

Notes

Differential Geometry

Semester

6th

Written by

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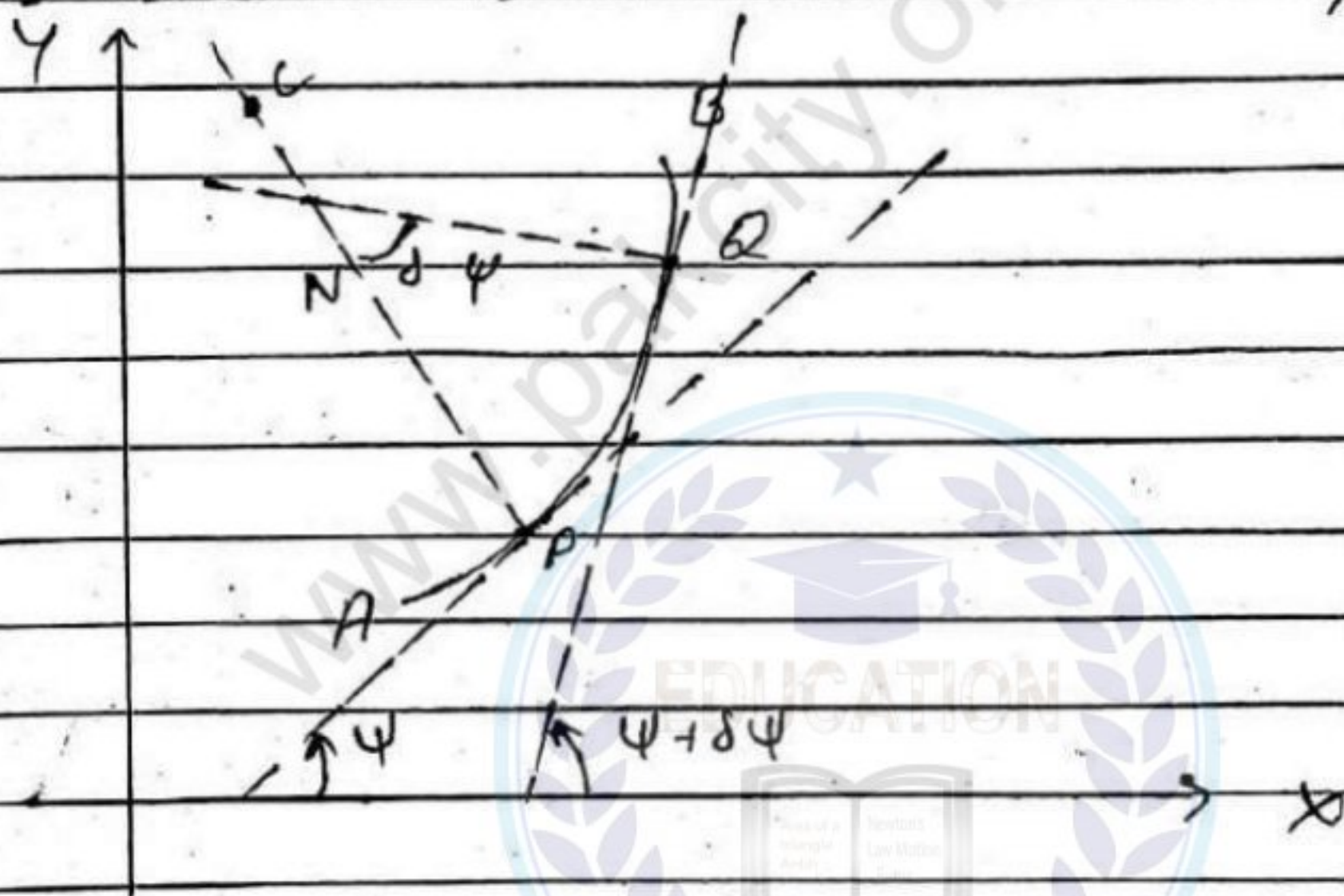
Curvature and Radius of curvature

"Curvature is a numerical measure of bending of the curve."

Curvature represented by k and Radius of curvature denoted by ρ curvature and Radius are inversely proportional to each other.

○ circle has constant curvature. $\odot \rho$

Explanation:-



Consider a curve AB on the xy. coordinates system. Choose two points P & Q on the curve and draw tangent on P and Q makes the angle with x-axis are ψ and $\psi + \delta\psi$ respectively. Draw normal on point P and Q. Both normal cut at angle $\delta\psi$.

where $AB = s$ and $PQ = \delta s$

If $Q \rightarrow P$ then $N \rightarrow C$ and $\delta\psi \rightarrow 0$

So

$$\lim_{\delta s \rightarrow 0} \frac{\partial \psi}{\partial s} = \frac{d\psi}{ds} = k$$

curvature at point P.

Let the tangent at any point of the curve makes an angle ψ with x-axis. Then by definition of derivative

$$\frac{dy}{dx} = \tan \psi$$

Diff. w.r.t x

$$\frac{d^2y}{dx^2} = \sec^2 \psi \frac{d\psi}{dx}$$

$$= (1 + \tan^2 \psi) \frac{d\psi}{ds} \frac{ds}{dx}$$

$$\frac{d^2y}{dx^2} = \left(1 + \left(\frac{dy}{dx}\right)^2\right) \frac{d\psi}{ds} \frac{ds}{dx}$$

$$= \left(1 + \left(\frac{dy}{dx}\right)^2\right) K \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{d^2y}{dx^2} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2} K$$

$$\frac{1}{K} = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}{d^2y/dx^2}$$

$$\rho = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}{d^2y/dx^2}$$

$$\begin{aligned} \Delta s^2 &= \Delta x^2 + \Delta y^2 \\ &= \Delta x^2 \left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right) \end{aligned}$$

$$\frac{\Delta s}{\Delta x} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Question:-

Show that Radius of curvature for a circle is constant.

Sol.

Let the equation of circle is

$$x^2 + y^2 = a^2$$

we know that Radius of curvature is

$$f = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

Diff. w.r.t. x eq. (1)

$$2x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{y(-1) + x \frac{dy}{dx}}{y^2} = \frac{-y + x(-\frac{x}{y})}{y^2}$$

$$= \frac{-y^2 - x^2}{y^3} = \frac{-(x^2 + y^2)}{y^3} = -\frac{a^2}{y^3}$$

putting values

$$f = \frac{\left(1 + \left(-\frac{x}{y}\right)^2\right)^{3/2}}{-a^2/y^3} = \frac{\left(1 + \frac{x^2}{y^2}\right)^{3/2}}{-a^2/y^3}$$

$$= \frac{\left(\frac{x^2 + y^2}{y^2}\right)^{3/2}}{-a^2/y^3} = \frac{(x^2 + y^2)^{3/2}}{-a^2} = \frac{-(a^2)^{3/2}}{a^2}$$

$$= -\frac{a^3}{a^2} = -a = a \quad \text{Ans}$$

Question:- Find the Radius of curvature at any point of cycloid

$$x = a(\theta + \sin\theta)$$

$$y = a(1 - \cos\theta)$$

Sol:-

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

where $x' = \frac{dx}{dt}$ $y' = \frac{dy}{dt}$

$$x'' = \frac{d^2x}{dt^2} \quad , \quad y'' = \frac{d^2y}{dt^2}$$

$$\begin{array}{l|l} x = a(\theta + \sin\theta) & y = a(1 - \cos\theta) \\ x' = \frac{dx}{d\theta} = a(1 + \cos\theta) & y' = \frac{dy}{d\theta} = a(\sin\theta) \end{array}$$

$$\begin{array}{l|l} x'' = \frac{d^2x}{d\theta^2} = a(-\sin\theta) & y'' = a \cos\theta \end{array}$$

$$\begin{aligned} x'^2 + y'^2 &= (a(1 + \cos\theta))^2 + (a\sin\theta)^2 \\ &= a^2(1 + \cos^2\theta + 2\cos\theta + \sin^2\theta) \\ &= a^2(2 + 2\cos\theta) \\ &= a^2 \cdot 2(1 + \cos\theta) \end{aligned}$$

$$(x'^2 + y'^2)^{3/2} = a^3(2)^{3/2}(1 + \cos\theta)^{3/2}$$

$$\begin{aligned} x'y'' - y'x'' &= a(1 + \cos\theta)(a \cos\theta) - a\sin\theta a(-\sin\theta) \\ &= a^2(\cos\theta + \cos^2\theta + \sin^2\theta) \\ &= a^2(1 + \cos\theta) \end{aligned}$$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{a^3(2)^{3/2}(1 + \cos\theta)^{3/2}}{a^2(1 + \cos\theta)}$$

$$= a(2)^{3/2}(1 + \cos\theta)^{1/2}$$

Question: Show that the Radius of curvature at any point of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ ($x = a \cos^3 \theta$, $y = a \sin^3 \theta$) is equal to three times of the length of the perpendicular from the origin to the tangent.

Soln

$$x = a \cos^3 \theta$$

$$x' = -3a \cos^2 \theta \sin \theta$$

$$x'' = -3a \cos^2 \theta \cos \theta - 3a \sin \theta \cdot 2a \cos \theta (-\sin \theta)$$

$$x'' = -3a \cos^3 \theta + 6a \sin^2 \theta \cos \theta$$

$$y = a \sin^3 \theta$$

$$y' = 3a \sin^2 \theta \cos \theta$$

$$y'' = -3a \sin^2 \theta \sin \theta + 3a \cos \theta \cdot 2 \sin \theta \cos \theta$$

$$= -3a \sin^3 \theta + 6a \cos^2 \theta \sin \theta$$

$$x'^2 + y'^2 = (-3a \cos^2 \theta \sin \theta)^2 + (3a \sin^2 \theta \cos \theta)^2$$

$$= 9a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^4 \theta \cos^2 \theta$$

$$= 9a^2 \sin^2 \theta \cos^2 \theta (\cos^2 \theta + \sin^2 \theta)$$

$$= 9a^2 \sin^2 \theta \cos^2 \theta$$

$$x'y'' - y'x'' = -3a \cos^2 \theta \sin \theta (-3a \sin^3 \theta + 6a \cos^2 \theta \sin \theta)$$

$$- 3a \sin^2 \theta \cos \theta (-3a \cos^3 \theta + 6a \sin^2 \theta \cos \theta)$$

$$= 9a^2 \cos^2 \theta \sin^4 \theta - 18a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^2 \theta \cos^4 \theta$$

$$- 18a^2 \sin^4 \theta \cos^2 \theta$$

$$= 9a^2 \cos^2 \theta \sin^2 \theta (\sin^2 \theta + \cos^2 \theta) - 18a^2 \sin^2 \theta \cos^2 \theta (\cos^2 \theta + \sin^2 \theta)$$

$$= 9a^2 \cos^2 \theta \sin^2 \theta - 18a^2 \sin^2 \theta \cos^2 \theta$$

$$= -9a^2 \sin^2 \theta \cos^2 \theta$$

$$r = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{(9a^2 \sin^2 \theta \cos^2 \theta)^{3/2}}{-9a^2 \sin^2 \theta \cos^2 \theta}$$

$$= (9a^2 \sin^2 \theta \cos^2 \theta)^{1/2} = 3a \sin \theta \cos \theta$$

Now we need to find Equation of tangent at any point on the curve

$$y - y_1 = m(x - x_1)$$

$$m = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{y'}{x'} = \frac{3a \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta} = -\tan \theta$$

So

$$y - a \sin^3 \theta = -\tan \theta (x - a \cos^3 \theta)$$

$$y - a \sin^3 \theta = -\frac{\sin \theta}{\cos \theta} (x - a \cos^3 \theta)$$

$$y \cos \theta - a \sin^3 \theta \cos \theta = -x \sin \theta + a \cos^3 \theta \sin \theta$$

$$x \sin \theta + y \cos \theta - a \sin^3 \theta \cos \theta - a \cos^3 \theta \sin \theta = 0$$

$$x \sin \theta + y \cos \theta - a \sin \theta \cos \theta (\sin^2 \theta + \cos^2 \theta) = 0$$

$$x \sin \theta + y \cos \theta - a \sin \theta \cos \theta = 0$$

$$d = \frac{|0 \cdot \sin \theta + 0 \cdot \cos \theta - a \sin \theta \cos \theta|}{\sqrt{\sin^2 \theta + \cos^2 \theta}}$$

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$d = a \sin \theta \cos \theta$$

It is clear that ρ is equal to three times of length.

Question:- If p and p' are the radii of curvature at the extremities of two conjugate diameters of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ prove that

$$(p^{2/3} + p'^{2/3})(ab)^{2/3} = a^2 + b^2$$

Sol:-

We know that, the parametric Equation of ellipse is

$$x = a \cos \theta \quad \text{and} \quad y = b \sin \theta$$

$$x' = -a \sin \theta$$

$$y' = b \cos \theta$$

$$x'' = -a \cos \theta$$

$$y'' = -b \sin \theta$$

$$x'^2 + y'^2 = (-a \sin \theta)^2 + (b \cos \theta)^2 = a^2 \sin^2 \theta + b^2 \cos^2 \theta$$

$$\begin{aligned} x'y'' - y'x'' &= (-a \sin \theta)(-b \sin \theta) - (b \cos \theta)(-a \cos \theta) \\ &= ab \sin^2 \theta + ab \cos^2 \theta = ab(\sin^2 \theta + \cos^2 \theta) \\ &= ab \end{aligned}$$

$$p = \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab} \quad \text{--- (1)}$$

For the radius of curvature at the extremity of other conjugate diameter is obtained by Replacing θ by $\theta + \pi/2$

$$p' = \frac{(a^2 \sin^2(\theta + \pi/2) + b^2 \cos^2(\theta + \pi/2))^{3/2}}{ab}$$

$$p' = \frac{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)^{3/2}}{ab}$$

$$p^{2/3} + p'^{2/3} = \left(\frac{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)^{3/2}}{ab} \right)^{2/3} + \left(\frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab} \right)^{2/3}$$

$$\rho^{2/3} + \rho'^{2/3} = \frac{a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta)}{(ab)^{2/3}}$$

$$(\rho^{2/3} + \rho'^{2/3})(ab)^{2/3} = a^2 + b^2$$

Hence proved.

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X ————— X ————— X

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Radius of curvature in polar form

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - r r_2}$$

$$\text{where } r_1 = \frac{dr}{d\theta}, \quad r_2 = \frac{d^2r}{d\theta^2}$$

Question:-

Prove that for the cardioid $r = a(1 + \cos \theta)$ ρ is constant.

Solution:- Here $r = a(1 + \cos \theta)$

$$r_1 = \frac{dr}{d\theta} = a(-\sin \theta)$$

$$r_2 = \frac{d^2r}{d\theta^2} = a(-\cos \theta)$$

$$\begin{aligned} r^2 + r_1^2 &= (a(1 + \cos \theta))^2 + (a(-\sin \theta))^2 \\ &= a^2(1 + \cos^2 \theta + 2\cos \theta) + a^2 \sin^2 \theta \\ &= a^2(1 + \cos^2 \theta + \sin^2 \theta + 2\cos \theta) \\ &= a^2(1 + 1 + 2\cos \theta) \\ &= a^2(2 + 2\cos \theta) = 2a^2(1 + \cos \theta) \end{aligned}$$

$$\begin{aligned}
 2r_1^2 + r^2 - r r_2 &= 2(a(1-\sin\theta))^2 + (a(1+\cos\theta))^2 - a(1+\cos\theta) \quad (a(1-\cos\theta)) \\
 &= 2a^2 \sin^2\theta + a^2(1+\cos^2\theta+2\cos\theta) + a^2(\cos^2\theta+\cos\theta) \\
 &= a^2(2\sin^2\theta + 1 + \cos^2\theta + 2\cos\theta + \cos^2\theta + \cos\theta) \\
 &= a^2(2\sin^2\theta + 2\cos^2\theta + 1 + 3\cos\theta) \\
 &= a^2(2(\sin^2\theta + \cos^2\theta) + 1 + 3\cos\theta) \\
 &= a^2(2 + 1 + 3\cos\theta) \\
 &= a^2(3 + 3\cos\theta) = 3a^2(1 + \cos\theta)
 \end{aligned}$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - r r_2} = \frac{(2a^2(1+\cos\theta))^{3/2}}{3a^2(1+\cos\theta)}$$

$$\rho = \frac{\sqrt[3]{8a^3(1+\cos\theta)}}{3}$$

$$\rho^2 = \frac{8}{9} a^2(1+\cos\theta) = \frac{8}{9} a \cdot r$$

$$\frac{\rho^2}{r} = \frac{8}{9} a \quad \text{Ans. which is a constant.}$$

Question:-

Find the Radius of curvature for $\sqrt{\frac{x}{a}} - \sqrt{\frac{y}{b}} = 1$ at the points where it touches the coordinate axis.

Solution:-

Given curve is

$$\sqrt{\frac{x}{a}} - \sqrt{\frac{y}{b}} = 1$$

Diff. w.r.t x .

$$\frac{1}{2} \left(\frac{x}{a}\right)^{-1/2} \cdot \frac{1}{a} - \frac{1}{2} \left(\frac{y}{b}\right)^{-1/2} \cdot \frac{1}{b} \frac{dy}{dx} = 0$$

$$\frac{1}{2} \left(\frac{x}{a}\right)^{-1/2} \cdot \frac{1}{a} = \frac{1}{2} \left(\frac{y}{b}\right)^{-1/2} \cdot \frac{1}{b} \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{b}{a} \left(\frac{x}{a}\right)^{-1/2} \left(\frac{y}{b}\right)^{1/2}$$

$$\frac{dy}{dx} = \left(\frac{b}{a}\right)^{1/2} x^{-1/2} y^{1/2}$$

Diff w.r.t x again

$$\frac{d^2y}{dx^2} = \left(\frac{b}{a}\right)^{1/2} \left[-\frac{1}{2} x^{-3/2} y^{1/2} + x^{-1/2} \cdot \frac{1}{2} y^{-1/2} \frac{dy}{dx} \right]$$

$$\frac{d^2y}{dx^2} = \sqrt{\frac{b}{a}} \left[-\frac{1}{2} x^{-3/2} y^{1/2} + \frac{1}{2} x^{-1/2} y^{-1/2} \left(\left(\frac{b}{a}\right)^{1/2} x^{-1/2} y^{1/2} \right) \right]$$

$$= \sqrt{\frac{b}{a}} \left[-\frac{1}{2} x^{-3/2} y^{1/2} + \frac{1}{2} \left(\frac{b}{a}\right)^{1/2} x^{-1} y^0 \right]$$

$$= \sqrt{\frac{b}{a}} \left[\sqrt{\frac{b}{a}} \cdot \frac{1}{2x} - \frac{1}{2x} \cdot \sqrt{\frac{y}{x}} \right]$$

At for y -axis

$$x = 0$$

$$\sqrt{\frac{y}{b}} = 1 \Rightarrow \frac{y}{b} = 1$$

$$y = b$$

$$\Rightarrow (0, b)$$

At for x -axis

$$y = 0$$

$$\sqrt{\frac{x}{a}} = 1 \Rightarrow \frac{x}{a} = 1$$

$$x = a$$

$$\Rightarrow (a, 0)$$

At $(a, 0)$

or x -axis

$$\frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} = \sqrt{\frac{b}{a}} \cdot \left(\sqrt{\frac{b}{a}} \cdot \frac{1}{2a} \right) = \frac{b}{a \cdot 2a} = \frac{b}{2a^2}$$

$$\rho = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+0)^{3/2}}{b/2a^2} //$$

$$\rho = \frac{2a^2}{b}$$

At (0, b) or y-axis

$$\frac{dx}{dy} = \left(\frac{b}{a}\right)^{1/2} \cdot x^{1/2} \cdot y^{-1/2} = 0$$

$$\frac{dx}{dy} = \frac{a}{2b^2}$$

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{dx}{dy}} = \frac{2b^2}{a}$$

Question:-

Find the curvature of $x = 4 \cos t$, $y = 3 \sin t$. At what point on this ellipse does not the curvature have the greatest and least values. what are the magnitudes??

Solution:-

$$x = 4 \cos t$$

$$x' = -4 \sin t$$

$$x'' = -4 \cos t$$

$$y = 3 \sin t$$

$$y' = 3 \cos t$$

$$y'' = -3 \sin t$$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{((-4 \sin t)^2 + (3 \cos t)^2)^{3/2}}{(-4 \sin t)(-3 \sin t) - 3 \cos t(-4 \cos t)}$$

$$\rho = \frac{(16 \sin^2 t + 9 \cos^2 t)^{3/2}}{12 \sin^2 t + 12 \cos^2 t} = \frac{1}{12} (9 \cos^2 t + 16 \sin^2 t)^{3/2}$$

$$\rho^{2/3} = \left(\frac{1}{12}\right)^{2/3} (9 \cos^2 t + 16 \sin^2 t)$$

Diff. w.r.t t

$$\frac{2}{3} p^{-1/3} \frac{dp}{dt} = \left(\frac{1}{12}\right)^{2/3} (18 \cos t (-\sin t) + 32 \sin t \cos t)$$

$$\text{put } \frac{dp}{dt} = 0$$

$$-18 \cos t \sin t + 32 \sin t \cos t = 0$$

$$14 \sin t \cos t = 0$$

$$\sin 2t = 0$$

$$2t = 0, \pi, 2\pi, \dots$$

$$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \dots$$

At $t = 0$

$$p = \frac{1}{12} (9 \cos^2(0) + 16 \sin^2(0))^{3/2}$$

$$= \frac{1}{12} (9)^{3/2} = \frac{1}{12} \cdot 27 = \frac{9}{4}$$

$$\frac{1}{p} = \frac{4}{9}$$

At $t = \frac{\pi}{2}$

$$p = \frac{1}{12} (9 \cos^2(\frac{\pi}{2}) + 16 \sin^2(\frac{\pi}{2}))^{3/2}$$

$$p = \frac{1}{12} (16)^{3/2} = \frac{64}{12} = \frac{16}{3}$$

$$\frac{1}{p} = \frac{3}{16}$$

least value is $\frac{3}{16}$ and greatest value $\frac{4}{9}$

Centre of curvature:-

Let $y = f(x)$ be a curve and (α, β) be the centre of curvature. Then at any point $P(x, y)$, the centre of curvature is defined as

$$\alpha = x_1 - \frac{\frac{dy}{dx} (1 + (\frac{dy}{dx})^2)}{d^2y/dx^2}$$

Similarly

$$\beta = y_1 + \frac{(1 + (\frac{dy}{dx})^2)}{d^2y/dx^2}$$

Circle of curvature with centre (α, β) and Radius " ρ " is written as
 $(x - \alpha)^2 + (y - \beta)^2 = \rho^2$

Question:- Find the centre of curvature at point (x, y) of $y^2 = 4ax$

Solution:-

We need to find (α, β) at point (x, y)

$$y^2 = 4ax \quad \text{--- (1)}$$

Diff. w.r.t x

$$2y \frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = 2a y^{-1}$$

Again Diff w.r.t x

$$\frac{d^2y}{dx^2} = 2a(-1)y^{-2} \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = -2a y^{-2} \cdot 2a y^{-1} = -\frac{4a^2}{y^3}$$

$$\alpha = x_1 - \frac{\frac{dy}{dx} (1 + (\frac{dy}{dx})^2)}{d^2y/dx^2}$$

$$\alpha = x - \frac{2ay'(1 + 4a^2y^2)}{-4a^2y^3} = x - \frac{2ay' + 8a^3y^3}{-4a^2y^3}$$

$$\alpha = x - (-\frac{1}{2ay^2} - 2a) = x + \frac{y^2}{2a} + 2a = x + \frac{4ax + 2a}{2a}$$

$$\alpha = x + 2x + 2a = 3x + 2a$$

$$\beta = y_1 + \frac{(1 + (\frac{dy}{dx})^2)}{d^2y/dx^2}$$

$$\beta = y + \frac{(1 + 4a^2y^2)}{-4a^2y^3} = y + (-\frac{1}{4a^2y^3} - \frac{1}{y})$$

$$\beta = y - \frac{y^3}{4a^2} - y = -\frac{y^3}{4a^2}$$

$$(\alpha, \beta) = (3x + 2a, -\frac{y^3}{4a^2}) \quad \text{Ans}$$

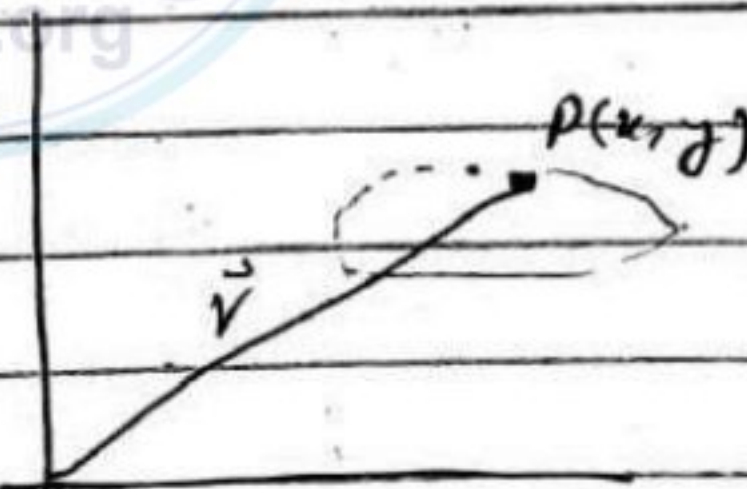
Formulas For Curvature (k), Radius of curvature And Torsion (Twist) = τ in vector notation: (Parametrized form)

$$k = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

$$\text{Radius of curvature} = \rho = \frac{1}{k}$$

$$\text{Torsion (Twist)} = \tau = \frac{(\vec{v} \times \vec{a}) \cdot \vec{a}'}{|\vec{v} \times \vec{a}|^2}$$

$$\text{Jerk} = \vec{a}' = \frac{d^3\vec{R}}{dt^3}$$



Question-

Find curvature (k), Radius of curvature (P) and Torsion (τ) ??

Solution-

we know

$$k = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

Given $\vec{r}(t) = \cos t \vec{i} + \sin t \vec{j} + t^2 \vec{k}$

$$\vec{v} = \frac{d\vec{r}}{dt} = -\sin t \vec{i} + \cos t \vec{j} + 2t \vec{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -\cos t \vec{i} - \sin t \vec{j} + 2 \vec{k}$$

$$\vec{a}' = \text{jerk} = \frac{d^3 \vec{r}}{dt^3} = \sin t \vec{i} - \cos t \vec{j}$$

$$|\vec{v}| = \sqrt{\sin^2 t + \cos^2 t + 4t^2} = \sqrt{1 + 4t^2}$$

$$|\vec{v} \times \vec{a}| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 2t \\ -\cos t & -\sin t & 2 \end{vmatrix}$$

$$= |(2 \cos t + 2t \sin t) \vec{i} - (-2 \sin t + 2t \cos t) \vec{j} + (\sin^2 t + \cos^2 t) \vec{k}|$$

$$= \sqrt{(2 \cos t + 2t \sin t)^2 + (-2 \sin t + 2t \cos t)^2 + 1}$$

$$= \sqrt{4 \cos^2 t + 4t^2 \sin^2 t + 8t \cos t \sin t + 4 \sin^2 t + 4t^2 \cos^2 t - 8t \cos t \sin t + 1}$$

$$= \sqrt{4 + 4t^2 + 1} = \sqrt{5 + 4t^2}$$

$$k = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{\sqrt{5 + 4t^2}}{(1 + 4t^2)^{3/2}}$$

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$$\rho = \frac{1}{k} = \frac{(1+4t^2)^{3/2}}{\sqrt{5+4t^2}}$$

$$\begin{aligned}\vec{v} \times \vec{a} \cdot \vec{a}' &= \sin t(2\cos t + 2t\sin t) + \cos t(-2\sin t + 2t\cos t) \\ &= 2\sin t\cos t + 2t\sin^2 t - 2\sin t\cos t + 2t\cos^2 t \\ &= 2t(\sin^2 t + \cos^2 t) = 2t\end{aligned}$$

$$\tau = \frac{(\vec{v} \times \vec{a}) \cdot \vec{a}'}{|\vec{v} \times \vec{a}|^2} = \frac{2t}{(\sqrt{5+4t^2})^2} = \frac{2t}{5+4t^2} \text{ Ans}$$

Note

- If the value of τ is positive then the curve is right handed.
- If the value of τ is negative then the curve is left handed.

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Frenet Serret Formula:

The magnitude of $T'(s)$ is called the curvature k at any point given by the vector $R(s)$

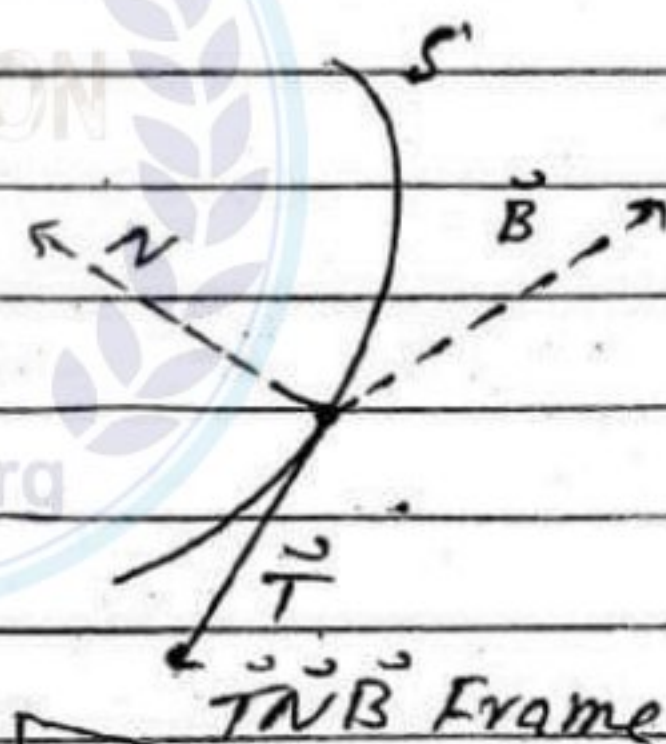
$$k = |T'(s)| \longrightarrow \textcircled{1}$$

$$\hat{T} \cdot \hat{T} = 1$$

Diff. w.r.t s

$$\frac{d\hat{T}}{ds} \cdot \hat{T} + \hat{T} \cdot \frac{d\hat{T}}{ds} = 0$$

$$2\hat{T} \cdot \frac{d\hat{T}}{ds} = 0$$



$\hat{T}$ = tangent
 $\hat{N}$ = Normal
 $\hat{B}$ = Binormal

If $a \cdot b = 0$ Then $a \perp b$

$$\vec{T} \cdot \frac{d\vec{T}}{ds} = 0$$

$$\vec{T} \cdot \vec{T}' = 0$$

$\Rightarrow \vec{T}$ and $\vec{T}'$ are perpendicular to each other

$$N(s) = \frac{\vec{T}'(s)}{|\vec{T}'(s)|} = \frac{\vec{T}'(s)}{K}$$

$$\vec{T}'(s) = K N(s) \quad \text{--- (2)}$$

The vector $N(s)$ is called the principal normal vector

$$\vec{B} = \vec{T} \times \vec{N}$$

$\vec{B}'$ is a measure on how the curve without in the plane

$$\vec{B} = \vec{T} \times \vec{N}$$

$$\frac{d\vec{B}}{ds} = \vec{T} \times \frac{d\vec{N}}{ds} + \frac{d\vec{T}}{ds} \times \vec{N}$$

$$\vec{B}' = \vec{T} \times \vec{N}' + \vec{T}' \times \vec{N}$$

By using eq. (2) we get

$$\vec{B}' = \vec{T} \times \vec{N}' + K \vec{N} \times \vec{N}$$

$$\vec{B}' = \vec{T} \times \vec{N}'$$

we know

$$\vec{B} \cdot \vec{B} = 1$$

Diff w.r.t s

$$\frac{d\vec{B}}{ds} \cdot \vec{B} + \vec{B} \cdot \frac{d\vec{B}}{ds} = 0$$

$$2 \frac{d\vec{B}}{ds} \cdot \vec{B} = 0$$

$$\frac{d\vec{B}}{ds} \cdot \vec{B} = 0$$

Note

$$\vec{N} = \vec{B} \times \vec{T}$$

$$\vec{B} = \vec{T} \times \vec{N}$$

$$\vec{T} = \vec{N} \times \vec{B}$$

$$\vec{B} \cdot \vec{B} = 1 \quad | \quad \vec{B} \cdot \vec{T} = 0$$

$$\vec{T} \cdot \vec{T} = 1 \quad | \quad \vec{T} \cdot \vec{N} = 0$$

$$\vec{N} \cdot \vec{N} = 1 \quad | \quad \vec{N} \cdot \vec{B} = 0$$

$\vec{B}$ and $\vec{B}'$ are perpendicular to each other. This means that $\vec{B}'(s)$ is perpendicular to $\vec{T}(s)$. But since $\vec{B}(s)$ is a unit vector, we have $\vec{B}'(s)$ is perpendicular to $\vec{B}(s)$ as well. Thus we have seen that $\vec{B}'(s)$ has to be parallel to $\vec{N}(s)$. Thus we get

$$\vec{B}'(s) = -\tau(s)\vec{N}(s) \quad (3)$$

For some scalar function $\tau(s)$ where $\tau(s)$ is the torsion of the curve at point $R(s)$.

By taking dot product of equation (3) with $\vec{N}(s)$ and using $\vec{N} \cdot \vec{N} = 1$

$$\begin{aligned} \vec{N}(s) \cdot \vec{B}'(s) &= -\tau(s)\vec{N}(s) \cdot \vec{N}(s) \\ \vec{N}(s) \cdot \vec{B}'(s) &= -\tau(s) \end{aligned}$$

$$\tau(s) = -\vec{N}(s) \cdot \vec{B}'(s) \quad (4)$$

where $\tau(s)$ is torsion of the curve at point $R(s)$.

Question: The helix

$$\vec{R}(s) = a \cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) \hat{i} + a \sin\left(\frac{s}{\sqrt{a^2+b^2}}\right) \hat{j} + \frac{bs}{\sqrt{a^2+b^2}} \hat{k}$$

is parametrized by arclength.

Solution:

$$\vec{R}(s) = a \cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) \hat{i} + a \sin\left(\frac{s}{\sqrt{a^2+b^2}}\right) \hat{j} + \frac{bs}{\sqrt{a^2+b^2}} \hat{k}$$

$$\vec{T}(s) = \vec{R}'(s) =$$

$$\begin{aligned} &= -a \sin\left(\frac{s}{\sqrt{a^2+b^2}}\right) \cdot \frac{1}{\sqrt{a^2+b^2}} \hat{i} + a \cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) \cdot \frac{1}{\sqrt{a^2+b^2}} \hat{j} \\ &\quad + \frac{b}{\sqrt{a^2+b^2}} \hat{k} \end{aligned}$$

$$\vec{T}'(s) = -\frac{a}{a^2+b^2} \cos \frac{s}{\sqrt{a^2+b^2}} \hat{i} - \frac{a}{a^2+b^2} \sin \frac{s}{\sqrt{a^2+b^2}} \hat{j} + 0 \hat{k} \quad 19$$

$$K = |\vec{T}'(s)|$$

$$K = \sqrt{\frac{a^2}{(a^2+b^2)^2} \left(\cos^2 \left(\frac{s}{\sqrt{a^2+b^2}} \right) + \sin^2 \left(\frac{s}{\sqrt{a^2+b^2}} \right) \right)}$$

$$K = \sqrt{\frac{a^2}{(a^2+b^2)^2}} = \frac{a}{a^2+b^2}$$

$$\vec{N}(s) = \frac{1}{K(s)} \cdot \vec{T}'(s) = \frac{a^2+b^2}{a} \left(-\frac{a}{a^2+b^2} \cos \frac{s}{\sqrt{a^2+b^2}} \hat{i} - \frac{a}{a^2+b^2} \sin \frac{s}{\sqrt{a^2+b^2}} \hat{j} \right)$$

$$\vec{N}(s) = -\cos \frac{s}{\sqrt{a^2+b^2}} \hat{i} - \sin \frac{s}{\sqrt{a^2+b^2}} \hat{j}$$

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{a}{\sqrt{a^2+b^2}} \sin \frac{s}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \cos \frac{s}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\cos \frac{s}{\sqrt{a^2+b^2}} & -\sin \frac{s}{\sqrt{a^2+b^2}} & 0 \end{vmatrix}$$

$$= \hat{i} \left(0 + \frac{b}{\sqrt{a^2+b^2}} \sin \frac{s}{\sqrt{a^2+b^2}} \right) - \hat{j} \left(0 + \frac{b}{\sqrt{a^2+b^2}} \cos \frac{s}{\sqrt{a^2+b^2}} \right) + \hat{k} \left(\frac{a}{\sqrt{a^2+b^2}} \sin^2 \frac{s}{\sqrt{a^2+b^2}} + \frac{a}{\sqrt{a^2+b^2}} \cos^2 \frac{s}{\sqrt{a^2+b^2}} \right)$$

$$\vec{B}(s) = \frac{b}{\sqrt{a^2+b^2}} \sin \frac{s}{\sqrt{a^2+b^2}} \hat{i} - \frac{b}{\sqrt{a^2+b^2}} \cos \frac{s}{\sqrt{a^2+b^2}} \hat{j} + \frac{a}{\sqrt{a^2+b^2}} \hat{k}$$

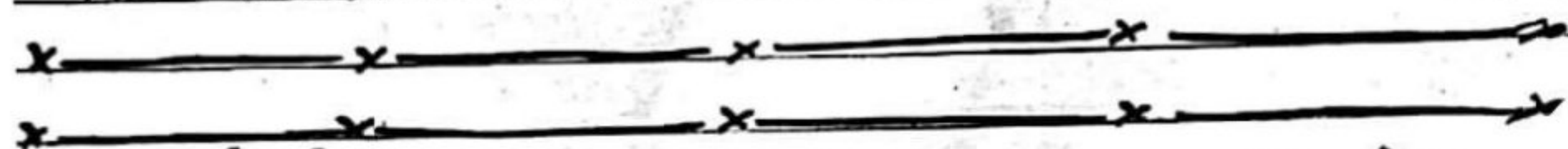
$$\vec{B}'(s) = \frac{b}{a^2+b^2} \cos \frac{s}{\sqrt{a^2+b^2}} \hat{i} + \frac{b}{a^2+b^2} \sin \frac{s}{\sqrt{a^2+b^2}} \hat{j}$$

$$\tau(s) = -\vec{N}(s) \cdot \vec{B}'(s) = -\left(\cos \frac{s}{\sqrt{a^2+b^2}} \hat{i} - \sin \frac{s}{\sqrt{a^2+b^2}} \hat{j} \right) \cdot \left(\frac{b}{a^2+b^2} \cos \frac{s}{\sqrt{a^2+b^2}} \hat{i} + \frac{b}{a^2+b^2} \sin \frac{s}{\sqrt{a^2+b^2}} \hat{j} \right)$$

$$\tau(s) = \frac{b}{a^2+b^2} \left(\cos \frac{s}{\sqrt{a^2+b^2}} + \sin^2 \frac{s}{\sqrt{a^2+b^2}} \right)$$

$$\tau(s) = \frac{b}{a^2+b^2}$$

The torsion is constant here.



22/02/19

Frenet Frame:

Now we need to find $\frac{d\vec{N}}{ds}$. Since $\vec{T}$, $\vec{N}$ and $\vec{B}$ are mutually perpendicular unit vectors we can express

$$\frac{d\vec{N}}{ds} = C_1(s)\vec{T} + C_2(s)\vec{N} + C_3(s)\vec{B} \quad (5)$$

with coefficients $C_1(s)$, $C_2(s)$ and $C_3(s)$

Here we need to find all $C_i(s)$. By taking dot product with $\vec{T}$

$$\vec{T} \cdot \frac{d\vec{N}}{ds} = C_1(s)\vec{T} \cdot \vec{T} + C_2(s)\vec{T} \cdot \vec{N} + C_3(s)\vec{T} \cdot \vec{B}$$

$$\vec{T} \cdot \frac{d\vec{N}}{ds} = C_1(s)$$

we know

$$\vec{N} \cdot \vec{T} = 0$$

Diff w.r.t s

$$\frac{d\vec{N}}{ds} \cdot \vec{T} + \vec{N} \cdot \frac{d\vec{T}}{ds} = 0$$

$$\vec{T} \cdot \frac{d\vec{N}}{ds} = -\vec{N} \cdot \frac{d\vec{T}}{ds} = -\vec{N} \cdot \vec{T}' = -\vec{N} \cdot \kappa \vec{N} = -\kappa$$

$$\Rightarrow C_1(s) = \vec{T} \cdot \frac{d\vec{N}}{ds} = -\kappa$$

By taking dot product with $\vec{N}$

$$\vec{N} \cdot \frac{d\vec{N}}{ds} = C_1(s)\vec{N} \cdot \vec{T} + C_2(s)\vec{N} \cdot \vec{N} + C_3(s)\vec{N} \cdot \vec{B}$$

$$\vec{N} \cdot \frac{d\vec{N}}{ds} = C_2(s)$$

Since $\vec{N} \perp \frac{d\vec{v}}{ds}$
 $C_2(s) = 0$

By taking dot product with $\vec{B}$

$$\vec{B} \cdot \frac{d\vec{v}}{ds} = C_1(s) \vec{T} \cdot \vec{B} + C_2(s) \vec{B} \cdot \vec{v} + C_3(s) \vec{B} \cdot \vec{B}$$

$$C_3(s) = \vec{B} \cdot \frac{d\vec{v}}{ds}$$

we know

$$\vec{v} \cdot \vec{B} = 0$$

Diff w.r.t s

$$\frac{d\vec{v}}{ds} \cdot \vec{B} + \vec{v} \cdot \frac{d\vec{B}}{ds} = 0$$

$$\vec{B} \cdot \frac{d\vec{v}}{ds} = -\vec{v} \cdot \vec{B}'(s) = -\vec{v} \cdot (-\tau \vec{v}) = \tau \vec{v} \cdot \vec{v} = \tau$$

$$C_3(s) = \tau$$

Replace C_1 , C_2 and C_3 in eq (5).

$$\frac{d\vec{v}}{ds} = -k\vec{T} + 0 \cdot \vec{v} + \tau \vec{B}$$

$$\vec{v}'(s) = -k\vec{T} + \tau \vec{B} \quad \text{--- (6)}$$

Question:

$$\vec{r}(t) = t^2 \hat{i} + \frac{8}{3} t^{3/2} \hat{j} + 4t \hat{k}$$

convert in parameter of s .

Solve:

$$\vec{r}(t) = t^2 \hat{i} + \frac{8}{3} t^{3/2} \hat{j} + 4t \hat{k}$$

$$x(t) = t^2$$

$$x'(t) = 2t$$

$$y(t) = \frac{8}{3} t^{3/2}$$

$$y'(t) = \frac{8}{3} \cdot \frac{3}{2} t^{3/2-1} = 4t^{1/2}$$

$$z(t) = 4t \quad ; \quad z'(t) = 4$$

Formula to convert parameter of s .

$$s(t) = \int_0^t \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

$$S(t) = \int_0^t \sqrt{4t^2 + 16t + 16} dt = 2 \int_0^t \sqrt{t^2 + 4t + 2} dt$$

$$= 2 \int_0^t \sqrt{(t+2)^2} dt = 2 \int_0^t (t+2) dt$$

$$S(t) = 2 \left(\frac{t^2}{2} + 2t \right) \Big|_0^t = t^2 + 4t$$

$$t^2 + 4t - S = 0$$

$$t = \frac{-4 \pm \sqrt{16 + 4S}}{2} = \frac{-4 \pm 2\sqrt{4+S}}{2}$$

$$= -2 \pm \sqrt{4+S}$$

$$= -2 + \sqrt{4+S}$$

$$\vec{r}(S) = (-2 + \sqrt{4+S})^2 \hat{i} + \frac{8}{3} (-2 + \sqrt{4+S})^{3/2} \hat{j} + 4(-2 + \sqrt{4+S}) \hat{k}$$

Assignment:

Convert into parameter of s .

1) $(a \cos t, a \sin t)$

$$\vec{r}(t) = a \cos t \hat{i} + a \sin t \hat{j}$$

$$x = a \cos t$$

$$x'(t) = -a \sin t$$

$$y(t) = a \sin t$$

$$y'(t) = a \cos t$$

we know

$$S(t) = \int_0^t \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

$$= \int_0^t \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} dt = \int_0^t \sqrt{a^2 (\sin^2 t + \cos^2 t)} dt$$

$$= \int_0^t a dt = at \Big|_0^t = at$$

$$s = at$$

$$\Rightarrow \frac{s}{a} = t$$

so

$$\vec{r}(s) = a \cos\left(\frac{s}{a}\right) \hat{i} + a \sin\left(\frac{s}{a}\right) \hat{j}$$

21 $a \cos t, b \sin t$

$$\vec{r}(t) = a \cos t \hat{i} + b \sin t \hat{j}$$

$$x(t) = a \cos t$$

$$x'(t) = -a \sin t$$

$$y(t) = b \sin t$$

$$y'(t) = b \cos t$$

$$s(t) = \int_0^t \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_0^t \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt$$

4

$$\vec{r}(s) = a \cos\left(\frac{s}{a}\right) \hat{i} + a \sin\left(\frac{s}{a}\right) \hat{j}$$

$$T = \frac{dr}{ds} = -a \sin\left(\frac{s}{a}\right) \cdot \frac{1}{a} \hat{i} + a \cos\left(\frac{s}{a}\right) \cdot \frac{1}{a} \hat{j}$$

$$= -\sin\left(\frac{s}{a}\right) \hat{i} + \cos\left(\frac{s}{a}\right) \hat{j}$$

$$T' = \frac{dT}{ds} = -\cos\left(\frac{s}{a}\right) \cdot \frac{1}{a} \hat{i} - \sin\left(\frac{s}{a}\right) \cdot \frac{1}{a} \hat{j}$$

$$= -\frac{1}{a} \cos\left(\frac{s}{a}\right) \hat{i} - \frac{1}{a} \sin\left(\frac{s}{a}\right) \hat{j}$$

$$k = |T'| = \sqrt{\left(-\frac{1}{a} \cos\left(\frac{s}{a}\right)\right)^2 + \left(-\frac{1}{a} \sin\left(\frac{s}{a}\right)\right)^2}$$

$$= \sqrt{\frac{1}{a^2} \cos^2 \frac{s}{a} + \frac{1}{a^2} \sin^2 \left(\frac{s}{a}\right)}$$

$$= \sqrt{\frac{1}{a^2} (\cos^2 \frac{s}{a} + \sin^2 \frac{s}{a})}$$

$$= \sqrt{\frac{1}{a^2}} = \frac{1}{a}$$

$$k = \frac{1}{a}$$

$$\rho = \frac{1}{k} = \frac{1}{1/a} = a$$

$$N = \frac{T'}{|T'|} = -\frac{1}{a} \cos\left(\frac{s}{a}\right) \hat{i} - \frac{1}{a} \sin\left(\frac{s}{a}\right) \hat{j}$$

$$N(s) = -\cos\left(\frac{s}{a}\right) \hat{i} - \sin\left(\frac{s}{a}\right) \hat{j} \quad \gamma_a$$

$$\vec{B} = T \times N = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\frac{s}{a} & \cos\frac{s}{a} & 0 \\ -\cos\frac{s}{a} & -\sin\frac{s}{a} & 0 \end{vmatrix}$$

$$\vec{B} = \hat{i}(0-0) - \hat{j}(0-1-\cos^2\frac{s}{a}) + \hat{k}(\sin^2\frac{s}{a} + \cos^2\frac{s}{a})$$

$$\vec{B} = \hat{k}$$

$$\vec{B}' = 0$$

$$\tau(s) = -N(s) \cdot \vec{B}'(s) = (\cos\left(\frac{s}{a}\right) \hat{i} + \sin\left(\frac{s}{a}\right) \hat{j}) \cdot 0$$

$$\tau(s) = 0$$

$$\vec{N}'(s) = \frac{1}{a} \sin\left(\frac{s}{a}\right) \hat{i} - \frac{1}{a} \cos\left(\frac{s}{a}\right) \hat{j}$$

$$\vec{T}' = -\frac{1}{a} \cos\left(\frac{s}{a}\right) \hat{i} - \frac{1}{a} \sin\left(\frac{s}{a}\right) \hat{j}$$

$$\vec{B}' = 0$$

$$\tau(s) = 0$$

$$k = \gamma_a, \quad \rho = a$$

Hence the Required result.

Question:

parabola Equation

$$x = at^2, \quad y = 2at$$

$$\text{or } \vec{r}(t) = at^2 \hat{i} + 2at \hat{j}$$

Solution:

$$x = at^2, \quad y = 2at$$

$$x' = 2at, \quad y' = 2a$$

$$s(t) = \int_0^t \sqrt{4a^2 t^2 + 4a^2} dt = 2a \int_0^t \sqrt{t^2 + 1} dt$$

Question:- Prove that

$$i) T'' = r''' = k'N - k^2T + k\tau B$$

$$ii) T''' = r'''' = (k'' - k^3 - k\tau^2)N - 3kk'T + (2k'\tau + k\tau')B$$

Solution:-

i) we know that

$$\vec{r} = \vec{r}(s)$$

Diff w.r.t s

$$\frac{d\vec{r}}{ds} = \vec{T}$$

$$r' = \vec{T}$$

$$\frac{d^2\vec{r}}{ds^2} = r'' = T' = kN(s)$$

$$\frac{d^3\vec{r}}{ds^3} = r''' = T'' = \frac{d}{ds}(kN(s)) = kN' + k'N$$

$$T'' = r''' = k(-kT + \tau B) + k'N = k'N - k^2T + k\tau B$$

ii)

$$r'''' = T''' = k'N - k^2T + k\tau B$$

$$r'''' = T''' = k'N + k'N' - 2kk'T - k^2T' + k\tau B' + k\tau'B + k'\tau B$$

$$= k''N + k'(kT + \tau B) - 2kk'T - k^2(kN) + k\tau(-\tau N)$$

$$+ k\tau'B + k'\tau B$$

$$= k''N - kk'T + k'\tau B - 2kk'T - k^3N - k\tau^2N + k\tau'B + k'\tau B$$

$$= (k'' - k\tau^2 + k^3)N - 3kk'T + (k\tau' + 2k'\tau)B$$

Hence proved.

Question:-

Prove that

$$\tau = \frac{1}{k^2} |\vec{r}', \vec{r}'', \vec{r}''|$$

Solution:-

$$\vec{r} = \vec{r}(s)$$

$$r' = \frac{d\vec{r}}{ds} = \vec{T}$$

$$r'' = \frac{d^2\vec{r}}{ds^2} = T' = kN$$

$$r''' = \frac{d^3\vec{r}}{ds^3} = T'' = kN' + k'N$$

$$= k(-kT + \tau B) + k'N$$

$$= -k^2T + k\tau B + k'N$$

$$[\vec{r}', \vec{r}'', \vec{r}'''] = [T, kN, -k^2T + k\tau B + k'N]$$

$$(\vec{a}, \vec{b}, \vec{c}) = \vec{a} \times \vec{b} \cdot \vec{c} \text{ using}$$

$$= (T \times kN) \cdot (-k^2T + k\tau B + k'N)$$

$$= k(T \times N) \cdot (-k^2T + k\tau B + k'N)$$

$$= kB \cdot (-k^2T + k\tau B + k'N)$$

$$= k(-k^2B \cdot T + k\tau B \cdot B + k'B \cdot N)$$

$$= k(-k^2(0) + k\tau(1) + k'(0))$$

$$= k(k\tau) = k^2\tau$$

$$\Rightarrow \tau = \frac{1}{k} [\vec{r}', \vec{r}'', \vec{r}''']$$

Question:

(1) Find $\vec{B}''$ and $\vec{B}'''$ if $\vec{B}' = -\tau \vec{N}$

(2) Find $\vec{N}''$ and $\vec{N}'''$ if $\vec{N}' = \tau B - kT$

Solution:

1)

$$\vec{B}' = -\tau \vec{N}$$

$$\vec{B}'' = \frac{d}{ds} (-\tau \vec{N}) = -\tau' \vec{N} - \tau \vec{N}'$$

$$= -\tau'(-kT + \tau B) - \tau \vec{N}'$$

$$= k\tau' T + \tau' \tau B + \tau \vec{N}'$$

$$\vec{B}''' = k\tau' T' + k\tau'' T + k'\tau T + 2\tau' \tau' B + \tau'' B - \tau' \vec{N}' - \tau \vec{N}''$$

$$= k\tau'(kN) + k\tau'' T + k'\tau T + 2\tau' \tau' B + \tau''(-\tau \vec{N}) - \tau(-kT + \tau B)$$

$$= k\tau' N + k\tau'' T + k'\tau T + 2\tau' \tau' B + \tau'' \tau N + k\tau T - \tau \tau' B$$

$$= k\tau' N + 2k\tau'' T + k'\tau T + 3\tau' \tau' B + \tau'' \tau N + k(1 - \tau' \tau) N$$

$$= (k\tau' + \tau'' - \tau') N + (2k\tau'' + k'\tau) T + 3\tau' \tau' B$$

$$\begin{aligned}
 \underline{2} \quad \vec{N}' &= \tau B - k \vec{T} \\
 N'' &= \tau B' + \tau' B - k \tau' - k' \tau \\
 &= \tau(-\tau N) + \tau' B - k(kN) - k' \tau \\
 &= -\tau^2 N + \tau' B - k^2 N - k' \tau \\
 &= (-\tau^2 - k^2) N + \tau' B - k' \tau
 \end{aligned}$$

$$\begin{aligned}
 N''' &= (-\tau^2 - k^2) N' + (-2\tau\tau' - 2kk') N + \tau' B' + \tau'' B - k'\tau' - k''\tau \\
 &= (-\tau^2 - k^2)(-k\tau + \tau B) + (-2\tau\tau' - 2kk') N + \tau'' B - k'(kN) - k''\tau \\
 &\quad - k\tau^2 + k^3\tau - \tau^3 B - k^2\tau B + (-2\tau\tau' - 2kk') N - \tau\tau' N \\
 &\quad + \tau'' B - k'kN - k''\tau \\
 &= -3(\tau\tau' + kk') N + (-\tau^3 - k^3\tau + \tau'') B + (-k\tau^2 + k^3 - k'') \tau
 \end{aligned}$$

Hence proved.

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06/03/019

Envelop, Evolute and Involute

Envelop =

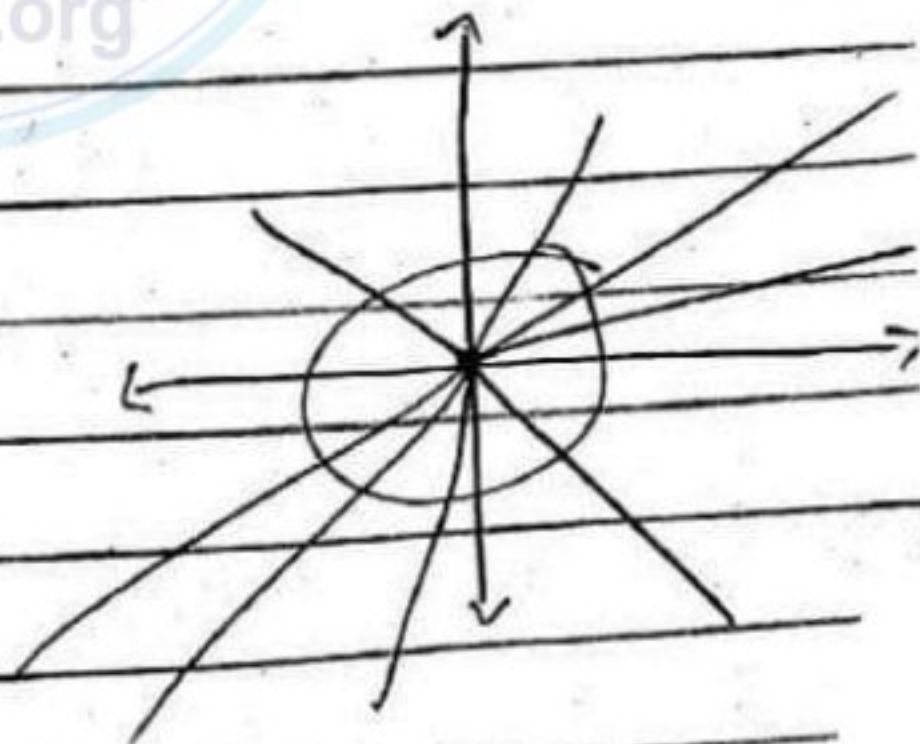
Definition

The envelop of a family of curve is the curve which touches each member of the family.

Method to find the envelop.

Let $y = mx + c$ — (1)

First diff. w.r.t parameter and Eliminate the value of parameter and put in parent equations.



Question: $y = mx + \frac{a}{m}$; Find Envelope
for above family of equations.

Solution:

Here m is the parameter

$$y = mx + \frac{a}{m} \quad \text{--- (1)}$$

Diff w.r.t m

$$0 = x - \frac{a}{m^2} \Rightarrow x = \frac{a}{m^2}$$

$$\Rightarrow m^2 = \frac{a}{x} \Rightarrow m = \pm \sqrt{\frac{a}{x}}$$

put in eq. (1)

$$y = \sqrt{\frac{a}{x}} \cdot x + \frac{a}{\sqrt{\frac{a}{x}}}$$

$$= \sqrt{ax} + \sqrt{ax} = 2\sqrt{ax}$$

$$y^2 = 4ax \quad \text{Ans}$$

Question:

Find the envelop of family of circles
given by

$$(x-c)^2 + (y-c)^2 = 1 \quad \text{--- (1)}$$

Solution:

$$(x-c)^2 + (y-c)^2 = 1 \quad \text{--- (1)}$$

Here c is parameter

Diff w.r.t c

$$2(x-c)(-1) + 2(y-c)(-1) = 0$$

$$-x + c - y + c = 0$$

$$2c = x + y$$

$$c = \frac{x+y}{2} \quad \text{--- (2)}$$

put in eq. (1)

$$\left(x - \frac{x+y}{2}\right)^2 + \left(y - \frac{x+y}{2}\right)^2 = 1$$

$$\left(\frac{2x-x-y}{2}\right)^2 + \left(\frac{2y-x-y}{2}\right)^2 = 1$$

$$\left(\frac{x-y}{2}\right)^2 + \left(\frac{y-x}{2}\right)^2 = 1$$

$$(x-y)^2 + (y-x)^2 = 4$$

$$(x-y)^2 + (x-y)^2 = 4$$

$$2(x-y)^2 = 4$$

$$(x-y)^2 = 2 \Rightarrow x-y = \pm\sqrt{2}$$

$$y = x \pm \sqrt{2}$$

Question: Find the envelop of the family of straight lines

$$y = mx + am^p$$

Solution: Here m is parameter

$$y = mx + am^p \quad \text{--- (1)}$$

Diff w.r.t m

$$0 = x + p a m^{p-1} \quad \text{--- (2)}$$

multiply by m

$$0 = mx + p a m^p$$

$$mx = -p a m^p$$

put in eq. (1)

$$y = -p a m^p + a m^p$$

$$y = -p a m^p + a m^p$$

$$= a m^p (1-p)$$

$$m^p = \frac{y}{a(1-p)}$$

Taking $1/p$ power on both sides

$$m = \frac{y^{1/p}}{(a(1-p))^{1/p}}$$

Put in eq. (1)

$$y = \frac{y^{1/p} x}{(a(1-p))^{1/p}} + a \frac{y}{a(1-p)} = \frac{y^{1/p} x}{a^{1/p}(1-p)^{1/p}} + \frac{y}{1-p}$$

$$1 = \frac{y^{1/p-1} x}{a^{1/p}(1-p)^{1/p}} + \frac{1}{1-p}$$

$$= \frac{y^{1/p-1} x (1-p) + a^{1/p}(1-p)^{1/p}}{a^{1/p}(1-p)^{1/p}(1-p)}$$

$$a^{1/p}(1-p)^{1/p}(1-p) = x y^{1/p-1} (1-p)^{1/p} + a^{1/p}(1-p)^{1/p}$$

$$a^{1/p}(1-p)^{1/p}(1-p) = (1-p)^{1/p} [x y^{1/p-1} (1-p)^{1/p} + a^{1/p}]$$

$$a^{1/p}(1-p) = x y^{1/p-1} (1-p)^{1/p} + a^{1/p}$$

Question:

$$x \cos \alpha + y \sin \alpha - p = 0$$

Find envelop

Solution:

$$x \cos \alpha + y \sin \alpha - p = 0 \quad \text{--- (1)}$$

Here α is parameter

Diff w.r.t α

$$x(-\sin \alpha) + y \cos \alpha = 0$$

$$y \cos \alpha = x \sin \alpha$$

$$\frac{y}{x} = \tan \alpha$$

Express $\sin \alpha$ & $\cos \alpha$ in terms of $\tan \alpha$

$$1 + \tan^2 \alpha = \sec^2 \alpha$$

$$1 + \tan^2 \alpha = \frac{1}{\cos^2 \alpha}$$

$$\cos^2 \alpha = \frac{1}{1 + \tan^2 \alpha} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} = \frac{x^2}{x^2 + y^2}$$

$$\cos \alpha = \pm \frac{x}{\sqrt{x^2 + y^2}}$$

$$1 + \tan^2 \alpha = \sec^2 \alpha$$

$$\sin^2 \alpha (1 + \tan^2 \alpha) = \frac{\sin^2 \alpha}{\cos^2 \alpha}$$

$$\sin^2 \alpha = \frac{\tan^2 \alpha}{1 + \tan^2 \alpha} = \frac{(y/x)^2 \cdot x^2}{x^2 + y^2} = \frac{y^2}{x^2 + y^2}$$

$$\sin \alpha = \pm \frac{y}{\sqrt{x^2 + y^2}}$$

put in eq. ①

$$x \left(\frac{x}{\sqrt{x^2 + y^2}} \right) + y \left(\frac{y}{\sqrt{x^2 + y^2}} \right) - p = 0$$

$$\frac{x^2}{\sqrt{x^2 + y^2}} + \frac{y^2}{\sqrt{x^2 + y^2}} - p = 0$$

$$\frac{x^2 + y^2}{\sqrt{x^2 + y^2}} - p = 0$$

$$\pm \sqrt{x^2 + y^2} = p$$

$$\Rightarrow p^2 = x^2 + y^2$$

Other method to find

$\sin \alpha, \cos \alpha$

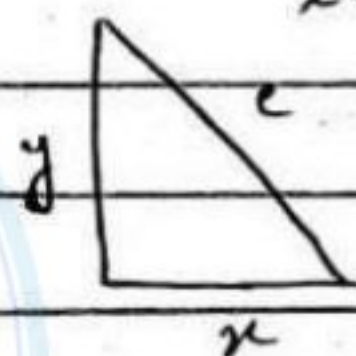
$$c^2 = x^2 + y^2$$

$$c = \sqrt{x^2 + y^2}$$

$$\sin \alpha = \pm \frac{y}{\sqrt{x^2 + y^2}}$$

$$\cos \alpha = \pm \frac{x}{\sqrt{x^2 + y^2}}$$

$$\tan \alpha = \frac{y}{x}$$



Question

Find the envelope of the family of ellipse defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

provided that $a^2 + b^2 = 1$

Solution:

$$\frac{x^2}{a^2} + \frac{y^2}{1-a^2} = 1 \quad \text{--- ①}$$

Here parameter is a

Diff w.r.t a

$$\frac{-2x^2}{a^3} + \frac{-y^2(-2a)}{(1-a^2)^2} = 0$$

$$\frac{2ay^2}{(1-a^2)^2} - \frac{2x^2}{a^3} = 0 \Rightarrow \frac{ay^2}{(1-a^2)^2} = \frac{x^2}{a^3}$$

$$y^2 a^4 = x^2 (1-a^2)^2$$

$$\Rightarrow y a^2 = x (1-a^2)$$

$$y a^2 = x - x a^2$$

$$y a^2 + x a^2 = x$$

$$\Rightarrow a^2 = \frac{x}{y+x}$$

put in eq. (1)

$$\frac{x^2}{\frac{x}{y+x}} + \frac{y^2}{1 - \frac{x}{y+x}} = 1$$

$$\frac{x^2(y+x)}{x} + \frac{y^2(y+x)}{y} = 1$$

$$x(y+x) + y(y+x) = 1 \Rightarrow xy + x^2 + y^2 + xy = 1$$

$$\Rightarrow (x+y)^2 = 1$$

$$\Rightarrow x+y = \pm 1$$

ignore negative sign

$$\Rightarrow x+y = 1 \quad \text{Ans.}$$

Question:

Find the envelop of the family of parabola

Solution:

Standard Form of parabola

$$(x-h)^2 = 4p(y-k)$$

let $h=k=c$

$$\Rightarrow (x-c)^2 = 4p(y-c) \quad \text{--- (1)}$$

Here parameter is c

Diff w.r.t c

$$2(x-c)(-1) = 4p(0-1)$$

$$-2(x-c) = -4p$$

$$x - c = 2p$$

$$x - 2p = c \quad \text{--- (2)}$$

Put in eq. (1)

$$(x - (x - 2p))^2 = 4p(y - (x - 2p))$$

$$(x - x + 2p)^2 = 4p(y - x + 2p)$$

$$(2p)^2 = 4p(y - x + 2p)$$

$$4p^2 = 4p(y - x + 2p)$$

$$p = y - x + 2p$$

$$x - y = 2p - p$$

$$x - y = p$$

Ans

Question

Find the envelop of the family of hyperbola defined by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

provided that

$$a^2 + b^2 = 1$$

Solution:

$$\frac{x^2}{a^2} - \frac{y^2}{1+a^2} = 1 \quad \text{--- (1)}$$

Here parameter is a

Diff w.r.t a

$$\frac{-2x^2}{a^3} + \frac{y^2(+2a)}{(1+a^2)^2} = 0$$

$$\frac{-2x^2}{a^3} + \frac{2ay^2}{(1+a^2)^2} = 0$$

$$\frac{-2x^2}{a^3} = \frac{-2ay^2}{(1+a^2)^2}$$

$$-x^2(1+a^2)^2 = -a^4y^2$$

$$a^2y = x(1+a^2) \quad \Rightarrow \quad a^2y = -x + a^2x$$

$$a^2(y+x) = x \quad \Rightarrow \quad x = a^2(x-y)$$

$$a^2 = \frac{x}{x-y} \quad \Rightarrow \quad \frac{x}{x-y} = a^2$$

put in eq. (1)

$$\frac{x^2}{(-y+x)^2} - \frac{y^2}{\frac{x}{-y+x} - 1} = 1$$

$$\frac{x^2(-y+x)}{x} - \frac{y^2(-y+x)}{x+y-x} = 1$$

$$-xy + x^2 + \frac{y^2(-y+x)}{y} = 1$$

$$-xy + x^2 + y(-y+x) = 1 \Rightarrow -xy + x^2 + y^2 + xy = 1$$

$$\Rightarrow x^2 + y^2 + 2xy = 1$$

$$(x+y)^2 = 1$$

$$\Rightarrow x+y = \pm 1$$

$$\Rightarrow x-y = \pm 1 \quad \text{Ans.}$$

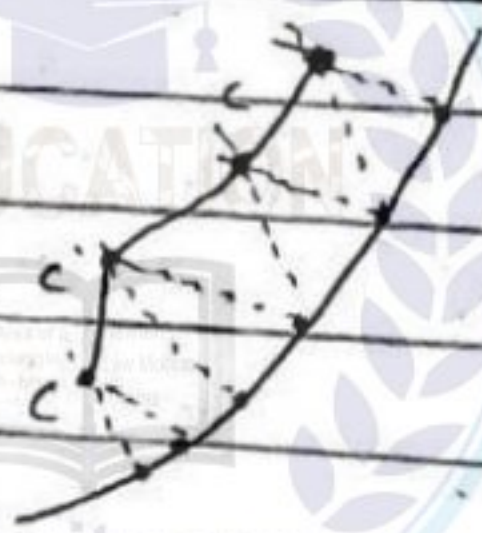
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Evolute and Involute:-

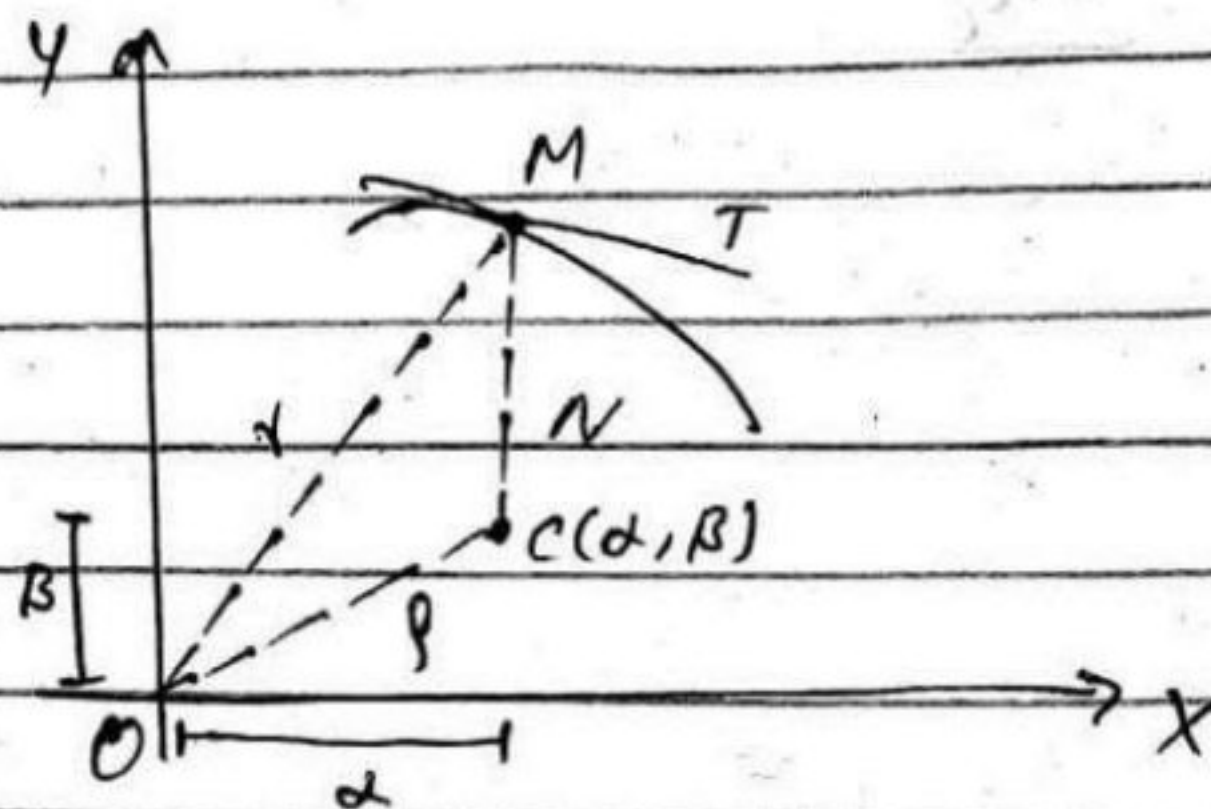
Let a plane curve C be given by the Equation $r = r(s)$

where the parameter " s " means the arc length of the curve. Suppose that at each point, the curvature of the curve is not zero, $k(s) \neq 0$.

Then at any point M , we can define a finite radius of curvature $R = R(s) = \frac{1}{k(s)}$ on the normal N we draw the segment MC equal to the radius of curvature R at the point M as shown in fig.



Path of C is called Evolute. Original curve is Involute



The point c is called the centre of curvature of the curve c at point M . If Radius vector is of the centre of curvature and is denoted by ρ . Then

$$\rho = OM + MC$$

$$= r + PN$$

where N is normal vector $N = \frac{T'}{K}$

$$\rho = r + R \cdot \frac{T'}{K}$$

$$= r + R \cdot R T'$$

$$= r + R^2 T' = r + R^2 \frac{dr}{ds^2} \quad \text{--- (1)}$$

We can find the centre of curvature. The position of the centre of curvature corresponding to the point M is calculated by eq. (1)

We denote the centre of curvature by point c with coordinates (α, β) . These coordinates are defined as

$$\alpha = x - \frac{y'(x'^2 + y'^2)}{x'y'' - x''y'}$$

$$\beta = y + \frac{x'(x'^2 + y'^2)}{x'y'' - x''y'}$$

$$\alpha = x - \frac{1 + (y')^2}{y''} y'$$

$$\beta = y + \frac{1 + (x')^2}{x''} x'$$

Question:

Find the Evolute of Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

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Solution:

$$\begin{aligned} x &= a \cos t, & y &= b \sin t \\ x' &= -a \sin t & y' &= b \cos t \\ x'' &= -a \cos t & y'' &= -b \sin t \end{aligned}$$

$$\alpha = x - \frac{y'(x'^2 + y'^2)}{x'y'' - x''y'} = x - \frac{b \cos t (a^2 \sin^2 t + b^2 \cos^2 t)}{-a \sin t (-b \sin t) - (-a \cos t)(b \cos t)}$$

$$\alpha = a \cos t - \frac{b \cos t (a^2 \sin^2 t + b^2 \cos^2 t)}{ab(\sin^2 t + \cos^2 t)}$$

$$\begin{aligned} a\alpha &= a^2 \cos t - a^2 \cos t (1 - \cos^2 t) + b^2 \cos^3 t \\ &= a^2 \cos t - a^2 \cos t + a^2 \cos^3 t + b^2 \cos^3 t \end{aligned}$$

$$a\alpha = (a^2 + b^2) \cos^3 t \quad (1)$$

$$\beta = y + \frac{x'(x'^2 + y'^2)}{x'y'' - x''y'} = b \sin t + \frac{a \sin t (a^2 \sin^2 t + b^2 \cos^2 t)}{ab(\sin^2 t + \cos^2 t)}$$

$$= b \sin t + \frac{\sin t (a^2 \sin^2 t + b^2 (1 - \sin^2 t))}{b}$$

$$\begin{aligned} b\beta &= b^2 \sin t + a^2 \sin^3 t - b^2 \sin t + b^2 \sin^3 t \\ b\beta &= (b^2 + a^2) \sin^3 t \end{aligned}$$

(1) $\Rightarrow$

$$\frac{a\alpha}{a^2 + b^2} = \cos^3 t$$

(2)

(3)

② $\Rightarrow$

$$\frac{-b\beta}{a^2-b^2} = \sin^2 t \quad \text{--- ⑥}$$

Taking $2/3$ power of eq. ③ & ⑥. Then adding

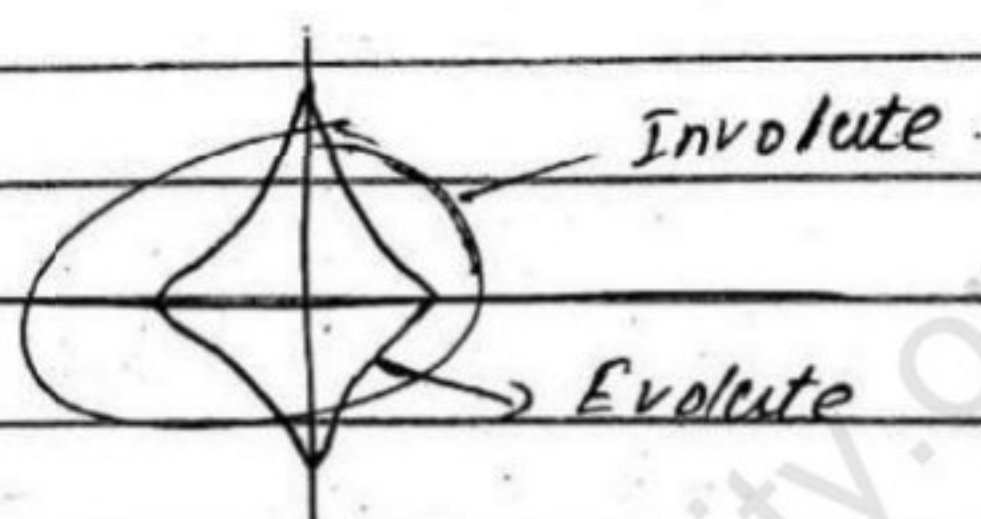
$$\left(\frac{a\alpha}{a^2-b^2}\right)^{2/3} + \left(\frac{b\beta}{a^2-b^2}\right)^{2/3} = \cos^2 t + \sin^2 t$$

$$(a\alpha)^{2/3} + (b\beta)^{2/3} = (a^2-b^2)^{2/3}$$

Let $x = a\alpha$, $y = b\beta$, $A = a^2-b^2$

$$x^{2/3} + y^{2/3} = A^{2/3}$$

Astroid. Ans.



Question:

Find the equation of the evolute of the hyperbola $y = \frac{1}{x}$

Solution:

$$y = \frac{1}{x}$$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

$$\alpha = x - \frac{1 + (y')^2}{y''} y'$$

$$= x - \frac{1 + \left(-\frac{1}{x^2}\right)^2 \left(-\frac{1}{x^2}\right)}{\frac{2}{x^3}}$$

$$= x + \frac{x^3 \left(1 + \frac{1}{x^4}\right)}{2x^2} = x + \frac{x + \frac{1}{x^3}}{2}$$

$$\alpha = \frac{2x^4 + x^4 + 1}{2x^3} = \frac{3x^4 + 1}{2x^3}$$

$$\beta = y + \frac{1 + (y')^2}{y''} = y + \frac{1 + \left(-\frac{1}{x^2}\right)^2}{\frac{2}{x^3}}$$

$$\beta = \frac{1}{x} + \frac{x^3(1 + \frac{1}{x^4})}{2} = \frac{1}{x} + \frac{x(x^4+1)}{2x^5}$$

$$\beta = \frac{x^4+3}{2x}$$

$$\alpha + \beta = \frac{3x^4+1}{2x^3} + \frac{x^4+3}{2x} = \frac{3x^4+1+x^6+3x^2}{2x^3}$$

$$= \frac{x^6+3x^4+3x^2+1}{2x^3} = \frac{(x^2+1)^3}{2x^3} = \frac{1}{2} \left(\frac{x^2+1}{x^3} \right)^3$$

$$= \frac{1}{2} \left(x + \frac{1}{x} \right)^3$$

$$2(\alpha + \beta) = \left(x + \frac{1}{x} \right)^3 \quad \text{--- (1)}$$

$$\alpha - \beta = \frac{3x^4+1}{2x^3} - \frac{x^4+3}{2x} = \frac{3x^4+1-x^6-3x^2}{2x^3}$$

$$(\alpha - \beta) = \frac{-(x^6-3x^4+3x^2-1)}{2x^3} = \frac{-(x^2-1)^3}{2x^3}$$

$$2(\alpha - \beta) = -\left(x - \frac{1}{x} \right)^3 \quad \text{--- (2)}$$

Taking $\sqrt[3]{}$ power of (1) & (2) and adding subtracting second from the first

$$(4)^{\frac{1}{3}} \left[(\alpha + \beta)^{\frac{2}{3}} - (\alpha - \beta)^{\frac{2}{3}} \right] = \left(x + \frac{1}{x} \right)^2 - \left(x - \frac{1}{x} \right)^2$$

$$= x^2 + \frac{1}{x^2} + 2 - x^2 + \frac{1}{x^2} - 2$$

$$(\alpha + \beta)^{\frac{2}{3}} - (\alpha - \beta)^{\frac{2}{3}} = \frac{4}{(4)^{\frac{1}{3}}}$$

Evolute of the hyperbola:

Q#2 Find the equation of the evolute

$$x = t - \sin t, \quad y = 1 - \cos t$$

Solution

$$x = t - \sin t$$

$$y = 1 - \cos t$$

$$x' = 1 - \cos t$$

$$y' = \sin t$$

$$x'' = \sin t$$

$$y'' = \cos t$$

$$\begin{aligned} \alpha &= x - \frac{y'(x'^2 + y'^2)}{x'y'' - x''y'} = t - \sin t - \frac{\sin t(1 + \cos^2 t - 2\cos t + \sin^2 t)}{\cos t(1 - \cos t) - \sin^2 t} \\ &= t - \sin t - \frac{2\sin t(1 - \cos t)}{-(1 - \cos t)} = t - \sin t + 2\sin t \\ &= t + \sin t \end{aligned}$$

$$\alpha = t + \sin t$$

$$\begin{aligned} \beta &= y + \frac{x'(x'^2 + y'^2)}{x'y'' - x''y'} = 1 - \cos t + \frac{(1 - \cos t)(1 + \cos^2 t - 2\cos t + \sin^2 t)}{-(1 - \cos t)} \\ &= 1 - \cos t + 2(1 - \cos t)(1 - \cos t) \\ &= 1 - \cos t + 2 - 2\cos t = \cos t - 1 \end{aligned}$$

$$\beta = 1 - \cos t + 2 + 2\cos t = \cos t - 1$$

$$\alpha = t + \sin t$$

$$\beta = \cos t - 1$$

α, β in terms of τ

$$\alpha = \tau + \pi + \sin(\tau + \pi) = \tau - \sin \tau + \pi$$

$$\beta = \cos(\tau + \pi) - 1 = -\cos \tau - 1 = (1 - \cos \tau) + 2$$

Evolute of the cycloid is also cycloid.

Q#3 Find evolute of the logarithmic spiral $r = e^\theta$

Solution:-

This curve in cartesian coordinates is

$$x = r \cos \theta = e^\theta \cos \theta$$

$$y = r \sin \theta = e^\theta \sin \theta$$

$$x' = -e^\theta \sin \theta + e^\theta \cos \theta$$

$$x'' = -e^\theta \cos \theta - e^\theta \sin \theta - e^\theta \sin \theta + e^\theta \cos \theta$$

$$x'' = -2e^\theta \sin \theta$$

$$y' = e^\theta \cos \theta + e^\theta \sin \theta$$

$$y'' = -e^\theta \sin \theta + e^\theta \cos \theta + e^\theta \cos \theta + e^\theta \sin \theta$$

$$y'' = 2e^\theta \cos \theta$$

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$$\alpha = x - y' \frac{(x'^2 + y'^2)}{x'y'' - x''y'} = e^\theta \cos \theta - \frac{e^\theta (\cos \theta + \sin \theta)}{e^\theta (\cos \theta - \sin \theta) - (-2e^\theta \sin \theta)(\cos \theta + \sin \theta)}$$

$$\begin{aligned} x'^2 + y'^2 &= (e^\theta (\cos \theta - \sin \theta))^2 + (e^\theta (\cos \theta + \sin \theta))^2 \\ &= e^{2\theta} (\cos^2 \theta + \sin^2 \theta - 2 \sin \theta \cos \theta + \cos^2 \theta + \sin^2 \theta + 2 \sin \theta \cos \theta) \\ &= 2e^{2\theta} \end{aligned}$$

$$\begin{aligned} x'y'' - x''y' &= e^\theta (\cos \theta - \sin \theta) \cdot 2e^\theta \cos \theta - (-2e^\theta \sin \theta)(\cos \theta + \sin \theta)e^\theta \\ &= e^{2\theta} (2 \cos^2 \theta - 2 \sin \theta \cos \theta + 2 \sin \theta \cos \theta + 2 \sin^2 \theta) \\ &= 2e^{2\theta} \end{aligned}$$

$$\alpha = e^\theta \cos \theta - (e^\theta \cos \theta + e^\theta \sin \theta) \cdot \frac{2e^{2\theta}}{2e^{2\theta}} = -e^\theta \sin \theta = -y$$

$$\beta = e^\theta \sin \theta + (e^\theta \cos \theta - e^\theta \sin \theta) \cdot \frac{2e^{2\theta}}{2e^{2\theta}} = e^\theta \cos \theta = x$$

$$\alpha = -e^\theta \sin \theta, \quad \beta = e^\theta \sin \theta$$

The evolute of the logarithmic spiral $r = e^\theta$ is again the same curve rotated counterclockwise by the angle $\frac{\pi}{2}$.

Q#4 Find the evolute of the parabola

$$y = x^2$$

Solution:-

$$\frac{dy}{dx} = 2x \quad ; \quad \frac{d^2y}{dx^2} = 2$$

$$\alpha = x - \frac{y'(1 + (y')^2)}{y''} = x - \frac{2x(1 + 4x^2)}{2}$$

$$\alpha = x - x + 4x^3 = -4x^3 \quad \text{--- (1)}$$

$$\beta = y + \frac{1 + (y')^2}{y''} = x^2 + \frac{1 + 4x^2}{2} = x^2 + 2x^2 + \frac{1}{2}$$

$$\beta = 3x^2 + \frac{1}{2} \quad \text{--- (2)}$$

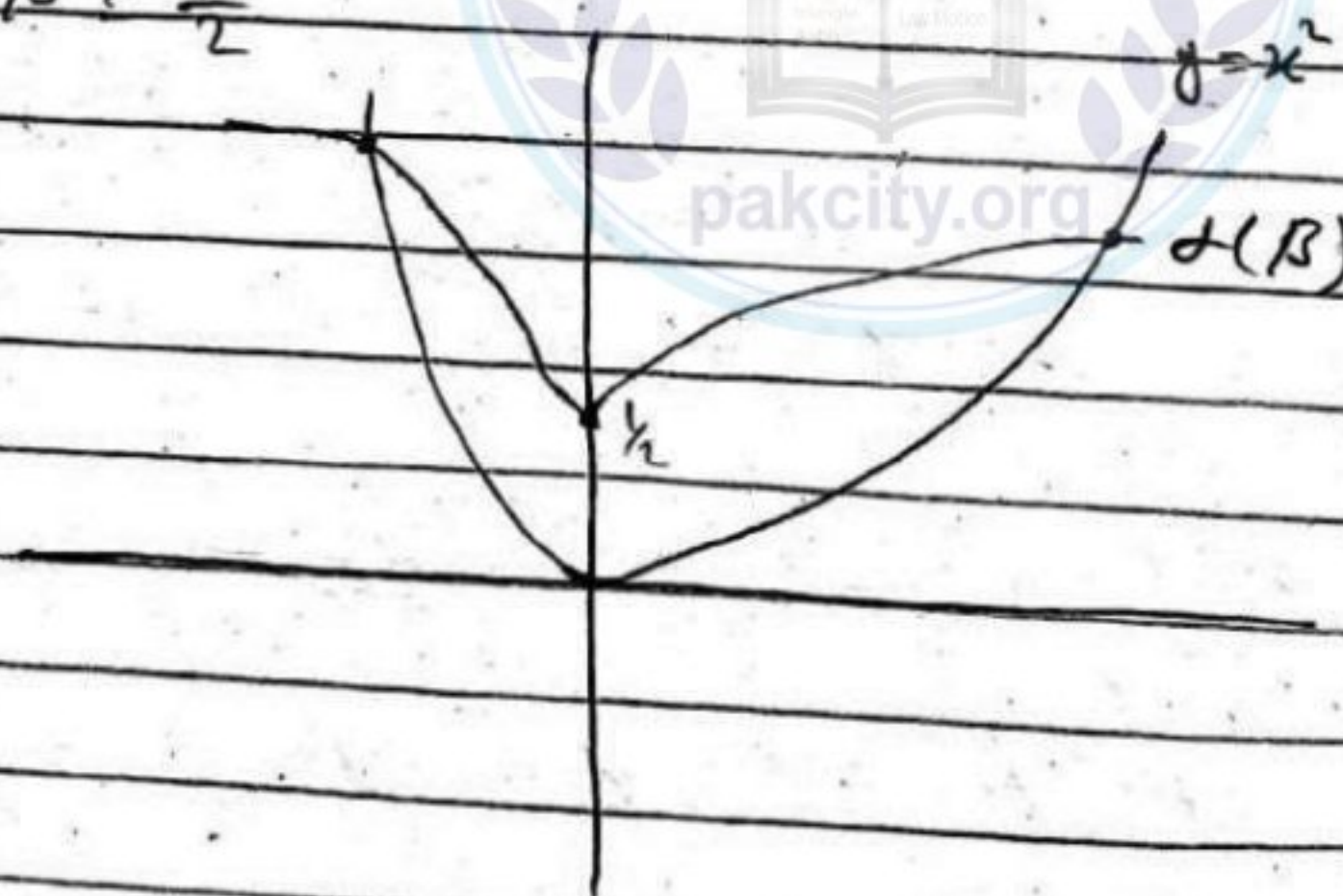
$$\beta - \frac{1}{2} = 3x^2$$

$$\frac{\beta}{3} - \frac{1}{6} = x^2 \Rightarrow x = \pm \left(\frac{\beta}{3} - \frac{1}{6} \right)^{1/2}$$

put in eq. (1)

$$\alpha = -4 \left[\pm \left(\frac{\beta}{3} - \frac{1}{6} \right)^{1/2} \right]^3 = \pm 4 \left(\frac{\beta}{3} - \frac{1}{6} \right)^{3/2}$$

where $\beta \geq \frac{1}{2}$



Q#5 Prove that the curve given by the equations.

$$x = R(\cos t + t \sin t)$$

$$y = R(\sin t - t \cos t)$$

is the involute of the circle of radius R centered at origin.

Solution=

Diff. w.r.t t

$$x = R(\cos t + t \sin t)$$

$$x' = R(-\sin t + \sin t + t \cos t)$$

$$x' = R(t \cos t)$$

$$x'' = R(\cos t - t \sin t)$$

$$y = R(\sin t - t \cos t)$$

$$y' = R(\cos t - \cos t + t \sin t)$$

$$y' = R(t \sin t)$$

$$y'' = R(\sin t + t \cos t)$$

$$\alpha = x - \frac{y'(x'^2 + y'^2)}{x'y'' - x''y'} = R(\cos t + t \sin t) - \frac{R(t \sin t) R t (\cos^2 t + \sin^2 t)}{R t \cos t (R \sin t + t \cos t) - R(\cos t - t \sin t) (R t \sin t)}$$

$$= R(\cos t + t \sin t) - \frac{R t^2 \sin t}{R t^2 (\cos^2 t + \sin^2 t - \cos t \sin t + t \sin^2 t)}$$

$$= R(\cos t + t \sin t) - \frac{R t^2 \sin t}{t(\sin^2 t + \cos^2 t)} = R \cos t + R t \sin t - R t \sin t$$

$$\alpha = R \cos t = x \quad \text{--- (1)}$$

$$\beta = y + \frac{x'(x'^2 + y'^2)}{x'y'' - x''y'} = y + \frac{R(t \cos t) (R t (\cos^2 t + \sin^2 t))}{R t^2}$$

$$\beta = R \sin t - R t \cos t + R t \cos t = R \sin t = y \quad \text{--- (2)}$$

Taking square eq. (1) & (2) and adding

$$\tilde{x}^2 + \tilde{y}^2 = R^2 (\cos^2 t + \sin^2 t)$$

$$\tilde{x}^2 + \tilde{y}^2 = R^2 \quad \text{Hence proved}$$

Q#6 Find the evolute of given equations 43

$$x^2 + y^2 = R^2$$

Solution:-

Parametric form of circle

$$x = R \cos t$$

$$y = R \sin t$$

$$x' = -R \sin t$$

$$y' = R \cos t$$

$$x'' = -R \cos t$$

$$y'' = -R \sin t$$

$$\alpha = x - \frac{y'(x'^2 + y'^2)}{x'y'' - x''y'} = R \cos t - \frac{R \cos t (R^2 (\sin^2 t + \cos^2 t))}{R^2 (\sin^2 t + \cos^2 t)}$$

$$\alpha = R \cos t - R \cos t = 0$$

$$\beta = y + \frac{x'(x'^2 + y'^2)}{x'y'' - y'x''} = R \sin t + \frac{-R \sin t (R^2)}{R^2}$$

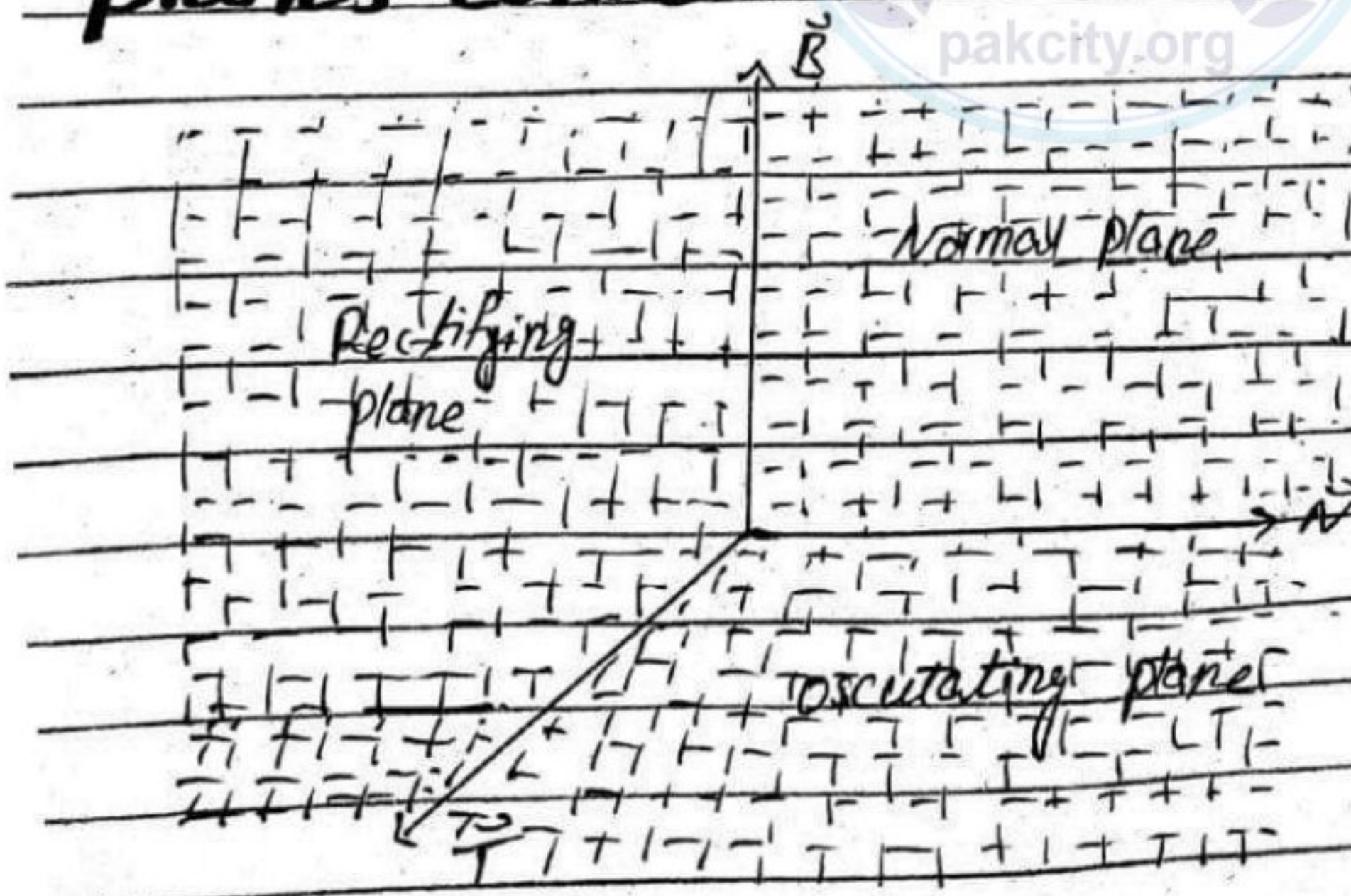
$$\beta = R \sin t - R \sin t = 0$$

$$\beta = 0, \alpha = 0$$



14/03/09

Planes Connected with TBN Frame



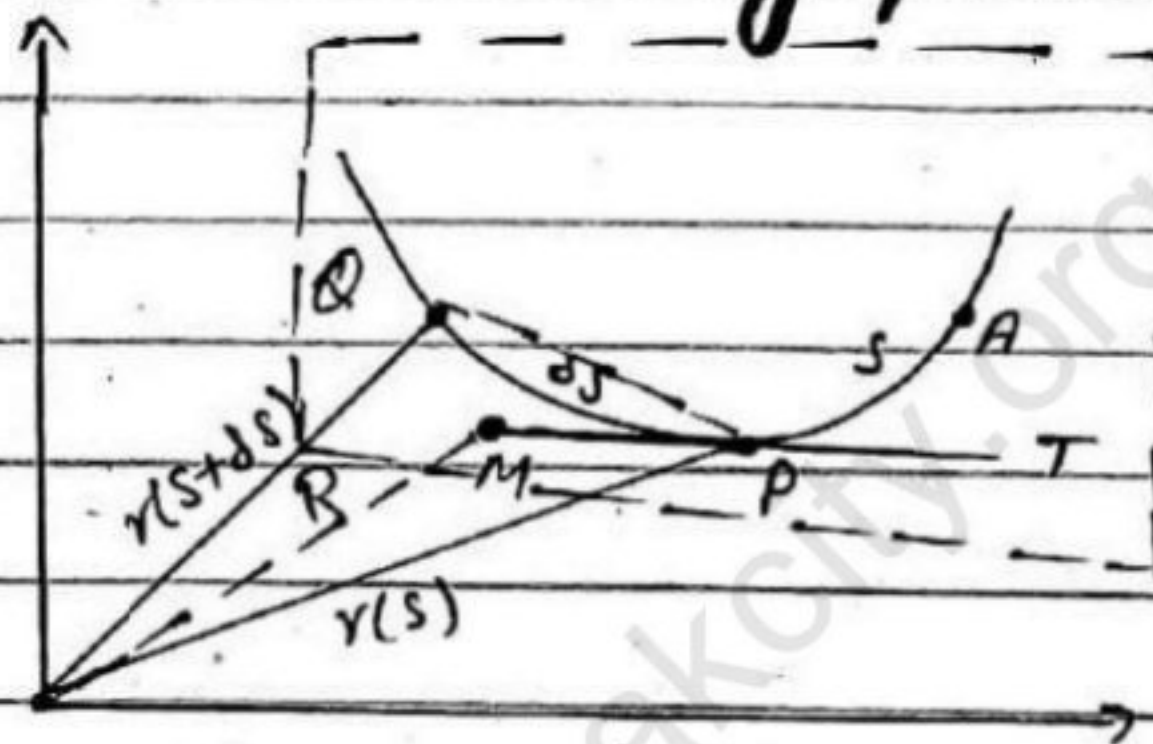
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A plane containing $\vec{N}$ and $\vec{B}$ is called normal plane. The vector $\vec{T}$ is normal vector on this plane.

The plane containing $\vec{N}$ and $\vec{T}$ is called osculating plane. The vector $\vec{B}$ is normal vector on this plane.

The plane containing $\vec{B}$ and $\vec{T}$ is called a rectifying plane. The vector $\vec{N}$ is the normal vector on this plane.

Equation of osculating plane:



Let C be a space curve with parameter s . Let P and Q be two neighbouring points on the curve C . Let a tangent unit vector $\vec{T}$ be at point P . The arc length from A to P be s and from P to Q be δs .

$$\widehat{AP} = s, \quad \widehat{PQ} = \delta s, \quad \widehat{AQ} = s + \delta s$$

The position vectors of P and Q are $\vec{OP} = r(s)$, $\vec{OQ} = r(s + \delta s)$.

The osculating plane at point P is defined as

$$OSP \text{ at point } P = \lim_{Q \rightarrow P} \text{plane determined by } T \text{ \& } Q$$

$$= \lim_{\delta s \rightarrow 0} \text{plane determined by } T \text{ \& } Q$$

Let τ be the osculating plane at point P . Suppose that R be the position vector for point M on the osculating plane. Then the vectors $\vec{PM}$, $\vec{\tau}$ and $\vec{PQ}$ are coplanar, by definition of scalar tripple products,

$$[\vec{PM}, \vec{\tau}, \vec{PQ}] = 0$$

$$[R - r(s), r'(s), r(s + \delta s) - r(s)] = 0$$

Now the equation for osculating plane

$$\lim_{\delta s \rightarrow 0} [R - r(s), r'(s), r(s + \delta s) - r(s)] = 0 \quad \text{--- (1)}$$

By taking and using Tayler theorem

$$r(s + \delta s) = r(s) + \delta s r'(s) + \frac{\delta s^2}{2!} r''(s) + \frac{\delta s^3}{3!} r'''(s) + \dots$$

$$r(s + \delta s) - r(s) = \delta s r'(s) + \frac{\delta s^2}{2!} r''(s) + \frac{\delta s^3}{3!} r'''(s) + \dots$$

(1) $\Rightarrow$

$$\lim_{\delta s \rightarrow 0} [R - r(s), r'(s), \delta s r'(s) + \frac{\delta s^2}{2!} r''(s) + \dots] = 0$$

$$\lim_{\delta s \rightarrow 0} [R - r(s), r'(s), \delta s r'(s)] + \lim_{\delta s \rightarrow 0} [R - r(s), r'(s), \frac{\delta s^2}{2!} r''(s) + \dots] = 0$$

$$0 + \lim_{\delta s \rightarrow 0} \frac{\delta s^2}{2} [R - r(s), r'(s), r''(s) + \text{Terms containing } \delta s] = 0$$

$$\lim_{\delta s \rightarrow 0} [R - r(s), r'(s), r''(s) + \text{Terms containing } \delta s] = 0$$

$$[R - r(s), r'(s), r''(s)] = 0$$

Equation for osculating plane

$x - x$	$y - y$	$z - z$	
$\dot{x}$	$\dot{y}$	$\dot{z}$	$= 0$
$\ddot{x}$	$\ddot{y}$	$\ddot{z}$	

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Question= Find the Equation for osculating plane at the point on the helix

Solution=

$$\begin{aligned} x &= a \cos t, & y &= a \sin t, & z &= bt \\ \dot{x} &= -a \sin t, & \dot{y} &= a \cos t, & \dot{z} &= b \\ \ddot{x} &= -a \cos t, & \ddot{y} &= -a \sin t, & \ddot{z} &= 0 \end{aligned}$$

$x - a \cos t$	$y - a \sin t$	$z - bt$	
$-a \sin t$	$a \cos t$	b	$= 0$
$-a \cos t$	$-a \sin t$	0	

$$(x - a \cos t)(0 + a b \sin t) - (y - a \sin t)(0 + a b \cos t) + (z - bt)(a^2 \sin^2 t + \cos^2 t) = 0$$

$$a b \sin t x - a^2 b \sin t \cos t - a b \cos t y + a^2 b \sin t \cos t + z a^2 - b t a^2 = 0$$

$$a b \sin t x - a b \cos t y + a^2 z = a^2 b t$$

$$b \sin t x - b \cos t y + a z = a b t$$

Ans

Question= Find the Equation for osculating plane at the point

$$x = 3t, \quad y = 3t^2, \quad z = 2t^3$$

Solution=

$$\begin{aligned} x &= 3t, & y &= 3t^2, & z &= 2t^3 \\ \dot{x} &= 3, & \dot{y} &= 6t, & \dot{z} &= 6t^2 \\ \ddot{x} &= 0, & \ddot{y} &= 6, & \ddot{z} &= 12t \end{aligned}$$

$x - 3t$	$y - 3t^2$	$z - 2t^3$	
3	$6t$	$6t^2$	$= 0$
0	6	$12t$	

$$(x - 3t)(72t - 36t^2) - (y - 3t^2)(36t - 0) + (z - 2t^3)(18 - 0) = 0$$

$$36t(x - 3t) - 36t^2 y + 108t^3 + 18z - 36t^3 = 0$$

$$36t^2 x - 108t^3 - 36t^2 y + 108t^3 + 18z - 36t^3 = 0$$

$$36t^2x - 36ty + 18z = 36t^3$$

$$18(2t^2x - 2ty + z) = 36t^3$$

$$2t^2x - 2ty + z = 2t^3 \quad \text{Ans}$$

15/03/09

Question= Find Equation of normal, rectifying and osculating plane for

$$r(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$$

corresponding to $t=2$ and passing from point $(2, 4, 8)$

Solution=

Formula

$(a\hat{i} + b\hat{j} + c\hat{k})$ normal vector on plane

$$(a\hat{i} + b\hat{j} + c\hat{k}) \cdot ((x-x_0)\hat{i} + (y-y_0)\hat{j} + (z-z_0)\hat{k}) = 0$$

Equation for normal plane

Normal plane is determined by $\vec{B}$ and $\vec{N}$.

$\vec{T}$ is the perpendicular vector on normal plane.

$$\vec{T} = r'(t) = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

$$\vec{T}|_{t=2} = r'(2) = \hat{i} + 4\hat{j} + 12\hat{k}$$

$\hat{i} + 4\hat{j} + 12\hat{k}$ is perpendicular on normal plane

Eqn. for normal plane at $(2, 4, 8)$ is

$$(\hat{i} + 4\hat{j} + 12\hat{k}) \cdot ((x-2)\hat{i} + (y-4)\hat{j} + (z-8)\hat{k}) = 0$$

$$(x-2) + 4(y-4) + 12(z-8) = 0$$

$$x-2 + 4y-16 + 12z-96 = 0$$

$$x + 4y + 12z - 114 = 0$$

$$x + 4y + 12z = 114 \quad \text{equation of normal plane.}$$

Equation for rectifying plane.

This plane is determined by $\vec{B}$ and $\vec{T}$. The vector $\vec{N}$ is $\perp$ on this plane.

We need to find $\vec{N}$

$$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}$$

$$\vec{T} = i + 2t\hat{j} + 3t^2\hat{k}$$

$$\hat{T} = \frac{i + 2t\hat{j} + 3t^2\hat{k}}{\sqrt{1 + 4t^2 + 9t^4}} = (1 + 4t^2 + 9t^4)^{-1/2} (i + 2t\hat{j} + 3t^2\hat{k})$$

$$\hat{T}' = -\frac{1}{2}(1 + 4t^2 + 9t^4)^{-3/2} (8t + 36t^3) (i + 2t\hat{j} + 3t^2\hat{k}) +$$

$$(1 + 4t^2 + 9t^4)^{-1/2} (2\hat{j} + 6t\hat{k})$$

$$\hat{T}'|_{t=2} = -\frac{1}{2}(1 + 16 + 144)^{-3/2} (16 + 288) (i + 4\hat{j} + 12\hat{k}) + (1 + 16 + 144)^{-1/2} (2\hat{j} + 12\hat{k})$$

$$= -\frac{1}{2}(161)^{-3/2} (304) (i + 4\hat{j} + 12\hat{k}) + (161)^{-1/2} (2\hat{j} + 12\hat{k})$$

$$\hat{T}'|_{t=2} = \frac{1}{(161)^{3/2}} (-152i - 608\hat{j} - 1824\hat{k} + 322\hat{j} + 1932\hat{k})$$

$$\hat{T}' = \frac{1}{(161)^{3/2}} (-152i - 286\hat{j} + 108\hat{k})$$

$$|\hat{T}'| = \frac{1}{(161)^{3/2}} (\sqrt{152^2 + 286^2 + 108^2}) = \frac{1}{(161)^{3/2}} (341.4147)$$

$$N = \frac{\hat{T}'}{|\hat{T}'|} = \frac{(161)^{3/2}}{(161)^{3/2} (341.4147)} (-152i - 286\hat{j} + 108\hat{k})$$

$$\vec{N} = \frac{-2}{341.4147} (76i + 143\hat{j} - 54\hat{k})$$

$$\vec{N} = 76i + 143\hat{j} - 54\hat{k}$$

Equation for rectifying plane at (2, 4, 8) is

$$(76i + 143\hat{j} - 54\hat{k}) \cdot ((x-2)i + (y-4)\hat{j} + (z-8)\hat{k}) = 0$$

$$76(x-2) + 143(y-4) - 54(z-8) = 0$$

$$76x - 152 + 143y - 572 - 54z + 432 = 0$$

$$76x + 143y - 54z - 292 = 0$$

$$76x + 143y - 54z = 292$$

Equation for osculating plane.

A plane determined by $\vec{r}$ and $\vec{T}$ is called osculating plane. Perpendicular vector on osculating plane is $\vec{B}$.

$$\vec{B} = \vec{T} \times \vec{N} = (\hat{i} + 4\hat{j} + 12\hat{k}) \times (76\hat{i} + 143\hat{j} - 54\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 12 \\ 76 & 143 & -54 \end{vmatrix}$$

$$= \hat{i}(-216 - 1716) - \hat{j}(-54 - 912) + \hat{k}(143 - 304)$$

$$= -1932\hat{i} + 966\hat{j} - 161\hat{k}$$

$$= -161(12\hat{i} - 6\hat{j} + \hat{k})$$

$$\vec{B} = 12\hat{i} - 6\hat{j} + \hat{k}$$

Equation of osculating plane at $(2, 4, 8)$ is

$$(12\hat{i} - 6\hat{j} + \hat{k}) \cdot ((x-2)\hat{i} + (y-4)\hat{j} + (z-8)\hat{k}) = 0$$

$$12(x-2) - 6(y-4) + (z-8) = 0$$

$$12x - 24 - 6y + 24 + z - 8 = 0$$

$$12x - 6y + z = 8 \quad \text{Ans}$$

20/03/09

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Question: Write down the Equation of osculating, normal and rectifying planes also find curvature and torsion for the following curves.

① $r(t) = 3\sin t \hat{i} + 3\cos t \hat{j} + 4t \hat{k} \rightarrow t = \frac{\pi}{2}$

② $r(t) = \cos t \hat{i} + \sin t \hat{j} + t \hat{k} \quad t = \pi$

③ $r(t) = 3\cos 2t \hat{i} + e^{2t} \hat{j} + (2t^2 - t) \hat{k} \quad t = 0$

④ $r(t) = (t - \sin t) \hat{i} + (1 - \cos t) \hat{j} + t \hat{k}$

⑤ $r(t) = (\cos t + \sin t) \hat{i} + (\sin t + \cos t) \hat{j} + \sin 2t \hat{k}$

Solution:-

$$1) \quad r(t) = 3 \sin t \hat{i} + 3 \cos t \hat{j} + 4t \hat{k} \quad t = \frac{\pi}{2}$$

$$x = 3 \sin t, \quad y = 3 \cos t, \quad z = 4t$$

$$x = 3, \quad y = 0, \quad z = 2\pi$$

$$(x, y, z) = (3, 0, 2\pi)$$

$$r'(t) = T = 3 \cos t \hat{i} - 3 \sin t \hat{j} + 4 \hat{k}$$

$$r'(t)|_{t=\frac{\pi}{2}} = T = 0 \hat{i} - 3 \hat{j} + 4 \hat{k}$$

Since T is perpendicular on normal plane.

→ Equation of normal plane is

$$(0 \hat{i} - 3 \hat{j} + 4 \hat{k}) \cdot ((x-3) \hat{i} + (y-0) \hat{j} + (z-2\pi) \hat{k}) = 0$$

$$0(x-3) - 3(y-0) + 4(z-2\pi) = 0$$

$$0 - 3y + 4z - 8\pi = 0$$

$$3y - 4z + 8\pi = 0$$

$$3y - 4z = -8\pi \quad \text{Equation of normal plane.}$$

Now Rectifying plane, The N vector is perpendicular on rectifying plane.

$$N = \frac{T'}{|T'|}$$

$$T = 3 \cos t \hat{i} - 3 \sin t \hat{j} + 4 \hat{k}$$

$$T' = -3 \sin t \hat{i} - 3 \cos t \hat{j} + 0 \hat{k}$$

$$|T'| = \sqrt{9(\sin^2 t + \cos^2 t)} = 3$$

$$N = \frac{T'}{|T'|} = \frac{-3 \sin t \hat{i} - 3 \cos t \hat{j} + 0 \hat{k}}{3}$$

$$N = -\sin t \hat{i} - \cos t \hat{j} + 0 \hat{k}$$

$$N|_{t=\frac{\pi}{2}} = -\hat{i} - 0 \hat{j} + 0 \hat{k}$$

→ Equation of rectifying plane is

$$(-\hat{i} + 0 \hat{j} + 0 \hat{k}) \cdot ((x-3) \hat{i} + (y-0) \hat{j} + (z-0) \hat{k}) = 0$$

$$-(x-3) = 0$$

$$x = 3$$

Now osculating plane, The B vector is perpendicular on osculating plane.

$$B = T \times N$$

$$B = \begin{vmatrix} i & j & k \\ 0 & -3 & 4 \\ -1 & 0 & 0 \end{vmatrix} = 0\vec{i} - 4\vec{j} + \vec{k}(-3)$$

$$B = 0\vec{i} - 4\vec{j} - 3\vec{k}$$

$\Rightarrow$ equation of osculating plane is

$$(0\vec{i} - 4\vec{j} - 3\vec{k}) \cdot ((x-3)\vec{i} + (y-0)\vec{j} + (z-2\pi)\vec{k}) = 0$$

$$0(x-3) - 4(y-0) - 3(z-2\pi) = 0$$

$$-4y - 3z + 6\pi = 0$$

$$4y + 3z = 6\pi \quad \text{Ans}$$

Now

$$K = |T'| = \sqrt{9(\sin^2 t + \cos^2 t)} = 3$$

$$T' = -3\sin t \vec{i} - 3\cos t \vec{j} + 0\vec{k}$$

$$T'|_{t=\frac{\pi}{2}} = -3\vec{i} - 0\vec{j} + 0\vec{k}$$

$$K = |T'| = 3$$

Torsion

$$\tau = -B' \cdot N$$

$$\text{Also } \tau = \frac{r'(t) \times r''(t) \cdot r'''(t)}{|r'(t) \times r''(t)|^2}$$

$$B = \begin{vmatrix} i & j & k \\ \cos t & -3\sin t & 4 \\ -\sin t & -\cos t & 0 \end{vmatrix} = T \times N$$

$$B = \vec{i}(-3\cos^2 t) - \vec{j}(3\sin^2 t) + \vec{k}(0 + 4\cos t) - \vec{j}(0 + 4\sin t) + \vec{k}(-3\cos^2 t - 3\sin^2 t)$$

$$B = 4\cos t \vec{i} - 4\sin t \vec{j} - 3\vec{k}$$

$$B' = 4\sin t \vec{i} - 4\cos t \vec{j} + 0\vec{k}$$

$$B'|_{t=\frac{\pi}{2}} = 4\vec{i} + 0\vec{j} + 0\vec{k}$$

$$N|_{t=\frac{\pi}{2}} = -\vec{i} - 0\vec{j} + 0\vec{k}$$

$$\tau = -N \cdot B' = -(-\vec{i} + 0\vec{j} + 0\vec{k}) \cdot (4\vec{i} + 0\vec{j} + 0\vec{k})$$

$$\tau = -4 + 0 + 0 = -4 \quad \text{Ans}$$

$$\underline{2} \quad r(t) = \cos t \hat{i} + \sin t \hat{j} + t \hat{k} \quad t = \pi$$

$$x = \cos t \quad y = \sin t \quad z = t$$

$$x = -1 \quad y = 0 \quad z = \pi$$

$$(x, y, z) = (-1, 0, \pi)$$

Since T is perpendicular on normal plane.

$$r'(t) = T = -\sin t \hat{i} + \cos t \hat{j} + \hat{k}$$

$$r'(t)|_{t=\pi} = T = 0\hat{i} - \hat{j} + \hat{k}$$

Equation of normal plane is

$$(0\hat{i} - \hat{j} + \hat{k}) \cdot ((x+1)\hat{i} + (y-0)\hat{j} + (z-\pi)\hat{k}) = 0$$

$$0(x+1) - 1(y-0) + (z-\pi) = 0$$

$$-y + z - \pi = 0$$

$$y - z = -\pi$$

Now Rectifying plane, The N vector is perpendicular on rectifying plane.

$$N = \frac{T'}{|T'|}$$

$$T' = -\cos t \hat{i} - \sin t \hat{j} + 0\hat{k}$$

$$|T'| = \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} = 1$$

$$N = \frac{-\cos t \hat{i} - \sin t \hat{j} + 0\hat{k}}{1} = -\cos t \hat{i} - \sin t \hat{j} + 0\hat{k}$$

Equation of rectifying plane is

$$(-\cos t \hat{i} - \sin t \hat{j} + 0\hat{k}) \cdot ((x+1)\hat{i} + (y-0)\hat{j} + (z-\pi)\hat{k}) = 0$$

$$N|_{t=\pi} = \hat{i} - 0\hat{j} + 0\hat{k}$$

$$(\hat{i} + 0\hat{j} + 0\hat{k}) \cdot ((x+1)\hat{i} + (y-0)\hat{j} + (z-\pi)\hat{k}) = 0$$

$$1 \cdot (x+1) + 0 \cdot (y-0) + 0 \cdot (z-\pi) = 0$$

$$x+1 = 0 \Rightarrow \boxed{x = -1}$$

Osculating plane, The B vector is perpendicular on osculating plane.

$$B = T \times N$$

$$B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$\vec{B} = i(0-0) - j(-1) + k(0+1) = 0\vec{i} + \vec{j} + \vec{k}$$

Equation of osculating plane is:

$$(0\vec{i} + \vec{j} + \vec{k}) \cdot ((x+1)\vec{i} + (y-0)\vec{j} + (z-\pi)\vec{k}) = 0$$

$$0(x+1) + 1(y-0) + 1(z-\pi) = 0$$

$$y + z - \pi = 0$$

$$y + z = \pi$$

Now

$$K = |\vec{T}'| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = \sqrt{\cos^2 t + \sin^2 t}$$

$$K = 1$$

$$B = \begin{vmatrix} i & j & k \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = i(0 + \sin t) - j(0 + \cos t) + k(\sin^2 t + \cos^2 t)$$

$$B = \sin t \vec{i} - \cos t \vec{j} + k$$

$$B' = \cos t \vec{i} + \sin t \vec{j} + 0 \vec{k}$$

$$\tau = -N \cdot B' = -(-\cos t \vec{i} - \sin t \vec{j} + 0 \vec{k}) \cdot (\cos t \vec{i} + \sin t \vec{j} + 0 \vec{k})$$

$$= \cos^2 t + \sin^2 t + 0$$

$$\tau = 1 \quad \text{Ans}$$

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$$r(t) = 3\cos 2t \vec{i} + e^{2t+1} \vec{j} + (2t^2 - t) \vec{k} \quad t=0$$

$$x = 3\cos 2t \quad y = e^{2t+1} \quad z = 2t^2 - t$$

$$x = 3 \quad y = e \quad z = 0$$

$$(x, y, z) = (3, e, 0)$$

$$r'(t) = T = -6\sin 2t \vec{i} + 2e^{2t+1} \vec{j} + (4t-1) \vec{k}$$

$$r'(t)|_{t=0} = T = 0\vec{i} + 2e\vec{j} - \vec{k}$$

Since T is perpendicular on normal plane.

Equation of normal plane is

$$(0\vec{i} + 2e\vec{j} - \vec{k}) \cdot ((x-3)\vec{i} + (y-e)\vec{j} + (z-0)\vec{k}) = 0$$

$$0(x-3) + 2e(y-e) - 1(z-0) = 0$$

$$2ey - 2e^2 - z = 0$$

$$2ey - z = 2e^2$$

Rectifying plane: The N vector is perpendicular on rectifying plane.

$$N = \frac{\vec{T}'}{|\vec{T}'|}$$

$$\vec{T} = -6\sin 2t \hat{i} + 2e^{2t+1} \hat{j} + (4t-1) \hat{k}$$

$$\vec{T}' = -12\cos 2t \hat{i} + 4e^{2t+1} \hat{j} + 4\hat{k}$$

$$|\vec{T}'| = \sqrt{144\cos^2 2t + 16e^{4t+2} + 16}$$

$$\vec{T}'|_{t=0} = -12\hat{i} + 4e\hat{j} + 4\hat{k}$$

$$|\vec{T}'|_{t=0} = \sqrt{144 + 16e^2 + 16} = \sqrt{160 + 16e^2}$$

$$= \sqrt{16(10 + e^2)} = 4\sqrt{10 + e^2}$$

$$N = \frac{-12\hat{i} + 4e\hat{j} + 4\hat{k}}{4\sqrt{10 + e^2}} = \frac{-3}{\sqrt{10 + e^2}} \hat{i} + \frac{e}{\sqrt{10 + e^2}} \hat{j} + \frac{1}{\sqrt{10 + e^2}} \hat{k}$$

⇒ Equation of rectifying plane is

$$\left(\frac{-3}{\sqrt{10 + e^2}} \hat{i} + \frac{e}{\sqrt{10 + e^2}} \hat{j} + \frac{1}{\sqrt{10 + e^2}} \hat{k} \right) \cdot ((x-3)\hat{i} + (y-e)\hat{j} + (z-0)\hat{k}) = 0$$

$$\frac{-3(x-3)}{\sqrt{10 + e^2}} + \frac{e(y-e)}{\sqrt{10 + e^2}} + \frac{1(z-0)}{\sqrt{10 + e^2}} = 0$$

$$-3x + 9 + ey - e^2 + z = 0$$

$$-3x + ey + z = e^2 - 9$$

Osculating plane: The B vector is perpendicular on osculating plane

$$B = \vec{T} \times N$$

$$B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ T_x & T_y & T_z \\ N_x & N_y & N_z \end{vmatrix}$$

$$\begin{array}{lll}
 x = 3 \cos 2t & y = e^{2t+1} & z = 2t^2 - t \\
 \dot{x} = -6 \sin 2t & \dot{y} = 2e^{2t+1} & \dot{z} = 4t - 1 \\
 \dot{x}|_{t=0} = 0 & \dot{y}|_{t=0} = 2e & \dot{z}|_{t=0} = -1 \\
 \ddot{x} = -12 \cos 2t & \ddot{y} = 4e^{2t+1} & \ddot{z} = 2 = 2 \\
 \ddot{x}|_{t=0} = -12 & \ddot{y}|_{t=0} = 4e &
 \end{array}$$

$$x|_{t=0} = 3 \quad y|_{t=0} = e \quad z|_{t=0} = 0$$

$$\begin{vmatrix} x-x & y-y & z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = \begin{vmatrix} x-3 & y-e & z-0 \\ 0 & 2e & -1 \\ -12 & 4e & 2 \end{vmatrix} = 0$$

$$(x-3)(8e+4e) - (y-e)(0-12) + (z-0)(0+24e) = 0$$

$$(8e+4e)x - 24e - 12e + 12y - 12e + 24ez = 0$$

$$4e(e+e)x + 12y + 24ez - 24e - 24e = 0$$

$$4e(e+e)x + 12y + 24ez = 24e + 24e$$

$$12(e+e)x + 12y + 24ez = 48e \quad \text{Ans}$$

$$\text{Now } ex + y + 2ez = 4e \quad \text{Ans}$$

$$\text{Now } k = |T'|$$

$$k = \sqrt{144 + 16e^2 + 16} = 4\sqrt{10+e^2}$$

$$\text{Torsion}$$

$$r'(t) = -6 \sin 2t \hat{i} + 2e^{2t+1} \hat{j} + (4t-1) \hat{k}$$

$$r'(t)|_{t=0} = 0 \hat{i} + 2e \hat{j} - \hat{k}$$

$$r''(t) = -12 \cos 2t \hat{i} + 4e^{2t+1} \hat{j} + 4 \hat{k}$$

$$r''(t)|_{t=0} = -12 \hat{i} + 4e \hat{j} + 4 \hat{k}$$

$$r'''(t) = -24 \sin 2t \hat{i} + 8e^{2t+1} \hat{j} + 0 \hat{k}$$

$$r'''(t)|_{t=0} = 0 \hat{i} + 8e \hat{j} + 0 \hat{k}$$

$$\tau = \frac{r'(t) \times r''(t) \times r'''(t)}{|r'(t) \times r''(t)|^2}$$

$$r'(t) \times r''(t) \times r'''(t) = \begin{vmatrix} 0 & 2e & -1 \\ -12 & 4e & 4 \\ 0 & 8e & 0 \end{vmatrix}$$

$$= 0(0 - 32e) - 2e(0 - 0) + (-1)(-96e - 0)$$

$$= 96e$$

$$|r'(t) \times r''(t)|^2 = \begin{vmatrix} i & j & k \\ -0 & 2e & -1 \\ -12 & 4e & 4 \end{vmatrix} = i(-32e) - j(0) + k(-96e)$$

$$= i(12e) - j(-12) + k(24e)$$

$$= |12e i + 12j + 24ek|$$

$$= (\sqrt{144e^2 + 144 + 576e^2})^2$$

$$= \sqrt{1024e^2 + 4216e^2} = 64e^2 + 16$$

$$= 720e^2 + 144$$

$$\tau = \frac{96e}{720e^2 + 144} \text{ Ans}$$

$$\tau = \frac{96e}{32e\sqrt{10}} = \frac{3}{\sqrt{10}} \text{ Ans}$$

4

$$r(t) = (t - \sin t) \hat{i} + (1 - \cos t) \hat{j} + t \hat{k}$$

$$T = r'(t) = (1 - \cos t) \hat{i} + \sin t \hat{j} + \hat{k}$$

Since T is perpendicular vector on normal plane. Equation of normal plane is

$$((1 - \cos t) \hat{i} + \sin t \hat{j} + \hat{k}) \cdot ((x - (t - \sin t)) \hat{i} + (y - (1 - \cos t)) \hat{j} + (z - t) \hat{k}) = 0$$

$$(1 - \cos t)(x - t + \sin t) + \sin t(y - 1 + \cos t) + z - t = 0$$

$$(1 - \cos t)x - t(1 - \cos t) + \sin t(1 - \cos t) + \sin t y - \sin t + \sin t \cos t + z - t = 0$$

$$(1 - \cos t)x - t + t \cos t + \sin t - \sin t \cos t + y \sin t - \sin t + \sin t \cos t + z - t = 0$$

$$(1 - \cos t)x + y \sin t + z - 2t + t \cos t = 0$$

Rectifying plane: The N vector is perpendicular on rectifying plane

$$N = \frac{T'}{|T'|}$$

$$T = (1 - \cos t)\hat{i} + \sin t\hat{j} + \hat{k}$$

$$T' = \sin t\hat{i} + \cos t\hat{j} + 0\hat{k}$$

$$|T'| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$N = \frac{\sin t\hat{i} + \cos t\hat{j} + 0\hat{k}}{1} = \sin t\hat{i} + \cos t\hat{j} + 0\hat{k}$$

Equation of rectifying plane is

$$(\sin t\hat{i} + \cos t\hat{j} + 0\hat{k}) \cdot ((x-t+\sin t)\hat{i} + (y-1+\cos t)\hat{j} + (z-t)\hat{k}) = 0$$

$$\sin t(x-t+\sin t) + \cos t(y-1+\cos t) + 0(z-t) = 0$$

$$x\sin t - t\sin t + \sin^2 t + y\cos t - \cos t + \cos^2 t = 0$$

$$x\sin t + y\cos t - \cos t - t\sin t + 1 = 0$$

$$x\sin t + y\cos t = \cos t + t\sin t - 1$$

Osculating plane: The B vector is perpendicular on osculating plane

$$B = T \times N$$

$$B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1-\cos t & \sin t & 1 \\ \sin t & \cos t & 0 \end{vmatrix} = \hat{i}(0 - \cos t) - \hat{j}(0 - \sin t) + \hat{k}(\cos t - \cos^2 t - \sin^2 t)$$

$$B = -\cos t\hat{i} + \sin t\hat{j} + (\cos t - 1)\hat{k}$$

Equation of osculating plane is

$$(-\cos t\hat{i} + \sin t\hat{j} + (\cos t - 1)\hat{k}) \cdot ((x-t+\sin t)\hat{i} + (y-1+\cos t)\hat{j} + (z-t)\hat{k}) = 0$$

$$-\cos t(x-t+\sin t) + \sin t(y-1+\cos t) + (\cos t - 1)(z-t) = 0$$

$$-x\cos t + t\cos t - \sin t\cos t + y\sin t - \sin t + \sin t\cos t + (\cos t - 1)z - t\cos t + t = 0$$

$$-x\cos t + y\sin t + (\cos t - 1)z + t - \sin t = 0$$

$$x\cos t - y\sin t + (1 - \cos t)z = t - \sin t$$

Now

$$\vec{T}' = \sin t \hat{i} + \cos t \hat{j} + 0 \hat{k}$$

$$K = |\vec{T}'| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

Torsion.

$$\tau = -\vec{B}' \cdot \vec{N}$$

$$\vec{B} = -\cos t \hat{i} + \sin t \hat{j} + (\cos t - 1) \hat{k}$$

$$\vec{B}' = \sin t \hat{i} + \cos t \hat{j} - \sin t \hat{k}$$

$$\tau = -\vec{B}' \cdot \vec{N} = -(\sin t \hat{i} + \cos t \hat{j} - \sin t \hat{k}) \cdot (\sin t \hat{i} + \cos t \hat{j} + 0 \hat{k})$$

$$= -(\sin^2 t + \cos^2 t + 0)$$

$$\tau = -1$$

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$$\vec{r}(t) = (\cos t + \sin t) \hat{i} + (\sin t + \cos t) \hat{j} + \sin t \hat{k}$$

$$\vec{T} = \vec{r}'(t) = (-\sin t + \cos t) \hat{i} + (\cos t - \sin t) \hat{j} + 2 \cos 2t \hat{k}$$

Since $\vec{T}$ is perpendicular on normal plane.

Equation of normal plane is

$$((-\sin t + \cos t) \hat{i} + (\cos t - \sin t) \hat{j} + 2 \cos 2t \hat{k}) \cdot ((x - (\cos t + \sin t)) \hat{i} + (y - (\sin t + \cos t)) \hat{j} + (z - \sin t) \hat{k}) = 0$$

$$(x - (\cos t + \sin t))(\cos t - \sin t) + (y - (\sin t + \cos t))(\cos t - \sin t) + (z - \sin t)(2 \cos 2t) = 0$$

$$x(\cos t - \sin t) - (\cos t + \sin t)(\cos t - \sin t) + y(\cos t - \sin t) - (\sin t + \cos t)(\cos t - \sin t) + 2z \cos 2t - 2 \sin t \cos 2t = 0$$

$$x(\cos t - \sin t) + y(\cos t - \sin t) - 2(\cos^2 t - \sin^2 t) - 2 \sin t \cos 2t + 2z \cos 2t = 0$$

$$x(\cos t - \sin t) + y(\cos t - \sin t) = 2(\cos^2 t - \sin^2 t) + \sin 4t - 2z \cos 2t$$

$$x + y = 2(\cos t + \sin t) + \frac{\sin 4t}{\cos t - \sin t} - \frac{2z \cos 2t}{\cos t - \sin t}$$

$$x + y + \frac{2z \cos 2t}{\cos t - \sin t} = 2(\cos t + \sin t) + \frac{\sin 4t}{\cos t - \sin t}$$

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Rectifying plane: The N vector is perpendicular on rectifying plane.

$$N = \frac{T'}{|T'|}$$

$$T = (\cos t - \sin t)\hat{i} + (\cos t - \sin t)\hat{j} + 2\cos 2t\hat{k}$$

$$T' = (-\sin t - \cos t)\hat{i} + (-\sin t - \cos t)\hat{j} - 4\sin 2t\hat{k}$$

$$T' = -(\sin t + \cos t)\hat{i} - (\sin t + \cos t)\hat{j} - 4\sin 2t\hat{k}$$

$$|T'| = \sqrt{(\sin t + \cos t)^2 + (\sin t + \cos t)^2 + (4\sin 2t)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t + 2\sin t \cos t + \sin^2 t + \cos^2 t + 2\sin t \cos t + 16\sin^2 2t}$$

$$= \sqrt{2 + 2\sin 2t + 16\sin^2 2t}$$

$$N = \frac{1}{\sqrt{2 + 2\sin 2t + 16\sin^2 2t}} ((\sin t + \cos t)\hat{i} + (\sin t + \cos t)\hat{j} + 4\sin 2t\hat{k})$$

Equation of ~~osculating~~ ^{rectifying} plane is

$$\frac{-1}{\sqrt{2 + 2\sin 2t + 16\sin^2 2t}} ((\sin t + \cos t)\hat{i} + (\sin t + \cos t)\hat{j} + 4\sin 2t\hat{k}) \cdot ((x - (\cos t + \sin t))\hat{i} + (y - (\sin t + \cos t))\hat{j} + (z - \sin 2t)\hat{k}) = 0$$

$$(\sin t + \cos t)(x - (\cos t + \sin t)) + (\sin t + \cos t)(y - (\sin t + \cos t)) + 4\sin 2t(z - \sin 2t) = 0$$

$$x(\sin t + \cos t) - (\sin t + \cos t)^2 + y(\sin t + \cos t) - (\sin t + \cos t)^2 + 4z\sin 2t - 4\sin^2 2t = 0$$

$$x(\sin t + \cos t) + y(\sin t + \cos t) = 2(\sin t + \cos t)^2 + 4\sin^2 2t - 4z\sin 2t$$

$$x(\sin t + \cos t) + y(\sin t + \cos t) + 4z\sin 2t = 2(\sin t + \cos t)^2 + 4\sin^2 2t$$

now

osculating plane is

$$x = \cos t + \sin t$$

$$y = \sin t + \cos t$$

$$z = \sin 2t$$

$$\dot{x} = -\sin t + \cos t$$

$$\dot{y} = \cos t - \sin t$$

$$\dot{z} = 2\cos 2t$$

$$\ddot{x} = -\cos t - \sin t$$

$$\ddot{y} = -\sin t - \cos t$$

$$\ddot{z} = -4\sin 2t$$

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$X-x$	$Y-y$	$Z-z$	$X-\cos t-\sin t$	$Y-\sin t-\cos t$	$Z-\sin 2t$
$\dot{x}$	$\dot{y}$	$\dot{z}$	$-\sin t+\cos t$	$-\sin t+\cos t$	$2\cos 2t$
$\ddot{x}$	$\ddot{y}$	$\ddot{z}$	$-\cos t-\sin t$	$-\cos t-\sin t$	$-4\sin 2t$

$$\begin{aligned} & (X-(\cos t+\sin t))(-4\sin 2t(\cos t-\sin t)+2\cos 2t(\cos t+\sin t)) \\ & - (Y-(\sin t+\cos t))(-4\sin 2t(-\sin t+\cos t)+2\cos 2t(\cos t+\sin t)) \\ & + (Z-\sin 2t)((\cos t+\sin t)(\cos t-\sin t)+(-\sin t+\cos t)(\cos t+\sin t)) \\ & = 0 \end{aligned}$$

$$\begin{aligned} & (X-(\cos t+\sin t))(-4\sin 2t\cos t+4\sin 2t\sin t+2\cos 2t\cos t+2\cos 2t\sin t) - \\ & (Y-(\sin t+\cos t))(+4\sin 2t\sin t-4\sin 2t\cos t+2\cos 2t\cos t+2\cos 2t\sin t) \\ & + (Z-\sin 2t)(0) = 0 \end{aligned}$$

$$X - \cos t + \sin t - (Y - (\sin t + \cos t)) = 0$$

$$X - \cos t - \sin t - Y + \sin t + \cos t = 0$$

$$X - Y = 0$$

Now

$$\vec{T}' = -(\sin t + \cos t)\hat{i} - (\sin t + \cos t)\hat{j} - 4\sin 2t\hat{k}$$

$$K = |\vec{T}'| = \sqrt{2 + 4\sin^2 t + 16\sin^2 2t}$$

and

$$\vec{r}'(t) = (\cos t - \sin t)\hat{i} + (\cos t - \sin t)\hat{j} + 2\cos 2t\hat{k}$$

$$\vec{r}''(t) = -(\sin t + \cos t)\hat{i} - (\sin t + \cos t)\hat{j} - 4\sin 2t\hat{k}$$

$$\vec{r}'''(t) = -(\cos t - \sin t)\hat{i} - (\cos t - \sin t)\hat{j} - 8\cos 2t\hat{k}$$

$$= (\sin t - \cos t)\hat{i} + (\sin t - \cos t)\hat{j} - 8\cos 2t\hat{k}$$

$\vec{r}'(t) \times \vec{r}''(t) \times \vec{r}'''(t)$	$\cos t - \sin t$	$\cos t - \sin t$	$2\cos 2t$
	$-(\sin t + \cos t)$	$-(\sin t + \cos t)$	$-4\sin 2t$
	$\sin t - \cos t$	$\sin t - \cos t$	$-8\cos 2t$

$$= (\cos t - \sin t)(8\cos 2t(\sin t + \cos t) + 4\sin 2t(\sin t - \cos t))$$

$$= (\cos t - \sin t)(8\cos 2t(\sin t + \cos t) + 4\sin 2t(\sin t - \cos t)) + 2\cos 2t(-(\sin t + \cos t)(\sin t - \cos t) + (\sin t + \cos t)(\sin t - \cos t))$$

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$$= (\cos t - \sin t)(8\cos 2t \sin t + 8\cos 2t \cos t + 4\sin 2t \sin t - 4\sin 2t \cos t) \\ - (\cos t - \sin t)(8\cos 2t \sin t + 8\cos 2t \cos t + 4\sin 2t - 4\sin 2t \cos t)$$

$$= (\cos t - \sin t)(8\cos 2t \sin t + 8\cos 2t \cos t + 4\sin 2t \sin t - 4\sin 2t \cos t \\ - 8\cos 2t \sin t - 8\cos 2t \cos t - 4\sin 2t + 4\sin 2t \cos t) \\ = 0$$

$$\tau = \frac{r' \times r'' \times r'''}{|r' \times r''|^2} = \frac{0}{|r' \times r''|^2} = 0 \quad \text{Ans}$$

$$K = |r' \times r''| = \begin{vmatrix} i & j & k \\ \cos t - \sin t & \cos t - \sin t & 2\cos 2t \\ -(\sin t + \cos t) & -(\sin t + \cos t) & -4\sin t \end{vmatrix}$$

$$= i(-4\sin 2t \cos t + 4\sin t \sin 2t + 2\cos 2t \sin t + 2\cos t \cos 2t) \\ - j(-4\sin 2t \cos t + 4\sin t \sin 2t + 2\cos 2t \sin t + 2\cos t \cos 2t) \\ + k(-(\sin t + \cos t)(\cos t - \sin t) + (\sin t + \cos t)(\cos t - \sin t))$$

$$= \sqrt{(-4\sin 2t \cos t + 4\sin t \sin 2t + 2\cos 2t \sin t + 2\cos t \cos 2t)^2 + (-4\sin t \cos t + 4\sin t \sin 2t + 2\cos 2t \sin t + 2\cos t \cos 2t)^2}$$

$$= \sqrt{2(-4\sin 2t \cos t + 4\sin t \sin 2t + 2\cos 2t \sin t + 2\cos t \cos 2t)^2}$$

$$K = \sqrt{2}(-4\sin 2t \cos t + 4\sin t \sin 2t + 2\cos 2t \sin t + 2\cos t \cos 2t) \quad \text{Ans}$$

$$K = \frac{|v \times a|}{|v|^3} = \frac{|v'(t) \times v''(t)|}{|v'(t)|^3}$$

$$\tau = -\frac{v \cdot a'}{|v'|^3}$$

$$K = \frac{|v'|}{|v'|^3}$$

Surface=

A surface in $\mathbb{R}^3$ is the subset of $\mathbb{R}^3$, i.e. a certain collection of points on $\mathbb{R}^3$, it is not necessary that all subsets are surfaces. we requires smooth surfaces in two dimensional in our course.

"A surface is said to be smooth if it does not have any singular points," In other words it have a unique tangent plane at every point.

Example smooth surface

$$x^n + y^n + z^n = a^n \quad (\text{Lame surface})$$

Question=

$$x = a \cos^3 u \cos^3 v ; \quad y = a \sin^3 u \cos^3 v$$

$$z = a \sin^3 v \quad \text{ellipsoid}$$

Find surface equation or $f(x, y, z) = ??$

Solution=

$$x = a \cos^3 u \cos^3 v \quad (1)$$

$$z = a \sin^3 v \quad (3)$$

$$x^2 = a^2 \cos^6 u \cos^6 v \quad (A)$$

$$z^2 = a^2 \sin^6 v \quad (C)$$

$$y = a \sin^3 u \cos^3 v \quad (2)$$

$$y^2 = a^2 \sin^6 u \cos^6 v \quad (B)$$

Take cube root and

Adding (A), (B) & (C), we get

$$(x^2)^{1/3} + (y^2)^{1/3} + (z^2)^{1/3} = (a^2 \cos^6 u \cos^6 v)^{1/3} + (a^2 \sin^6 u \cos^6 v)^{1/3} + (a^2 \sin^6 v)^{1/3}$$

$$= a^{2/3} (\cos^2 u \cos^2 v + \sin^2 u \cos^2 v + \sin^2 v)$$

$$= a^{2/3} (\cos^2 v (\cos^2 u + \sin^2 u) + \sin^2 v)$$

$$= a^{2/3} (\cos^2 v + \sin^2 v)$$

$$x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$$

Question: The parametric form of ellipsoid is

$$x = a \cos u \cos v; \quad y = b \cos u \sin v; \quad z = c \sin v$$

Find surface equation?

Solution:

$$x = a \cos u \cos v \quad \text{--- (1)}$$

$$\frac{x}{a} = \cos u \cos v$$

$$\left(\frac{x}{a}\right)^2 = \cos^2 u \cos^2 v \quad \text{--- (A)}$$

$$y = b \cos u \sin v \quad \text{--- (2)}$$

$$z = c \sin v$$

$$\frac{y}{b} = \cos u \sin v$$

$$\left(\frac{z}{c}\right) = \sin v$$

$$\left(\frac{y}{b}\right)^2 = \cos^2 u \sin^2 v \quad \text{--- (B)}$$

$$\left(\frac{z}{c}\right)^2 = \sin^2 v \quad \text{--- (C)}$$

Adding (A), (B) & (C), we get

$$\begin{aligned} \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 &= \cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 v \\ &= \cos^2 u (\cos^2 v + \sin^2 v) + \sin^2 v \\ &= \cos^2 u + \sin^2 v \end{aligned}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{Ans}$$

Note:

A surface is the locus of the points $P(x, y, z)$ whose coordinates are functions of two independent parameters u and v

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v)$$

These equations are parametric form of a surface. If we eliminate u and v from above equations, we get $F(x, y, z) = 0$ as equation of surface.

Tangent plane and normal plane at any point P on a surface:

Equation of tangent plane at P on a surface $F(x, y, z) = 0$ is defined as

$$(X-x)F_x + (Y-y)F_y + (Z-z)F_z = 0$$

And Equation of normal plane is defined as

$$\frac{X-x}{F_x} = \frac{Y-y}{F_y} = \frac{Z-z}{F_z} \quad \text{at any point } P(x, y, z)$$

note
$F_x = \frac{\partial}{\partial x}$
$F_y = \frac{\partial}{\partial y}$
$F_z = \frac{\partial}{\partial z}$

Question:

Show that the sum of the squares of the intercepts on the coordinate axes by the tangent plane to the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$ is constant.

Solution:

$$F(x, y, z) = x^{2/3} + y^{2/3} + z^{2/3} - a^{2/3}$$

$$F_x = \frac{2}{3}x^{-1/3}; \quad F_y = \frac{2}{3}y^{-1/3}; \quad F_z = \frac{2}{3}z^{-1/3}$$

$$(X-x)F_x + (Y-y)F_y + (Z-z)F_z = 0$$

$$(X-x)\frac{2}{3}x^{-1/3} + (Y-y)\frac{2}{3}y^{-1/3} + (Z-z)\frac{2}{3}z^{-1/3} = 0$$

$$(X-x)x^{-1/3} + (Y-y)y^{-1/3} + (Z-z)z^{-1/3} = 0$$

$$Xx^{2/3} - x^{2/3} + Yy^{2/3} - y^{2/3} + Zz^{2/3} - z^{2/3} = 0$$

$$Xx^{2/3} + Yy^{2/3} + Zz^{2/3} - a^{2/3} = 0$$

Equation of tangent plane.

Now

For X-intercept point $Y = Z = 0$

$$Xx^{2/3} - a^{2/3} = 0$$

$$x = x^{1/3} a^{2/3}$$

For y-intercept point $x = z = 0$

$$y y^{1/3} - a^{2/3} = 0$$

$$y = y^{1/3} a^{2/3}$$

For z-intercept point $y = x = 0$

$$z z^{1/3} - a^{2/3} = 0$$

$$z = z^{1/3} a^{2/3}$$

Take square of intercepts and adding.

$$x^2 + y^2 + z^2 = a^{4/3} (x^{2/3} + y^{2/3} + z^{2/3}) = a^{4/3} \cdot a^{2/3}$$

$$x^2 + y^2 + z^2 = a^2 \text{ is constant.}$$

Equation of normal plane

$$\frac{x-x}{F_x} = \frac{y-y}{F_y} = \frac{z-z}{F_z}$$

$$\frac{x-x}{\frac{2}{3} x^{1/3}} = \frac{y-y}{\frac{2}{3} y^{1/3}} = \frac{z-z}{\frac{2}{3} z^{1/3}}$$

$$(x-x) x^{1/3} = (y-y) y^{1/3} = (z-z) z^{1/3}$$

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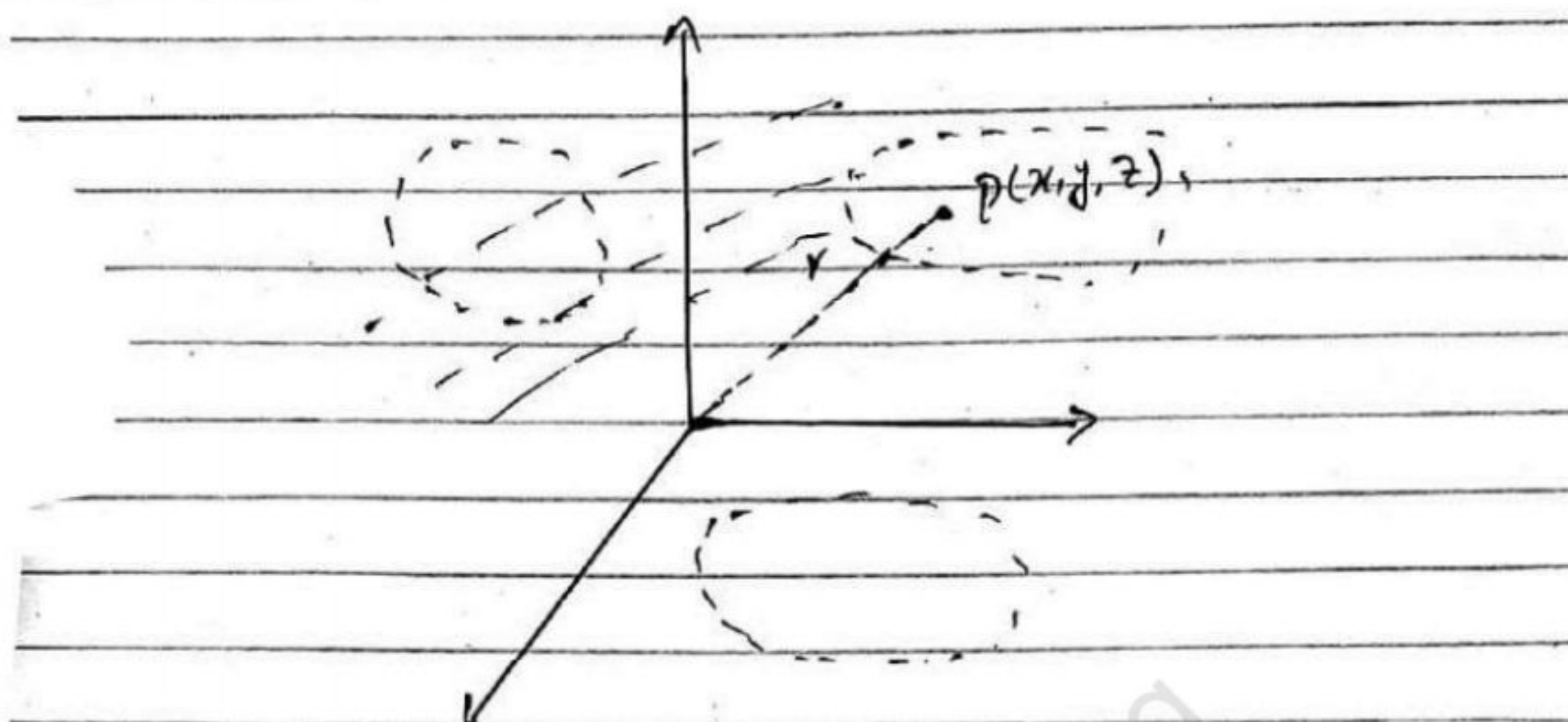


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Surfaces:



Def: Let $D \subset \mathbb{R}^2$ denote an open subset. A smooth vector function $r: D \rightarrow \mathbb{R}^3$ of two variables is called a parametrization or coordinate patch for the surface $S \subset \mathbb{R}^3$ consisting of all points p with $\vec{OP} = r(u, v)$ with $(u, v) \in D$ if r is a one-to-one injective map i.e. Every point in S corresponding to a unique point in D .

The partial derivative of

$$r_u(u_0, v_0) = [x_u(u_0, v_0), y_u(u_0, v_0), z_u(u_0, v_0)]$$

$$= \left[\frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0) \right]$$

$$r_v(u_0, v_0) = \left[\frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0), \frac{\partial z}{\partial v}(u_0, v_0) \right]$$

are linearly independent at every point $(u_0, v_0) \in D$.

“A subset $S \subset \mathbb{R}^3$ that has coordinate patch r is called a regular surface.”

$$\vec{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$$

$$\frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j} + \frac{\partial z}{\partial u}\hat{k}$$

$$\frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial v}\hat{i} + \frac{\partial y}{\partial v}\hat{j} + \frac{\partial z}{\partial v}\hat{k}$$

$\vec{r}_u \times \vec{r}_v = 0$ singular form which has no perpendicular vector, does not find tangent.

$\vec{r}_u \times \vec{r}_v \neq 0$ non-singular form which has perpendicular vector, so we find tangent Equation.

Question=

$$\vec{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$$

if $a=b=c$

we get sphere

$$\vec{r}(u, v) = a \cos u \cos v \hat{i} + b \cos u \sin v \hat{j} + c \sin u \hat{k}$$

$$\vec{r}_u = -a \sin u \cos v \hat{i} - b \sin u \sin v \hat{j} + c \cos u \hat{k}$$

$$\vec{r}_v = -a \cos u \sin v \hat{i} + b \cos u \cos v \hat{j} + 0 \hat{k}$$

Since

$\vec{r}_u \times \vec{r}_v$ is $\perp$ on tangent plane

$$u = \frac{\pi}{4}, v = \frac{\pi}{2}$$

$$(x, y, z) = (0, \frac{b}{\sqrt{2}}, \frac{c}{\sqrt{2}})$$

$$\vec{r}_u = 0\hat{i} - \frac{b}{\sqrt{2}}\hat{j} + \frac{c}{\sqrt{2}}\hat{k}$$

$$\vec{r}_v = -\frac{a}{\sqrt{2}}\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -\frac{b}{\sqrt{2}} & \frac{c}{\sqrt{2}} \\ -\frac{a}{\sqrt{2}} & 0 & 0 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(\frac{ac}{\sqrt{2}}) + \hat{k}(-\frac{ab}{\sqrt{2}})$$

$$= 0\hat{i} - \frac{ac}{\sqrt{2}}\hat{j} - \frac{ab}{\sqrt{2}}\hat{k}$$

Equation of tangent plane on given surface
 r is at $P(0, \frac{b}{\sqrt{2}}, \frac{c}{\sqrt{2}})$

$$(0\hat{i} - \frac{ac}{2}\hat{j} - \frac{ab}{2}\hat{k}) \cdot ((x-0)\hat{i} + (y-\frac{b}{\sqrt{2}})\hat{j} + (z-\frac{c}{\sqrt{2}})\hat{k}) = 0$$

$$0 - \frac{ac}{2}y + \frac{acb}{2\sqrt{2}} - \frac{ab}{2}z + \frac{abc}{2\sqrt{2}} = 0$$

$$acy + abz = \sqrt{2} abc$$

$$\frac{y}{b} + \frac{z}{c} = \sqrt{2}$$

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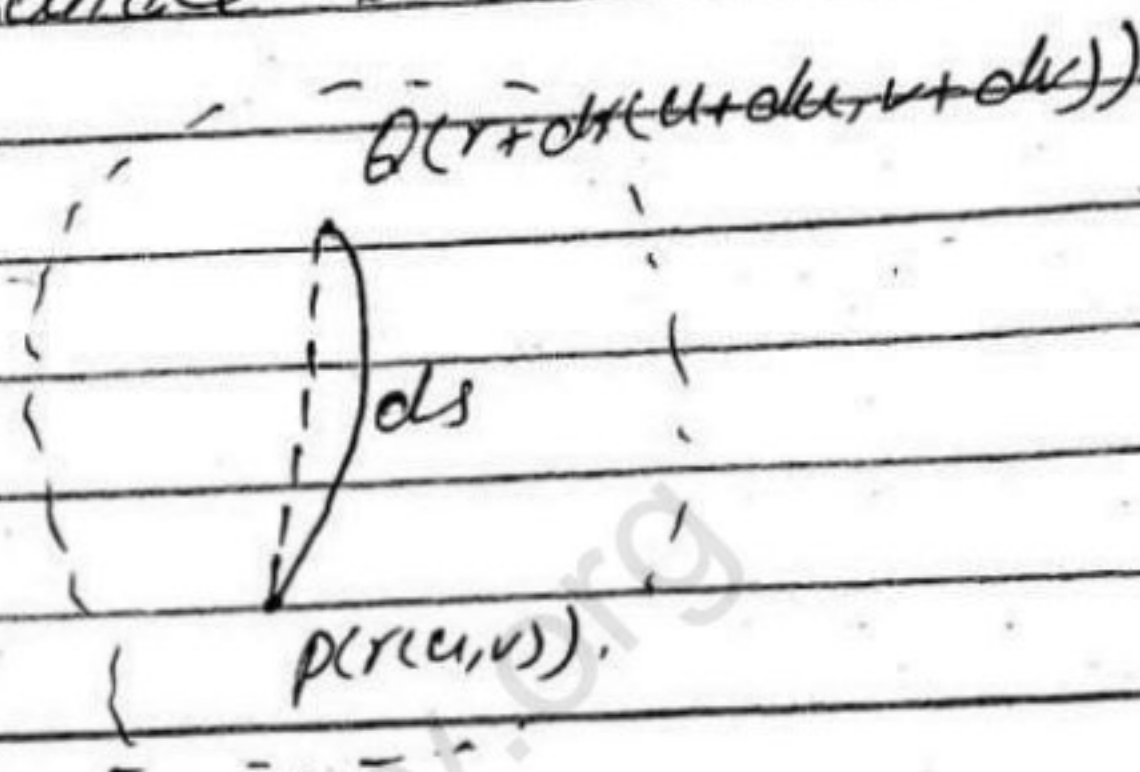
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①

First Fundamental form:-

$$E(du)^2 + 2Fdu dv + G(dv)^2$$

Let there be a surface S with Equation $\vec{r} = r(u, v)$ and let $P(r(u, v))$ and $Q(r + dr(u + du, v + dv))$ are any two neighbouring points on the surface S .



Then

$$|PQ| = ds = |r + dr - r|$$

$$ds = |dr|$$

Since r is function of (u, v)

$$dr = \frac{\partial r}{\partial u} du + \frac{\partial r}{\partial v} dv$$

$$= r_u du + r_v dv$$

$$= |r_u du + r_v dv|$$

$$ds = \sqrt{(r_u du + r_v dv) \cdot (r_u du + r_v dv)}$$

$$= \sqrt{r_u r_u (du)^2 + r_u r_v du dv + r_v r_u dv du + r_v r_v (dv)^2}$$

$$ds = \sqrt{r_u r_u (du)^2 + 2r_u r_v du dv + r_v r_v (dv)^2}$$

$$(ds)^2 = E(du)^2 + 2F du dv + G(dv)^2$$

where

$$E = r_u r_u, \quad F = r_u r_v, \quad G = r_v r_v$$

E , F and G are coefficient of first fundamental form.

Question:

②

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Find the arc length of the curve $u=t, v=t, 0 \leq t \leq 1$ on a hyperboloid

$$r(u, v) = u\hat{i} + v\hat{j} + uv\hat{k}$$

where $0 \leq u, v \leq 1$

Solution:

$$(ds)^2 = E(du)^2 + 2Fdu dv + G(dv)^2$$

$$\frac{\partial r}{\partial u} = r_u = 1\hat{i} + v\hat{k} = \hat{i} + v\hat{k}$$

$$E = r_u \cdot r_u = 1 + v^2$$

$$\frac{\partial r}{\partial v} = r_v = 1\hat{j} + u\hat{k} = \hat{j} + u\hat{k}$$

$$F = r_u \cdot r_v = uv$$

$$G = r_v \cdot r_v = 1 + u^2$$

note

If $F=0$
then surface
is orthogonal.

$$(ds)^2 = (1+v^2)(du)^2 + 2uv du dv + (1+u^2)(dv)^2$$

$$= (1+t^2)(dt)^2 + 2t^2(dt)^2 + (1+t^2)(dt)^2$$

$$= (1+t^2 + 2t^2 + 1+t^2)(dt)^2$$

$$(ds)^2 = (2 + 4t^2)(dt)^2$$

$$ds = \sqrt{2 + 4t^2} dt$$

integrate

$$s = \int_0^1 \sqrt{1 + 2t^2} dt$$

$$s = \frac{\sqrt{3}}{2} + \frac{1}{2} \log(\sqrt{3} + 3) \text{ Ans.}$$

Q

Suppose we have a smooth surface expressed as $r(u, v)$, suppose that a smooth curve parametrized as

$r(t) = r(u(t), v(t))$ along this surface with " t " ranging from " a " to " b ".

Then the arc length is

$$S = \int_a^b |r'(t)| dt = \int_a^b \sqrt{r'(t) \cdot r'(t)} dt$$

③

$$\begin{aligned}
 &= \int_a^b \int_c^d (\mathbf{r}_u \mathbf{u}'(t) + \mathbf{r}_v \mathbf{v}'(t)) \cdot (\mathbf{r}_u \mathbf{u}'(t) + \mathbf{r}_v \mathbf{v}'(t)) \, dt \\
 &= \int_a^b \int_c^d \left[\mathbf{r}_u \cdot \mathbf{r}_u (\mathbf{u}'(t))^2 + 2 \mathbf{r}_u \cdot \mathbf{r}_v \mathbf{u}'(t) \mathbf{v}'(t) + \mathbf{r}_v \cdot \mathbf{r}_v (\mathbf{v}'(t))^2 \right] \, dt \\
 &= \int_a^b \int_c^d \left[E (\mathbf{u}'(t))^2 + 2F \mathbf{u}'(t) \mathbf{v}'(t) + G (\mathbf{v}'(t))^2 \right] \, dt
 \end{aligned}$$

Definition (Area of a surface)

Let $\mathbf{r}: D \rightarrow \mathbb{R}^3$ denote a smooth parametrization for the surface S and let $R \subset S$ denote the subset whose boundary is piecewise smooth curve $C \subset S$. The area A of region R is given as

$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv = \iint_R \sqrt{EG - F^2} \, du \, dv$$

Question:

Find the area of the region R on a sphere with radius ρ with spherical coordinates having parametrization as

$$\mathbf{r}(u, v) = \rho \sin u \cos v \hat{i} + \rho \sin u \sin v \hat{j} + \rho \cos u \hat{k}$$

where $0 \leq u < \pi$ and $0 \leq v < 2\pi$

Solution:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \rho \cos u \cos v \hat{i} + \rho \cos u \sin v \hat{j} - \rho \sin u \hat{k}$$

$$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = -\rho \sin u \sin v \hat{i} + \rho \sin u \cos v \hat{j}$$

$$E = \mathbf{r}_u \cdot \mathbf{r}_u = \rho^2 (\cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 u)$$

$$= \rho^2 (\cos^2 u (\cos^2 v + \sin^2 v) + \sin^2 u) = \rho^2 (\cos^2 u + \sin^2 u)$$

$$E = \rho^2$$

$$F = \rho^2 (-\cos u \cos v \sin u \sin v + \cos u \sin v \sin u \cos v)$$

$$F = 0$$

$$G = \rho^2 \sin^2 u$$

④

$$\begin{aligned}
 A &= \iint_R \sqrt{\rho^2 \cdot \rho^2 \sin^2 u - 0} \, du \, dv = \rho^2 \iint_R \sin u \, du \, dv \\
 &= \rho^2 \int_0^{2\pi} \int_0^{\pi} \sin u \, du \, dv = -\rho^2 \int_0^{2\pi} [\cos u]_0^{\pi} \, dv \\
 &= -\rho^2 \int_0^{2\pi} (\cos \pi - \cos 0) \, dv = -\rho^2 \int_0^{2\pi} (-1 - 1) \, dv \\
 &= 2\rho^2 \int_0^{2\pi} dv = 2\rho^2 v \Big|_0^{2\pi} = 2\rho^2 (2\pi - 0) \\
 &= 4\pi \rho^2 \quad \text{Ans}
 \end{aligned}$$

24/04/19

Question: Let $r: \mathbb{R}^2 \rightarrow S$ be a surface patch

$$r(u, v) = ui + vj + \frac{1}{2}(u^2 - v^2)k$$

i) Compute first fundamental form of S

ii) Let $c: [0, 1] \rightarrow S$ be the curve

$c(t) = r(\sinh t, \sinh t)$, Find the length of curve.

Solution:

i) $r_u = i + uk$

$r_v = j - vk$

$E = r_u \cdot r_u = (i + uk) \cdot (i + uk) = 1 + u^2$

$G = r_v \cdot r_v = (j - vk) \cdot (j - vk) = 1 + v^2$

$F = r_u \cdot r_v = (i + uk) \cdot (j - vk) = -uv$

We know that

$$(ds)^2 = E \, du^2 + 2F \, du \, dv + G \, dv^2$$

$$= (1 + u^2) \, du^2 - 2uv \, du \, dv + (1 + v^2) \, dv^2$$

ii) $c(t) = (\sinh t, \sinh t)$ ①

$u(t) = \sinh t$

$v(t) = \sinh t$

$du = \cosh t \, dt$

$dv = \cosh t \, dt$

Putting in eq. (1)

$$\begin{aligned}
 (ds)^2 &= (1 + \sinh^2 t)(\cosh t)^2 - 2 \sinh t \sinh t \cosh^2 t + (1 + \sinh^2 t)(\cosh t)^2 \, dt^2 \\
 &= (\cosh^2 t \cosh^2 t - 2 \sinh^2 t \cosh^2 t + \cosh^2 t \cosh^2 t) \, dt^2
 \end{aligned}$$

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$$= \cosh^2 t (\cosh^2 t - 2 \sinh^2 t + \cosh^2 t) dt^2$$

$$= \cosh^2 t (2 \cosh^2 t - 2 \sinh^2 t) dt^2$$

$$= 2 \cosh^2 t (\cosh^2 t - \sinh^2 t) dt^2$$

$$= 2 \cosh^2 t dt^2$$

$$ds = \sqrt{2} \cosh t dt$$

integrate

$$S = \int_0^1 \sqrt{2} \cosh t dt = \sqrt{2} \int_0^1 \cosh t dt$$

$$= \sqrt{2} \sinh t \Big|_0^1 = \sqrt{2} (\sinh(1) - \sinh(0))$$

$$S = \sqrt{2} \sinh(1) \text{ Ans.}$$

Question: Consider the parametrization of torus
 $r(u,v) = (R + r \cos v) \cos u \hat{i} + (R + r \cos v) \sin u \hat{j} + r \sin v \hat{k}$
 where $0 \leq u < 2\pi$, $0 \leq v < 2\pi$. Find the Area of surface patch.

Solution:

$$r(u,v) = (R + r \cos v) \cos u \hat{i} + (R + r \cos v) \sin u \hat{j} + r \sin v \hat{k}$$

$$r_u = -(R + r \cos v) \sin u \hat{i} + (R + r \cos v) \cos u \hat{j} + 0 \hat{k}$$

$$r_v = -(r \sin v) \cos u \hat{i} + (-r \sin v) \sin u \hat{j} + r \cos v \hat{k}$$

$$r_v = -r \sin v \cos u \hat{i} - r \sin v \sin u \hat{j} + r \cos v \hat{k}$$

$$E = r_u \cdot r_u = (R + r \cos v)^2 \sin^2 u + (R + r \cos v)^2 \cos^2 u$$

$$= (R + r \cos v)^2 (\sin^2 u + \cos^2 u)$$

$$E = (R + r \cos v)^2$$

$$G = r_v \cdot r_v = r^2 \sin^2 v \cos^2 u + r^2 \sin^2 v \sin^2 u + r^2 \cos^2 v$$

$$= r^2 (\sin^2 v (\cos^2 u + \sin^2 u) + \cos^2 v)$$

$$= r^2 (\sin^2 v + \cos^2 v) = r^2$$

$$G = r^2$$

$$F = r_u \cdot r_v = (R + r \cos v) \sin u \cdot (-r \sin v \cos u) - (R + r \cos v) \cos u \cdot (-r \sin v \sin u)$$

$$F = 0$$

$$A = \iint_R \sqrt{EG - F^2} du dv = \iint_R \sqrt{(R + r \cos v)^2 \cdot r^2 - 0} du dv$$

⑥

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$$= \int_0^{2\pi} \int_0^{2\pi} r(R + r \cos v) du dv$$

$$= \int_0^{2\pi} r(R + r \cos v) u \Big|_0^{2\pi} dv = \int_0^{2\pi} r(R + r \cos v) 2\pi dv$$

$$= 2\pi r \int_0^{2\pi} (R + r \cos v) dv = 2\pi r (Rv + r \sin v) \Big|_0^{2\pi}$$

$$= 2\pi r (R(2\pi) + r \sin(2\pi) - 0)$$

$$= 2\pi r (R2\pi)$$

$$= 4\pi^2 r R \quad \text{Ans.}$$

Question: Let S be a half cylinder parametrized by $r(u, v) = u\hat{i} + v\hat{j} + \sqrt{1-v^2}\hat{k}$ where $u, v \in (-1, 1) \times (-1, 1)$. Find the Area.

Solution:

$$r(u, v) = u\hat{i} + v\hat{j} + \sqrt{1-v^2}\hat{k}$$

$$r_u = \hat{i}$$

$$r_v = \hat{j} + \frac{1}{2\sqrt{1-v^2}}(-2v)\hat{k} = \hat{j} + \frac{-v}{\sqrt{1-v^2}}\hat{k}$$

$$F = r_u \cdot r_u = 1$$

$$G = r_v \cdot r_v = 1 + \frac{v^2}{1-v^2}$$

$$F = r_u \cdot r_v = (\hat{i}) \cdot \left(\hat{j} + \frac{v}{1-v^2}\hat{k}\right) = 0$$

We know that

$$A = \iint_R \sqrt{F \cdot G - F^2} du dv$$

$$= \iint_R \sqrt{1 + \frac{v^2}{1-v^2}} du dv$$

$$= \iint_R \sqrt{\frac{1-v^2+v^2}{1-v^2}} du dv = \iint_R \frac{1}{\sqrt{1-v^2}} du dv$$

$$= \int_{-1}^1 \int_{-1}^1 \frac{1}{\sqrt{1-v^2}} du dv = \int_{-1}^1 \frac{1}{\sqrt{1-v^2}} \cdot u \Big|_{-1}^1 dv$$

$$= \int_{-1}^1 \frac{1}{\sqrt{1-v^2}} (1+1) dv = 2 \int_{-1}^1 \frac{1}{\sqrt{1-v^2}} dv$$

(7)

$$\begin{aligned}
 &= 2 \sin^{-1} v \Big|_{-1}^1 = 2(\sin^{-1}(1) - \sin^{-1}(-1)) \\
 &= 2(\sin^{-1}(1) - \sin^{-1}(-1)) = 2\left(\frac{\pi}{2} + \frac{\pi}{2}\right) \\
 &= 2\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right) = \pi + \pi = 2\pi \text{ Ang}
 \end{aligned}$$

The Second Fundamental Form:

Suppose we have a surface $r(u, v)$ and we push or pull the surface along its normal vector. Then we get a family of surfaces $R(u, v, t) = r(u, v) - t n(u, v)$. The first fundamental form encodes something about arc length on a surface, while the 2nd fundamental form encodes how the arc length changes as surface moves along its normal vector i.e. how the first fundamental form of the surface changes as t changes.

Let $E(t) du^2 + 2F(t) du dv + G(t) dv^2$ be the first fundamental form of the family of surfaces

$$R(u, v, t) = r(u, v) - t n(u, v) \text{ where}$$

$$E(t) = R_u \cdot R_u, \quad F(t) = R_u \cdot R_v, \quad G(t) = R_v \cdot R_v.$$

Then the 2nd fundamental form of $r(u, v) = R(u, v, 0)$ is

$$\frac{1}{2} \frac{\delta}{\delta t} (E(t) du^2 + 2F(t) du dv + G(t) dv^2)_{t=0} \quad \text{--- (A)}$$

$$\begin{aligned}
 E(t) &= R_u \cdot R_u = (r_u - t n_u) \cdot (r_u - t n_u) \\
 &= r_u \cdot r_u - t r_u \cdot n_u - t n_u \cdot r_u + t^2 n_u \cdot n_u \\
 &= r_u \cdot r_u - 2t r_u \cdot n_u + t^2 n_u \cdot n_u
 \end{aligned}$$

$$\begin{aligned}
 F(t) &= R_u \cdot R_v = (r_u - t n_u) \cdot (r_v - t n_v) \\
 &= r_u \cdot r_v - t r_u \cdot n_v - t n_u \cdot r_v + t^2 n_u \cdot n_v
 \end{aligned}$$

$$\begin{aligned}
 G(t) &= R_v \cdot R_v = (r_v - t n_v) \cdot (r_v - t n_v) \\
 &= r_v \cdot r_v - 2t n_v \cdot r_v + t^2 n_v \cdot n_v
 \end{aligned}$$

(8)

$$\frac{\delta E}{\delta t} = 0 - 2r_u \cdot n_u + 2t n_u \cdot n_u$$

$$\frac{\delta E}{\delta t} \Big|_{t=0} = -2r_u \cdot n_u + 0 = -2r_u \cdot n_u$$

$$\frac{\delta F}{\delta t} = 0 - r_u \cdot n_v - n_u \cdot r_v + 2t n_u \cdot n_v$$

$$\frac{\delta F}{\delta t} \Big|_{t=0} = -(r_u \cdot n_v + n_u \cdot r_v)$$

$$\frac{\delta G}{\delta t} = -2n_v \cdot r_u + 2t n_v \cdot n_v$$

$$\frac{\delta G}{\delta t} \Big|_{t=0} = -2n_v \cdot r_u = -2n_v \cdot r_u$$

Since r_u and r_v are always orthogonal to n therefore we can write

$$r_u \cdot n = 0$$

Diff w.r.t u

$$r_u \cdot n + r_u \cdot n_u = 0$$

$$-r_u \cdot n_u = r_u \cdot n$$

$$-2r_u \cdot n_u = 2r_u \cdot n$$

$$\frac{\delta E}{\delta t} \Big|_{t=0} = 2r_u \cdot n \quad \text{--- (B)}$$

Again

$$r_u \cdot n = 0$$

Diff w.r.t v

$$r_u \cdot n + r_u \cdot n_v = 0 \quad \text{--- (1)}$$

Adding (1) & (2)

$$2r_u \cdot n + r_u \cdot n_v + r_u \cdot n_u = 0$$

$$2r_u \cdot n = -(r_u \cdot n_u + r_u \cdot n_v)$$

$$\frac{\delta F}{\delta t} \Big|_{t=0} = 2r_u \cdot n \quad \text{--- (D)}$$

$$r_v \cdot n = 0$$

Diff w.r.t v

$$r_v \cdot n + r_v \cdot n_v = 0$$

$$-r_v \cdot n_v = r_v \cdot n$$

$$-2r_v \cdot n_v = 2r_v \cdot n$$

$$\frac{\delta G}{\delta t} \Big|_{t=0} = 2r_v \cdot n \quad \text{--- (C)}$$

$$r_v \cdot n = 0$$

Diff w.r.t u

$$r_v \cdot n + r_v \cdot n_u = 0 \quad \text{--- (2)}$$

(9)

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using A, B, C and D, we have

$$r_{uu}^n (du)^2 + 2r_{uv}^n du dv + r_{vv}^n (dv)^2$$

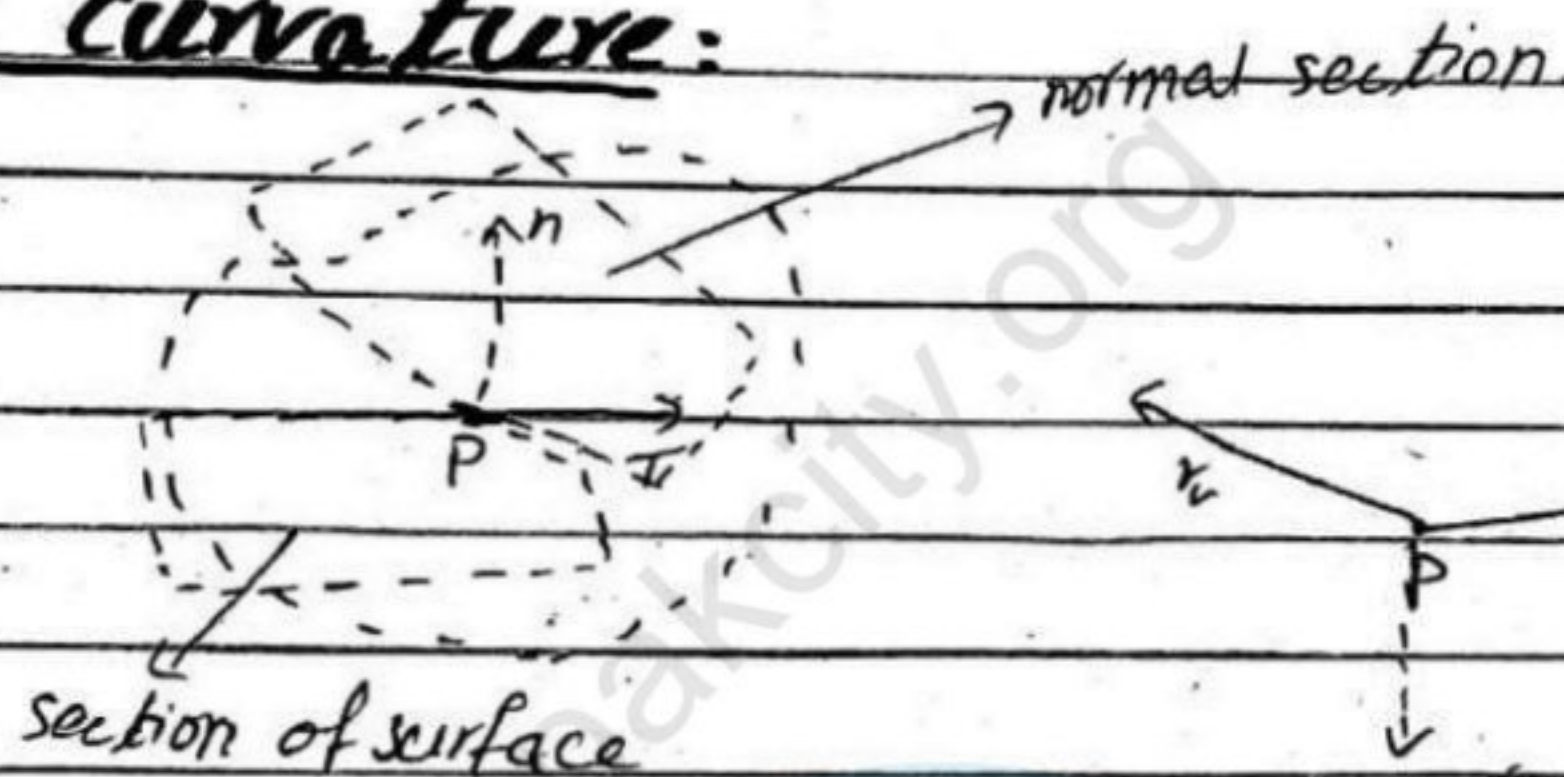
$$L (du)^2 + 2M du dv + N (dv)^2$$

where $L = r_{uu}^n$, $M = r_{uv}^n$, $N = r_{vv}^n$

and L, M, N are coefficient of second fundamental form.

02/05/019

Normal curvature:



we know

$$r''(t) = k N(t)$$

$$\hat{n} = \frac{r_u \times r_v}{|r_u \times r_v|}$$

If we want to find the curvature for normal section of the surface.

$$r''(t) = k_n \hat{n}$$

Taking dot product with $\hat{n}$

$$\hat{n} \cdot r''(t) = k_n$$

$$k_n = r''(t) \cdot \hat{n}$$

k_n can be expressed in terms of Fundamental forms

$$r = r(u, v)$$

Diff w.r.t s

$$\frac{\partial r}{\partial s} = \frac{\partial r}{\partial u} \frac{du}{ds} + \frac{\partial r}{\partial v} \frac{dv}{ds}$$

$$r' = r_u(u) + r_v(v)$$

Again Diff w.r.t s .

(10)

$$r'' = r_u(\ddot{u}) + \dot{u} \left[\frac{\partial r_u}{\partial u} \frac{du}{ds} + \frac{\partial r_u}{\partial v} \frac{dv}{ds} \right] + r_v(\ddot{v}) + \dot{v} \left[\frac{\partial r_v}{\partial u} \frac{du}{ds} + \frac{\partial r_v}{\partial v} \frac{dv}{ds} \right]$$

$$= r_u(\ddot{u}) + \dot{u} [r_{uu}(\dot{u}) + r_{uv}(\dot{v})] + r_v(\ddot{v}) + \dot{v} [r_{vu}(\dot{u}) + r_{vv}(\dot{v})]$$

$$= r_u(\ddot{u}) + r_{uu}(\dot{u})^2 + r_{uv}(\dot{u}\dot{v}) + r_v(\ddot{v}) + r_{vu}(\dot{u}\dot{v}) + r_{vv}(\dot{v})^2$$

$$= r_u(\ddot{u}) + r_{uu}(\dot{u})^2 + 2r_{uv}(\dot{u}\dot{v}) + r_v(\ddot{v}) + r_{vv}(\dot{v})^2$$

$$r'' \cdot n = n \cdot [r_u(\ddot{u}) + r_{uu}(\dot{u})^2 + 2r_{uv}(\dot{u}\dot{v}) + r_v(\ddot{v}) + r_{vv}(\dot{v})^2]$$

Since $n \perp u$ & $n \perp v \Rightarrow n \cdot r_u = 0 = n \cdot r_v$. So

$$k_n = (r_{uu} \cdot n) \left(\frac{du}{ds} \right)^2 + 2(r_{uv} \cdot n) \frac{du}{ds} \frac{dv}{ds} + (r_{vv} \cdot n) \left(\frac{dv}{ds} \right)^2$$

$$= \frac{L \left(\frac{du}{ds} \right)^2 + 2M \frac{du}{ds} \frac{dv}{ds} + N \left(\frac{dv}{ds} \right)^2}{\left(\frac{ds}{ds} \right)^2}$$

$$= \frac{L \left(\frac{du}{ds} \right)^2 + 2M \frac{du}{ds} \frac{dv}{ds} + N \left(\frac{dv}{ds} \right)^2}{E \left(\frac{du}{ds} \right)^2 + 2F \frac{du}{ds} \frac{dv}{ds} + G \left(\frac{dv}{ds} \right)^2}$$

where L, M, N, E, F & G are coefficients of Fundamental form.

Question: Find the normal curvature of the circle $\alpha(t) = (\cos t, \sin t, 1)$ on the elliptic paraboloid $r(u,v) = u\hat{i} + v\hat{j} + (u^2 + v^2)\hat{k}$

Solution:

$$r(u,v) = u\hat{i} + v\hat{j} + (u^2 + v^2)\hat{k}$$

$$r_u = \hat{i} + 2u\hat{k} \Rightarrow r_{uu} = 2\hat{k}$$

$$r_v = \hat{j} + 2v\hat{k} \Rightarrow r_{vv} = 2\hat{k}$$

$$r_{uv} = 0$$

$$\vec{n} = r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = \hat{i}(-2u) - \hat{j}(2v) + \hat{k}(1) = -2u\hat{i} - 2v\hat{j} + \hat{k}$$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{-2u\hat{i} - 2v\hat{j} + \hat{k}}{\sqrt{4u^2 + 4v^2 + 1}}$$

(11)

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$$M = \mathbf{r}_{uv} \cdot \mathbf{n} = 0$$

$$N = \mathbf{r}_{vv} \cdot \mathbf{n} = 2\hat{k} \cdot \left(\frac{-2u\hat{i} - 2v\hat{j} + \hat{k}}{\sqrt{4u^2 + 4v^2 + 1}} \right) = \frac{2}{\sqrt{4u^2 + 4v^2 + 1}}$$

$$L = \mathbf{r}_{uu} \cdot \mathbf{n} = 2\hat{k} \cdot \left(\frac{-2u\hat{i} - 2v\hat{j} + \hat{k}}{\sqrt{4u^2 + 4v^2 + 1}} \right) = \frac{2}{\sqrt{4u^2 + 4v^2 + 1}}$$

$$E = \mathbf{r}_u \cdot \mathbf{r}_u = 1 + 4u^2$$

$$G = \mathbf{r}_v \cdot \mathbf{r}_v = 1 + 4v^2$$

$$F = \mathbf{r}_u \cdot \mathbf{r}_v = 4uv$$

$$k_n = \frac{L(du)^2 + 2M du dv + N(dv)^2}{E(du)^2 + 2F du dv + G(dv)^2} = \frac{2(du)^2 + 2(0) du dv + 2(dv)^2}{(1+4u^2)(du)^2 + 8uv du dv + (1+4v^2)(dv)^2} \times \frac{1}{\sqrt{4u^2 + 4v^2 + 1}}$$

put $u = \cos t$, $v = \sin t$ or $\mathbf{r}(u, v) = \mathbf{r}(\cos t, \sin t)$

$$\Rightarrow du = -\sin t dt, dv = \cos t dt$$

$$k_n = \frac{2(\sin^2 t + \cos^2 t) dt^2}{\underbrace{(1+4\cos^2 t)}_{\sin^2 t} + 8\cos t \sin t (-\sin t)(\cos t) + \underbrace{(1+4\sin^2 t)}_{\cos^2 t} dt^2} \times \frac{1}{\sqrt{5}}$$

$$= \frac{2}{\sin^2 t + 8\cos^2 t \sin^2 t - 8\cos^2 t \sin^2 t + \cos^2 t} \times \frac{1}{\sqrt{5}}$$

$$k_n = \frac{2}{\sqrt{5}}$$

other method:

$$k_n = \mathbf{r}''(t) \cdot \hat{n}(t)$$

$$\hat{n} = \frac{-2u\hat{i} - 2v\hat{j} + \hat{k}}{\sqrt{4u^2 + 4v^2 + 1}}$$

$$\hat{n}(t) = \frac{-2\cos t \hat{i} - 2\sin t \hat{j} + \hat{k}}{\sqrt{5}}$$

$$\mathbf{r}(u, v) = \cos t \hat{i} + \sin t \hat{j} + \hat{k}$$

$$\mathbf{r}' = -\sin t \hat{i} + \cos t \hat{j}$$

$$\mathbf{r}'' = -\cos t \hat{i} - \sin t \hat{j}$$

$$k_n = r'' \cdot \hat{n} = (-\cos t \hat{i} - \sin t \hat{j}) \cdot \frac{(-2 \cos t \hat{i} - 2 \sin t \hat{j} + \hat{k})}{\sqrt{5}}$$

$$= \frac{2 \cos^2 t + 2 \sin^2 t}{\sqrt{5}} = \frac{2}{\sqrt{5}} \text{ Ans.}$$

Question: Find the normal curvature for the surface $r(\theta, \phi) = \cos \theta \sin \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \phi \hat{k}$

$$r(t) = \sqrt{1-a^2} \cos t \hat{i} + \sqrt{1-a^2} \sin t \hat{j} + a \hat{k}$$

Solution:

$$r(\theta, \phi) = \cos \theta \sin \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \phi \hat{k}$$

$$r_\theta = -\sin \theta \sin \phi \hat{i} + \cos \theta \sin \phi \hat{j}$$

$$r_{\theta\theta} = -\cos \theta \sin \phi \hat{i} - \sin \theta \sin \phi \hat{j}$$

$$r_\phi = \cos \theta \cos \phi \hat{i} + \sin \theta \cos \phi \hat{j} - \sin \phi \hat{k}$$

$$r_{\phi\phi} = -\cos \theta \sin \phi \hat{i} - \sin \theta \sin \phi \hat{j} - \cos \phi \hat{k}$$

$$r_{\theta\phi} = -\sin \theta \cos \phi \hat{i} + \cos \theta \cos \phi \hat{j}$$

$$\hat{n} = r_\theta \times r_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin \theta \sin \phi & \cos \theta \sin \phi & 0 \\ \cos \theta \cos \phi & \sin \theta \cos \phi & -\sin \phi \end{vmatrix}$$

$$= \hat{i}(-\cos \theta \sin^2 \phi - 0) - \hat{j}(\sin \theta \sin^2 \phi - 0) + \hat{k}(-\sin \theta \sin \phi \cos \phi - \cos \theta \sin \phi \cos \phi)$$

$$= -\cos \theta \sin^2 \phi \hat{i} - \sin \theta \sin^2 \phi \hat{j} + \sin \phi \cos \phi \hat{k}$$

$$\hat{n} = \frac{-\cos \theta \sin^2 \phi \hat{i} - \sin \theta \sin^2 \phi \hat{j} - \sin \phi \cos \phi \hat{k}}{\sqrt{(\cos^2 \theta \sin^4 \phi) + (\sin^2 \theta \sin^4 \phi) + \sin^2 \phi \cos^2 \phi}}$$

$$= \frac{-\cos \theta \sin^2 \phi \hat{i} - \sin \theta \sin^2 \phi \hat{j} - \sin \phi \cos \phi \hat{k}}{\sqrt{\sin^4 \phi (\cos^2 \theta + \sin^2 \theta) + \sin^2 \phi \cos^2 \phi}}$$

$$= \frac{-\cos \theta \sin^2 \phi \hat{i} - \sin \theta \sin^2 \phi \hat{j} - \sin \phi \cos \phi \hat{k}}{\sqrt{\sin^4 \phi (\cos^2 \theta + \sin^2 \theta) + \sin^2 \phi \cos^2 \phi}}$$

$$= \frac{-\cos \theta \sin^2 \phi \hat{i} - \sin \theta \sin^2 \phi \hat{j} - \sin \phi \cos \phi \hat{k}}{\sqrt{\sin^4 \phi (\sin^2 \phi + \cos^2 \phi)}}$$

$$= \frac{-\cos \theta \sin^2 \phi \hat{i} - \sin \theta \sin^2 \phi \hat{j} - \sin \phi \cos \phi \hat{k}}{\sin \phi}$$

$$= -\cos \theta \sin \phi \hat{i} - \sin \theta \sin \phi \hat{j} - \cos \phi \hat{k}$$

$$= -\cos \theta \sin \phi \hat{i} - \sin \theta \sin \phi \hat{j} - \cos \phi \hat{k}$$

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$$\hat{n} = -\cos\theta\sin\varphi\hat{i} - \sin\theta\sin\varphi\hat{j} - \cos\varphi\hat{k}$$

$$E = r_\theta \cdot r_\theta = \sin^2\theta\sin^2\varphi + \cos^2\theta\sin^2\varphi = \sin^2\varphi(\sin^2\theta + \cos^2\theta)$$

$$E = \sin^2\varphi$$

$$G = r_\varphi \cdot r_\varphi = \cos^2\theta\cos^2\varphi + \sin^2\theta\cos^2\varphi + \sin^2\varphi$$

$$= \cos^2\varphi(\cos^2\theta + \sin^2\theta) + \sin^2\varphi = \cos^2\varphi + \sin^2\varphi$$

$$= 1$$

$$F = r_\theta \cdot r_\varphi = -\sin\theta\sin\varphi \cdot \cos\theta\cos\varphi + \cos\theta\sin\varphi \cdot \sin\theta\cos\varphi$$

$$= 0$$

$$M = r_\theta \cdot \hat{n} = (-\sin\theta\cos\varphi\hat{i} + \cos\theta\cos\varphi\hat{j}) \cdot (-\cos\theta\sin\varphi\hat{i} - \sin\theta\sin\varphi\hat{j} - \cos\varphi\hat{k})$$

$$= \sin\theta\cos\theta\cos\varphi\sin\varphi - \sin\theta\cos\theta\cos\varphi\sin\varphi$$

$$= 0$$

$$L = r_\theta \cdot \hat{n} = (-\cos\theta\sin\varphi\hat{i} - \sin\theta\sin\varphi\hat{j}) \cdot (-\cos\theta\sin\varphi\hat{i} - \sin\theta\sin\varphi\hat{j} - \cos\varphi\hat{k})$$

$$= \cos^2\theta\sin^2\varphi + \sin^2\theta\sin^2\varphi$$

$$= (\cos^2\theta + \sin^2\theta)\sin^2\varphi = \sin^2\varphi$$

$$N = r_\varphi \cdot \hat{n} = (-\cos\theta\sin\varphi\hat{i} - \sin\theta\sin\varphi\hat{j} - \cos\varphi\hat{k}) \cdot (-\cos\theta\sin\varphi\hat{i} - \sin\theta\sin\varphi\hat{j} - \cos\varphi\hat{k})$$

$$= \cos^2\theta\sin^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\varphi$$

$$= \sin^2\varphi(\cos^2\theta + \sin^2\theta) + \cos^2\varphi = \sin^2\varphi + \cos^2\varphi$$

$$= 1$$

$$k_n = \frac{L(du)^2 + 2M du dv + N(dv)^2}{E(du)^2 + 2F du dv + G(dv)^2}$$

$$k_n = \frac{\sin^2\varphi (d\theta)^2 + 0 + (d\varphi)^2}{\sin^2\varphi (d\theta)^2 + 0 + (d\varphi)^2} = \frac{\sin^2\varphi (d\theta)^2 + (d\varphi)^2}{\sin^2\varphi (d\theta)^2 + (d\varphi)^2}$$

$$\theta = \sqrt{1-a^2} \cos t, \quad \varphi = \sqrt{1-a^2} \sin t$$

$$d\theta = -\sqrt{1-a^2} \sin t dt, \quad d\varphi = \sqrt{1-a^2} \cos t dt$$

$$k_n = \frac{(\sin^2\varphi(1-a^2)\sin^2 t) + (1-a^2)\cos^2 t}{(\sin^2\varphi(1-a^2)\sin^2 t) + (1-a^2)\cos^2 t} dt^2$$

$$k_n = \frac{\sin^2(\sqrt{1-a^2} \sin t)(1-a^2)\sin^2 t + (1-a^2)\cos^2 t}{\sin^2(\sqrt{1-a^2} \sin t)(1-a^2)\sin^2 t + (1-a^2)\cos^2 t} = 1$$

$$k_n = r''(t) \cdot \hat{n}(t)$$

$$\hat{n} = -\cos\theta\sin\phi\hat{i} - \sin\theta\sin\phi\hat{j} - \cos\phi\hat{k}$$

$$\hat{n}(t) = -\cos(\sqrt{1-a^2}\cos t)\sin(\sqrt{1-a^2}\sin t)\hat{i} - \sin(\sqrt{1-a^2}\cos t)\sin(\sqrt{1-a^2}\sin t)\hat{j} - \cos(\sqrt{1-a^2}\sin t)\hat{k}$$

$$r(\theta, \phi) = \cos(\sqrt{1-a^2}\cos t)\sin(\sqrt{1-a^2}\sin t)\hat{j} + \sin(\sqrt{1-a^2}\cos t)\sin(\sqrt{1-a^2}\sin t)\hat{j} + \cos(\sqrt{1-a^2}\sin t)\hat{k}$$

$$r' = -\sin(\sqrt{1-a^2}\cos t) \cdot \sqrt{1-a^2} \sin t$$

This method is very difficult so we can not choose this. previous method is best.

Question: Find the normal curvature for

$$r(u, v) = \cos v \hat{i} + \sin v \hat{j} + u \hat{k}$$

$$\alpha(t) = \cos t \hat{i} + \sin t \hat{j} + h(t) \hat{k}$$

Solution:

$$r(u, v) = \cos v \hat{i} + \sin v \hat{j} + u \hat{k}$$

$$r_u = \hat{k}$$

$$r_{uu} = 0$$

$$r_v = -\sin v \hat{i} + \cos v \hat{j}$$

$$r_{vv} = -\cos v \hat{i} - \sin v \hat{j}$$

$$r_{uv} = 0$$

$$\vec{n} = r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ -\sin v & \cos v & 0 \end{vmatrix} = \hat{i}(-\cos v) - \hat{j}(\sin v) + \hat{k}(0) = -\cos v \hat{i} - \sin v \hat{j}$$

$$\hat{n} = \frac{-\cos v \hat{i} - \sin v \hat{j}}{\sqrt{\cos^2 v + \sin^2 v}} = -\cos v \hat{i} - \sin v \hat{j}$$

$$E = r_u \cdot r_u = 1$$

$$G = r_v \cdot r_v = \sin^2 v + \cos^2 v = 1$$

$$F = r_u \cdot r_v = 0$$

$$L = r_{uu} \cdot n = 0$$

$$M = r_{uv} \cdot n = 0$$

$$N = r_{vv} \cdot n = \cos^2 v + \sin^2 v = 1$$

$$k_n = \frac{L(du)^2 + 2Mdu dv + N(dv)^2}{E(du)^2 + 2Fdu dv + G(dv)^2}$$

$$= \frac{0 + 0 + (dv)^2}{(du)^2 + 0 + (dv)^2} = \frac{(dv)^2}{(du)^2 + (dv)^2}$$

$$u = \cos t, \quad v = \sin t \quad r(u, v) = r(\cos t, \sin t)$$

$$du = -\sin t dt, \quad dv = \cos t dt$$

$$k_n = \frac{\cos^2 t dt^2}{(\sin^2 t + \cos^2 t) dt^2} = \frac{\cos^2 t}{1} = \cos^2 t$$

03/05/09

Curvature of principal Direction, Principal curvature, Mean curvature and Gaussian curvature:

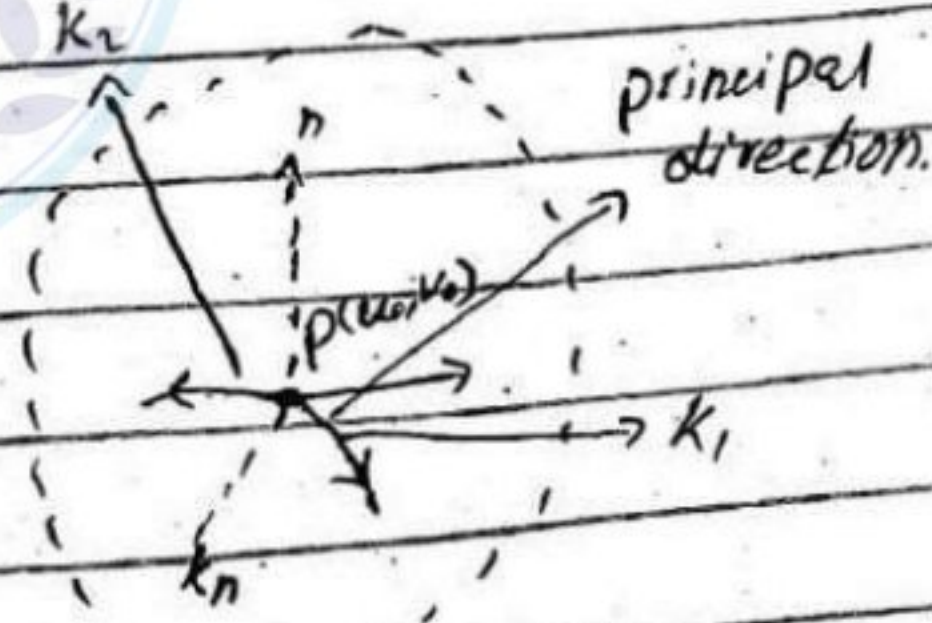
We are going to find two principal curvatures at any point p on a surface S such that all other normal curvatures $k_n(t)$ are sandwiched between these two principal curvatures. Also we use the expression of normal curvature as a quotient of fundamental forms. We need to address the maximum or minimum values for this expression.

$$E = E(u_0, v_0), \quad F = F(u_0, v_0)$$

$$G = G(u_0, v_0)$$

Let there be a curve lying on surface S . Let p be a point on the curve.

at any point p , there are two directions along which normal curvature be maximum or minimum. The direction where normal curvature be maximum or minimum are known as principal directions."



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We know that at any point "p" the normal curvature k_n in the direction of du and dv is given by

$$\frac{k_n}{1} = \frac{L(du)^2 + 2M du dv + N(dv)^2}{E(du)^2 + 2F du dv + G(dv)^2}$$

comparing

$$k_n = \frac{L(du)^2 + 2M du dv + N(dv)^2}{E(du)^2 + 2F du dv + G(dv)^2} \quad \text{--- (1)}$$

$$1 = \frac{E(du)^2 + 2F du dv + G(dv)^2}{E(du)^2 + 2F du dv + G(dv)^2} \quad \text{--- (2)}$$

We need to find the maximum or minimum normal curvature we use the Lagrange maximum or minimum condition for k_n
set $dk_n = 0$

$$dk_n = d(L(du)^2 + 2M du dv + N(dv)^2)$$

$$= 2L du \cdot d(du) + 2M du \cdot d(dv) + 2M dv \cdot d(du) + 2N dv \cdot d(dv)$$

$$0 = 2L du \cdot d^2u + 2M du d^2v + 2M dv d^2u + 2N dv d^2v$$

$$(L du + M dv) d^2u + (M du + N dv) d^2v = 0 \quad \text{--- (3)}$$

From equation (2)

$$0 = 2E du \cdot d(du) + 2F du \cdot d(dv) + 2F dv \cdot d(du) + 2G dv \cdot d(dv)$$

$$(E du + F dv) d^2u + (F du + G dv) d^2v = 0 \quad \text{--- (4)}$$

Multiply equation (4) by λ & add in eq. (3)

$$(L du + M dv) d^2u + (M du + N dv) d^2v + \lambda (E du + F dv) d^2u + \lambda (F du + G dv) d^2v = 0$$

$$[(L du + M dv) + \lambda (E du + F dv)] d^2u + [(M du + N dv) + \lambda (F du + G dv)] d^2v = 0$$

From here we get two Equations.

$$L du + M dv + \lambda (E du + F dv) = 0 \quad \text{--- (5)}$$

$$M du + N dv + \lambda (F du + G dv) = 0 \quad \text{--- (6)}$$

Multiplying Eq. (5) by du and (6) by dv

$$L(du)^2 + M du \cdot dv + \lambda (E(du)^2 + F du dv) = 0 \quad \text{--- (7)}$$

$$M du \cdot dv + N(dv)^2 + \lambda (F du dv + G(dv)^2) = 0 \quad \text{--- (8)}$$

Adding (7) & (8)

(17)

$$(L(du)^2 + Mdu dv + N(dv)^2) + \lambda(E(du)^2 + 2Fdu dv + G(dv)^2) = 0$$

$$(L(du)^2 + 2Mdu dv + N(dv)^2) + \lambda(E(du)^2 + 2Fdu dv + G(dv)^2) = 0$$

using Eq. (1) & (2)

$$k_n + \lambda(1) = 0$$

$$\lambda = -k_n \quad (9)$$

put Eq. (9) in (5) & (6)

$$(Ldu + Mdv) - k_n(Edu + Fdv) = 0$$

$$(Mdu + Ndv) - k_n(Fdu + Gdv) = 0$$

$$Ldu + Mdv = k_n(Edu + Fdv) \quad (10)$$

$$Mdu + Ndv = k_n(Fdu + Gdv) \quad (11)$$

From (10) & (11) we get

$$(L - Ek_n)du = (k_n F - M)dv \quad (12)$$

$$(M - Fk_n)du = (k_n G - N)dv \quad (13)$$

From principal directions divide Eq. (10) by (11)

$$\frac{Ldu + Mdv}{Mdu + Ndv} = \frac{Fdu + Fdv}{Fdu + Gdv}$$

$$(EM - 2F)(du)^2 + (EN - GL)du dv + (FN - GM)(dv)^2 = 0 \quad (14)$$

Eq. (14) is an Equation for principal direction.

Dividing Eq. (12) by (13)

$$\frac{L - Ek_n}{M - Fk_n} = \frac{k_n F - M}{k_n G - N}$$

$$LGk_n + ENk_n - EGk_n^2 - LN = MFk_n - M^2 - Fk_n^2 + FMk_n$$

$$(EG - F^2)k_n^2 - (EN + GL - 2FM)k_n + LN - M^2 = 0 \quad (15)$$

Eq. (15) is quadratic in k_n and we can determine the principal curvature from this Eq.suppose it have roots k_1 and k_2

$$k_n^2 - \left(\frac{EN + GL - 2FM}{EG - F^2} \right) k_n + \frac{LN - M^2}{EG - F^2} = 0 \quad (16)$$

$$k_1 + k_2 = -\frac{b}{a} = \frac{EN + GL - 2FM}{EG - F^2}$$

$$\frac{k_1 + k_2}{2} = \frac{1}{2} \left(\frac{EN + GL - 2FM}{EG - F^2} \right)$$

(18)

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$$H = \frac{1}{2} \frac{EN + GL - 2FM}{EG - F^2}$$

Here "H" is known as mean curvature

$$K_1 K_2 = \frac{C}{a} = \frac{LN - M^2}{EG - F^2}$$

$$K = \frac{LN - M^2}{EG - F^2}$$

where K is known as Gaussian curvature.

(16) $\Rightarrow$

$$k_n^2 - 2Hk_n + K = 0$$

$$k_n = \frac{-(-2H) \pm \sqrt{4H^2 - 4K}}{2(1)} = H \pm \sqrt{H^2 - K}$$

$$k_1 = H + \sqrt{H^2 - K}, \quad k_2 = H - \sqrt{H^2 - K}$$

k_1 & k_2 are known as principal curvature.

08/05/09

Question: Find mean curvature and Gaussian curvature
 $r(u, v) = f(u) \cos v \hat{i} + f(u) \sin v \hat{j} + g(u) \hat{k}; f > 0, f'^2 + g'^2 = 1$

Sol: $r(u, v) = f(u) \cos v \hat{i} + f(u) \sin v \hat{j} + g(u) \hat{k}$

$$r_u = f' \cos v \hat{i} + f' \sin v \hat{j} + g' \hat{k}$$

$$r_{uu} = f'' \cos v \hat{i} + f'' \sin v \hat{j} + g'' \hat{k}$$

$$r_v = -f \sin v \hat{i} + f \cos v \hat{j} + 0 \hat{k}$$

$$r_{vv} = -f \cos v \hat{i} - f \sin v \hat{j}$$

$$r_{uv} = -f' \sin v \hat{i} + f' \cos v \hat{j} + 0 \hat{k}$$

$$\vec{n} = r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ f' \cos v & f' \sin v & g' \\ -f \sin v & f \cos v & 0 \end{vmatrix} = \hat{i}(-g'f \cos v) - \hat{j}(g'f \sin v) + \hat{k}(ff' \cos^2 v + ff' \sin^2 v)$$

$$\vec{n} = -g'f \cos v \hat{i} - g'f \sin v \hat{j} + ff' \hat{k}$$

$$\hat{n} = \frac{-g'f \cos v \hat{i} - g'f \sin v \hat{j} + ff' \hat{k}}{\sqrt{(g'f)^2 + (g'f)^2 + (ff')^2}} = \frac{-g'f \cos v \hat{i} - g'f \sin v \hat{j} + ff' \hat{k}}{f \sqrt{g'^2 + g'^2 + f'^2}}$$

(19)

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$$\hat{n} = -g' \cos v \hat{i} - g' \sin v \hat{j} + f \hat{k}$$

$$E = \mathbf{r}_u \cdot \mathbf{r}_u = f'^2 \cos^2 v + f'^2 \sin^2 v + g'^2 = f'^2 + g'^2 = 1$$

$$G = \mathbf{r}_v \cdot \mathbf{r}_v = f^2 \sin^2 v + f^2 \cos^2 v = f^2$$

$$F = \mathbf{r}_u \cdot \mathbf{r}_v = -ff' \cos v \sin v + ff' \cos v \sin v = 0$$

$$L = \mathbf{r}_{uu} \cdot \hat{n} = (f'' \cos v \hat{i} + f'' \sin v \hat{j} + g'' \hat{k}) \cdot (-g' \cos v \hat{i} - g' \sin v \hat{j} + f \hat{k})$$

$$L = \frac{-g' f'' \cos v - g' f'' \sin v + g'' f}{\sqrt{g'^2 + 1}} = \frac{-g' f'' + g'' f}{\sqrt{g'^2 + 1}}$$

$$M = \mathbf{r}_{uv} \cdot \hat{n} = (-f' \sin v \hat{i} + f' \cos v \hat{j}) \cdot \frac{-g' \cos v \hat{i} - g' \sin v \hat{j} + f \hat{k}}{\sqrt{g'^2 + 1}}$$

$$M = \frac{f' g' \cos v \sin v - g' f' \cos v \sin v}{\sqrt{g'^2 + 1}} = 0$$

$$N = \mathbf{r}_{vv} \cdot \hat{n} = (-f \cos v \hat{i} - f \sin v \hat{j}) \cdot \frac{-g' \cos v \hat{i} - g' \sin v \hat{j} + f \hat{k}}{\sqrt{g'^2 + 1}}$$

$$N = \frac{g' f \cos^2 v + g' f \sin^2 v}{\sqrt{g'^2 + 1}} = \frac{g' f}{\sqrt{g'^2 + 1}}$$

Gaussian curvature

$$K = \frac{LN - M^2}{EG - F^2} = \frac{(-g' f'' + g'' f)(g' f) - 0}{1 \cdot f^2 - 0} \times \frac{1}{(\sqrt{g'^2 + 1})^2}$$

$$K = \frac{-g'^2 f'' f + g' g'' f^2}{f^2 (\sqrt{g'^2 + 1})^2} = \frac{-g'^2 f'' + g' g'' f}{f (\sqrt{g'^2 + 1})^2} = \frac{-g'^2 f'' + g' g'' f}{f (g'^2 + 1)}$$

$$H = \frac{1}{2} \frac{EN + GL - 2FM}{EG - F^2} = \frac{1}{2} \frac{1 \cdot g' f + f \cdot (g' f'' - g'' f) - 0}{f^2}$$

$$H = \frac{1}{2} \frac{g' + f (g'' f - g' f'')}{f \sqrt{g'^2 + 1}}$$

(20)

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Question: Find Gaussian curvature and Mean curvature

$$r(x,y) = x\hat{i} + y\hat{j} + f(x,y)\hat{k}$$

Solution:

$$r(x,y) = x\hat{i} + y\hat{j} + f(x,y)\hat{k}$$

$$r_x = \hat{i} + f_x \hat{k}$$

$$r_{xx} = f_{xx} \hat{k}$$

$$r_y = \hat{j} + f_y \hat{k}$$

$$r_{yy} = f_{yy} \hat{k}$$

$$r_{xy} = f_{xy} \hat{k}$$

$$n = r_x \times r_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \hat{i}(-f_x) - \hat{j}(f_y) + \hat{k}(1)$$

$$\hat{n} = -f_x \hat{i} - f_y \hat{j} + \hat{k}$$

$$\hat{n} = \frac{-f_x \hat{i} - f_y \hat{j} + \hat{k}}{\sqrt{f_x^2 + f_y^2 + 1}}$$

$$E = r_x \cdot r_x = 1 + f_x^2, \quad G = r_y \cdot r_y = 1 + f_y^2$$

$$F = r_x \cdot r_y = f_x f_y$$

$$L = r_{xx} \cdot \hat{n} = \frac{f_{xx}}{\sqrt{f_x^2 + f_y^2 + 1}}$$

$$M = r_{xy} \cdot \hat{n} = \frac{f_{xy}}{\sqrt{f_x^2 + f_y^2 + 1}}; \quad N = r_{yy} \cdot \hat{n} = \frac{f_{yy}}{\sqrt{f_x^2 + f_y^2 + 1}}$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{f_{xx} \cdot f_{yy} - (f_{xy})^2}{(1 + f_x^2)(1 + f_y^2) - (f_x f_y)^2} \times \frac{1}{f_x^2 + f_y^2 + 1}$$

$$= \frac{f_{xx} f_{yy} - f_{xy}^2}{f_y^2 + 1 + f_x^2 f_y^2 + f_x^2 - f_x^2 f_y^2} \times \frac{1}{f_x^2 + f_y^2 + 1}$$

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(f_x^2 + f_y^2 + 1)^2}$$

(2)

$$H = \frac{1}{2} \frac{EN + GL - 2FM}{EG - F^2} = \frac{1}{2} \frac{f_{xy}(1+f_x^2) + (1+f_y^2)f_{xx} - 2f_x f_y f_{xy}}{(f_x^2 + f_y^2 + 1)^{3/2}}$$

$$= \frac{1}{2} \frac{f_{xy}(1+f_x^2) + (1+f_y^2)f_{xx} - 2f_x f_y f_{xy}}{(f_x^2 + f_y^2 + 1)^{3/2}}$$

Pr

Question: Calculate the principal curvature, Gaussian curvature and mean curvatures.

1) $r(u, v) = R \sin v \cos u \hat{i} + R \sin v \sin u \hat{j} + R \cos v \hat{k}$

2) $r(h, \theta) = R \cos \theta \hat{i} + R \sin \theta \hat{j} + h \hat{k}$ cylinder

3) $r(h, \theta) = kh \cos \theta \hat{i} + kh \sin \theta \hat{j} + h \hat{k}$ cone

4) $r(u, v) = u \hat{i} + v \hat{j} + f(u, v) \hat{k}$

Solution:

1) $r(u, v) = R \sin v \cos u \hat{i} + R \sin v \sin u \hat{j} + R \cos v \hat{k}$

$$r_u = -R \sin v \sin u \hat{i} + R \sin v \cos u \hat{j} + 0 \hat{k}$$

$$r_{uu} = -R \sin v \cos u \hat{i} - R \sin v \sin u \hat{j}$$

$$r_v = R \cos v \cos u \hat{i} + R \cos v \sin u \hat{j} - R \sin v \hat{k}$$

$$r_{uv} = -R \sin v \sin u \hat{i} + R \sin v \cos u \hat{j} - R \cos v \hat{k}$$

$$r_{vv} = -R \cos v \sin u \hat{i} - R \cos v \cos u \hat{j} - R \sin v \hat{k}$$

	$\hat{i}$	$\hat{j}$	$\hat{k}$
$n = r_u \times r_v =$	$-R \sin v \sin u$	$-R \sin v \cos u$	0
	$R \cos v \cos u$	$R \cos v \sin u$	$-R \sin v$

$$= \hat{i} (R \sin^2 v \sin u) - \hat{j} (R \sin^2 v \cos u) + \hat{k} (-R \sin v \cos u \cos u + R \sin v \sin u \cos u)$$

$$n = R \sin^2 v \sin u \hat{i} - R \sin^2 v \cos u \hat{j}$$

$$\hat{n} = \frac{R \sin^2 v \sin u \hat{i} - R \sin^2 v \cos u \hat{j}}{\sqrt{R^2 \sin^4 v \sin^2 u + R^2 \sin^4 v \cos^2 u}} = \frac{R \sin^2 v \sin u \hat{i} - R \sin^2 v \cos u \hat{j}}{\sqrt{2} R \sin^2 v}$$

$$\hat{n} = \frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{j}$$

$$E = r_u \cdot r_u = R^2 \sin^2 v \sin^2 u + R^2 \sin^2 v \cos^2 u = 2R^2 \sin^2 v$$

$$G = r_v \cdot r_v = R^2 \cos^2 v \cos^2 u + R^2 \cos^2 v \sin^2 u + R^2 \sin^2 v = 2R^2 \cos^2 v + R^2 \sin^2 v$$

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$$F = \mathbf{r}_u \cdot \mathbf{r}_v = (-R \sin v \sin u \hat{i} - R \sin v \sin u \hat{j} + 0 \hat{k}) \cdot (R \cos v \cos u \hat{i} + R \cos v \cos u \hat{j} - R \sin v \hat{k})$$

$$= -R^2 \sin v \sin u \cos v \cos u - R^2 \sin v \sin u \cos v \cos u + 0$$

$$F = -2R^2 \sin v \sin u \cos v \cos u$$

$$L = \mathbf{r}_{uu} \cdot \hat{n} = (-R \sin v \cos u \hat{i} - R \sin v \cos u \hat{j}) \cdot \left(\frac{1}{R} \hat{i} - \frac{1}{R} \hat{j} \right)$$

$$L = -\frac{1}{R} R \sin v \cos u + \frac{1}{R} R \sin v \cos u = 0$$

$$M = \mathbf{r}_{uv} \cdot \hat{n} = (-R \cos v \sin u \hat{i} - R \cos v \sin u \hat{j}) \cdot \left(\frac{1}{R} \hat{i} - \frac{1}{R} \hat{j} \right)$$

$$= -\frac{1}{R} R \cos v \sin u + \frac{1}{R} R \cos v \sin u = 0$$

$$N = \mathbf{r}_{vv} \cdot \hat{n} = (-R \sin v \cos u \hat{i} - R \sin v \cos u \hat{j} - R \cos v \hat{k}) \cdot \left(\frac{1}{R} \hat{i} - \frac{1}{R} \hat{j} \right)$$

$$= -\frac{1}{R} R \sin v \cos u + \frac{1}{R} R \sin v \cos u = 0$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{0(0) - (0)^2}{EG - F^2} = 0$$

$$H = \frac{1}{2} \frac{EN + GL - 2FM}{EG - F^2} = \frac{1}{2} \frac{E(0) + G(0) - 2F(0)}{EG - F^2} = \frac{0}{EG - F^2}$$

$$H = 0$$

principal curvatures

$$k_1 = 0, \quad k_2 = 0$$

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$$\mathbf{r}(h, \theta) = R \cos \theta \hat{i} + R \sin \theta \hat{j} + h \hat{k}$$

$$\mathbf{r}_h = \hat{k}$$

$$\mathbf{r}_{hh} = 0$$

$$\mathbf{r}_\theta = -R \sin \theta \hat{i} + R \cos \theta \hat{j}$$

$$\mathbf{r}_{\theta\theta} = -R \cos \theta \hat{i} - R \sin \theta \hat{j}$$

$$\mathbf{r}_{h\theta} = 0$$

$$\hat{n} = \mathbf{r}_h \times \mathbf{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ -R \sin \theta & R \cos \theta & 0 \end{vmatrix}$$

$$= \hat{i}(-R \cos \theta) - \hat{j}(R \sin \theta) + 0 \hat{k}$$

$$= -R \cos \theta \hat{i} - R \sin \theta \hat{j} + 0 \hat{k}$$

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$$\hat{n} = \frac{-R\cos\theta\hat{i} - R\sin\theta\hat{j}}{\sqrt{R^2(\cos^2\theta + \sin^2\theta)}} = -\cos\theta\hat{i} - \sin\theta\hat{j}$$

$$E = r_h \cdot r_h = 1, \quad G = r_\theta \cdot r_\theta = R^2\sin^2\theta + R^2\cos^2\theta = R^2$$

$$F = r_h \cdot r_\theta = 0$$

$$L = r_{hh} \cdot \hat{n} = 0$$

$$M = r_{h\theta} \cdot \hat{n} = 0, \quad N = r_{\theta\theta} \cdot \hat{n} = (-R\cos\theta\hat{i} - R\sin\theta\hat{j}) \cdot (-\cos\theta\hat{i} - \sin\theta\hat{j})$$

$$N = R\cos^2\theta + R\sin^2\theta = R$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{0(R) - 0}{1 \cdot R^2 - 0} = 0$$

$$H = \frac{1}{2} \frac{EN + GL - 2FM}{EG - F^2} = \frac{1}{2} \frac{1 \cdot R + R^2(0) - 0}{1 \cdot R^2 - 0}$$

$$H = \frac{1}{2} \frac{R}{R^2} = \frac{1}{2R}$$

Principal curvature

$$k_1 = H + \sqrt{H^2 - K} = \frac{1}{2R} + \frac{1}{2R} = \frac{1}{R}$$

$$k_2 = H - \sqrt{H^2 - K} = \frac{1}{2R} - \frac{1}{2R} = 0$$

$$\mathbf{r}(h, \theta) = kh\cos\theta\hat{i} + kh\sin\theta\hat{j} + h\hat{k}$$

$$r_h = k\cos\theta\hat{i} + k\sin\theta\hat{j} + \hat{k}$$

$$r_{hh} = 0$$

$$r_\theta = -k\sin\theta\hat{i} + k\cos\theta\hat{j} + 0\hat{k}$$

$$r_{\theta\theta} = -k\cos\theta\hat{i} - k\sin\theta\hat{j} + 0\hat{k}$$

$$r_{h\theta} = -k\sin\theta\hat{i} + k\cos\theta\hat{j} + 0\hat{k}$$

$$n = r_h \times r_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ k\cos\theta & k\sin\theta & 1 \\ -k\sin\theta & k\cos\theta & 0 \end{vmatrix}$$

$$= \hat{i}(-kh\cos\theta) - \hat{j}(kh\sin\theta) + \hat{k}(k^2h\cos^2\theta + k^2h\sin^2\theta)$$

$$= -kh\cos\theta\hat{i} - kh\sin\theta\hat{j} + k^2h\hat{k}$$

$$\hat{n} = \frac{-kh\cos\theta\hat{i} - kh\sin\theta\hat{j} + k^2h\hat{k}}{\sqrt{k^2h^2 + k^2h^2}} = \frac{-kh\cos\theta\hat{i} - kh\sin\theta\hat{j} + k^2h\hat{k}}{kh\sqrt{1 + k^2}}$$

$$= \frac{-\cos\theta\hat{i} - \sin\theta\hat{j} + k\hat{k}}{\sqrt{1 + k^2}}$$

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$$L = r_{hh} \cdot \vec{n} = 0 \quad E = k^2 + 1, \quad G = k^2 h^2, \quad F = 0$$

$$M = r_{h\theta} \cdot \vec{n} = 0 = \frac{k^2 h \sin \theta \cos \theta - k^2 h \sin \theta \cos \theta}{\sqrt{1+k^2}} = 0$$

$$N = r_{\theta\theta} \cdot \vec{n} = \frac{k h}{\sqrt{1+k^2}}$$

$$k = \frac{LN - M^2}{EG - F^2} = \frac{0(N) - 0}{EG - F^2} = 0$$

$$H = \frac{1}{2} \frac{EN + GL - 2FM}{EG - F^2} = \frac{1}{2} \frac{(k^2+1) \cdot k h}{(k^2+1) \cdot k^2 h^2 - 0} = \frac{1}{2kh\sqrt{k^2+1}}$$

Principal curvature

$$k_1 = H + \sqrt{H^2 - K} = \frac{1}{2kh\sqrt{k^2+1}} + \frac{1}{2kh\sqrt{k^2+1}} = \frac{1}{kh\sqrt{k^2+1}}$$

$$k_2 = H - \sqrt{H^2 - K} = \frac{1}{2kh} - \frac{1}{2kh} = 0$$

09/05/019

Question: $r(u,v) = p \sin u \cos v \hat{i} + p \sin u \sin v \hat{j} + p \cos u \hat{k}$
Find Mean curvature, Gaussian curvature and Principal curvature.

Sol: $r(u,v) = p \sin u \cos v \hat{i} + p \sin u \sin v \hat{j} + p \cos u \hat{k}$

$$r_u = p \cos u \cos v \hat{i} + p \cos u \sin v \hat{j} - p \sin u \hat{k}$$

$$r_{uu} = -p \sin u \cos v \hat{i} - p \sin u \sin v \hat{j} - p \cos u \hat{k}$$

$$r_v = -p \sin u \sin v \hat{i} + p \sin u \cos v \hat{j} + 0 \hat{k}$$

$$r_{uv} = -p \sin u \cos v \hat{i} - p \sin u \sin v \hat{j}$$

$$r_{vv} = +p \cos u \sin v \hat{i} + p \cos u \cos v \hat{j} + 0 \hat{k}$$

	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\vec{n} = r_u \times r_v =$	$p \cos u \cos v$	$p \cos u \sin v$	$-p \sin u$
	$-p \sin u \sin v$	$p \sin u \cos v$	0

$$= \hat{i} (p^2 \sin^2 u \cos v) - \hat{j} (-p^2 \sin^2 u \sin v) + \hat{k} (p^2 \sin u \cos u \cos v + p^2 \sin u \sin u \cos v \sin v)$$

$$= \hat{i} (p^2 \sin^2 u \cos v) + \hat{j} (p^2 \sin^2 u \sin v) + \hat{k} (p^2 \sin u \cos u (\cos^2 v + \sin^2 v))$$

$$= p^2 \sin^2 u \cos v \hat{i} + p^2 \sin^2 u \sin v \hat{j} + p^2 \sin u \cos u \hat{k}$$

$$\hat{n} = \frac{\rho \sin u \cos v \hat{i} + \rho \sin u \sin v \hat{j} + \rho \sin u \cos u \hat{k}}{\sqrt{\rho^2 \sin^2 u \cos^2 v + \rho^2 \sin^2 u \sin^2 v + \rho^2 \sin^2 u \cos^2 u}}$$

$$= \frac{\rho \sin u \cos v \hat{i} + \rho \sin u \sin v \hat{j} + \rho \sin u \cos u \hat{k}}{\rho \sin u \sqrt{\sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u}}$$

$$= \frac{\sin u \cos v \hat{i} + \sin u \sin v \hat{j} + \cos u \hat{k}}{\sqrt{1}}$$

$$E = \mathbf{r}_u \cdot \mathbf{r}_u = \rho^2 \cos^2 u \cos^2 v + \rho^2 \cos^2 u \sin^2 v + \rho^2 \sin^2 u$$

$$= \rho^2 (\cos^2 u (\cos^2 v + \sin^2 v) + \sin^2 u) = \rho^2 (\cos^2 u + \sin^2 u) = \rho^2$$

$$G = \mathbf{r}_v \cdot \mathbf{r}_v = \rho^2 \sin^2 u \sin^2 v + \rho^2 \sin^2 u \cos^2 v$$

$$= \rho^2 \sin^2 u (\sin^2 v + \cos^2 v) = \rho^2 \sin^2 u$$

$$F = \mathbf{r}_u \cdot \mathbf{r}_v = -\rho^2 \sin u \cos u \cos v \sin v + \rho^2 \sin u \cos u \cos v \cos v$$

$$F = 0$$

$$L = \mathbf{r}_u \cdot \hat{n} = \rho \cos u \sin u \cos^2 v + \rho \cos u \sin u \sin^2 v = \rho \sin u \cos u$$

$$= (\rho \sin u \cos v \hat{i} - \rho \sin u \sin v \hat{j} - \rho \cos u \hat{k}) \cdot (\sin u \cos v \hat{i} + \sin u \sin v \hat{j} + \cos u \hat{k})$$

$$= -\rho \sin^2 u \cos^2 v - \rho \sin^2 u \sin^2 v - \rho \cos^2 u$$

$$= -\rho (\sin^2 u (\cos^2 v + \sin^2 v) + \cos^2 u) = -\rho (\sin^2 u + \cos^2 u)$$

$$= -\rho$$

$$M = \mathbf{r}_{uv} \cdot \hat{n} = (-\rho \cos u \sin v \hat{i} + \rho \cos u \cos v \hat{j} + 0 \hat{k}) \cdot (\sin u \cos v \hat{i} + \sin u \sin v \hat{j} + \cos u \hat{k})$$

$$= -\rho \sin u \cos u \sin v \cos v + \rho \sin u \cos u \sin v \sin v = 0$$

$$N = \mathbf{r}_v \cdot \hat{n} = (-\rho \sin u \cos v \hat{i} - \rho \sin u \sin v \hat{j}) \cdot (\sin u \cos v \hat{i} + \sin u \sin v \hat{j} + \cos u \hat{k})$$

$$= -\rho \sin^2 u \cos^2 v - \rho \sin^2 u \sin^2 v = -\rho \sin^2 u (\cos^2 v + \sin^2 v)$$

$$= -\rho \sin^2 u$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\rho^2 \sin^2 u - 0}{\rho^4 \sin^2 u - 0} = \frac{1}{\rho^2}$$

$$H = \frac{1}{2} \frac{EN + GL - 2FM}{EG - F^2} = \frac{1}{2} \frac{-\rho \cdot \rho \sin^2 u + \rho \sin^2 u \cdot (-\rho) - 0}{\rho^4 \sin^2 u}$$

$$H = \frac{1}{2} \frac{-2 \rho^2 \sin^2 u}{\rho^4 \sin^2 u} = \frac{1}{2} \times \frac{-2}{\rho} = -\frac{1}{\rho}$$

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$$k_1 = H + \sqrt{H^2 - K} = -\frac{1}{\rho} + \sqrt{\frac{1}{\rho^2} - \frac{1}{\rho^2}} = -\frac{1}{\rho}$$

$$= -\frac{2}{\rho} + \frac{\sqrt{3}}{\rho} = \frac{\sqrt{3}-2}{\rho}$$

$$k_2 = H - \sqrt{H^2 - K} = -\frac{1}{\rho} - \sqrt{\frac{1}{\rho^2} - \frac{1}{\rho^2}} = -\frac{1}{\rho}$$

$$k_2 = \frac{\sqrt{3}}{\rho} - \frac{1}{\rho} \Rightarrow k_1 = k_2 = -\frac{1}{\rho}$$

Definition: A point $p \in S$ is called elliptic if $k(p) > 0$,

Hyperbolic if $k(p) < 0$,

parabolic if $k(p) = 0$ and $k_1(p) \neq 0$ or $k_2(p) \neq 0$

and planar if $k_1(p) = k_2(p) = 0$

Definition:

Let $f: D \rightarrow \mathbb{R}$, $D \subset \mathbb{R}^2$ denote a smooth function with critical points $P_0(u_0, v_0) \in D$

i.e. $f_u(P_0) = 0$, $f_v(P_0) = 0$. The function f has

a local maxima or local minima at P_0 if

$$f_{uu}f_{vv} - (f_{uv})^2 > 0 \text{ or } f_{uu}f_{vv} - (f_{uv})^2 < 0$$

Question: Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ denote the function

$$f(x, y) = \cos x + \cos y - 2 \text{ with parametrization}$$

$$r(u, v) = u\hat{i} + v\hat{j} + (\cos u + \cos v - 2)\hat{k}$$

Solution:

$$r(u, v) = u\hat{i} + v\hat{j} + (\cos u + \cos v - 2)\hat{k}$$

$$r_u = \hat{i} - \sin u \hat{k} \quad r_{uu} = -\cos u \hat{k}$$

$$r_v = \hat{j} - \sin v \hat{k} \quad r_{vv} = -\cos v \hat{k}$$

$$r_{uv} = 0$$

$$\vec{n} = r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -\sin u \\ 0 & 1 & -\sin v \end{vmatrix}$$

$$= \hat{i}(\sin u) - \hat{j}(-\sin v) + \hat{k}(1)$$

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$$\vec{n} = \frac{\sin u \hat{i} + \sin v \hat{j} + \hat{k}}{\sqrt{1 + \sin^2 u + \sin^2 v}}$$

$$E = \vec{r}_u \cdot \vec{r}_u = 1 + \sin^2 u, \quad G = 1 + \sin^2 v$$

$$F = \vec{r}_u \cdot \vec{r}_v = \sin u \sin v$$

$$L = \vec{r}_{uu} \cdot \vec{n} = \frac{-\cos u}{\sqrt{1 + \sin^2 u + \sin^2 v}}$$

$$M = \vec{r}_{uv} \cdot \vec{n} = 0$$

$$N = \vec{r}_{vv} \cdot \vec{n} = \frac{-\cos v}{\sqrt{1 + \sin^2 u + \sin^2 v}}$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\cos u \cos v - 0}{(1 + \sin^2 u)(1 + \sin^2 v) - \sin^2 u \sin^2 v} \times \frac{1}{(1 + \sin^2 u + \sin^2 v)}$$

$$K = \frac{\cos u \cos v}{1 + \sin^2 v + \sin^2 u + \sin^2 u \sin^2 v - \sin^2 u \sin^2 v} \times \frac{1}{(1 + \sin^2 u + \sin^2 v)}$$

$$K = \frac{\cos u \cos v}{(1 + \sin^2 u + \sin^2 v)^2} \quad \text{--- (1)}$$

For points

$$f(x, y) = \cos x + \cos y - 2$$

$$f_x = -\sin x = 0$$

$$\sin x = 0$$

$$x = 0, \pi, 2\pi, \dots$$

$$x = l\pi$$

$$f_y = -\sin y = 0$$

$$\sin y = 0$$

$$y = 0, \pi, 2\pi, \dots$$

$$y = k\pi$$

Let $p(0, 0)$ put in Eq. (1)

$$K(0, 0) = \frac{\cos(0)\cos(0)}{1 + \sin^2 0 + \sin^2 0} = 1 > 0$$

$K(0, 0) > 0$ The point is elliptic.

Case-I When l and k both are even

i.e. $p(l\pi, k\pi)$ in (1)

$$\Rightarrow K(l\pi, k\pi) = 1 > 0$$

point is elliptic when l and k are even.

Case-II When k and l both are odd

$$K(l\pi, k\pi) = 1 > 0$$

point is elliptic

Case-III When k is even and l is odd

$$K(l\pi, k\pi) = -1 < 0$$

point is hyperbolic

Case-IV When k is odd and l is even

$$K(l\pi, k\pi) = -1$$

point is hyperbolic

Now we check $f(x, y)$ is maxima or minima

$$f(x, y) = \cos x + \cos y - 2$$

$$f_x = -\sin x$$

$$f_y = -\sin y$$

$$f_{xx} = -\cos x$$

$$f_{yy} = -\cos y$$

$$f_{xx}(0, 0) = -1$$

$$f_{yy}(0, 0) = -1$$

$$f_{xy} = 0$$

$$f_{xx}f_{yy} - (f_{xy})^2 = (-1)(-1) - (0)^2 = 1 > 0$$

$\Rightarrow f(x, y)$ is local maxima.

10/05/019

Minimal surface:

1) If $H=0$ surface is minimal

2) Equation of minimal surface

$$r(x, y) = (x, y, z(x, y)) \quad \text{when}$$

$$(1 + z_y^2)z_{xx} - 2z_xz_yz_{xy} + (1 + z_x^2)z_{yy} = 0$$

Example: The Scherk's doubly periodic surface discovered in 1830 is given as

$$f(x, y) = (x, y, \ln \frac{\cos x}{\cos y}) = x\hat{i} + y\hat{j} + \ln \frac{\cos x}{\cos y} \hat{k}$$

Solution:

$$f(x, y) = \ln \frac{\cos x}{\cos y}$$

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$$f_x = \frac{1}{\frac{\cos x}{\cos y}} \frac{d}{dx} \left(\frac{\cos x}{\cos y} \right) = \frac{\cos y}{\cos x} \cdot \frac{-\sin x}{\cos y} = \frac{-\sin x}{\cos x} = -\tan x$$

$$f_{xx} = -\sec^2 x$$

$$f_y = \frac{1}{\frac{\cos x}{\cos y}} \frac{d}{dy} \left(\frac{\cos x}{\cos y} \right) = \frac{\cos y}{\cos x} \cdot \cos x \cdot \sec^2 y \tan y$$

$$f_y = \tan y, \quad f_{xy} = 0$$

$$f_{yy} = \sec^2 y$$

$$\begin{aligned} (1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy} &= (1 + \tan^2 y)(-\sec^2 x) - 2(0) \\ &\quad + (1 + \tan^2 x)(\sec^2 y) \\ &= -\sec^2 y \sec^2 x + \sec^2 x \sec^2 y \\ &= 0 \end{aligned}$$

Other Method

$$E(x, y) = x \hat{i} + y \hat{j} + \ln \frac{\cos x}{\cos y} \hat{k}$$

$$E_x = \hat{i} - \tan x \hat{k}, \quad E_y = \hat{j} + \tan y \hat{k}$$

$$E_{xx} = -\sec^2 x \hat{k}, \quad E_{yy} = \sec^2 y \hat{k}$$

$$E_{xy} = 0$$

$$\hat{n} = E_x \times E_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -\tan x \\ 0 & 1 & \tan y \end{vmatrix} = \hat{i}(-\tan x) - \hat{j}(\tan y) + \hat{k}(1)$$

$$\vec{n} = -\tan x \hat{i} - \tan y \hat{j} + \hat{k}$$

$$\hat{n} = \frac{-\tan x \hat{i} - \tan y \hat{j} + \hat{k}}{\sqrt{\tan^2 x + \tan^2 y + 1}}$$

$$E = E_x \cdot E_x = 1 + \tan^2 x = \sec^2 x$$

$$G = E_y \cdot E_y = 1 + \tan^2 y = \sec^2 y$$

$$F = E_x \cdot E_y = \tan x \tan y$$

$$L = E_{xx} \cdot \hat{n} = \frac{-\sec^2 x \hat{k}}{\sqrt{\tan^2 x + \tan^2 y + 1}}, \quad M = E_{xy} \cdot \hat{n} = 0$$

$$N = E_{yy} \cdot \hat{n} = \frac{\sec^2 y \hat{k}}{\sqrt{\tan^2 x + \tan^2 y + 1}}$$

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$$H = \frac{1}{2} \frac{EN + GL - 2FM}{EG - F^2} = \frac{\sec^2 x \sec^2 y - \sec^2 x \sec^2 y}{\sec^2 x \sec^2 y - 0} = 0 \times \frac{1}{\sqrt{\tan^2 x + \sec^2 y}}$$

$$H = 0, \quad K = \frac{-\sec^2 x \sec^2 y}{\sqrt{\sec^2 x \sec^2 y - \tan^2 x \tan^2 y}} \times \frac{1}{\sqrt{\tan^2 x + \sec^2 y}} < 0$$

surface is minimal

Question: Helicoid $R(r, z) = r \cos z \hat{i} + r \sin z \hat{j} + cz \hat{k}$, Find Minimal??**Solution:** $R(r, z) = r \cos z \hat{i} + r \sin z \hat{j} + cz \hat{k}$

$$R_r = \cos z \hat{i} + \sin z \hat{j} + 0 \hat{k}$$

$$R_{rr} = 0$$

$$R_{rz} = -\sin z \hat{i} + \cos z \hat{j} + 0 \hat{k}$$

$$R_z = -r \sin z \hat{i} + r \cos z \hat{j} + c \hat{k}$$

$$R_{zz} = -r \cos z \hat{i} - r \sin z \hat{j} + 0 \hat{k}$$

$$\vec{n} = R_r \times R_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos z & \sin z & 0 \\ -r \sin z & r \cos z & c \end{vmatrix} = \hat{i}(c \sin z) - \hat{j}(c \cos z) + \hat{k}(r \cos^2 z + r \sin^2 z)$$

$$\vec{n} = c \sin z \hat{i} - c \cos z \hat{j} + r \hat{k}$$

$$\hat{n} = \frac{c \sin z \hat{i} - c \cos z \hat{j} + r \hat{k}}{\sqrt{c^2 + r^2}}$$

$$E = R_r \cdot R_r = \cos^2 z + \sin^2 z = 1$$

$$G = R_z \cdot R_z = r^2 \sin^2 z + r^2 \cos^2 z + c^2 = r^2 + c^2$$

$$F = R_r \cdot R_z = -r \sin z \cos z + r \sin z \cos z + 0 = 0$$

$$L = R_{rr} \cdot \hat{n} = 0$$

$$M = R_{rz} \cdot \hat{n} = \frac{-c \sin^2 z - c \cos^2 z}{\sqrt{c^2 + r^2}} = \frac{-c}{\sqrt{c^2 + r^2}}$$

$$N = R_{zz} \cdot \hat{n} = \frac{-r^2 \cos z \sin z + r \sin z \cos z + 0}{\sqrt{c^2 + r^2}} = 0$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{0(N) - \frac{c^2}{r^2 + c^2}}{r^2 + c^2 - 0} \times \frac{1}{r^2 + c^2}$$

$$K = -\frac{c^2}{r^2 + c^2} < 0$$

points is hyperbolic

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$$H = \frac{1}{2} \frac{EN + GZ - 2FM}{EG - F^2} = \frac{1 \cdot 0 + (r^2 + c^2) \cdot 0 - 2(0)(-c)}{r^2 + c^2 - 0} \times \frac{1}{\sqrt{c^2 + r^2}}$$

$H = 0$ surface is Minimal.

Enneper's minimal surface:

Alfred Enneper 1863.

$$R(r, \varphi) = (r \cos \varphi - \frac{1}{3} r^3 \cos 3\varphi) \hat{i} + (r \sin \varphi - \frac{1}{3} r^3 \sin 3\varphi) \hat{j} + r^2 \cos 2\varphi \hat{k}.$$

Solution:

$$R_r = (\cos \varphi - r^2 \cos 3\varphi) \hat{i} + (\sin \varphi - r^2 \sin 3\varphi) \hat{j} + 2r \cos 2\varphi \hat{k}$$

$$R_{rr} = -2r \cos 3\varphi \hat{i} - 2r \sin 3\varphi \hat{j} - 2 \cos 2\varphi \hat{k}$$

$$R_{r\varphi} = (-\sin \varphi + \frac{2}{3} r^3 \sin 3\varphi) \hat{i} + (\cos \varphi - \frac{2}{3} r^3 \cos 3\varphi) \hat{j} - 4r \sin 2\varphi \hat{k}$$

$$R_{\varphi} = (-r \sin \varphi + r^3 \sin 3\varphi) \hat{i} + (r \cos \varphi - r^3 \cos 3\varphi) \hat{j} + 2r^2 \sin 2\varphi \hat{k}$$

$$R_{\varphi\varphi} = (-r \cos \varphi + 3r^3 \cos 3\varphi) \hat{i} + (-r \sin \varphi - 3r^3 \sin 3\varphi) \hat{j} - 4r^2 \cos 2\varphi \hat{k}$$

$$\vec{n} = R_r \times R_{\varphi} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \varphi - r^2 \cos 3\varphi & \sin \varphi - r^2 \sin 3\varphi & 2r \cos 2\varphi \\ -r \sin \varphi + r^3 \sin 3\varphi & r \cos \varphi - r^3 \cos 3\varphi & -2r^2 \sin 2\varphi \end{vmatrix}$$

$$= \hat{i} (-2r^2 \sin 2\varphi \sin \varphi + 2r^4 \sin 2\varphi \sin 3\varphi - (2r^2 \cos 2\varphi \cos \varphi - 2r^4 \cos 2\varphi \cos 3\varphi)) \\ - \hat{j} (-2r^2 \sin 2\varphi \cos \varphi + 2r^4 \sin 2\varphi \cos 3\varphi - (2r^2 \cos 2\varphi \sin \varphi - 2r^4 \cos 2\varphi \sin 3\varphi)) \\ + \hat{k} (r \cos^2 \varphi - r^3 \cos \varphi \cos 3\varphi - r^3 \cos 3\varphi \cos \varphi + r \cos^3 \varphi - (-r \sin^2 \varphi + r^3 \sin \varphi \sin 3\varphi + r^3 \sin 3\varphi \sin \varphi - r \sin^3 \varphi))$$

$$= \hat{i} (-2r^2 \cos(2\varphi - \varphi) + 2r^4 \cos(2\varphi + \varphi)) - \hat{j} (2r^2 \cos(2\varphi - \varphi) - 2r^4 \sin(3\varphi - 2\varphi)) \\ + \hat{k} (r - r^3 \cos(3\varphi - \varphi) - r^3 \cos(3\varphi - \varphi) + r^6)$$

$$\vec{n} = (-2r^2 \cos \varphi + 2r^4 \cos \varphi) \hat{i} + (2r^2 \sin \varphi + 2r^4 \sin \varphi) \hat{j} + (r + r^6 - 2r^3 \cos 2\varphi) \hat{k}$$

$$|R_r \times R_{\varphi}| = \sqrt{4r^4 \cos^2 \varphi + 4r^8 \cos^2 \varphi - 8r^6 \cos^2 \varphi + 4r^4 \sin^2 \varphi + 4r^8 \sin^2 \varphi - 8r^6 \sin^2 \varphi + (r^2 + r^6 - 2r^3 \cos 2\varphi)^2}$$

$$= \sqrt{4r^4 + 4r^8 + 8r^6(\sin^2 \varphi - \cos^2 \varphi) + (r^2 + r^6 - 2r^3 \cos 2\varphi)^2}$$

Enneper's minimal surface:

Alfred Enneper 1863

$$R(r, \phi) = (r \cos \phi - \frac{1}{3} r^3 \cos 3\phi) \hat{i} + (r \sin \phi - \frac{1}{3} r^3 \sin 3\phi) \hat{j} + r^2 \cos 2\phi \hat{k}$$

Solution:

$$R_r = (\cos \phi - r^2 \cos 3\phi) \hat{i} + (-\sin \phi - r^2 \sin 3\phi) \hat{j} + 2r \cos 2\phi \hat{k}$$

$$R_\phi = (-\sin \phi + r^3 \sin 3\phi) \hat{i} + (-\cos \phi - r^3 \cos 3\phi) \hat{j} - 4r \sin 2\phi \hat{k}$$

$$R_{rr} = -2r \cos 3\phi \hat{i} - 2r \sin 3\phi \hat{j} + 2 \cos 2\phi \hat{k}$$

$$R_\phi\phi = (-r \sin \phi + r^3 \sin 3\phi) \hat{i} + (-r \cos \phi - r^3 \cos 3\phi) \hat{j} - 2r^2 \sin 2\phi \hat{k}$$

$$R_{\phi\phi} = (-r \cos \phi + 3r^3 \cos 3\phi) \hat{i} + (r \sin \phi + 3r^3 \sin 3\phi) \hat{j} - 4r^2 \cos 2\phi \hat{k}$$

$$R_r \times R_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \phi - r^2 \cos 3\phi & -\sin \phi - r^2 \sin 3\phi & 2r \cos 2\phi \\ -r \sin \phi + r^3 \sin 3\phi & -r \cos \phi - r^3 \cos 3\phi & -2r^2 \sin 2\phi \end{vmatrix}$$

$$\begin{aligned} &= \hat{i}(-2r^2 \sin 2\phi(-\sin \phi - r^2 \sin 3\phi) - 2r \cos 2\phi(-r \cos \phi - r^3 \cos 3\phi)) \\ &\quad - \hat{j}(-2r^2 \sin 2\phi(\cos \phi - r^2 \cos 3\phi) - 2r \cos 2\phi(-r \sin \phi + r^3 \sin 3\phi)) \\ &\quad + \hat{k}((-r \cos \phi - r^3 \cos 3\phi)(\cos \phi - r^2 \cos 3\phi) - (-\sin \phi - r^2 \sin 3\phi)(-r \sin \phi + r^3 \sin 3\phi)) \\ &= \hat{i}(2r^2 \sin 2\phi \sin \phi + 2r^4 \sin 2\phi \sin 3\phi + 2r^2 \cos 2\phi \cos \phi + 2r^4 \cos 2\phi \cos 3\phi) \\ &\quad - \hat{j}(-2r^2 \sin 2\phi \cos \phi + 2r^4 \sin 2\phi \cos 3\phi + 2r^2 \cos 2\phi \sin \phi - 2r^4 \cos 2\phi \sin 3\phi) \\ &\quad + \hat{k}(-r \cos^2 \phi + r^3 \cos \phi \cos 3\phi - r^3 \cos 3\phi \cos \phi + r^5 \cos^3 \phi - (r \sin^2 \phi - r^3 \sin \phi \sin 3\phi \\ &\quad \quad \quad + r^3 \sin \phi \sin 3\phi - r^5 \sin^3 \phi)) \\ &= \hat{i}(2r^2(\cos 2\phi \cos \phi + \sin 2\phi \sin \phi) + 2r^4(\cos 3\phi \cos 2\phi + \sin 3\phi \sin 2\phi)) \\ &\quad - \hat{j}(-2r^2(\sin 2\phi \cos \phi - \cos 2\phi \sin \phi) - 2r^4(-\sin 2\phi \cos 3\phi + \sin 3\phi \cos 2\phi)) \\ &\quad + \hat{k}(-r + r^5) \\ &= \hat{i}(2r^2 \cos \phi + 2r^4 \cos \phi) - \hat{j}(-2r^2 \sin \phi - 2r^4 \sin \phi) + \hat{k}(-r + r^5) \\ &= 2r^2 \cos \phi(1+r^2) \hat{i} + 2r^2 \sin \phi(1+r^2) \hat{j} - r(1-r^4) \hat{k} \end{aligned}$$

$$\begin{aligned} |R_r \times R_\phi| &= \sqrt{4r^4(1+r^2)^2(\cos^2 \phi + \sin^2 \phi) + (r(1-r^4)(1+r^2))^2} \\ &= \sqrt{4r^4(1+r^2)^2 + r^2(1-r^2)^2(1+r^2)^2} \end{aligned}$$

$$= (1+r^2) \sqrt{4r^4 + r^2(1+r^2-2r^2)}$$

$$= (1+r^2) \sqrt{4r^4 + r^2 + r^6 - 2r^4}$$

$$= (1+r^2) \sqrt{r^2 + r^6 + 2r^4} = (1+r^2) \cdot r \sqrt{1+r^4+2r^2}$$

$$= r(1+r^2)^2$$

$$\hat{n} = \frac{2r^2 \cos \phi (1+r^2) \hat{i} + 2r^2 \sin \phi (1+r^2) \hat{j} - r(1-r^2)(1+r^2) \hat{k}}{r(1+r^2)^2}$$

$$\hat{n} = \frac{2r \cos \phi \hat{i} + 2r \sin \phi \hat{j} - (1-r^2) \hat{k}}{1+r^2}$$

$$\begin{aligned} E = R_r \cdot R_r &= (\cos \phi - r^2 \cos 3\phi)^2 + (-\sin \phi - r^2 \sin 3\phi)^2 + (2r \cos 2\phi)^2 \\ &= \cos^2 \phi + r^4 \cos^2 3\phi - 2r^2 \cos 3\phi \cos \phi + \sin^2 \phi + r^4 \sin^2 3\phi + 2r \sin \phi \sin 3\phi + 4r^2 \cos^2 2\phi \\ &= 1 + r^4 - 2r^2 \cos 4\phi + 4r^2 \cos^2 2\phi \\ &= 1 + r^4 - 2r^2 \cos 4\phi + 4r^2 (1 + \cos 4\phi) \\ &= 1 + r^4 - 2r^2 \cos 4\phi + 2r^2 + 2r^2 \cos 4\phi \end{aligned}$$

$$E = (r^2 + 1)$$

$$F = R_r \cdot R_\phi = (\cos \phi - r^2 \cos 3\phi)(-r \sin \phi + r^3 \sin 3\phi) + (\sin \phi + r^2 \sin 3\phi)(r \cos \phi + r^3 \cos 3\phi) - 4r^3 \sin 2\phi \cos 2\phi$$

$$\begin{aligned} F &= -r \cos \phi \sin \phi + r^3 \sin 3\phi \cos \phi + r^3 \cos 3\phi \sin \phi - r^5 \cos 3\phi \sin 3\phi + r \sin \phi \cos \phi \\ &\quad + r^3 \cos 3\phi \sin \phi + r^3 \sin 3\phi \cos \phi + r^5 \sin 3\phi \cos 3\phi - 4r^3 \sin 2\phi \cos 2\phi \end{aligned}$$

$$= 2r^3 \sin 3\phi \cos \phi + 2r^3 \sin \phi \cos 3\phi - 4r^3 \sin 2\phi \cos 2\phi$$

$$= 2r^3 \sin 4\phi - 4r^3 \sin 2\phi \cos 2\phi$$

$$= 4r^3 \sin 2\phi \cos 2\phi - 4r^3 \sin 2\phi \cos 2\phi = 0$$

$$F = 0$$

$$\begin{aligned} G = R_\phi \cdot R_\phi &= (-r \sin \phi + r^3 \sin 3\phi)^2 + (-r \cos \phi + r^3 \cos 3\phi)^2 + 4r^4 \sin^2 \phi \\ &= r^2 \sin^2 \phi + r^6 \sin^2 3\phi - 2r^4 \sin 3\phi \sin \phi + r^2 \cos^2 \phi + r^6 \cos^2 3\phi + 2r^4 \cos 3\phi \cos \phi + 4r^4 \sin^2 \phi \end{aligned}$$

$$= r^2 + r^6 + 2r^4 \cos 4\phi + 4r^4 (1 - \cos 4\phi)$$

$$= r^2 + r^6 + 2r^4 \cos 4\phi + 2r^4 - 2r^4 \cos 4\phi$$

$$= r^2 (1 + r^4 + 2r^2)$$

$$G = r^2 (1 + r^2)^2$$

$$L = R_{rx} \cdot \hat{n} = \frac{-4r^2 \cos 3\phi \cos \phi - 4r^2 \sin 3\phi \sin \phi - 2 \cos 2\phi (1-r^2)}{1+r^2}$$

$$L = \frac{-4r^2 \cos 2\phi - 2 \cos 2\phi (1-r^2)}{1+r^2} = \frac{-2 \cos 2\phi (1-r^2+2r^2)}{1+r^2} = \frac{-2 \cos 2\phi (1+r^2)}{1+r^2}$$

$$L = -2 \cos 2\phi$$

$$\begin{aligned}
 M = R_{\phi} \cdot \hat{n} &= \frac{2r \cos \phi (-\sin \phi + 3r^2 \sin 3\phi) + 2r \sin \phi (-\cos \phi - 3r^2 \cos 3\phi) + 4r \sin 2\phi (1-r^2)}{1+r^2} \\
 &= \frac{-2r \cos \phi \sin \phi + 6r^3 \cos \phi \sin 3\phi - 2r \sin \phi \cos \phi - 6r^3 \sin \phi \cos 3\phi + 4r \sin 2\phi (1-r^2)}{1+r^2} \\
 &= \frac{-4r \sin \phi \cos \phi + 6r^3 \sin 2\phi + 4r \sin 2\phi (1-r^2)}{1+r^2} \\
 &= \frac{-2r \sin 2\phi + 6r^3 \sin 2\phi + 4r \sin 2\phi (1-r^2)}{1+r^2} \\
 &= \frac{(-2r + 4r^3 + 6r^3 + 4r) \sin 2\phi}{1+r^2} = \frac{(2r + 2r^3) \sin 2\phi}{1+r^2} \\
 &= \frac{2r(1+r^2) \sin 2\phi}{1+r^2} = 2r \sin 2\phi
 \end{aligned}$$

$$\begin{aligned}
 N = R_{\phi} \cdot \hat{n} &= \frac{2r \cos \phi (-r \cos \phi + 3r^3 \cos 3\phi) + 2r \sin \phi (r \sin \phi + 3r^3 \sin 3\phi) + 4r^2 \cos 2\phi (1-r^2)}{1+r^2} \\
 &= \frac{-2r^2 \cos^2 \phi + 6r^4 \cos 3\phi \cos \phi + 2r^2 \sin^2 \phi + 6r^4 \sin 3\phi \sin \phi + 4r^2 \cos 2\phi (1-r^2)}{1+r^2} \\
 &= \frac{-2r^2 \cos^2 \phi + 2r^2 \sin^2 \phi + 6r^4 \cos 2\phi + 4r^2 \cos 2\phi (1-r^2)}{1+r^2} \\
 N &= \frac{-2r^2 \left(\frac{1+\cos 2\phi}{2}\right) + 2r^2 \left(\frac{1-\cos 2\phi}{2}\right) + (6r^4 + 4r^2 - 4r^4) \cos 2\phi}{1+r^2} \\
 &= \frac{-r^2(1+\cos 2\phi) + r^2(1-\cos 2\phi) + (2r^4 + 4r^2) \cos 2\phi}{1+r^2} \\
 &= \frac{-r^2 + r^2 + (-2r^2 + 2r^4 + 4r^2) \cos 2\phi}{1+r^2} = \frac{(2r^4 + 2r^2) \cos 2\phi}{1+r^2} \\
 &= \frac{2r(1+r^2) \cos 2\phi}{1+r^2} = 2r \cos 2\phi
 \end{aligned}$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-2 \cos 2\phi \cdot 2r \cos 2\phi - (2r \sin 2\phi)^2}{r^2(1+r^2)^2} = 0$$

$$= \frac{-4r^2 (\cos^2 2\phi + \sin^2 2\phi)}{r^2(1+r^2)^2} = \frac{-4}{(1+r^2)^2} < 0$$

$$H = \frac{1}{2} \frac{EN + GL - 2FM}{EG - F^2} = \frac{1}{2} \frac{2r^2(r^2+1) \cos 2\phi - 2r^2(r^2+1) \cos 2\phi}{r^2(r^2+1)^2} = 0$$

$$H = 0$$

Q

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Question: $x(u,v) = a \cosh v \cos u \hat{i} + a \cosh v \sin u \hat{j} + av \hat{k}$
catenoid surface $a < k$, $0 \leq u < 2\pi$, $-\infty < v < \infty$

Solution:

$$x(u,v) = a \cosh v \cos u \hat{i} + a \cosh v \sin u \hat{j} + av \hat{k}$$

$$x_u = -a \cosh v \sin u \hat{i} + a \cosh v \cos u \hat{j} + 0 \hat{k}$$

$$x_{uu} = -a \cosh v \cos u \hat{i} - a \cosh v \sin u \hat{j} + 0 \hat{k}$$

$$x_{uv} = -a \sinh v \sin u \hat{i} + a \sinh v \cos u \hat{j} + 0 \hat{k}$$

$$x_v = a \sinh v \cos u \hat{i} + a \sinh v \sin u \hat{j} + a \hat{k}$$

$$x_{vv} = a \cosh v \cos u \hat{i} + a \cosh v \sin u \hat{j} + 0 \hat{k}$$

	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\vec{n} = x_u \times x_v =$	$-a \cosh v \sin u$	$a \cosh v \cos u$	0
	$a \sinh v \cos u$	$a \sinh v \sin u$	a

$$= \hat{i} (a \cosh v \cos u) - \hat{j} (-a \cosh v \sin u) + \hat{k} (-a \sinh v \cosh v \sin u - a \cosh v \sinh v \cos u)$$

$$= a \cosh v \cos u \hat{i} + a \cosh v \sin u \hat{j} - a \sinh v \cosh v \hat{k}$$

$$\vec{n} = \frac{a \cosh v \cos u \hat{i} + a \cosh v \sin u \hat{j} - a \sinh v \cosh v \hat{k}}{a^2 (\cosh^2 v (\cos^2 u + \sin^2 u) + \sinh^2 v \cosh^2 v)}$$

$$= \frac{a \cosh v \cos u \hat{i} + a \cosh v \sin u \hat{j} - a \sinh v \cosh v \hat{k}}{a^2 \cosh^2 v (1 + \sinh^2 v)}$$

$$= \frac{a \cosh v \cos u \hat{i} + a \cosh v \sin u \hat{j} - a \sinh v \cosh v \hat{k}}{a^2 \cosh^2 v}$$

$$= \frac{a \cosh v \cos u \hat{i} + a \cosh v \sin u \hat{j} - a \sinh v \cosh v \hat{k}}{a^2 \cosh^2 v}$$

$$= \frac{a \cosh v \cos u \hat{i} + a \cosh v \sin u \hat{j} - a \sinh v \cosh v \hat{k}}{a^2 \cosh^2 v}$$

$$= \frac{a \cosh v \cos u \hat{i} + a \cosh v \sin u \hat{j} - a \sinh v \cosh v \hat{k}}{a^2 \cosh^2 v}$$

$$= \frac{a \cosh v \cos u \hat{i} + a \cosh v \sin u \hat{j} - a \sinh v \cosh v \hat{k}}{a^2 \cosh^2 v}$$

$$= \frac{a \cosh v \cos u \hat{i} + a \cosh v \sin u \hat{j} - a \sinh v \cosh v \hat{k}}{a^2 \cosh^2 v}$$

$$= \frac{a \cosh v \cos u \hat{i} + a \cosh v \sin u \hat{j} - a \sinh v \cosh v \hat{k}}{a^2 \cosh^2 v}$$

$$= \frac{a \cosh v \cos u \hat{i} + a \cosh v \sin u \hat{j} - a \sinh v \cosh v \hat{k}}{a^2 \cosh^2 v}$$

$$E = x_u \cdot x_u = a^2 \cosh^2 v, \quad G = x_v \cdot x_v = a^2 \sinh^2 v + a^2 = a^2 \cosh^2 v$$

$$F = x_u \cdot x_v = 0$$

$$L = x_{uu} \cdot \vec{n} = \frac{-a \cosh^3 v}{\cosh^2 v} = -a \cosh v$$

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$$M = \chi_{uv} \cdot \hat{n} = 0$$

$$N = \chi_{vv} \cdot \hat{n} = a$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-a^2 - 0}{a^4 \cos^4 uv} = \frac{-1}{a^2 \cos^4 uv} \neq 0$$

Hyperbolic at $-\pi/2 < uv < \pi/2$.

$$H = \frac{1}{2} \frac{EN + GL - 2FM}{EG - F^2} = \frac{a^3 \cos^3 uv - a^3 \cos^3 uv - 0}{a^4 \cos^4 uv - 0}$$

$$H = 0$$

Theorem. Let α be a closed curve in space bounding the graphs given by the variation $\tilde{z}(x, y) = z(x, y) + \varepsilon \eta(x, y)$ and let c be the projection of α onto the xy -plane enclosing a domain D . If the area of the graph of $z(x, y)$ is the minimum area of all graphs, the function of area given by

$$I(\varepsilon) = \iint_D \sqrt{1 + \tilde{z}_x^2 + \tilde{z}_y^2} \, dx \, dy$$

must be a minimum when $\varepsilon = 0$ and z must satisfy the minimal surface equation.

$$(1 + z_y^2) z_{xx} - 2 z_x z_y z_{xy} + (1 + z_x^2) z_{yy} = 0.$$

Proof

Assume that $z(x, y)$ gives the graph with minimum area bounded of all graphs bounded by α . Then $I(\varepsilon)$ must be at minimum when $\varepsilon = 0$, then the graph of $z(x, y)$ must give minimal area of all graphs bounded by α . Now $z(x, y)$ must be shown to satisfy the minimal surface Equation. By Differentiate $I(\varepsilon)$ w.r.t. ε we get

$$\begin{aligned}\frac{\partial}{\partial \epsilon} I(\epsilon) &= \frac{\partial}{\partial \epsilon} \iint_D \sqrt{1 + z_x^2 + z_y^2} \, dx dy \\ &= \frac{\partial}{\partial \epsilon} \iint_D \sqrt{1 + (z_x + \epsilon f_x)^2 + (z_y + \epsilon f_y)^2} \, dx dy\end{aligned}$$

$$\frac{\partial}{\partial \epsilon} I(\epsilon) = \iint_D \frac{2(z_x + \epsilon f_x) \cdot f_x + 2(z_y + \epsilon f_y) \cdot f_y}{\sqrt{1 + (z_x + \epsilon f_x)^2 + (z_y + \epsilon f_y)^2}} \, dx dy$$

$$\frac{\partial}{\partial \epsilon} I(\epsilon) = \iint_D \frac{z_x f_x + z_y f_y}{\sqrt{1 + z_x^2 + z_y^2}} \, dx dy$$

$$= \iint_D \frac{z_x f_x}{\sqrt{1 + z_x^2 + z_y^2}} \, dx dy + \iint_D \frac{z_y f_y}{\sqrt{1 + z_x^2 + z_y^2}} \, dx dy \quad (2)$$

In order to evaluate the double integrals as iterated integrals, integration limits need to be found for the integrals w.r.t x & y . Let us define a function

$$g(x, y) = \begin{cases} \frac{z_x f_x}{\sqrt{1 + z_x^2 + z_y^2}} & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \in \mathbb{R}^2 - D \end{cases}$$

and letting

$$A_y = \{(x, y) \mid \forall x \in \mathbb{R}\} \cap D$$

For the first integral of Eq. (2), now evaluating the integral w.r.t x

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \, dx dy = \int_{-\infty}^{\infty} \left(\int_D \frac{z_x f_x}{\sqrt{1 + z_x^2 + z_y^2}} \, dx \right) dy$$

$$= \int_{-\infty}^{\infty} \left(\int_{A_y} \frac{z_x f_x}{\sqrt{1 + z_x^2 + z_y^2}} \, dx \right) dy$$

$$u = \frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}, \quad v = f_x$$

$$= \int_{-\infty}^{\infty} \left[\int_{\partial A_y} \frac{z_x}{\sqrt{1+z_x^2+z_y^2}} - \int_{A_y} \frac{\partial}{\partial x} \frac{z_x}{\sqrt{1+z_x^2+z_y^2}} dx \right] dy$$

where ∂A_y is the boundary of A_y

$$= \int_{-\infty}^{\infty} \left[0 - \int_{A_y} \frac{\partial}{\partial x} \frac{z_x}{\sqrt{1+z_x^2+z_y^2}} dx \right] dy$$

$$= - \int_{-\infty}^{\infty} \int_D \frac{\partial}{\partial x} \frac{z_x}{\sqrt{1+z_x^2+z_y^2}} dx dy$$

$$= - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \frac{z_x}{\sqrt{1+z_x^2+z_y^2}} dx dy$$

Similarly 2nd part of integral

$$= - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial}{\partial y} \frac{z_y}{\sqrt{1+z_x^2+z_y^2}} dx dy$$

$$I'(0) = - \iint_D \left(\frac{\partial}{\partial x} \frac{z_x}{\sqrt{1+z_x^2+z_y^2}} + \frac{\partial}{\partial y} \frac{z_y}{\sqrt{1+z_x^2+z_y^2}} \right) dx dy$$

Since $I'(0)$ must be at minimum when $\epsilon=0$

Since $f(x,y)$ be any differentiable function with compact support over D . Then

the sum of partial derivative must be zero, which gives

$$\frac{\partial}{\partial x} \left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}} \right) + \frac{\partial}{\partial y} \frac{z_y}{\sqrt{1+z_x^2+z_y^2}} = 0$$

$$\frac{\partial}{\partial x} (z_x (1+z_x^2+z_y^2)^{-1/2}) = z_{xx} (1+z_x^2+z_y^2)^{-1/2} - \frac{1}{2} z_x (1+z_x^2+z_y^2)^{-3/2} (2z_x z_{xx} + 2z_y z_{xy})$$

$$= (1+z_x^2+z_y^2)^{-3/2} [z_{xx} (1+z_x^2+z_y^2) - z_x^2 z_{xx} - z_x z_y z_{xy}]$$

$$= (1+z_x^2+z_y^2)^{-3/2} [z_{xx} + z_x^2 z_{xx} + z_y^2 z_{xx} - z_x^2 z_{xx} - z_x z_y z_{xy}]$$

$$= (1 + z_x^2 + z_y^2)^{-\frac{3}{2}} [z_{xx} + z_y^2 z_{xx} - z_x z_y z_{xy}]$$

$$= (1 + z_x^2 + z_y^2)^{-\frac{3}{2}} [(1 + z_y^2) z_{xx} - z_x z_y z_{xy}]$$

similarly

$$\frac{\partial}{\partial y} (z_y (1 + z_x^2 + z_y^2)^{-\frac{1}{2}}) = (1 + z_x^2 + z_y^2)^{-\frac{3}{2}} (z_{yy} (1 + z_x^2) - z_x z_y z_{xy})$$

so

$$(1 + z_x^2 + z_y^2)^{-\frac{3}{2}} [z_{xx} (1 + z_y^2) - 2 z_x z_y z_{xy} + z_{yy} (1 + z_x^2)] = 0$$

$$z_{xx} (1 + z_y^2) - 2 z_x z_y z_{xy} + z_{yy} (1 + z_x^2) = 0$$

which is Equation for minimal surface.

Hence proved.

Compact support function-

A function with compact support is a function that vanishes outside a compact set.



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