 MATHEMATICS 1^{st} YEAR

UNIT # 01

NUMBER SYSTEM

Muhammad Salman Sherazi
M.Phil (Math)

 www.pakcity.org

Contents

Sherazi Mathematics

اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Rational number:-

A number which can be written in the form $\frac{p}{q}$ where $p, q \in Z \wedge q \neq 0$, is called rational number. e.g., $\sqrt{16}, 4, 3, \frac{3}{5}$

Irrational number:-

A number which cannot be written in the form $\frac{p}{q}$ where $p, q \in Z \wedge q \neq 0$ is called irrational number. e.g., $\sqrt{2}, \sqrt{3}, \frac{2}{\sqrt{5}}$, 1.141142..

Decimal Representation of Rational and Irrational Numbers

1) Terminating decimals:-

A decimal that contains finite number of digits in its decimal part is called Terminating decimal.

e.g., 2.02.04, 0.000415

* Every terminating decimal can be converted to a common fraction.
* Every terminating decimal represents a rational number.

2) Non-Terminating decimals:-

A decimal having infinite number of digits in its decimal part is called non-terminating decimal.

e.g., 0.428571...., 0.33333....

$\sqrt{2} = 1.414213....$

There are TWO types of non-terminating decimals:

i) Recurring decimals

ii) Non-recurring decimals

i) Recurring decimals:-

A recurring or periodic or cyclic decimal is a decimal in which one or more digits repeat indefinitely.

e.g., $2.\overline{3} = 2.3333....$ (rational no.)

$\frac{1}{3} = 0.33333....$ (rational no.)

$\frac{3}{7} = 0.4285714285714....$ (rational no.)

* Every recurring decimal can be converted to a common fraction.
* Every recurring decimal represents a rational number.

ii) Non-recurring decimals:-

A decimal which neither terminates nor it is recurring is called non-recurring decimal. e.g.,

$\sqrt{2} = 1.414213562...$, $\sqrt{7} = 2.645751...$

* Every non-recurring decimal can not be converted to a common fraction.
* Every non-terminating and non-recurring decimal represents an irrational number.

Example 1.

i) $0.25 = \frac{25}{100}$ rational no.

ii) $0.333.... = \frac{1}{3}$ recurring decimal (rational no.)

iii) $2.\overline{3} = 2.333....$ rational no.

iv) $0.142857142857... = \frac{1}{7}$ rational no.

v) 0.01001000100001.... non-terminating, non-periodic so irrational no.

vi) 214.121122111222.... irrational no.

vii) 1.4142135... is an irrational no.

viii) 7.3205080... irrational no.

ix) 1.709975947... irrational no.

x) 3.141592654.... important irrational number called π (Pi) and

$$\pi = \frac{\text{circumference of any circle}}{\text{length of its diameter}}$$

Example 2. Prove that $\sqrt{2}$ is an irrational number.

Solution:-

Suppose $\sqrt{2}$ is a rational number.

then $\sqrt{2} = \frac{p}{q}$ where $p, q \in Z \wedge q \neq 0$

If $HCF(p,q) \neq 1$, then by dividing p and q by $HCF(p,q)$, $\sqrt{2}$ can be reduced as

$\sqrt{2} = \frac{a}{b}$ where $HCF(a,b) = 1 \longrightarrow (i)$

$\rightarrow \sqrt{2}\, b = a$

$\rightarrow 2b^2 = a^2$

$\rightarrow a^2$ is divisible by 2

$\rightarrow a$ is divisible by 2 $\longrightarrow (ii)$

$\rightarrow a = 2c$, where c is an integer

$\therefore \sqrt{2}\, b = 2c$

$2b^2 = 4c^2$

$\rightarrow b^2 = 2c^2$

$\rightarrow b^2$ is divisible by 2

$\rightarrow b$ is divisible by $2 \longrightarrow (iii)$

From (ii) and (iii), 2 is a common factor of a and b, which contrdicts (i)

So $\sqrt{2}$ is an irrational number.

Example 3. Prove that $\sqrt{3}$ is an irrational number.

Solution:-

Suppose $\sqrt{3}$ is a rational number.

then $\sqrt{3} = \frac{p}{q}$ where $p, q \in Z \wedge q \neq 0$

If $HCF(p,q) \neq 1$, then by dividing p and q by $HCF(p,q)$, $\sqrt{3}$ can be reduced as

$\sqrt{3} = \frac{a}{b}$ where $HCF(a,b) = 1 \longrightarrow (i)$

$\rightarrow \sqrt{3}\, b = a$

$\rightarrow 3b^2 = a^2$

$\rightarrow a^2$ is divisible by 3

$\rightarrow a$ is divisible by $3 \longrightarrow (ii)$

$\rightarrow a = 3c$, where c is an integer.

$\therefore \sqrt{3}\, b = 3c$

$\rightarrow 3b^2 = 9c^2$

$\rightarrow b^2 = 3c^2$

$\rightarrow b^2$ is divisible by 3

$\rightarrow b$ is divisible by $3 \longrightarrow (iii)$

from (ii) and (iii), 3 is a common factor of a and b, which contradicts (i)

So $\sqrt{3}$ is an irrational number.

Note:- Using the same method we can prove that the irrationality of $\sqrt{5}, \sqrt{7}, \ldots, \sqrt{n}$ where $n \in P$.

Properties of Real Numbers

Binary Operation:- A binary operation in a set A is a rule usually denoted by $*$ that assigns to any pair of elements of A, taken in a definite order, another element of A.

* Two binary operations addition (+) and multiplication (. or ×) in a set of real numbers ($\mathbb{R}$) are important.

1. Addition Laws

i) Closure Law

$\forall a, b \in \mathbb{R},\ a+b \in \mathbb{R}$

ii) Associative Law

$\forall a, b, c \in \mathbb{R},\ a+(b+c) = (a+b)+c$

iii) Additive Identity

$\forall a \in \mathbb{R}, \exists\ 0 \in \mathbb{R}$ such that

$a+0 = 0+a = a$ 0(zero) is called identity element of addition.

iv) Additive Inverse

$\forall a \in \mathbb{R}, \exists (-a) \in \mathbb{R}$ such that

$a+(-a) = 0 = (-a)+a$

v) Commutative Law

$\forall a, b \in \mathbb{R},\ a+b = b+a$

2. Multiplication Laws

vi) Closure Law

$\forall a,b \in \mathbb{R}$, $a.b \in \mathbb{R}$

($a.b$ is usually written as ab)

vii) Associative Law

$\forall a,b,c \in \mathbb{R}$, $a(bc) = (ab)c$

viii) Multiplicative Identity

$\forall a \in \mathbb{R}$ $\exists$ $1 \in \mathbb{R}$ such that

$a.1 = 1.a = a$ 1 is called multiplicative identity.

ix) Multiplicative Inverse

$\forall$ $a(\neq 0) \in \mathbb{R}$, $\exists$ $a^{-1} \in \mathbb{R}$ such that

$a.a^{-1} = a^{-1}.a = 1$ (a^{-1} is also written as $\frac{1}{a}$)

x) Commutative Law

$\forall a,b \in \mathbb{R}$, $ab = ba$

3. Multiplication - Addition Law

xi) $\forall a,b,c \in \mathbb{R}$

$a(b+c) = ab+ac$ Left Distributive

$(a+b)c = ac+bc$ Right Distributive

4. Properties of Equality

i) Reflexive property:-

$\forall a \in \mathbb{R}$, $a = a$

ii) Symmetric property:-

$\forall a,b \in \mathbb{R}$, $a = b \rightarrow b = a$

iii) Transitive property:-

$\forall a,b,c \in \mathbb{R}$, $a = b \wedge b = c \rightarrow a = c$

iv) Additive property:-

$\forall a,b,c \in \mathbb{R}$, $a = b \rightarrow a+c = b+c$

v) Multiplicative property:-

$\forall a,b,c \in \mathbb{R}$, $a = b \rightarrow ac = bc$
$\wedge\ ca = cb$

vi) Cancellation property w.r.t addition:-

$\forall a,b,c \in \mathbb{R}$, $a+c = b+c \rightarrow a = b$

vii) Cancellation property w.r.t Multiplication:-

$\forall a,b,c \in \mathbb{R}$, $ac = bc \rightarrow a = b$, $c \neq 0$

5. Properties of Inequalities

1) Trichotomy Property:-

$\forall a,b \in \mathbb{R}$ either $a = b$ or $a > b$ or $a < b$

2) Transitive property:-

$\forall a,b,c \in \mathbb{R}$

i) $a > b \wedge b > c \rightarrow a > c$

ii) $a < b \wedge b < c \rightarrow a < c$

3) Additive property:-

$\forall a,b,c,d$ $\mathbb{R}$

a) i) $a > b \rightarrow a+c > b+c$

ii) $a < b \rightarrow a+c < b+c$

b) i) $a > b \wedge c > d \rightarrow a+c > b+d$

ii) $a < b \wedge c < d \rightarrow a+c < b+d$

4) Multiplicative properties:-

a) $\forall a,b,c \in \mathbb{R}$ and $c > 0$

i) $a > b \rightarrow ac > bc$

ii) $a < b \rightarrow ac < bc$

b) $\forall a,b,c \in \mathbb{R}$ and $c < 0$

i) $a > b \rightarrow ac < bc$

ii) $a < b \rightarrow ac > bc$

c) $\forall a,b,c,d \in \mathbb{R}$ and a,b,c,d are all positive

i) $a > b \wedge c > d \rightarrow ac > bd$

ii) $a < b \wedge c < d \rightarrow ac < bd$

Note:-

* a and $(-a)$ are additive inverse of each other.
  ∴ by def. inverse of $-a$ is a
  $\therefore -(-a) = a$
* a and $\frac{1}{a}$ are 'x' inverses of each other.
  ∵ inverse of $\frac{1}{a}$ is a (i.e., inverse of a^{-1} is a)
  $\therefore (a^{-1})^{-1} = a$ or $\frac{1}{\frac{1}{a}} = a$

Example 4. Prove that for any real numbers a, b

i) $a.0 = 0$ ii) $ab = 0 \Rightarrow a = 0 \vee b = 0$

Solution:- i) $a.0 = 0$

L.H.S $= a.0$

$= a[1 + (-1)]$ (additive inverse property)

$= a(1 - 1)$ (Def. of subtraction)

$= a(1) - a(1)$ (Left Dist. Law)

$= a - a$ (Mult. identity prop.)

$= a + (-a)$ (Def. of subtraction)

$= 0$ (Additive inverse prop.)

$=$ R.H.S

ii) Given that

$ab = 0$ suppose $a \neq 0$ then $\frac{1}{a}$ exists

so $\frac{1}{a}(ab) = \frac{1}{a}(0)$ ('x' property)

$\Rightarrow (\frac{1}{a}.a)b = \frac{1}{a}.0$ (Assoc. Law of x)

$\Rightarrow 1.b = 0$ ('x' inverse prop.)

$\Rightarrow b = 0$ (x identity prop.)

Similarly

Given that

$ab = 0$ suppose $b \neq 0$ then $\frac{1}{b}$ exists

so $(ab)\frac{1}{b} = (0)\frac{1}{b}$ (x prop.)

$\Rightarrow a(b.\frac{1}{b}) = 0.\frac{1}{b}$ (Assoc. Law)

$a.1 = 0$ (x inverse prop.)

$\Rightarrow a = 0$ (x identity prop.)

Hence $ab = 0 \Rightarrow a = 0 \vee b = 0$

Example 5. For real numbers a, b show the following by stating the properties used.

i) $(-a)b = a(-b) = -ab$

ii) $(-a)(-b) = ab$

Solution:-

i) Consider

$(-a)b + ab = (-a + a)b$ (Dist. Law)

$= 0.b = 0$ (additive inverse property)

$\therefore (-a)b + ab = 0$

i.e., $(-a)b$ and ab are additive inverse of each other. so $(-a)b = -ab \rightarrow (i)$

Now consider

$a(-b) + ab = a(-b + b)$ (Dist. Law)

$= a(0)$ (Additive inverse Prop.)

$= 0$

$\therefore a(-b) + ab = 0$

i.e., $a(-b)$ and ab are additive inverse of each other so $a(-b) = -ab \rightarrow (ii)$

Hence from (i) and (ii)

$(-a)b = a(-b) = -ab$

ii) consider

$(-a)(-b) - ab = (-a)(-b) + (-ab)$

$(-a)(-b) - ab = (-a)(-b) + (-a)(b)$

$= (-a)(-b + b)$ (Dist. Law)

$= (-a)(0)$ (Additive inverse Prop)

$(-a)(-b) - ab = 0$

i.e., $(-a)(-b)$ and $-ab$ are additive inverse of each other.

Hence $(-a)(-b) = ab$

Example 6. Prove that

i) $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$

(principle of equality of fractions)

Solution:-

Suppose $\frac{a}{b} = \frac{c}{d}$

$\Rightarrow \frac{a}{b}\left(\frac{d}{d}\right) = \frac{c}{d}\left(\frac{b}{b}\right)$ (Golden rule of fraction)

$\Rightarrow \frac{ad}{bd} = \frac{bc}{bd}$ ('x' prop.)

$ad \cdot \frac{1}{\cancel{bd}} = bc \cdot \frac{1}{\cancel{bd}}$ $\left(\because \frac{a}{b} = a \cdot \frac{1}{b}\right)$

$\Rightarrow ad = bc$ (By Cancellation Law)

Conversely,

$ad = bc$

$\Rightarrow ad \times \frac{1}{bd} = bc \times \frac{1}{bd}$ ('x' prop.)

$\Rightarrow \frac{a}{b}\left(d \times \frac{1}{d}\right) = \frac{c}{d}\left(b \times \frac{1}{b}\right)$

$\Rightarrow \frac{a}{b}(1) = \frac{c}{d}(1)$ ('x' inverse prop.)

$\Rightarrow \frac{a}{b} = \frac{c}{d}$ ('x' identity prop.)

Hence proved

$\Rightarrow \frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$

ii) $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$

Solution:-

$L.H.S = \frac{1}{a} \cdot \frac{1}{b}$

$\because \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

$\Rightarrow 6 \times \frac{1}{2} \cdot \frac{1}{3} = 1$

$\Rightarrow 6 \times \frac{1}{2} \cdot \frac{1}{3} = 1 \times \frac{1}{1}$

Consider that

$ab\left(\frac{1}{a} \cdot \frac{1}{b}\right)$

$= ba\left(\frac{1}{a} \cdot \frac{1}{b}\right)$ (closure prop.)

$= b\left(a \times \frac{1}{a}\right)\frac{1}{b}$ (Assoc. Law)

$= \cancel{b} \times 1 \cdot \frac{1}{\cancel{b}}$

$= 1$

so ab & $\frac{1}{a} \cdot \frac{1}{b}$ are multiplicative inverse of each other.

Hence $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$

iii) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

(Rule for product of fractions)

Solution:-

$L.H.S = \frac{a}{b} \cdot \frac{c}{d}$

$= a \times \frac{1}{b} \cdot c \times \frac{1}{d}$

$= a\left(\frac{1}{b} \times c\right)\frac{1}{d}$ (Assoc. Law)

$= a\left(c \times \frac{1}{b}\right)\frac{1}{d}$ (commutative Law)

$= ac \times \frac{1}{bd}$

$= \frac{ac}{bd} = R.H.S$ Hence proved.

iv) $\frac{a}{b} = \frac{ka}{kb}$, $(k \neq 0)$

(Golden rule of fractions)

Solution:-

$L.H.S = \frac{a}{b}$

$= \frac{a}{b} \times 1$ ('x' identity)

$= \frac{a}{b}\left(k \times \frac{1}{k}\right)$ ('x' inverse prop)

$= \frac{ka}{kb} = R.H.S$

Hence proved.

v) $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$

(Rule for quotient of fractions)

Solution:-

$L.H.S = \dfrac{\frac{a}{b}}{\frac{c}{d}}$

$$= \frac{\frac{a}{b}\times\left(\frac{d}{d}\right)}{\frac{c}{d}\left(\frac{b}{b}\right)} \quad \text{(Golden rule of fraction)}$$

$$= \frac{\frac{ad}{bd}}{\frac{bc}{bd}} \quad \text{('x' prop.)}$$

$$= \frac{ad\cdot\cancel{\frac{1}{bd}}}{bc\cdot\cancel{\frac{1}{bd}}} \quad \left(\because \frac{a}{b} = a\cdot\frac{1}{b}\right)$$

$$= \frac{ad}{bc} = R.H.S$$

Hence proved.

Example 7. Does the set $\{1,-1\}$ possess closure property with respect to (i) addition . (ii) Multiplication?

Solution:- (i) addition

+	1	-1
1	2	0
-1	0	-2

It is clear from the table given set {1,-1} have no closure property w.r.t "+"

(ii) multiplication

x	1	-1
1	1	-1
-1	-1	1

It is clear from the table given set {1,-1} have closure property w.r.t "x"

Exercise 1.1

Q1. Which of the following sets have closure property w.r.t addition and multiplication?

i) $\{0\}$ ii) $\{1\}$ iii) $\{0,-1\}$ iv) $\{1,-1\}$

Solution:- i) $\{0\}$

+	0
0	0

It is clear from the table given set {0} have a closure proprty w.r.t "+".

x	0
0	0

It is clear from the table given set {0} have a closure proprty w.r.t "x".

ii) $\{1\}$

+	1
1	2

It is clear from the table given set {1} is have no closure property w.r.t "+"

x	1
1	1

It is clear from the table given set {1} is have a closure property w.r.t "x"

iii) $\{0,-1\}$

+	0	-1
0	0	-1
-1	-1	-2

It is clear from the table given set {0,-1} have no closure property w.r.t "+"

x	0	-1
0	0	0
-1	0	1

It is clear from the table given set {0,-1} have no closure property w.r.t "x"

iv) $\{1,-1\}$

+	1	-1
1	2	0
-1	0	-2

It is clear from the table given set {1,-1} have no closure property w.r.t "+"

x	1	-1
1	1	-1
-1	-1	1

It is clear from the table given set {1,-1} have closure property w.r.t "x"

Q2. Name the properties used in the following equations. (Letters, where used, represent real numbers).

Solution:-

i) $4+9 = 9+4$

Commutative property w.r.t '+'

ii) $(a+1)+\frac{3}{4} = a+\left(1+\frac{3}{4}\right)$

Associative property w.r.t +'

iii) $(\sqrt{3}+\sqrt{5})+\sqrt{7}=\sqrt{3}+(\sqrt{5}+\sqrt{7})$

Associative property w.r.t '+'

iv) $100+0=100$

Additive identity property

v) $1000\times 1=1000$

Multiplicative identity property

vi) $4.1+(-4.1)=0$

Additive inverse property

vii) $a-a=0$

Additive inverse property

viii) $\sqrt{2}\times\sqrt{5}=\sqrt{5}\times\sqrt{2}$

commutative property w.r.t 'x'

ix) $a.(b-c)=ab-ac$

Left Distributive property

x) $(x-y)z=xz-yz$

Right Distributive property

xi) $4\times(5\times 8)=(4\times 5)\times 8$

Associative property w.r.t 'x'

xii) $a(b+c-d)=ab+ac-ad$

Left Distributive property

Q3. Name the properties used in the following inequalities:

Solution:-

i) $-3<-2 \rightarrow 0<1$

(Add 3 both sides)

Additive property

ii) $-5<-4 \rightarrow 20>16$

(Multiplying by -4)

Multiplicative property

iii) $1>-1 \rightarrow -3>-5$

(Add -4 both sides)

Additive property

iv) $a<0 \rightarrow -a>0$

(Multiply by -1)

Multiplicative property

v) $a>b \rightarrow \frac{1}{a}<\frac{1}{b}$

('x' by $\frac{1}{ab}$)

Multiplicative property or inverse

vi) $a>b \rightarrow -a<-b$

(Multiply by -1)

Multiplicative property

Q4. Prove the following rules of addition:-

i) $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$

Solution:-

$L.H.S=\frac{a}{c}+\frac{b}{c}$

$=a.\frac{1}{c}+b.\frac{1}{c}$ $\quad(\because \frac{a}{b}=a.\frac{1}{b})$

$=(a+b)\frac{1}{c}$ $\quad$ (Dist. Law)

$=\frac{a+b}{c}=R.H.S$ $\quad$ ('x' Prop.)

Hence proved

ii) $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$

Solution:-

$L.H.S=\frac{a}{b}+\frac{c}{d}$

$=\frac{a}{b}\left(\frac{d}{d}\right)+\frac{c}{d}\left(\frac{b}{b}\right)$ $\quad$ (Golden Rule of fraction)

$=\frac{ad}{bd}+\frac{bc}{bd}$ $\quad$ ('x' prop)

$=ad.\frac{1}{bd}+bc.\frac{1}{bd}$ $\quad(\because \frac{a}{b}=a.\frac{1}{b})$

$=(ad+bc)\frac{1}{bd}$ $\quad$ (Dist. Law)

$=\frac{ad+bc}{bd}$ $\quad$ ('x' prop.)

$=R.H.S$

Hence proved

Q5. Prove that $\frac{-7}{12}-\frac{5}{18}=\frac{-21-10}{36}$

Solution:-

$L.H.S = \frac{-7}{12}-\frac{5}{18}$

$= -\frac{7}{12}\left(\frac{3}{3}\right)-\frac{5}{18}\left(\frac{2}{2}\right)$ (Golden Rule)

$= -\frac{21}{36}-\frac{10}{36}$ ('x' prop.)

$= -21\cdot\frac{1}{36}-10\cdot\frac{1}{36}$ $\left(\because \frac{a}{b}=a\cdot\frac{1}{b}\right)$

$= (-21-10)\frac{1}{36}$ (Dist. Law)

$= \frac{-21-10}{36} = R.H.S$

Hence proved.

Q6. Simplify by justify each step.

i) $\frac{4+16x}{4}$

Solution:-

$\frac{4+16x}{4}$

$= \frac{1}{4}(4+16x)$ $\left(\because \frac{a}{b}=\frac{1}{b}\cdot a\right)$

$= \frac{1}{4}[4\cdot1+4\cdot4x]$ ('x' prop.)

$= \frac{1}{4}[4(1+4x)]$ (Dist. Law)

$= \left(\frac{1}{4}\times4\right)(1+4x)$ (Associative Law)

$= 1+4x$

ii) $\dfrac{\frac{1}{4}+\frac{1}{5}}{\frac{1}{4}-\frac{1}{5}}$

Solution:-

$\dfrac{\frac{1}{4}+\frac{1}{5}}{\frac{1}{4}-\frac{1}{5}}$

$= \dfrac{\frac{1}{4}\left(\frac{5}{5}\right)+\frac{1}{5}\left(\frac{4}{4}\right)}{\frac{1}{4}\left(\frac{5}{5}\right)-\frac{1}{5}\left(\frac{4}{4}\right)}$ (Golden Rule)

$= \dfrac{\frac{5}{20}+\frac{4}{20}}{\frac{5}{20}-\frac{4}{20}}$ ('x' prop)

$= \dfrac{5\cdot\frac{1}{20}+4\cdot\frac{1}{20}}{5\cdot\frac{1}{20}-4\cdot\frac{1}{20}}$ $\left(\because \frac{a}{b}=a\cdot\frac{1}{b}\right)$

$= \dfrac{(5+4)\frac{1}{20}}{(5-4)\frac{1}{20}}$ (Dist. Law)

$= \dfrac{5+4}{5-4}$

$= \frac{9}{1}$

$= 9$

iii) $\dfrac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}$

Solution:-

$\dfrac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}$

$= \dfrac{\frac{a}{b}\left(\frac{d}{d}\right)+\frac{c}{d}\left(\frac{b}{b}\right)}{\frac{a}{b}\left(\frac{d}{d}\right)-\frac{c}{d}\left(\frac{b}{b}\right)}$ (Golden Rule)

$= \dfrac{\frac{ad}{bd}+\frac{bc}{bd}}{\frac{ad}{bd}-\frac{bc}{bd}}$ ('x' prop.)

$= \dfrac{ad\cdot\frac{1}{bd}+bc\cdot\frac{1}{bd}}{ad\cdot\frac{1}{bd}-bc\left(\frac{1}{bd}\right)}$ $\left(\because \frac{a}{b}=a\cdot\frac{1}{b}\right)$

$= \dfrac{(ad+bc)\frac{1}{bd}}{(ad-bc)\frac{1}{bd}}$ (Distributive Law)

$= \dfrac{ad+bc}{ad-bc}$

iv) $\dfrac{\frac{1}{a}-\frac{1}{b}}{1-\frac{1}{a}\cdot\frac{1}{b}}$

Solution:-

$\dfrac{\frac{1}{a}-\frac{1}{b}}{1-\frac{1}{a}\cdot\frac{1}{b}}$

$$= \frac{\frac{1}{a} - \frac{1}{b}}{1 - \frac{1}{ab}} \quad (\text{'x' prop.})$$

$$= \frac{\frac{1}{a}\left(\frac{b}{b}\right) - \frac{1}{b}\left(\frac{a}{a}\right)}{1\left(\frac{ab}{ab}\right) - \frac{1}{ab}\left(\frac{1}{1}\right)} \quad (\text{Golden Rule})$$

$$= \frac{\frac{b}{ab} - \frac{a}{ab}}{\frac{ab}{ab} - \frac{1}{ab}} \quad (\text{'x' prop})$$

$$= \frac{b\cdot\frac{1}{ab} - a\cdot\frac{1}{ab}}{ab\cdot\frac{1}{ab} - 1\cdot\frac{1}{ab}} \quad \left(\because \frac{a}{b} = a\cdot\frac{1}{b}\right)$$

$$= \frac{(b-a)\frac{1}{ab}}{(ab-1)\frac{1}{ab}} \quad (\text{Dist. Law})$$

$$= \frac{b-a}{ab-1}$$

Complex Numbers

The numbers of the form $x+iy$, where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$, are called complex numbers.

Here x is called real part and y is called imaginary part of the complex number. e.g., $3+4i$, $2-\frac{5}{7}i$.

★ Every real number is a complex number with 0 as its imaginary part.

Consider the equation

$$x^2 + 1 = 0$$
$$\Rightarrow x^2 = -1$$
$$\Rightarrow x = \pm\sqrt{-1}$$

$\sqrt{-1} \notin \mathbb{R}$, for convenience call it imaginary number and denot it by i (read as iota).

Powers of i:-

$i^2 = -1$ (by def)

$i^3 = i^2 . i = (-1)i = -i$

$i^4 = i^2 \times i^2 = (-1)\times(-1) = 1$

$i^{13} = i^{12} . i = (i^2)^6 . i = (-1)^6 . i = i$

$i^6 = (i^2)^3 = (-1)^3 = -1$

Thus any power of i must be equal to $i, -i, 1$ or -1.

Operations on Complex Numbers

1) $a+bi = c+di \Rightarrow a=c \wedge b=d$

2) Addition:
$(a+bi)+(c+di) = (a+c)+(b+d)i$

3) $k(a+bi) = ka + kbi$

4) $(a+bi)-(c+di) = (a-c)+(b-d)i$

5) $(a+bi).(c+di) = ac+adi+bci+bdi^2$
$= (ac-bd) + (ad+bc)i$

6) **Conjugate Complex Numbers:**
For $z = a+ib$ then its conjugate is denoted by $\bar{z}$ and defined as
$\bar{z} = \overline{a+ib} = a-ib$

★ A real number is self-conjugate

Complex Numbers as Ordered Pairs of Real Numbers

i) $(a,b) = (c,d) \Leftrightarrow a=c \wedge b=d$

ii) $(a,b)+(c,d) = (a+c, b+d)$

iii) If k is any real number, then
$k(a,b) = (ka, kb)$

iv) $(a,b)(c,d) = (ac-bd, ad+bc)$

v) $(a,b)-(c,d) = (a-c, b-d)$

Example 1. Find the sum, difference and product of the complex numbers $(8,9)$ and $(5,-6)$.

Solution:- Sum $= (8+5, 9-6) = (13, 3)$

Difference $= (8-5, 9-(-6))$
$= (3, 15)$

Product $= ((8)(5)-(9)(-6), (9)(5)+(-6)(8))$
$= (40+54, 45-48)$
$= (94, -3)$

Properties of the Fundamental Operations on Complex Numbers

i) The additive identity in C is $(0,0)$

ii) Every complex number (a,b) has the additive inverse $(-a,-b)$
i.e., $(a,b)+(-a,-b)=(0,0)$

iii) The multiplicative identity is $(1,0)$
i.e., $(a,b)(1,0) = (a.1-b.0, b.1+a.0)$
$= (a,b) = (1,0)(a,b)$

iv) Every non zero complex number i.e., number not equal to $(0,0)$ has a multiplicative inverse.

Q. Prove that the multiplicative inverse of (a,b) is $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

Proof:-

Let $z=(a,b)$ or $z=a+ib$

$$z^{-1} = \frac{1}{a+ib}$$

$$= \frac{1}{a+ib} \times \frac{a-ib}{a-ib}$$

$$z^{-1} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a-ib}{a^2-i^2b^2}$$

$$\Rightarrow z^{-1} = \frac{a-ib}{a^2-(-1)b^2} = \frac{a-ib}{a^2+b^2}$$

$$\Rightarrow z^{-1} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

Hence $z^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$

v) $(a,b)[(c,d)\pm(e,f)]$
$= (a,b)(c,d) \pm (a,b)(e,f)$

A Special Subset of C

For all $(a,0), (b,0) \in C$

$(a,0)+(b,0) = (a+b, 0)$

$(a,0)\cdot(b,0) = (ac, 0)$

$k(a,0) = (ka, 0)$

Multiplicative inverse of $(a,0)$ is $\left(\frac{1}{a}, 0\right)$, provided $a \neq 0$

Exercise 1.2

Q1. Verify the addition properties of complex numbers

Solution:-

i. Closure

Let $a+ib, c+id \in C$ then
$(a+ib)+(c+id) = (a+c)+i(b+d) \in C$

ii. Associative

Let $a+ib, c+id, e+if \in C$

$[(a+ib)+(c+id)]+(e+if)$
$= [(a+c)+i(b+d)]+(e+if)$
$= (a+c+e)+i(b+d+f)$
$= (a+ib)+[(c+e)+i(d+f)]$
$= (a+ib)+[(c+id)+(e+if)]$

iii. Additive Identity

$0+i0, (a+ib) \in C$ then
$(a+ib)+(0+i0) = (a+0)+i(b+0)$
$= a+ib \in C$

Also $(0+i0)+(a+ib) = (0+a)+i(0+b)$
$= a+ib \in C$

iv. Additive Inverse

$(a+ib), (-a-ib) \in C$ then
$(a+ib)+(-a-ib) = (a-a)+i(b-b)$
$= 0+i0 \in C$

Also $(-a-ib)+(a+ib) = (-a+a)+i(-b+b)$
$= 0+i0 \in C$

v. Commutative

$(a+ib), (c+id) \in C$ then

$(a+ib)+(c+id)$
$= (a+c)+i(b+d)$
$= (c+a)+i(d+b) = (c+id)+(a+ib)$

Q2. Verify the multiplication properties of the complex numbers.

Solution:-

i. Close w.r.t 'x'

$(a+ib),(c+id) \in C$ then

$(a+ib)\cdot(c+id) = ac+iad+ibc+i^2bd$
$= ac+i(ad+bc)-bd$
$= (ac-bd)+i(ad+bc) \in C$

ii. Associative w.r.t 'x'

$(a+ib),(c+id),(e+if) \in C$

$[(a+ib)(c+id)](e+if)$
$= [(ac-bd)+i(bc+ad)](e+if)$
$= [e(ac-bd)-f(bc+ad)]$
$+i[f(ac-bd)+e(bc+ad)]$
$= [ace-ebd-fbc-fad]$
$+i[fac-fbd+ebc+aed]$
$= [a(ec-df)-b(cf+de)]$
$+i[a(cf+de)+b(ec-df)]$
$= (a+ib)[(ec-df)+i(cf+de)]$
$= (a+ib)[(c+id)(e+if)]$

iii. Identity

$(a+ib),(1+i0) \in C$ then
$(a+ib)(1+i0) = a+0+ib+0$
$= a+ib \in C$

iv. Inverse

$(a+ib),\left(\frac{a}{a^2+b^2}-\frac{ib}{a^2+b^2}\right) \in C$ then

$(a+ib)\left(\frac{a}{a^2+b^2}-\frac{ib}{a^2+b^2}\right)$
$= (a+ib)\left(\frac{a-ib}{a^2+b^2}\right)$

$= \frac{a^2-(ib)^2}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1 = 1+0i$

v. Commutative

$(a+ib),(c+id) \in C$
$(a+ib)(c+id) = (ac-bd)+i(ad+bc)$
$= (ca-db)+i(da+cb)$
$= (c+id)(a+ib)$

Q3. Verify the distributive law of complex numbers

$(a,b)[(c,d)+(e,f)] = (a,b)(c,d)+(a,b)(e,f)$

Solution:-

$L.H.S = (a,b)[(c,d)+(e,f)]$
$= (a,b)(c+e,\ d+f)$
$= (a(c+e)-b(d+f),\ a(d+f)+b(c+e))$
$= (ac+ae-bd-bf,\ ad+af+bc+be)$

$R.H.S = (a,b)(c,d)+(a,b)(e,f)$
$= (ac-bd,\ ad+bc)+(ae-bf,\ af+be)$
$= (ac+ae-bd-bf,\ ad+af+bc+be)$

Hence proved

Q4. Simplify the following

i) i^9 ii) i^{14} iii) $(-i)^{19}$ iv) $(-1)^{-\frac{21}{2}}$

Solution:-

i) $i^9 = i^8\cdot i = (i^2)^4\cdot i = (-1)^4\cdot i = 1\cdot i = i$

ii) $i^{14} = (i^2)^7 = (-1)^7 = -1$

iii) $(-i)^{19} = (-1)^{19} i^{19} = (-1)i^{18}\cdot i = (-1)(i^2)^9\cdot i$
$= (-1)(-1)^9 i = (-1)(-1)i = i$

iv) $(-1)^{-\frac{21}{2}} = \left[(-1)^{\frac{1}{2}}\right]^{-21} = \left[\sqrt{-1}\right]^{-21}$
$= (i)^{-21} = i\cdot i^{-22} = i(i^2)^{-11}$
$= i(-1)^{-11} = -i$

Q5. Write in terms of i

i) $\sqrt{-1}\,b$ ii) $\sqrt{-5}$ iii) $\sqrt{\frac{-16}{25}}$

iv) $\sqrt{\frac{1}{-4}}$

Solution:-

i) $\sqrt{-1}\,b = i\,b = bi \quad \because i = \sqrt{-1}$

ii) $\sqrt{-5} = \sqrt{-1}\sqrt{5} = i\sqrt{5} = \sqrt{5}\,i \quad \because i = \sqrt{-1}$

iii) $\sqrt{\frac{-16}{25}} = \sqrt{-1}\sqrt{\frac{16}{25}} = i\frac{4}{5} = \frac{4}{5}i$

iv) $\sqrt{\frac{1}{-4}} = \sqrt{-1}\sqrt{\frac{1}{4}} = i\frac{1}{2} = \frac{1}{2}i$

Simplify the following:

Q6. $(7,9)+(3,-5)$

Solution:-

$(7,9)+(3,-5) = (7+3, 9-5) = (10,4)$

Q7. $(8,-5)-(-7,4)$

Solution:-

$(8,-5)-(-7,4) = (8+7, -5-4) = (15,-9)$

Q8. $(2,6)(3,7)$

Solution:- $(2,6)(3,7)$

$= (2.3-6.7,\ 2.7+6.3)$

$= (6-42,\ 14+18) = (-36,32)$

Q9. $(5,-4)(-3,-2)$

Solution:- $(5,-4)(-3,-2)$

$= (5(-3)-(-4)(-2),\ 5(-2)+(-4)(-3))$

$= (-15-8,\ -10+12) = (-23,2)$

Q10. $(0,3)(0,5)$

Solution:- $(0,3)(0,5)$

$= (0.3-3.5,\ 0.5+3.0) = (0-15,\ 0+0)$

$= (-15,0)$

Q11. $(2,6)\div(3,7)$

Solution:-

$(2,6)\div(3,7)$

$= \frac{(2,6)}{(3,7)} = \frac{2+6i}{3+7i}$

$= \frac{2+6i}{3+7i} \times \frac{3-7i}{3-7i}$

$= \frac{2(3)+2(-7i)+(6i)(3)+6i(-7i)}{(3)^2-(7i)^2}$

$= \frac{6-14i+18i-42i^2}{9-49i^2} \quad (\because i^2=-1)$

$= \frac{6+4i+42}{9+49} = \frac{48+4i}{58}$

$= \frac{48}{58} + \frac{4}{58}i = \frac{24}{29} + \frac{2}{29}i = \left(\frac{24}{29}, \frac{2}{29}\right)$

Q12. $(5,-4)\div(-3,-8)$

Solution:-

$(5,-4)\div(-3,-8)$

$= \frac{(5,-4)}{(-3,-8)} = \frac{5-4i}{-3-8i}$

$= \frac{5-4i}{-3-8i} \times \frac{-3+8i}{-3+8i}$

$= \frac{5(-3)+(5)(8i)+(-4i)(-3)+(-4i)(8i)}{(-3)^2-(8i)^2}$

$= \frac{-15+40i+12i-32i^2}{9-64i^2}$

$= \frac{-15+52i+32}{9+64} \quad (\because i^2=-1)$

$= \frac{17+52i}{73} = \frac{17}{73} + \frac{52}{73}i = \left(\frac{17}{73}, \frac{52}{73}\right)$

Q13. Prove that the sum as well as the product of any two conjugate complex numbers is a real number.

Solution:-

Let $z = a+ib$, $\bar{z} = \overline{a+ib}$

$\bar{z} = a-ib$

Now $z+\bar{z} = a+ib+a-ib$

$= 2a \in \mathbb{R}$

Also $z\bar{z} = (a+ib)(a-ib)$

$= (a)^2-(ib)^2$

$= a^2 - i^2b^2$

$= a^2-(-1)b^2 = a^2+b^2 \in \mathbb{R}$

Hence proved.

Q14. Find the multiplicative inverse of the following numbers:

i) $(-4,7)$ ii) $(\sqrt{2},-\sqrt{5})$ iii) $(1,0)$

Solution:- i) $(-4,7)$

$\because z=(a,b)$ so $z^{-1}=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)$

Now let $z=(-4,7)$

$\rightarrow z^{-1}=\left(\frac{-4}{(-4)^2+(7)^2},\frac{-7}{(-4)^2+(7)^2}\right)$

$=\left(\frac{-4}{16+49};\frac{-7}{16+49}\right)$

$z^{-1}=\left(\frac{-4}{65},\frac{-7}{65}\right)$

ii) $(\sqrt{2},-\sqrt{5})$

$\because z=(a,b) \rightarrow z^{-1}=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)$

Now let $z=(\sqrt{2},-\sqrt{5})$

$\rightarrow z^{-1}=\left(\frac{\sqrt{2}}{(\sqrt{2})^2+(-\sqrt{5})^2},\frac{\sqrt{5}}{(\sqrt{2})^2+(-\sqrt{5})^2}\right)$

$=\left(\frac{\sqrt{2}}{2+5},\frac{\sqrt{5}}{2+5}\right)$

$z^{-1}=\left(\frac{\sqrt{2}}{7},\frac{\sqrt{5}}{7}\right)$

iii) $(1,0)$

$\because z=(a,b) \rightarrow z^{-1}=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)$

Now let $z=(1,0)$

$\rightarrow z^{-1}=\left(\frac{1}{(1)^2+(0)^2},\frac{-0}{(1)^2+(0)^2}\right)$

$z^{-1}=\left(\frac{1}{1+0},\frac{0}{1+0}\right)=(1,0)$

Q15. Factorize the following:

i) a^2+4b^2

Solution:-

$a^2+4b^2=a^2-(-1)4b^2$

$=a^2-i^2 4b^2=(a)^2-(2bi)^2$

$=(a+2bi)(a-2bi)$

ii) $9a^2+16b^2$

Solution:-

$9a^2+16b^2=9a^2-(-1)16b^2$

$=9a^2-i^2 16b^2=(3a)^2-(4bi)^2$

$=(3a+4bi)(3a-4bi)$

iii) $3x^2+3y^2$

Solution:-

$3x^2+3y^2=3x^2-(-1)3y^2$

$=3[x^2-i^2y^2]=3[(x)^2-(yi)^2]$

$=3(x+yi)(x-yi)$

Q16. Separate into real and imaginary parts (write as a simple complex number):-

i) $\frac{2-7i}{4+5i}$

Solution:- $\frac{2-7i}{4+5i}$

$=\frac{2-7i}{4+5i}\times\frac{4-5i}{4-5i}$

$=\frac{2(4)-2(5i)-7i(4)+7i(5i)}{(4)^2-(5i)^2}$

$=\frac{8-10i-28i+35i^2}{16-25i^2}$ $(\because i^2=-1)$

$=\frac{8-38i-35}{16+25}$

$=\frac{-27-38i}{41}=-\frac{27}{41}-\frac{38}{41}i$

ii) $\frac{(-2+3i)^2}{(1+i)}$

Solution:- $\frac{(-2+3i)^2}{1+i}$

$=\frac{(-2)^2+(3i)^2+2(-2)(3i)}{1+i}$

$=\frac{4+9i^2-12i}{1+i}$

$=\frac{4-9-12i}{1+i}=\frac{-5-12i}{1+i}$

$=\frac{-5-12i}{1+i}\times\frac{1-i}{1-i}$

 www.pakcity.org

$$= \frac{-5+5i-12i+12i^2}{(1)^2-(i)^2}$$

$$= \frac{-5-7i-12}{1-i^2} = \frac{-17-7i}{1+1}$$

$$= \frac{-17-7i}{2} = \frac{-17}{2} - \frac{7}{2}i$$

iii) $\frac{i}{1+i}$

Solution:-

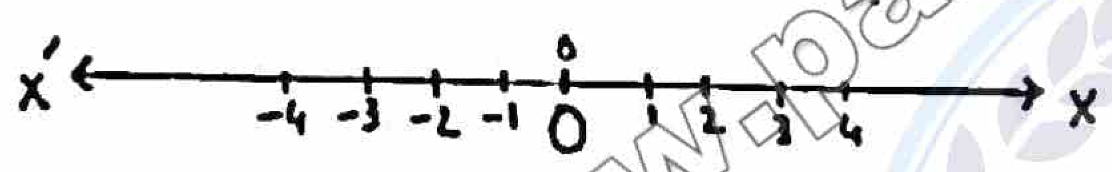
$$\frac{i}{1+i} = \frac{i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{i-i^2}{1-i^2} = \frac{i-(-1)}{1-(-1)} \quad (\because i^2=-1)$$

$$= \frac{i+1}{1+1} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i$$

The Real Line

The set of real numbers are represented by a straight line XOX' as shown.



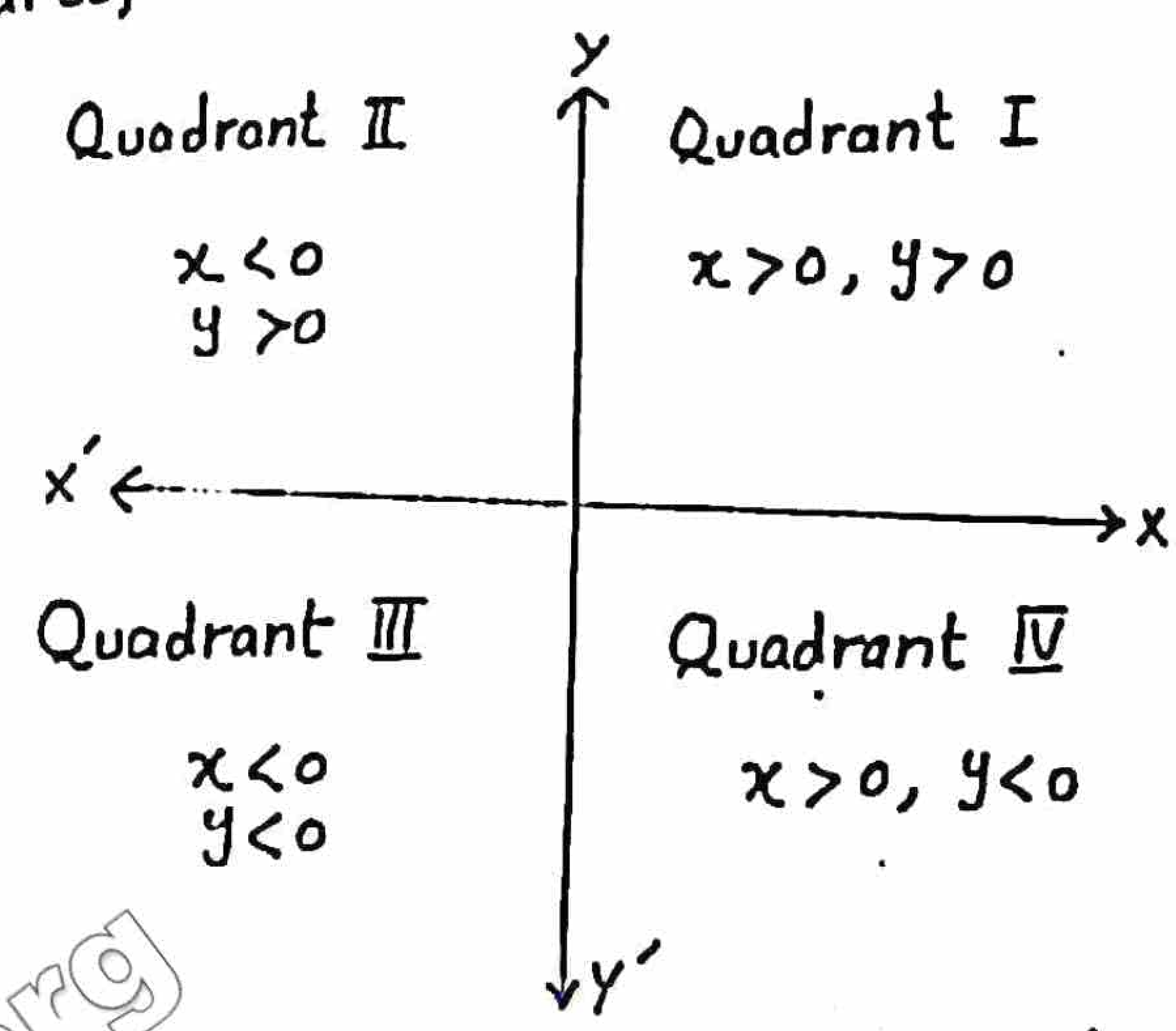
The Real Plane/coordinate plane

The plane made by two mutually ⊥ar lines is called coordinate plane. Let us draw two mutually ⊥ar lines XX' and YY' such as O be their point of intersection.

The lines XX' and YY' are together coordinate axes. The common point O is called origin or initial point.

XOX' is called X-axis, which is horizontal line and YOY' is called Y-axis, which is vertical line. Thus the plane made by both x-axis and y-axis is called xy-plane/coordinate plane or Real plane.

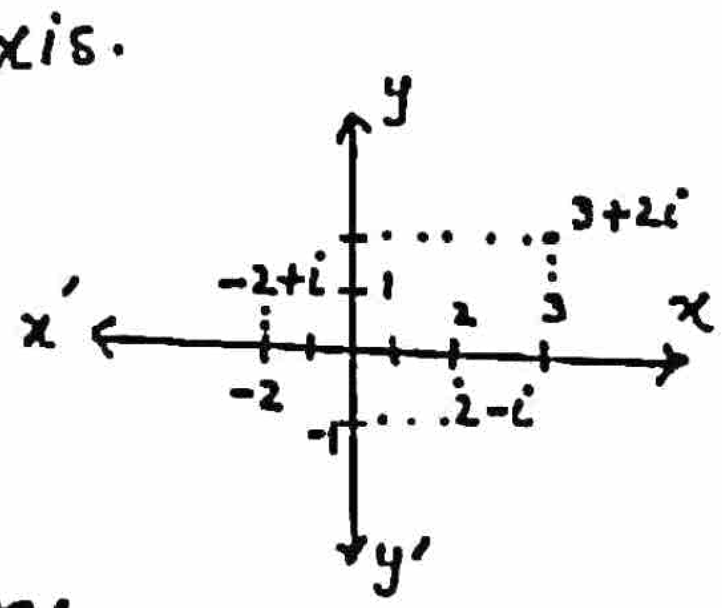
If (a,b) are coordinates of a point P then a is called x-coordinate or **abscissa** of point P and b is called y-coordinate or **ordinate** of point P. The coordinate axes divide the coordinate plane into four equal parts, called quadrants.



Geometrical Representation of Complex Numbers

Geometrically, a complex number $a+ib$ is represented by a point $P(a,b)$ on the coordinate plane. When we represent a complex number on coordinate plane, then the coordinate plane is called complex plane or z-plane.

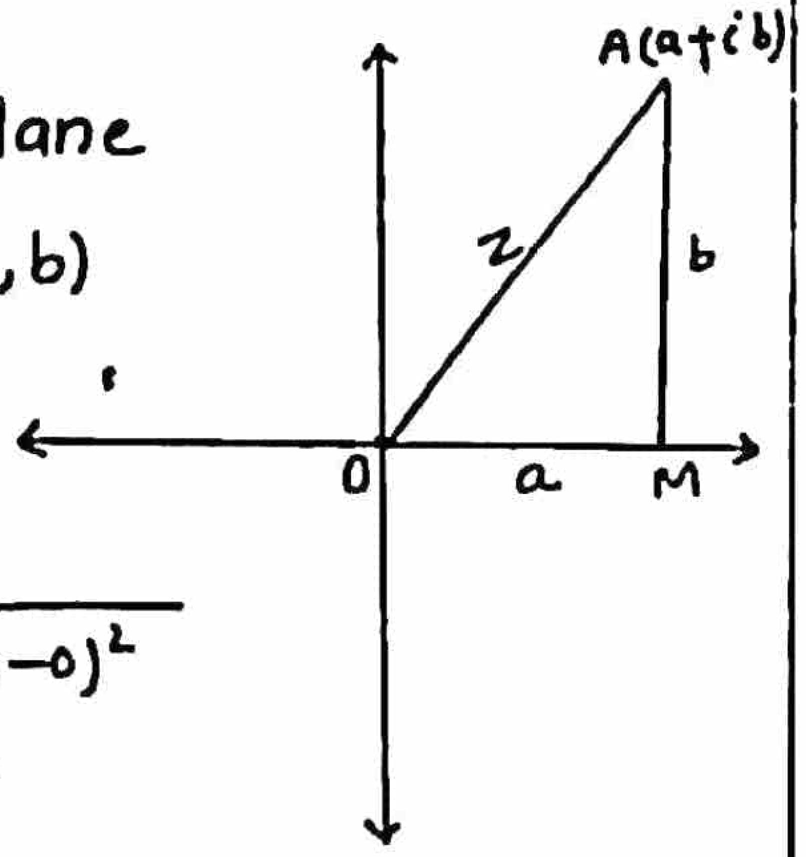
* In the representation the x-axis is called real axis and y-axis is called imaginary axis.



Argand diagram:-

Figure representing one or more complex numbers on complex plane is called Argand diagram.

Modulus of the Complex Number $a+bi$



In cartesian plane distance of $A(a,b)$ from origin $O(0,0)$

$|OP| = \sqrt{(a-0)^2+(b-0)^2}$

$= \sqrt{a^2+b^2}$

Draw ⊥ar AM from A on x-axis then $|OM| = a$, $|AM| = b$, $|OA| = z$

By Pathagoras theorem on ΔAOM

$|OA|^2 = |OM|^2 + |AM|^2$

$\rightarrow z^2 = a^2 + b^2 \rightarrow |z| = \sqrt{a^2+b^2}$

Thus the modulus of a complex number is the distance of the point (representing complex number) from the origin.

Example 1. Find moduli of the following complex numbers:

i) $1-i\sqrt{3}$ ii) 3 iii) $-5i$ iv) $3+4i$

Solution:- i) $1-i\sqrt{3}$

Let $z = 1-i\sqrt{3} \rightarrow z = (1,-\sqrt{3})$

$\rightarrow |z| = \sqrt{(1)^2+(-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4}$

$|z| = 2$

ii) 3

Let $z = 3 \rightarrow z = (3,0)$

$|z| = \sqrt{(3)^2+(0)^2} = \sqrt{9} = 3$

iii) $-5i$

Let $z = -5i \rightarrow z = (0,-5)$

$|z| = \sqrt{(0)^2+(-5)^2} = \sqrt{0+25} = 5$

iv) $3+4i$

Let $z = 3+4i \rightarrow z = (3,4)$

$\rightarrow z = \sqrt{(3)^2+(4)^2}$

$\rightarrow z = \sqrt{9+16} = \sqrt{25} = 5$

Theorems

i) $|-z| = |z| = |\bar{z}| = |-\bar{z}|$ $\forall z \in C$

Proof:-

Let $z = a+ib \rightarrow |z| = \sqrt{a^2+b^2} \longrightarrow (i)$

$\bar{z} = a-ib \rightarrow |\bar{z}| = \sqrt{a^2+(-b)^2}$

$\rightarrow |\bar{z}| = \sqrt{a^2+b^2} \longrightarrow (ii)$

$-z = -a-ib \rightarrow |-z| = \sqrt{(-a)^2+(-b)^2}$

$\rightarrow |-z| = \sqrt{a^2+b^2} \longrightarrow (iii)$

$-\bar{z} = -a+ib \rightarrow |-\bar{z}| = \sqrt{(-a)^2+b^2}$

$\rightarrow |-\bar{z}| = \sqrt{a^2+b^2} \longrightarrow (iv)$

from (i), (ii), (iii) and (iv)

$|-z| = |z| = |\bar{z}| = |-\bar{z}|$

ii) $\forall z \in C$ $\bar{\bar{z}} = z$

Proof:-

Let $z = a+ib \longrightarrow (i) \rightarrow \bar{z} = a-ib$

$\rightarrow \bar{\bar{z}} = a+ib \longrightarrow (ii)$

from (i) and (ii), we have

$\bar{\bar{z}} = z$

iii) $\forall z \in C$ $z.\bar{z} = |z|^2$

Proof:-

Let $z = a+ib \rightarrow \bar{z} = a-ib$

L.H.S $= z.\bar{z}$

$= (a+ib)(a-ib)$

$= (a)^2-(ib)^2 = a^2 - i^2b^2$

$= a^2+b^2$

R.H.S $= |z|^2 = \left(\sqrt{a^2+b^2}\right)^2 = a^2+b^2$

Hence L.H.S = R.H.S

iv) $\forall z_1, z_2 \in C$ $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

Proof:-

Let $z_1 = a+ib$, $z_2 = c+id$

$\rightarrow z_1+z_2 = (a+ib)+(c+id)$

$z_1+z_2=(a+c)+i(b+d)$

$\rightarrow \overline{z_1+z_2}=a+c-i(b+d) \longrightarrow (i)$

Also $\bar{z}_1+\bar{z}_2=(a-ib)+(c-id)$

$=(a+c)-i(b+d) \longrightarrow (ii)$

from (i) and (ii), we have

v) $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2} \quad \forall z_1,z_2 \in C$

Proof:-

Let $z_1=a+ib$, $z_2=c+id$

$\rightarrow \bar{z}_1=a-ib$, $\bar{z}_2=c-id$

$\left(\frac{z_1}{z_2}\right)=\frac{a+ib}{c+id}\times\frac{c-id}{c-id}$

$=\frac{ac-aid+ibc-i^2bd}{(c)^2-(id)^2}$

$=\frac{ac-adi+ibc+bd}{c^2-i^2d^2}$

$\left(\frac{z_1}{z_2}\right)=\frac{(ac+bd)-i(ad-bc)}{c^2+d^2}$

$\rightarrow \overline{\left(\frac{z_1}{z_2}\right)}=\frac{(ac+bd)+i(ad-bc)}{c^2+d^2} \longrightarrow (i)$

Now

$\frac{\bar{z}_1}{\bar{z}_2}=\frac{a-ib}{c-id}\times\frac{c+id}{c+id}$

$=\frac{ac+adi-bci-bdi^2}{c^2-(id)^2}$

$=\frac{ac+adi-bci+bd}{c^2-i^2d^2}$

$\frac{\bar{z}_1}{\bar{z}_2}=\frac{(ac+bd)+i(ad-bc)}{c^2+d^2} \longrightarrow (ii)$

From (i) and (ii) $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2}$

vi) $\forall z_1,z_2 \in C \quad |z_1\cdot z_2|=|z_1||z_2|$

Proof:-

Let $z_1=a+ib$, $z_2=c+id$

$\rightarrow |z_1|=\sqrt{a^2+b^2}$, $|z_2|=\sqrt{c^2+d^2}$

L.H.S $=|z_1\cdot z_2|$

$=|(a+ib)(c+id)|$

$=|ac+aid+ibc+i^2bd|$

$=|ac+aid+ibc-bd|$

$=|(ac-bd)+i(ad+bc)|$

$=\sqrt{(ac-bd)^2+(ad+bc)^2}$

$=\sqrt{a^2c^2+b^2d^2-\cancel{2abcd}+a^2d^2+b^2c^2+\cancel{2abcd}}$

$=\sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}$

$=\sqrt{a^2(c^2+d^2)+b^2(c^2+d^2)}$

$=\sqrt{(a^2+b^2)(c^2+d^2)}$

$=\sqrt{a^2+b^2}\sqrt{c^2+d^2}$

$=|z_1|\cdot|z_2|=$ R.H.S

Hence proved.

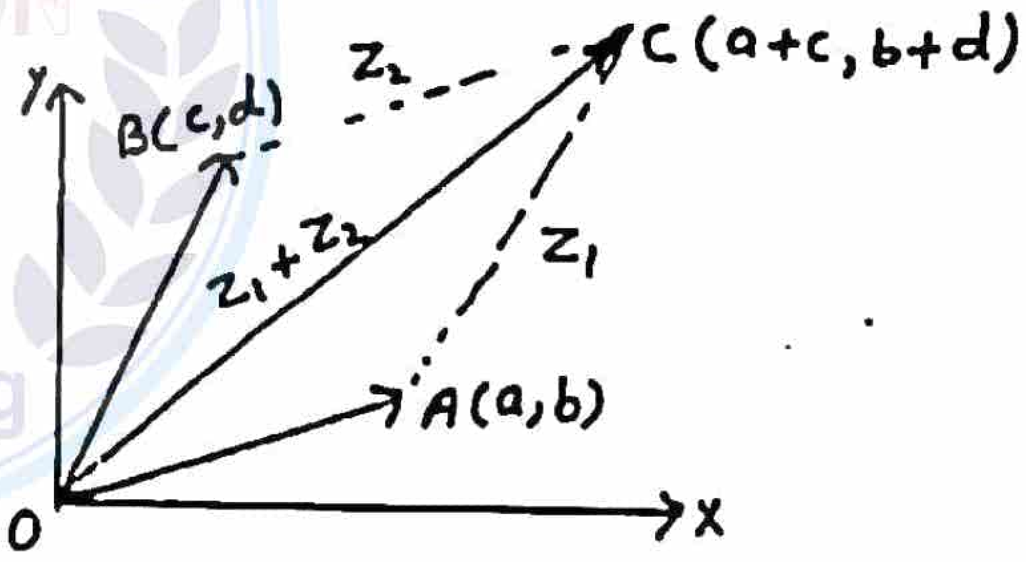
vii) $\forall z_1,z_2 \in C$

$|z_1|-|z_2|\le|z_1+z_2|\le|z_1|+|z_2|$

Proof:-

Let $z_1=a+ib$, $z_2=c+id$

$\rightarrow z_1+z_2=(a+c)+i(b+d)$

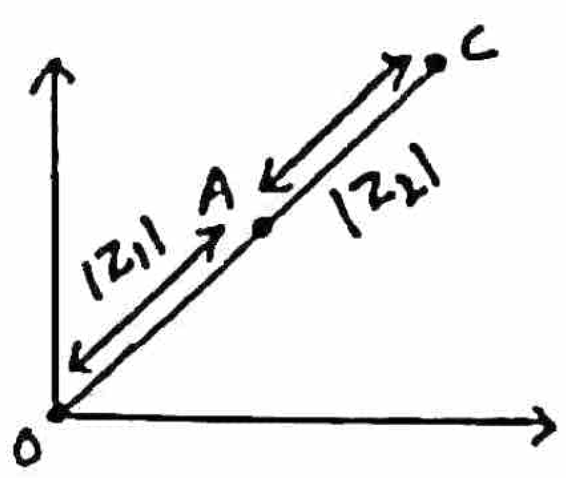


$|z_1|=|\overline{OA}|$, $|z_2|=|\overline{OB}|$

$|z_1+z_2|=|\overline{OC}|$

$|\overline{OA}|+|\overline{AC}|>|\overline{OC}|$

$|z_1|+|z_2|>|z_1+z_2| \longrightarrow (i)$



For collinear points

$|\overline{OA}|+|\overline{AC}|=|\overline{OC}|$

$\rightarrow |z_1|+|z_2|=|z_1+z_2| \longrightarrow (ii)$

By (i) and (ii)

$$|z_1| + |z_2| \geq |z_1 + z_2| \longrightarrow \text{(iii)}$$

Now

$$|z_1| = |z_1 + z_2 - z_2|$$

$$|z_1| \leq |z_1 + z_2| + |-z_2|$$

$$|z_1| \leq |z_1 + z_2| + |z_2|$$

$$|z_1| - |z_2| \leq |z_1 + z_2| \longrightarrow \text{(iv)}$$

By (iii) and (iv)

$$|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1 + z_2|$$

Hence proved

Example 2. If $z_1 = 2+i$, $z_2 = 3-2i$, $z_3 = 1+3i$ then express $\dfrac{\overline{z_1}\,\overline{z_2}}{z_2}$ in the form $a+ib$

Solution:-

$z_1 = 2+i \;\rightarrow\; \overline{z_1} = 2-i$

$z_3 = 1+3i \;\rightarrow\; \overline{z_3} = 1-3i$

Now

$$\frac{\overline{z_1}\,\overline{z_3}}{z_2} = \frac{(2-i)(1-3i)}{3-2i}$$

$$= \frac{2-6i-i+3i^2}{3-2i} = \frac{2-7i-3}{3-2i}$$

$$\frac{\overline{z_1}\,\overline{z_3}}{z_2} = \frac{-1-7i}{3-2i}$$

$$\rightarrow \frac{\overline{z_1}\,\overline{z_3}}{z_2} = \frac{-1-7i}{3-2i} \times \frac{3+2i}{3+2i}$$

$$= \frac{-3-2i-21i-14i^2}{(3)^2-(2i)^2}$$

$$= \frac{-3-23i+14}{9-4i^2} = \frac{11-23i}{9+4}$$

$$= \frac{11-23i}{13} = \frac{11}{13} - \frac{23}{13}i$$

$$= \frac{11}{13} + \left(-\frac{23}{13}\right)i$$

Example 3. Show that $\forall\, z_1, z_2 \in \mathbb{C}$, $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

Solution:-

Let $z_1 = a+bi \;\rightarrow\; \overline{z_1} = a-bi$

$z_2 = c+di \;\rightarrow\; \overline{z_2} = c-di$

$$z_1 z_2 = (a+bi)(c+di)$$

$$= ac + adi + bci + bdi^2$$

$$= ac + adi + bci - bd$$

$$z_1 z_2 = (ac - bd) + i(ad+bc)$$

$$\rightarrow \overline{z_1 z_2} = (ac-bd) - i(ad+bc) \longrightarrow \text{(i)}$$

$$\overline{z_1}\,\overline{z_2} = (a-bi)(c-di)$$

$$= ac - adi - bci + bdi^2$$

$$= ac - adi - bci - bd$$

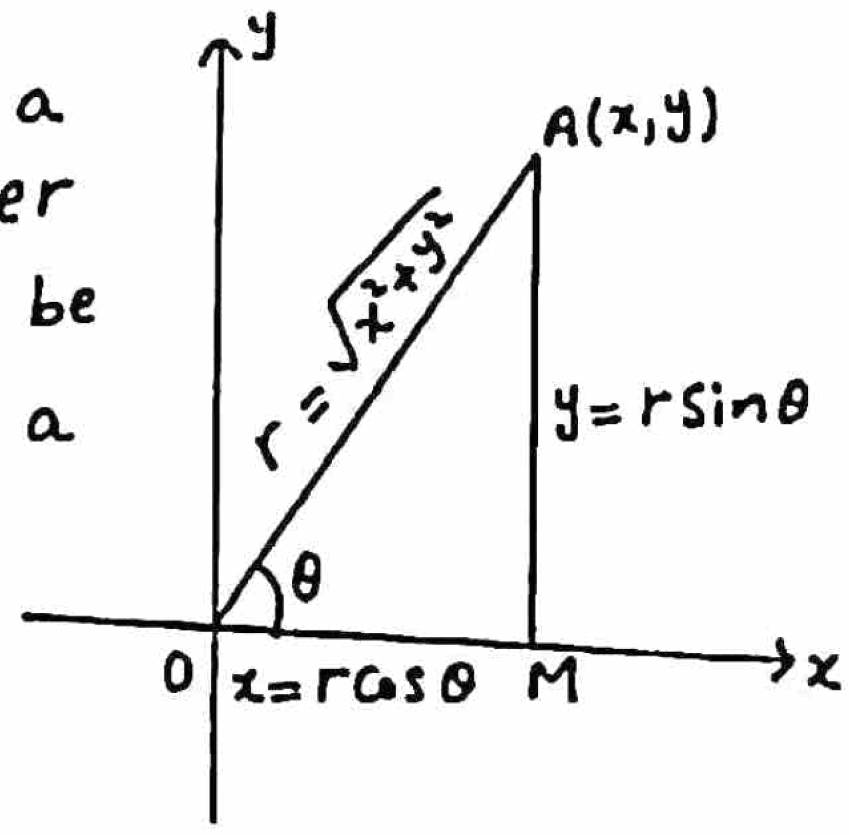
$$= (ac-bd) - i(ad+bc) \longrightarrow \text{(ii)}$$

from (i) and (ii)

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$ Hence proved

Polar form of a Complex Number:-

We know that a complex number $z = x+iy$ can be expressed by a point $A(x,y)$ in coordinate system.



Similarly,

A complex number $z = x+iy$ can be expressed in polar coordinates (r,θ)

From fig.,

$\cos\theta = \dfrac{x}{r}$, $\quad \sin\theta = \dfrac{y}{r}$

$\rightarrow x = r\cos\theta \longrightarrow \text{(i)}$, $\quad \rightarrow y = r\sin\theta \longrightarrow \text{(ii)}$

so $z = x+iy$

$$\rightarrow z = r\cos\theta + i\,r\sin\theta$$

$\rightarrow z = r(\cos\theta + i\sin\theta)$

This is called "Polar form of a complex number".

Now by squaring and adding (i) and (ii)

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

$x^2+y^2 = r^2(1)$

$\rightarrow r^2 = x^2+y^2$

or $r = \sqrt{x^2+y^2}$

and by $\frac{(ii)}{(i)} \rightarrow \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}$

$\rightarrow \tan\theta = \frac{y}{x}$

or $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Example 4. Express the complex number $1+i\sqrt{3}$ in polar form.

Solution:-

Let $z = 1+i\sqrt{3} \longrightarrow (i)$

In polar form $z = r\cos\theta + ir\sin\theta \longrightarrow (ii)$

By (i) and (ii)

$r\cos\theta = 1 \longrightarrow (iii)$

and $r\sin\theta = \sqrt{3} \longrightarrow (iv)$

By $(iii)^2 + (iv)^2 \rightarrow r^2\cos^2\theta + r^2\sin^2\theta = (1)^2 + (\sqrt{3})^2$

$\rightarrow r^2(\cos^2\theta + \sin^2\theta) = 1+3$

$\rightarrow r^2(1) = 4 \rightarrow r^2 = 4$ or

$r = 2$

By $\frac{(iv)}{(iii)} \rightarrow \frac{r\sin\theta}{r\cos\theta} = \frac{\sqrt{3}}{1} \rightarrow \tan\theta = \sqrt{3}$

or $\theta = \tan^{-1}\sqrt{3}$ so

$\rightarrow \theta = 60^\circ$

Now $z = 1+\sqrt{3}i$

$\rightarrow 2\cos 60^\circ + i\,2\sin 60^\circ = 1+i\sqrt{3}$

De Moivre's Theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$\forall n \in Z$

Proof of this theorem is beyond the scope of this book.

To find real and imaginary parts of

i) $(x+iy)^n$ ii) $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n$, $x_2+iy_2 \neq 0$

i) $(x+iy)^n$, $n = \pm1, \pm2, \pm3, \ldots$

Let $z = x+iy$

In polar form,

$\rightarrow x+iy = r\cos\theta + ir\sin\theta$

$\rightarrow (x+iy)^n = (r\cos\theta + ir\sin\theta)^n$

$= r^n(\cos\theta + i\sin\theta)^n$

$(x+iy)^n = r^n(\cos n\theta + i\sin n\theta)$ (By De Moivre's theorem)

$\rightarrow (x+iy)^n = r^n\cos n\theta + ir^n\sin n\theta$

Real part of $z = Re(z) = r^n\cos n\theta$

Imaginary part of $z = Im(z) = r^n\sin n\theta$

where $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

ii) $\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n$, $x_2+iy_2 \neq 0$

$n = \pm1, \pm2, \pm3, \ldots\ldots$

Let $x_1 = r_1\cos\theta$, $y_1 = r_1\sin\theta$

$x_2 = r_2\cos\theta$, $y_2 = r_2\sin\theta$

$\left(\frac{x_1+iy_1}{x_2+iy_2}\right)^n = \left(\frac{r_1\cos\theta_1 + ir_1\sin\theta_1}{r_2\cos\theta_2 + ir_2\sin\theta_2}\right)^n$

$= \frac{r_1^n}{r_2^n}\cdot\frac{(\cos\theta_1 + i\sin\theta_1)^n}{(\cos\theta_2 + i\sin\theta_2)^n}$

$= \frac{r_1^n}{r_2^n}(\cos\theta_1 + i\sin\theta_1)^n(\cos\theta_2 + i\sin\theta_2)^{-n}$

By De Moivre's theorem

$$= \frac{r_1^n}{r_2^n}(\cos n\theta_1 + i\sin n\theta_1)(\cos(-n)\theta_2 + i\sin(-n)\theta_2)$$

$$= \frac{r_1^n}{r_2^n}(\cos n\theta_1 + i\sin n\theta_1)(\cos n\theta_2 - i\sin n\theta_2)$$

$\because \cos(-\theta) = \cos\theta$

$\sin(-\theta) = -\sin\theta$

$$= \frac{r_1^n}{r_2^n}(\cos n\theta_1 \cdot \cos n\theta_2 - i\cos n\theta_1 \sin n\theta_2 + i\cos n\theta_2 \sin n\theta_1 - i^2 \sin n\theta_1 \sin n\theta_2)$$

$$= \frac{r_1^n}{r_2^n}(\cos n\theta_1 \cos n\theta_2 - i\cos n\theta_1 \sin n\theta_2 + i\cos n\theta_2 \sin\theta_1 + \sin n\theta_1 \sin n\theta_2)$$

$$= \frac{r_1^n}{r_2^n}\left[(\cos n\theta_1 \cos n\theta_2 + \sin n\theta_1 \sin n\theta_2) + i(\sin n\theta_1 \sin n\theta_2 - \cos n\theta_1 \sin n\theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n}\left[\cos(n\theta_1 - n\theta_2) + i\sin(n\theta_1 - n\theta_2)\right]$$

$\because \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)$

$\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)$

$$= \frac{r_1^n}{r_2^n}\left[\cos n(\theta_1-\theta_2) + i\sin n(\theta_1-\theta_2)\right]$$

$$= \frac{r_1^n}{r_2^n}\cos n(\theta_1-\theta_2) + i\frac{r_1^n}{r_2^n}\sin n(\theta_1-\theta_2)$$

Thus

Real part $= \frac{r_1^n}{r_2^n}\cos n(\theta_1-\theta_2)$

Imaginary part $= \frac{r_1^n}{r_1^n}\sin n(\theta_1-\theta_2)$

where $r_1 = \sqrt{x_1^2 + y_1^2}$

$\theta_1 = \tan^{-1}\left(\frac{y_1}{x_1}\right)$, $r_2 = \sqrt{x_2^2 + y_2^2}$

$\theta_2 = \tan^{-1}\left(\frac{y_2}{x_2}\right)$

Example 5. Find out real and imaginary parts of each of the following complex numbers.

i) $(\sqrt{3}+1)^3$ ii) $\left(\frac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^5$

Solution:- i) $(\sqrt{3}+1)^3$

Let $r\cos\theta = \sqrt{3}$, $r\sin\theta = 1$

where

$r^2\cos^2\theta + r^2\sin^2\theta = (\sqrt{3})^2 + (1)^2$

$\rightarrow r^2(\cos^2\theta + \sin^2\theta) = 3+1$

$r^2 = 4$ or $r = 2$

Now $\frac{r\sin\theta}{r\cos\theta} = \frac{1}{\sqrt{3}} \rightarrow \tan\theta = \frac{1}{\sqrt{3}}$

$\rightarrow \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$ so

$(\sqrt{3}+1)^3 = [r(\cos\theta + i\sin\theta)]^3$

$= r^3(\cos\theta + i\sin\theta)^3$

$= 2^3(\cos 30° + i\sin 30°)^3$

$= 8(\cos 3(30°) + i\sin 3(30°))$

$= 8(\cos 90° + i\sin 90°)$

$= 8(0 + i \cdot 1) = 0 + 8i$

Real part $= 0$, Imaginary part $= 8$

ii) $\left(\frac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^5$

Let $r_1\cos\theta_1 = 1$, $r_1\sin\theta_1 = -\sqrt{3}$

$\rightarrow r_1 = \sqrt{(1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

and $\theta_1 = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -60°$

Also $r_2\cos\theta_2 = 1$, $r_2\sin\theta_2 = \sqrt{3}$

$\rightarrow r_2 = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$

and $\theta_2 = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60°$

So

$$\left(\frac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^5 = \frac{[2\cos(-60°) + i2\sin(-60°)]^5}{[2\cos(60°) + i2\sin(60°)]^5}$$

$$= \frac{\cancel{x^8}}{\cancel{x^8}}\left[\cos(-60^\circ)+i\sin(-60^\circ)\right]^5\left[\cos 60^\circ+i\sin 60^\circ\right]^{-5}$$

$$= \left[\cos(-300^\circ)+i\sin(-300^\circ)\right]\left[\cos(300^\circ)+i\sin(300^\circ)\right]$$

$$= \left[\cos 300^\circ - i\sin 300^\circ\right]\left[\cos 300^\circ+i\sin 300^\circ\right] \longrightarrow (i)$$

$$\because \cos 300^\circ = \cos(3\times 90^\circ+30^\circ) = \sin 30^\circ = \frac{1}{2}$$

$$\sin 300^\circ = \sin(3\times 90^\circ+30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$= \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 \qquad (i) \text{ becomes}$$

$$\rightarrow \left(\frac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^5 = \frac{1}{4} + 2\left(\frac{1}{2}\right)\left(i\frac{\sqrt{3}}{2}\right) + \left(i\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{1}{4} + \frac{\sqrt{3}}{2}i - \frac{3}{4}$$

$$= \frac{1-3}{4} + \frac{\sqrt{3}}{2}i$$

$$= -\frac{2}{4} + \frac{\sqrt{3}}{2}i$$

$$\rightarrow \left(\frac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^5 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Real part $= -\frac{1}{2}$, Imaginary part $= \frac{\sqrt{3}}{2}$

Exercise 1.3

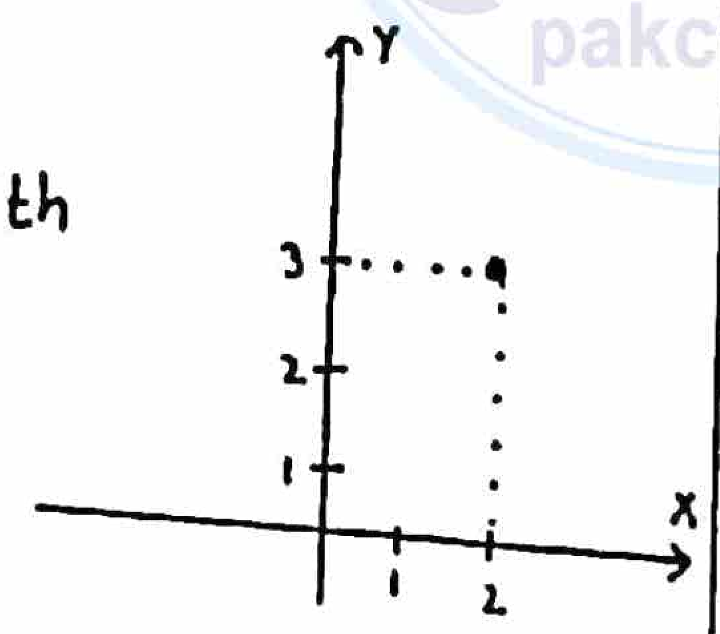
Q1. Graph the following numbers on the complex plane:

i) $2+3i$

Solution:-

$2+3i$ compare with $x+iy$ Here

$x = 2,\ y = 3$



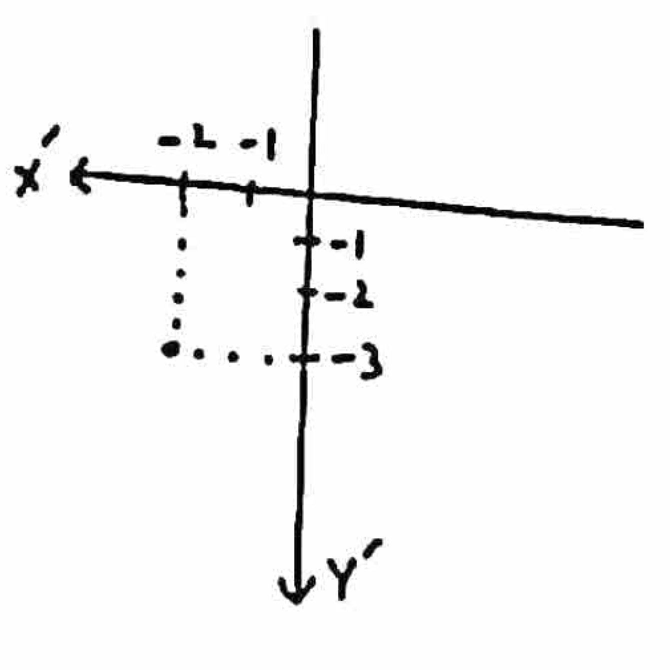
ii) $-2-3i$

Solution:-

$-2-3i$ compare with $x+iy$

Here $x = -2$

$y = -3$

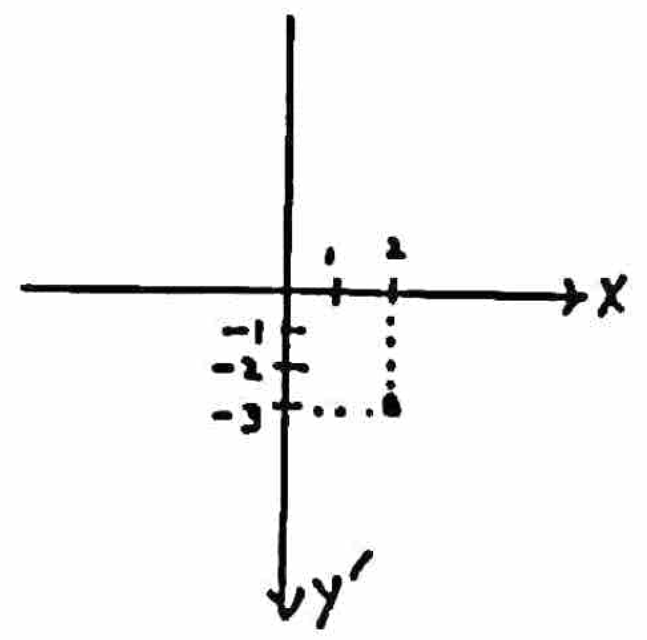


iii) $2-3i$

Solution:-

$2-3i$ compare with $x+iy$

Here $x = 2,\ y = -3$

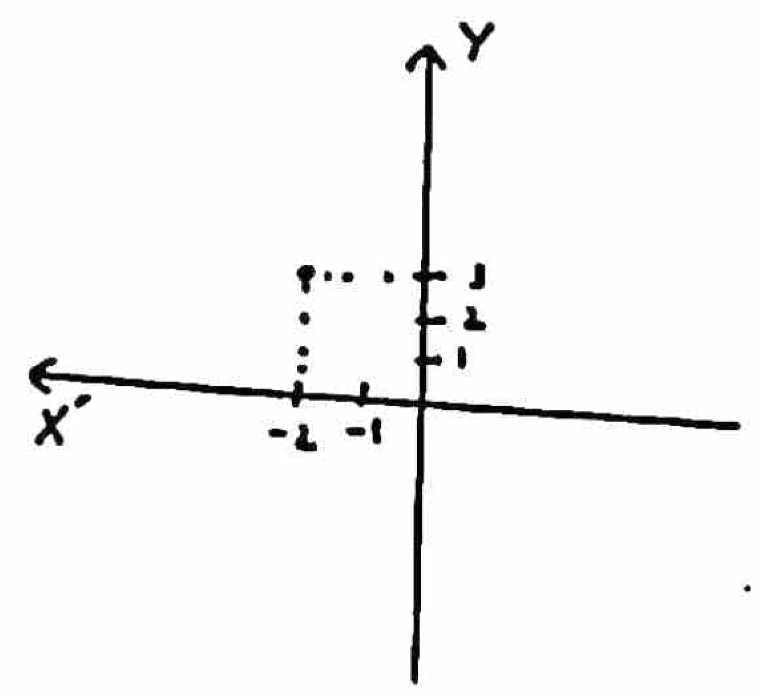


iv) $-2+3i$

Solution:-

$-2+3i$ compare with $x+iy$

Here $x = -2$ and $y = 3$

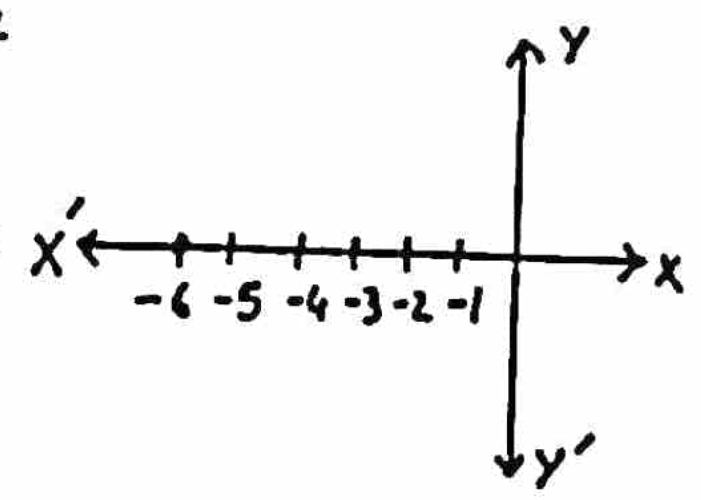


v) -6

Solution:-

$-6+0i$ compare with $x+iy$

Here $x = -6,\ y = 0$

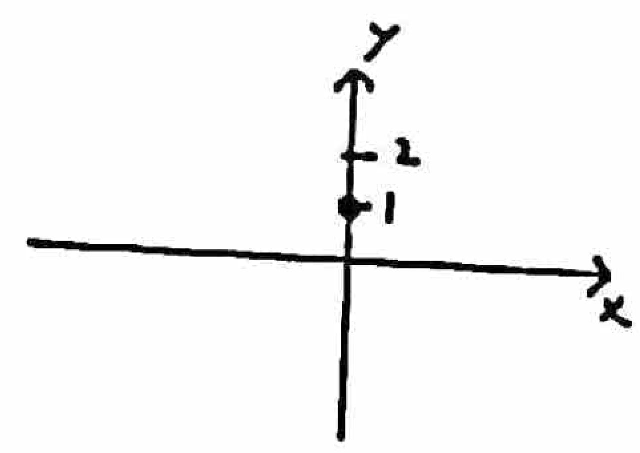


vi) i

Solution:-

$i = 0+1i$

Compare with $x+iy \rightarrow x = 0,\ y = 1$

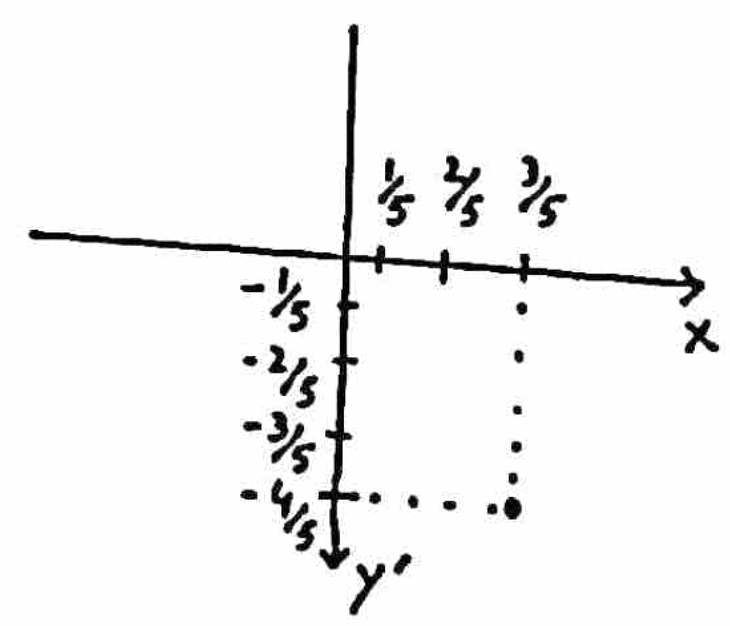


vii) $\frac{3}{5} - \frac{4}{5}i$

Solution:-

$\frac{3}{5} - \frac{4}{5}i$ compare with $x+iy \rightarrow$

$x = \frac{3}{5}\ ,\ y = -\frac{4}{5}$



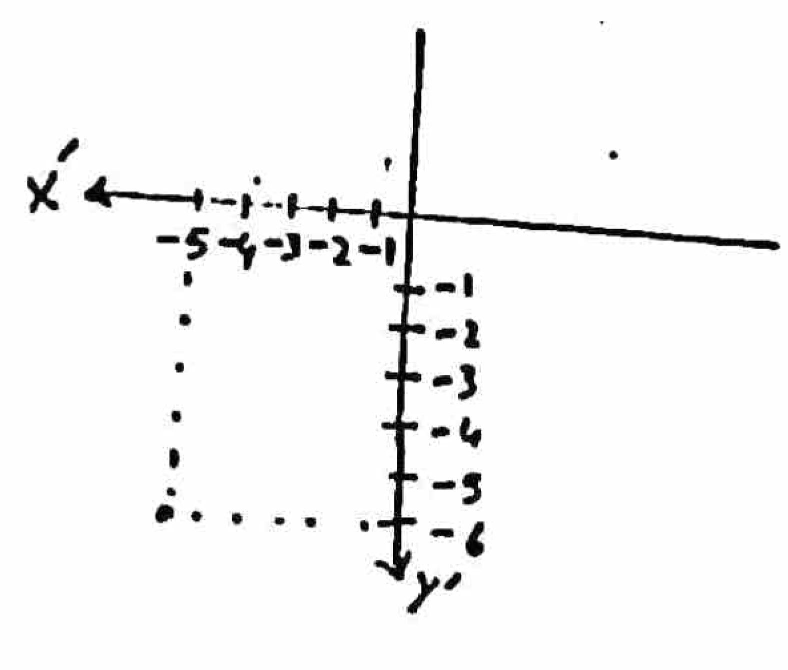
viii) $-5-6i$

Solution:-

$-5-6i$ compare with $x+iy \rightarrow$

$x = -5$

$y = -6$



Q2. Find the multiplicative inverse of each of the following numbers.

i) $-3i$

Solution:– Let $z=-3i = 0-3i = (0,-3)$

$\because$ For $z=(a,b)$

$\rightarrow z^{-1} = \left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right)$

so for $z=(0,-3)$

$\rightarrow z^{-1} = \left(\dfrac{0}{(0)^2+(-3)^2}, \dfrac{-(-3)}{(0)^2+(-3)^2}\right)$

$= \left(0, \dfrac{3}{9}\right) = \left(0, \dfrac{1}{3}\right)$

$= 0+\dfrac{1}{3}i$

ii) $1-2i$

Solution:– Let $z = 1-2i = (1,-2)$

$\because$ For $z=(a,b) \rightarrow z^{-1} = \left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right)$

so for $z=(1,-2) \rightarrow z^{-1} = \left(\dfrac{1}{(1)^2+(-2)^2}, \dfrac{-(-2)}{(1)^2+(-2)^2}\right)$

$\rightarrow z^{-1} = \left(\dfrac{1}{1+4}, \dfrac{2}{1+4}\right)$

$\rightarrow z^{-1} = \left(\dfrac{1}{5}, \dfrac{2}{5}\right) = \dfrac{1}{5}+\dfrac{2}{5}i$

iii) $-3-5i$

Solution:– Let $z=-3-5i = (-3,-5)$

$\because$ For $z=(a,b) \rightarrow z^{-1} = \left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right)$

so for $z=(-3,-5)$

$\rightarrow z^{-1} = \left(\dfrac{-3}{(-3)^2+(-5)^2}, \dfrac{-(-5)}{(-3)^2+(-5)^2}\right)$

$= \left(\dfrac{-3}{9+25}, \dfrac{5}{9+25}\right)$

$z^{-1} = \left(\dfrac{-3}{34}, \dfrac{5}{34}\right) = \dfrac{-3}{34}+\dfrac{5}{34}i$

iv) $(1,2)$

Solution:– Let $z=(1,2)$

$\because$ For $z=(a,b)$

$\rightarrow z^{-1} = \left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right)$

so for $z=(1,2)$

$\rightarrow z^{-1} = \left(\dfrac{1}{(1)^2+(2)^2}, \dfrac{-2}{(1)^2+(2)^2}\right)$

$= \left(\dfrac{1}{1+4}, \dfrac{-2}{1+4}\right)$

$z^{-1} = \left(\dfrac{1}{5}, \dfrac{-2}{5}\right)$

Q3. Simplify

i) i^{101}

Solution:–

$i^{101} = i^{100}\cdot i = (i^2)^{50}\cdot i = (-1)^{50}\cdot i$

$= (1)i = i$

ii) $(-ai)^4, \ a\in\mathbb{R}$

Solution:–

$(-ai)^4 = (-a)^4 i^4 = a^4(i^2)^2 = a^4(-1)^2$

$= a^4(1) = a^4$

iii) i^{-3}

Solution:–

$i^{-3} = \dfrac{1}{i^3} = \dfrac{1}{i^2\cdot i} = \dfrac{1}{(-1)i} = \dfrac{1}{-i}$

$= -\dfrac{1}{i}\times\dfrac{i}{i} = -\dfrac{i}{i^2} = -\dfrac{i}{(-1)} = i$

i^{-10}

Solution:–

$i^{-10} = \dfrac{1}{i^{10}} = \dfrac{1}{(i^2)^5} = \dfrac{1}{(-1)^5} = \dfrac{1}{-1} = -1$

Q4. Prove that $\bar{z}=z$ iff z is real.

Solution:–

Suppose $\bar{z} = z$ Now

let $z = a+ib \longrightarrow (i)$

$\rightarrow \bar{z} = a-ib$

$\because \bar{z}=z \rightarrow a-ib = a+ib$

$\rightarrow a-a = ib+ib$

$\rightarrow 0 = 2ib \rightarrow b=0$

so (i) $\rightarrow z = a+0$

$\rightarrow z = a \rightarrow z$ is real

Conversely,

Suppose z is real

so $z = a \longrightarrow$ (ii)

$\rightarrow \bar{z} = \bar{a}$

$\bar{z} = a \longrightarrow$ (iii)

By (ii) and (iii), $z = \bar{z}$

Hence proved

Q5. Simplify by expressing in the form $a+bi$

i) $5 + 2\sqrt{-4}$

Solution:-

$5+2\sqrt{-4} = 5+2\sqrt{(-1)4}$

$= 5+2i\sqrt{4} = 5+2i(2)$

$= 5+i4$

ii) $(2+\sqrt{-3})(3+\sqrt{-3})$

Solution:-

$(2+\sqrt{-3})(3+\sqrt{-3})$

$= (2+i\sqrt{3})(3+i\sqrt{3})$

$= 6+2i\sqrt{3}+3i\sqrt{3}+i^2(3)$

$= 6+5\sqrt{3}i-3 \qquad (\because i^2=-1)$

$= 3+5\sqrt{3}i$

iii) $\dfrac{2}{\sqrt{5}+\sqrt{-8}}$

Solution:-

$\dfrac{2}{\sqrt{5}+\sqrt{-8}} = \dfrac{2}{\sqrt{5}+\sqrt{8}i} \times \dfrac{\sqrt{5}-\sqrt{8}i}{\sqrt{5}-\sqrt{8}i}$

$= \dfrac{2(\sqrt{5}-\sqrt{8}i)}{(\sqrt{5})^2-(\sqrt{8}i)^2}$

$= \dfrac{2\sqrt{5}-2\sqrt{8}i}{5-8i^2} = \dfrac{2\sqrt{5}-2\sqrt{8}i}{5+8}$

$= \dfrac{2\sqrt{5}-2\sqrt{8}i}{13} = \dfrac{2\sqrt{5}}{13} - i\dfrac{2\sqrt{8}}{13}$

iv) $\dfrac{3}{\sqrt{6}-\sqrt{-12}}$

Solution:-

$\dfrac{3}{\sqrt{6}-\sqrt{-12}} = \dfrac{3}{\sqrt{6}-i\sqrt{12}}$

$= \dfrac{3}{\sqrt{6}-i\sqrt{12}} \times \dfrac{\sqrt{6}+i\sqrt{12}}{\sqrt{6}+i\sqrt{12}}$

$= \dfrac{3(\sqrt{6}+i\sqrt{12})}{(\sqrt{6})^2-(\sqrt{12}i)^2}$

$= \dfrac{3\sqrt{6}+3\sqrt{12}i}{6-12i^2}$

$= \dfrac{3\sqrt{6}+3\sqrt{12}i}{6+12} = \dfrac{3\sqrt{6}+3\sqrt{12}i}{18}$

$= \dfrac{3\sqrt{6}}{18} + \dfrac{3\sqrt{12}}{18}i$

$= \dfrac{\sqrt{6}}{6} + \dfrac{\sqrt{12}}{6}i$

$= \dfrac{\sqrt{6}}{\sqrt{6}\sqrt{6}} + \dfrac{\sqrt{4\times3}}{6}i = \dfrac{1}{\sqrt{6}} + \dfrac{2\sqrt{3}}{6}i$

$= \dfrac{1}{\sqrt{6}} + \dfrac{\sqrt{3}}{3}i = \dfrac{1}{6} + \dfrac{\sqrt{3}}{\sqrt{3}\sqrt{3}}i$

$= \dfrac{1}{6} + \dfrac{1}{\sqrt{3}}i$

Q6. Show that $\forall z \in C$

i) $z^2+\bar{z}^2$ is a real number

ii) $(z-\bar{z})^2$ is a real number

Solution:-

i) $z^2+\bar{z}^2$ is a real number

Take $z = a+ib \rightarrow \bar{z} = a-ib$

Now $z^2+\bar{z}^2 = (a+ib)^2+(a-ib)^2$

$= a^2+2abi+(ib)^2+a^2-2abi+(ib)^2$

$= a^2-b^2+a^2-b^2$

$= 2a^2-2b^2$ which is real

ii) $(z-\bar{z})^2$ is a real number

Take $z = a+ib \rightarrow \bar{z} = a-ib$

Now $(z-\bar{z})^2 = [a+ib-(a-ib)]^2$

$= [a+ib-a+ib]^2$

$= (2ib)^2 = 4b^2i^2$

$= 4b^2(-1) = -4b^2$

which is real.

Q7. Simplify the following

i) $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$

Solution:-

$\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$

$=\left(-\frac{1}{2}\right)^3+3\left(-\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2}i\right)+3\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}i\right)^2+\left(\frac{\sqrt{3}}{2}i\right)^3$

$=-\frac{1}{8}+3\left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{2}i\right)-\left(\frac{3}{2}\right)\left(-\frac{3}{4}\right)+\frac{3\sqrt{3}}{8}(-i)$

$=-\frac{1}{8}+\cancel{\frac{3\sqrt{3}}{8}i}+\frac{9}{8}-\cancel{\frac{3\sqrt{3}}{8}i}$

$=-\frac{1}{8}+\frac{9}{8}=\frac{-1+9}{8}=\frac{8}{8}=1$

ii) $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)^3$

Solution:-

$\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)^3$

$=\left(-\frac{1}{2}\right)^3+3\left(-\frac{1}{2}\right)^2\left(-\frac{\sqrt{3}}{2}i\right)+3\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}i\right)^2+\left(-\frac{\sqrt{3}}{2}i\right)^3$

$=-\frac{1}{8}+3\left(\frac{1}{4}\right)\left(-\frac{\sqrt{3}}{2}i\right)-\frac{3}{2}\left(-\frac{3}{4}\right)+\left(-\frac{3\sqrt{3}}{8}i^3\right)$

$=-\frac{1}{8}+\frac{3}{4}\left(-\frac{\sqrt{3}}{2}i\right)+\frac{9}{8}+\left(-\frac{3\sqrt{3}}{8}(-i)\right)$

$=-\frac{1}{8}-\cancel{\frac{3\sqrt{3}}{8}i}+\frac{9}{8}+\cancel{\frac{3\sqrt{3}}{8}i}$

$=-\frac{1}{8}+\frac{9}{8}=\frac{-1+9}{8}=\frac{8}{8}=1$

iii) $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-2}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$

Solution:-

$\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-2}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$

$=\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-2+1}$

$=\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-1}$

$=\left(\frac{-1-\sqrt{3}i}{2}\right)^{-1}$

$=\frac{2}{-1-\sqrt{3}i}=\frac{2}{[-(1+\sqrt{3}i)]}$

$=\frac{-2}{1+\sqrt{3}i}\times\frac{1-\sqrt{3}i}{1-\sqrt{3}i}$

$=\frac{-2(1-\sqrt{3}i)}{(1)^2-(\sqrt{3}i)^2}$

$=\frac{-2(1-\sqrt{3}i)}{1-3i^2}=\frac{-2+2\sqrt{3}i}{1+3}$

$=\frac{2(-1+\sqrt{3}i)}{4}=\frac{-1+\sqrt{3}i}{2}$

iv) $(a+bi)^2$

Solution:-

$(a+bi)^2=(a)^2+(bi)^2+2(a)(bi)$

$=a^2+b^2i^2+2abi$

$=a^2-b^2+2abi$

v) $(a+bi)^{-2}$

Solution:-

$(a+bi)^{-2}=\frac{1}{(a+bi)^2}$

$=\frac{1}{a^2+b^2i^2+2abi}=\frac{1}{a^2-b^2+2abi}$

$=\frac{1}{(a^2-b^2)+2abi}\times\frac{(a^2-b^2)-2abi}{(a^2-b^2)-2abi}$

$=\frac{a^2-b^2-2abi}{(a^2-b^2)^2-(2abi)^2}=\frac{a^2-b^2-2abi}{a^4+b^4-2a^2b^2+4a^2b^2}$

$=\frac{a^2-b^2-2abi}{a^4+b^4+2a^2b^2}=\frac{a^2-b^2-2abi}{(a^2+b^2)^2}$

$=\frac{a^2-b^2}{(a^2+b^2)^2}-\frac{2ab}{(a^2+b^2)^2}i$

vi) $(a+bi)^3$

Solution:-

$(a+bi)^3=a^3+(bi)^3+3a^2(bi)+3a(bi)^2$

$=a^3+b^3i^3+3a^2bi+3ab^2i^2$

$=a^3-b^3i+3a^2bi-3ab^2$

$=(a^3-3ab^2)+(3a^2b-b^3)i$

vii) $(a-bi)^3$

Solution:-

$$(a-bi)^3 = a^3 - 3a^2(bi) + 3a(bi)^2 - (bi)^3$$

$$= a^3 - 3a^2bi - 3ab^2 - b^3i^3$$

$$= a^3 - 3a^2bi - 3ab^2 + b^3$$

$\because i^3 = i^2 i$
$= (-1)i = -i$

$\because (a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

$(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$

viii) $(3-\sqrt{-4})^{-3}$

Solution:-

$$(3-\sqrt{-4})^{-3} = \frac{1}{(3-\sqrt{-4})^3}$$

$$= \frac{1}{(3-\sqrt{4}i)^3}$$

$$= \frac{1}{(3-2i)^3} = \frac{1}{(3)^3 - (2i)^3 - 3(3)^2(2i) + 3(3)(2i)^2}$$

$$= \frac{1}{27 - 8i^3 - 27(2i) + 9(4i^2)}$$

$$= \frac{1}{27 + 8i - 54i - 36}$$

$\because i^2 = -1$
$i^3 = i^2 \cdot i = (-1)i = -i$

$$= \frac{1}{-9-46i} \times \frac{-9+46i}{-9+46i}$$

$$= \frac{-9+46i}{(-9)^2 - (46i)^2} = \frac{-9+46i}{81-(-2116)} = \frac{-9+46i}{81+2116}$$

$$= \frac{-9+46i}{2197} = \frac{-9}{2197} + \frac{46}{2197}i$$