1st Year Physics

Chapter # 02

Vectors and Equilibrium

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Chapter : 02

Vectors And Equilibrium

Physical Quantities:

All the measurable quantities are called physical quantities. For example, length, mass, time, force, velocity, momentum etc.

Types: There are two types of physical quantities

i- scalars ii- vectors

i- Scalars:

The physical quantities that are described by magnitude only are called scalars.

For example, length, mass time, distance etc.

ii- Vectors:

The physical quantities that are described by magnitude and direction are called vectors.

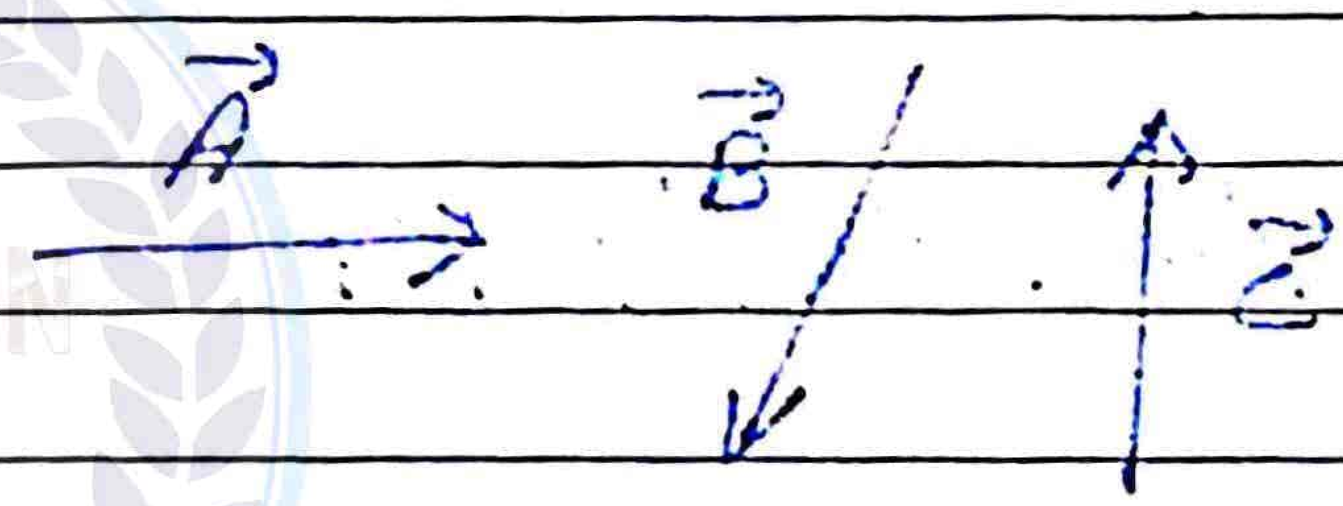
For example, force, velocity, momentum, torque etc.

→ How we differentiate vectors from scalars?

To differentiate vectors from scalars we use bold letters **A**, **F** or place an arrow over the symbol $\vec{A}$, $\vec{F}$.

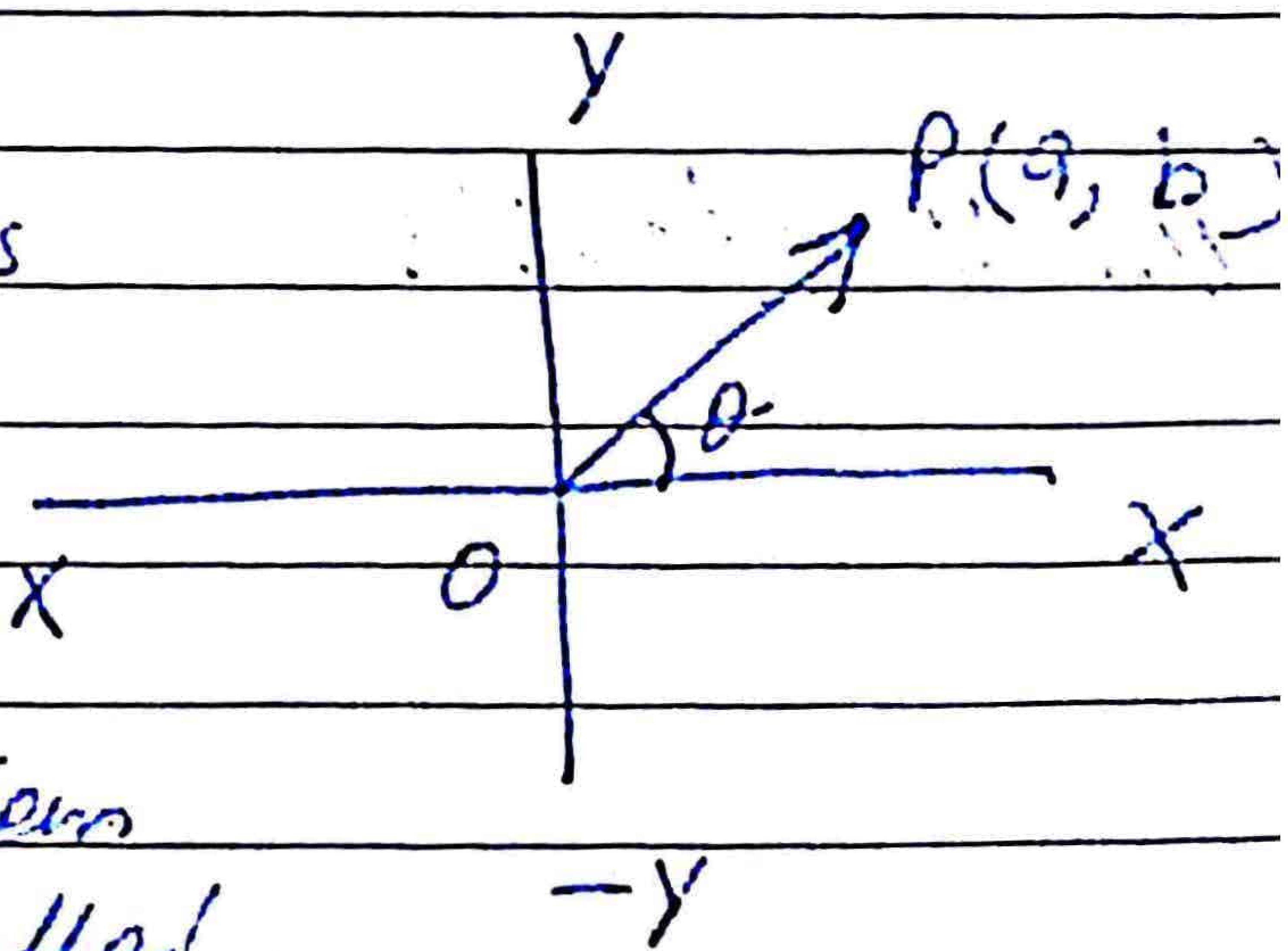
→ How are vectors represented graphically?

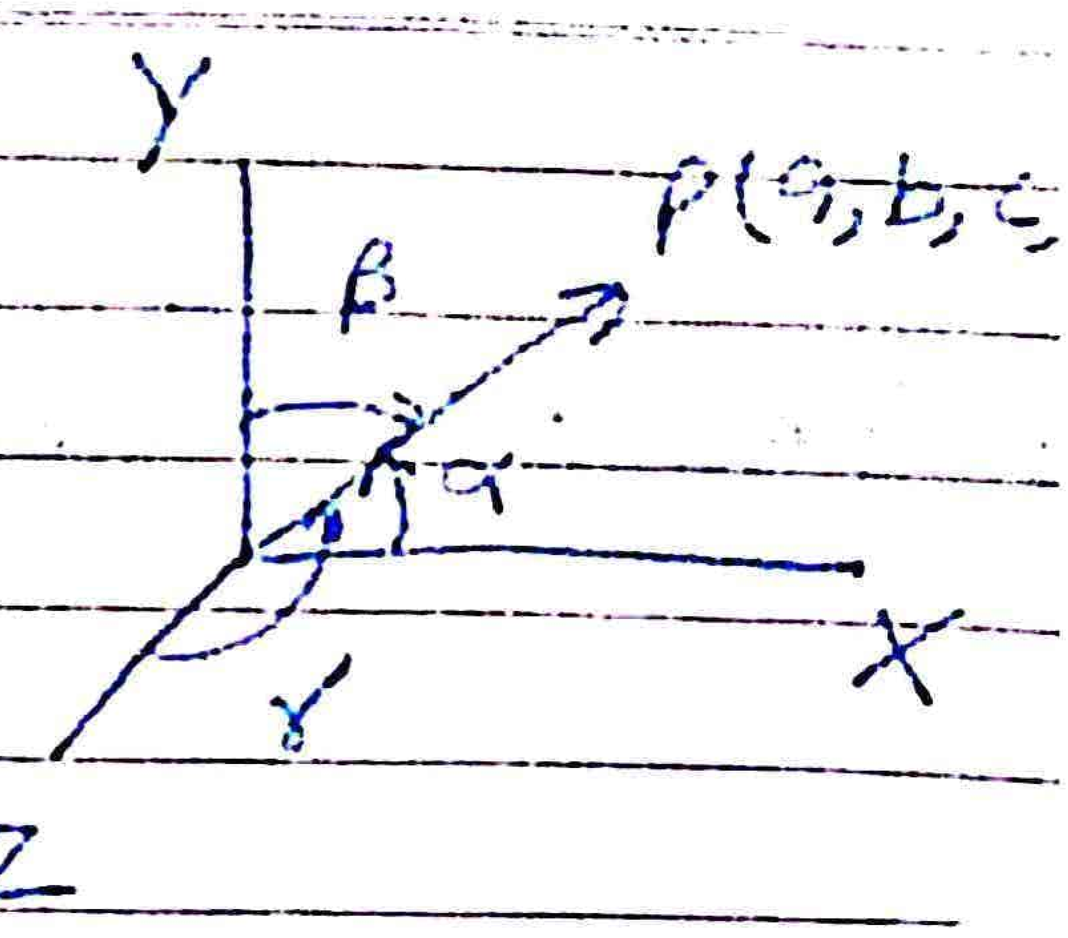
Graphically, vectors are represented by a straight line having an arrow over its one end. The length of the line shows the magnitude of vector and arrow shows its direction.



Coordinate Axes / Origin / Rectangular coordinate system:

Two lines drawn at right angles to each other ~~as shown~~ in fig are known as coordinate axes and their point of intersection is known as origin. This system of coordinate axes is called as cartesian or rectangular coordinate system.

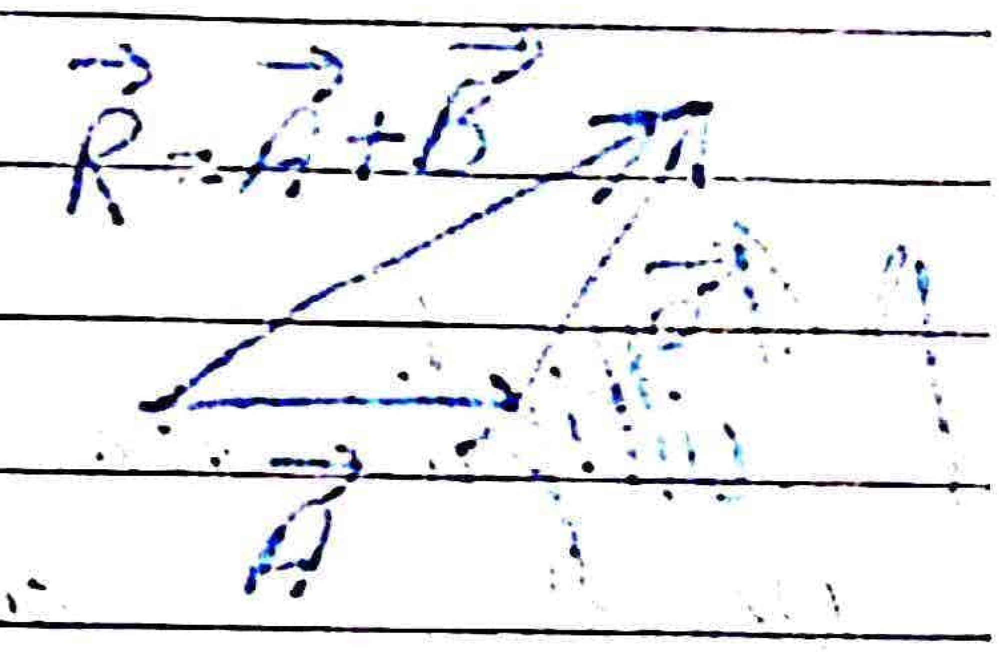


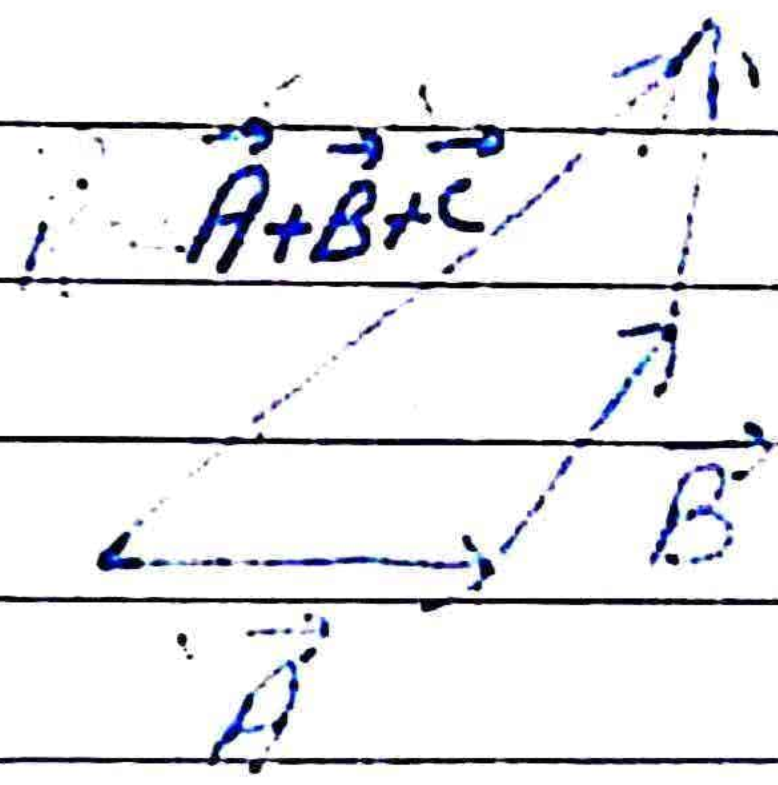


Head to tail rule:

Head to tail rule is the graphical method to add two or more vectors.

We place the tail of second vector with the head of first vector and so on. Finally, we draw the resultant such that its tail meets with the tail of first vector and its head meets with the head of last vector.

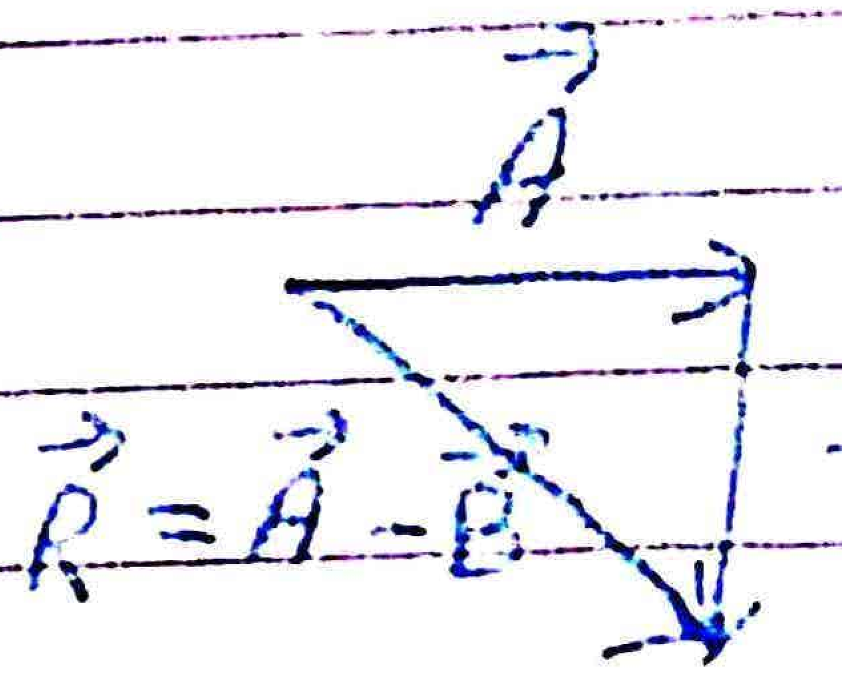




→ Resultant vector:-

The resultant of a number of similar vectors is that single vector which would have the same effect as all the original vectors taken together.

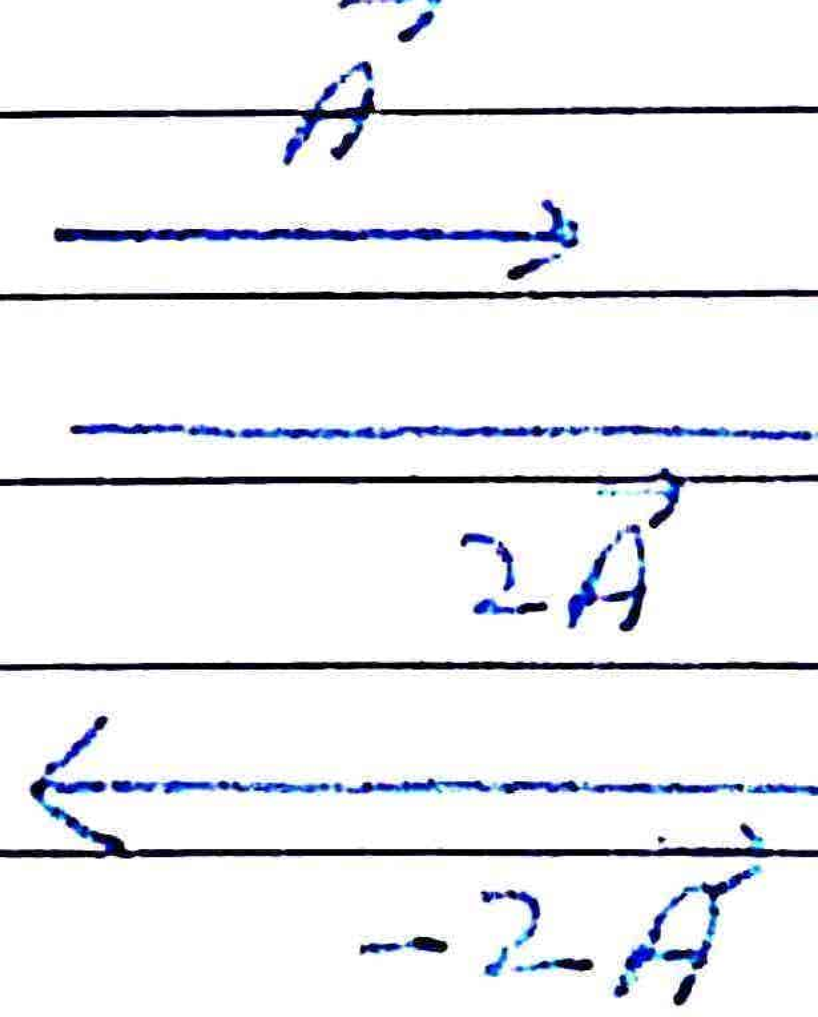
→ Vector Subtraction:-



The subtraction of a vector is equivalent to the addition of the same vector with its direction reversed. Thus, to subtract vector $\vec{B}$ from vector $\vec{A}$, reverse the direction of $\vec{B}$ and Add it to $\vec{A}$.

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

⇒ Multiplication of a vector by a scalar:-

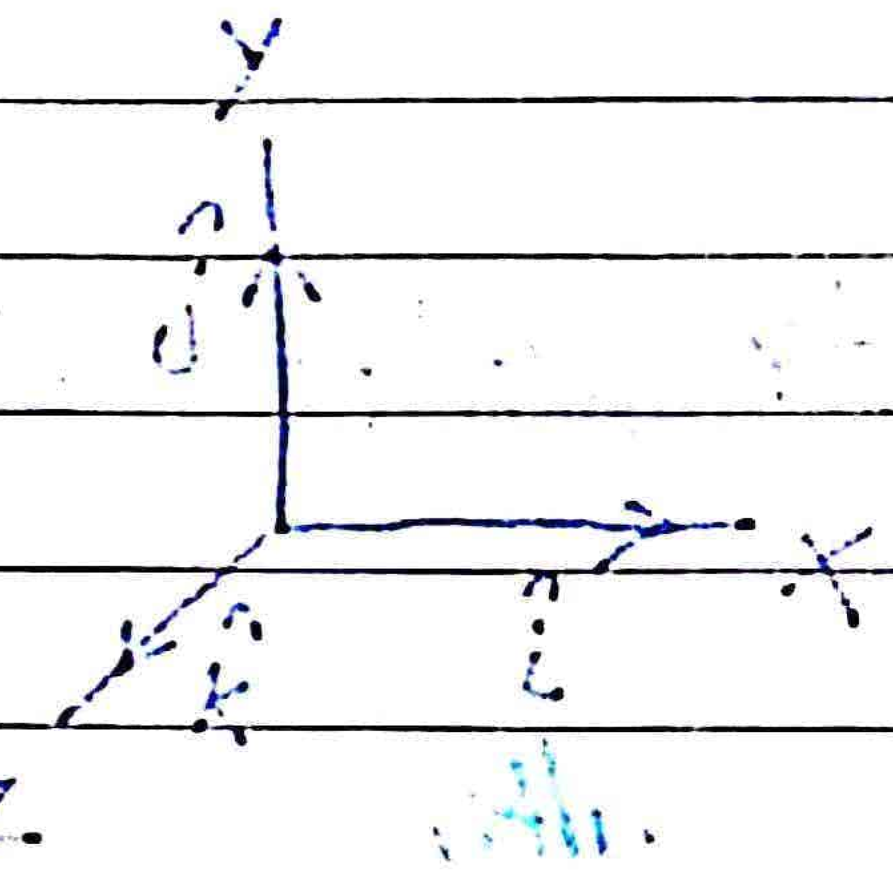


The product of a vector $\vec{A}$ and a number $n > 0$ is defined to be a new vector $n\vec{A}$ having the same direction as $\vec{A}$ but a magnitude n times the magnitude of $\vec{A}$ as illustrated in fig. if the vector is multiplied by a negative number, then its direction is reversed.

Unit Vector:

The vector whose magnitude is one is called unit vector. Unit vector is represented by placing a hat over the symbol. For a vector $\vec{A}$, unit vector is represented as $\hat{A}$. mathematically:

$$\hat{A} = \frac{\vec{A}}{A}$$



The unit vector along x-axis is $\hat{i}$, along y-axis is $\hat{j}$ and along z-axis is $\hat{k}$.

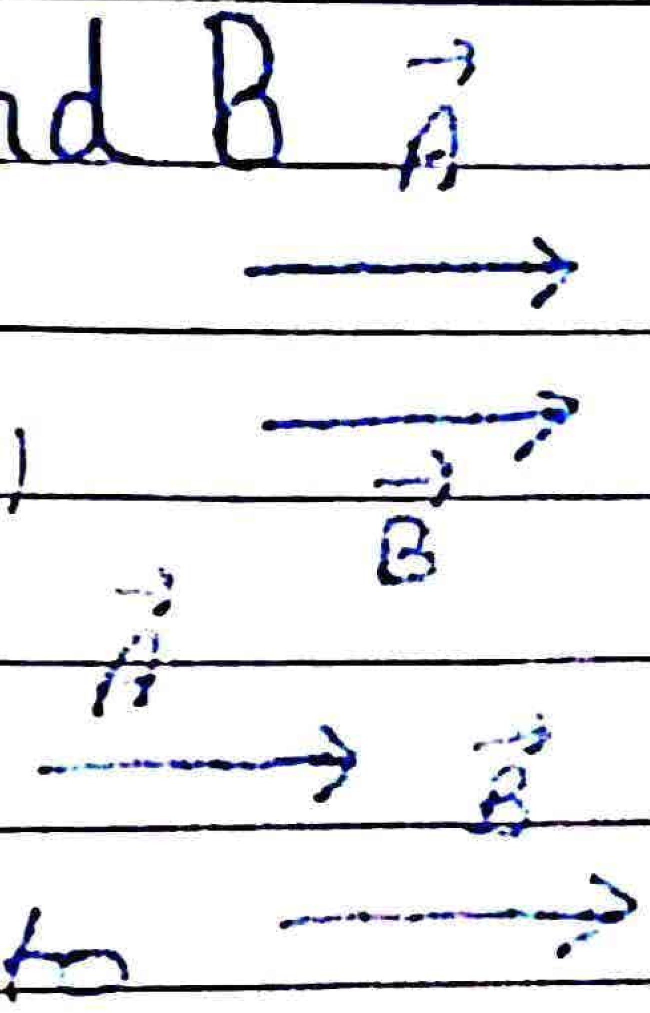
Null Vector:

It is a vector of zero magnitude and arbitary (with no specific direction) direction. E.g. the sum of a vector and its negative vector is a null vector.

$$A + (-A) = 0$$

Equal vector:-

Two vector A and B are said to be equal if they have the same magnitude and direction, regardless of the position of their initial points. This means that parallel vectors of the same magnitude are equal to each other.



Rectangular Components Of A Vector:

Component:

A component of a vector is its effective value in a given direction. For example, for a vector $\vec{A}$

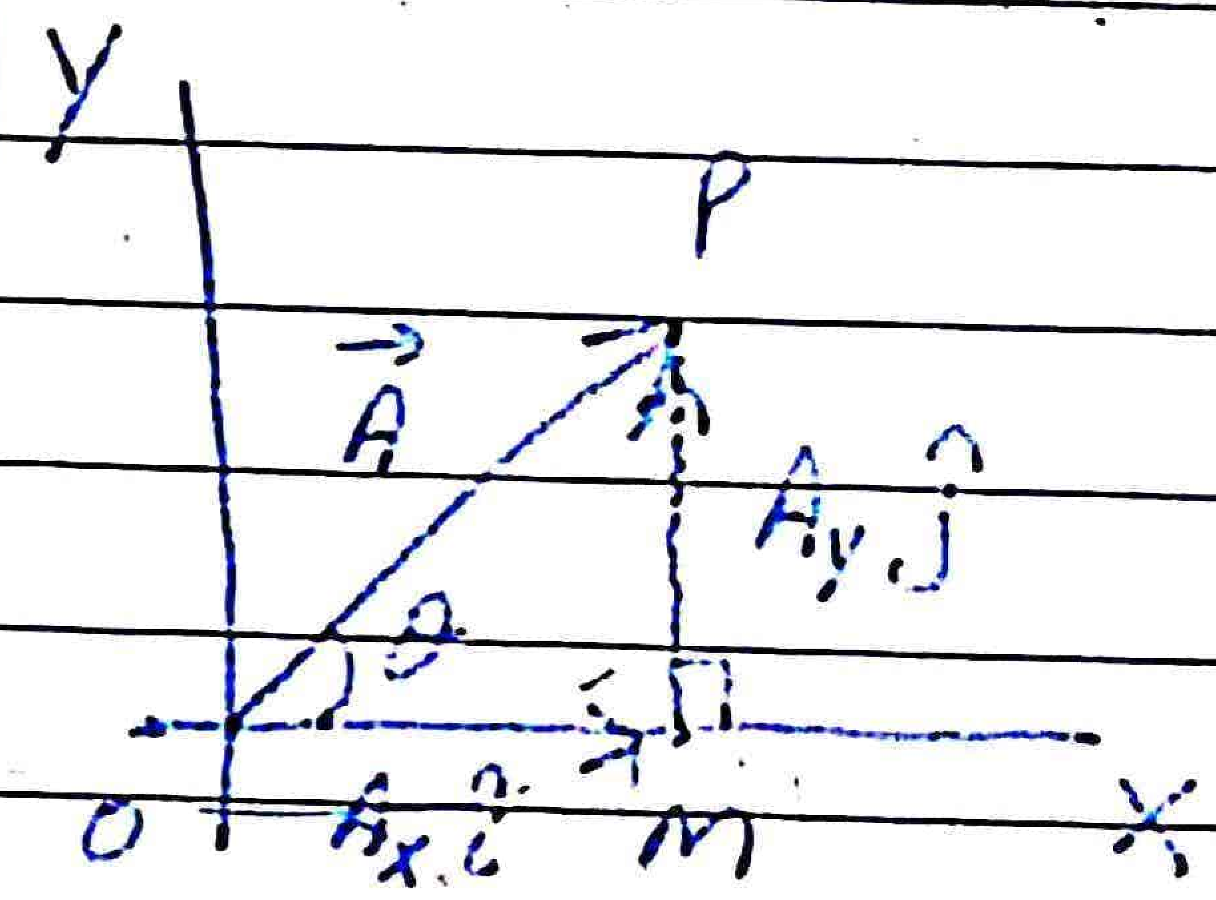
$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

Here A_x, A_y and A_z are the components of vector $\vec{A}$.

Rectangular Components:

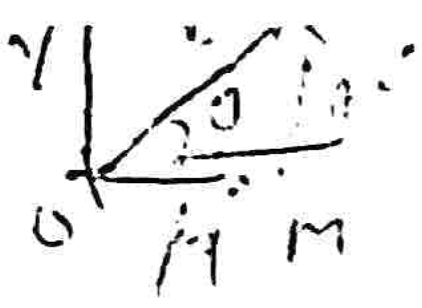
The mutually perpendicular components of a vector are called its rectangular component.

Explanation:



Consider a vector $\vec{A}$ represented by $|OP|$ making angle θ with x-axis. We draw a perpendicular on x-axis and get the rectangular components of vector $\vec{A}$. Here $A_x\hat{i}$ is the horizontal component along x-axis and $A_y\hat{j}$ is the vertical component along y-axis. According to head to tail rule:

$$\therefore \quad A_x\hat{i} + A_y\hat{j}$$



For the right triangle OPM

$$\sin\theta = \frac{P}{H}$$

$$= \frac{|MP|}{|OP|} = \frac{A_y}{A}$$

$$\sin\theta = \frac{A_y}{A}$$

$$A\sin\theta = A_y \Rightarrow \boxed{A_y = A\sin\theta}$$

and

$$\cos\theta = \frac{B}{H}$$

$$= \frac{|OM|}{|OP|}$$

$$\cos\theta = \frac{A_x}{A}$$

$$A\cos\theta = A_x \Rightarrow \boxed{A_x = A\cos\theta}$$

According to Pythagorean's theorem:

$$(hyp)^2 = (base)^2 + (perp)^2$$

$$|OP|^2 = |OM|^2 + |MP|^2$$

$$A^2 = A_x^2 + A_y^2$$

$$\sqrt{A^2} = \sqrt{A_x^2 + A_y^2}$$

$$\boxed{A = \sqrt{A_x^2 + A_y^2}}$$

and

$\tan\theta = \frac{A_y}{A_x}$

$$\tan\theta = \frac{P}{B}$$

$\theta = \tan^{-1}$

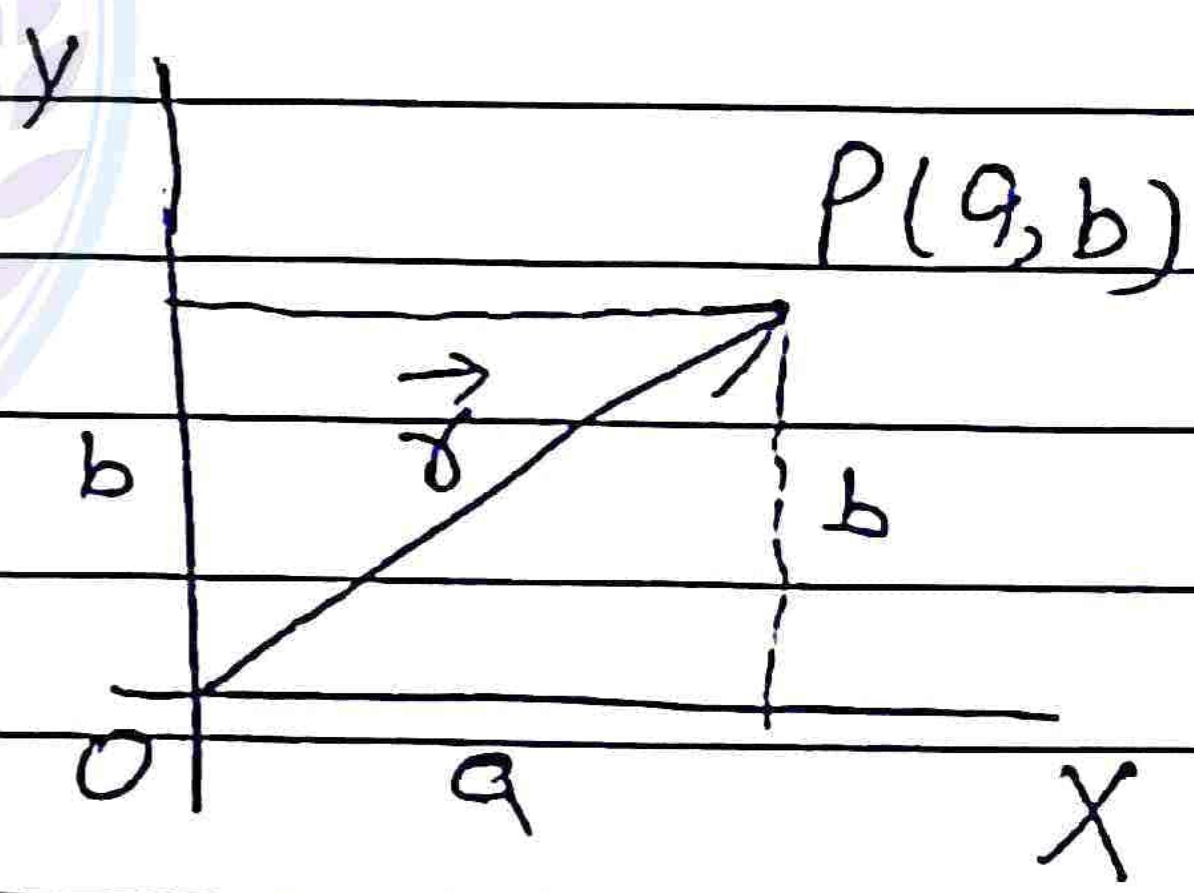
$$= \frac{|mP|}{|om|}$$

$$\tan\theta = \frac{A_y}{A_x}$$

$$\boxed{\theta = \tan^{-1}\frac{A_y}{A_x}}$$

Position vector:

The position vector $\vec{r}$ is a vector that describes the location of a point with respect to the origin.



Representation :-

It is represented by a straight line drawn in such a way that its tail conicides with origin and the head with point P(a,b).

$r = ai + bj$ and $r = \sqrt{a^2}$

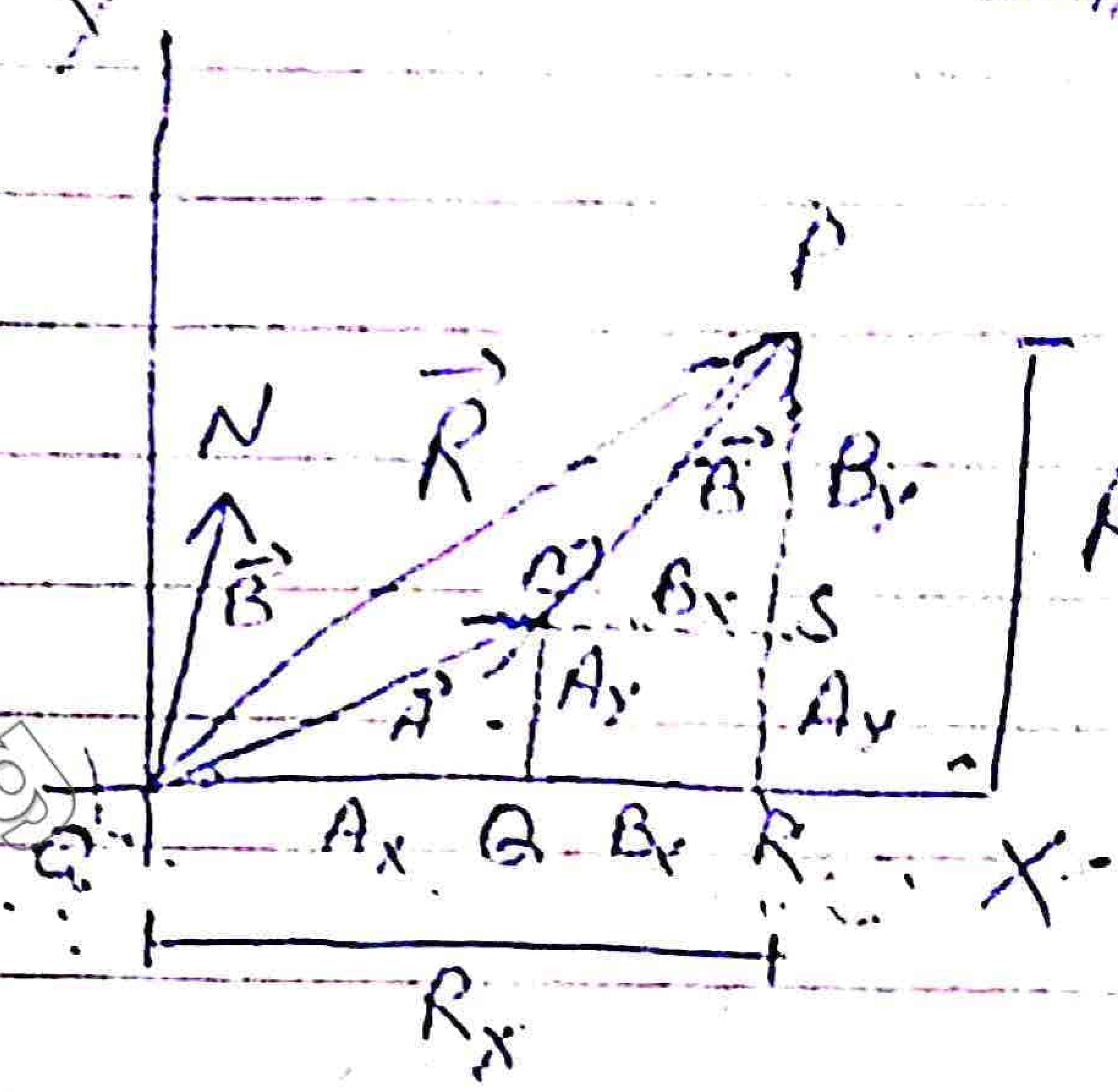
$\vec{r} = a\hat{i} + b\hat{j}$ and $r = \sqrt{a^2 + b^2}$ ....

$\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ and $r = \sqrt{a^2 + b^2 + c^2}$ ....

Vector Addition By Rectangular Components:

Rectangular components: The mutually perpendicular components of a vector is called its rectangular components.

Consider two vectors $\vec{A}$ and $\vec{B}$ represented by $|OM|$ and $|ON|$ respectively. We place the tail of vector $\vec{B}$ on the head of vector $\vec{A}$ and get the resultant $\vec{R}$ by head to tail rule. Now we resolve vectors $\vec{A}$, $\vec{B}$ and $\vec{R}$ into their rectangular components A_x, A_y, B_x, B_y and R_x, R_y as shown in the figure.



As

$$|OR| = |OQ| + |QR|$$

$$R_x = A_x + B_x$$

It means that sum of x-components of the vectors is equal to x-component of the resultant.

Similarly,

$$|RP| = |RS| + |SP|$$

$$R_y = A_y + B_y$$

It means that the sum of Y-components of the vectors is equal to the Y-component of the resultant.

R_x and R_y are the rectangular components of the resultant vector. So;

$$\vec{R} = R_x \hat{i} + R_y \hat{j}$$

so,

$$\vec{R} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$

The magnitude of resultant vector will be

$$\boxed{R = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}}$$

The angle made by resultant with x-axis is:

$$\theta = \tan^{-1} \frac{R_y}{R_x}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{A_y + B_y}{A_x + B_x}\right)}$$

For any number of coplaner vectors $\vec{A}, \vec{B}, \vec{C}, \vec{D}, ----$

$$R = \sqrt{(A_x + B_x + C_x + -----)^2 + (A_y + B_y + C_y + ----)^2}$$

The angle can be written as:

$$\theta = \tan^{-1}\left(\frac{A_y + B_y + C_y + \cdots}{A_x + B_x + C_x + \cdots}\right)$$

$\vec{R} = 2\hat{i} - 3\hat{j}$

IV

$\vec{R} = -3\hat{i} - 5\hat{j}$

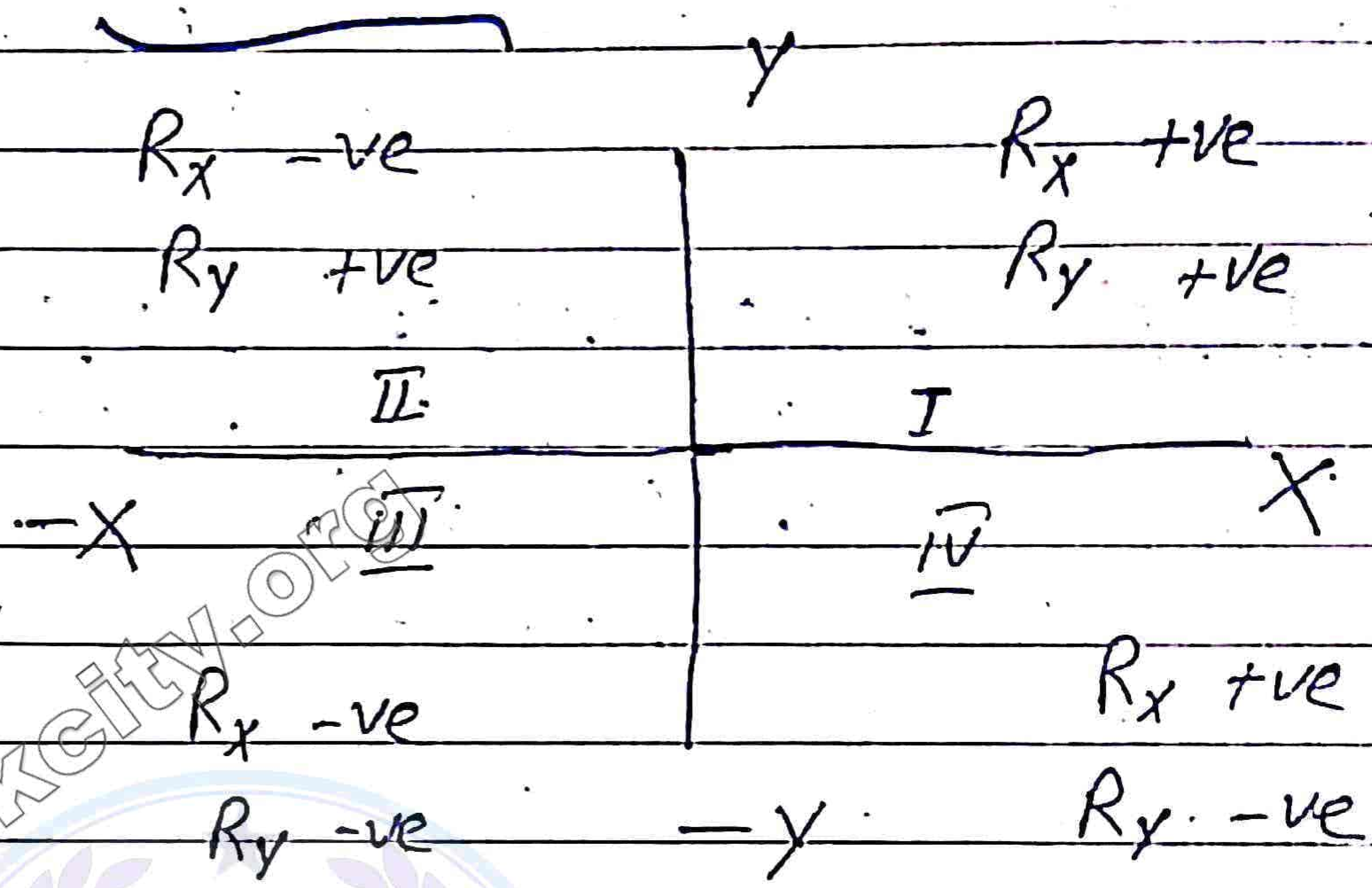
III

$\vec{R} = -3\hat{j} + 5\hat{i}$

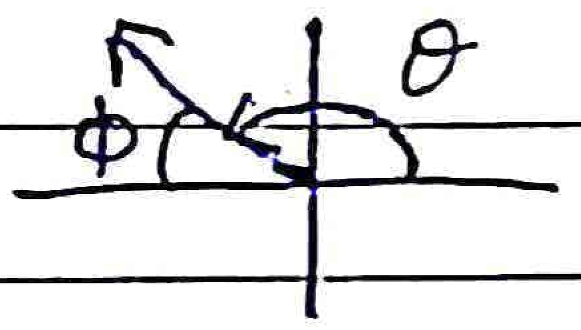
IV

$\vec{R} = 2\hat{i}$ (on x-axis)

$\vec{R} = -3\hat{j}$ (on -y-axis)

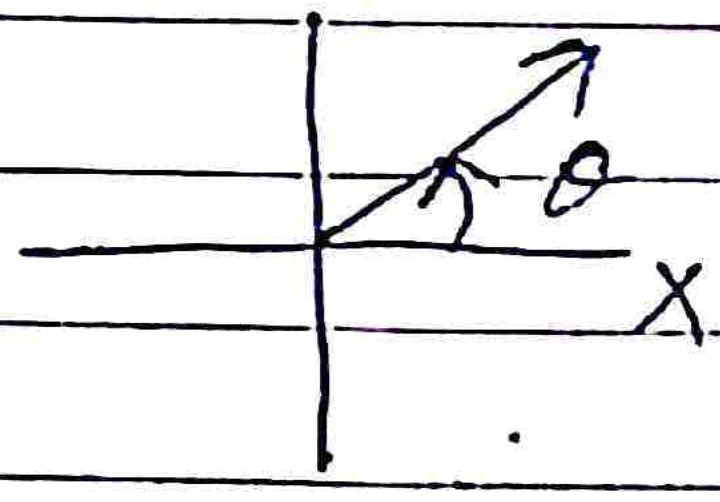


II quad



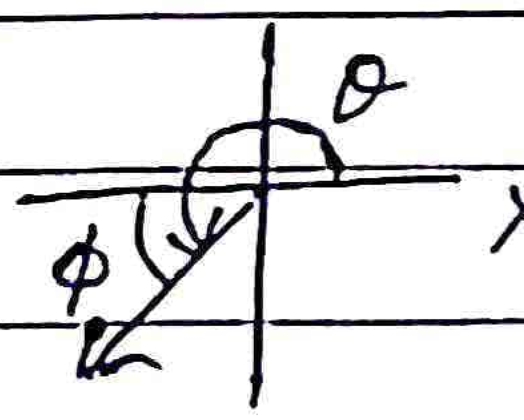
$\theta = 180^\circ - \phi$

I quad



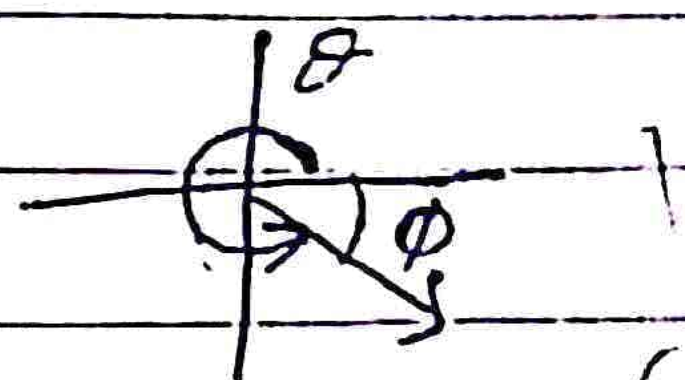
$\theta = \phi$

III quad



$\theta = 180^\circ + \phi$

IV quad



$\theta = 360^\circ - \phi$

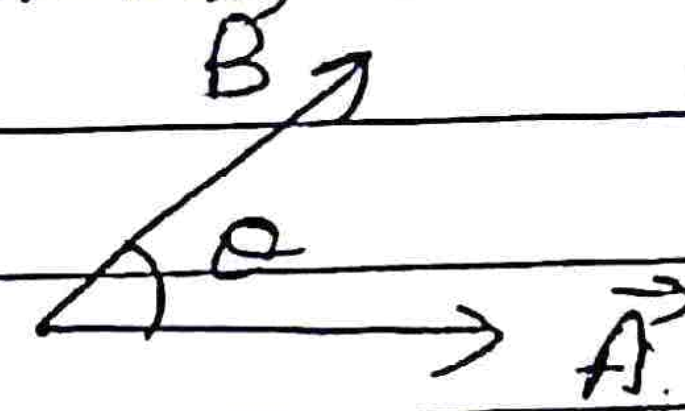
Product Of Two Vectors:

There are two methods to multiply two vectors.

i- Scalar product / Dot product
ii- vector product / Cross product

Scalar Product / Dot Product:

If the product of two vectors results into a scalar quantity then this product is called scalar product. It is also called dot product. For two vectors $\vec{A}$ and $\vec{B}$ their dot product is written as:



$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

Examples:

i- Work is the dot product of force $\vec{F}$ and displacement $\vec{d}$.

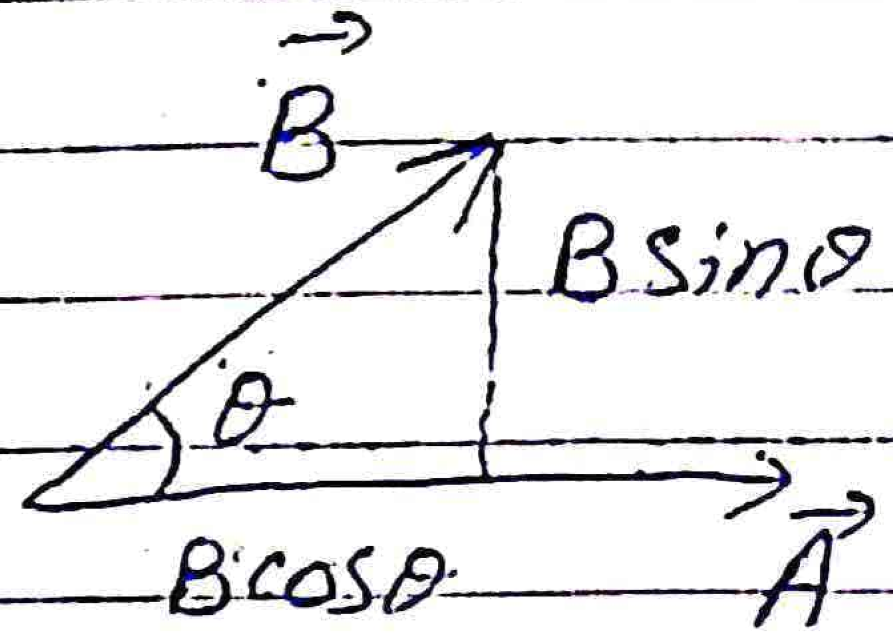
$$W = \vec{F} \cdot \vec{d}$$

ii- Power is the dot product of force $\vec{F}$ and velocity $\vec{v}$.

$$P = \vec{F} \cdot \vec{V}$$

Explanation:

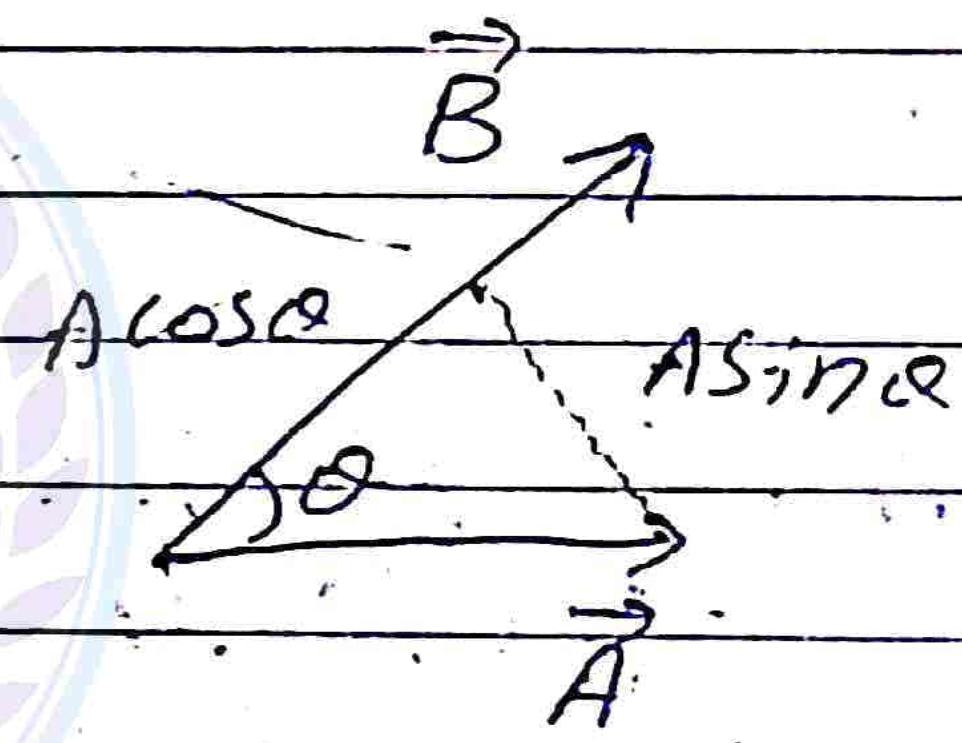
consider two vectors $\vec{A}$ and $\vec{B}$, θ is the angle between them. we draw a perpendicular from $\vec{B}$ to $\vec{A}$ and get the projection of $\vec{B}$ on $\vec{A}$. that is along the vector $\vec{A}$



$$\vec{A}\cdot\vec{B} = A\,(\text{projection of } \vec{B} \text{ on } \vec{A})$$

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

If we draw a perpendicular from $\vec{A}$ to $\vec{B}$ and get the projection of $\vec{A}$ along the vector $\vec{B}$. Now



$$\vec{B}\cdot\vec{A} = B\,(\text{projection of } \vec{A} \text{ on } \vec{B})$$

$$\vec{B}\cdot\vec{A} = BA\cos\theta$$

Characteristics:

Some characteristics of scalar product are given below:

i- Commutative Property:

For two vectors $\vec{A}$ and $\vec{B}$.

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

and

$$\vec{B} \cdot \vec{A} = BA\cos\theta$$

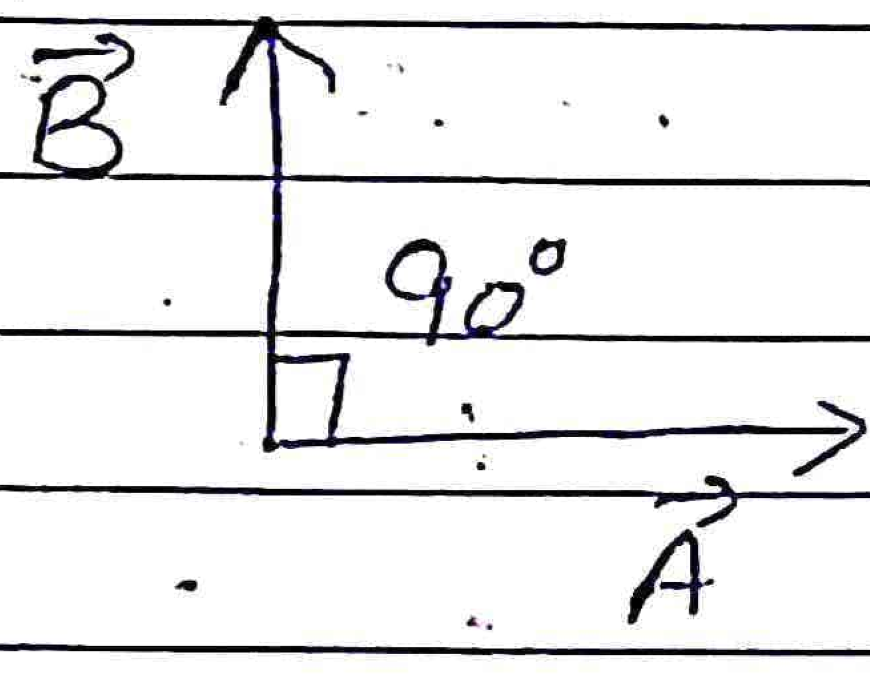
$$= AB\cos\theta$$

$$\vec{B} \cdot \vec{A} = \vec{A} \cdot \vec{B}$$

So, commutative property holds in scalar product.

ii- Scalar product of two perpendicular vectors:

For two perpendicular vectors $\vec{A}$ and $\vec{B}$, $\theta = 90°$



$$\vec{A} \cdot \vec{B} = AB\cos 90°$$

$$= AB(0)$$

$$\vec{A} \cdot \vec{B} = 0$$

For unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

$$\hat{i} \cdot \hat{j} = |\hat{i}||\hat{j}| \cos 90^\circ$$

$$= (1)(1)(0)$$

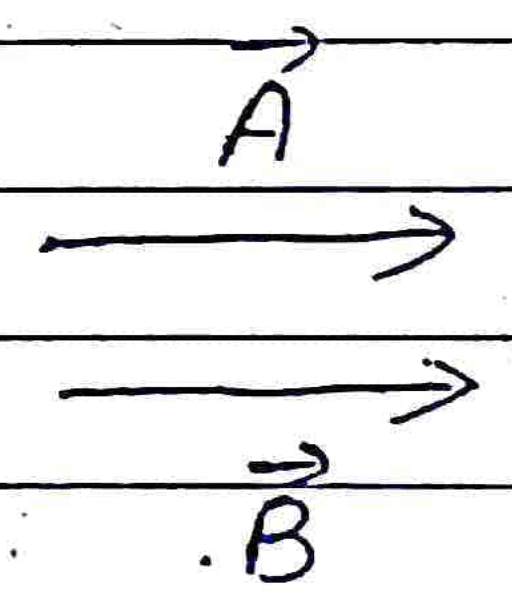
$$\hat{i} \cdot \hat{j} = 0$$

Similarly,

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

iii- Scalar product of two parallel vectors:

For two parallel vectors $\vec{A}$ and $\vec{B}$, $\theta = 0^\circ$



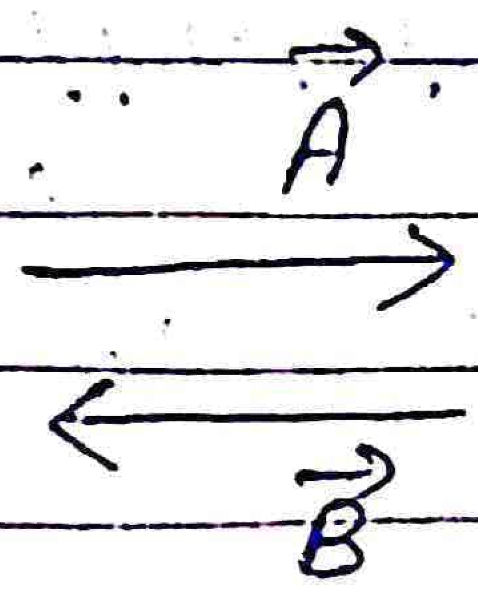
$$\vec{A} \cdot \vec{B} = AB \cos 0^\circ$$

$$= AB(1)$$

$$\vec{A} \cdot \vec{B} = AB$$

iv- Scalar product of two anti-parallel vectors:

For two anti-parallel vectors $\vec{A}$ and $\vec{B}$, $\theta = 180^\circ$



$$\vec{A} \cdot \vec{B} = AB \cos 180^\circ$$

$$= AB(-1)$$

$$\vec{A} \cdot \vec{B} = -AB$$

v- Self Scalar product:

If the scalar product of a vector is taken with itself. Then

$$\vec{A} \cdot \vec{B} = AB \cos\theta$$

here

$$\vec{A} \cdot \vec{A} = AA \cos 0^\circ$$

$$= A^2 (1)$$

$$\vec{A} \cdot \vec{A} = A^2$$

For unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

$$\hat{i} \cdot \hat{i} = |\hat{i}|\,|\hat{i}| \cos 0^\circ$$

$$= (1)(1)(1)$$

$$\hat{i} \cdot \hat{i} = 1$$

Similarly,

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

vi- Scalar product in terms of rectangular components:

The vectors $\vec{A}$ and $\vec{B}$ in terms of rectangular components will be:

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

and

$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$$\vec{A} \cdot \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

$$= A_x B_x (\hat{i} \cdot \hat{i}) + A_y B_y (\hat{j} \cdot \hat{j}) + A_z B_z (\hat{k} \cdot \hat{k})$$

$$AB \cos\theta = A_x B_x (1) + A_y B_y (1) + A_z B_z (1)$$

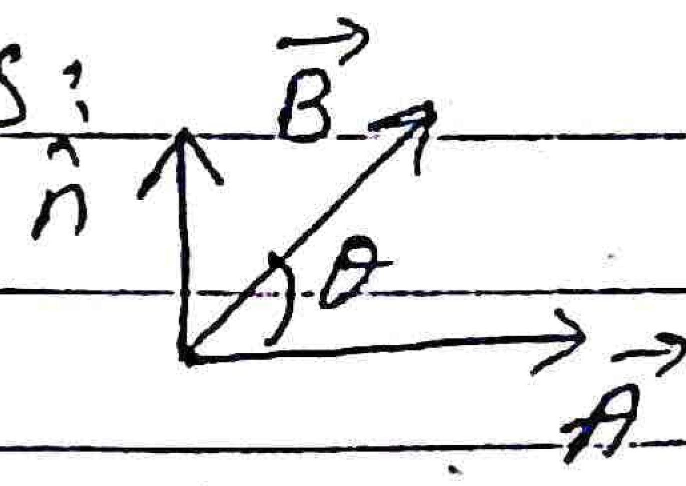
$$\left(\begin{array}{l} \because \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \\ \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0 \end{array} \right)$$

$$AB \cos\theta = A_x B_x + A_y B_y + A_z B_z$$

$$\cos\theta = \frac{A_x B_x + A_y B_y + A_z B_z}{AB}$$

Vector Product / Cross Product:

If the product of two vectors results into a vector quantity then this product is called vector product. It is also called cross product. For two vectors $\vec{A}$ and $\vec{B}$ vector product is written as:



$$\vec{A} \times \vec{B} = AB \sin\theta \, \hat{n}$$

Examples:

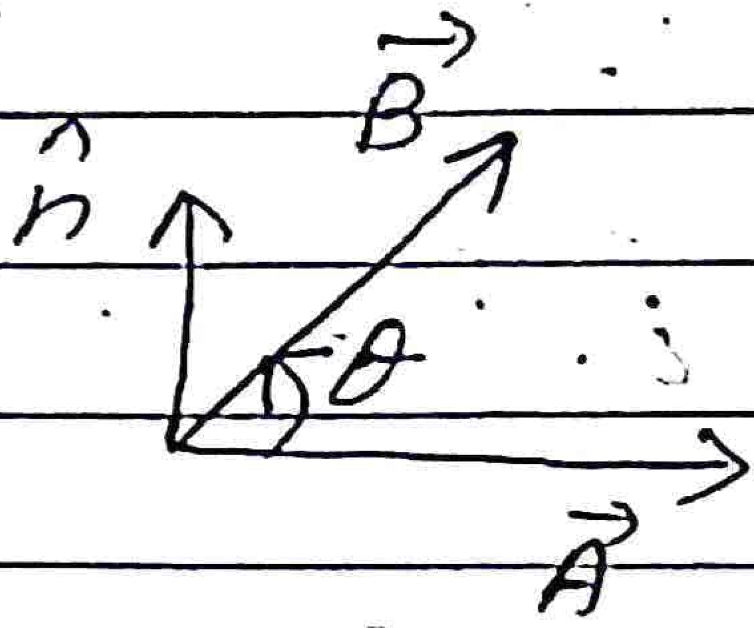
i- Torque $(\vec{\tau})$ is the cross product of moment arm $(\vec{r})$ and force $(\vec{F})$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Angular momentum $\vec{L}$ is the cross product of position vector $\vec{r}$ and linear momentum $\vec{P}$.

$$\vec{L} = \vec{r} \times \vec{P}$$

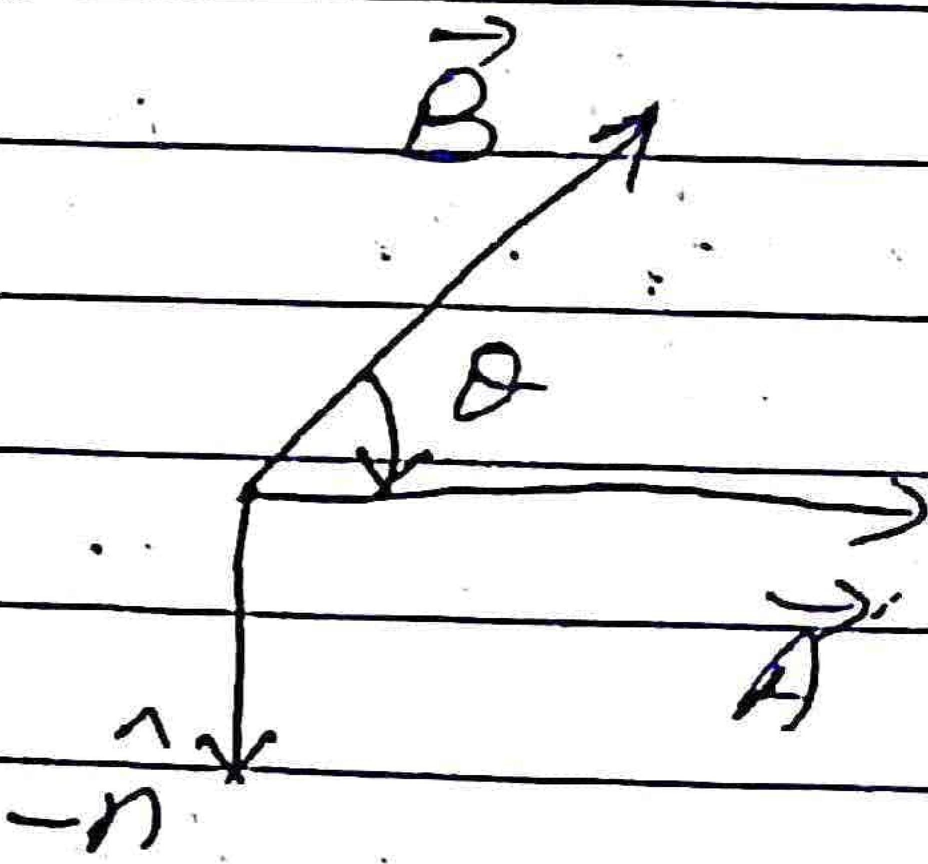
Direction of $\hat{n}$:



For the cross product θ is the angle between vectors $\vec{A}$ and $\vec{B}$ and $\hat{n}$ is the unit vector that shows the direction of $\vec{A} \times \vec{B}$.

Curl the fingers of right hand from $\vec{A}$ to $\vec{B}$ then erect thumb will show the direction of $\hat{n}$, that is directed outward.

$$\vec{A} \times \vec{B} = AB \sin\theta \ \hat{n}$$



If we find the direction of $\vec{B} \times \vec{A}$ from the same rule. it is inward.

$$\vec{B} \times \vec{A} = BA \sin\theta(-\hat{n})$$

Characteristics:

Some characteristics of vector product are given below:

i- commutative property:

For two vectors $\vec{A}$ and $\vec{B}$.

$$\vec{A} \times \vec{B} = AB \sin\theta \, \hat{n}$$

and

$$\vec{B} \times \vec{A} = BA \sin\theta \, (-\hat{n})$$

$$= -AB \sin\theta \, \hat{n}$$

$$\vec{B} \times \vec{A} = -(\vec{A} \times \vec{B})$$

So vector product does not hold commutative property.

ii- vector product of two perpendicular vectors:

For two perpendicular vectors $\vec{A}$ and $\vec{B}$, $\theta = 90^\circ$

$$\vec{A} \times \vec{B} = AB \sin 90^\circ \, \hat{n}$$

$$= AB (1) \, \hat{n}$$

$$\vec{A} \times \vec{B} = AB \, \hat{n}$$

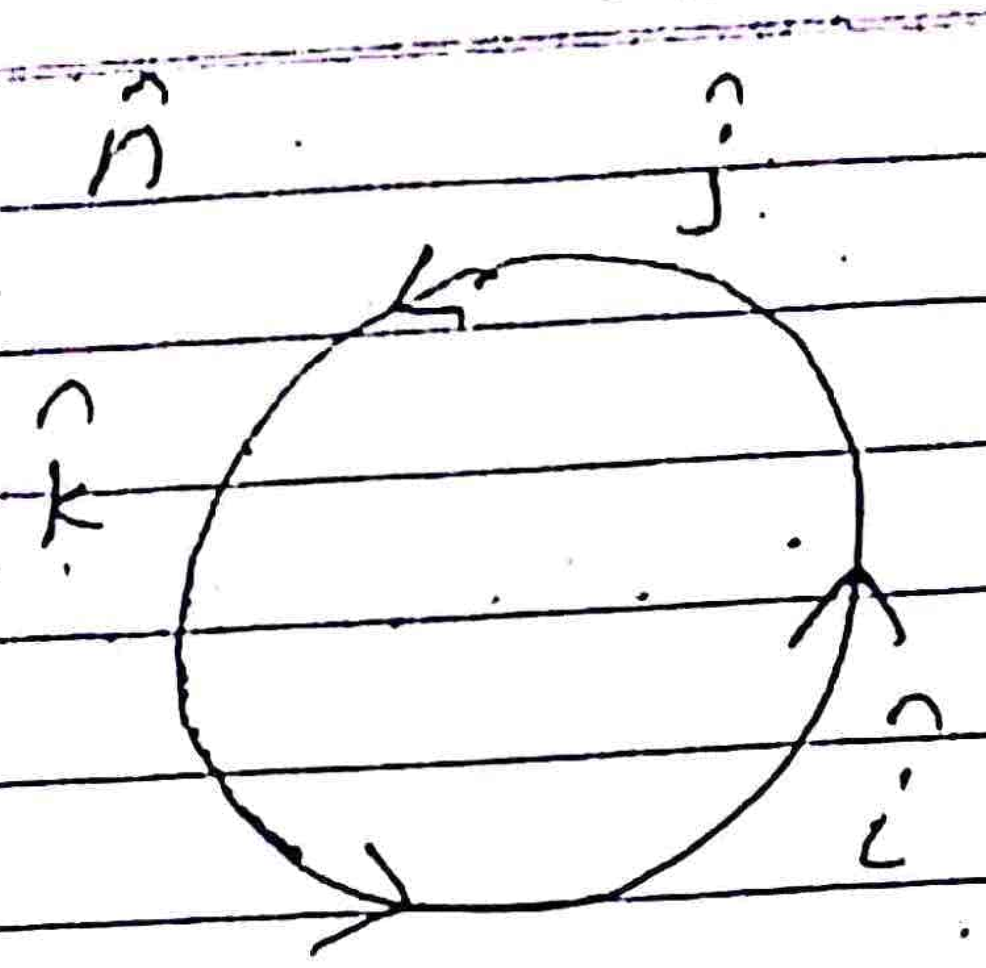
For unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

$$\hat{i} \times \hat{j} = |\hat{i}|\,|\hat{j}| \sin 90^\circ \, \hat{n}$$

$$= (1)(1)(1)\, \hat{n}$$

$$= \hat{n}$$

$$\hat{i} \times \hat{j} = \hat{k}$$



Similarly,

$\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$

and

$\hat{j} \times \hat{i} = -\hat{k}$, $\hat{k} \times \hat{j} = -\hat{i}$, $\hat{i} \times \hat{k} = -\hat{j}$

iii- Vector product of two parallel vectors:

For two parallel vectors $\vec{A}$ and $\vec{B}$, $\theta = 0^\circ$

$$\vec{A} \times \vec{B} = AB \sin 0^\circ \, \hat{n}$$

$$= AB(0)\, \hat{n}$$

$$\vec{A} \times \vec{B} = 0$$

iv- Vector product of two anti-parallel vectors:

For two anti-parallel vecto[rs] $\vec{A}$ and $\vec{B}$, $\theta = 180^\circ$

$$\vec{A} \times \vec{B} = AB \sin 180^\circ \, \hat{n}$$

$$= AB(0)\, \hat{n}$$

$$\vec{A} \times \vec{B} = 0$$

v- Self Vector Product:

If the vector product of a vector is taken with itself:

Then

$$\vec{A} \times \vec{B} = AB \sin\theta \, \hat{n}$$

here

$$\vec{A} \times \vec{A} = AA \sin 0^\circ \, \hat{n}$$

$$= A^2 (0) \, \hat{n}$$

$$\vec{A} \times \vec{A} = 0$$

For unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

$$\hat{i} \times \hat{i} = |\hat{i}||\hat{i}| \sin 0^\circ \, \hat{n}$$

$$= (1)(1)(0) \, \hat{n}$$

$$\hat{i} \times \hat{i} = 0$$

Similarly,

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

vi- Vector Product In Terms Of Rectangular Components:

The vectors $\vec{A}$ and $\vec{B}$ in terms of rectangular components will be

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

and

$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{j} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

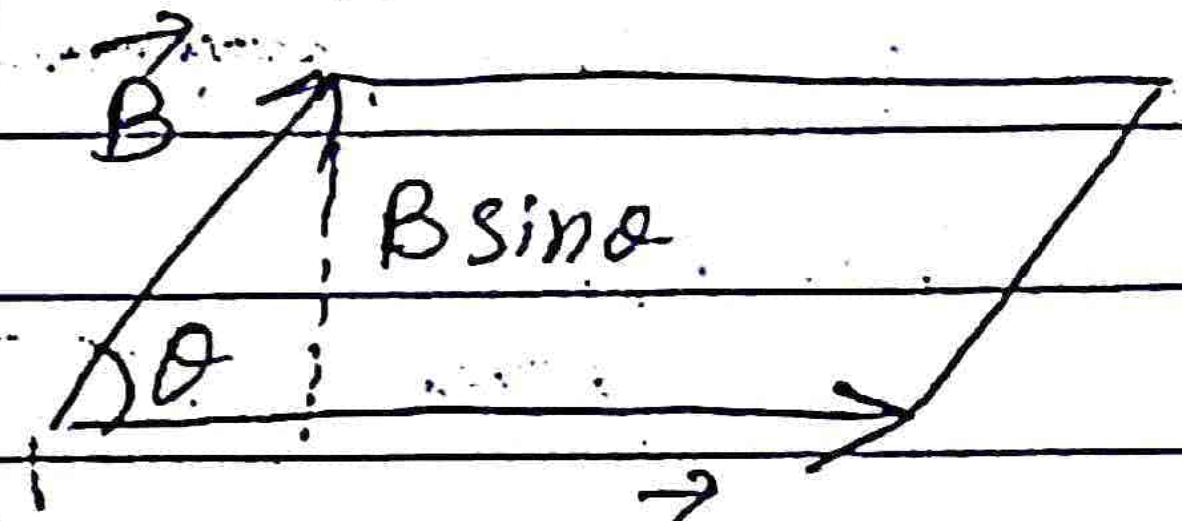
$$= \hat{i}(A_y B_z - A_z B_y) - \hat{j}(A_x B_z - A_z B_x) + \hat{k}(A_x B_y - A_y B_x)$$

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$

vii- Area Of Parallelogram:

The four sided shape whose opposite sides are parallel is called parallelogram.

If the two sides of parallelogram shows the two vectors $\vec{A}$ and $\vec{B}$, then the magnitude of $\vec{A}$ cross product shows the area.



Area = (length)(width)

$= (A)(B\sin\theta)$

Area $= |\vec{A} \times \vec{B}|$

Torque:

The turning effect of force is called torque.

OR

The cross product of moment arm $\vec{r}$ and force $\vec{F}$ is called torque.

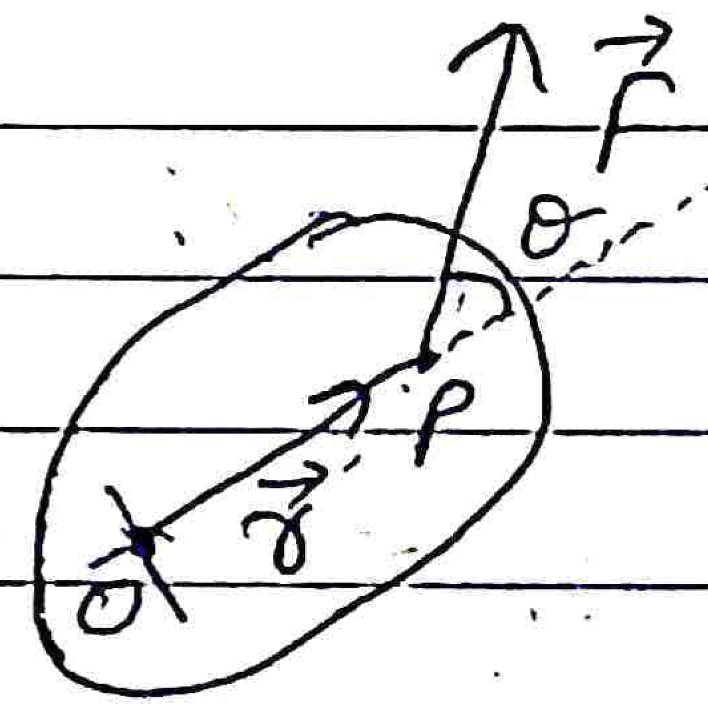
mathematically,

$$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\theta\,\hat{n}$$

Torque is a vector quantity. Its SI unit is (Nm).

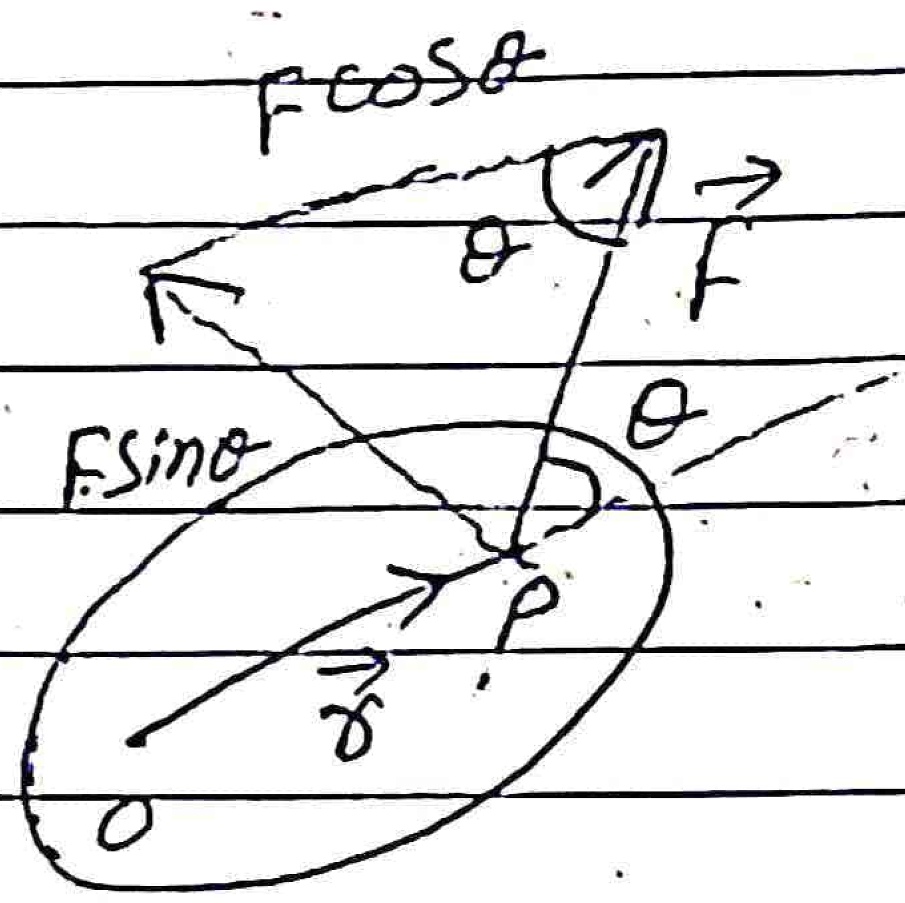
Direction: we can find the direction of torque by right hand rule. Curl the fingers of right hand from $\vec{r}$ to $\vec{F}$ then erect thumb will show the direction of torque.

Explanation:



Consider a rigid body whose point O is axis of rotation about which the body rotates. while point P lies on the line of action of force. The angle between applied force and ~~the~~ ~~of~~ moment arm is θ. Moment arm is the distance between axis of rotation and line of action of force. Now we will resolve force and moment arm to get the formula for torque.

Resolution of force:



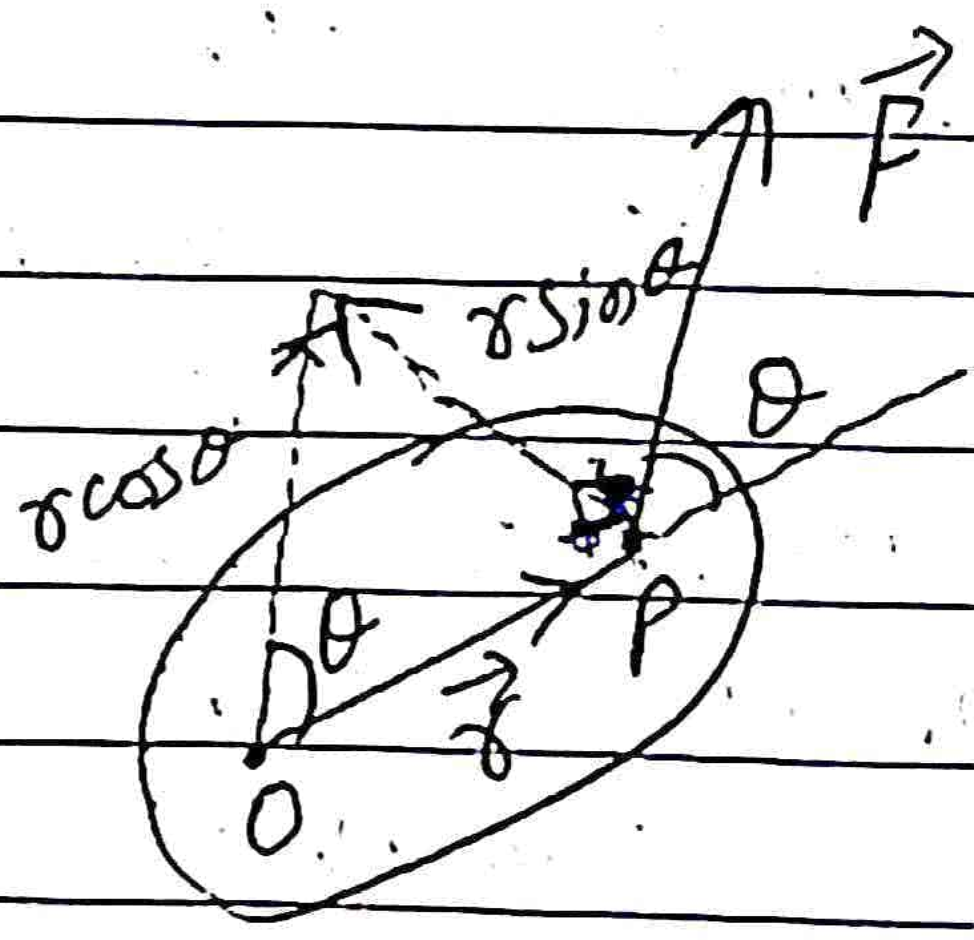
We resolve the force F into its rectangular components Fsinθ and Fcosθ. Here Fsinθ acts along the direction of rotation of the body and it is effective component while Fcosθ is uneffective.

$$\text{Torque} = (\text{moment arm})(\text{component of force})$$

$$\tau = r(F\sin\theta)$$

$$\tau = rF\sin\theta$$

Resolution of moment arm:



We resolve the moment arm into its rectangular components rsinθ and rcosθ. Here rsinθ acts along the direction of rotation of the body and it is effective component. while rcosθ is uneffective component because it passes through the axis of rotation. So,

$$\text{torque} = \begin{pmatrix}\text{component of} \\ \text{moment arm}\end{pmatrix}(\text{force})$$

$$\tau = (r\sin\theta)(F)$$

$$\tau = rF\sin\theta$$

Vector form;

$$\vec{\tau} = rF\sin\theta\,\hat{n}$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Here $\hat{n}$ is the unit vector that shows the direction of torque.

Comparison of force and torque:

Force produces linear motion and linear acceleration in the body while torque produces angular motion and angular acceleration.

Equilibrium of forces:-

If a body, under the action of a number of forces, is at rest or moving with uniform velocity, it is said to be in equilibrium.

First condition of Equilibrium: A body at rest or moving with uniform velocity has zero acceleration. From Newton's law of motion the vector sum of all force acting on it must be zero.

This is known as first condition of equilibrium. Using the mathematical symbol Σ for the sum of all forces we can write.

$$\Sigma F = 0$$
$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$

It may be noted that if the rightward forces are taken as positive then leftward forces are negative. Similarly if upward forces are taken as positive then downward forces are negative.

Equilibrium of Torques

Second Condition of equilibrium:-

For a body in equilibrium, the vector sum of all the torques acting on it about any arbitrary axis should be zero. This is known as second condition of equilibrium.

Mathematically it is written as:

$$\Sigma \tau = 0$$

By convention, the counter clockwise torque are taken as positive and clockwise torques as negative.

↳ **Translational equilibrium:-** When 1st condition is satisfied, there is no linear acceleration and body will be in translational equilibrium.

↳ **Rotational equilibrium:-** When 2nd condition is satisfied, there is no angular acceleration and body will be in rotational equilibrium.

↳ **Static equilibrium:-** If body is at rest, it is said to be in static equilibrium.

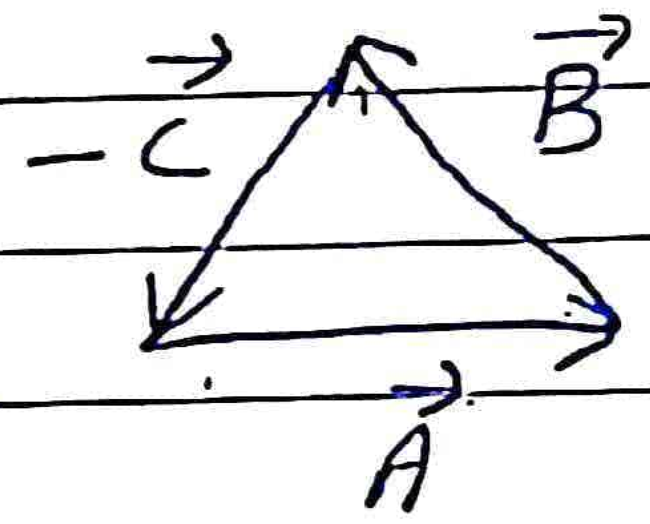
↳ **Dynamic equilibrium:-** If body is moving with uniform velocity or rotating with uniform angular velocity, it is said to be in dynamic equilibrium

↳ **Coplanar:** All the forces which lie in a common plane are said to be coplanar. We will also assume that these forces lie in the xy-plane.

Short Questions

2.2: If the three vectors act along the sides of a triangle in a cyclic order then the vector sum of these vectors will give a zero resultant.

For three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$



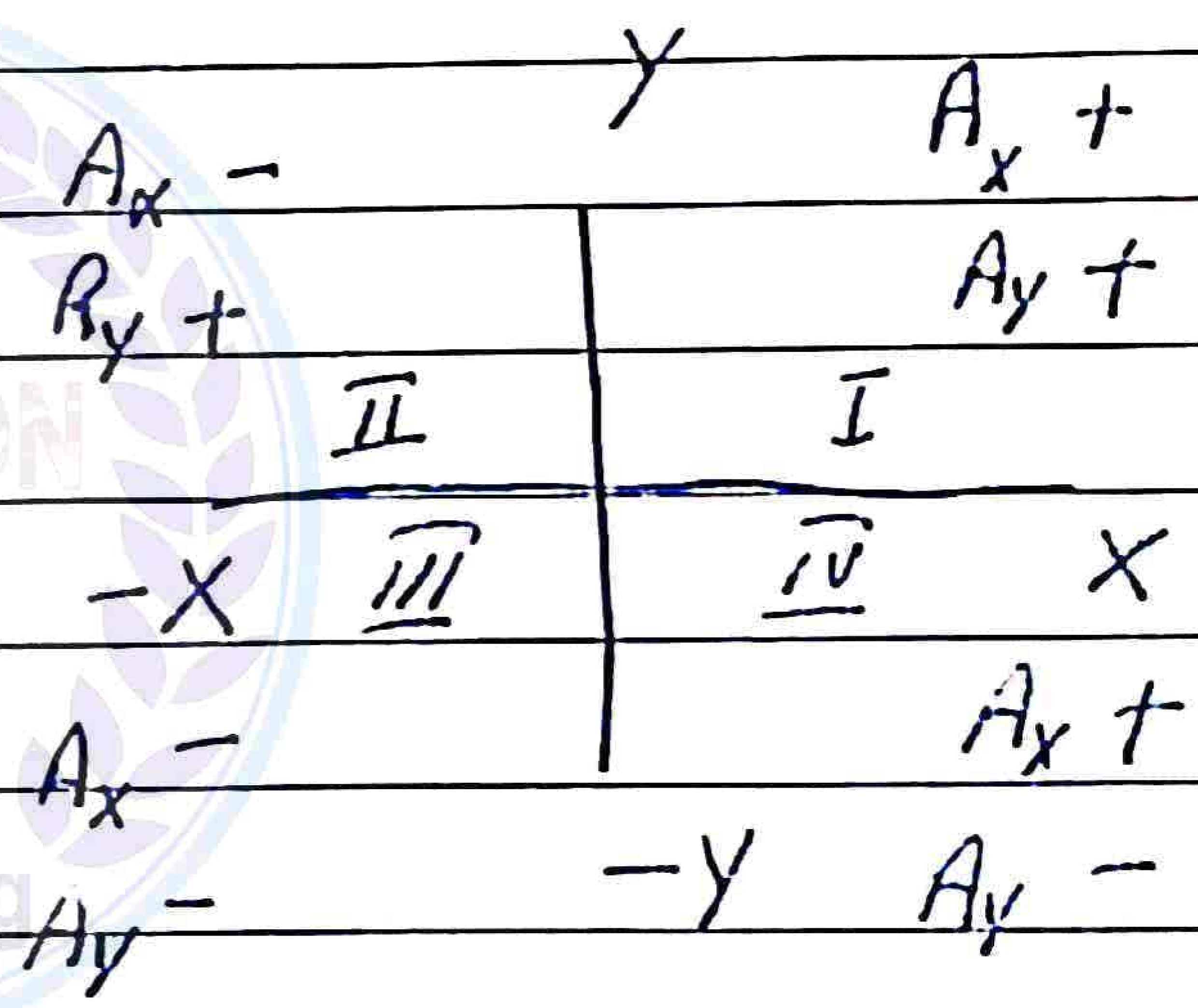
$$\vec{A} + \vec{B} = -\vec{C}$$

$$\vec{A} + \vec{B} + \vec{C} = 0$$

2.3:

In the III quadrant, both of the rectangular components will be negative.

For II and IV quadrants the components have opposite signs.



2.4: NO; the magnitude can not be zero if one of the rectangular components of a vector is not zero. For example, for vector $\vec{A}$

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

Its magnitude will be

$$A = \sqrt{A_x^2 + A_y^2}$$

If $A_x = 0$
then $A = \sqrt{0 + A_y^2} = \sqrt{A_y^2}$

$$A = A_y \neq 0$$

if $A_y = 0$
then
$A = \sqrt{A_x^2 + 0} \Rightarrow A = A_x \neq 0$

2.5: No, a vector can not have a component greater than the vector's magnitude. Because component of a vector is the part of that vector. For example

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

its magnitude is:

$$A = \sqrt{A_x^2 + A_y^2}$$

$$A \geq A_x \quad \text{and} \quad A \geq A_y$$

2.6: No, the magnitude of a vector can not have a negative value. Because the square of a negative number will also become positive. For vector

$$\vec{B} = A_x\hat{i} + A_y\hat{j}$$

Its magnitude is

$$A = \sqrt{A_x^2 + A_y^2}$$

2.7:

$$\vec{A} + \vec{B} = \vec{0}$$

$$A_x\hat{i} + A_y\hat{j} + B_x\hat{i} + B_y\hat{j} = 0\hat{i} + 0\hat{j}$$

$$(A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} = 0\hat{i} + 0\hat{j}$$

comparing both sides:

$$A_x + B_x = 0 \quad ; \quad A_y + B_y = 0$$

$$A_x = -B_x \quad ; \quad A_y = -B_y$$

So, the components of the two vectors are equal and opposite.

2.8: A vector would have components that are equal in magnitude for an angle $\boxed{45^\circ}$. For example, vector $\vec{A}$ have components A_x and A_y.

$$A_x = A\cos\theta \quad , \quad A_y = A\sin\theta$$

For $\theta = 45^\circ$

$$A_x = A\cos 45^\circ \qquad A_y = A\sin 45^\circ$$

$$A_x = A(0.707) \qquad A_y = A(0.707)$$

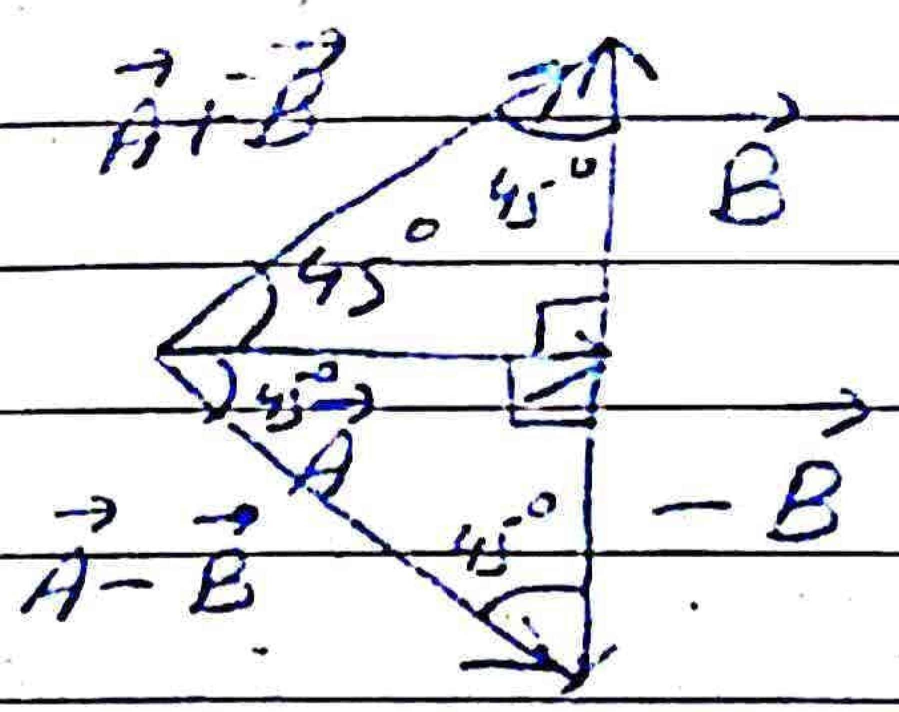
2.9: No, it is not possible to add a vector quantity to a scalar quantity. Because these are two different quantities. Scalars are described by magnitude only while vectors are described by magnitude and direction. Scalars are added by simple math rules while vectors are added by head to tail rule.

2.10: No, we can not add zero to a null vector. Because ~~scalar~~ zero is a scalar quantity while null vector is a vector quantity. Scalars are described by magnitude while vectors are described by magnitude and direction.

2.11: No, the vector sum can not be zero if two vectors have unequal magnitudes. The sum can only be zero if the vectors are equal in magnitude and opposite in direction. mathematically

$$\vec{A} + (-\vec{A}) = 0$$

2.12: Consider two perpendicular vectors of equal length $\vec{A}$ and $\vec{B}$. We get their sum $\vec{A}+\vec{B}$ and difference $\vec{A}-\vec{B}$ by head to tail rule. From the figure we can see that $\vec{A}+\vec{B}$ and $\vec{A}-\vec{B}$ are perpendicular to each other.



now

$$|\vec{A}+\vec{B}| = \sqrt{A^2+B^2}$$

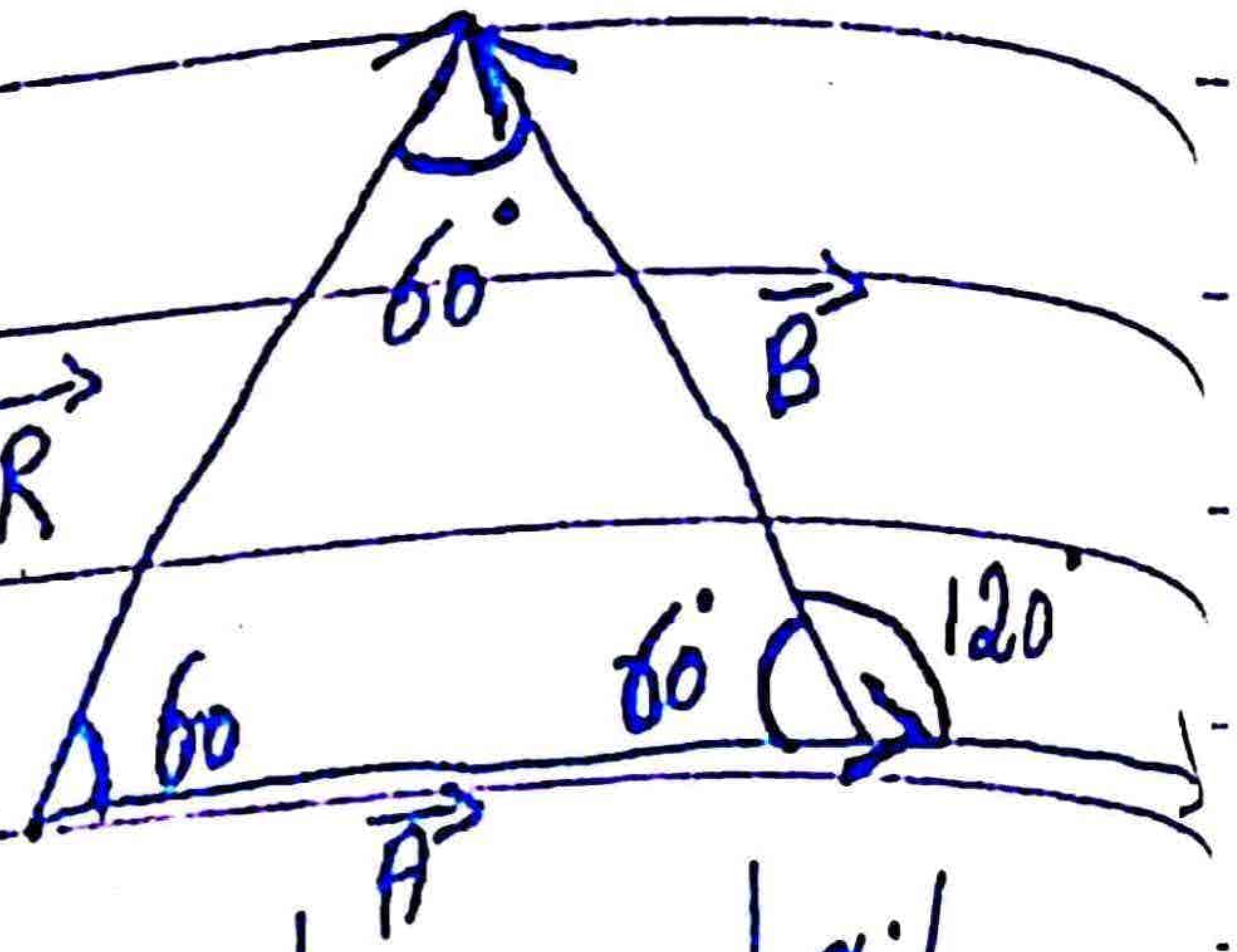
$$|\vec{A}-\vec{B}| = \sqrt{A^2+(-B)^2}$$

$$= \sqrt{A^2+B^2}$$

Therefore, the sum and difference of two perpendicular vectors of equal lengths are also perpendicular and of same length.

length.

Ans 2.13



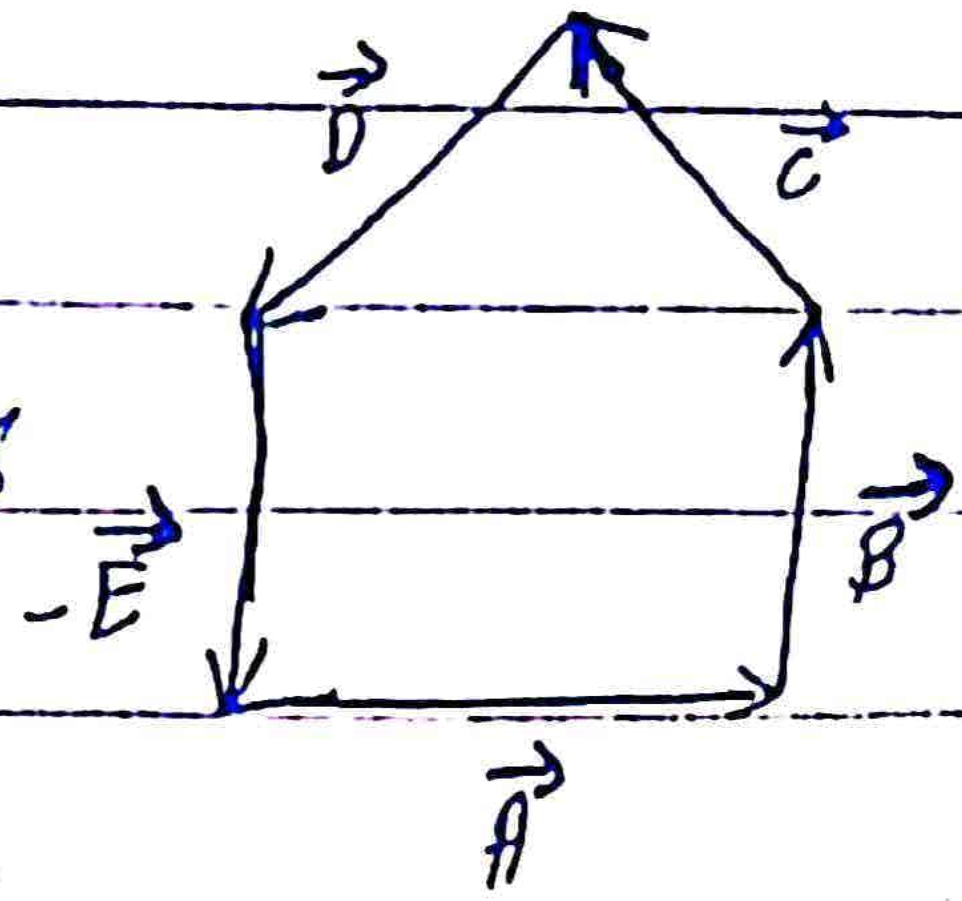
* We have two equal vectors $\vec{A}$ and $\vec{B}$ and according to head to tail rule we have to make resultant R also having same magnitude.

$$|\vec{A}| = |\vec{B}| = |\vec{R}|$$

In this way there will be an equilateral triangle so, angle between vectors $\vec{A}$ and $\vec{B}$ will be:

$$180^\circ - 60^\circ = 120^\circ$$

(Ans 2.15)



* Consider a close polygon whose sides represents vectors $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ and $\vec{E}$.

According to head to tail rule:

$$\vec{A} + \vec{B} + \vec{C} + \vec{D} = -\vec{E}$$

$$\vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E} = 0$$

Therefore, sum of the vectors is equal to zero.

(Ans 2.17)

* There will be no change in $\vec{A}_1 \times \vec{A}_2$. If all the components of vectors $\vec{A}_1$ and $\vec{A}_2$ are reversed.

Mathematically:

$$\vec{A}_1 \times \vec{A}_2 = (-\vec{A}_1) \times (-\vec{A}_2)$$

$$\vec{A}_1 \times \vec{A}_2 = \vec{A}_1 \times \vec{A}_2$$

(Ans 2.18)

* $\vec{A_1} \times \vec{A_2} = 0$

$A_1 A_2 \sin\theta \, \hat{n} = 0$

This can be possible is:

i) $\vec{A_1}$ is null vector

ii) $\vec{A_2}$ is null vector

iii) $\vec{A_1}$ and $\vec{A_2}$ are parallel

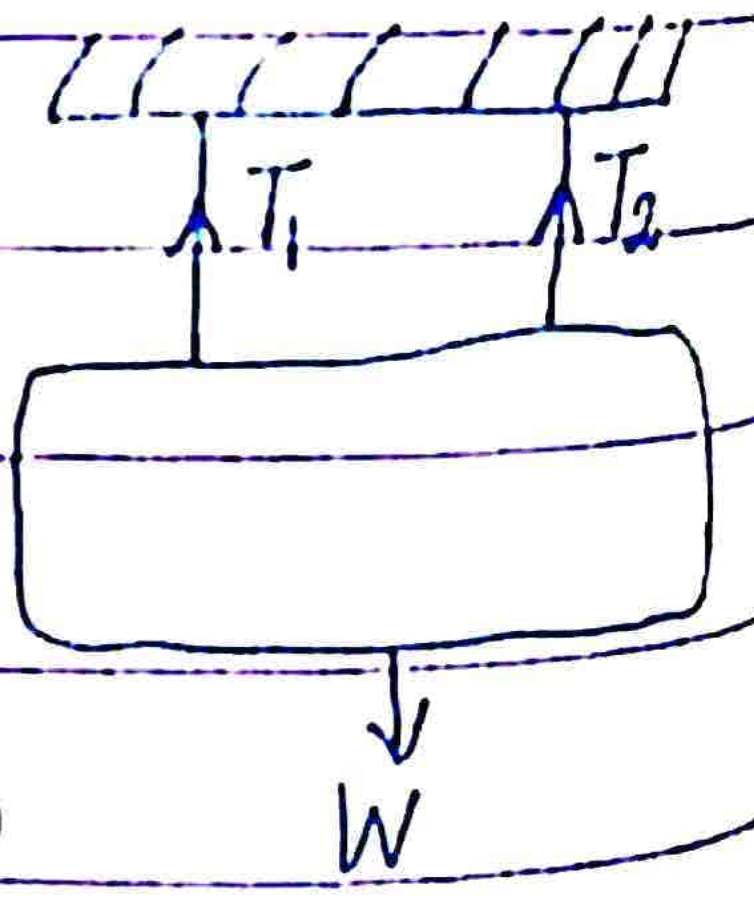
$\sin 0 = 0$

iv) $\vec{A_1}$ and $\vec{A_2}$ are anti-parallel

$\sin 180^\circ = 0$

(Ans 2.20)

* If the strings are attached vertically with the picture.



Then the tension in strings will be minimum. Sum of tension in strings will be equal to the weight of picture.

$T_1 + T_2 = W$

(Ans 2.21)

* No, a body cannot rotate about its centre of gravity under the action of its weight. Because moment arm will become zero in this case that makes torque zero

$$\tau = r \times F \sin\theta$$

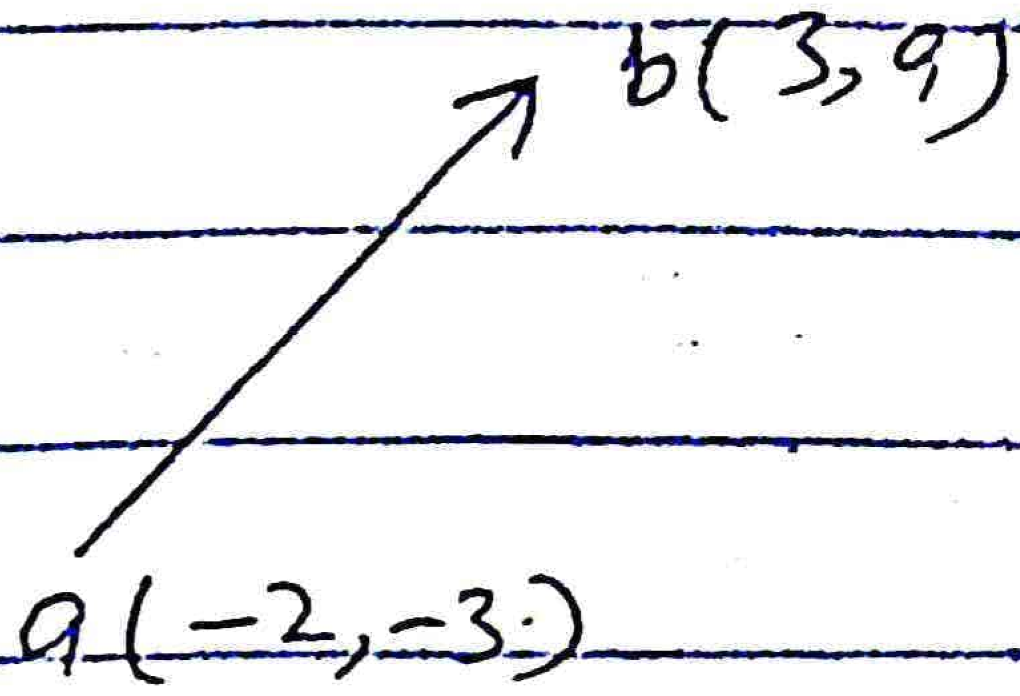
Here, $r = 0$

$$\tau = (0) F \sin\theta$$

$$\tau = 0.$$

Numerical Problems

2.1 Data



$a(-2,-3)$

$\vec{oa} = -2\hat{i} - 3\hat{j}$

$b(3,9)$

$\vec{ob} = 3\hat{i} + 9\hat{j}$

$\vec{ab} = \vec{ob} - \vec{oa}$

$= 3\hat{i} + 9\hat{j} - (-2\hat{i} - 3\hat{j})$

$= 3\hat{i} + 9\hat{j} + 2\hat{i} + 3\hat{j}$

$\vec{ab} = 5\hat{i} + 12\hat{j}$

$|\vec{ab}| = ?$

Solution: $\vec{ab} = 5\hat{i} + 12\hat{j}$

$$|\vec{ab}| = \sqrt{(5)^2 + (12)^2}$$

$$= \sqrt{25 + 144}$$

$$= \sqrt{169}$$

$$|\vec{ab}| = 13 \text{ units}$$

2.2 Data

$r(2,1)$

$\vec{Or} = 2\hat{i} + \hat{j}$

$|\vec{Or}| = ?$

Solution

$\vec{Or} = 2\hat{i} + \hat{j}$

$|\vec{Or}| = \sqrt{(2)^2 + (1)^2}$

$= \sqrt{4+1}$

$= \sqrt{5}$

$\boxed{|\vec{Or}| = 2.2m}$

2.3 Data

$\hat{A} = ?$, $\vec{A} = 4\hat{i} + 3\hat{j}$

Solution:

$\hat{A} = \frac{\vec{A}}{A}$

$\vec{A} = 4\hat{i} + 3\hat{j}$

$|\vec{A}| = A = \sqrt{(4)^2 + (3)^2}$

$$A = \sqrt{16+9} = \sqrt{25}$$

$$A = 5$$

So,

$$\boxed{\hat{A} = \frac{4\hat{i}+3\hat{j}}{5}}$$

2.4 Data

$$\vec{r}_1 = 3\hat{i}+7\hat{j} \quad , \quad \vec{r}_2 = -2\hat{i}+3\hat{j}$$

$$|\vec{r}_2 - \vec{r}_1| = ? , \quad \theta = ?$$

Solution:

$$\vec{r}_2 - \vec{r}_1 = -2\hat{i}+3\hat{j} - (3\hat{i}+7\hat{j})$$

$$= -2\hat{i}+3\hat{j}-3\hat{i}-7\hat{j}$$

$$\vec{r}_2 - \vec{r}_1 = -5\hat{i} - 4\hat{j}$$

$$|\vec{r}_2 - \vec{r}_1| = \sqrt{(-5)^2+(-4)^2}$$

$$= \sqrt{25+16}$$

$$= \sqrt{41}$$

$$\boxed{|\vec{r}_2 - \vec{r}_1| = 6.4}$$

Date: / /20

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\left(\frac{4}{5}\right)$$

$$\phi = 39^\circ$$

As the vector lies in III quadrant so,

$$\theta = 180^\circ + \phi$$

$$= 180^\circ + 39^\circ$$

$$\boxed{\theta = 219^\circ}$$

2.5 Data

$$\vec{A} + \vec{B} = 6\hat{i} + \hat{j}$$

$$\vec{A} - \vec{B} = -4\hat{i} + 7\hat{j}$$

$$|\vec{A}| = ?$$

Sol:

$$\vec{A} + \cancel{\vec{B}} = 6\hat{i} + \hat{j}$$

$$\vec{A} - \cancel{\vec{B}} = -4\hat{i} + 7\hat{j}$$

$$2\vec{A} = 2\hat{i} + 8\hat{j}$$

$$\cancel{2}\vec{A} = \cancel{2}(\hat{i} + 4\hat{j})$$

$$\vec{A} = \hat{i} + 4\hat{j}$$

$$|\vec{A}| = \sqrt{(1)^2 + (4)^2}$$

$$= \sqrt{1+16} = \sqrt{17}$$

$$\boxed{A = 4.1}$$

2.6 Data

$\vec{A} = 2\hat{i} + 3\hat{j}$, $\vec{B} = 3\hat{i} - 4\hat{j}$

a) $\vec{C} = \vec{A} + \vec{B}$

$|\vec{C}| = ?$, $\theta = ?$

b) $\vec{D} = 3\vec{A} - 2\vec{B}$

$|\vec{D}| = ?$, $\theta = ?$

Solution:

a) $\vec{C} = \vec{A} + \vec{B}$

$$\vec{C} = 2\hat{i} + 3\hat{j} + 3\hat{i} - 4\hat{j}$$

$$\vec{C} = 5\hat{i} - \hat{j}$$

$$|\vec{C}| = \sqrt{(5)^2 + (-1)^2}$$

$$= \sqrt{25+1} = \sqrt{26}$$

$$\boxed{C = 5.1}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\left(\frac{1}{5}\right)$$

$$\phi = 11^\circ$$

As vector lies in IV quad.

$$\theta = 360^\circ - \phi$$

$$= 360^\circ - 11^\circ$$

$$\boxed{\theta = 349^\circ}$$

b) $\vec{D} = 3\vec{A} - 2\vec{B}$

$$\vec{D} = 3(2\hat{i} + 3\hat{j}) - 2(3\hat{i} - 4\hat{j})$$

$$= \cancel{6\hat{i}} + 9\hat{j} - \cancel{6\hat{i}} + 8\hat{j}$$

$$\vec{D} = 0\hat{i} + 17\hat{j}$$

$|\vec{D}| = \sqrt{(0)^2 + (17)^2}$

$= \sqrt{(17)^2}$

$\boxed{D = 17}$

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$

$= \tan^{-1}\left(\frac{17}{0}\right)$

$= \tan^{-1}(\infty)$

$\phi = 90^\circ$

As vector lies in Ist quad.

$\theta = \phi$

$\boxed{\theta = 90^\circ}$

2.7 Data

$\vec{A} = 5\hat{i} + \hat{j}$, $\vec{B} = 2\hat{i} + 4\hat{j}$

$\theta = ?$

Sol:

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

$$\frac{\vec{A} \cdot \vec{B}}{AB} = \cos\theta$$

$$\vec{A} \cdot \vec{B} = (5\hat{i} + \hat{j}) \cdot (2\hat{i} + 4\hat{j})$$

$$= (5)(2) + (1)(4)$$

$$= 10 + 4$$

$$\vec{A} \cdot \vec{B} = 14$$

$$\vec{A} = 5\hat{i} + \hat{j}$$

$$A = \sqrt{(5)^2 + (1)^2} = \sqrt{25+1}$$

$$= \sqrt{26}$$

$$A = 5.1$$

$$\vec{B} = 2\hat{i} + 4\hat{j}$$

$$B = \sqrt{(2)^2 + (4)^2} = \sqrt{4+16} = \sqrt{20}$$

$B = 4.5$

Now

$$\cos\theta = \frac{14}{(5.1)(4.5)}$$

$$\cos\theta = 0.61$$

$$\theta = \cos^{-1}(0.61)$$

$$\boxed{\theta = 52^\circ}$$

2.8 Data

work done $= W = ?$

force $= \vec{F} = 3\hat{i} + 2\hat{j}$

$a(2, -1)$, $b(6, 4)$

$\vec{oa} = 2\hat{i} - \hat{j}$, $\vec{ob} = 6\hat{i} + 4\hat{j}$

displacement $= \vec{d} = \vec{ab} = \vec{ob} - \vec{oa}$

$$\vec{d} = 6\hat{i} + 4\hat{j} - (2\hat{i} - \hat{j})$$

$$= 6\hat{i} + 4\hat{j} - 2\hat{i} + \hat{j}$$

$$\vec{d} = 4\hat{i} + 5\hat{j}$$

Sol

$$W = \vec{F} \cdot \vec{d}$$

$$= (3\hat{i} + 2\hat{j}) \cdot (4\hat{i} + 5\hat{j})$$

$$= (3)(4) + (2)(5)$$

$$= 12 + 10$$

$$\boxed{W = 22 \text{ units}}$$

2.9 Data

$\vec{A} = \hat{i} + \hat{j} + \hat{k}$, $\vec{B} = 2\hat{i} - 3\hat{j} + \hat{k}$

$\vec{C} = 4\hat{i} + \hat{j} - 5\hat{k}$

Show that these vectors are mutually perpendicular.

Solution

$\vec{A}\cdot\vec{B} = (\hat{i} + \hat{j} + \hat{k})\cdot(2\hat{i} - 3\hat{j} + \hat{k})$

$= (1)(2) + (1)(-3) + (1)(1)$

$= 2 - 3 + 1$

$= -1 + 1$

$\vec{A}\cdot\vec{B} = 0$

So, $\vec{A}$ and $\vec{B}$ are perpendicular.

$\vec{B}\cdot\vec{C} = (2\hat{i} - 3\hat{j} + \hat{k})\cdot(4\hat{i} + \hat{j} - 5\hat{k})$

$= (2)(4) + (-3)(1) + (1)(-5)$

$= 8 - 3 - 5$

$= 5 - 5$

$\vec{B}\cdot\vec{C} = 0$

So, $\vec{B}$ and $\vec{C}$ are perpendicular.

$\vec{C}\cdot\vec{A} = (4\hat{i}+\hat{j}-5\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})$

$= (4)(1)+(1)(1)+(-5)(1)$

$= 4+1-5$

$= 5-5$

$\vec{C}\cdot\vec{A} = 0$

So, vectors $\vec{C}$ and $\vec{A}$ are perpendicular.

Therefore, given three vectors are mutually perpendicular.

2.10 Data

$\vec{A} = \hat{i}-2\hat{j}+3\hat{k}$

$\vec{B} = 3\hat{i}-4\hat{k}$

projection of $\vec{A}$ on $\vec{B}$

$= A\cos\theta = ?$

Solution

$\vec{A}\cdot\vec{B} = AB\cos\theta$

$\dfrac{\vec{A}\cdot\vec{B}}{B} = A\cos\theta$

$\vec{A}\cdot\vec{B} = (\hat{i}-2\hat{j}+3\hat{k})\cdot(3\hat{i}+0\hat{j}-4\hat{k})$

$$\vec{A}\cdot\vec{B} = (1)(3) + (-2)(0) + (3)(-4)$$
$$= 3 + 0 - 12$$
$$\vec{A}\cdot\vec{B} = -9$$

$$\vec{B} = 3\hat{i} - 4\hat{k}$$

$$B = \sqrt{(3)^2 + (-4)^2}$$

$$= \sqrt{9+16} = \sqrt{25}$$

$$B = 5$$

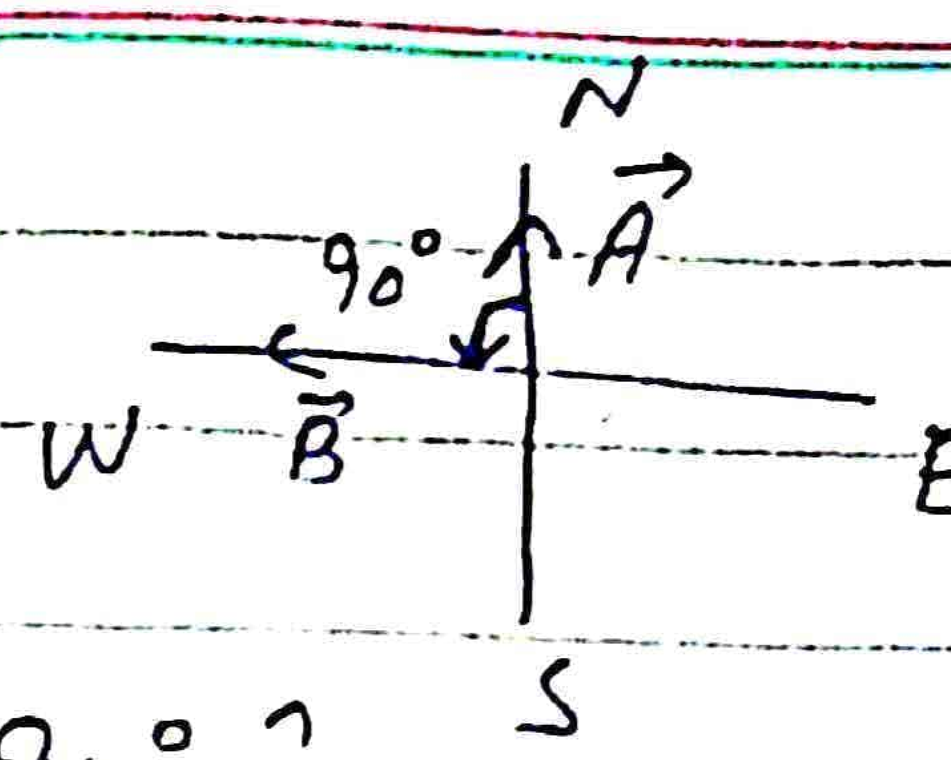
so,
$$\boxed{A\cos\theta = \frac{-9}{5}}$$

2.11 Data

$\vec{A} = 4$ units north
$\vec{B} = 3$ units west
$\vec{C} = 8$ units east
$\vec{A}\times\vec{B} = ?$
$\vec{A}\times\vec{C} = ?$, $\vec{B}\times\vec{C} = ?$

Solution

$$\vec{A}\times\vec{B} = AB\sin\theta\ \hat{n}$$

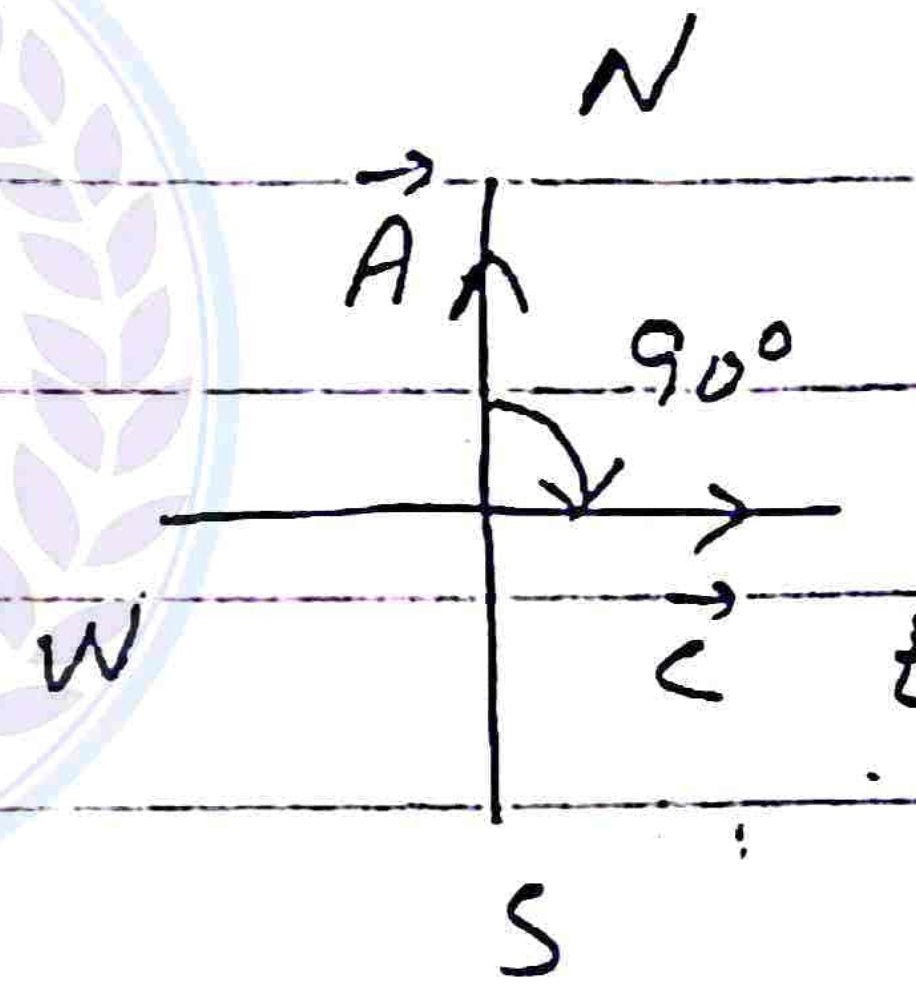


$$\vec{A} \times \vec{B} = (4)(3)\sin 90^\circ \hat{n}$$

$$= 12(1)\hat{n} = 12\hat{n}$$

$$\boxed{\vec{A} \times \vec{B} = 12 \text{ units vertically up}}$$

$$\vec{A} \times \vec{C} = AC \sin\theta \, \hat{n}$$



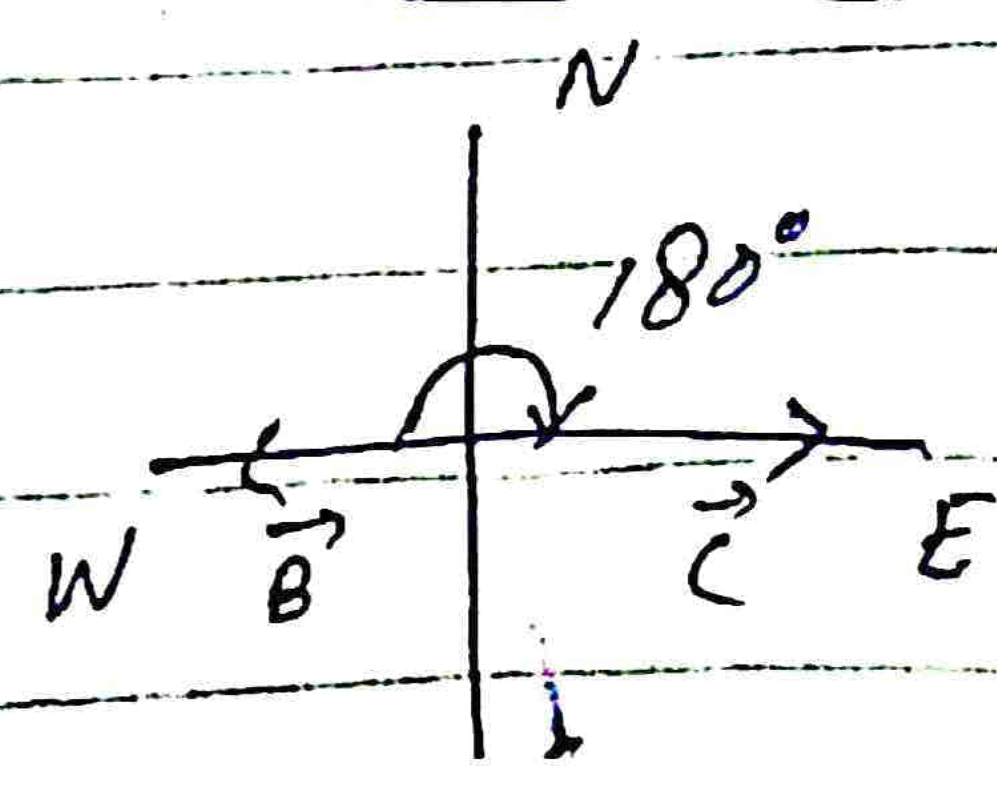
$$= (4)(8)\sin 90^\circ \hat{n}$$

$$= 32(1)\hat{n}$$

$$\vec{A} \times \vec{C} = 32\hat{n}$$

$$\boxed{\vec{A} \times \vec{C} = 32 \text{ units vertically down}}$$

$$\vec{B} \times \vec{C} = BC \sin\theta \, \hat{n}$$



$$= (3)(8)\sin 180^\circ \hat{n}$$

$$\boxed{\vec{B} \times \vec{C} = 0}$$

2.12 Data

$$\vec{F} = -3\hat{i} + \hat{j} + 5\hat{k}$$

$$\vec{r} = 7\hat{i} + 3\hat{j} + \hat{k}$$

$$\vec{\tau} = ?$$

Solution

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 3 & 1 \\ -3 & 1 & 5 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} - \hat{j}\begin{vmatrix} 7 & 1 \\ -3 & 5 \end{vmatrix} + \hat{k}\begin{vmatrix} 7 & 3 \\ -3 & 1 \end{vmatrix}$$

$$= \hat{i}(15-1) - \hat{j}(35-(-3)) + \hat{k}(7-(-9))$$

$$= \hat{i}(14) - \hat{j}(35+3) + \hat{k}(7+9)$$

$$\boxed{\vec{\tau} = (14\hat{i} - 38\hat{j} + 16\hat{k})\ Nm}$$

2.13 Data

$\vec{F} = \hat{i} - 2\hat{j}$, $\vec{r} = -\hat{j} + \hat{k}$

a) $\vec{\tau} = ?$

b) $r_1 = \hat{i} + \hat{k}$, $\vec{\tau} = ?$

Solution

a.) $\vec{\tau} = \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -1 & 1 \\ -2 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 0 & -1 \\ 1 & -2 \end{vmatrix}$$

$$\vec{\tau} = \hat{i}(0-(-2)) - \hat{j}(0-1) + \hat{k}(0-(-1))$$

$$= \hat{i}(2) - \hat{j}(-1) + \hat{k}(1)$$

$$\boxed{\vec{\tau} = 2\hat{i} + \hat{j} + \hat{k}}$$

b) $\vec{r}_1 = \hat{i} + \hat{k}$, $\vec{r}_2 = -\hat{j} + \hat{k}$

$$\vec{r} = \vec{r}_2 - \vec{r}_1$$

$$= -\hat{j} + \hat{k} - (\hat{i} + \hat{k})$$

$$= -\hat{j} + \cancel{\hat{k}} - \hat{i} - \cancel{\hat{k}}$$

$$\vec{r} = -\hat{i} - \hat{j}$$

$$\vec{\tau} = \vec{r} \times \vec{f}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 0 \\ 1 & -2 & 0 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -1 & 0 \\ -2 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} -1 & 0 \\ 1 & 0 \end{vmatrix}$$

$$+\hat{k}\begin{vmatrix} -1 & -1 \\ 1 & -2 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2-(-1))$$

$$= 0 - 0 + \hat{k}(2+1)$$

$$\boxed{\vec{r} = 3\hat{k}}$$

2.14 Data

Let the vectors are $\vec{A}$ and $\vec{B}$.

$|\vec{A}\cdot\vec{B}| = 6\sqrt{3}$

$|\vec{A}\times\vec{B}| = 6$

angle $= \theta = ?$

Solution

$$\frac{|\vec{A}\times\vec{B}|}{|\vec{A}\cdot\vec{B}|} = \frac{\cancel{6}}{\cancel{6}\sqrt{3}}$$

$$\frac{\cancel{AB}\sin\theta}{\cancel{AB}\cos\theta} = \frac{1}{\sqrt{3}}$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\boxed{\theta = 30^\circ}$$

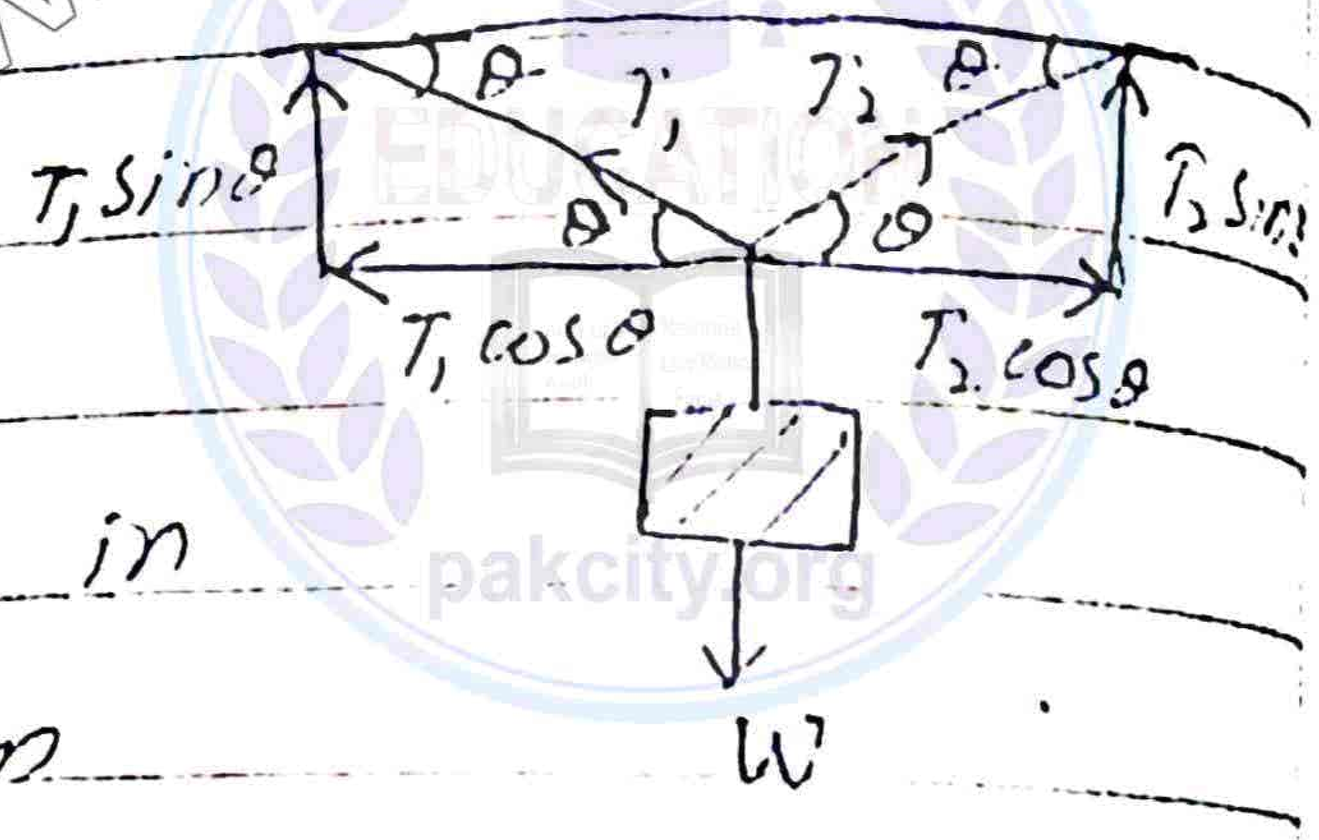
2.15 Data

weight $= W = 10\,N$

angle $= \theta = 15^\circ$

tension $= T = ?$

Solution



As the system is in equilibrium state. So,

sum of all the forces along x-axis will be zero

$$T_1 \cos\theta = T_2 \cos\theta$$

$$T_1 = T_2 = T$$

Now comparing forces of y-axis:

$$T_1 \sin\theta + T_2 \sin\theta = W$$

$$T \sin\theta + T \sin\theta = W$$

$$2T \sin\theta = W$$

$$T = \frac{W}{2 \sin\theta}$$

$$= \frac{10}{2 \sin 15^\circ}$$

$$\boxed{T = 19.3\ N}$$