - 3(5-0) - 1(-3-2)$$

$$|B| = 10 - 15 + 5 = 0$$

$\Rightarrow B$ is singular

iii) $\begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{bmatrix}$

Solution:-

Let $C = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{bmatrix}$

$$\Rightarrow |C| = \begin{vmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & 4 \\ 0 & -4 & -3 & 7 \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - 3R_1 \end{matrix}$$

$$= 1\begin{vmatrix} 1 & -3 & -2 \\ 1 & -3 & 4 \\ -4 & -3 & 7 \end{vmatrix} - 0 + 0 - 0 \quad \text{Expand by } C_1$$

$$= \begin{vmatrix} 1 & -3 & -2 \\ 0 & 0 & 6 \\ 0 & -15 & -1 \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 + 4R_1 \end{matrix}$$

$$= 1\begin{vmatrix} 0 & 6 \\ -15 & -1 \end{vmatrix} - 0 + 0 \quad \text{Expand by } C_1$$

$$|C| = 0 + 90 = 90 \neq 0$$

$\Rightarrow C$ is not singular

Q13. Find the inverse of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$ and show that $A^{-1}A = I_3$

Solution:-

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$$

$$= 2(5-0) - 1(5-0) = 10 - 5 = 5$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix} = (5-0) = 5$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} = -(5-0) = -5$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = (-3-2) = -5$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix} = -(5-0) = -5$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 0 \\ 2 & 5 \end{vmatrix} = (10-0) = 10$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} = -(-6-2) = 8$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = (0-0) = 0$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} = -(0-0) = 0$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (2-1) = 1$$

$$\text{Cofactor of } A = \begin{bmatrix} 5 & -5 & -5 \\ -5 & 10 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Adj}A = \begin{bmatrix} 5 & -5 & 0 \\ -5 & 10 & 0 \\ -5 & 8 & 1 \end{bmatrix}$$

$$\because A^{-1} = \frac{\text{adj}A}{|A|} = \frac{1}{5}\begin{bmatrix} 5 & -5 & 0 \\ -5 & 10 & 0 \\ -5 & 8 & 1 \end{bmatrix}$$

Now

$$A^{-1}A = \frac{1}{5}\begin{bmatrix} 5 & -5 & 0 \\ -5 & 10 & 0 \\ -5 & 8 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 10-5+0 & 5-5+0 & 0-0+0 \\ -10+10+0 & -5+10-0 & 0+0+0 \\ -10+8+2 & -5+8-3 & 0+0+5 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \frac{5}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Hence $A^{-1}A = I_3$

Q14. Verify that $(AB)^{-1} = B^{-1}A^{-1}$ if

i) $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix}$

Solution:-

we know that

$$(AB)^{-1} = \frac{\text{adj}(AB)}{|AB|}$$

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -3+8 & 1-2 \\ 3+0 & -1-0 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 3 & -1 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 5 & -1 \\ 3 & -1 \end{vmatrix} = -5+3 = -2 \neq 0$$

$$\text{adj}(AB) = \begin{bmatrix} -1 & 1 \\ -3 & 5 \end{bmatrix}$$

$$L.H.S = (AB)^{-1} = \frac{\text{adj}(AB)}{|AB|}$$

$$L.H.S = \frac{1}{-2}\begin{bmatrix} -1 & 1 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

Now for B^{-1}

$$|B| = \begin{vmatrix} -3 & 1 \\ 4 & -1 \end{vmatrix} = 3-4 = -1 \neq 0$$

$$\text{adj } B = \begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix}$$

$$\Rightarrow B^{-1} = \frac{\text{adj}B}{|B|} = \frac{\begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix}}{-1}$$

$$B^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}$$

For A^{-1}, $|A| = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 0+2 = 2 \neq 0$

$$\text{adj } A = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}A}{|A|} = \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

$$R.H.S = B^{-1}A^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0+1 & -2+1 \\ 0+3 & -8+3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 3 & -5 \end{bmatrix}$$

$$R.H.S = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{5}{2} \end{bmatrix}$$

Hence $L.H.S = R.H.S$

ii) $A=\begin{bmatrix}5 & 1\\ 2 & 2\end{bmatrix}$, $B=\begin{bmatrix}4 & 3\\ 2 & 1\end{bmatrix}$

Solution:-

$\because (AB)^{-1} = \dfrac{adj(AB)}{|AB|}$

$AB = \begin{bmatrix}5 & 1\\ 2 & 2\end{bmatrix}\begin{bmatrix}4 & 3\\ 2 & 1\end{bmatrix}$

$AB = \begin{bmatrix}20+2 & 15+1\\ 8+4 & 6+2\end{bmatrix} = \begin{bmatrix}22 & 16\\ 12 & 8\end{bmatrix}$

$|AB| = \begin{vmatrix}22 & 16\\ 12 & 8\end{vmatrix} = 176-192=-16 \neq 0$

L.H.S:=

$(AB)^{-1} = \dfrac{adj(AB)}{|AB|} = \dfrac{1}{-16}\begin{bmatrix}8 & -16\\ -12 & 22\end{bmatrix}$

$(AB)^{-1} = \begin{bmatrix}8/-16 & -16/-16\\ -12/-16 & 22/-16\end{bmatrix} = \begin{bmatrix}-1/2 & 1\\ \frac{3}{4} & -11/8\end{bmatrix}$

Now for B,

$|B| = \begin{vmatrix}4 & 3\\ 2 & 1\end{vmatrix} = 4-6=-2\neq 0$

$adj\, B = \begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}$

$B^{-1} = \dfrac{adj\, B}{|B|} = \dfrac{1}{-2}\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}$

For A, $|A| = \begin{vmatrix}5 & 1\\ 2 & 2\end{vmatrix} = 10-2=8\neq 0$

$adj\, A = \begin{bmatrix}2 & -1\\ -2 & 5\end{bmatrix}$

$A^{-1} = \dfrac{adj\, A}{|A|} = \dfrac{1}{8}\begin{bmatrix}2 & -1\\ -2 & 5\end{bmatrix}$

$R.H.S = B^{-1}A^{-1}$

$= \dfrac{1}{-2}\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}\cdot\dfrac{1}{8}\begin{bmatrix}2 & -1\\ -2 & 5\end{bmatrix}$

$= \dfrac{1}{-16}\begin{bmatrix}1 & -3\\ -2 & 4\end{bmatrix}\begin{bmatrix}2 & -1\\ -2 & 5\end{bmatrix}$

$= \dfrac{1}{-16}\begin{bmatrix}2+6 & -1-15\\ -4-8 & 2+20\end{bmatrix}$

$= \dfrac{-1}{16}\begin{bmatrix}8 & -16\\ -12 & 22\end{bmatrix}$

$= \begin{bmatrix}-8/16 & -16/16\\ -12/-16 & 22/-16\end{bmatrix} = \begin{bmatrix}-1/2 & 1\\ \frac{3}{4} & -11/8\end{bmatrix}$

Hence L.H.S = R.H.S

Q15. Verify that $(AB)^t = B^tA^t$ if

$A=\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 1\\ 3 & 2\\ 0 & -1\end{bmatrix}$

Solution:-

$L.H.S = (AB)^t$

$= \left(\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 3 & 2\\ 0 & -1\end{bmatrix}\right)^t$

$= \begin{bmatrix}1-3+0 & 1-2-2\\ 0+9+0 & 0+6-1\end{bmatrix}^t$

$= \begin{bmatrix}-2 & -3\\ 9 & 5\end{bmatrix}^t$

$L.H.S = \begin{bmatrix}-2 & 9\\ -3 & 5\end{bmatrix}$

$R.H.S = B^t\, A^t$

$= \begin{bmatrix}1 & 1\\ 3 & 2\\ 0 & -1\end{bmatrix}^t\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{bmatrix}^t$

$= \begin{bmatrix}1 & 3 & 0\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 3\\ 2 & 1\end{bmatrix}$

$R.H.S = \begin{bmatrix}1-3+0 & 0+9+0\\ 1-2-2 & 0+6-1\end{bmatrix} = \begin{bmatrix}-2 & 9\\ -3 & 5\end{bmatrix}$

Hence L.H.S = R.H.S

Q16. If $A=\begin{bmatrix}2 & -1\\ 3 & 1\end{bmatrix}$ verify that $(A^{-1})^t = (A^t)^{-1}$

Solution:-

$A = \begin{bmatrix}2 & -1\\ 3 & 1\end{bmatrix}$

$|A| = \begin{vmatrix}2 & -1\\ 3 & 1\end{vmatrix} = 2+3=5\neq 0$

$$\text{adj} A = \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}$$

Thus $A^{-1} = \frac{\text{adj} A}{|A|} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 1/5 & 1/5 \\ -3/5 & 2/5 \end{bmatrix}$$

$$\rightarrow (A^{-1})^t = \begin{bmatrix} 1/5 & -3/5 \\ 1/5 & 2/5 \end{bmatrix} \longrightarrow (I)$$

Now $A^t = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$

$$|A^t| = \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 2+3 = 5$$

$$\text{adj}(A^t) = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$(A^t)^{-1} = \frac{1}{|A^t|} \text{adj}(A^t)$$

$$= \frac{1}{5}\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$(A^t)^{-1} = \begin{bmatrix} 1/5 & -3/5 \\ 1/5 & 2/5 \end{bmatrix} \longrightarrow (II)$$

By (I) and (II),

L.H.S = R.H.S

Q17. If A and B are non-singular matrices, then show that

i) $(AB)^{-1} = B^{-1}A^{-1}$ ii) $(A^{-1})^{-1} = A$

Solution:- i) $(AB)^{-1} = B^{-1}A^{-1}$

we know that

$$(AB)(AB)^{-1} = I$$

Pre-multiplying by A^{-1}

$$A^{-1}(AB).(AB)^{-1} = A^{-1}I$$

$$(A^{-1}A)B(AB)^{-1} = A^{-1} \quad \text{(Associative Law)}$$

$$IB(AB)^{-1} = A^{-1}$$

$$B(AB)^{-1} = A^{-1}$$

Pre-multiplying by B^{-1}

$$B^{-1}.B(AB)^{-1} = B^{-1}A^{-1}$$

$$(B^{-1}B)(AB)^{-1} = B^{-1}A^{-1}$$

$$I(AB)^{-1} = B^{-1}A^{-1}$$

$$\rightarrow (AB)^{-1} = B^{-1}A^{-1}$$

Hence proved

ii) $(A^{-1})^{-1} = A$

we know that

$$I = AA^{-1}$$

Post-multiplying by $(A^{-1})^{-1}$

$$I(A^{-1})^{-1} = (AA^{-1})(A^{-1})^{-1}$$

$$(A^{-1})^{-1} = A[A^{-1}(A^{-1})^{-1}] \quad \text{(Associative Law)}$$

$$\rightarrow (A^{-1})^{-1} = A(I)$$

$$\rightarrow (A^{-1})^{-1} = A$$

Hence proved

Elementary Row and Column Operations on a Matrix

Row operation:-

The following three operations on a matrix are called elementary row operations.

i) Interchange of any two rows.

ii) Multiplication of a row by any non-zero number.

iii) Addition of any multiple of one row to another row.

Column operation:-

The following three operations on a matrix are called elementary column operation.

i) Interchange of any two columns.

ii) Multiplication of a column by a non-zero number.

iii) Addition of any multiple of one column to another column.

Upper Triangular Matrix:-

A square matrix $A=[a_{ij}]$ is called upper triangular matrix if all elements below the main diagonal are zero.

i.e., $a_{ij}=0$ for all $i>j$

e.g., $\begin{bmatrix}1&2&3\\0&1&4\\0&0&6\end{bmatrix}$

Lower Triangular Matrix:-

A square matrix $A=[a_{ij}]$ is called a lower Triangular matrix if all elements above the main diagonal are zero.

i.e., $a_{ij}=0$ for all $i<j$

e.g., $\begin{bmatrix}1&0&0\\2&1&0\\3&4&6\end{bmatrix}$

Triangular Matrix:-

A square matrix A is said to be triangular matrix if it is upper triangular or lower triangular. while both upper and lower are called triangular matrix

* Diagonal matrices are both upper triangular and lower triangular.

Symmetric Matrix:-

A square matrices $A=[a_{ij}]_{n\times n}$ is called symmetric matrix if $A^t=A$ e.g.,

If $A=\begin{bmatrix}1&3\\3&2\end{bmatrix} \rightarrow A^t=\begin{bmatrix}1&3\\3&2\end{bmatrix}=A$

Skew-Symmetric Matrix:-

A square matrix $A=[a_{ij}]_{n\times n}$ is called skew-symmetric matrix. if $A^t=-A$

e.g., $A=\begin{bmatrix}0&-4&1\\4&0&-3\\-1&3&0\end{bmatrix}$

$A^t=\begin{bmatrix}0&4&-1\\-4&0&3\\1&-3&0\end{bmatrix}=-\begin{bmatrix}0&-4&1\\4&0&-3\\-1&3&0\end{bmatrix}$

$\rightarrow A^t=-A$

Hermitian Matrix:-

A square matrix A is said to be hermitian matrix if $(\bar{A})^t=A$ e.g.,

$A=\begin{bmatrix}1&1+i\\1-i&2\end{bmatrix}$, $\bar{A}=\begin{bmatrix}1&1-i\\1+i&2\end{bmatrix}$

$(\bar{A})^t=\begin{bmatrix}1&1+i\\1-i&2\end{bmatrix}=A$

Skew-hermitian Matrix:-

A square matrix A is said to be skew-hermitian matrix if $(\bar{A})^t=-A$ e.g.,

$A=\begin{bmatrix}0&2-3i\\-2+3i&0\end{bmatrix}$, $\bar{A}=\begin{bmatrix}0&2+3i\\-2+3i&0\end{bmatrix}$

$(\bar{A})^t=\begin{bmatrix}0&-2+3i\\2+3i&0\end{bmatrix}=-\begin{bmatrix}0&2-3i\\-2-3i&0\end{bmatrix}$

$(\bar{A})^t=-A$

Zero Row:- If all entries of a row are zero then this row is called zero row, otherwise non-zero row. e.g.,

$\begin{bmatrix}1&2&3\\0&0&0\\0&0&3\end{bmatrix}$ R_2 is zero row. R_1 and R_3 are non zero rows.

Leading Entry:- In any non-zero row, the first non-zero element is the leading entry of that row. e.g.,

$\begin{bmatrix}1&0&5&-4\\0&2&3&9\\0&0&8&7\end{bmatrix}$ In R_1 leading entry is 1. In R_2 leading entry is 2

In R_3 leading entry is 8.

Leading Zeros:- The zeros before the leading entry of a row are called leading zeros.

e.g., $\begin{bmatrix} 1 & 0 & 5 & 4 \\ 0 & 2 & 3 & 9 \\ 0 & 0 & 8 & 7 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

In R_1 there is no leading zero. In R_2 only one zero is leading zero. In R_3 two zeros are leading zeros. In R_4 only one zero is leading zero.

Echelon form of a Matrix

A matrix is said to be Echelon form if

i) 1 is the leading entry of each non-zero row.

ii) In each row, the number of leading zeros is greater than the preceding row. e.g.,

$$A = \begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A and B are in Echelon form.

$$C = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

C is not in Echelon form $\because$ leading zero in R_1 > leading zero in R_2.

D is not in Echelon form $\because$ leading entry in R_2 and R_3 is not 1.

Reduced Echelon form of Matrix

A matrix is said to be in Reduced Echelon form if

i) it is in Echelon form.

ii) In the column of leading entry all elements above and below leading entry (1) must be zero.

e.g., $A = \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A and B are in Reduced Echelon form.

$C = \begin{bmatrix} 1 & -4 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

C and D are in Echelon form but not in Reduced Echelon form.

Example 1. Reduce the following matrix to (row) echelon and reduced (row) echelon form.

$$\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}$$

Solution:- Given matrix is

$$\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}$$

Echelon form:-

$$\underset{R}{\sim} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 3 & -1 & 9 \\ 3 & 1 & 3 & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\underset{R}{\sim} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 2-2 & 3+2 & -1-4 & 9+6 \\ 3-3 & 1+3 & 3-6 & 2+9 \end{bmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\underset{R}{\sim} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -5 & 15 \\ 0 & 4 & -3 & 11 \end{bmatrix}$$

$$\sim\begin{bmatrix}1&-1&2&-3\\0&1&-1&3\\0&4&-3&11\end{bmatrix}\ \frac{1}{5}R_2$$

$$\sim\begin{bmatrix}1&-1&2&-3\\0&1&-1&3\\0&4-4&-3+4&11-12\end{bmatrix}\ R_3-4R_2$$

$$\sim\begin{bmatrix}1&-1&2&-3\\0&1&-1&3\\0&0&1&-1\end{bmatrix}$$

This is Echelon form.

Reduced Echelon form:-

$$\sim\begin{bmatrix}1&-1+1&2-1&-3+3\\0&1&-1&3\\0&0&1&-1\end{bmatrix}\ R_1+R_2$$

$$\sim\begin{bmatrix}1&0&1&0\\0&1&-1&3\\0&0&1&-1\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&1-1&0+1\\0&1&-1+1&3-1\\0&0&1&-1\end{bmatrix}\ \begin{matrix}R_1-R_3\\R_2+R_3\end{matrix}$$

$$\sim\begin{bmatrix}1&0&0&1\\0&1&0&2\\0&0&1&-1\end{bmatrix}$$

This is Reduced Echelon form.

Example 2. Find the inverse of the matrix $A=\begin{bmatrix}2&5&-1\\3&4&2\\1&2&-2\end{bmatrix}$

Solution:-

> **Note:-** A^{-1} can be find by three different methods
> i) By Adjoint method
> ii) By Row operation
> iii) By Column operation
> we solve A^{-1} by all three methods one by one

$$A=\begin{bmatrix}2&5&-1\\3&4&2\\1&2&-2\end{bmatrix}$$

$$|A|=\begin{vmatrix}2&5&-1\\3&4&2\\1&2&-2\end{vmatrix}$$

Expanding by C_1

$$=2\begin{vmatrix}4&2\\2&-2\end{vmatrix}-3\begin{vmatrix}5&-1\\2&-2\end{vmatrix}+1\begin{vmatrix}5&-1\\4&2\end{vmatrix}$$

$$=2(-8-4)-3(-10+2)+1(10+4)$$

$$=2(-12)-3(-8)+1(14)$$

$$=-24+24+14=14$$

$\rightarrow |A|=14\neq 0$ so, A^{-1} exists

By Adjoint Method

$$A_{11}=(-1)^{1+1}\begin{vmatrix}4&2\\2&-2\end{vmatrix}=(-8-4)=-12$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix}3&2\\1&-2\end{vmatrix}=-(-6-2)=8$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}3&4\\1&2\end{vmatrix}=(6-4)=2$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}5&-1\\2&-2\end{vmatrix}=-(-10+2)=8$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}2&-1\\1&-2\end{vmatrix}=(-4+1)=-3$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}2&5\\1&-2\end{vmatrix}=-(4-5)=1$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}5&-1\\4&2\end{vmatrix}=(10+4)=14$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}2&-1\\3&2\end{vmatrix}=-(4+3)=-7$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}2&5\\3&4\end{vmatrix}=(8-15)=-7$$

$$\text{adj } A=\begin{bmatrix}-12&8&14\\8&-3&-7\\2&1&-7\end{bmatrix}$$

$$\because A^{-1}=\frac{1}{|A|}\text{adj } A$$

$$A^{-1}=\frac{1}{14}\begin{bmatrix}-12&8&14\\8&-3&-7\\2&1&-7\end{bmatrix}$$

$$\rightarrow A^{-1}=\begin{bmatrix}-12/14&8/14&14/14\\8/14&-3/14&-7/14\\2/14&1/14&-7/14\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}-6/7&4/7&1\\4/7&-3/14&-1/2\\1/7&1/14&1/2\end{bmatrix}$$

By Row operation

As $A = A\cdot I$

$$\rightarrow A = \begin{bmatrix}2&5&-1\\3&4&2\\1&2&2\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\underset{R}{\sim}\begin{bmatrix}1&2&-2\\3&4&2\\2&5&-1\end{bmatrix}\begin{bmatrix}0&0&0\\0&1&0\\1&0&1\end{bmatrix} \quad R_{1\leftrightarrow 3}$$

$$\underset{R}{\sim}\begin{bmatrix}1&2&-2\\0&-2&8\\0&1&3\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&-3\\1&0&-2\end{bmatrix} \quad \begin{matrix}R_2-3R_1\\R_3-2R_1\end{matrix}$$

$$\underset{R}{\sim}\begin{bmatrix}1&2&-2\\0&1&3\\0&-2&8\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&-2\\1&0&-3\end{bmatrix} \quad R_{2\leftrightarrow 3}$$

$$\underset{R}{\sim}\begin{bmatrix}1&0&-8\\0&1&3\\0&0&14\end{bmatrix}\begin{bmatrix}-2&0&5\\1&0&-2\\2&1&-7\end{bmatrix} \quad \begin{matrix}R_1-2R_2\\R_3+2R_2\end{matrix}$$

$$\underset{R}{\sim}\begin{bmatrix}1&0&-8\\0&1&3\\0&0&1\end{bmatrix}\begin{bmatrix}-2&0&5\\1&0&-2\\\frac{1}{7}&\frac{1}{14}&-\frac{1}{2}\end{bmatrix} \quad \frac{1}{14}R_3$$

$$\underset{R}{\sim}\begin{bmatrix}1&0&-8+8\\0&1&3-3\\0&0&1\end{bmatrix}\begin{bmatrix}-2+\frac{8}{7}&0+\frac{8}{14}&5-8/2\\1-3/7&0-\frac{3}{14}&-2+3/2\\\frac{1}{7}&\frac{1}{14}&-1/2\end{bmatrix} \quad \begin{matrix}R_1+8R_3\\R_2-3R_3\end{matrix}$$

$$\underset{R}{\sim}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\frac{-14+8}{7}&\frac{8}{14}&\frac{10-8}{2}\\\frac{7-3}{7}&-3/14&-\frac{4+3}{2}\\\frac{1}{7}&1/14&-1/2\end{bmatrix}$$

$$\underset{R}{\sim}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-6/7&4/7&1\\4/7&-3/14&-1/2\\1/7&1/14&-1/2\end{bmatrix}$$

$$\rightarrow A^{-1} = \begin{bmatrix}-6/7&4/7&1\\4/7&-3/7&-1/2\\1/7&1/14&-1/2\end{bmatrix}$$

By Column Operation

As $A = AI$

$$A = \begin{bmatrix}2&5&-1\\3&4&2\\1&2&2\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\underset{C}{\sim}\begin{bmatrix}-1&5&2\\2&4&3\\-2&2&1\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} \quad C_{1\leftrightarrow 3}$$

$$\underset{C}{\sim}\begin{bmatrix}1&5&2\\-2&4&3\\2&2&1\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix} \quad (-1)C_1$$

$$\underset{C}{\sim}\begin{bmatrix}1&0&0\\-2&14&7\\2&-8&-3\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\-1&5&2\end{bmatrix} \quad \begin{matrix}C_2-5C_1\\C_3-2C_1\end{matrix}$$

$$\underset{C}{\sim}\begin{bmatrix}1&0&0\\-2&1&7\\2&-\frac{4}{7}&-3\end{bmatrix}\begin{bmatrix}0&0&1\\0&1/14&0\\-1&5/14&2\end{bmatrix} \quad \frac{1}{14}C_2$$

$$\underset{C}{\sim}\begin{bmatrix}1&0&0\\-2+2&1&7-7\\2-\frac{8}{7}&-\frac{4}{7}&-3+4\end{bmatrix}\begin{bmatrix}0&0&1\\\frac{2}{14}&\frac{1}{14}&-7/14\\-1+\frac{10}{14}&\frac{5}{14}&2-\frac{5}{2}\end{bmatrix} \quad \begin{matrix}C_1+2C_2\\C_3-7C_2\end{matrix}$$

$$\underset{C}{\sim}\begin{bmatrix}1&0&0\\0&1&0\\\frac{6}{7}&-\frac{4}{7}&1\end{bmatrix}\begin{bmatrix}0&0&1\\1/7&1/14&-1/2\\-2/7&\frac{5}{14}&-1/2\end{bmatrix}$$

$$\underset{C}{\sim}\begin{bmatrix}1&0&0\\0&1&0\\\frac{6}{7}-\frac{6}{7}&-\frac{4}{7}+\frac{4}{7}&1\end{bmatrix}\begin{bmatrix}0-6/7&0+4/7&1\\\frac{1}{7}+\frac{3}{7}&\frac{1}{14}-\frac{2}{7}&-1/2\\-\frac{2}{7}+\frac{3}{7}&\frac{5}{14}-\frac{2}{7}&-1/2\end{bmatrix}$$

$$C_1-\frac{6}{7}C_3 \qquad C_2+\frac{4}{7}C_3$$

$$\underset{C}{\sim}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-6/7&4/7&1\\4/7&-3/14&-1/2\\1/7&1/14&-1/2\end{bmatrix}$$

$$\rightarrow A^{-1} = \begin{bmatrix}-6/7&4/7&1\\4/7&-3/14&-1/2\\1/7&1/14&-1/2\end{bmatrix}$$

Rank of a Matrix:-

The number of non-zero rows in Echelon form or reduced Echelon form of a matrix is called rank of a matrix.

A^{-1} by Row operation

* For a non-singular matrix A if $A = \begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$ we find A^{-1} using row operation as

Make in C_1

First $a_{11} = 1$ then $a_{21} = 0$ and $a_{31} = 0$

Make in C_2

First $a_{22} = 1$ then $a_{12} = 0$ and $a_{32} = 0$

Make in C_3

First $a_{33}=1$ then $a_{13}=0$ and $a_{23}=0$

A^{-1} by Column Operation

★ For a non-singular matrix A

if $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then we

find A^{-1} using Column operation as

Make in R_1

First $a_{11}=1$ then $a_{12}=0$ and $a_{13}=0$

Make in R_2

First $a_{22}=1$ then $a_{21}=0$ and $a_{23}=0$

Make in R_3

First $a_{33}=1$ then $a_{31}=0$ and $a_{32}=0$

Example:- Find the rank of the matrix $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 0 & 7 & -7 \\ 3 & 1 & 12 & -11 \end{bmatrix}$

Solution:-

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 0 & 7 & -7 \\ 3 & 1 & 12 & -11 \end{bmatrix}$$

$$R \begin{bmatrix} 1 & -1 & 2 & -3 \\ 2-2 & 0+2 & 7-4 & -7+6 \\ 3-3 & 1+3 & 12-6 & -11+9 \end{bmatrix} \begin{matrix} R_2-2R_1 \\ R_3-3R_1 \end{matrix}$$

$$R \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 2 & 3 & -1 \\ 0 & 4 & 6 & -2 \end{bmatrix}$$

$$R \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 4 & 6 & -2 \end{bmatrix} \frac{1}{2}R_2$$

$$R \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 4-4 & 6-6 & -2+2 \end{bmatrix} R_3-4R_2$$

$$R \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in Echelon form.

No. of non-zero rows $=2$

so Rank $=2$

Note:- If we reduce it into reduced echelon form then no change occurs in the rank of matrix as

$$R \begin{bmatrix} 1 & -1+1 & 2+3/2 & -3-1/2 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1+R_2$$

$$R \begin{bmatrix} 1 & 0 & 7/2 & -7/2 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in reduced Echelon form. No. of non-zero rows $=2$

so Rank $=2$

Exercise 3.4

Q1. If $A=\begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -1 \\ 5 & -1 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} -3 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$ then show that $A+B$ is symmetric.

Solution:-

$$A+B=\begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -1 \\ 5 & -1 & 0 \end{bmatrix}+\begin{bmatrix} -3 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$A+B=\begin{bmatrix} 1-3 & -2+1 & 5-2 \\ -2+1 & 3+0 & -1-1 \\ 5-2 & -1-1 & 0+2 \end{bmatrix}=\begin{bmatrix} -2 & -1 & 3 \\ -1 & 3 & -2 \\ 3 & -2 & 2 \end{bmatrix}$$

$$(A+B)^t=\begin{bmatrix} -2 & -1 & 3 \\ -1 & 3 & -2 \\ 3 & -2 & 2 \end{bmatrix}^t=\begin{bmatrix} -2 & -1 & 3 \\ -1 & 3 & -2 \\ 3 & -2 & 2 \end{bmatrix}=A+B$$

Hence $(A+B)$ is symmetric.

Q2. If $A=\begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}$, show that

i) $A+A^t$ is symmetric

ii) $A-A^t$ is skew symmetric

Solution:- i) $A+A^t$ is symmetric

$$A+A^t=\begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}+\begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}^t$$

$$=\begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}+\begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 1+1 & 2+3 & 0-1 \\ 3+2 & 2+2 & -1+3 \\ -1+0 & 3-1 & 2+2 \end{bmatrix}$$

$$A+A^t = \begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

$$\Rightarrow (A+A^t)^t = \begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix}^t$$

$$(A+A^t)^t = \begin{bmatrix} 2 & 5 & -1 \\ 5 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix} = A+A^t$$

Hence $A+A^t$ is symmetric

ii) $A-A^t$ is skew symmetric

$$A-A^t = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix}^t$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 2-3 & 0+1 \\ 3-2 & 2-2 & -1-3 \\ -1-0 & 3+1 & 2-2 \end{bmatrix}$$

$$A-A^t = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}$$

$$\Rightarrow (A-A^t)^t = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}^t$$

$$= - \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 4 \\ 1 & -4 & 0 \end{bmatrix}$$

$$\Rightarrow (A-A^t)^t = - \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix} = -(A-A^t)$$

Hence $A-A^t$ is skew symmetric

Q3. If A is any square matrix of order 3, show that

i) $A+A^t$ is symmetric and

ii) $A-A^t$ is skew symmetric

Solution:- i) $A+A^t$ is symmetric

We have to prove that $A+A^t$ is symmetric.

$$(A+A^t)^t = A^t + (A^t)^t$$
$$= A^t + A$$
$$= A+A^t$$

Since $(A^t)^t = A$

Hence $A+A^t$ is symmetric.

ii) $A-A^t$ is skew - symmetric

Solution:-

We have to prove that $A-A^t$ is skew symmetric.

$$(A-A^t)^t = A^t - (A^t)^t$$
$$= A^t - A$$
$$= -A + A^t$$
$$= -(A - A^t)$$

Since $(A^t)^t = A$

Hence $A-A^t$ is skew symmetric.

Q4. If the matrices A and B are symmetric and $AB = BA$, show that AB is symmetric.

Solution:-

Given $A^t = A$, $B^t = B$, $AB = BA$

Now $(AB)^t = B^t A^t$

$= BA \quad \because B^t = B, A^t = A$

$\Rightarrow (AB)^t = AB \quad \because BA = AB$

Hence AB is symmetric.

Q5. Show that AA^t and A^tA are symmetric for any matrix of order 2×3.

Solution:-

By the def., of symmetric matrix,

$A^t = A$

$(AA^t)^t = (A^t)^t A^t$ Since $(A^t)^t = A$

$= A A^t$

Hence $A A^t$ is symmetric.

Also,

$(A^t A)^t = A^t (A^t)^t$ Since $(A^t)^t = A$

$= A^t A$

Hence $A^t A$ is symmetric.

Q6. If $A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$, show that

i) $A + (\bar{A})^t$ is hermitian

ii) $A - (\bar{A})^t$ is skew-hermitian

Solution:- i) $A + (\bar{A})^t$ is hermitian

$A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$, $\bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & i \end{bmatrix}$

$(\bar{A})^t = \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$

$A + (\bar{A})^t = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} + \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$

$A + (\bar{A})^t = \begin{bmatrix} i-i & 1+i+1 \\ 1+1-i & -i+i \end{bmatrix} = \begin{bmatrix} 0 & 2+i \\ 2 & 0 \end{bmatrix}$

$\overline{A + (\bar{A})^t} = \begin{bmatrix} 0 & 2-i \\ 2 & 0 \end{bmatrix}$

$\left(\overline{A + (\bar{A})^t}\right)^t = \begin{bmatrix} 0 & 2 \\ 2-i & 0 \end{bmatrix} = A + (\bar{A})^t$

So $A + (\bar{A})^t$ is hermitian

ii) $A - (\bar{A})^t$ is skew-hermitian

$A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$, $\bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & i \end{bmatrix}$

$\Rightarrow (\bar{A})^t = \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$

$A - (\bar{A})^t = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} - \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$

$A - (\bar{A})^t = \begin{bmatrix} i+i & 1+i-1 \\ 1-1+i & -i-i \end{bmatrix} = \begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$

$\overline{A - (\bar{A})^t} = \begin{bmatrix} -2i & -i \\ -i & 2i \end{bmatrix}$

$\left(\overline{A - (\bar{A})^t}\right)^t = \begin{bmatrix} -2i & -i \\ -i & 2i \end{bmatrix} = -\begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$

$= -(A - (\bar{A})^t)$

Hence $A - (\bar{A})^t$ is skew-hermitian.

Q7. If A is symmetric or skew symmetric, show that A^2 is symmetric.

Solution:-

Given $A^t = A$ or $A^t = -A$

Now $(A^2)^t = (A \cdot A)^t$ Also $(A^2)^t = (A \cdot A)^t$

$= A^t \cdot A^t$ or $= A^t \cdot A^t$

$= A \cdot A$ $(\because A^t = A)$ or $= (-A)(-A)$ $(\because A^t = -A)$

$= A^2$ or $= A^2$

$\rightarrow A^2$ is symmetric $\rightarrow A^2$ is skew-symmetric

Q8. If $A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$, find $A(\overline{A})^t$

Solution:-

$A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$, $\overline{A} = \begin{bmatrix} 1 \\ 1-i \\ -i \end{bmatrix}$.

$(\overline{A})^t = \begin{bmatrix} 1 & 1-i & -i \end{bmatrix}$

$A(\overline{A})^t = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix} \begin{bmatrix} 1 & 1-i & -i \end{bmatrix}$

$= \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 1-i^2 & -i-i^2 \\ i & i-i^2 & -i^2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 1-(-1) & -i-(-1) \\ i & i-(-1) & -(-1) \end{bmatrix}$

$A(\overline{A})^t = \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 2 & 1-i \\ i & 1+i & 1 \end{bmatrix}$

Q9. Find the inverses of the following matrices. Also find their inverses by row and column operations.

i) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$

Solution:-

Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{vmatrix}$

Expand by R_2

$= -0 + (-2)\begin{vmatrix} 1 & -3 \\ -2 & 2 \end{vmatrix} - 0$

$|A| = -2(2-6) = -2(-4) = 8$

$\rightarrow |A| \neq 0$ so A^{-1} exists

By Adjoint method

$A_{11} = (-1)^{1+1}\begin{vmatrix} -2 & 0 \\ -2 & 2 \end{vmatrix} = (-4+0) = -4$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ -2 & 2 \end{vmatrix} = -(0+0) = 0$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -3 \\ -2 & 2 \end{vmatrix} = (0-4) = -4$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -3 \\ -2 & 2 \end{vmatrix} = -(4-6) = 2$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -3 \\ -2 & 2 \end{vmatrix} = (2-6) = -4$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -2 & -2 \end{vmatrix} = -(-2+4) = -2$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -3 \\ -2 & 0 \end{vmatrix} = (0-6) = -6$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix} = -(0+0) = 0$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = +(-2-0) = -2$

$\text{Adj } A = \begin{bmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & -2 & -2 \end{bmatrix}$

$\because A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{8}\begin{bmatrix} -4 & 2 & -6 \\ 0 & -4 & 0 \\ -4 & -2 & -2 \end{bmatrix}$

$\rightarrow A^{-1} = \begin{bmatrix} -4/8 & \frac{2}{8} & -\frac{6}{8} \\ 0/8 & -\frac{4}{8} & \frac{0}{8} \\ -4/8 & -\frac{2}{8} & -\frac{2}{8} \end{bmatrix}$

$\rightarrow A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & -3/4 \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -1/4 \end{bmatrix}$

By Row operation

$\because A = AI$

$\rightarrow A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$R \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2+2 & -2+4 & 2-6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0+2 & 0 & 1 \end{bmatrix} R_3+2R_1$

$R \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ 0 & 2 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$R \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 2 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 2 & 0 & 1 \end{bmatrix} -\frac{1}{2}R_2$

$R \begin{bmatrix} 1 & 2-2 & -3 \\ 0 & 1 & 0 \\ 0 & 2-2 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0+1 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 2 & 0+1 & 1 \end{bmatrix} \begin{matrix} R_1-2R_2 \\ R_3-2R_2 \end{matrix}$

$$\underset{\sim}{R}\begin{bmatrix}1&0&-3\\0&1&0\\0&0&-4\end{bmatrix}\begin{bmatrix}1&1&0\\0&-\frac{1}{2}&0\\2&1&1\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1&0&-3\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&-\frac{1}{2}&0\\-\frac{1}{2}&-\frac{1}{4}&-\frac{1}{4}\end{bmatrix}\ -\frac{1}{4}R_3$$

$$\underset{\sim}{R}\begin{bmatrix}1&0&-3+3\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1-\frac{3}{2}&1-\frac{3}{4}&-\frac{3}{4}\\0&-\frac{1}{2}&0\\-\frac{1}{2}&-\frac{1}{4}&-\frac{1}{4}\end{bmatrix}\ R_1+3R_3$$

$$\underset{\sim}{R}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-\frac{1}{2}&\frac{1}{4}&-\frac{3}{4}\\0&-\frac{1}{2}&0\\-\frac{1}{2}&-\frac{1}{4}&-\frac{1}{4}\end{bmatrix}$$

$$\rightarrow A^{-1}=\begin{bmatrix}-\frac{1}{2}&\frac{1}{4}&-\frac{3}{4}\\0&-\frac{1}{2}&0\\-\frac{1}{2}&-\frac{1}{4}&-\frac{1}{4}\end{bmatrix}$$

By Column Operation

$\because A = AI$

$$A=\begin{bmatrix}1&2&-3\\0&-2&0\\-2&-2&2\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\underset{\sim}{C}\begin{bmatrix}1&2-2&-3+3\\0&-2&0\\-2&-2+4&2-6\end{bmatrix}\begin{bmatrix}1&0-2&0+3\\0&1&0\\0&0&1\end{bmatrix}\ \begin{matrix}C_2-2C_1\\C_3+3C_1\end{matrix}$$

$$\underset{\sim}{C}\begin{bmatrix}1&0&0\\0&-2&0\\-2&2&-4\end{bmatrix}\begin{bmatrix}1&-2&3\\0&1&0\\0&0&1\end{bmatrix}$$

$$\underset{\sim}{C}\begin{bmatrix}1&0&0\\0&1&0\\-2&-1&-4\end{bmatrix}\begin{bmatrix}1&1&3\\0&-\frac{1}{2}&0\\0&0&1\end{bmatrix}\ -\frac{1}{2}C_2$$

$$\underset{\sim}{C}\begin{bmatrix}1&0&0\\0&1&0\\-2&-1&1\end{bmatrix}\begin{bmatrix}1&1&-\frac{3}{4}\\0&-\frac{1}{2}&0\\0&0&-\frac{1}{4}\end{bmatrix}\ -\frac{1}{4}C_3$$

$$\underset{\sim}{C}\begin{bmatrix}1&0&0\\0&1&0\\-2+2&-1+1&1\end{bmatrix}\begin{bmatrix}1-\frac{3}{2}&1-\frac{3}{4}&-\frac{3}{4}\\0&-\frac{1}{2}&0\\0-\frac{1}{2}&0-\frac{1}{4}&-\frac{1}{4}\end{bmatrix}\ \begin{matrix}C_1+2C_3\\C_2+C_3\end{matrix}$$

$$\underset{\sim}{C}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-\frac{1}{2}&\frac{1}{4}&-\frac{3}{4}\\0&-\frac{1}{2}&0\\-\frac{1}{2}&-\frac{1}{4}&-\frac{1}{4}\end{bmatrix}$$

$$\rightarrow A^{-1}=\begin{bmatrix}-\frac{1}{2}&\frac{1}{4}&-\frac{3}{4}\\0&-\frac{1}{2}&0\\-\frac{1}{2}&-\frac{1}{4}&-\frac{1}{4}\end{bmatrix}$$

ii) Let $A=\begin{bmatrix}1&2&-1\\0&-1&3\\1&0&2\end{bmatrix}$

$$|A|=\begin{vmatrix}1&2&-1\\0&-1&3\\1&0&2\end{vmatrix}$$

$$|A|=\begin{vmatrix}1&2&-1\\0&-1&3\\0&2&3\end{vmatrix}\ R_3-R_1$$

$$=1\begin{vmatrix}-1&3\\-2&3\end{vmatrix}-0+0 \quad \text{Expand by } C_1$$

$|A| = -3+6 = 3 \neq 0$ so A^{-1} exists

By Adjoint method

$$A_{11}=(-1)^{1+1}\begin{vmatrix}-1&3\\0&2\end{vmatrix}=(-2-0)=-2$$

$$A_{12}=(-1)^{1+2}\begin{vmatrix}0&3\\1&2\end{vmatrix}=-(0-3)=3$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}0&-1\\1&0\end{vmatrix}=(0+1)=1$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}2&-1\\0&2\end{vmatrix}=-(4+0)=-4$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}1&-1\\1&2\end{vmatrix}=(2+1)=3$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}1&2\\1&0\end{vmatrix}=-(0-2)=2$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}2&-1\\-1&3\end{vmatrix}=(6-1)=5$$

$$A_{32}=(-1)^{3+2}\begin{vmatrix}1&-1\\0&3\end{vmatrix}=-(3+0)=-3$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix}1&2\\0&-1\end{vmatrix}=(-1-0)=-1$$

$$\text{adj}\,A=\begin{bmatrix}-2&-4&5\\3&3&-3\\1&2&-1\end{bmatrix}$$

$$\because A^{-1}=\frac{1}{|A|}\text{adj}\,A=\frac{1}{3}\begin{bmatrix}-2&-4&5\\3&3&-3\\1&2&-1\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}-\frac{2}{3}&-\frac{4}{3}&\frac{5}{3}\\1&1&-1\\\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}\end{bmatrix}$$

By Row Operation

$\because A = AI$

$$\rightarrow A=\begin{bmatrix}1&2&-1\\0&-1&3\\1&0&2\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & -1\\ 0 & -1 & 3\\ 1-1 & 0-2 & 2+1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0-1 & 0 & 1\end{bmatrix} \quad R_3 - R_1$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & -1\\ 0 & -1 & 3\\ 0 & -2 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2 & -1\\ 0 & 1 & -3\\ 0 & -2 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ -1 & 0 & 1\end{bmatrix} \quad (-1)R_2$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 2-2 & -1+6\\ 0 & 1 & -3\\ 0 & -2+2 & 3-6\end{bmatrix}\begin{bmatrix}1 & 0+2 & 0\\ 0 & -1 & 0\\ -1 & 0-2 & 1\end{bmatrix} \quad \begin{matrix}R_1-2R_2\\ R_3+2R_2\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 5\\ 0 & 1 & -3\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 0 & -1 & 0\\ -1 & -2 & 1\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 5\\ 0 & 1 & -3\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 0 & -1 & 0\\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3}\end{bmatrix} \quad -\frac{1}{3}R_3$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 5-5\\ 0 & 1 & -3+3\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1-\frac{5}{3} & 2-\frac{10}{3} & 0+\frac{5}{3}\\ 0+1 & -1+2 & 0-1\\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3}\end{bmatrix} \quad \begin{matrix}R_1-5R_3\\ R_2+3R_3\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-\frac{2}{3} & -\frac{4}{3} & \frac{5}{3}\\ 1 & 1 & -1\\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix}$$

$$\rightarrow A^{-1} = \begin{bmatrix}-\frac{2}{3} & -\frac{4}{3} & \frac{5}{3}\\ 1 & 1 & -1\\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix}$$

By Column Operation

$\because A = AI$

$$\rightarrow A = \begin{bmatrix}1 & 2 & -1\\ 0 & -1 & 3\\ 1 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\underset{\sim}{C}\begin{bmatrix}1 & 2-2 & -1+1\\ 0 & -1 & 3\\ 1 & 0-2 & 2+1\end{bmatrix}\begin{bmatrix}1 & 0-2 & 0+1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad \begin{matrix}C_2-2C_1\\ C_3+C_1\end{matrix}$$

$$\underset{\sim}{C}\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 3\\ 1 & -2 & 3\end{bmatrix}\begin{bmatrix}1 & -2 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\underset{\sim}{C}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 3-3\\ 1 & 2 & 3-6\end{bmatrix}\begin{bmatrix}1 & 2 & 1-6\\ 0 & -1 & 0+3\\ 0 & 0 & 1\end{bmatrix} \quad C_3-3C_2$$

$$\underset{\sim}{C}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 2 & -3\end{bmatrix}\begin{bmatrix}1 & 2 & -5\\ 0 & -1 & 3\\ 0 & 0 & 1\end{bmatrix}$$

$$\underset{\sim}{C}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & \frac{5}{3}\\ 0 & -1 & -1\\ 0 & 0 & -\frac{1}{3}\end{bmatrix} \quad -\frac{1}{3}C_3$$

$$\underset{\sim}{C}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1-1 & 2-2 & 1\end{bmatrix}\begin{bmatrix}1-\frac{5}{3} & 2-\frac{10}{3} & \frac{5}{3}\\ 0+1 & -1+2 & -1\\ 0+\frac{1}{3} & 0+\frac{2}{3} & -\frac{1}{3}\end{bmatrix} \quad \begin{matrix}C_1-C_3\\ C_2-2C_3\end{matrix}$$

$$\underset{\sim}{C}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-\frac{2}{3} & -\frac{4}{3} & \frac{5}{3}\\ 1 & 1 & -1\\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix}$$

$$\rightarrow A^{-1} = \begin{bmatrix}-\frac{2}{3} & -\frac{4}{3} & \frac{5}{3}\\ 1 & 1 & -1\\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix}$$

iii) Let $A = \begin{bmatrix}1 & -3 & 2\\ 2 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}$

Solution:-

$$|A| = \begin{vmatrix}1 & -3 & 2\\ 2 & 1 & 0\\ 0 & -1 & 1\end{vmatrix}$$

$$= \begin{vmatrix}1 & -1 & 2\\ 2 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} \quad C_2 + C_3$$

Expand by R_3

$$= 0 - 0 + 1\begin{vmatrix}1 & -1\\ 2 & 1\end{vmatrix}$$

$$|A| = 1(1+2) = 3 \neq 0$$

so A^{-1} exists

By Adjoint Method

$$A_{11} = (-1)^{1+1}\begin{vmatrix}1 & 0\\ -1 & 1\end{vmatrix} = (1+0) = 1$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix}2 & 0\\ 0 & 1\end{vmatrix} = -(2-0) = -2$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix}2 & 1\\ 0 & -1\end{vmatrix} = (-3+2) = -1$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix}-3 & 2\\ -1 & 1\end{vmatrix} = -(-3+2) = 1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix} = (1-0) = 1$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -3 \\ 0 & -1 \end{vmatrix} = -(-1+0) = 1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -3 & 2 \\ 1 & 0 \end{vmatrix} = (0-2) = -2$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -(0-4) = 4$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = (1+6) = 7$$

$$\text{adj}\, A = \begin{bmatrix} 1 & 1 & -2 \\ -2 & 1 & 4 \\ -2 & 1 & 7 \end{bmatrix}$$

$$\because A^{-1} = \frac{1}{|A|}\text{adj}\, A$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & -2 \\ -2 & 1 & 4 \\ -2 & 1 & 7 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/3 & 1/3 & -2/3 \\ -2/3 & 1/3 & 4/7 \\ -2/3 & 1/3 & 7/3 \end{bmatrix}$$

By Row Operation

$$\because A = AI$$

$$\rightarrow A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -3 & 2 \\ 2-2 & 1+6 & 0-4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0-2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2-2R_1$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -3 & 2 \\ 0 & 7 & -4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 1 \\ 0 & 7 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} R_{2\leftrightarrow 3}$$

$$\underset{\sim}{R}\begin{bmatrix} 1+0 & -3+3 & 2-3 \\ 0 & 1 & -1 \\ 0 & 7-7 & -4+7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0-3 \\ 0 & 0 & -1 \\ -2 & 1 & 0+7 \end{bmatrix} \begin{matrix} R_1+3R_2 \\ R_3-7R_2 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & -1 \\ -2 & 1 & 7 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & -1 \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix} \frac{1}{3}R_3$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & -1+1 \\ 0 & 1 & -1+1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1-\frac{2}{3} & 0+\frac{1}{3} & -3+\frac{7}{3} \\ 0-\frac{2}{3} & 0+\frac{1}{3} & -1+\frac{7}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{7}{3} \end{bmatrix} \begin{matrix} R_1+R_3 \\ R_2+R_3 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/3 & 1/3 & -2/3 \\ -2/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & 7/3 \end{bmatrix}$$

$$\rightarrow A^{-1} = \begin{bmatrix} 1/3 & 1/3 & -2/3 \\ -2/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & 7/3 \end{bmatrix}$$

By Column Operation

$$\because A = AI$$

$$\rightarrow A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & -3+3 & 2-2 \\ 2 & 1+6 & 0-4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0+3 & 0-2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} C_2+3C_1 \\ C_3-2C_1 \end{matrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 7 & -4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 7-8 & -4 \\ 0 & -1+2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3-4 & -2 \\ 0 & 1 & 0 \\ 0 & 0+2 & 1 \end{bmatrix} C_2+2C_3$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & -4 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 0 \\ 0 & -2 & 1 \end{bmatrix} (-1)C_2$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 2-2 & 1 & -4+4 \\ 0+2 & -1 & 1-4 \end{bmatrix}\begin{bmatrix} 1-2 & 1 & -2+4 \\ 0+2 & -1 & 0-4 \\ 0+4 & -2 & 1-8 \end{bmatrix} \begin{matrix} C_1-2C_2 \\ C_3+4C_2 \end{matrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & -3 \end{bmatrix}\begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -4 \\ 4 & -2 & -7 \end{bmatrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & -2/3 \\ 2 & -1 & 4/3 \\ 4 & -2 & 7/3 \end{bmatrix} -\frac{1}{3}C_3$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2-2 & -1+1 & 1 \end{bmatrix}\begin{bmatrix} -1+4/3 & 1-2/3 & -2/3 \\ 2-8/3 & -1+4/3 & 4/3 \\ 2-14/3 & -2+7/3 & 7/3 \end{bmatrix} \begin{matrix} C_1-2C_3 \\ C_2+C_3 \end{matrix}$$

$$\underset{\sim}{C}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/3 & 1/3 & -2/3 \\ -2/3 & 1/3 & 4/3 \\ -2/3 & 1/3 & 7/3 \end{bmatrix}$$

$$\rightarrow A^{-1} = \begin{bmatrix} 1/3 & 1/3 & -2/3 \\ -2/3 & 1/3 & 4/3 \\ -2/3 & 1/3 & 7/3 \end{bmatrix}$$

Q10. Find the rank of the following matrices.

i) $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -6 & 5 & 1 \\ 3 & 5 & 4 & -3 \end{bmatrix}$

Solution:-

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -6 & 5 & 1 \\ 3 & 5 & 4 & -3 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2-2 & -6+2 & 5-4 & 1-2 \\ 3-3 & 5+3 & 4-6 & -3-3 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & -4 & 1 & -1 \\ 0 & 8 & -2 & 6 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 8 & -2 & 6 \end{bmatrix} -\tfrac{1}{4}R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 8-8 & -2+2 & -6-2 \end{bmatrix} R_3 - 8R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1/4 & 1/4 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & -1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix} -\tfrac{1}{8}R_3$$

which is in Echelon form

No. of non-zero rows = 3

so, Rank = 3

ii) $\begin{bmatrix} 1 & -4 & -7 \\ 2 & -5 & 1 \\ 1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}$

Solution:-

$$\begin{bmatrix} 1 & -4 & -7 \\ 2 & -5 & 1 \\ 1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -4 & -7 \\ 2-2 & -5+8 & 1 \\ 1-1 & -2+4 & \\ 3-3 & -7+12 & \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 - 3R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -4 & -7 \\ 0 & 3 & 15 \\ 0 & 2 & 10 \\ 0 & 5 & 25 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -4 & -7 \\ 0 & 1 & 5 \\ 0 & 2 & 10 \\ 0 & 5 & 25 \end{bmatrix} \tfrac{1}{3}R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -4 & -7 \\ 0 & 1 & 5 \\ 0 & 2-2 & 10-10 \\ 0 & 5-5 & 25-25 \end{bmatrix} \begin{matrix} R_3 - 2R_2 \\ R_4 - 5R_2 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -4 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is in Echelon form

No. of non-zero rows = 2

so, rank = 2

iii) $\begin{bmatrix} 3 & -1 & 3 & -1 \\ 1 & 2 & -1 & -2 \\ 2 & 3 & 4 & 5 \\ 2 & 5 & -2 & 3 \end{bmatrix}$

Solution:-

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 3 & -1 & 3 & 0 & -1 \\ 2 & 3 & 4 & 2 & 5 \\ 2 & 5 & -2 & -3 & 3 \end{bmatrix} R_{1 \leftrightarrow 2}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 3-3 & -1-6 & 3+3 & 0+9 & -1+6 \\ 2-2 & 3-4 & 4+2 & 2+6 & 5+4 \\ 2-2 & 5-4 & -2+2 & -3+9 & 3+4 \end{bmatrix} \begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_1 \\ R_4 - 2R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & -7 & 6 & 9 & 5 \\ 0 & -1 & 6 & 8 & 9 \\ 0 & 1 & 0 & 3 & 7 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & -1 & 6 & 8 & 9 \\ 0 & -7 & 6 & 9 & 5 \end{bmatrix} R_{2 \leftrightarrow 4}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & -1+1 & 6+0 & 8+3 & 9+7 \\ 0 & -7+7 & 6 & 9+21 & 5+49 \end{bmatrix} \begin{matrix} R_3 + R_2 \\ R_4 + 7R_2 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 11 & 16 \\ 0 & 0 & 6 & 30 & 54 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 6 & 11 & 16 \\ 0 & 0 & 1 & 5 & 9 \end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1&2&-1&-3&-2\\0&1&0&3&7\\0&0&1&5&9\\0&0&6&11&16\end{bmatrix} R_{3\leftrightarrow 4}$$

$$\underset{\sim}{R}\begin{bmatrix}1&2&-1&-3&-2\\0&1&0&3&7\\0&0&1&5&9\\0&0&6-6&11-30&16-54\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1&2&-1&-3&-2\\0&1&0&3&7\\0&0&1&5&9\\0&0&0&-19&-38\end{bmatrix}$$

$$\underset{\sim}{R}\begin{bmatrix}1&2&-1&-3&-2\\0&1&0&3&7\\0&0&1&5&9\\0&0&0&1&2\end{bmatrix} \frac{-1}{19}R_4$$

which is in Echelon form

No. of non-zero rows = 4

so, Rank = 4

System of Linear Equations

i) The equation $ax+by=k$ where $a\neq 0$, $b\neq 0$, $k\neq 0$ is called a non-homogeneous linear equation in two variables x and y.

If $ax+by=0$ then it is called Homogeneous linear equation.

ii) The equations

$$\left.\begin{aligned}a_1x+b_1y&=k_1\\a_2x+b_2y&=k_2\end{aligned}\right\}$$ are called

system of non-homogeneous linear equations in two variables x and y. If k_1, k_2 are not both zero or at least one of k_1 or k_2 is non-zero.

If $$\left.\begin{aligned}a_1x+b_1y&=0\\a_2x+b_2y&=0\end{aligned}\right\}$$ then

it is called system of homogeneous linear equations.

iii) The equations

$$\left.\begin{aligned}a_1x+b_1y+c_1z&=k_1\\a_2x+b_2y+c_2z&=k_2\\a_3x+b_3y+c_3z&=k_3\end{aligned}\right\}$$

are called system of non-homogeneous linear equations in three variables x, y and z.

If k_1, k_2 and k_3 are not at all zero. If

$$\left.\begin{aligned}a_1x+b_1y+c_1z&=0\\a_2x+b_2y+c_2z&=0\\a_3x+b_3y+c_3z&=0\end{aligned}\right\}$$ then

it is called system of homogeneous linear equations.

System of Homogeneous linear equations

consider

$$a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1$$
$$a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2$$
$$a_{31}x_1+a_{32}x_3+a_{33}x_3=b_3$$

In matrix form

$$\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

then $AX=B$

where $A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$, $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

$B=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$

Here A is matrix of coefficients.

and $A_b=\left[\begin{array}{ccc|c}a_{11}&a_{12}&a_{13}&b_1\\a_{21}&a_{22}&a_{23}&b_2\\a_{31}&a_{32}&a_{33}&b_3\end{array}\right]$

Here A_b is called augmented matrix.

Consistency of a System

A system of linear equations is said to be consistent if

i) It has unique solution or
ii) It has an unlimited number of solution.

Inconsistency of a System

A system of linear equations is said to be inconsitent if it has no solution.

Remember some important note

i) If a system of linear equations is consitent and has unique solution then

$$\text{Rank}(A) = \text{Rank}(A_b)$$

ii) If a system of linear equations is consistent and has unlimited solutions. then

$$\text{Rank}(A) = \text{Rank}(A_b)$$

also $\text{Rank}(A) <$ No. of variables used in the system

iii) If a system of linear equations is inconsitent i-e., it has no solution. then $\text{Rank}(A) \neq \text{Rank}(A_b)$

Trivial Solution

If we solve a system and get values of all variables, zero, then the solution is called trivial solution.

For trivial solution $|A| \neq 0$

Non-trivial Solution

Solutions in which at least one of the variables has a value different from zero is called non-trivial solution.

For non-trivial solution $|A| = 0$ also

$\text{Rank}(A) <$ No. of variables used in the system

Example:- (Page # 128)

Solve
$$2x + 5y - z = 5$$
$$3x + 4y + 2z = 11$$
$$x + 2y - 2z = -3$$
by reducing the augmented matrix into reduced echelon form.

Solution:- The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & 5 & -1 & 5 \\ 3 & 4 & 2 & 11 \\ 1 & 2 & -2 & -3 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 2 & -2 & -3 \\ 3 & 4 & 2 & 11 \\ 2 & 5 & -1 & 5 \end{array}\right] R_1 \leftrightarrow 3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 2 & -2 & -3 \\ 3-3 & 4-6 & 2+6 & 11+9 \\ 2-2 & 5-4 & -1+4 & 5+6 \end{array}\right] \begin{array}{l} R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 2 & -2 & -3 \\ 0 & -2 & 8 & 20 \\ 0 & 1 & 3 & 11 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 2 & -2 & -3 \\ 0 & 1 & -4 & -10 \\ 0 & 1 & 3 & 11 \end{array}\right] -\tfrac{1}{2} R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 2-2 & -2+8 & -3+20 \\ 0 & 1 & -4 & -10 \\ 0 & 1-1 & 3+4 & 11+10 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ R_3 - R_2 \end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 6 & 17 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 7 & 21 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 6 & 17 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{array}\right] \tfrac{1}{7} R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 6-6 & 17-18 \\ 0 & 1 & -4+4 & -10+12 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_1 - 6R_3 \\ R_2 + 4R_3 \end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Thus $x = -1$, $y = 2$, $z = 3$

$\because$ system has unique solution so system is Consitent.

Example:- (Page # 129)

Solve
$$x+y+2z=1$$
$$2x-y+7z=11$$
$$3x+5y+4z=-3$$

By reducing the augmented matrix into reduced echelon form.

Solution:-

The augmented matrix of the system is

$$A_b=\left[\begin{array}{ccc:c} 1 & 1 & 2 & 1 \\ 2 & -1 & 7 & 11 \\ 3 & 5 & 4 & -3 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 1 & 2 & 1 \\ 2-2 & -1-2 & 7-4 & 11-2 \\ 3-3 & 5-3 & 4-6 & -3-3 \end{array}\right] \begin{array}{l} R_2-2R_1 \\ R_3-3R_1 \end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 1 & 2 & 1 \\ 0 & -3 & 3 & 9 \\ 0 & 2 & -2 & -6 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & -3 \\ 0 & 2 & -2 & -6 \end{array}\right] -\tfrac{1}{3}R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 1-1 & 2+1 & 1+3 \\ 0 & 1 & -1 & -3 \\ 0 & 2-2 & -2+2 & -6+6 \end{array}\right] \begin{array}{l} R_1-R_2 \\ R_3-2R_2 \end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & 3 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow x+3z=-4 \longrightarrow (i)$$
$$y-z=-3 \longrightarrow (ii)$$
$$0z=0 \longrightarrow (iii)$$

The equation (iii) is satisfied for any value of z.

Let $z=t,\ t\in R$ then (i) and (ii) becomes

$$x+3t=4 \quad , \quad y-t=-3$$
$$x=4-3t \quad , \quad y=t-3$$

so $x=4-3t,\ y=t-3,\ z=t$

$\therefore$ system has unlimited solution so system is consistent.

Example:- (Page # 130)

Solve
$$x-y+2z=1$$
$$2x-6y+5z=7$$
$$3x+5y+4z=-3$$

By reducing the augmented matrix into reduced echelon form.

Solution:-

The augmented matrix of the system is

$$A_b=\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & -6 & 5 & 7 \\ 3 & 5 & 4 & -3 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2-2 & -6+2 & 5-4 & 7-2 \\ 3-3 & 5+3 & 4-6 & -3-3 \end{array}\right] \begin{array}{l} R_2-2R_1 \\ R_3-3R_1 \end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & -4 & 1 & 5 \\ 0 & 8 & -2 & -6 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1/4 & -5/4 \\ 0 & 8 & -2 & -6 \end{array}\right] -\tfrac{1}{4}R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & -1+1 & 2-\frac{1}{4} & 1-5/4 \\ 0 & 1 & -1/4 & -5/4 \\ 0 & 8-8 & -2+2 & -6+10 \end{array}\right] \begin{array}{l} R_1+R_2 \\ R_3-8R_2 \end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 7/4 & -1/4 \\ 0 & 1 & -1/4 & -5/4 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

$$\Rightarrow x+\tfrac{7}{4}z=-\tfrac{1}{4} \longrightarrow (i)$$
$$y-\tfrac{1}{4}z=-\tfrac{5}{4} \longrightarrow (ii)$$
$$0z=4 \longrightarrow (iii)$$

The equation (iii) is not satisfied for any value of z. So system has no solution.

$\therefore$ system is inconsistent.

Example:- (Page # 131)

Solve the system of Homogeneous linear equations

$$x_1+x_2+x_3=0 \longrightarrow (i)$$
$$x_1-x_2+3x_3=0 \longrightarrow (ii)$$
$$x_1+3x_2-x_3=0 \longrightarrow (iii)$$

Solution:- In matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\because |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= 1\begin{vmatrix} -2 & 2 \\ 2 & -2 \end{vmatrix} - 0 + 0 \quad \text{Expanding by } C_1$$

$$\rightarrow |A| = 4 - 4 = 0$$

$\because |A| = 0$ so system has non-trivial solution.

Now we solve (i) and (ii) to find x_1 and x_2

By (i) + (ii) $\rightarrow 2x_1 + 4x_3 = 0$

$\rightarrow x_1 = -2x_3$

By (ii) − (i) $\rightarrow -2x_2 + 2x_3 = 0$

$\rightarrow x_2 = x_3$

Put $x_1 = -2x_3$ and $x_2 = x_3$ in (iii)

$$-2x_3 + 3(x_3) - x_3 = 0$$

$$-3x_3 + 3x_3 = 0$$

$$\rightarrow 0 = 0$$

Eq (iii) is satisfied.

Let $x_3 = t$, $t \in R$

then $x_2 = t$ and $x_1 = -2t$

so $x_1 = -2t$, $x_2 = t$, $x_3 = t$

Hence the system has unlimited solutions.

Example:- (Page # 152)

Solve the system of Homogeneous linear equations

$$x_1 + x_2 + x_3 = 0 \longrightarrow (i)$$

$$x_1 - x_2 + 3x_3 = 0 \longrightarrow (ii)$$

$$x_1 + 3x_2 - 2x_3 = 0 \longrightarrow (iii)$$

Solution:-

In matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & 3 & -2 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\because |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 2 & -3 \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= 1\begin{vmatrix} -2 & 2 \\ 2 & -3 \end{vmatrix} - 0 + 0 \quad \text{Expanding by } C_1$$

$$|A| = 1(6-4) = 2$$

$\because |A| \neq 0$ Thus system has trivial solution.

Now we solve (i) and (ii) to find x_1 and x_2

By (i) + (ii) $\rightarrow 2x_1 + 4x_3 = 0$

$\rightarrow x_1 = -2x_3$

By (ii) − (i) $\rightarrow -2x_2 + 2x_3 = 0$

$\rightarrow x_2 = x_3$

Put $x_1 = -2x_3$ and $x_2 = x_3$ in (iii)

$$-2x_3 + 3(x_3) - 2x_3 = 0$$

$$\rightarrow 3x_3 = 0$$

$$\rightarrow x_3 = 0$$

Eq (iii) is not satisfied. it is only satisfied if $x_3 = 0$ Now if $x_3 = 0$ then $x_1 = 0$ and $x_2 = 0$

$\because$ values of all variables are zero, so system has trivial solution.

★ How to Solve Non-Homogeneous Linear Equations ★

Non-homogeneous linear equations can be solved by following three methods.

i) Using matrices

ii) Using Echelon and Reduced echelon form

iii) Using Crammer's Rule

Example 1. Use matrices to solve the system

$$x_1 - 2x_2 + x_3 = -4$$
$$2x_1 - 3x_2 + 2x_3 = -6$$
$$2x_1 + 2x_2 + x_3 = 5$$

Solution:- In matrix form

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix}$$

$$A \quad X = B \quad \text{where}$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix}$$

Now

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -2 & 1 \\ 0 & -3 & 2 \\ 1 & 2 & 1 \end{vmatrix} \quad C_1 - C_3$$

$$= 0 - 0 + \begin{vmatrix} -2 & 1 \\ -3 & 2 \end{vmatrix} \quad \text{Expanding by } C_1$$

$$|A| = -4 + 3 = -1$$

$$\Rightarrow |A| = -1 \neq 0 \quad \text{so } A^{-1} \text{ exists}$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -3 & 2 \\ 2 & 1 \end{vmatrix} = (-3-4) = -7$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = -(2-4) = 2$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -3 \\ 2 & 2 \end{vmatrix} = (4+6) = 10$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} = -(-2-2) = 4$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = (1-2) = -1$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix} = -(2+4) = -6$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ -3 & 2 \end{vmatrix} = (-4+3) = -1$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = -(2-2) = 0$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} = (-3+4) = 1$$

$$\text{adj } A = \begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix}$$

$$\because \text{Now } A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix}$$

$$\because AX = B$$

$$\Rightarrow X = A^{-1}B$$

$$\Rightarrow X = \begin{bmatrix} 7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -28+24+5 \\ 8-6+0 \\ 40-36-5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow x_1 = 1, \quad x_2 = 2, \quad x_3 = -1$$

Example 2. Solve the system

$$\left.\begin{aligned} x_1 + 3x_2 + 2x_3 &= 3 \\ 4x_1 + 5x_2 - 3x_3 &= -3 \\ 3x_1 - 2x_2 + 17x_3 &= 42 \end{aligned}\right\}$$

by reducing its augmented matrix to the echelon form and reduced echelon form.

Solution:- Solution by Echelon form

The augmented matrix is

$$A_b = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 4 & 5 & -3 & -3 \\ 3 & -2 & 17 & 42 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & 3 & 2 & 3\\ 4-4 & 5-12 & -3-8 & -3-12\\ 3-3 & -2-9 & 17-6 & 42-9\end{array}\right] \begin{array}{l}R_2-4R_1\\ R_3-3R_1\end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & 3 & 2 & 3\\ 0 & -7 & -11 & -15\\ 0 & -11 & 11 & 33\end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & 3 & 2 & 3\\ 0 & -7 & -11 & -15\\ 0 & 1 & -1 & -3\end{array}\right] \frac{-1}{11}R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & 3 & 2 & 3\\ 0 & 1 & -1 & -3\\ 0 & -7 & -11 & -15\end{array}\right] R_2 \longleftrightarrow 3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & 3 & 2 & 3\\ 0 & 1 & -1 & -3\\ 0 & -7+7 & -11-7 & -15-21\end{array}\right] R_3+7R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & 3 & 2 & 3\\ 0 & 1 & -1 & -3\\ 0 & 0 & -18 & -36\end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & 3 & 2 & 3\\ 0 & 1 & -1 & -3\\ 0 & 0 & 1 & 2\end{array}\right] -\frac{1}{18}R_3$$

$\Rightarrow x_1 + 3x_2 + 2x_3 = 3 \longrightarrow$ (i)

$x_2 - x_3 = -3 \longrightarrow$ (ii)

$x_3 = 2 \longrightarrow$ (iii)

Put $x_3 = 2$ in (i) and (ii)

$x_2 - 2 = -3 \Rightarrow x_2 = -1$

Put $x_3 = 2, x_2 = -1$ in (ii)

$x_1 + 3(-1) + 2(2) = 3$

$\Rightarrow x_1 - 3 + 4 = 3$

$\Rightarrow x_1 = 2$

Thus $x_1 = 2,\ x_2 = -1,\ x_3 = 2$

Solution by Reduced Echelon form

we reduce $\left[\begin{array}{ccc|c}1 & 3 & 2 & 3\\ 0 & 1 & -1 & -3\\ 0 & 0 & 1 & 2\end{array}\right]$

into reduced echelon form

$$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & 3-3 & 2+3 & 3+9\\ 0 & 1 & -1 & -3\\ 0 & 0 & 1 & 2\end{array}\right] R_1-2R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & 0 & 5 & 12\\ 0 & 1 & -1 & -3\\ 0 & 0 & 1 & 2\end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & 0 & 5-5 & 12-10\\ 0 & 1 & -1+1 & -3+2\\ 0 & 0 & 1 & 2\end{array}\right] \begin{array}{l}R_1-5R_3\\ R_2+R_3\end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c}1 & 0 & 0 & 2\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 2\end{array}\right]$$

$\Rightarrow x_1 = 2,\ x_2 = -1,\ x_3 = 2$

Cramer's Rule

consider a system

$$\left.\begin{array}{l}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1\\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2\\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3\end{array}\right\}$$

In matrix form

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$$

$A\,X = B \Rightarrow X = A^{-1}B$

$\because A^{-1}$ exists if $|A| \neq 0$ Now

$$A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix},\ X = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix},\ B = \begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$$

$$\text{Matrix of cofactor} = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{bmatrix}$$

$$\text{so } X = \frac{1}{|A|}\begin{bmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{bmatrix}\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \frac{1}{|A|}\begin{bmatrix}b_1A_{11} + b_2A_{21} + b_3A_{31}\\ b_1A_{12} + b_2A_{22} + b_3A_{32}\\ b_1A_{13} + b_2A_{23} + b_3A_{33}\end{bmatrix}$$

$$\Rightarrow x_1 = \frac{b_1A_{11} + b_2A_{21} + b_3A_{31}}{A}$$

or $x_1 = \dfrac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|A|}$

or $x_1 = \dfrac{|A_1|}{|A|}$, $A_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{23} & a_{33} \end{vmatrix}$

also, $x_2 = \dfrac{b_1 A_{12} + b_2 A_{22} + b_3 A_{22}}{|A|}$

$x_2 = \dfrac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|A|}$

or $x_2 = \dfrac{|A_2|}{|A|}$, $|A_2| = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$

and $x_3 = \dfrac{b_1 A_{13} + b_2 A_{23} + b_3 A_{33}}{|A|}$

$x_3 = \dfrac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|A|}$

or $x_3 = \dfrac{|A_3|}{|A|}$, $|A_3| = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$

Thus $x_1 = \dfrac{|A_1|}{|A|}$, $x_2 = \dfrac{|A_2|}{|A|}$, $x_3 = \dfrac{|A_3|}{|A|}$

Example 3. Use Cramer's rule to solve the system

$3x_1 + x_2 - x_3 = -4$
$x_1 + x_2 - x_3 = -4$
$-x_1 + 2x_2 - x_3 = 1$

Solution:- In matrix form

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 1 \end{bmatrix}$$

$A \; X = B$ where

$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & 2 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} -4 \\ -4 \\ 1 \end{bmatrix}$

$|A| = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & 2 & -1 \end{vmatrix}$

Expanding by R_1

$|A| = 3\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$

$|A| = 3(-1+4) - 1(-1-2) - 1(2+1)$
$= 3(3) - 1(-3) - 1(3)$

$|A| = 9 + 3 - 3 = 9 \neq 0$

Now $|A_1| = \begin{vmatrix} -4 & 1 & -1 \\ -4 & 1 & -2 \\ 1 & 2 & -1 \end{vmatrix}$

$= \begin{vmatrix} 0 & 1 & -1 \\ 0 & 1 & -2 \\ 9 & 2 & -1 \end{vmatrix} \quad C_1 + 4C_2$

$|A_1| = 0 - 0 + 9\begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} = 9(-2+1) = -9$

$\because x_1 = \dfrac{|A_1|}{|A|} = \dfrac{-9}{9} = -1$

$|A_2| = \begin{vmatrix} 3 & -4 & -1 \\ 1 & -4 & -2 \\ -1 & 1 & -1 \end{vmatrix}$

$= \begin{vmatrix} -1 & -4 & -5 \\ -3 & -4 & -6 \\ 0 & 1 & 0 \end{vmatrix} \quad \begin{matrix} C_1 + C_2 \\ C_3 + C_2 \end{matrix}$

$|A_2| = 0 - 1\begin{vmatrix} -1 & -5 \\ -3 & -6 \end{vmatrix} + 0$

$= -1(6-15) = 9$

$\because x_2 = \dfrac{|A_2|}{|A|} = \dfrac{9}{9} = 1$

$|A_3| = \begin{vmatrix} 3 & 1 & -4 \\ 1 & 1 & -4 \\ -1 & 2 & 1 \end{vmatrix}$

$= \begin{vmatrix} 3 & 1 & -4 \\ 1 & 1 & -4 \\ 0 & 3 & -3 \end{vmatrix} \quad R_3 + R_2$

$= 3\begin{vmatrix} 1 & -4 \\ 3 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & -4 \\ 3 & -3 \end{vmatrix} + 0$

$= 3(-3+12) - 1(-3+12) + 0$

$|A_3| = 3(9) - 9 = 18$

$\because x_3 = \dfrac{|A_3|}{|A|} = \dfrac{18}{9} = 2$

Hence $x_1 = -1$, $x_2 = 1$, $x_3 = 2$

Exercise 3.5

Q1. Solve the following systems of linear equations by Cramer's Rule.

i) $2x + 2y + z = 3$
$3x - 2y - 2z = 1$
$5x + y - 3z = 2$

Solution:- In matrix form

$$\begin{bmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$A \quad X = B$ where

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{vmatrix}$$

$$= 2\begin{vmatrix} -2 & -2 \\ 1 & -3 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ 5 & -3 \end{vmatrix} + 1\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}$$

$$= 2(6+2) - 2(-9+10) + 1(3+10)$$

$$= 2(8) - 2(1) + (13)$$

$$|A| = 16 - 2 + 13 = 27 \neq 0$$

so solution exists. Now

$$|A_1| = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & -3 \end{vmatrix}$$

$$= 3\begin{vmatrix} -2 & -2 \\ 1 & -3 \end{vmatrix} - 2\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$$

$$= 3(6+2) - 2(-3+4) + 1(1+4)$$

$$= 3(8) - 2(1) + 1(5)$$

$$|A_1| = 24 - 2 + 5 = 27$$

$$\because x_1 = \frac{|A_1|}{|A|} = \frac{27}{27} = 1$$

$$|A_2| = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} - 3\begin{vmatrix} 3 & -2 \\ 5 & -3 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix}$$

$$= 2(-3+4) - 3(-9+10) + 1(6-5)$$

$$= 2(1) - 3(1) + 1(1)$$

$$|A_2| = 2 - 3 + 1 = 0$$

$$\because y = \frac{|A_2|}{|A|} = \frac{0}{27} = 0$$

$$|A_3| = \begin{vmatrix} 2 & 2 & 3 \\ 3 & -2 & 1 \\ 5 & 1 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} + 3\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}$$

$$= 2(-4-1) - 2(6-5) + 3(3+10)$$

$$= 2(-5) - 2(1) + 3(13)$$

$$|A_3| = -10 - 2 + 39 = 27$$

$$\because z = \frac{|A_3|}{|A|} = \frac{27}{27} = 1$$

Hence $x = 1$, $y = 0$, $z = 1$

ii) $2x_1 - x_2 + x_3 = 5$
$4x_1 + 2x_2 + 3x_3 = 8$
$3x_1 - 4x_2 - x_3 = 3$

Solution:- In matrix form

$$\begin{bmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

$A \quad X = B$ where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 3 \\ -4 & -1 \end{vmatrix} - 4\begin{vmatrix} -1 & 1 \\ -4 & -1 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 2(-2+12) - 4(1+4) + 3(-3-2)$$

$$= 2(10) - 4(5) + 3(-5)$$

$$|A| = 20 - 20 - 15 = -15 \neq 0$$

so solution exists. Now

$$|A_1| = \begin{vmatrix} 5 & -1 & 1 \\ 8 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}$$

$$= 5\begin{vmatrix} 2 & 3 \\ -4 & -1 \end{vmatrix} - 8\begin{vmatrix} -1 & 1 \\ -4 & -1 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 5(-2+12) - 8(1+4) + 3(-3-2)$$

$$= 5(10) - 8(5) + 3(-5)$$

$$|A_1| = 50 - 40 - 15 = -5$$

$$\because x_1 = \frac{|A_1|}{|A|} = \frac{-5}{-15} = \frac{1}{3}$$

$$|A_2| = \begin{vmatrix} 2 & 5 & -1 \\ 4 & 8 & 3 \\ 3 & 3 & -1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 8 & 3 \\ 3 & -1 \end{vmatrix} - 4\begin{vmatrix} 5 & 1 \\ 3 & -1 \end{vmatrix} + 3\begin{vmatrix} 5 & 1 \\ 8 & 3 \end{vmatrix}$$

$$= 2(-8-9) - 4(-5-3) + 3(15-8)$$

$$= 2(-17) - 4(-8) + 3(7)$$

$$|A_2| = -34 + 32 + 21 = 19$$

$$\because x_2 = \frac{|A_2|}{|A|} = \frac{19}{-15}$$

$$|A_3| = \begin{vmatrix} 2 & -1 & 5 \\ 4 & 2 & 8 \\ 3 & -4 & 3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 8 \\ -4 & 3 \end{vmatrix} - 4\begin{vmatrix} -1 & 5 \\ -4 & 3 \end{vmatrix} + 3\begin{vmatrix} -1 & 5 \\ 2 & 8 \end{vmatrix}$$

$$= 2(6+32) - 4(-3+20) + 3(-8-10)$$

$$= 2(38) - 4(17) + 3(-18)$$

$$|A_3| = 76 - 68 - 54 = 76 - 122 = -46$$

$$\because x_3 = \frac{|A_3|}{|A|} = \frac{-46}{-15} = \frac{46}{15}$$

Hence $x_1 = \frac{1}{3}$, $x_2 = -\frac{19}{15}$, $x_3 = \frac{46}{15}$

iii) $2x_1 - x_2 + x_3 = 8$
$x_1 + 2x_2 + 2x_3 = 6$
$x_1 - 2x_2 - x_3 = 1$

Solution:- In matrix form

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 1 \end{bmatrix}$$

$A \; X = B$ where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ B = \begin{bmatrix} 8 \\ 6 \\ 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= 2(-2+4) - 1(1+2) + 1(-2-2)$$

$$= 2(2) - 1(3) + 1(-4)$$

$$|A| = 4 - 3 - 4 = -3 \neq 0$$

so solution exists. Now

$$|A_1| = \begin{vmatrix} 8 & -1 & 1 \\ 6 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= 8\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - 6\begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= 8(-2+4) - 6(1+2) + 1(-2-2)$$

$$= 8(2) - 6(3) + 1(-4)$$

$$|A_1| = 16 - 18 - 4 = -6$$

$$\because x_1 = \frac{|A_1|}{|A|} = -\frac{6}{-3} = 2$$

$$|A_2| = \begin{vmatrix} 2 & 8 & 1 \\ 1 & 6 & 2 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 6 & 2 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 8 & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 8 & 1 \\ 6 & 2 \end{vmatrix}$$

$$= 2(-6-2) - 1(-8-1) + 1(16-6)$$

$$= 2(-8) - 1(-9) + 1(10)$$

$$|A_2| = -16 + 9 + 10 = 3$$

$$\because x_2 = \frac{|A_2|}{|A|} = \frac{3}{-3} = -1$$

$$|A_3| = \begin{vmatrix} 2 & -1 & 8 \\ 1 & 2 & 6 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 6 \\ -2 & 1 \end{vmatrix} - 1\begin{vmatrix} -1 & 8 \\ -2 & 1 \end{vmatrix} + 1\begin{vmatrix} -1 & 8 \\ 2 & 6 \end{vmatrix}$$

$$= 2(2+12) - 1(-1+16) + 1(-6-16)$$

$$= 2(14) - 1(15) + 1(-22)$$

$$|A_3| = 28 - 15 - 22 = -9$$

$$\because x_3 = \frac{|A_3|}{|A|} = \frac{-9}{-3} = 3$$

Hence $x_1 = 2$, $x_2 = -1$, $x_3 = 3$

Q2. Use matrices to solve the following systems:

i) $x - 2y + z = -1$
$3x + y - 2z = 4$
$y - z = 1$

Solution:- In matrix form

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= 1\begin{vmatrix}1 & -2\\ 1 & -1\end{vmatrix} - 3\begin{vmatrix}-2 & 1\\ 1 & -1\end{vmatrix} + 0$$

$$= 1(-1+2) - 3(2-1) = 1(1) - 3(1)$$

$|A| = 1 - 3 = -2 \neq 0$ so solution exists

$\because AX = B$

$\Rightarrow X = A^{-1}B \longrightarrow (i)$

For A^{-1},

$$A_{11} = (-1)^{1+1}\begin{vmatrix}1 & -2\\ 1 & -1\end{vmatrix} = (-1+2) = 1$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix}3 & -2\\ 0 & 1\end{vmatrix} = -(-3+0) = 3$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix}3 & 1\\ 0 & 1\end{vmatrix} = (3-0) = 3$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix}-2 & 1\\ 1 & -1\end{vmatrix} = -(2-1) = -1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix}1 & 1\\ 0 & -1\end{vmatrix} = (-1-0) = -1$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix}1 & -2\\ 0 & -1\end{vmatrix} = -(1+0) = -1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix}-2 & 1\\ 1 & -2\end{vmatrix} = (4-1) = 3$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix}1 & 1\\ 3 & -2\end{vmatrix} = -(-2-3) = 5$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix}1 & -2\\ 3 & 1\end{vmatrix} = (1+6) = 7$$

$$\text{Matrix of cofactor} = \begin{bmatrix}1 & 3 & 3\\ -1 & -1 & -1\\ 3 & 5 & 7\end{bmatrix}$$

$\text{adj } A = (\text{matrix of cofactor})^t$

$$\text{adj } A = \begin{bmatrix}1 & -1 & 3\\ 3 & -1 & 5\\ 3 & -1 & 7\end{bmatrix}$$

$$\because A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-2}\begin{bmatrix}1 & -1 & 3\\ 3 & -1 & 5\\ 3 & -1 & 7\end{bmatrix}$$

So (i)

$$X = \frac{1}{-2}\begin{bmatrix}1 & -1 & 3\\ 3 & -1 & 5\\ 3 & -1 & 7\end{bmatrix}\begin{bmatrix}-1\\ 4\\ 1\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}-1-4+3\\ -3-4+5\\ -3-4+7\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}-2\\ -2\\ 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$\Rightarrow x = 1, \; y = 1, \; z = 0$

ii) $2x_1 + x_2 + 3x_3 = 3$

$x_1 + x_2 - 2x_3 = 0$

$-3x_1 - x_2 + 2x_3 = -4$

Solution:- In matrix form

$$\begin{bmatrix}2 & 1 & 3\\ 1 & 1 & -2\\ -3 & -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}3\\ 0\\ -4\end{bmatrix}$$

where

$$A = \begin{bmatrix}2 & 1 & 3\\ 1 & 1 & -2\\ -3 & -1 & 2\end{bmatrix}, \; X = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}, \; B = \begin{bmatrix}3\\ 0\\ -4\end{bmatrix}$$

$$|A| = \begin{vmatrix}2 & 1 & 3\\ 1 & 1 & -2\\ -3 & -1 & 2\end{vmatrix}$$

$$= 2\begin{vmatrix}1 & -2\\ -1 & 2\end{vmatrix} - 1\begin{vmatrix}1 & -2\\ -3 & 2\end{vmatrix} + 3\begin{vmatrix}1 & 1\\ -3 & -1\end{vmatrix}$$

$$= 2(2-2) - 1(2-6) + 3(-1+3)$$

$$= 2(0) - 1(-4) + 3(2)$$

$|A| = 4 + 6 = 10 \neq 0$, solution exists.

$\because AX = B$

$\Rightarrow X = A^{-1}B \longrightarrow (i)$

For A^{-1},

$$A_{11} = (-1)^{1+1}\begin{vmatrix}1 & -2\\ -1 & 2\end{vmatrix} = (2-2) = 0$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix}1 & -2\\ -3 & 2\end{vmatrix} = -(2-6) = 4$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix}1 & 1\\ -3 & -1\end{vmatrix} = (-1+3) = 2$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix}1 & 3\\ -1 & 2\end{vmatrix} = -(2+3) = -5$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix}2 & 3\\ -3 & 2\end{vmatrix} = (4+9) = 13$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix}2 & 1\\ -3 & -1\end{vmatrix} = -(-2+3) = -1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix}1 & 3\\ 1 & -2\end{vmatrix} = (-2-3) = -5$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix}2 & 3\\ 1 & -2\end{vmatrix} = -(-4-3) = 7$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix} = (2-1) = 1$$

$$\text{Matrix of cofactor} = \begin{bmatrix}0 & 4 & 2\\ -5 & 13 & -1\\ -5 & 7 & 1\end{bmatrix}$$

$\text{adj } A = (\text{matrix of cofactor})^t$

$$\text{adj } A = \begin{bmatrix}0 & -5 & -5\\ 4 & 13 & 7\\ 2 & -1 & 1\end{bmatrix}$$

$\because A^{-1} = \dfrac{adj A}{|A|} = \dfrac{1}{10}\begin{bmatrix} 1 & -5 & -5 \\ 4 & 13 & 7 \\ 2 & -1 & -1 \end{bmatrix}$

so (i) $X = \dfrac{1}{10}\begin{bmatrix} 0 & -5 & -5 \\ 4 & 13 & 7 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \\ -4 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \dfrac{1}{10}\begin{bmatrix} 0-0+20 \\ -12+0-28 \\ -6-0-4 \end{bmatrix} = \dfrac{1}{10}\begin{bmatrix} 20 \\ -40 \\ -10 \end{bmatrix}$

$\rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}$ so $x_1 = 2,\ x_2 = -4$, $x_3 = -1$

iii) $x + y = 2$
$2x - z = 1$
$2y - 3z = -1$

Solution:- In matrix form

$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

where

$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{vmatrix}$

$= 1\begin{vmatrix} 0 & -1 \\ 2 & -3 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} + 0$

$|A| = (0+2) - 2(-3-0) = 2+6 = 8 \neq 0$

so solution exists.

$\because AX = B$

$\rightarrow X = A^{-1}B \longrightarrow (i)$

For A^{-1},

$A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & -1 \\ 2 & -3 \end{vmatrix} = (0+2) = 2$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} = -(-6+0) = 6$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = (4-0) = 4$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = -(-3-0) = 3$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} = (-3-0) = -3$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -(2-0) = -2$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = (-1-0) = -1$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -(-1-0) = 1$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = (0-2) = -2$

Matrix of cofactor $= \begin{bmatrix} 2 & 6 & 4 \\ 3 & -3 & -2 \\ -1 & 1 & -2 \end{bmatrix}$

adj A = (matrix of cofactor)^t

adj $A = \begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}$

$\because A^{-1} = \dfrac{1}{|A|} adj A = \dfrac{1}{8}\begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}$

so (i) $X = \dfrac{1}{8}\begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{1}{8}\begin{bmatrix} 4+3+1 \\ 12-3-1 \\ 8-2+2 \end{bmatrix} = \dfrac{1}{8}\begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ so $x = 1,\ y = 1$, $z = 1$

Q3. Solve the following systems by reducing their augmented matrices to echelon form and the reduced echelon form.

i) $x_1 - 2x_2 - 2x_3 = -1$
$2x_1 + 3x_2 + x_3 = 1$
$5x_1 - 4x_2 - 3x_3 = 1$

Solution:- Solution by Echelon form

The augmented matrix is

$A_b = \begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 2 & 3 & 1 & : & 1 \\ 5 & -4 & -3 & : & 1 \end{bmatrix}$

$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 2-2 & 3+4 & 1+4 & : & 1+2 \\ 5-5 & -4+10 & -3+10 & : & 1+5 \end{bmatrix}$ $R_2 - 2R_1$, $R_3 - 5R_1$

$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 7 & 5 & : & 3 \\ 0 & 6 & 7 & : & 6 \end{bmatrix}$

$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 7-6 & 5-7 & : & 3-6 \\ 0 & 6 & 7 & : & 6 \end{bmatrix}$ $R_2 - R_3$

$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & -2 & : & -3 \\ 0 & 6-6 & 7+12 & : & 6+18 \end{bmatrix}$ $R_3 - 6R_2$

$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & -2 & : & -3 \\ 0 & 0 & 19 & : & 24 \end{bmatrix}$

$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & -2 & : & -3 \\ 0 & 0 & 1 & : & 24/19 \end{bmatrix}$ $1/19\, R_3$

$\Rightarrow x_1 - 2x_2 - 2x_3 = -1 \longrightarrow (i)$

$x_2 - 2x_3 = -3 \longrightarrow (ii)$

$x_3 = \frac{24}{19} \longrightarrow (iii)$

Put $x_3 = \frac{24}{19}$ in (ii)

$x_2 - 2\left(\frac{24}{19}\right) = -3 \Rightarrow x_2 = -3 + \frac{48}{19}$

$\Rightarrow x_2 = \frac{-57+48}{19} = \frac{-9}{19}$

Put $x_3 = \frac{24}{19}$ and $x_2 = -\frac{9}{19}$ in (i)

$x_1 - 2\left(\frac{-9}{19}\right) - 2\left(\frac{24}{19}\right) = -1$

$x_1 + \frac{18}{19} - \frac{48}{19} + 1 = 0$

$\Rightarrow x_1 + \frac{18-48+19}{19} = 0$

$\Rightarrow x_1 - \frac{11}{19} = 0 \Rightarrow x_1 = \frac{11}{19}$

Hence $x_1 = \frac{11}{19}$, $x_2 = \frac{-9}{19}$, $x_3 = \frac{24}{19}$

Solution by Reduced Echelon form

we reduce $\begin{bmatrix} 1 & -2 & -2 & : & -1 \\ 0 & 1 & -2 & : & -3 \\ 0 & 0 & 1 & : & \frac{24}{9} \end{bmatrix}$ into reduced echelon form

$\underset{\sim}{R} \begin{bmatrix} 1 & -2+2 & -2-4 & : & -1-6 \\ 0 & 1 & -2 & : & -3 \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix} R_1 + 2R_2$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & -6 & : & -7 \\ 0 & 1 & -2 & : & -3 \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix}$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & -6+6 & : & -7+6\left(\frac{24}{19}\right) \\ 0 & 1 & -2+2 & : & -3+2\left(\frac{24}{19}\right) \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix} \begin{matrix} R_1 + 6R_3 \\ R_2 + 2R_3 \end{matrix}$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & : & \frac{-133+144}{19} \\ 0 & 1 & 0 & : & \frac{-57+48}{19} \\ 0 & 0 & 1 & : & \frac{24}{19} \end{bmatrix}$

$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & : & 11/19 \\ 0 & 1 & 0 & : & -9/19 \\ 0 & 0 & 1 & : & 24/19 \end{bmatrix}$

$\Rightarrow x_1 = 11/19$, $x_2 = -9/19$

$x_3 = \frac{24}{19}$

ii) $x + 2y + z = 2$

$2x + y + 2z = -1$

$2x + 3y - z = 9$

Solution:– Solution by Echelon form

The augmented matrix is

$A_b = \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 2 & 1 & 2 & : & -1 \\ 2 & 3 & -1 & : & 9 \end{bmatrix}$

$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 2-2 & 1-3 & 2-2 & : & -1-4 \\ 2-2 & 3-4 & -1-2 & : & 9-4 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 2R_1 \end{matrix}$

$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & -2 & 0 & : & -5 \\ 0 & -1 & -3 & : & 5 \end{bmatrix}$

$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & -1 & -3 & : & 5 \\ 0 & -2 & 0 & : & -5 \end{bmatrix} R_{2 \leftrightarrow 3}$

$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 3 & : & -5 \\ 0 & -3 & 0 & : & -5 \end{bmatrix} (-1)R_2$

$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 3 & : & -5 \\ 0 & -3+3 & 0+9 & : & -5-15 \end{bmatrix} R_3 + 3R_2$

$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 3 & : & -5 \\ 0 & 0 & 9 & : & -20 \end{bmatrix}$

$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 1 & 3 & : & -5 \\ 0 & 0 & 1 & : & -20/9 \end{bmatrix} \frac{1}{9}R_3$

$\Rightarrow x + 2y + z = 2 \longrightarrow (i)$

$y + 3z = -5 \longrightarrow (ii)$

$z = \frac{-20}{9} \longrightarrow (iii)$

Put $z = -\frac{20}{9}$ in (ii)

$y + 3\left(\frac{-20}{9}\right) = -5 \Rightarrow y = -5 + \frac{60}{9}$

$y = \frac{-45+60}{9} = \frac{15}{9} = \frac{5}{3}$

Put $z = -\frac{20}{9}$ and $y = \frac{5}{3}$ in (i)

$x + 2\left(\frac{5}{3}\right) - \frac{20}{9} = 2$

$x + \frac{10}{3} - \frac{20}{9} - 2 = 0$

$x + \frac{30-20-18}{9} = 0 \Rightarrow x - \frac{8}{9} = 0$

$\Rightarrow x = 8/9$

Hence $x = 8/9$, $y = \frac{5}{3}$, $z = \frac{-20}{9}$

Solution by Reduced Echelon form

we reduce $\left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -\frac{20}{9} \end{array}\right]$

into reduced echelon form

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 2-2 & 1-6 & 2+10 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -\frac{20}{9} \end{array}\right] R_1-2R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & -5 & 12 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -\frac{20}{9} \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & -5+5 & 12+5(-\frac{20}{9}) \\ 0 & 1 & 3-3 & -5-3(-\frac{20}{9}) \\ 0 & 0 & 1 & -\frac{20}{9} \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & 0 & \frac{108-100}{9} \\ 0 & 1 & 0 & \frac{-45+60}{9} \\ 0 & 0 & 1 & -\frac{20}{9} \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & 0 & 8/9 \\ 0 & 1 & 0 & 5/9 \\ 0 & 0 & 1 & -20/9 \end{array}\right]$$

$\Rightarrow x_1 = \frac{8}{9}, \quad x_2 = \frac{5}{9}, \quad x_3 = \frac{-20}{9}$

iii) $x_1 + 4x_2 + 2x_3 = 2$
$2x_1 + x_2 - 2x_3 = 9$
$3x_1 + 2x_2 - 2x_3 = 12$

Solution:- Solution by Echelon form

The augmented matrix is

$$A_b = \left[\begin{array}{ccc:c} 1 & 4 & 2 & 2 \\ 2 & 1 & -2 & 9 \\ 3 & 2 & -2 & 12 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 4 & 2 & 2 \\ 2-2 & 1-8 & -2-4 & 9-4 \\ 3-3 & 2-12 & -2-6 & 12-6 \end{array}\right] \begin{matrix} R_2-2R_1 \\ R_3-3R_1 \end{matrix}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 4 & 2 & 2 \\ 0 & -7 & -6 & 5 \\ 0 & -10 & -8 & 6 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 4 & 2 & 2 \\ 0 & 21 & 18 & -15 \\ 0 & -20 & -16 & 12 \end{array}\right] \begin{matrix} (-3)R_2 \\ 2R_3 \end{matrix}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 4 & 2 & 2 \\ 0 & 1 & 2 & -3 \\ 0 & -20 & -16 & 12 \end{array}\right] R_2+R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 4 & 2 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & -20+20 & -16+40 & 12-60 \end{array}\right] R_3+20R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 4 & 2 & 2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 24 & -48 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 4 & 2 & 2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right] \frac{1}{24}R_3$$

$x_1 + 4x_2 + 2x_3 = 2 \longrightarrow (i)$
$x_2 + 2x_3 = -3 \longrightarrow (ii)$
$x_3 = -2 \longrightarrow (iii)$

Put $x_3 = -2$ in (ii)

$x_2 + 2(-2) = -3 \Rightarrow x_2 = -3+4$
$x_2 = 1$

Put $x_2 = 1$ and $x_3 = -2$ in (i)

$x_1 + 4(1) + 2(-2) = 2$
$x_1 + 4 - 4 = 2 \Rightarrow x_1 = 2$

Hence $x_1 = 2, \quad x_2 = 1, \quad x_3 = -2$

Solution by Reduced Echelon form

we reduce $\left[\begin{array}{ccc:c} 1 & 4 & 2 & 2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right]$ into reduce echelon form.

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 4-4 & 2-8 & 2+12 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right] R_1-4R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & -6 & 14 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & -6+6 & 14-12 \\ 0 & 1 & 2-2 & -3+4 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{matrix} R_1+6R_3 \\ R_2-2R_3 \end{matrix}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$\Rightarrow x_1 = 2, \quad x_2 = 1, \quad x_3 = -2$

Q4. Solve the following systems of homogeneous linear equations.

i) $x + 2y - 2z = 0 \longrightarrow (i)$
$2x + y + 5z = 0 \longrightarrow (ii)$
$5x + 4y + 8z = 0 \longrightarrow (iii)$

Solution:-

In matrix form

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where $A=\begin{bmatrix}1 & 2 & -2\\2 & 1 & 5\\5 & 4 & 8\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$

$O=\begin{bmatrix}0\\0\\0\end{bmatrix}$ Now

$$|A|=\begin{vmatrix}1 & 2 & -2\\2 & 1 & 5\\5 & 4 & 8\end{vmatrix}$$

$$=1\begin{vmatrix}1 & 5\\4 & 8\end{vmatrix}-2\begin{vmatrix}2 & -2\\4 & 8\end{vmatrix}+5\begin{vmatrix}2 & -2\\1 & 5\end{vmatrix}$$

$$=1(8-20)-2(16+8)+5(10+2)$$

$$=1(-12)-2(24)+5(12)$$

$$|A|=-12-48+60=0$$

$\Rightarrow |A|=0 \Rightarrow$ We cannot find A^{-1}

Hence system has non-trivial solution. Now we solve (i) and (ii) to find x_1 and x_2

By (i)-2(ii) $\Rightarrow$

$$\begin{array}{r} x+y-2z=0\\ \underline{\mp 4x \mp 2y \mp 10z=0}\\ -3x-12z=0 \end{array}$$

$\Rightarrow 3x=-12z \Rightarrow x=-4z$

By (ii) − 2(i) $\Rightarrow$

$$\begin{array}{r} 2x+y+5z=0\\ \underline{\mp 2x \mp 4y \pm 4z=0}\\ -3y+9z=0 \end{array}$$

$-3y=-9z$

$\Rightarrow \therefore y=3z$

Put $x=-4z$ and $y=3z$ in (iii)

$$5(-4z)+4(3z)+8z=0$$

$$-20z+12z+8z=0$$

$$\Rightarrow 0=0$$

Eq (iii) is satisfied.

Let $z=t$, $t\in R$ then

$x=-4t$ and $y=3t$

so $x=-4t$, $y=3t$, $z=t$

Hence ... the system has unlimited solutions.

ii) $x_1+4x_2+2x_3=0 \longrightarrow$ (i)

$2x_1+x_2-3x_3=0 \longrightarrow$ (ii)

$3x_1+2x_2-4x_3=0 \longrightarrow$ (iii)

Solution:- In matrix form

$$\begin{bmatrix}1 & 4 & 2\\2 & 1 & -3\\3 & 2 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

where $A=\begin{bmatrix}1 & 4 & 2\\2 & 1 & -3\\3 & 2 & -4\end{bmatrix}$, $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, $O=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$|A|=\begin{vmatrix}1 & 4 & 2\\2 & 1 & -3\\3 & 2 & -4\end{vmatrix}=1\begin{vmatrix}1 & -3\\2 & -4\end{vmatrix}-2\begin{vmatrix}4 & 2\\2 & -4\end{vmatrix}+3\begin{vmatrix}4 & 2\\1 & -3\end{vmatrix}$$

$$=1(-4+6)-2(-16-4)+3(-12-2)$$

$$=2-2(-20)+3(-14)=2+40-42=0$$

$\Rightarrow |A|=0 \Rightarrow$ We cannot find A^{-1}

So system has non-trivial solution.

Now we solve (i) and (ii) to find x_1 and x_2

By (i) − 4(ii) $\Rightarrow$

$$\begin{array}{r} x_1+4x_2+2x_3=0\\ \underline{-8x_1 - 4x_2 + 12x_3=0}\\ -7x_1+14x_3=0 \end{array}$$

$\Rightarrow x_1=2x_3$

By (ii) − 2(i) $\Rightarrow$

$$\begin{array}{r} 2x_1+x_2-3x_3=0\\ \underline{-2x_1 - 8x_2 - 4x_3=0}\\ -7x_2-7x_3=0 \end{array}$$

$\Rightarrow x_2=-x_3$

Put $x_1=2x_3$ and $x_2=-x_3$ in (iii)

$$3(2x_3)+2(-x_3)-4x_3=0$$

$$6x_3-2x_3-4x_3=0$$

$\Rightarrow 0=0$ Eq (iii) is satisfied.

Let $x_3=t$, $t\in R$ then $x_1=2t$ and $x_2=-t$

Hence $x_1=2t$, $x_2=-t$, $x_3=t$

$\therefore$ system has unlimited solution.

iii) $x_1-2x_2-x_3=0 \longrightarrow$ (i)

$x_1+x_2+5x_3=0 \longrightarrow$ (ii)

$2x_1-x_2+4x_3=0 \longrightarrow$ (iii)

Solution:- In matrix form

$$\begin{bmatrix}1 & -2 & -1\\1 & 1 & 5\\2 & -1 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

∴ where $A = \begin{bmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 5 \\ -1 & 4 \end{vmatrix} - 1\begin{vmatrix} -2 & -1 \\ -1 & 4 \end{vmatrix} + 2\begin{vmatrix} -2 & -1 \\ 1 & 5 \end{vmatrix}$$

$$= (4+5) - 1(-8-1) + 2(-10+1)$$

$$|A| = 9 + 9 - 18 = 0$$

$\rightarrow |A| = 0$ Hence system has non-trivial solutions.

Now we solve (i) and (ii) to find x_1 and x_2.

By (i) + 2(ii) $\quad x_1 - 2x_2 - x_3 = 0$

$$\underline{2x_1 + 2x_2 + 10x_3 = 0}$$

$$3x_1 + 9x_3 = 0$$

$$\rightarrow x_1 = -3x_3$$

By (ii) − (i) → $\quad x_1 + x_2 + 5x_3 = 0$

$$\underline{\begin{matrix} x_1 - 2x_2 - x_3 = 0 \\ - \quad + \quad + \end{matrix}}$$

$$3x_2 + 6x_3 = 0$$

$$x_2 = -2x_3$$

Put $x_1 = -2x_3$ and $x_2 = -2x_3$ in (iii)

$$2(-3x_3) - (-2x_3) + 4x_3 = 0$$

$$-6x_3 + 2x_3 + 4x_3 = 0$$

$$\rightarrow 0 = 0$$

Eq (iii) is satisfied.

Let $x_3 = t$, $t \in R$ then $x_1 = -3t$ and $x_2 = -2t$

Hence $x_1 = -3t$, $x_2 = -2t$, $x_3 = t$

∴ system has unlimited solutions

Q5. Find the value of λ for which the following systems have non-trivial solution. Also solve the system for the value of λ.

i) $x + y + z = 0$
$2x + y - \lambda z = 0$
$x + 2y - 2z = 0$

Solution:- In matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

∵ system has non-trivial solution

so $|A| = 0$

$$\rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{vmatrix} = 0$$

$$1\begin{vmatrix} 1 & -\lambda \\ 2 & -2 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$-2 + 2\lambda - 2(-2-2) + 1(-\lambda - 1) = 0$$

$$-2 + 2\lambda + 8 + \lambda - 1 = 0$$

$$\lambda + 5 = 0 \rightarrow \lambda = -5$$

For $\lambda = -5$ given system becomes

$x + y + z = 0 \longrightarrow$ (i)
$2x + y + 5z = 0 \longrightarrow$ (ii)
$x + 2y - 2z = 0 \longrightarrow$ (iii)

we solve (i) and (ii) to find x and y

By (ii) − (i) → $\quad 2x + y + 5z = 0$

$$\underline{\underset{-}{x} \pm y \pm z = 0}$$

$$x + 4z = 0$$

$$\rightarrow x = -4z$$

By (ii) − 2(i) → $\quad 2x + y + 5z = 0$

$$\underline{\underset{-}{2x} \underset{-}{+2y} \underset{-}{+2z} = 0}$$

$$-y + 3z = 0$$

$$\rightarrow y = 3z$$

Put $x = -4z$, $y = 3z$ in (iii)

$$-4z + 2(3z) - 2z = 0$$

$$-4z + 6z - 2z = 0$$

$$\rightarrow 0 = 0$$

Eq (iii) is satisfied.

Let $z = t$, $t \in R$ then $x = -4t$, $y = 3t$

Hence $x = -4t$, $y = 3t$, $z = t$

∴ system has unlimited solutions.

ii) $x_1 + 4x_2 + \lambda x_3 = 0$

$2x_1 + x_2 - 3x_3 = 0$

$3x_1 + \lambda x_2 - 4x_3 = 0$

Solution:- In matrix form

$$\begin{bmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Since the system has non-trivial solution so $|A| = 0$

$$\Rightarrow \begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{vmatrix} = 0$$

$$1\begin{vmatrix} 1 & -3 \\ \lambda & -4 \end{vmatrix} - 2\begin{vmatrix} 4 & \lambda \\ \lambda & -4 \end{vmatrix} + 3\begin{vmatrix} 4 & \lambda \\ 1 & -3 \end{vmatrix} = 0$$

$$(-4 + 3\lambda) - 2(-16 - \lambda^2) + 3(-12 - \lambda) = 0$$

$$-4 + \cancel{3\lambda} + 32 + 2\lambda^2 - 36 - \cancel{3\lambda} = 0$$

$$2\lambda^2 - 8 = 0$$

$$\Rightarrow \lambda^2 = 4$$

$$\Rightarrow \lambda = \pm 2$$

For $\lambda = 2$

given system becomes as

$x_1 + 4x_2 - 2x_3 = 0 \longrightarrow$ (i)

$2x_1 + x_2 - 3x_3 = 0 \longrightarrow$ (ii)

$3x_1 + 2x_2 - 4x_3 = 0 \longrightarrow$ (iii)

we solve (i) and (ii) to find x_1 and x_2

By (i) − 4(ii) $\Rightarrow$

$$\begin{array}{r} x_1 + 4x_2 + 2x_3 = 0 \\ \underset{-}{8x_1} \underset{-}{+4x_2} \underset{+}{-12x_3} = 0 \\ \hline -7x_1 + 14x_3 = 0 \end{array}$$

$$\Rightarrow x_1 = 2x_3$$

By (ii) − 2(i) $\Rightarrow$

$$\begin{array}{r} 2x_1 + x_2 - 3x_3 = 0 \\ \underset{-}{2x_1} \underset{-}{+8x_2} \underset{-}{+4x_3} = 0 \\ \hline -7x_2 - 7x_3 = 0 \end{array}$$

$$\Rightarrow x_2 = -x_3$$

Put $x_1 = 2x_3$, $x_2 = -x_3$ in (iii)

$$3(2x_3) + 2(-x_3) - 4x_3 = 0$$

$$6x_3 - 2x_3 - 4x_3 = 0$$

$$\Rightarrow 0 = 0$$

Eq (iii) is satisfied.

Let $x_3 = t$, $t \in R$ so $x_1 = 2t$ and $x_2 = -t$

Hence $x_1 = 2t$, $x_2 = -t$, $x_3 = t$

$\therefore$ system has unlimited solutions

For $\lambda = -2$

given system becomes as

$x_1 + 4x_2 - 2x_3 = 0 \longrightarrow$ (i)

$2x_1 + x_2 - 3x_3 = 0 \longrightarrow$ (ii)

$3x_1 - 2x_2 - 4x_3 = 0 \longrightarrow$ (iii)

we solve (i) and (ii) for x_1 and x_2

By (i) − 4(ii) $\Rightarrow$

$$\begin{array}{r} x_1 + 4x_2 - 2x_3 = 0 \\ \underset{-}{8x_1} \underset{-}{+4x_2} \underset{+}{-12x_3} = 0 \\ \hline -7x_1 + 10x_3 = 0 \end{array}$$

$$\Rightarrow x_1 = \frac{10}{7}x_3$$

By (ii) − 2(i) $\Rightarrow$

$$\begin{array}{r} 2x_1 + x_2 - 3x_3 = 0 \\ \underset{-}{2x_1} \underset{-}{+8x_2} \underset{+}{-4x_3} = 0 \\ \hline -7x_2 + x_3 = 0 \end{array}$$

$$\Rightarrow x_2 = \frac{1}{7}x_3$$

Put $x_1 = \frac{10}{7}x_3$ and $x_2 = \frac{1}{7}x_3$ in (iii)

$$3\left(\frac{10}{7}x_3\right) - 2\left(\frac{1}{7}x_3\right) - 4x_3 = 0$$

$$\frac{30}{7}x_3 - \frac{2}{7}x_3 - 4x_3 = 0$$

$$\frac{30x_3 - 2x_3 - 28x_3}{7} = 0$$

$$\Rightarrow 0 = 0$$

Eq (iii) is satisfied.

Let $x_3 = t$, $t \in R$ then $x_1 = \frac{10}{7}t$, $x_2 = \frac{1}{7}t$

Hence $x_1 = \frac{10}{7}t$, $x_2 = \frac{1}{7}t$, $x_3 = t$

$\therefore$ System has unlimited Solution.

Important note:-

If a system does not possess unique solution it means that it has unlimited solutions.
we know already a system has unlimited solution if

Rank $(A) = $ Rank(A_b) and

Rank $(A) <$ no. of variables used in the system

Q6. Find the value of λ for which the following system does not possess unique solution. Also solve the system for the value of λ.

$$x_1 + 4x_2 + \lambda x_3 = 2$$
$$2x_1 + x_2 - 2x_3 = 11$$
$$3x_1 + 2x_2 - 2x_3 = 16$$

Solution:-

Augmented matrix is

$$A_b = \begin{bmatrix} 1 & 4 & \lambda & : & 2 \\ 2 & 1 & -2 & : & 11 \\ 3 & 2 & -2 & : & 16 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & \lambda & : & 2 \\ 2-2 & 1-8 & -2-2\lambda & : & 11-4 \\ 3-3 & 2-12 & -2-3\lambda & : & 16-6 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & \lambda & : & 2 \\ 0 & -7 & -2(1+\lambda) & : & 7 \\ 0 & -10 & -(2+3\lambda) & : & 10 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & \lambda & : & 2 \\ 0 & 1 & \frac{2(1+\lambda)}{7} & : & -1 \\ 0 & -10 & -(2+3\lambda) & : & 10 \end{bmatrix} -\frac{1}{7}R_2$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & \lambda & : & 2 \\ 0 & 1 & \frac{2}{7}(1+\lambda) & : & -1 \\ 0 & 0 & \frac{6-\lambda}{7} & : & 0 \end{bmatrix} R_3 + 10R_2 \longrightarrow (A)$$

$\because$ system does not possess unique solution for

$$\frac{6-\lambda}{7} = 0$$
$$\Rightarrow 6 - \lambda = 0$$
$$\Rightarrow \lambda = 6$$

For $\lambda = 6$ (A) becomes

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 6 & : & 2 \\ 0 & 1 & \frac{2+2(6)}{7} & : & -1 \\ 0 & 0 & \frac{6-6}{7} & : & 0 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 6 & : & 2 \\ 0 & 1 & 2 & : & -1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$x_1 + 4x_2 + 6x_3 = 2 \longrightarrow (i)$$
$$x_2 + 2x_3 = -1$$
$$\Rightarrow x_2 = -1 - 2x_3 \text{ put in (i)}$$

$$x_1 + 4(-1-2x_3) + 6x_3 = 2$$
$$x_1 - 4 - 8x_3 + 6x_3 = 2$$
$$x_1 - 2x_3 = 6$$
$$\Rightarrow x_1 = 2x_3 + 6$$

Let $x_3 = t$, then $x_1 = 2t + 6$, $t \in R$ and $x_2 = -2t - 1$

Hence $x_1 = 2t + 6$, $x_2 = -2t - 1$ and $x_3 = t$

$\therefore$ System has unlimited solutions.