Unit NO. 2:-

"Sets, Functions and Groups"

Definitions

Set:-

A **set** is generally defined as a collection of distinct and well-defined objects.

For examples:-

$$A = \{1, 2, 3, \ldots, 10\}$$

Members:-

The objects in a set are called its **members** or **elements**.

For examples:-

$A = \{a, b, c\}$, then a, b and c are the members of set A.

Order of a Set:-

The number of elements present in a set is called order of a set.

For examples:-

A set may have 0, 1, 2, 3 or any other elements in a set.

Singleton Sets:-

A set having only one element is called singleton set.

For example:-

A positive even integer between 2 and 5.

Empty Set:-

A set with no element is called **empty set** or **null set**. The empty set is denoted by the symbol ϕ and $\{\}$.

For example:-

A set of natural numbers between 1 and 2.

Equal Sets:-

Two sets A and B are **equal sets** i.e $A = B$, if and only if they have same elements that is, if and only if every element of each set is the element of the other set.

For example:-

The sets $A = \{1, 2, 3\}$

and $B = \{2, 1, 3\}$ are equal sets.

Equivalent Sets:-

If elements of two sets A and B can be paired in such a way that each element of A is paired with one and only one element of B and vice versa, then such a pairing is called one-to-one correspondence between A and B.

Subset:-

If every element of set A is an element of set B, then A is a subset of B. Symbolically this is written as $A \subseteq B$.

For example:-

If $A = \{a, b, c, d\}$ and $B = \{a, b, c, d\}$ then $A \subseteq B$.

Proper Subset:-

If A is the subset of B and B contains at least one element which is not an element of A, then A is said to be proper subset of B. In such a case, we write $A \subset B$.

For examples:-

$A = \{a, b, c\}$; $B = \{a, b, c, d\}$

$A \subset B$

Improper Subset:-

If A is a subset of B and A = B, then we say that A is an improper subset of B. From this definition it also follows that every set A is an improper subset of itself.

For example:-

$A = \{a, b, c\}$; $B = \{a, b, c\}$

Finite or Infinite Sets:-

If a set is equivalent to the to the set $\{1, 2, 3, \ldots, n\}$ for some fixed natural number n, then the set is said to be finite otherwise infinite.

Sets of numbers N, Z, Z' etc, mentioned earlier are infinite sets.

The set $\{1, 3, 5, \ldots, 9999\}$ is a finite set but the set $\{1, 3, 5, \ldots\}$

which is the set of all positive odd natural numbers is an infinite set.

*Exercise 2.1

Q.NO.1:-

Write the fallowing sets in set buider notation.

(i) $\{1, 2, 3, \ldots, 1000\}$

$= \{x / x \in N \wedge x \leq 1000\}$

(ii) $\{0, 1, 2, \ldots, 100\}$

$= \{x / x \in W \wedge x \leq 100\}$

(iii) $\{0, \pm 1, \pm 2, \ldots, \pm 1000\}$

$= \{x / x \in Z \wedge -1000 \leq x \leq 1000\}$

(iv) $\{0, -1, -2, \ldots, -500\}$

$= \{x / x \in Z \wedge -500 \leq x \leq 0\}$

(v) $\{100, 101, 102, \ldots, 400\}$

$= \{x / x \in N \wedge 100 \leq x \leq 400\}$

(vi) $\{-100, -101, -102, \ldots, -500\}$

$= \{x / x \in Z \wedge -500 \leq x \leq -100\}$

(vii) {Peshawar, Lahore, Karachi, Quetta}

= {x/x is the capital of a province of Pakistan}.

(viii) {January, June, July}

= {x/x is the month of the year}

(ix) The set of all odd natural numbers.
$=\{x \mid x \text{ is an odd natural number}\}$.

(x) The set of all rational numbers.
$=\{x \mid x \in Q\}$

(xi) The set of all real numbers between 1 and 2.
$=\{x \mid x \in \mathbb{R} \wedge 1 < x < 2\}$.

(xii) The set of all integers between -100 and 1000.
$=\{x \mid x \in Z \wedge -100 < x < 1000\}$

Q. NO. 2:-

Write each of the following sets in the descriptive and tabular form.

(i) $\{x \mid x \in N \wedge x \leq 10\}$
The set of first ten natural numbers.
$=\{1, 2, 3, \ldots, 10\}$

(ii) $\{x \mid x \in N \wedge 4 < x < 12\}$
The set of natural numbers between 4 and 12.
$=\{5, 6, 7, \ldots, 11\}$

(iii) $\{x \mid x \in Z \wedge -5 < x < 5\}$
The set of integers between -5 and 5.
$=\{-4, -3, -2, \ldots, 4\}$

(iv) $\{x \mid x \in E \wedge 2 < x \leq 4\}$

The set of even integers between 2 and 5.

$= \{4\}$

(v) $\{x \mid x \in P \wedge x < 12\}$

The set of prime numbers less than 12.

$= \{2, 3, 5, 7, 11\}$

(vi) $\{x \mid x \in O \wedge 3 < x < 12\}$

The set of odd integers between 3 and 12.

$= \{5, 7, 9, 11\}$

(vii) $\{x \mid x \in E \wedge 4 \leq x \leq 10\}$

The set of even integers from 4 to 10.

$= \{4, 6, 8, 10\}$

(viii) $\{x \mid x \in E \wedge 4 < x < 6\}$

The set of even integers between 4 & 6.

$= \{\ \}$

(ix) $\{x \mid x \in O \wedge 5 \leq x \leq 7\}$

The set of odd integers between 5 & 7.

$= \{5, 7\}$

(x) $\{x \mid x \in O \wedge 5 \leq x < 7\}$

The set of odd integers from 5 to 7.

$= \{5\}$

(xi) $\{x \mid x \in N \wedge x + 4 = 0\}$

The set of natural numbers x satisfying $x + 4 = 0$

$=\{\ \}$

(xii) $\{x \mid x \in Q \wedge x^2 = 2\}$

The set of rational numbers x satisfying $x^2 = 2$

$=\{\ \}$

(xiii) $\{x \mid x \in \mathbb{R} \wedge x = x\}$

The set of real numbers x satisfying $x = x$ set of real numbers.

(xiv) $\{x \mid x \in Q \wedge x = -x\}$

The set of rational numbers x satisfying $x = -x$

$=\{0\}$

(xv) $\{x \mid x \in \mathbb{R} \wedge x \neq x\}$

The set of real numbers x satisfying $x \neq x$

$=\{\ \}$

(xvi) $\{x \mid x \in \mathbb{R} \wedge x \notin Q\}$

The set of real numbers x which are not rational, Q'

Q.NO.3:-

Which of the following sets are finite and which of these are infinite?

(i) The set of students of your class.
Finite

(ii) The set of all schools in Pakistan.
Finite

(iii) The set of natural numbers between 3 and 10.
Finite

(iv) The set of rational numbers between 3 and 10.
Infinite

(v) The set of real numbers between 0 and 1.
Infinite

(vi) The set of rationales between 0 & 1.
Infinite

(vii) The set of whale numbers between 0 and 1.
Finite

(viii) The set of all leaves of trees in Pakistan.
Infinite

(ix) $P(N)$
Infinite

(x) $P\{a, b, c\}$
Finite

(xi) $\{1, 2, 3, 4, \ldots\}$
Infinite

(xii) $\{1, 2, 3, \ldots, 100000000\}$
Finite

(xiii) $\{x \mid x \in \mathbb{R} \wedge x \neq x\}$

(xiv) $\{x \mid x \in \mathbb{R} \wedge x^2 = -16\}$
Finite

(xv) $\{x \mid x \in \mathbb{Q} \wedge x^2 = 5\}$
Finite

(xvi) $\{x \mid x \in \mathbb{Q} \wedge 0 \leq x \leq 1\}$
Infinite

Q.NO.6:-

What is the difference between $\{a, b\}$ and $\{\{a, b\}\}$?

Answer:-

$\{a, b\}$ is a set containing two elements a and b, but $\{\{a, b\}\}$ is a singleton containing only one element $\{a, b\}$.

Q.NO.4:-

Write two proper subsets of each of the following sets:

(i) $\{a, b, c\}$

$= \{a\}\{b\}$

(ii) $\{0, 1\}$

$= \{0\}\{1\}$

(iii) N

$= \{1\}\{2\}$

(iv) Z

$= \{1\}\{-2\}$

(v) Q

$= \{\frac{1}{2}\}\{\frac{3}{2}\}$

(vi) $\mathbb{R}$

$= \{1\}\{\frac{2}{3}\}$

(vii) W

$= \{0\}\{1\}$

(viii) $\{x / x \in Q \wedge 0 < x \leq 12\}$

$= \{1\}\{2\}$

Q.NO.5:-

Is there any set which has no proper subset? If so name that set.

$\{\ \}$

Q.NO.9:-

Write down the power set of each of the following sets:

(i) $\{9, 11\}$

$= \{\phi, \{9\}, \{11\}, \{9, 11\}\}$

(ii) $\{+, -, \times, \div\}$

$= \{\phi, \{+\}, \{-\}, \{\times\}, \{\div\}, \{+, -\}, \{+, \times\}, \{+, \div\}, \{-, \times\}, \{-, \div\}, \{\times, \div\}, \{+, -, \times\}, \{+, -, \div\}, \{+, \times, \div\}, \{-, \times, \div\}, \{+, -, \times, \div\}\}$

(iii) $\{\phi\}$

$= \{\phi, \{\phi\}\}$

(iv) $\{a, \{b, c\}\}$

$= \{\phi, \{a\}, \{\{b, c\}\}, \{a \{b, c\}\}\}$

Definitions

Union of two sets:-

The **union** of two sets A and B, denoted by $A \cup B$, is the set of all the elements, which belong to A or B.

Symbolically:-

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

For example:-

if $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$, then

$A \cup B = \{1, 2, 3, 4, 5\}$

Intersection of two sets:-

The intersection of two sets A and B, denoted by $A \cap B$, is the set of all elements, which belong to both A and B.

Symbolically:-

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

For example:-

If $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$, then $A \cap B = \{2, 3\}$.

Disjoint Sets:-

If the intersection of two sets is the empty set then the sets are said to be disjoint sets.

For example:-

If S_1 = The set of odd natural numbers and S_2 = The set of even natural numbers, then S_1 and S_2 are disjoint sets.

Overlapping Sets:-

If the intersection of two sets is non-empty but neither is a subset of the other, the sets are called overlapping sets.

For example:-

If $L = \{2, 3, 4, 5, 6\}$ and $M = \{5, 6, \ldots, 10\}$, then L and M are two overlapping sets.

Complement of a Set:-

The complement of a set A, denoted by A' or A^c relative to the universal set is the set of all elements of U, which do not belong to A.

Symbolically:-

$$A' = \{x \mid x \in U \wedge x \notin A\}$$

For example:-

if $U = N$, then $E' = O$ and $O' = E$

Difference of two sets:-

The difference set of two sets A and B denoted by $A - B$ consist of all elements which belong to A but do not belong to B.

The Difference set of two sets B and A denoted by $B - A$ consist of all the elements which belong to B but do not belong to A.

Symbolically:-

$A-B=\{x \mid x\in A \wedge x\notin B\}$ and

$B-A=\{x \mid x\in B \wedge x\notin A\}$

*Exercise 2.3

Q.NO.1:-

Verify the commutative properties of union and intersection for the following pairs of sets:

(i) $A=\{1,2,3,4,5\}$, $B=\{4,6,8,10\}$

Commutative law of union

$$A\cup B = B\cup A$$

$$\{1,2,3,4,5\}\cup\{4,6,8,10\}=\{4,6,8,10\}\cup\{1,2,3,4,5\}$$

$$\{1,2,3,4,5,6,8,10\}=\{1,2,3,4,5,6,8,10\}$$

Hence proved

Commutative law of intersection

$$A\cap B = B\cap A$$

$$\{1,2,3,4,5\}\cap\{4,6,8,10\}=\{4,6,8,10\}\cap\{1,2,3,4,5\}$$

$$\{4\} = \{4\}$$

Hence proved

(ii) N, Z

$N=\{1,2,3,\cdots\}$

$Z=\{0,\pm1,\pm2,\cdots\}$

Commutative law of union

$$N \cup Z = Z \cup N$$

$$\{1,2,3,\ldots\} \cup \{0,\pm 1,\pm 2,\ldots\} = \{0,\pm 1,\pm 2,\ldots\} \cup \{1,2,3,\ldots\}$$

$$\{0,\pm 1,\pm 2,\ldots\} = \{0,\pm 1,\pm,\ldots\}$$

$$Z = Z$$

Commutative law of intersection

$$N \cap Z = Z \cap N$$

$$\{1,2,3,\ldots\} \cap \{0,\pm 1,\pm 2,\ldots\} = \{0,\pm 1,\pm 2,\ldots\} \cap \{1,2,3,\ldots\}$$

$$\{1,2,3,\ldots\} = \{1,2,3,\ldots\}$$

$$N = N$$

(iii) $A = \{x \mid x \in \mathbb{R} \wedge x \geq 0\}$, $B = \mathbb{R}$

$\mathbb{R}$ = real numbers all Q and Q'

(Not imaginary)

Commutative law of union

$$A \cup B = \{x \mid x \in \mathbb{R} \wedge x \geq 0\} \cup \mathbb{R}$$
$$= \mathbb{R}$$

$$B \cup A = \mathbb{R} \cup \{x \mid x \in \mathbb{R} \wedge x \geq 0\}$$
$$= \mathbb{R}$$

So, $A \cup B = B \cup A$

Commutative law of intersection

$$A \cap B = \{x \mid x \in \mathbb{R} \wedge x \geq 0\} \cap \mathbb{R}$$
$$= \{x \mid x \in \mathbb{R} \wedge x \geq 0\}$$

$$B \cap A = \mathbb{R} \cap \{x \mid x \in \mathbb{R} \wedge x \geq 0\}$$
$$= \{x \mid x \in \mathbb{R} \wedge x \geq 0\}$$

Q.NO.2:-

Verify the properties for the set A, B and C given below:

~~Answer:~~

(i) Associativity of Union

$A\cup(B\cup C) = (A\cup B)\cup C$

(a) $A=\{1,2,3,4\}$, $B=\{3,4,5,6,7,8\}$

$C=\{5,6,7,9,10\}$

L.H.S:-

$B\cup C=\{3,4,5,6,7,8\}\cup\{5,6,7,9,10\}$

$=\{3,4,5,6,7,8,9,10\}$

$A\cup(B\cup C)=\{1,2,3,4\}\cup\{3,4,5,6,7,8,9,10\}$

$=\{1,2,3,\dots,10\}$

R.H.S:-

$A\cup B=\{1,2,3,4\}\cup\{3,4,5,6,7,8\}$

$=\{1,2,3,4,5,6,7,8\}$

$(A\cup B)\cup C=\{1,2,3,4,5,6,7,8\}\cup\{5,6,7,9,10\}$

$=\{1,2,3,\dots,10\}$

So, L.H.S = R.H.S

(b) $A=\phi=\{\}$, $B=\{0\}$, $C=\{0,1,2\}$

$A\cup(B\cup C) = (A\cup B)\cup C$

L.H.S:-

$B\cup C=\{0\}\cup\{0,1,2\}$

$=\{0,1,2\}$

$A\cup(B\cup C) = \{\ \}\cup\{0,1,2\}$

$= \{0,1,2\}$

R.H.S:-

$A\cup B = \{\ \}\cup\{0\}$

$= \{0\}$

$(A\cup B)\cup C = \{0\}\cup\{0,1,2\}$

$= \{0,1,2\}$

So L.H.S = R.H.S

(c) N, Z, Q

$N\cup(Z\cup Q) = (N\cup Z)\cup Q$

L.H.S:-

$Z\cup Q = Q$

$N\cup(Z\cup Q) = N\cup Q$

$= Q$

R.H.S:-

$N\cup Z = Z$

$(N\cup Z)\cup Q = Z\cup Q$

$= Q$

So L.H.S = R.H.S

(ii) Associativity of Intersection

$A\cap(B\cap C) = (A\cap B)\cap C$

(a) $A = \{1,2,3,4\}$, $B = \{3,4,5,6,7,8\}$

$C = \{5,6,7,9,10\}$

$B \cap C = \{3,4,5,6,7,8\} \cap \{5,6,7,9,10\}$
$= \{5,6\}$
$A \cap (B \cap C) = \{1,2,3,4\} \cap \{5,6\}$
$= \{\}$
R.H.S:-
$(A \cap B) = \{1,2,3,4\} \cap \{3,4,5,6,7,8\}$
$= \{3,4\}$
$(A \cap B) \cap C = \{3,4\} \cap \{5,6,7,9,10\}$
$= \{\}$
So L.H.S = R.H.S

(b) $A = \phi = \{\}$, $B = \{0\}$, $C = \{0,1,2\}$
$A \cap (B \cap C) = (A \cap B) \cap C$
L.H.S:-
$B \cap C = \{0\} \cap \{0,1,2\}$
$= \{0\}$
$A \cap (B \cap C) = \{\} \cap \{0\}$
$= \{\}$
R.H.S:-
$A \cap B = \{\} \cap \{0\}$
$= \{\}$
$(A \cap B) \cap C = \{\} \cap \{0,1,2\}$
$= \{\}$
So L.H.S = R.H.S

(c) N, Z, Q

$$N \cap (Z \cap Q) = (N \cap Z) \cap Q$$

L.H.S:-

$$Z \cap Q = Z$$

$$N \cap (Z \cap Q) = N \cap Z$$

$$= N$$

R.H.S:-

$$N \cap Z = N$$

$$(N \cap Z) \cap Q = N \cap Q$$

$$= N$$

So L.H.S = R.H.S

(iii) Distributivity of Intersection over union

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(a) $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7, 8\}$

$C = \{5, 6, 7, 9, 10\}$

L.H.S:-

$$B \cup C = \{3,4,5,6,7,8\} \cap \{5,6,7,9,10\}$$

$$= \{3,4,5,6,7,8,9,10\}$$

$$A \cap (B \cup C) = \{1,2,3,4\} \cap \{3,4,5,6,7,8,9,10\}$$

$$= \{3,4\}$$

R.H.S:-

$$A \cap B = \{1,2,3,4\} \cap \{3,4,5,6,7,8\}$$

$$= \{3,4\}$$

$A \cap C = \{1, 2, 3, 4\} \cap \{5, 6, 7, 9, 10\}$
$= \{\ \}$

$(A \cap B) \cup (A \cap C) = \{3, 4\} \cup \{\ \}$
$= \{3, 4\}$

(b) $A = \phi = \{\ \}$, $B = \{0\}$, $C = \{0, 1, 2\}$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

L.H.S:-

$B \cup C = \{0\} \cup \{0, 1, 2\}$
$= \{0, 1, 2\}$

$A \cap (B \cup C) = \{\ \} \cap \{0, 1, 2\}$
$= \{\ \}$

R.H.S:-

$A \cap B = \{\ \} \cap \{0\}$
$= \{\ \}$

$A \cap C = \{\ \} \cap \{0, 1, 2\}$
$= \{\ \}$

$(A \cap B) \cup (A \cap C) = \{\ \} \cup \{\ \}$
$= \{\ \}$

(c) N, Z, Q

$N \cap (Z \cup Q) = (N \cap Z) \cup (N \cap Q)$

L.H.S:-

$Z \cup Q = Q$

$N \cap (Z \cup Q) = N \cap Q$
$= N$

R.H.S:-

$$N \cap Z = N$$

$$N \cap Q = N$$

$$(N \cap Z) \cup (N \cap Q) = N \cup N$$

$$= N$$

(iv) Distributivity of union over intersection

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(a) $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7, 8\}$

$C = \{5, 6, 7, 9, 10\}$

L.H.S:-

$$B \cap C = \{3, 4, 5, 6\} \cap \{5, 6, 7, 9, 10\}$$

$$= \{5, 6, 7\}$$

$$A \cup (B \cap C) = \{1, 2, 3, 4\} \cup \{5, 6, 7\}$$

$$= \{1, 2, 3, 4, 5, 6, 7\}$$

L.H.S:-

$$A \cup B = \{1, 2, 3, 4\} \cup \{3, 4, 5, 6, 7, 8\}$$

$$= \{1, 2, 3, 4, 5, 6, 7, 8\}$$

$$A \cup C = \{1, 2, 3, 4\} \cup \{5, 6, 7, 9, 10\}$$

$$= \{1, 2, 3, 4, 5, 6, 7, 9, 10\}$$

$$(A \cup B) \cap (A \cup C) = \{1, 2, 3, \cdots, 8\} \cap \{1, 2, 3, 4, 5, 6, 7, 9, 10\}$$

$$= \{1, 2, 3, 4, 5, 6, 7\}$$

(b) $A = \phi = \{\}$, $B = \{0\}$, $C = \{0, 1, 2\}$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

L.H.S:-

$B \cap C = \{0\} \cap \{0, 1, 2\}$

$= \{0\}$

$A \cup (B \cap C) = \{\ \} \cup \{0\}$

$= \{0\}$

R.H.S:-

$A \cup B = \{\ \} \cup \{0\}$

$= \{0\}$

$A \cup C = \{\ \} \cup \{0, 1, 2\}$

$= \{0, 1, 2\}$

$(A \cup B) \cap (A \cup C) = \{0\} \cap \{0, 1, 2\}$

$= \{0\}$

(c) N, Z, Q

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$N \cup (Z \cap Q) = (N \cup Z) \cap (N \cup Q)$

L.H.S

$Z \cap Q = Z$

$N \cup (Z \cap Q) = N \cup Z$

$= Z$

R.H.S:-

$N \cup Z = Z$

$N \cup Q = Q$

$(N \cup Z) \cap (N \cup Q) = Z \cap Q$

$= Z$

Q.NO.3:-

Verify De-Morgan's Laws for the following sets:

$U = \{1, 2, 3, \ldots, 20\}$, $A = \{2, 4, 6, \ldots, 20\}$ and $B = \{1, 3, 5, \ldots, 19\}$

(i) $(A \cap B)' = A' \cup B'$

L.H.S:-

$A \cap B = \{2, 4, 6, \ldots, 20\} \cap \{1, 3, 5, \ldots, 19\}$
$= \{\}$

$(A \cap B)' = U - (A \cap B)$
$= \{1, 2, 3, \ldots, 20\} - \{\}$
$= \{1, 2, 3, \ldots, 20\}$

R.H.S:-

$A' = U - A$

$A' = \{1, 2, 3, \ldots, 20\} - \{2, 4, 6, \ldots, 20\}$
$= \{1, 3, 5, \ldots, 19\}$

$B' = U - B$
$= \{1, 2, 3, \ldots, 20\} - \{1, 3, 5, \ldots, 19\}$
$= \{2, 4, 6, \ldots, 20\}$

$A' \cup B' = \{1, 3, 5, \ldots, 19\} \cup \{2, 4, 6, \ldots, 20\}$
$= \{1, 2, 3, \ldots, 20\}$

So L.H.S = R.H.S

(ii) $(A\cup B)' = A'\cap B'$

L.H.S:-

$A\cup B = \{2,4,6,\dots,20\}\cup\{1,3,5,\dots,19\}$

$= \{1,2,3,\dots,20\}$

$(A\cup B)' = U-(A\cup B)$

$= \{1,2,3,\dots,20\}-\{1,2,3,\dots,20\}$

$= \{\ \}$

R.H.S:-

$A' = U-A$

$= \{1,2,3,\dots,20\}-\{2,4,6,\dots,20\}$

$= \{1,3,5,\dots,19\}$

$B' = U-A$

$= \{1,2,3,\dots,20\}-\{1,3,5,\dots,19\}$

$= \{2,4,6,\dots,20\}$

$A'\cap B' = \{1,3,5,\dots,19\}\cap\{2,4,6,\dots,20\}$

$= \{\ \}$

Hence proved

Q.NO.4:-

Let U = The set of English alphabet

$A = \{x \mid x$ is a vowel$\}$, $B = \{y \mid y$ is consonant$\}$

Verify De-Morgan's law of these sets

(i) $(A\cup B)' = A'\cap B'$

L.H.S:-

$A\cup B = \{x \mid x$ is a vowel$\}\cup\{y \mid y$ is a consonant$\}$

$= \text{English Alphabets}$

$(A \cup B)' = U - (A \cup B)$

$= \text{English alphabet} - \text{English alphabet}$

$= \{\ \}$

R.H.S:-

$A' = U - A$

$= \text{English alphabet} - \{x \mid x \text{ is a vowel}\}$

$= \{y \mid y \text{ is a consonant}\}$

$B' = U - B$

$= \text{English alphabet} - \{y \mid y \text{ is a consonant}\}$

$= \{x \mid x \text{ is a vowel}\}$

$A' \cap B' = \{y \mid y \text{ is a consonant}\} \cap \{x \mid x \text{ is a vowel}\}$

$= \{\ \}$

Hence proved

(ii) $(A \cap B)' = A' \cup B'$

L.H.S:-

$A \cap B = \{x \mid x \text{ is a vowel}\} \cap \{y \mid y \text{ is consonant}\}$

$= \{\ \}$

$(A \cap B)' = U - (A \cap B)$

$= \text{English alphabet} - \{\ \}$

$= \text{English alphabet}$

R.H.S:-

$A' = U - A$

$= \text{English alphabet} - \{x \mid x \text{ is a vowel}\}$

$= \{y \mid y \text{ is a consonant}\}$

$B' = U - B$

$= \text{English alphabet} - \{y \mid y \text{ is a consonant}\}$

$= \{x \mid x \text{ is a vowel}\}$

$A' \cup B' = \{y \mid y \text{ is a consonant}\} \cup \{x \mid x \text{ is a vowel}\}$

$= \text{English alphabet}$

Q.NO.6:-

Taking any set, say $A = \{1,2,3,4,5\}$, verify the following:

(i) $A \cup \phi = A$

$A \cup \phi = \{1,2,3,4,5\} \cup \{\ \}$

$= \{1,2,3,4,5\}$

$= A$

$A \cup \phi = A$

(ii) $A \cup A = A$

$A \cup A = \{1,2,3,4,5\} \cup \{1,2,3,4,5\}$

$= \{1,2,3,4,5\}$

$= A$

$A \cup A = A$

(iii) $A \cap A = A$

$A \cap A = \{1,2,3,4,5\} \cap \{1,2,3,4,5\}$

$= \{1,2,3,4,5\}$

$A \cap A = A$

Q.NO.7:-

If $U = \{1, 2, 3, \cdots, 20\}$ and $A = \{1, 3, 5, \cdots, 19\}$, verify the following:

(i) $A \cup A' = U$

$A' = U - A$

$= \{1, 2, 3, \cdots, 20\} - \{1, 3, 5, \cdots, 19\}$

$= \{2, 4, 6, \cdots, 20\}$

$A \cup A' = \{1, 3, 5, \cdots, 19\} \cup \{2, 4, 6, \cdots, 20\}$

$= \{1, 2, 3, \cdots, 20\}$

$= U$

(ii) $A \cap U = A$

$A \cap U = \{1, 3, 5, \cdots, 17\} \cap \{1, 2, 3, \cdots, 20\}$

$= \{1, 3, 5, \cdots, 19\}$

$= A$

(iii) $A \cap A' = \phi$

$A' = U - A$

$= \{1, 2, 3, \cdots, 20\} - \{1, 3, 5, \cdots, 19\}$

$= \{2, 4, 6, \cdots, 20\}$

$A \cap A' = \{1, 3, 5, \cdots, 19\} \cap \{2, 4, 6, \cdots, 20\}$

$= \{\ \}$

$= \phi$

Q.NO.8:-

From suitable properties of union and intersection deduced the fallowing results:

(i) $A \cap (A \cup B) = A \cup (A \cap B)$

L.H.S

$= A \cap (A \cup B)$

$= (A \cap A) \cup (A \cap B)$ ∴ distributive property

$= A \cup (A \cap B)$

As $A \cap A = A$

So,

$A \cap (A \cup B) = A \cup (A \cap B)$

(ii) $A \cup (A \cap B) = A \cap (A \cup B)$

L.H.S:-

$A \cup (A \cap B)$

$= (A \cup A) \cap (A \cup B)$

∴ distributive property

$= A \cap (A \cup B)$ ∴ $A \cup A = A$

$=$ R.H.S

So,

$A \cup (A \cap B) = A \cap (A \cup B)$

*Exercise 2.2

Q.NO.3:-

Under what conditions on A and B are the fallowing statements are true?

(i) $A \cup B = A$
if $B \subseteq A$

(ii) $A \cup B = B$
if $A \subseteq B$

(iii) $A - B = A$
if $A \cap B = \phi$

(iv) $A \cap B = B$
if $B \subseteq A$

(v) $n(A \cup B) = n(A) + n(B)$
if $A \cap B = \phi$

(vi) $n(A \cap B) = n(A)$
if $A \subseteq B$

(vii) $A - B = A$
if $A \cap B = \phi$

(viii) $n(A \cap B) = 0$
if $A \cap B = \phi$

(ix) $n(A \cap B) = n(B)$
if $B \subseteq A$

(x) $A \cup B = B \cup A$

(xi) $A \cup B = U$
if $A = B^c$
or $B = A^c$

(xii) $U - A = \phi$

if $U = A$

Q.NO.4:-

Let $U = \{1, 2, 3, \cdots, 10\}$,

$A = \{2, 4, 6, 8, 10\}$, $B = \{1, 2, 3, 4, 5\}$

and $C = \{1, 3, 5, 7, 9\}$.

List the members of each of the following sets:

(i) A^c

$A^c = U - A$

$= \{1, 2, 3, \cdots, 10\} - \{2, 4, 6, 8, 10\}$

$= \{1, 3, 5, 7, 9\}$

(ii) B^c

$B^c = U - B$

$= \{1, 2, 3, \cdots, 10\} - \{1, 2, 3, 4, 5\}$

$= \{6, 7, 8, 9, 10\}$

(iii) $A \cup B$

$A \cup B = \{2, 4, 6, 8, 10\} \cup \{1, 2, 3, 4, 5\}$

$= \{1, 2, 3, 4, 5, 6, 8, 10\}$

(iv) $A - B$

$A - B = \{2, 4, 6, 8, 10\} - \{1, 2, 3, 4, 5\}$

$= \{6, 8, 10\}$

(v) $A \cap C$

$A \cap C = \{2,4,6,8,10\} \cap \{1,3,5,7,9\}$
$= \{\ \}$

(vi) $A^c \cup C^c$

$A^c = U - A$
$= \{1,2,3,\cdots,10\} - \{2,4,6,8,10\}$
$= \{1,3,5,7,9\}$

$C^c = U - C$
$= \{1,2,3,\cdots,10\} - \{1,3,5,7,9\}$
$= \{2,4,6,8,10\}$

$A^c \cup C^c = \{1,3,5,7,9\} \cup \{2,4,6,8,10\}$
$= \{1,2,3,\cdots,10\}$
$= U$

(vii) $A^c \cup C$

$A^c = U - A$
$= \{1,2,3,\cdots,10\} - \{2,4,6,8,10\}$
$= \{1,3,5,7,9\}$

$A^c \cup C = \{1,3,5,7,9\} \cup \{1,3,5,7,9\}$
$= \{1,3,5,7,9\}$
$= C$

(viii) U^c

$= U - U$
$= \{1,2,3,\cdots,10\} - \{1,2,3,\cdots,10\}$
$= \phi$

Definitions

Induction:-

We generally form opinions about others on the basis of a few contacts only. This way of drawing conclusions is called induction.

Deduction:-

The way of reasoning i.e., drawing conclusions from premises believed to be true, is called deduction.

Important Points:-

→ In daily life we often draw general conclusion from a limited number of observations or experiences.

→ On many occasions we have to adopt the opposite course. We have to draw conclusions from accepted or well-known facts.

Aristotlian logics:-

Deductive logic in which every statement is regarded as true or false and there is no other possibility.

is called Aristotlian logics-

Non-Aristotlian Logics:-

Logic in which there is scope for third or fourth possibility is called **non-aristotlian logic**.

Proposition:-

A daclarative statement which may be true or false but not both is called a **proposition**.

Negation:-

If p is any proposition its negation is denoted by ~p, read 'not p'. It follows from this definition that if is true, ~p false and if p is false, ~p is true. The adjoining table called **truth table**, gives the possible truth values of p and ~p.

Truth table:-

p	~p
T	F
F	T

Conjunction:-

Conjunction of two statements p and q is denoted symbalically as $p \wedge q$ (p and q). A conjunction is considered to be true if both its components are true.

Truth table:-

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction:-

Disjunction of p and q is p or q. It is symbolically written $p \vee q$. The disjunction $p \vee q$ is considered to be true when at least one of the components p and q is true. It is false when both of them are false.

Truth table:-

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Implication or Conditional:-

A compound statement of the form if p then q also written p implies q is called a conditional or an implication p is called the antecedent or hypothesis and q is called the consequent or the conclusion.

Truth table:-

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional: $p \leftrightarrow q$

The proposition $p \rightarrow q \wedge q \rightarrow p$ is shortly written $p \leftrightarrow q$ and is called the biconditional or equivalence.

It is read p iff q (iff stands for "if and only if").

Tautologies:-

A statement which is true for all the possible values of the variables involved in it is called a tautology, for example, $p \to q \leftrightarrow (\sim p \to \sim p)$ is a tautology.

Absurdity:-

A statement which is always false is called an absurdity or a contradiction.

For examples-

$$p \to \sim p$$

Contingency:-

A statement which can be true or false depending upon the truth values of the variable involved in it is called a contingency.

For example:-

$(p \to q) \wedge (p \vee q)$ is a contingency

Exercise 2.4

Q.No. 1:-

Write the converse, inverse and contrapositive of the following conditionals.

(i) $\sim p \rightarrow q$

Converse:- $q \rightarrow \sim p$

Inverse:- $p \rightarrow \sim q$

Contrapositive:- $\sim q \rightarrow p$

(ii) $q \rightarrow p$

Converse: $p \rightarrow q$

Inverse:- $\sim q \rightarrow \sim p$

Contrapositive:- $\sim p \rightarrow \sim q$

(iii) $\sim p \rightarrow \sim q$

Converse:- $\sim q \rightarrow \sim p$

Inverse:- $p \rightarrow q$

Contrapositive:- $q \rightarrow p$

(iv) $\sim q \rightarrow \sim p$

Converse:- $\sim p \rightarrow \sim q$

Inverse:- $q \rightarrow p$

Contrapositive:- $p \rightarrow q$

Q. NO. 2:-

Construct truth tables for the following statements:

(i) $(p \to \sim p) \vee (p \to q)$

p	q	$\sim p$	$p \to \sim p$	$p \to q$	$(p \to \sim p) \vee (p \to q)$
T	T	F	F	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	T	T	T

(ii) $(p \wedge \sim p) \to q$

p	q	$\sim p$	$p \wedge \sim p$	$(p \wedge \sim p) \to q$
T	T	F	F	F
T	F	F	F	T
F	T	T	F	F
F	F	T	F	T

(iii) $\sim(p \to q) \leftrightarrow (p \wedge \sim q)$

p	q	$p \to q$	$\sim(p \to q)$	$\sim q$	$(p \wedge \sim q)$	$\sim(p \to q) \leftrightarrow (p \wedge \sim q)$
T	T	T	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	F	T	F	T	F	T

Q.NO.3:-

Show that each of the following statements is a tautology.

(i) $(p \wedge q) \rightarrow p$

p	q	$(p \wedge q)$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T

(ii) $p \rightarrow (p \vee q)$

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

(iii) $\sim(p \rightarrow q) \rightarrow p$

p	q	$(p \rightarrow q)$	$\sim(p \rightarrow q)$	$\sim(p \rightarrow q) \rightarrow p$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

(iv) $\sim q \wedge (p \rightarrow q) \rightarrow \sim p$

p	q	$\sim q$	$\sim p$	$(p\to q)$	$\sim q\wedge(p\to q)$	$\sim q\wedge(p\to q)\to\sim p$
T	T	F	F	T	F	T
T	F	T	F	F	F	T
F	T	F	T	T	F	F
F	F	T	T	T	T	T

Q.NO. 4:-

Determine whether each of the following is a tautology, a contingency or an absurdity.

i) $p\wedge\sim p$

p	q	$\sim p$	$p\wedge\sim p$
T	T	F	F
T	F	F	F
F	T	T	F
F	F	T	F

ii) $p\to(q\to p)$

p	q	$q\to p$	$p\to(q\to p)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

iii) $q\vee(\sim q\vee p)$

p	q	$\sim q$	$\sim q \vee p$	$q \vee (\sim q \vee p)$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	T

Q.NO.5:-

Prove that.

$p \vee (\sim p \wedge \sim q) \vee (p \wedge q) = p \vee (\sim p \wedge \sim q)$

L.H.S

p	q	$\sim p$	$\sim q$	$(\sim p \wedge \sim q)$	$p \vee (\sim p \wedge \sim q)$	$p \wedge q$	$p \vee (\sim p \wedge \sim q) \vee (p \wedge q)$
T	T	F	F	F	T	T	T
T	F	F	T	F	T	F	T
F	T	T	F	F	F	F	F
F	F	T	T	T	T	F	T

R.H.S

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$p \vee (\sim p \wedge \sim q)$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	F
F	F	T	T	T	T

So L.H.S = R.H.S

* Exercise 2.5

Convert the following theorems to logical and prove them by constructing truth tables:

Q.NO.1:-

$$(A\cap B)' = A' \cup B'$$

$\sim(p\wedge q) = \sim p \vee \sim q$

L.H.S

p	q	$(p\wedge q)$	$\sim(p\wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

R.H.S

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

So L.H.S = R.H.S

Q.NO.2:-

$$(A\cup B)\cup C = A\cup(B\cup C)$$

$$(p\vee q)\vee r = p\vee(q\vee r)$$

L.H.S:-

p	q	r	$p \vee q$	$(p \vee q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

R.H.S

p	q	r	$q \vee r$	$p \vee (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

So L.H.S = R.H.S

Q.NO.3:-

$$(A \cap B) \cap C = A \cap (B \cap C)$$

$$(p \wedge q) \wedge r = p \wedge (q \wedge r)$$

L.H.S

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

R.H.S

p	q	r	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

L.H.S = R.H.S

Q.NO.4:-

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$p \wedge (q \wedge r) = (p \wedge q) \wedge (q \wedge r)$

L.H.S.

p	q	r	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

R.H.S

p	q	r	$p \wedge q$	$q \wedge r$	$(p \wedge q) \wedge (p \wedge r)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	F	T	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

So L.H.S = R.H.S

Definitions

Binary relation:-

Let A and B be two non-empty sets, then any subset of the Cartesian product $A \times B$ is called a **binary relation**, or simply a **relation**, from A to B. Ordinarily a relation will be denoted by the letter r.

Domain:-

The set of the first elements of the ordered pairs forming a relation is called its **domain**.

Range:-

The set of the second elements of the ordered pairs forming a relation is called its **range**.

Relation in A:-

If A is a non-empty set, any subset of $A \times A$ is called a **relation in A**. Some authors call it a **relation on A**.

Functions:-

A very important special type of relation is a function defined as below:-

Let A and B be two non-empty sets such that:

(i) f is a relation from A to B that is, f is a subset of $A \times B$.

(ii) Dom $f = A$

(iii) Domain doesn't repeat.

Into Function:-

If a function $f: A \rightarrow B$ is such that Ran $f \subset B$ i.e., Ran $f \neq B$, then f is said to be a function from A into B.

Onto (Surjective) function:-

If a function $f: A \rightarrow B$ is such that Ran $f = B$ i.e, every element of B is the image of some elements of A, then f is called is called an **onto** function or a surjective function.

(1-1) and into (Injective) function:-

If a function f from A into B is such that second elements of no two of its ordered pairs are equal, then it is called an **injective** (1-1 and into) function.

(1-1) and Onto function (bijective function).

If f is a function from A onto B such that second elements of no two of its ordered pairs are the same, then f is said to be (1-1) function from A onto B.

Such a function is also called a (1-1) correspondence between A and B. It is also called a **bijective function**.

Linear function:-

The function $\{(x,y) \mid y = mx + c\}$ is called a **linear function**, because its graph is a straight line.

Quadratic functions:-

The function $\{(x,y) / y = ax^2 + bx + c\}$ is called a **quadratic function** because it is defined by a quadratic (second degree) in x, y.

Inverse of a functions:-

The **inverse of each function** is exist, if it is bijective function.

Method:-

If a relation or a function is given in the tabular form i.e, as a set of ordered pairs, its inverse is obtained by interchanging the components of each ordered pairs. The inverse of r and f are denoted r^{-1} and f^{-1} respectively.

Exercise 2.6

Q.NO.1:-

For $A = \{1, 2, 3, 4\}$, find the following relations in A. State the domain and range of each relation. Also draw the graph of each.

i) ~~$\{(x, y) \mid y$~~

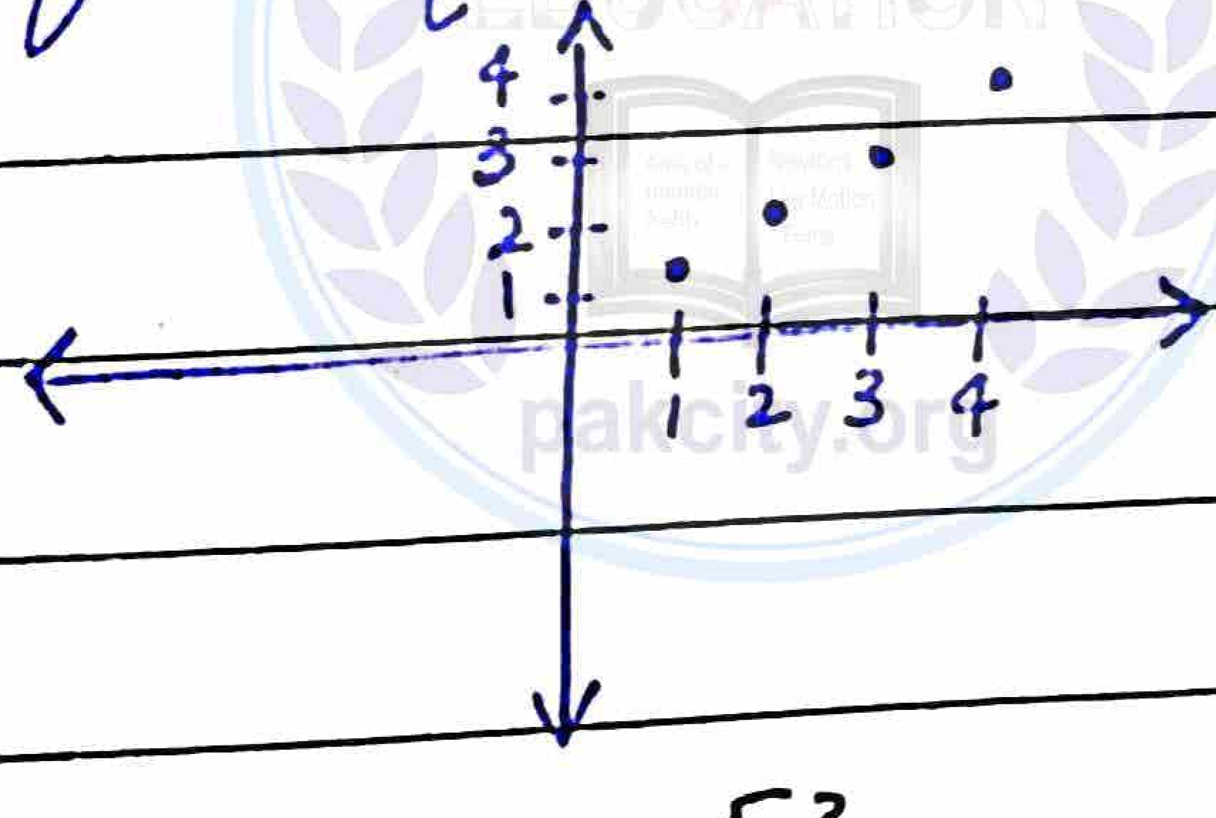
$A \times A = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$
$= \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$

i) $B = \{(x, y) \mid y = x\}$

$B = \{(1,1), (2,2), (3,3), (4,4)\}$

Dom of $B = \{1, 2, 3, 4\}$

Range of $B = \{1, 2, 3, 4\}$

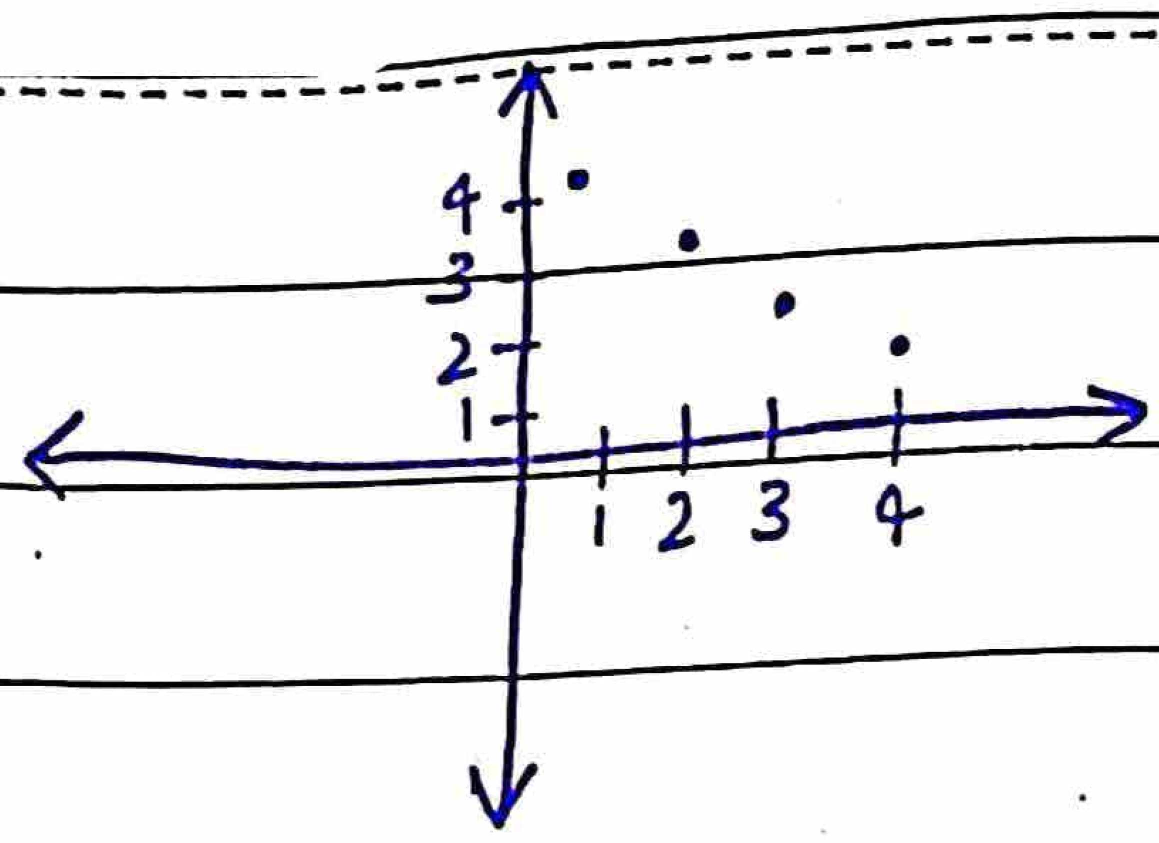


ii) $C = \{(x, y) \mid y + x = 5\}$

$C = \{(1,4), (2,3), (3,2), (4,1)\}$

Dom of $C = \{1, 2, 3, 4\}$

Range of $C = \{1, 2, 3, 4\}$

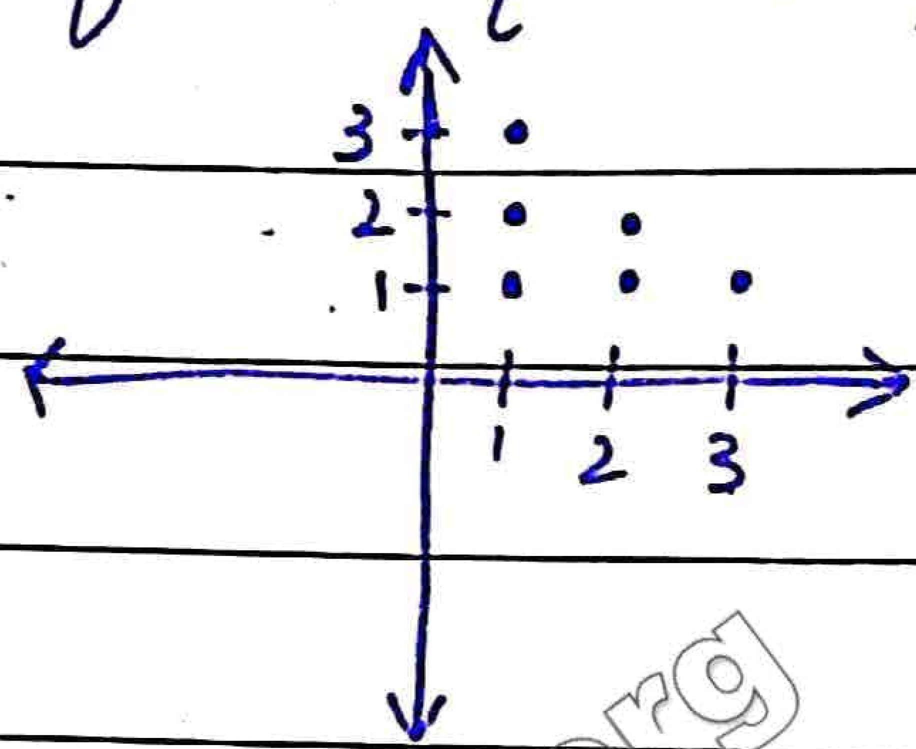


(iii) $D = \{(x, y) \mid x + y < 5\}$

$D = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}$

Dom of $D = \{1, 2, 3\}$

Range of $D = \{1, 2, 3\}$

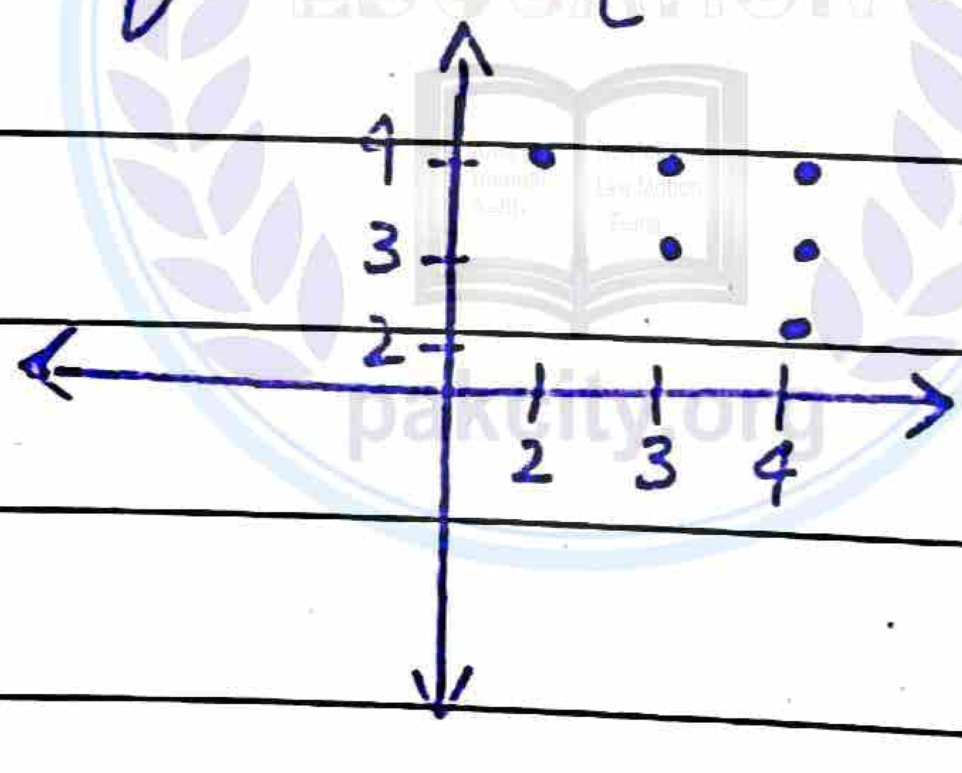


(iv) $E = \{(x, y) \mid x + y > 5\}$

$E = \{(3,3), (3,4), (4,3), (4,2), (2,4), (4,4)\}$

Dom of $E = \{2, 3, 4\}$

Range of $E = \{2, 3, 4\}$



Q.NO.4:-

Find the inverse of each of the following relations. Tell whether each relation and its inverse is a function or not.

i) $\{(2,1),(3,2),(4,3),(5,4),(6,5)\}$

$f = \{(2,1),(3,2),(4,3),(5,4),(6,5)\}$

$Dom(f) = \{2,3,4,5,6\}$

$f^{-1} = \{(1,2),(2,3),(3,4),(4,5),(5,6)\}$

$Dom(f^{-1}) = \{1,2,3,4,5\}$

Here both f and f^{-1} are functions.

ii) $\{(1,3),(2,5),(3,7),(4,9),(5,11)\}$

$f = \{(1,3),(2,5),(3,7),(4,9),(5,11)\}$

$Dom(f) = \{1,2,3,4,5\}$

$f^{-1} = \{(3,1),(5,2),(7,3),(9,4),(11,5)\}$

$Dom\ f^{-1} = \{3,5,7,9,11\}$

Here both f and f^{-1} are function.

iii) $E = \{(x,y) \mid y = 2x+3,\ x \in \mathbb{R}\}$

$E = \{(x,y) \mid y = 2x+3,\ x \in \mathbb{R}\}$

E is a function

$E^{-1} = \{(x,y) \mid x = 2y+3,\ x \in \mathbb{R}\}$

E^{-1} is a function

iv) $F = \{(x,y) \mid y^2 = 4ax,\ x \geq 0\}$

F is not a function

$F^{-1} = \{(x,y) \mid x^2 = 4ay,\ x \geq 0\}$

F^{-1} is a function

(v) $\{(x,y) \mid x^2+y^2=9, |x|\leq 3, |y|\leq 3\}$

$S=\{(x,y) \mid x^2+y^2=9, |x|\leq 3, |y|\leq 3\}$

S is not a function

$S^{-1}=\{(x,y) \mid y^2+x^2=9, |x|\leq 3, |y|\leq 3\}$

S^{-1} is not a function

Because it is an equation of circle

Definitions

Groupoid:-

A groupoid is non-empty set on which a binary operation $*$ is defined.

Some authors call the system $(S,*)$ a groupoid. But, for the sake of brevity and convenience we shall call S a groupoid, it being understood that an operation $*$ is defined on it.

In other words, a closed set with respect to an operation $*$ is called a groupoid.

Semi-group:-

A non-empty set is semi-group if;

i) It is closed with respect to an operation $*$ and
ii) The operation $*$ is associative.

Monoid:-

A semi-group having an identity is called a **monoid** i.e., a monoid is a set S;

i) which is closed w.r.t some operation.
ii) the operation $*$ is associative and
iii) it has an identity.

Definition of Groups-

A monoid having inverse of each of its elements under $*$ is called a group under $*$. That is a group under $*$ is a set G (say) if:

i) G is closed w.r.t some operation $*$.
ii) The operation on $*$ is associative.
iii) G has an identity element w.r.t $*$ and
iv) Every element of G has an inverse in G w.r.t $*$.

Abelian:-

If G satisfies the additional condition:

For every $a, b \in G$

$a * b = b * a$

then G is said to be an Abelian* or commutative group under $*$.

*Exercise 2.8

Q. NO. 1:-

Operation $\oplus$ performed on the two-member set $G = \{0, 1\}$ is shown in the adjoining table. Answer the questions:-

$\oplus$	0	1
0	0	1
1	1	0

(i) Name the identity element if it exists?

$0 + 0 = 0$

$1 + 0 = 0$

0 is an identity

ii) What is the inverse of 1?

$$0 + 0 = 0$$

$$1 + 1 = 0$$

Inverse of each elements exist

iii) Is the set G, under the given operation a group?
Abelian or non-Abelian?

$$0 + 1 = 1 + 0$$

$$1 = 1$$

It is an Abalien

Q. NO. 2:-

The operation ⊕ as performed on the set {0, 1, 2, 3} is shown in the adjoining table, show that the set is an Abelian group?

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

(i) Closure - law

From table, given S is closed w.r.t (+).

(ii) Associative law

$1, 2, 3 \in S$

$(1+2)+3 = 1+(2+3)$

$3+3 = 1+1$

$2 = 2$

Associative law hold w.r.t (+)

(iii) Identity

e.g:-

$0+0=0$

$1+0=1$

$2+0=2$

$3+0=3$

0 is an identity element.

(iv) Inverse

e.g:-

$0+0=0$

$1+3=0$

$3+1=0$

$2+2=0$

Inverse of each element exist

(v) Commutative law

$2, 3 \in S$

$2+3 = 3+2$

$1 = 1$

Hence, all axioms satisfied. So, the given set S is an abelian group under (+).

Q.NO.4:-

Show that the adjoining table represents the sums of the elements of the set $\{E, O\}$. What is the identity element of this set? Show that this is an abelian group?

+	E	O
E	E	O
O	O	E

(i) Closure-law

From table, given G is closed w.r.t (+).

(ii) Associative law

$E, O \in G$

$(E+O)+E = E+(O+E)$

$O+E = E+O$

$O = O$

Associative law hold w.r.t (+).

(iii) Identity

e.g:-

$$O + E = O$$
$$E + E = E$$

E is an identity

(iv) Inverse

e.g:-

$$E + E = E$$
$$O + O = E$$

Inverse of each element exist

(v) Commutative law

$$E, O \in G$$
$$E + O = O + E$$
$$O = O$$

Hence all axioms satisfied. So, the given set G is an abelian group under $(+)$.

Q.NO.5:-

Show that the set $\{1, \omega, \omega^2\}$ when $\omega^3 = 1$ is an Abelian group w.r.t ordinary multiplication.

$$G = \{1, \omega, \omega^2\}$$

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

(i) Closure-law

From table, clearly the given set G is closed under $(\cdot)$.

(ii) Associative law

$$1, \omega, \omega^2 \in G$$

$$(1\cdot\omega)\cdot\omega^2 = 1\cdot(\omega\cdot\omega^2)$$

$$\omega\cdot\omega^2 = 1\cdot 1$$

$$1 = 1$$

Associative law hold w.r.t $(\cdot)$.

(iii) Identity

e.g:-

$$1\cdot\omega = \omega$$

$$1\cdot\omega^2 = \omega^2$$

$$1\cdot 1 = 1$$

1 is an identity element

(iv) Inverse

e.g:-

$$1\cdot 1 = 1$$

$$\omega\cdot\omega^2 = 1$$

$$\omega^2\cdot\omega = 1$$

Inverse of each element exist

(v) Commutative law

$\omega, \omega^2 \in G$

$\omega \cdot \omega^2 = \omega^2 \cdot \omega$

$1 = 1$

All conditions are satisfied, then the given set G is Abelian group under $(\cdot)$.

Q.NO.6:-

If G is a group under the operation $\dot{\times}$ and $a, b \in G$, find the solutions of the equations

(i) $a \dot{\times} x = b$

Pre-multiply by a^{-1} on both sides

$a^{-1} \dot{\times} a \dot{\times} x = a^{-1} \dot{\times} b$

$a^{-1} \dot{\times} a = e$

$\therefore$ e is an identity

$e \dot{\times} x = a^{-1} \dot{\times} b$

$x = a^{-1} \dot{\times} b$

(ii) $x \dot{\times} a = b$

Post-multiply by a^{-1} on both sides

$x \dot{\times} a \dot{\times} a^{-1} = b \dot{\times} a^{-1}$

$\therefore$ e is an identity

$x \dot{\times} e = b \dot{\times} a^{-1}$

$\therefore$ e is an identity

$x = b \cdot x \cdot a^{-1}$

Examples (imp)

Example 12:

The set N w.r.t +

(i) Closure-law

$N = \{1, 2, 3, \dots\}$

The set N hold closure law.

(ii) Associative law

$1, 2, 3 \in N$

$1 + (2+3) = (1+2)+3$

$1 + 5 = 3+3$

$6 = 6$

(iii) Identity

Identity does not hold. So, it is a semi-group.

Example 13:

The set N w.r.t $\otimes$

$N = \{1, 2, 3 \dots\}$

(i) Closure law

Closure law hold in set N.

(ii) Associative law

$1 \times (2 \times 3) = (1 \times 2) \times 3$

$$1 \times 6 = 2 \times 3$$
$$6 = 6$$

Associative law hold w.r.t $(\cdot)$.

iii- Identity

$$2 \times 1 = 2$$
$$1 \times 1 = 1$$
$$3 \times 1 = 3$$
$$4 \times 1 = 4$$

So, the identity element is 1

iv- Inverse

Inverse of elements cannot exist.

So, it is a monoid.

Example 14:

Consider $S = \{0, 1, 2\}$ upon which operation $\oplus$ has been performed as shown in the following table. Show that S is an abelian group under $\oplus$.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

(i) Closure law

From given table, the given set

S is closed w.r.t (+).

(ii) Associative law

$$0+(1+2) = (0+1)+2$$
$$0+0 = 1+2$$
$$0 = 0$$

Associative law holds w.r.t (+).

(iii) Identity

$$0, 1, 2 \in S$$

e.g:- $0+1 = 1$

$$0+2 = 2$$
$$0+3 = 3$$

0 is an identity element.

(iv) Inverse

e.g:- $0+0 = 0$

$$1+2 = 0$$
$$2+1 = 0$$

Inverse of each element exist.

(v) Commutative law

$$0+1 = 1+0$$
$$1 = 1$$

As all the properties satisfied. So, the given set S is an abelian group.

Example 15:

Consider the set $S = \{1, -1, i, -i\}$. Set up its multiplication table and show that the set in an abelian group under multiplication.

$\times$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	+1	-1

(i) Closure law

From given set table, the set S is closed w.r.t $(\times)$.

(ii) Associative law

$1, -1, i \in S$

$1 \times (-1 \times i) = (1 \times -1) \times i$

$1 \times -i = -1 \times i$

$-i = -i$

Associative law holds w.r.t $(\cdot)$.

(iii) Identity

e.g $1 \times 1 = 1$

$1 \times i = i$

$1 \times -i = -i$

So, 1 is an identity element.

(iv) Inverse

e.g

$$1 \times 1 = 1$$
$$-1 \times -1 = 1$$
$$i \times -i = 1$$
$$-i \times i = 1$$

Inverse of each element exist

(v) Commutative law

$$1 \times -1 = -1 \times 1$$
$$-1 = -1$$

As, all the properties are satisfied. So, the given set S is an Abelian group under $(\cdot)$.

Theorems

Theorem 1:

If $(G, \dot{\times})$ is a group with e its identity, then e is unique.

Proof:

e is not unique

Let e' is another identity.

$e' \dot{\times} e = e \dot{\times} e' = e'$ $\because$ e is identity

$e' \ast e = e \ast e' = e \quad \because e'$ is identity

$e' = e$

Thus identity of a group is always unique.

Theorem 2:

If $(G, \ast)$ is a group and $a \in G$, there is unique inverse of a in G.

Proof:

$a' = a' \ast e \quad \because e$ is identity

$= a' \ast (a \ast a'')$

$= (a' \ast a) \ast a'' \quad \because$ Associative law

$= e \ast a''$

$a' = a''$

Thus inverse of a is unique in G