Question # 1

Find δy and dy in the following cases:

(i) $y=x^2-1$ when x changes from 3 to 3.02

(ii) $y=x^2+2x$ when x changes from 2 to 1.8

(iii) $y=\sqrt{x}$ when x changes from 4 to 4.41

Solution

(i) $y=x^2-1$ …… (i)

$x=3$ & $\delta x=3.02-3=0.02$

$$y+\delta y=(x+\delta x)^2-1$$

$$\Rightarrow \delta y=(x+\delta x)^2-1-x^2+1$$

$$=(x+\delta x)^2-x^2$$

Put $x=3$ & $\delta x=0.02$

$$\delta y=(3+0.02)^2-(3)^2$$

$$\Rightarrow \boxed{\delta y=0.1204}$$

Taking differential of (i)

$$dy=d\left(x^2-1\right)$$

$$\Rightarrow dy=2x\ dx$$

Put $x=3$ & $dx=\delta x=0.02$

$$dy=2(3)(0.02)$$

$$\Rightarrow \boxed{dy=0.12}$$

(ii) *Do yourself as above.*

(iii) $y=\sqrt{x} = x^{\frac{1}{2}}$ ……. (i)

$x=4$ & $\delta x=4.41-4=0.41$

$$y+\delta y=(x+\delta x)^{\frac{1}{2}}$$

$$\Rightarrow \delta y=(x+\delta x)^{\frac{1}{2}}-x^{\frac{1}{2}}$$

Put $x=4$ & $\delta x=0.41$

$$\delta y=(4+0.41)^{\frac{1}{2}}-(4)^{\frac{1}{2}}$$

$$=2.1-2$$

$$\Rightarrow \boxed{\delta y=0.1}$$

Taking differential of (i)

$$dy=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)dx$$

$$=\frac{1}{2}x^{-\frac{1}{2}}\ dx$$

$$=\frac{1}{2x^{\frac{1}{2}}}\ dx$$

Put $x=4$ & $dx=\delta x=0.41$

$$dy = \frac{1}{2(4)^{\frac{1}{2}}}(0.41)$$

$$=\frac{0.41}{4}$$

$$\Rightarrow \boxed{dy=0.1025}$$

Question # 2

Using differentials find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in the following equations.

(i) $xy+x = 4$ (ii) $x^2+2y^2 = 16$

(iii) $x^4+y^2 = xy^2$ (iv) $xy-\ln x = c$

Solution

(i) $xy+x = 4$

Taking differential on both sides

$$d(xy)+dx = d(4)$$

$$\Rightarrow xdy+ydx+dx = 0$$

$$\Rightarrow xdy+(y+1)dx = 0$$

$$\Rightarrow xdy = -(y+1)dx$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y+1}{x}$$

$$\& \quad \frac{dx}{dy}=-\frac{x}{y+1}$$

(ii) *Do yourself as above*

(iii) $x^4+y^2 = xy^2$

Taking differential

$$d\left(x^4\right)+d\left(y^2\right) = d\left(xy^2\right)$$

$$\Rightarrow 4x^3dx+2ydy = x\cdot 2ydy+y^2dx$$

$$\Rightarrow 2ydy-2xydy = y^2dx-4x^3dx$$

$$\Rightarrow 2y(1-x)dy = \left(y^2-4x^3\right)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2-4x^3}{2y(1-x)}$$

$$\& \quad \frac{dx}{dy} = \frac{2y(1-x)}{y^2-4x^3}$$

(iv) $xy-\ln x = c$

Taking differential

$$d(xy)-d(\ln x) = d(c)$$

$$\Rightarrow xdy+ydx-\frac{1}{x}dx = 0$$

$$\Rightarrow xdy = \frac{1}{x}dx-ydx$$

$$= \left(\frac{1}{x}-y\right)dx$$

$$\Rightarrow xdy = \left(\frac{1-xy}{x}\right)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-xy}{x^2}$$

$$\frac{dx}{dy} = \frac{x^2}{1-xy}$$

Question # 3

Using differentials to approximate the values of

(i) $\sqrt[4]{17}$ (ii) $(31)^{\frac{1}{5}}$

(iii) $\cos 29^\circ$ (iv) $\sin 61^\circ$

Solution

(i) Let $y=f(x)=\sqrt[4]{x}$

where $x=16$ and $\delta x=dx=1$

Taking differential of above

$$dy = d\left(\sqrt[4]{x}\right)$$

$$= d(x)^{\frac{1}{4}}$$

$$= \frac{1}{4}x^{\frac{1}{4}-1}dx$$

$$= \frac{1}{4}x^{-\frac{3}{4}}dx$$

$$= \frac{1}{4x^{\frac{3}{4}}}dx$$

Put $x=16$ and $dx=1$

$$dy = \frac{1}{4(16)^{\frac{3}{4}}}(1)$$

$$= \frac{1}{4\left(2^4\right)^{\frac{3}{4}}}$$

$$= \frac{1}{4(8)} = 0.03125$$

Now $f(x+dx) \approx y+dy$

$$= f(x)+dy$$

$$\because y=f(x)$$

$$\Rightarrow \sqrt[4]{16+1} \approx \sqrt[4]{16}+0.03125$$

$$\Rightarrow \sqrt[4]{17} \approx \left(2^4\right)^{\frac{1}{4}}+0.03125$$

$$= 2+0.03125$$

$$= 2.03125$$

(ii) Let $y = f(x) = x^{\frac{1}{5}}$

Where $x=32$ & $\delta x=dx=-1$

Try yourself as above.

(iii) Let $y=f(x)=\cos x$

Where $x=30^\circ$ & $\delta x=-1^\circ=-\dfrac{\pi}{180}$ rad

$=-0.01745$ rad

Now $dy = d(\cos x)$

$$= -\sin x\, dx$$

Put $x=30^\circ$ and $dx=\delta x=-0.01745$

$$dy = -\sin 30^\circ(-0.01745)$$

$$= -(0.5)(-0.01745) = 0.008725$$

Now $f(x+\delta x) \approx y+dy$

$$= f(x)+dy$$

$$\Rightarrow \cos(30-1) = \cos 30^\circ+0.008725$$

$$\Rightarrow \cos 29^\circ = 0.866+0.008725$$

$$= 0.8747$$

(iv) Let $y=f(x)=\sin x$

Where $x=60^\circ$ & $\delta x=1^\circ=\dfrac{\pi}{180}$ rad

$=0.01745$ rad

Now $dy = d(\sin x)$

$$= \cos x\, dx$$

Put $x=60^\circ$ and $dx=\delta x=0.01745$

$$dy = \cos 60^\circ(0.01745)$$

$$= (0.5)(0.01745) = 0.008725$$

Now $f(x+\delta x) \approx y+dy$

$$= f(x)+dy$$

$$\Rightarrow \sin(60+1) = \sin 60^\circ+0.008725$$

$$\Rightarrow \sin 61^\circ = 0.866+0.008725$$

$$= 0.8747$$

Question # 4

Find the approximate increase in the volume of a cube if the length of its each edge changes from 5 to 5.02...

Solution

Let x be the length of side of cube where

$$x=5 \ \& \ \delta x=5.02-5=0.02$$

Assume V denotes the volume of the cube.

Then $V = x \cdot x \cdot x$

$= x^3$

Taking differential

$$dV = 3x^2 dx$$

Put $x = 5$ & $dx = \delta x = 0.02$

$$dV = 3(5)^2 (0.02)$$

$= 1.5$

Hence increase in volume is 1.5 cubic unit.

Theorem on Anti-Derivatives

i) $\int cf(x)dx = c\int f(x)dx$ where c is constant.

ii) $\int [f(x)\pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Important Integral

Since $\frac{d}{dx}x^{n+1} = (n+1)x^n$

Taking integral w.r.t x

$$\int \frac{d}{dx}x^{n+1}dx = \int (n+1)x^n dx$$

$$\Rightarrow x^{n+1} = (n+1)\int x^n dx$$

$$\Rightarrow \boxed{\int x^n\ dx = \frac{x^{n+1}}{n+1}} \quad \text{where } n\neq -1$$

If $n=-1$ then

$$\int x^{-1}dx = \int \frac{1}{x}dx \quad (\text{here } x\neq 0)$$

Since $\frac{d}{dx}\ln x = \frac{1}{x}$

Therefore $\boxed{\int \frac{1}{x}dx = \ln|x|+c}$

Note: Since log of zero and negative numbers does not exist therefore in above formula mod assure that we are taking a log of +ive quantity.

Question # 1

Evaluate the following indefinite integrals.

(i) $\int (3x^2-2x+1)dx$

(ii) $\int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx, (x>0)$

(iii) $\int x\left(\sqrt{x}+1\right)dx, (x>0)$

(iv) $\int (2x+3)^{\frac{1}{2}}dx$

(v) $\int \left(\sqrt{x}+1\right)^2 dx, (x>0)$

(vi) $\int \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx, (x>0)$

(vii) $\int \frac{3x+2}{\sqrt{x}}dx, (x>0)$

(viii) $\int \frac{\sqrt{y}(y+1)}{y}dy, (y>0)$

(ix) $\int \frac{\left(\sqrt{\theta}-1\right)^2}{\sqrt{\theta}}d\theta, (\theta>0)$

(x) $\int \frac{\left(1-\sqrt{x}\right)^2}{\sqrt{x}}dx, (x>0)$

(xi) $\int \frac{e^{2x}+e^x}{e^x}dx$

Solution

(i) $\int (3x^2-2x+1)dx = 3\int x^2dx - 2\int xdx + \int dx$

$$= 3\cdot\frac{x^{2+1}}{2+1} - 2\cdot\frac{x^{1+1}}{1+1} + x + c$$

$$= 3\cdot\frac{x^3}{3} - 2\cdot\frac{x^2}{2} + x + c$$

$$= x^3 - x^2 + x + c$$

(ii) $\int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx = \int \left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)dx$

$$= \int x^{\frac{1}{2}}dx + \int x^{-\frac{1}{2}}dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$$

(iii) $\int x\left(\sqrt{x}+1\right)dx = \int x\left(x^{\frac{1}{2}}+1\right)dx$

$$= \int \left(x^{\frac{3}{2}}+x\right)dx$$

$$= \int x^{\frac{3}{2}}dx + \int x\,dx$$

$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{x^{1+1}}{1+1} + c$$

$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^2}{2} + c$$

Important Integral

Since $\frac{d}{dx}(ax+b)^{n+1} = (n+1)(ax+b)^n\cdot a$

Taking integral

$$\int \frac{d}{dx}(ax+b)^{n+1}\,dx = \int (n+1)(ax+b)^n \cdot a\;dx$$

$$\Rightarrow\ (ax+b)^{n+1} = (n+1)\cdot a\int (ax+b)^n\;dx$$

$$\Rightarrow\ \boxed{\int (ax+b)^n\;dx = \frac{(ax+b)^{n+1}}{(n+1)\cdot a}}$$

(iv) $$\int (2x+3)^{\frac{1}{2}}\,dx = \frac{(2x+3)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\cdot 2}+c$$

$$=\frac{(2x+3)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)\cdot 2}+c$$

$$=\frac{1}{3}(2x+3)^{\frac{3}{2}}+c$$

(v) $$\int\left(\sqrt{x}+1\right)^2 dx = \int\left((\sqrt{x})^2+2\sqrt{x}+1\right)dx$$

$$=\int\left(x+2(x)^{\frac{1}{2}}+1\right)dx$$

$$=\int x\,dx+2\int (x)^{\frac{1}{2}}\,dx+\int dx$$

$$=\frac{x^{1+1}}{1+1}+2\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+x+c$$

$$=\frac{x^2}{2}+2\frac{x^{\frac{3}{2}}}{3/2}+x+c$$

$$=\frac{x^2}{2}+\frac{4x^{\frac{3}{2}}}{3}+x+c$$

(vi) $$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx = \int\left(x+\frac{1}{x}-2\right)dx$$

$$=\int x dx+\int\frac{1}{x}dx-2\int dx$$

$$=\frac{x^2}{2}+\ln|x|-2x+c$$

(vii) $$\int\frac{3x+2}{\sqrt{x}}dx = \int\frac{3x+2}{x^{1/2}}dx$$

$$=\int\frac{3x}{x^{1/2}}+\frac{2}{x^{1/2}}dx$$

$$=\int\left(3x^{1/2}+2x^{-1/2}\right)dx$$

$$=3\int x^{1/2}dx+2\int x^{-1/2}dx$$

Now do yourself.

(viii) $$\int\frac{\sqrt{y}\,(y+1)}{y}dy$$

$$=\int\frac{\sqrt{y}\,(y+1)}{\left(\sqrt{y}\right)^2}dy = \int\frac{(y+1)}{\sqrt{y}}dy$$

$$=\int\left(\frac{y}{\sqrt{y}}+\frac{1}{\sqrt{y}}\right)dy = \int\left(y^{\frac{1}{2}}+y^{-\frac{1}{2}}\right)dy$$

$$=\int y^{\frac{1}{2}}dy+\int y^{-\frac{1}{2}}dy$$

$$=\frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1}+\frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c \quad =\frac{y^{\frac{3}{2}}}{\frac{3}{2}}+\frac{y^{\frac{1}{2}}}{\frac{1}{2}}+c$$

$$=\frac{2}{3}y^{\frac{3}{2}}+2y^{\frac{1}{2}}+c$$

(ix) $$\int\frac{\left(\sqrt{\theta}-1\right)^2}{\sqrt{\theta}}\,d\theta = \int\frac{\theta-2\sqrt{\theta}+1}{\sqrt{\theta}}\,d\theta$$

$$=\int\left(\frac{\theta}{\sqrt{\theta}}-\frac{2\sqrt{\theta}}{\sqrt{\theta}}+\frac{1}{\sqrt{\theta}}\right)d\theta$$

$$=\int\left(\theta^{\frac{1}{2}}-2+\theta^{-\frac{1}{2}}\right)d\theta$$

$$=\frac{\theta^{\frac{1}{2}+1}}{\frac{1}{2}+1}-2\theta+\frac{\theta^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$$

$$=\frac{\theta^{\frac{3}{2}}}{3/2}-2\theta+\frac{\theta^{\frac{1}{2}}}{1/2}+c$$

$$=\frac{2}{3}\theta^{\frac{3}{2}}-2\theta+2\theta^{\frac{1}{2}}+c \quad Ans$$

(x) $$\int\frac{\left(1-\sqrt{x}\right)^2}{\sqrt{x}}\,dx = \int\frac{1-2\sqrt{x}+x}{\sqrt{x}}\,dx$$

$$=\int\left(\frac{1}{\sqrt{x}}-\frac{2\sqrt{x}}{\sqrt{x}}+\frac{x}{\sqrt{x}}\right)dx$$

$$=\int\left(x^{-\frac{1}{2}}-2+x^{\frac{1}{2}}\right)dx$$

$$=\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}-2x+\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c$$

$$=\frac{x^{\frac{1}{2}}}{1/2}-3\theta+\frac{x^{\frac{3}{2}}}{3/2}+c$$

$$= 2x^{\frac{1}{2}} - 2x + \frac{2}{3}x^{\frac{3}{2}} + c \quad Ans$$

Important Integral

We know $\frac{d}{dx}e^{ax} = a\cdot e^{ax}$

Taking integral

$$\int \frac{d}{dx}e^{ax}\,dx = \int a\cdot e^{ax}dx$$

$$\Rightarrow \quad e^{ax} = a\int e^{ax}dx$$

$$\Rightarrow \quad \boxed{\int e^{ax}dx = \frac{e^{ax}}{a}}$$

Also note that $\int e^{(ax+b)}dx = \frac{e^{(ax+b)}}{a}$

(xi) $\int \frac{e^{2x}+e^{x}}{e^{x}}\,dx = \int\left(\frac{e^{2x}}{e^{x}} + \frac{e^{x}}{e^{x}}\right)dx$

$$= \int\left(e^{x}+1\right)\,dx$$

$$= \int e^{x}dx + \int dx$$

$$= e^{x} + x + c \qquad \underline{Ans}$$

Question # 2

Evaluate

(i) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}} \begin{pmatrix} x+a>0 \\ x+b>0 \end{pmatrix}$

(ii) $\int \frac{1-x^2}{1+x^2}dx$

(iii) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}, (x>0, a>0)$

(iv) $\int (a-2x)^{\frac{3}{2}}\,dx$

(v) $\int \frac{\left(1+e^{x}\right)^{3}}{e^{x}}dx$

(vi) $\int \sin(a+b)x\,dx$

(vii) $\int \sqrt{1-\cos 2x}\,dx, \left(1-\cos 2x>0\right)$

(viii) $\int \ln x \times \frac{1}{x}\,dx, \left(x>0\right)$

(ix) $\int \sin^2 x\ dx$

(x) $\int \frac{1}{1+\cos x}dx, \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$

(xi) $\int \frac{ax+b}{ax^2+2bx+c}dx$

(xii) $\int \cos 3x \sin 2x\ dx$

(xiii) $\int \frac{\cos 2x - 1}{1+\cos 2x}\,dx, \left(1+\cos 2x \neq 0\right)$

(xiv) $\int \tan^2 x\ dx$

Solution

(i) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

$$= \int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}} \cdot \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{x+a-x-b}\ dx$$

$$= \int \frac{(x+a)^{\frac{1}{2}}-(x+b)^{\frac{1}{2}}}{a-b}\ dx$$

$$= \frac{1}{a-b}\left[\int (x+a)^{\frac{1}{2}}\,dx - \int (x+b)^{\frac{1}{2}}dx\right]$$

$$= \frac{1}{a-b}\left[\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{(x+b)^{\frac{1}{2}}}{\frac{1}{2}+1}\right] + c$$

$$= \frac{1}{a-b}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right] + c$$

$$= \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right] + c \quad Ans.$$

Important Integral

Since $\frac{d}{dx}Tan^{-1}x = \frac{1}{1+x^2}$

Also $\frac{d}{dx}\left(-Cot^{-1}x\right) = \frac{1}{1+x^2}$

Therefore $\int \frac{1}{1+x^2}dx = Tan^{-1}x$ or $-Cot^{-1}x$

Similarly $\int \frac{1}{\sqrt{1-x^2}}dx = Sin^{-1}x$ or $-Cos^{-1}x$

$\int \frac{1}{x\sqrt{x^2-1}}dx = Sec^{-1}x$ or $-Csc^{-1}x$

(ii) $\int \frac{1-x^2}{1+x^2}dx$

$$\begin{array}{r} -1 \\ 1+x^2\,\overline{)\,1-x^2} \\ \underset{+\quad +}{-1-x^2} \\ \hline 2 \end{array}$$

$$= \int\left(-1 + \frac{2}{1+x^2}\right)dx$$

$$= -\int dx + 2\int \frac{1}{1+x^2}dx$$

$$= -x + 2Tan^{-1}x + c$$

(iii) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$

$$= \int \frac{dx}{\sqrt{x+a}+\sqrt{x}} \cdot \frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}$$

$$= \int \frac{\sqrt{x+a}-\sqrt{x}}{x+a-x}\,dx$$

$$= \int \frac{(x+a)^{\frac{1}{2}}-(x)^{\frac{1}{2}}}{a}\,dx$$

$$= \frac{1}{a}\left[\int (x+a)^{\frac{1}{2}}\,dx - \int (x)^{\frac{1}{2}}dx\right]$$

$$= \frac{1}{a}\left[\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{(x)^{\frac{1}{2}}}{\frac{1}{2}+1}\right]+c$$

$$= \frac{1}{a}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right]+c$$

$$= \frac{2}{3a}\left[(x+a)^{\frac{3}{2}} - x^{\frac{3}{2}}\right]+c \quad Ans.$$

(iv) $\int (a-2x)^{\frac{3}{2}}\,dx$

$$= \frac{(a-2x)^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right)\cdot(-2)}+c$$

$$= \frac{(a-2x)^{\frac{5}{2}}}{\left(\frac{5}{2}\right)\cdot(-2)}+c$$

$$= -\frac{(a-2x)^{\frac{5}{2}}}{5}+c$$

(v) $\int \frac{(1+e^{x})^{3}}{e^{x}}dx = \int \frac{(1+3e^{x}+3e^{2x}+e^{3x})}{e^{x}}dx$

$$= \int \left(\frac{1}{e^{x}}+\frac{3e^{x}}{e^{x}}+\frac{3e^{2x}}{e^{x}}+\frac{e^{3x}}{e^{x}}\right)dx$$

$$= \int \left(e^{-x}+3+3e^{x}+e^{2x}\right)dx$$

$$= \frac{e^{-x}}{-1}+3x+3e^{x}+\frac{e^{2x}}{2}+c$$

$$= -e^{-x}+3x+3e^{x}+\frac{1}{2}e^{2x}+c$$

Important Integrals

We know $\frac{d}{dx}\cos ax = -a\sin ax$

Taking integral

$$\int \frac{d}{dx}\cos ax\,dx = -\int a\sin ax\,dx$$

$$\Rightarrow \cos ax = -a\int \sin ax\ dx$$

$$\Rightarrow \boxed{\int \sin ax\,dx = -\frac{\cos ax}{a}}$$

Also $\frac{d}{dx}\sin ax = a\cdot\cos ax$

$$\therefore \boxed{\int \cos ax\,dx = \frac{\sin ax}{a}}$$

Similarly

$$\boxed{\int \sec^{2} ax\ dx = \frac{\tan ax}{a}}$$

$$\boxed{\int \operatorname{cosec}^{2} ax\ dx = -\frac{\cot ax}{a}}$$

$$\boxed{\int \sec ax \tan ax\ dx = \frac{\sec ax}{a}}$$

$$\boxed{\int \csc ax \cot ax\ dx = -\frac{\csc ax}{a}}$$

Also note that

$$\int \sin(ax+b)\,dx = -\frac{\cos(ax+b)}{a}$$

$$\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} \quad \text{and so on.}$$

(vi) $\int \sin(a+b)x\,dx = -\frac{\cos(a+b)x}{a+b}+c$

Do yourself

(vii) $\int \sqrt{1-\cos 2x}\,dx$

$$= \int \sqrt{2\sin^{2} x}\,dx \quad \because\ \sin^{2} x = \frac{1-\cos 2x}{2}$$

$$= \sqrt{2}\int \sin x\,dx \quad = \sqrt{2}(-\cos x)+c$$

$$= -\sqrt{2}\cos x+c$$

Important Formula

$$\because\ \frac{d}{dx}[f(x)]^{n+1} = (n+1)[f(x)]^{n}\frac{d}{dx}f(x)$$

$$\Rightarrow \frac{d}{dx}[f(x)]^{n+1} = (n+1)[f(x)]^{n}\,f'(x)$$

Taking integral

$$\int \frac{d}{dx}[f(x)]^{n+1}\,dx = \int (n+1)[f(x)]^{n}\,f'(x)dx$$

$$\Rightarrow [f(x)]^{n+1} = (n+1)\int [f(x)]^{n}\,f'(x)dx$$

$$\Rightarrow \boxed{\int [f(x)]^{n}\,f'(x)dx = \frac{[f(x)]^{n+1}}{(n+1)}} \quad ;\ n \neq -1$$

Also $\frac{d}{dx}\ln|f(x)| = \frac{1}{f(x)}\cdot f'(x)$

Taking integral

$$\ln|f(x)| = \int \frac{f'(x)}{f(x)}dx$$

i.e. $\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+c}$

(viii) Let $I=\int \ln x \times \frac{1}{x}\,dx$

Put $f(x)=\ln x \Rightarrow f'(x)=\frac{1}{x}$

So $I=\int [f(x)]\,f'(x)\,dx$

$$=\frac{[f(x)]^{1+1}}{1+1}+c=\frac{[f(x)]^{2}}{2}+c$$

$$=\frac{(\ln x)^{2}}{2}+c$$

(ix) $\int \sin^2 x\,dx=\int\left(\frac{1-\cos 2x}{2}\right)dx$

$$=\int\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)dx$$

$$=\frac{1}{2}\int dx-\frac{1}{2}\int\cos 2x\,dx$$

$$=\frac{1}{2}x-\frac{1}{2}\frac{\sin 2x}{2}+c$$

$$=\frac{1}{2}x-\frac{1}{4}\sin 2x+c$$

(x) $\int\frac{1}{1+\cos x}dx$

$=\int\frac{1}{2\cos^2\frac{x}{2}}dx \qquad \because\ \cos^2\frac{x}{2}=\frac{1+\cos x}{2}$

$$=\frac{1}{2}\int\sec^2\frac{x}{2}\,dx=\frac{1}{2}\frac{\tan\frac{x}{2}}{1/2}+c=\tan\frac{x}{2}+c$$

Alternative

$$\int\frac{1}{1+\cos x}dx=\int\frac{1}{1+\cos x}\times\frac{1-\cos x}{1-\cos x}dx$$

$$=\int\frac{1-\cos x}{1-\cos^2 x}dx$$

$$=\int\frac{1-\cos x}{\sin^2 x}dx$$

$$=\int\left(\frac{1}{\sin^2 x}-\frac{\cos x}{\sin^2 x}\right)dx$$

$$=\int\left(\operatorname{cosec}^2 x-\frac{\cos x}{\sin x\cdot\sin x}\right)dx$$

$$=\int\operatorname{cosec}^2 x\,dx-\int\operatorname{cosec}x\cot x\,dx$$

$$=-\cot x-(-\operatorname{cosec}x)+c$$

$$=\operatorname{cosec}x-\cot x+c$$

(xi) Let $I=\int\frac{ax+b}{ax^2+2bx+c}dx$

Put $f(x)=ax^2+2bx+c$

$\Rightarrow f'(x)=2ax+2b$

$\Rightarrow f'(x)=2(ax+b) \Rightarrow \frac{1}{2}f'(x)=ax+b$

So $I=\int\frac{\frac{1}{2}f'(x)}{f(x)}dx$

$$=\frac{1}{2}\int\frac{f'(x)}{f(x)}dx=\frac{1}{2}\ln|f(x)|+c_1$$

$$=\frac{1}{2}\ln\left|ax^2+2bx+c\right|+c_1$$

Review

- $2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$
- $2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta)$
- $2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$
- $-2\sin\alpha\sin\beta=\cos(\alpha+\beta)-\cos(\alpha-\beta)$

(xii) $\int\cos 3x\sin 2x\,dx$

$$=\frac{1}{2}\int 2\cos 3x\sin 2x\,dx$$

$$=\frac{1}{2}\int[\sin(3x+2x)-\sin(3x-2x)]\,dx$$

$$=\frac{1}{2}\int[\sin 5x-\sin x]\,dx$$

$$=\frac{1}{2}\left[-\frac{\cos 5x}{5}-(-\cos x)\right]+c$$

$$=-\frac{1}{2}\left[\frac{\cos 5x}{5}-\cos x\right]+c$$

(xiii) $\int\frac{\cos 2x-1}{1+\cos 2x}dx$

$=-\int\frac{1-\cos 2x}{1+\cos 2x}dx \qquad \because\ \sin^2 x=\frac{1-\cos 2x}{2}$

$=-\int\frac{2\sin^2 x}{2\cos^2 x}dx \qquad \cos^2 x=\frac{1+\cos 2x}{2}$

$$=-\int\tan^2 x\,dx=-\int(\sec^2 x-1)\,dx$$

$$=-\int\sec^2 x\,dx+\int dx$$

$$=-\tan x+x+c$$

(xiv) $\int\tan^2 x\,dx=\int(\sec^2 x-1)\,dx$

$$=\int\sec^2 x\,dx-\int dx$$

$$=\tan x-x+c$$

Important Integral

Since $\frac{d}{dx}\ln|ax+b|=\frac{1}{ax+b}\cdot\frac{d}{dx}(ax+b)$

$$\Rightarrow\ \frac{d}{dx}\ln|ax+b|=\frac{1}{ax+b}\cdot a$$

On Integrating

$$\Rightarrow\ \ln|ax+b|=a\int\frac{1}{ax+b}\,dx$$

$$\Rightarrow\ \boxed{\int\frac{1}{ax+b}\,dx=\frac{\ln|ax+b|}{a}}$$

Evaluate the following integrals:

Question # 1

$$\int \frac{-2x}{\sqrt{4-x^2}}dx$$

Solution

Let $I=\int \frac{-2x}{\sqrt{4-x^2}}dx$

Put $t=4-x^2 \quad \Rightarrow \quad dt=-2x\ dx$

So $I=\int \frac{dt}{\sqrt{t}} \quad =\int (t)^{-\frac{1}{2}}dt$

$$=\frac{(t)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c \quad =\frac{(t)^{\frac{1}{2}}}{\frac{1}{2}}+c$$

$$=2\sqrt{t}+c \quad =2\sqrt{4-x^2}+c$$

Important Integrals

Since $\frac{d}{dx}Tan^{-1}\left(\frac{x}{a}\right)=\frac{a}{a^2+x^2}$

By Integrating, we have

$$Tan^{-1}\left(\frac{x}{a}\right)=\int \frac{a}{a^2+x^2}dx$$

$$=a\cdot\int \frac{1}{a^2+x^2}dx$$

$$\Rightarrow \boxed{\int \frac{1}{a^2+x^2}dx=\frac{1}{a}Tan^{-1}\left(\frac{x}{a}\right)}$$

Similarly

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}}=Sin^{-1}\frac{x}{a}}$$

$$\boxed{\int \frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}Sec^{-1}\frac{x}{a}}$$

Question # 2

$$\int \frac{dx}{x^2+4x+13}$$

Solution

Let $I=\int \frac{dx}{x^2+4x+13}$

$$=\int \frac{dx}{x^2+2(x)(2)+(2)^2-(2)^2+13}$$

$$=\int \frac{dx}{(x+2)^2-4+13}$$

$$=\int \frac{dx}{(x+2)^2+9} \quad =\int \frac{dx}{(x+2)^2+(3)^2}$$

Put $t=x+2 \quad \Rightarrow \quad dt=dx$

So $I=\int \frac{dt}{t^2+3^2}$

$$=\frac{1}{3}Tan^{-1}\frac{t}{3}+c$$

$$=\frac{1}{3}Tan^{-1}\frac{x+2}{3}+c$$

Question # 3

$$\int \frac{x^2}{4+x^2}dx$$

Solution

$$\int \frac{x^2}{4+x^2}dx$$

$$\begin{array}{r} 1 \\ 4+x^2\overline{)\,x^2\qquad} \\ \underline{\underset{-}{x^2}+\underset{-}{4}} \\ -4 \end{array}$$

$$=\int\left(1-\frac{4}{4+x^2}\right)dx$$

$$=\int dx-4\int \frac{dx}{4+x^2}$$

$$=x-4\int \frac{dx}{(2)^2+x^2}$$

$$=x-4\cdot\frac{1}{2}Tan^{-1}\left(\frac{x}{2}\right)+c$$

$$=x-2\ Tan^{-1}\left(\frac{x}{2}\right)+c$$

Question # 4

$$\int \frac{1}{x\ln x}dx$$

Solution

Suppose $I=\int \frac{1}{x\ln x}dx$

$$=\int \frac{1}{\ln x}\cdot\frac{1}{x}\ dx$$

Put $t=\ln x \quad \Rightarrow \quad dt=\frac{1}{x}dx$

So $I=\int \frac{1}{t}dt \quad =\ln|t|+c$

$$=\ln|\ln x|+c$$

Question # 5

$$\int \frac{e^x}{e^x+3}dx$$

Solution

Suppose $I = \int \frac{e^x}{e^x+3}dx$

Put $t = e^x + 3 \quad \Rightarrow \quad dt = e^x dx$

So $I = \int \frac{dt}{t} = \ln|t| + c$

$$= \ln\left|e^x+3\right| + c$$

Question # 6

$$\int \frac{x+b}{\left(x^2+2bx+c\right)^{\frac{1}{2}}}dx$$

Solution

Let $I = \int \frac{x+b}{\left(x^2+2bx+c\right)^{\frac{1}{2}}}dx$

Put $t = x^2+2bx+c$

$\Rightarrow \; dt = (2x+2b)dx \quad \Rightarrow \quad dt = 2(x+b)dx$

$\Rightarrow \; \frac{1}{2}dt = (x+b)dx$

So $I = \int \frac{\frac{1}{2}dt}{t^{\frac{1}{2}}} \quad = \frac{1}{2}\int t^{-\frac{1}{2}}\,dt$

$$= \frac{1}{2}\cdot\frac{t^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)} + c_1 \quad = \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c_1$$

$$= \left(x^2+2bx+c\right)^{\frac{1}{2}} + c_1$$

$$= \sqrt{x^2+2bx+c} + c_1$$

Question # 7

$$\int \frac{\sec^2 x}{\sqrt{\tan x}}dx$$

Solution

Let $I = \int \frac{\sec^2 x}{\sqrt{\tan x}}dx$

Put $t = \tan x \quad \Rightarrow \quad dt = \sec^2 x\; dx$

So $I = \int \frac{dt}{\sqrt{t}} \quad = \int t^{-\frac{1}{2}}\,dt$

$$= \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c \quad = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2(\tan x)^{\frac{1}{2}} + c \quad = 2\sqrt{\tan x} + c$$

Important Integral

$$\int \sec\theta\; d\theta \;= \int \frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}d\theta$$

$$= \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}d\theta$$

$$= \int \frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta}d\theta$$

Take $t = \sec\theta + \tan\theta$

$\Rightarrow \; dt = \left(\sec^2\theta + \sec\theta\tan\theta\right)d\theta$

So $\int \sec\theta\; d\theta = \int \frac{1}{t}dt$

$$= \ln|t| + c$$

$$= \ln|\sec\theta+\tan\theta| + c$$

$$\Rightarrow \quad \boxed{\int \sec\theta\; d\theta = \ln|\sec\theta+\tan\theta| + c}$$

Similarly

$$\boxed{\int \operatorname{cosec}\theta\; d\theta = \ln|\operatorname{cosec}\theta-\cot\theta| + c}$$

See proof at page 133

Question # 8

(a) Show that

$$\frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + c$$

(b) Show that

$$\sqrt{a^2-x^2}dx = \frac{a^2}{2}Sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c$$

Solution

(a) Let $I = \frac{dx}{\sqrt{x^2-a^2}}$

Put $x = a\sec\theta \quad \Rightarrow \quad dx = a\sec\theta\tan\theta\; d\theta$

So $I = \int \frac{a\sec\theta\tan\theta\; d\theta}{\sqrt{(a\sec\theta)^2 - a^2}}$

$$= \int \frac{a\sec\theta\tan\theta\; d\theta}{\sqrt{a^2\left(\sec^2\theta-1\right)}}$$

$$= \int \frac{a\sec\theta\tan\theta\; d\theta}{\sqrt{a^2\tan^2\theta}}$$

$\because \; 1+\tan^2\theta = \sec^2\theta$

$$= \int \frac{a\sec\theta\tan\theta\; d\theta}{a\tan\theta} \quad = \int \sec\theta\; d\theta$$

$$= \ln|\sec\theta+\tan\theta| + c_1$$

$$= \ln\left|\sec\theta+\sqrt{\sec^2\theta-1}\right| + c_1$$

$$= \ln\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right|+c_1 \qquad \begin{array}{|l} x=a\sec\theta \\ \frac{x}{a}=\sec\theta \end{array}$$

$$= \ln\left|\frac{x}{a}+\sqrt{\frac{x^2-a^2}{a^2}}\right|+c_1$$

$$= \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+c_1$$

$$= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+c_1$$

$$= \ln\left|x+\sqrt{x^2-a^2}\right|-\ln a+c_1$$

$$= \ln\left|x+\sqrt{x^2-a^2}\right|+c$$

where $c=-\ln a+c_1$

(b) Let $I=\sqrt{a^2-x^2}\,dx$

Put $x=a\sin\theta \Rightarrow dx=a\cos\theta\,d\theta$

So $I=\int\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\;d\theta$

$$=\int\sqrt{a^2\left(1-\sin^2\theta\right)}\cdot a\cos\theta\;d\theta$$

$$=\int\sqrt{a^2\cos^2\theta}\cdot a\cos\theta\;d\theta \qquad \because 1-\sin^2\theta=\cos^2\theta$$

$$=\int a\cos\theta\cdot a\cos\theta\;d\theta$$

$$=a^2\int\cos^2\theta\,d\theta \qquad \because\ \cos^2\theta=\frac{1+\cos 2\theta}{2}$$

$$=a^2\int\frac{1+\cos 2\theta}{2}d\theta \quad =\frac{a^2}{2}\int\left(1+\cos 2\theta\right)d\theta$$

$$=\frac{a^2}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+c$$

$$=\frac{a^2}{2}\left(\theta+\frac{2\sin\theta\cos\theta}{2}\right)+c$$

$$=\frac{a^2}{2}\left(\theta+\sin\theta\sqrt{1-\sin^2\theta}\right)+c \qquad \begin{array}{|l} x=a\sin\theta \\ \frac{x}{a}=\sin\theta \\ \sin^{-1}\frac{x}{a}=\theta \end{array}$$

$$=\frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right)+c$$

$$=\frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}}\right)+c$$

$$=\frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}\right)+c$$

$$=\frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a^2}\sqrt{a^2-x^2}\right)+c$$

$$=\frac{a^2}{2}Sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c$$

Evaluate the following integrals:

Question # 9

$$\int\frac{dx}{\left(1+x^2\right)^{\frac{3}{2}}}$$

Solution

Let $I=\int\frac{dx}{\left(1+x^2\right)^{\frac{3}{2}}}$

Put $x=\tan\theta \qquad \Rightarrow\ dx=\sec^2\theta\,d\theta$

$$I=\int\frac{\sec^2\theta\;d\theta}{(1+\tan^2\theta)^{\frac{3}{2}}}$$

$$=\int\frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^{\frac{3}{2}}} \quad \because\ 1+\tan^2\theta=\sec^2\theta$$

$$=\int\frac{\sec^2\theta\,d\theta}{\sec^3\theta}$$

$$=\int\frac{d\theta}{\sec\theta}\ =\int\cos\theta\,d\theta\ =\sin\theta+c$$

$$=\frac{\sin\theta}{\cos\theta}\cdot\cos\theta+c=\tan\theta\cdot\frac{1}{\sec\theta}+c$$

$$=\tan\theta\cdot\frac{1}{\sqrt{1+\tan^2\theta}}+c$$

$$=\frac{x}{\sqrt{1+x^2}}+c \qquad \because\ x=\tan\theta$$

Question # 10

$$\int\frac{1}{(1+x^2)\,Tan^{-1}x}dx$$

Solution

Let $I=\int\frac{1}{(1+x^2)\,Tan^{-1}x}dx$

$$=\int\frac{1}{Tan^{-1}x}\cdot\frac{1}{(1+x^2)}dx$$

Put $t=Tan^{-1}x \qquad \Rightarrow\ dt=\frac{1}{1+x^2}dx$

So $I=\int\frac{1}{t}dt\ =\ln|t|+c$

$$=\ln\left|Tan^{-1}x\right|+c$$

Question # 11

$$\int\sqrt{\frac{1+x}{1-x}}dx$$

Solution

Let $I=\int\sqrt{\dfrac{1+x}{1-x}}\,dx$

Put $x=\sin\theta \quad \Rightarrow \quad dx=\cos\theta\,d\theta$

So $I=\int\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}\cdot\cos\theta\,d\theta$

$$=\int\sqrt{\frac{1+\sin\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}}\cdot\cos\theta\,d\theta$$

$$=\int\sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}}\cdot\cos\theta\,d\theta$$

$$=\int\sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}}\cdot\cos\theta\,d\theta$$

$$=\int\frac{1+\sin\theta}{\cos\theta}\cdot\cos\theta\,d\theta \quad =\int(1+\sin\theta)\,d\theta$$

$$=\theta-\cos\theta+c$$

$$=\theta-\sqrt{1-\sin^2\theta}+c$$

$$=Sin^{-1}x-\sqrt{1-x^2}+c$$

$\because\ x=\sin\theta$

$\therefore\ \sin^{-1}x=\theta$

Question # 12

$$\int\frac{\sin\theta}{1+\cos^2\theta}d\theta$$

Solution

Let $I=\int\dfrac{\sin\theta}{1+\cos^2\theta}d\theta$

Put $t=\cos\theta$

$\Rightarrow\ dt=-\sin\theta\,d\theta \quad \Rightarrow \quad -dt=\sin\theta\,d\theta$

So $I=\int\dfrac{-dt}{1+t^2}\ =-\int\dfrac{dt}{1+t^2}$

$$=-\tan^{-1}t+c$$

$$=-\tan^{-1}(\cos\theta)+c$$

Question # 13

$$\int\frac{ax}{\sqrt{a^2-x^4}}\,dx$$

Solution

Let $I=\int\dfrac{ax}{\sqrt{a^2-x^4}}\,dx$

$$=a\int\frac{x}{\sqrt{a^2-x^4}}\,dx$$

Put $t=x^2$ then $t^2=x^4$

$dt=2x\,dx \quad \Rightarrow \quad \dfrac{1}{2}dt=x\cdot dx$

So $I=a\int\dfrac{\frac{1}{2}dt}{\sqrt{a^2-t^2}}$

$$=\frac{a}{2}\int\frac{dt}{\sqrt{a^2-t^2}}$$

$$=\frac{a}{2}Sin^{-1}\left(\frac{t}{a}\right)+c \qquad \because\ \int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}$$

$$=\frac{a}{2}Sin^{-1}\left(\frac{x^2}{a}\right)+c$$

Question # 14

$$\int\frac{dx}{\sqrt{7-6x-x^2}}$$

Solution

Let $I=\int\dfrac{dx}{\sqrt{7-6x-x^2}}$

$$=\int\frac{dx}{\sqrt{-(x^2+6x-7)}}$$

$$=\int\frac{dx}{\sqrt{-\left(x^2+2(3)(x)+(3)^2-(3)^2-7\right)}}$$

$$=\int\frac{dx}{\sqrt{-\left((x+3)^2-16\right)}}$$

$$=\int\frac{dx}{\sqrt{16-(x+3)^2}}$$

Put $t=x+3 \quad \Rightarrow \quad dx=dt$

So $I=\dfrac{dt}{\sqrt{16-t^2}} \quad =\int\dfrac{dx}{\sqrt{(4)^2-(t)^2}}$

$$=Sin^{-1}\left(\frac{t}{4}\right)+c$$

$$=Sin^{-1}\left(\frac{x+3}{4}\right)+c$$

Question # 15

$$\int\frac{\cos x}{\sin x\cdot\ln\sin x}dx$$

Solution

Let $I=\int\dfrac{\cos x}{\sin x\cdot\ln\sin x}dx$

$$=\int\frac{1}{\ln\sin x}\cdot\frac{\cos x}{\sin x}dx$$

Put $t=\ln\sin x \quad \Rightarrow \quad dt=\dfrac{1}{\sin x}\cdot\cos x\,dx$

So $I=\int\dfrac{1}{t}dt \quad =\ln|t|+c$

$$=\ln|\ln\sin x|+c$$

Question # 16

$$\int \cos x\left(\frac{\ln\sin x}{\sin x}\right)dx$$

Solution

Let $I=\int \cos x\left(\frac{\ln\sin x}{\sin x}\right)dx$

$$=\int \ln\sin x\cdot\frac{\cos x}{\sin x}\,dx$$

Put $t=\ln\sin x \quad \Rightarrow \quad dt=\frac{1}{\sin x}\cdot\cos x\,dx$

No do yourself

Question # 17

$$\int\frac{x\,dx}{4+2x+x^2}$$

Solution

Let $I=\int\frac{x\,dx}{4+2x+x^2}$

$$=\frac{1}{2}\int\frac{2x\,dx}{x^2+2x+4}$$

+ing and −ing 2 in numerator.

$$\Rightarrow\ I=\frac{1}{2}\int\frac{(2x+2)-2}{x^2+2x+4}\,dx$$

$$=\frac{1}{2}\int\left(\frac{2x+2}{x^2+2x+4}-\frac{2}{x^2+2x+4}\right)dx$$

$$=\frac{1}{2}\int\frac{2x+2}{x^2+2x+4}\,dx-\frac{1}{2}\int\frac{2}{x^2+2x+4}\,dx$$

$$=\frac{1}{2}\int\frac{\frac{d}{dx}\left(x^2+2x+4\right)}{x^2+2x+4}\,dx-\frac{2}{2}\int\frac{dx}{x^2+2x+1+3}$$

$$=\frac{1}{2}\ln\left|x^2+2x+4\right|-\int\frac{dx}{(x+1)^2+\left(\sqrt{3}\right)^2}$$

$$=\frac{1}{2}\ln\left|x^2+2x+4\right|-\frac{1}{\sqrt{3}}Tan^{-1}\frac{x+1}{\sqrt{3}}+c$$

Question # 18

$$\int\frac{x}{x^4+2x^2+5}\,dx$$

Solution

Let $I=\int\frac{x}{x^4+2x^2+5}\,dx$

Put $t=x^2$ then $t^2=x^4$

$$dt=2x\,dx \quad\Rightarrow\quad \frac{1}{2}dt=x\,dx$$

So $I=\int\frac{\frac{1}{2}dt}{t^2+2t+5} \quad =\frac{1}{2}\int\frac{dt}{t^2+2t+1+4}$

$$=\frac{1}{2}\int\frac{dt}{(t+1)^2+(2)^2}$$

$$=\frac{1}{2}\cdot\frac{1}{2}Tan^{-1}\left(\frac{t+1}{2}\right)+c$$

$$=\frac{1}{4}Tan^{-1}\left(\frac{x^2+1}{2}\right)+c$$

Question # 19

$$\int\left[\cos\left(\sqrt{x}-\frac{x}{2}\right)\right]\times\left(\frac{1}{\sqrt{x}}-1\right)dx$$

Solution

Let $I=\int\left[\cos\left(\sqrt{x}-\frac{x}{2}\right)\right]\times\left(\frac{1}{\sqrt{x}}-1\right)dx$

Put $t=\sqrt{x}-\frac{x}{2}$

$$\Rightarrow\ dt=\left(\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}\right)dx\ \Rightarrow\ dt=\frac{1}{2}\left(\frac{1}{\sqrt{x}}-1\right)dx$$

$$\Rightarrow\ 2\,dt=\left(\frac{1}{\sqrt{x}}-1\right)dx$$

So $I=\int\cos t\cdot 2\,dt$

$$=2\int\cos t\,dt$$

$$=2\sin t+c$$

Question # 20

$$\int\frac{x+2}{\sqrt{x+3}}\,dx$$

Solution

Let $I=\int\frac{x+2}{\sqrt{x+3}}\,dx$

Put $t=x+3$ then $t-3=x$

$\Rightarrow\ dt=dx$

So $I=\int\frac{t-3+2}{\sqrt{t}}dx$

$$=\int\frac{t-1}{(t)^{\frac{1}{2}}}\,dx \quad =\int\left(\frac{t}{(t)^{\frac{1}{2}}}-\frac{1}{(t)^{\frac{1}{2}}}\right)dx$$

$$=\int\left((t)^{\frac{1}{2}}-(t)^{-\frac{1}{2}}\right)dx$$

$$=\frac{(t)^{\frac{1}{2}+1}}{\frac{1}{2}+1}-\frac{(t)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c=\frac{(t)^{\frac{3}{2}}}{3/2}-\frac{(t)^{\frac{1}{2}}}{1/2}+c$$

$$=\frac{2(x+3)^{\frac{3}{2}}}{3}-2(x+3)^{\frac{1}{2}}+c$$

$$=\frac{2(x+3)^{\frac{3}{2}}}{3}-2\sqrt{x+3}+c$$

Question # 21

$$\int\frac{\sqrt{2}}{\sin x+\cos x}dx$$

Solution

Let $I=\int\frac{\sqrt{2}}{\sin x+\cos x}dx$

$$=\int\frac{1}{\frac{1}{\sqrt{2}}(\sin x+\cos x)}dx$$

$$=\int\frac{1}{\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x}dx$$

Put $\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}=\sin\frac{\pi}{4}$

So $I=\int\frac{1}{\sin\frac{\pi}{4}\cdot\sin x+\cos\frac{\pi}{4}\cdot\cos x}dx$

$$=\int\frac{1}{\cos\left(x-\frac{\pi}{4}\right)}dx\quad=\int\sec\left(x-\frac{\pi}{4}\right)dx$$

$$=\ln\left|\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right|+c$$

Question # 22

$$\int\frac{dx}{\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x}$$

Solution

Let $I=\int\frac{dx}{\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x}$

$\because\ \cos\frac{\pi}{3}=\frac{1}{2}$ & $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

$\therefore\ I=\int\frac{dx}{\cos\frac{\pi}{3}\sin x+\sin\frac{\pi}{3}\cos x}$

$$=\int\frac{dx}{\sin\left(x+\frac{\pi}{3}\right)}=\int\operatorname{cosec}\left(x+\frac{\pi}{3}\right)dx$$

$$=\ln\left|\operatorname{cosec}\left(x+\frac{\pi}{3}\right)-\cot\left(x+\frac{\pi}{3}\right)\right|+c$$

Integration by Parts

If u and v are function of x, then

$$\int uv\,dx = u\int v dx - \int\left(\int v dx\right)\cdot u'\,dx$$

Question # 1

Evaluate the following integrals by parts add a word representing all the functions are defined.

$u = x$
$v = \sin x$

(i) $\int x\sin x\,dx$ (ii) $\int \ln x\,dx$

(iii) $\int x\ln x\ dx$ (iv) $\int x^2 \ln x\,dx$

(v) $\int x^3 \ln x\,dx$ (vi) $\int x^4 \ln x\,dx$

(vii) $\int \tan^{-1} x\ dx$ (viii) $\int x^2 \sin x\ dx$

(ix) $\int x^2 \tan^{-1} x\,dx$ (x) $\int x\tan^{-1} x\,dx$

(xi) $\int x^3 \tan^{-1} x\ dx$ (xii) $\int x^3 \cos x\,dx$

(xiii) $\int \sin^{-1} x\,dx$ (xiv) $\int x\ \sin^{-1} x\ dx$

(xv) $\int e^x \sin x\cos x\,dx$ (xvi) $\int x\sin x\cos x\,dx$

(xvii) $\int x\cos^2 x\,dx$ (xviii) $\int x\sin^2 x\ dx$

(xix) $\int (\ln x)^2\,dx$ (xx) $\int \ln(\tan x)\sec^2 x dx$

(xxi) $\int \dfrac{x\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

Solution

(i) Let $I = \int x\sin x\,dx$

Integration by parts

$$I = x\cdot(-\cos x) - \int(-\cos x)\cdot(1)\,dx$$
$$= -x\cos x + \int \cos x\,dx$$
$$= -x\cos x + \sin x + c$$

(ii) Let $I = \int \ln x\,dx$

$= \int \ln x\cdot 1\ dx$

$u = \ln x$
$v = 1$

Integrating by parts

$$I = \ln x\cdot x - \int x\cdot\frac{1}{x}\ dx$$
$$= x\ln x - \int dx$$
$$= x\ln x - x + c$$

(iii) Let $I = \int x\ln x\ dx$

Integrating by parts

$u = \ln x$
$v = x$

$$I = \ln x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{x}dx$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\ dx$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + c$$
$$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$$

(iv) *Do yourself*

(v) *Do yourself*

(vi) *Do yourself*

(vii) Let $I = \int \tan^{-1} x\ dx$

$= \int \tan^{-1} x\cdot 1\ dx$

$u = \tan^{-1} x$
$v = 1$

Integrating by parts

$$I = \tan^{-1} x\cdot x - \int x\cdot\frac{1}{1+x^2}dx$$
$$= x\tan^{-1} x - \frac{1}{2}\int\frac{2x}{1+x^2}dx$$
$$= x\tan^{-1} x - \frac{1}{2}\int\frac{\frac{d}{dx}(1+x^2)}{1+x^2}dx$$
$$= x\tan^{-1} x - \frac{1}{2}\ln\left|1+x^2\right| + c$$

(viii) Let $I = \int x^2 \sin x\ dx$

$u = x^2$
$v = \sin x$

Integrating by parts

$$I = x^2(-\cos x) - \int(-\cos x)\cdot 2x\ dx$$
$$= -x^2\cos x + 2\int x\ \cos x\,dx$$

$u = x$
$v = \cos x$

Again integrating by parts

$$I = -x^2\cos x + 2\left(x\ \sin x - \int \sin x\ (1)\ dx\right)$$
$$= -x^2\cos x + 2x\ \sin x - 2(-\cos x) + c$$
$$= -x^2\cos x + 2x\ \sin x + 2\cos x + c$$

(ix) Let $I = \int x^2 \tan^{-1} x\,dx$

Integrating by parts

$u = \tan^{-1} x$
$v = x^2$

$$I = \tan^{-1} x\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{1+x^2}dx$$
$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int\frac{x^3}{1+x^2}dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int\left(x-\frac{x}{1+x^2}\right)dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int x\,dx+-\frac{1}{3}\int\frac{x}{1+x^2}\,dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\cdot\frac{x^2}{2}+-\frac{1}{3}\cdot\frac{1}{2}\int\frac{2x}{1+x^2}\,dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{x^2}{6}+-\frac{1}{6}\int\frac{\frac{d}{dx}(1+x^2)}{1+x^2}\,dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{x^2}{6}+-\frac{1}{6}\ln\left|1+x^2\right|+c$$

(x) Let $I=\int x\tan^{-1}x\,dx$

Integrating by parts

$u=\tan^{-1}x$
$v=x$

$$I=\frac{x^2}{2}\tan^{-1}x-\int\frac{x^2}{2}\cdot\frac{1}{1+x^2}\,dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{1+x^2-1}{1+x^2}\,dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\left(\frac{1+x^2}{1+x^2}-\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int dx+\frac{1}{2}\int\frac{1}{1+x^2}\,dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+c \quad \textit{Ans.}$$

(xi) Let $I=\int x^3\tan^{-1}x\,dx$

Integrating by parts

$u=\tan^{-1}x$
$v=x^3$

$$\begin{array}{r|l} & x^2-1 \\ \hline 1+x^2 & x^4 \\ & \underset{-}{x^4}+\underset{-}{x^2} \\ \hline & -x^2 \\ & \underset{+}{-1}\underset{+}{-x^2} \\ \hline & 1 \end{array}$$

$$I=\tan^{-1}x\cdot\frac{x^4}{4}-\int\frac{x^4}{4}\cdot\frac{1}{1+x^2}\,dx$$

$$=\frac{x^4}{4}\tan^{-1}x--\frac{1}{4}\int\frac{x^4}{1+x^2}\,dx$$

$$=\frac{x^4}{4}\tan^{-1}x$$
$$-\frac{1}{4}\int\left(x^2-1+\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\int x^2dx+\frac{1}{4}\int dx-\frac{1}{4}\int\frac{1}{1+x^2}\,dx$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\frac{x^3}{3}+\frac{1}{4}x-\frac{1}{4}\tan^{-1}x+c$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{x^3}{12}+\frac{1}{4}x-\frac{1}{4}\tan^{-1}x+c$$

(xii) *Do yourself as Question # 1(viii).*

(xiii) $I=\int\sin^{-1}x\,dx$

$u=\sin^{-1}x$
$v=1$

$$=\int\sin^{-1}x\cdot 1\,dx$$

Integrating by parts

$$I=\sin^{-1}x\cdot x-\int x\cdot\frac{1}{\sqrt{1-x^2}}\,dx$$

$$=x\sin^{-1}x-\int(1-x^2)^{-\frac{1}{2}}(x)\,dx$$

$$=x\sin^{-1}x+\frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}(-2x)\,dx$$

$$=x\sin^{-1}x+\frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}\frac{d}{dx}(1-x^2)\,dx$$

$$=x\sin^{-1}x+\frac{1}{2}\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$$

$$=x\sin^{-1}x+\frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{1/2}+c$$

$$=x\sin^{-1}x+\sqrt{1-x^2}+c$$

(xiv) Let $I=\int x\sin^{-1}x\,dx$

Integrating by parts

$$I=\sin^{-1}x\cdot\frac{x^2}{2}-\int\frac{x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}\,dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}\,dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\left(\frac{1-x^2}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\left(\sqrt{1-x^2}-\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\sqrt{1-x^2}\,dx-\frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}\,dx$$

$$\Rightarrow \ I = \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}I_1 - \frac{1}{2}\sin^{-1}x \ \text{….. (i)}$$

Where $I_1 = \int\sqrt{1-x^2}\,dx$

Put $x = \sin\theta \quad \Rightarrow \quad dx = \cos\theta\, d\theta$

$$\Rightarrow \ I_1 = \int\sqrt{1-\sin^2\theta}\ \cos\theta\ d\theta$$

$$= \int\sqrt{\cos^2\theta}\ \cos\theta\ d\theta$$

$$= \int\cos^2\theta\ d\theta \ = \int\left(\frac{1+\cos 2\theta}{2}\right)\ d\theta$$

$$= \frac{1}{2}\int(1+\cos 2\theta)\ d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$$

$$= \frac{1}{2}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right] + c$$

$$= \frac{1}{2}\left[\ \theta + \sin\theta\sqrt{1-\sin^2\theta}\ \right] + c$$

$$= \frac{1}{2}\left[\ \sin^{-1}x + x\sqrt{1-x^2}\ \right] + c$$

Using value of I_1 in (i)

$$I = \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\left[\frac{1}{2}\left(\sin^{-1}x + x\sqrt{1-x^2} + c\right)\right] - \frac{1}{2}\sin^{-1}x$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{4}\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2} + \frac{1}{2}c - \frac{1}{2}\sin^{-1}x$$

$$\Rightarrow \ I = \frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2} + \frac{1}{2}c$$

(xv) Let $I = \int e^x\sin x\cos x\,dx$

$$= \frac{1}{2}\int e^x\cdot 2\sin x\cos x\ dx$$

$$= \frac{1}{2}\int e^x\sin 2x\ dx \ \because \ \sin 2x = 2\sin x\cos x$$

Integrating by parts

$$I = \frac{1}{2}\left[e^x\cdot\frac{-\cos 2x}{2} - \int\frac{-\cos 2x}{2}\cdot e^x dx\right]$$

$$= -\frac{1}{4}\ e^x\cos 2x + \frac{1}{4}\int e^x\cos 2x\ dx$$

Again integrating by parts

$$I = -\frac{1}{4}e^x\cos 2x + \frac{1}{4}\left(e^x\cdot\frac{\sin 2x}{2} - \int\frac{\sin 2x}{2}\ e^x\right)$$

$$= -\frac{1}{4}\ e^x\cos 2x + \frac{1}{4}\left(e^x\cdot\frac{\sin 2x}{2} - \frac{1}{2}\int\ e^x\sin 2x\right)$$

$$= -\frac{1}{4}\ e^x\cos 2x + \frac{1}{4}\left(e^x\cdot\frac{\sin 2x}{2} - I\right) + c$$

$$= -\frac{1}{4}\ e^x\cos 2x + \frac{1}{8}e^x\sin 2x - \frac{1}{4}I + c$$

$$\Rightarrow \ I + \frac{1}{4}I = -\frac{1}{4}\ e^x\cos 2x + \frac{1}{8}e^x\sin 2x + c$$

$$\Rightarrow \ \frac{5}{4}I = -\frac{1}{4}\ e^x\cos 2x + \frac{1}{8}e^x\sin 2x + c$$

$$\Rightarrow \ I = -\frac{1}{5}\ e^x\cos 2x + \frac{1}{10}e^x\sin 2x + \frac{4}{5}c$$

(xvi) Let $I = \int x\sin x\cos x\,dx$

$$= \frac{1}{2}\int x\cdot 2\sin x\cos x\ dx$$

$$= \frac{1}{2}\int x\cdot\sin 2x\ dx$$

$u = x$
$v = \sin 2x$

Integrating by parts

$$I = \frac{1}{2}\left[x\left(\frac{-\cos 2x}{2}\right) - \int\left(\frac{-\cos 2x}{2}\right)(1)dx\right]$$

(xvii)Let $I = \int x\cos^2 x\,dx$

$$= \int x\left(\frac{1+\cos 2x}{2}\right)dx$$

$$= \frac{1}{2}\int x(1+\cos 2x)\,dx$$

$$= \frac{1}{2}\int x\,dx + \frac{1}{2}\int x\cos 2x dx$$

$u = x$
$v = \cos 2x$

$$= \frac{1}{2}\cdot\frac{x^2}{2} + \frac{1}{2}\left[x\ \frac{\sin 2x}{2} - \int\frac{\sin 2x}{2}\cdot(1)dx\right]$$

$$= \frac{x^2}{4} + \frac{1}{4}x\cdot\sin 2x - \frac{1}{4}\int\sin 2x\ dx$$

$$= \frac{x^2}{4} + \frac{1}{4}x\cdot\sin 2x - \frac{1}{4}\left(\frac{-\cos 2x}{2}\right) + c$$

$$= \frac{x^2}{4} + \frac{1}{4}x\cdot\sin 2x + \frac{1}{8}\cos 2x + c$$

(xviii) Let $I = \int x\sin^2 x\ dx$

$$=\int x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$=\frac{1}{2}\int x(1-\cos 2x)\,dx$$

$$=\frac{1}{2}\int x\,dx-\frac{1}{2}\int x\cos 2x\,dx$$

$u=x$
$v=\cos 2x$

Integrating by parts

$$I=\frac{1}{2}\frac{x^2}{2}-\frac{1}{2}\left[x\cdot\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}.(1)\,dx\right]$$

$$=\frac{x^2}{4}-\frac{1}{4}x\sin 2x+\frac{1}{4}\int\sin 2x\,dx$$

$$=\frac{x^2}{4}-\frac{1}{4}x\sin 2x+\frac{1}{4}\left(\frac{-\cos 2x}{2}\right)+c$$

$$=\frac{x^2}{4}-\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x+c$$

(xix) Let $I=\int(\ln x)^2dx$

$$=\int(\ln x)^2\cdot 1\,dx$$

$u=(\ln x)^2$
$v=1$

Integrating by parts

$$I=(\ln x)^2\cdot x-\int x\cdot 2(\ln x)\cdot\frac{1}{x}\,dx$$

$$=x(\ln x)^2-2\int(\ln x)\,dx$$

Again integrating by parts

$$I=x(\ln x)^2-2\left[\ln x\cdot x-\int x\cdot\frac{1}{x}\,dx\right]$$

$$=x(\ln x)^2-2x\ln x+2\int dx$$

$$=x(\ln x)^2-2x\ln x+2x+c$$

(xx) Let $I=\int\ln(\tan x)\sec^2 x\,dx$

Integrating by parts

$$I=\ln(\tan x)\cdot\tan x-\int\tan x\cdot\frac{1}{\tan x}\cdot\sec^2 x\,dx$$

$$=\tan x\ln(\tan x)-\int\sec^2 x\,dx$$

$$=\tan x\ln(\tan x)-\tan x+c$$

(xxi) Let $I=\int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

$u=\sin^{-1}x$
$v=\left(1-x^2\right)^{-\frac{1}{2}}(-2x)$

$$=\int\sin^{-1}x\cdot\frac{1}{\sqrt{1-x^2}}(x)\,dx$$

$$=-\frac{1}{2}\int\sin^{-1}x\cdot\left(1-x^2\right)^{-\frac{1}{2}}(-2x)\,dx$$

Integrating by parts

$$I=-\frac{1}{2}\left[\sin^{-1}x\cdot\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}-\int\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\cdot\frac{1}{\sqrt{1-x^2}}\,dx\right]$$

$$=-\frac{1}{2}\left[\sin^{-1}x\cdot\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}-\int\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\cdot\frac{1}{\sqrt{1-x^2}}\,dx\right]$$

$$=-\frac{1}{2}\left[2(1-x^2)^{\frac{1}{2}}\sin^{-1}x-2\int dx\right]$$

$$=-\sqrt{1-x^2}\sin^{-1}x+\int dx$$

$$=-\sqrt{1-x^2}\sin^{-1}x+x+c$$

$$=x-\sqrt{1-x^2}\sin^{-1}x+c$$

Question # 2

Evaluate the following integrals.

(i) $\int\tan^4 x\,dx$ (ii) $\int\sec^4 x\,dx$

(iii) $\int e^x\sin 2x\cos x\,dx$ (iv) $\int\tan^3 x\cdot\sec x\,dx$

(iv) $\int\tan^3 x\cdot\sec x\,dx$ (v) $\int x^3e^{5x}dx$

(vi) $\int e^{-x}\sin 2x\,dx$ (vii) $\int e^{2x}\cdot\cos 3x\,dx$

(viii) $\int\operatorname{cosec}^3 x\,dx$

Solution

(i) Let $I=\int\tan^4 x\,dx$

$$=\int\tan^2 x\cdot\tan^2 x\,dx$$

$$=\int\tan^2 x\left(\sec^2 x-1\right)dx$$

$$=\int\left(\tan^2 x\sec^2 x-\tan^2 x\right)dx$$

$$=\int\tan^2 x\sec^2 x\,dx-\int\tan^2 x\,dx$$

$$=\int\tan^2 x\frac{d}{dx}(\tan x)\,dx-\int(\sec^2 x-1)\,dx$$

$$=\frac{\tan^{2+1}x}{2+1}-\int\sec^2 x\,dx+\int dx$$

$$=\frac{1}{3}\tan^3 x-\tan x+x+c$$

(ii) Let $I=\int \sec^4 x\ dx$

$$=\int (\sec^2 x)\cdot(\sec^2 x)\ dx$$

$$=\int (1+\tan^2 x)\cdot(\sec^2 x)\ dx$$

$$=\int \sec^2 x\ dx+\int \tan^2 x\sec^2 x\ dx$$

$$=\tan x+\int (\tan x)^2\frac{d}{dx}(\tan x)\ dx$$

$$=\tan x+\frac{\tan^3 x}{3}+c$$

(iii) Let $I=\int e^x\sin 2x\cos x\,dx$

$$=\frac{1}{2}\int e^x\left(2\sin 2x\cos x\right)dx$$

$$=\frac{1}{2}\int e^x\left(\sin(2x+x)+\sin(2x-x)\right)dx$$

$$=\frac{1}{2}\int e^x\left(\sin 3x+\sin x\right)dx$$

$$=\frac{1}{2}\int e^x\sin 3x\ dx+\frac{1}{2}\int e^x\sin x\,dx$$

$$=\frac{1}{2}I_1+\frac{1}{2}I_2\ \ \ldots\ldots\ldots\ \text{(i)}$$

Where $I_1=\int e^x\sin 3x\,dx$ and $I_2=\int e^x\sin x\,dx$

Solve I_1 and I_2 as in Q # 1(xv) and put value of I_1 and I_2 in (i).

(iv) $I=\int \tan^3 x\cdot\sec x\ dx$

$$=\int \tan^2 x\cdot\tan x\cdot\sec x\ dx$$

$$=\int\left(\sec^2 x-1\right)\cdot\sec x\tan x\ dx$$

Put $t=\sec x\quad\Rightarrow\quad dt=\sec x\tan x\,dx$

So $I=\int\left(t^2-1\right)dt$

$$=\frac{t^3}{3}-t+c$$

$$=\frac{\sec^3 x}{3}-\sec+c$$

(v) Let $I=\int x^3e^{5x}dx$

$u=x^3$
$v=e^x$

Integrating by parts

$$I=x^3\cdot\frac{e^{5x}}{5}-\int\frac{e^{5x}}{5}\cdot 3x^2dx$$

$$=\frac{1}{5}x^3e^{5x}-\frac{3}{5}\int x^2e^{5x}dx$$

$u=x^2$
$v=e^x$

Again integrating by parts

$$I=\frac{1}{5}x^3e^{5x}-\frac{3}{5}\left[x^2\cdot\frac{e^{5x}}{5}-\int\frac{e^{5x}}{5}\cdot 2x\,dx\right]$$

$$=\frac{1}{5}x^3e^{5x}-\frac{3}{25}x^2e^{5x}+\frac{6}{25}\int x\,e^{5x}\ dx$$

Again integrating by parts

$$I=\frac{1}{5}x^3e^{5x}-\frac{3}{25}x^2e^{5x}+\frac{6}{25}\left[x\cdot\frac{e^{5x}}{5}-\int\frac{e^{5x}}{5}\cdot(1)dx\right]$$

$$=\frac{1}{5}x^3e^{5x}-\frac{3}{25}x^2e^{5x}+\frac{6}{125}xe^{5x}-\frac{6}{125}\int e^{5x}\ dx$$

$$=\frac{1}{5}x^3e^{5x}-\frac{3}{25}x^2e^{5x}+\frac{6}{125}xe^{5x}-\frac{6}{125}\cdot\frac{e^{5x}}{5}+c$$

$$=\frac{e^{5x}}{5}\left(x^3-\frac{3}{5}x^2+\frac{6}{25}x-\frac{6}{125}\right)+c$$

(vi) Let $I=\int e^{-x}\sin 2x\,dx$

Integrating by parts

$u=e^{-x}$
$v=\sin 2x$

$$I=e^{-x}\cdot\frac{-\cos 2x}{2}-\int\frac{-\cos 2x}{2}\cdot e^{-x}(-1)\,dx$$

$$=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{2}\int e^{-x}\cos 2x\ dx$$

Again integrating by parts

$$I=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{2}\left[e^{-x}\cdot\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}\cdot e^{-x}(-1)\ dx\right]$$

$$=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{4}e^{-x}\sin 2x-\frac{1}{4}\int e^{-x}\sin 2x\,dx$$

$$\Rightarrow\ I=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{4}e^{-x}\sin 2x-\frac{1}{4}I+c$$

$$\Rightarrow\ I+\frac{1}{4}I=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{4}e^{-x}\sin 2x+c$$

$$\Rightarrow\ \frac{5}{4}I=-\frac{1}{2}e^{-x}\cos 2x-\frac{1}{4}e^{-x}\sin 2x+c$$

$$\Rightarrow\ I=-\frac{2}{5}e^{-x}\cos 2x-\frac{1}{5}e^{-x}\sin 2x+\frac{4}{5}c$$

$$=-\frac{1}{5}e^{-x}\left(2\cos 2x+\sin 2x\right)+\frac{4}{5}c$$

(vii) *Do yourself as above*

(viii) $I=\int \text{cosec}^3 x\,dx$

$u=\text{cosec}\ x$
$v=\text{cosec}^2\ x$

$$= \int \operatorname{cosec} x \cdot \operatorname{cosec}^2 x \, dx$$

Integrating by parts

$$I = \csc x(-\cot x) i\int(-\cot x)(-\csc x \cot x)\,dx$$

$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x \cot^2 x \, dx$$

$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x\left(\operatorname{cosec}^2 x - 1\right) dx$$

$$= -\operatorname{cosec} x \cot x - \int \left(\operatorname{cosec}^3 x - \operatorname{cosec} x\right) dx$$

$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec}^3 x \, dx + \int \operatorname{cosec} x \, dx$$

$$= -\operatorname{cosec} x \cot x - I + \ln\left|\operatorname{cosec} x - \cot x\right| + c$$

$$\Rightarrow \quad I + I = -\operatorname{cosec} x \cot x + \ln\left|\operatorname{cosec} x - \cot x\right| + c$$

$$\Rightarrow \quad 2I = -\operatorname{cosec} x \cot x + \ln\left|\operatorname{cosec} x - \cot x\right| + c$$

$$\Rightarrow \quad I = -\frac{1}{2}\csc x \cot x + \frac{1}{2}\ln\left|\csc x - \cot x\right| + \frac{1}{2}c$$

Question # 3

Show that

$$\int e^{ax} \sin bx \, dx = \frac{1}{\sqrt{a^2+b^2}}\, e^{ax} \sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

Solution

Let $I = \int e^{ax} \sin bx \, dx$ $\qquad u = e^{ax}$, $v = \sin bx$

Integrating by parts

$$I = e^{ax}\left(-\frac{\cos bx}{b}\right) - \int\left(-\frac{\cos bx}{b}\right) \cdot e^{ax}(a)\,dx$$

$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax} \cos bx \, dx$$

Again integrating by parts

$$I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\left[e^{ax}\frac{\sin bx}{b} - \int \frac{\sin bx}{b} \cdot e^{ax} a \, dx\right]$$

$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2}\int e^{ax} \sin bx \, dx$$

$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} I + c_1$$

$$\Rightarrow \quad I + \frac{a^2}{b^2} I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2} e^{ax} \sin bx + c_1$$

$$\Rightarrow \quad \left(\frac{b^2+a^2}{b^2}\right) I = \frac{e^{ax}}{b^2}(-b\cos bx + a \sin bx) + c_1$$

$$\Rightarrow \quad \left(b^2+a^2\right) I = e^{ax}(a \sin bx - b \cos bx) + b^2 c_1$$

Put $a = r\cos\theta$ & $b = r\sin\theta$

Squaring and adding

$$a^2 + b^2 = r^2\left(\cos^2\theta + \sin^2\theta\right)$$

$$\Rightarrow \quad a^2 + b^2 = r^2 (1) \qquad \Rightarrow \quad r = \sqrt{a^2+b^2}$$

Also

$$\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} \qquad \Rightarrow \quad \frac{b}{a} = \tan\theta$$

$$\Rightarrow \quad \theta = \tan^{-1}\frac{b}{a}$$

So

$$\left(b^2+a^2\right) I = e^{ax}\left(r\cos\theta \sin bx - r\sin\theta \cos bx\right) + b^2 c_1$$

$$\left(b^2+a^2\right) I = e^{ax} r\left(\sin bx \cos\theta - \cos bx \sin\theta\right) + b^2 c_1$$

$$\Rightarrow \quad \left(a^2+b^2\right) I = e^{ax} r \sin(bx - \theta) + b^2 c_1$$

Putting value of r and θ

$$\left(a^2+b^2\right) I = e^{ax}\sqrt{a^2+b^2}\,\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + b^2 c_1$$

$$\Rightarrow I = \frac{\sqrt{a^2+b^2}}{(a^2+b^2)}\, e^{ax} \sin\left(bx - \tan^{-1}\frac{b}{a}\right) + \frac{b^2}{a^2+b^2} c_1$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{a^2+b^2}}\, e^{ax} \sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

Where $c = \dfrac{b^2}{a^2+b^2} c_1$

Question # 4

Evaluate the following indefinite integrals.

(i) $\int \sqrt{a^2 - x^2}\, dx$ $\qquad$ (ii) $\int \sqrt{x^2 - a^2}\, dx$

(iii) $\int \sqrt{4 - 5x^2}\, dx$ $\qquad$ (iv) $\int \sqrt{3 - 4x^2}\, dx$

(v) $\int \sqrt{x^2 + 4}\, dx$ $\qquad$ (vi) $\int x^2 e^{ax} dx$

Solution

(i) Let $I = \int \sqrt{a^2 - x^2}\, dx$ $\qquad u = \sqrt{a^2 - x^2}$, $v = 1$

$$= \int \sqrt{a^2 - x^2} \cdot 1 \, dx$$

Integrating by parts

$$I = \sqrt{a^2 - x^2} \cdot x - \int x \cdot \frac{1}{2}\left(a^2 - x^2\right)^{-\frac{1}{2}} \cdot (-2x)\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\left(a^2 - x^2\right)^{\frac{1}{2}}}\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\left(a^2 - x^2\right)^{\frac{1}{2}}}\, dx$$

$$= x\sqrt{a^2-x^2} - \int \left(\frac{a^2-x^2}{\left(a^2-x^2\right)^{\frac{1}{2}}} - \frac{a^2}{\left(a^2-x^2\right)^{\frac{1}{2}}} \right) dx$$

$$= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\ dx + \int \frac{a^2}{\sqrt{a^2-x^2}}\, dx$$

$$\Rightarrow\ I = x\sqrt{a^2-x^2} - I + a^2 \int \frac{1}{\sqrt{a^2-x^2}}\, dx$$

$$\Rightarrow\ I + I = x\sqrt{a^2-x^2} + a^2 Sin^{-1}\frac{x}{a} + c$$

$$\Rightarrow\ 2I = x\sqrt{a^2-x^2} + a^2 Sin^{-1}\frac{x}{a} + c$$

$$\Rightarrow\ I = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2 Sin^{-1}\frac{x}{a} + \frac{1}{2}c$$

Review

- $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left| x + \sqrt{x^2-a^2} \right| + c$
- $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left| x + \sqrt{x^2+a^2} \right| + c$

(ii)Let $I = \int \sqrt{x^2-a^2}\ dx$

$= \int \sqrt{x^2-a^2} \cdot 1\ dx$

$u = \sqrt{x^2-a^2}$
$v = 1$

Integrating by parts

$$I = \sqrt{x^2-a^2} \cdot x - \int x \cdot \frac{1}{2}\left(x^2-a^2\right)^{-\frac{1}{2}} \cdot (2x)\ dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2}{\left(x^2-a^2\right)^{\frac{1}{2}}}\ dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2-a^2+a^2}{\left(x^2-a^2\right)^{\frac{1}{2}}}\ dx$$

$$= x\sqrt{x^2-a^2} - \int \left(\frac{x^2-a^2}{\left(x^2-a^2\right)^{\frac{1}{2}}} + \frac{a^2}{\left(x^2-a^2\right)^{\frac{1}{2}}} \right) dx$$

$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\ dx - \int \frac{a^2}{\sqrt{x^2-a^2}}\, dx$$

$$\Rightarrow\ I = x\sqrt{x^2-a^2} - I - a^2 \int \frac{1}{\sqrt{x^2-a^2}}\, dx$$

$$\Rightarrow\ I + I = x\sqrt{x^2-a^2} - a^2 \ln\left| x + \sqrt{x^2-a^2} \right| + c$$

$$\because\ \int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left| x + \sqrt{x^2-a^2} \right| + c$$

$$\Rightarrow\ 2I = x\sqrt{x^2-a^2} - a^2 \ln\left| x + \sqrt{x^2-a^2} \right| + c$$

$$\Rightarrow\ I = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left| x + \sqrt{x^2-a^2} \right| + \frac{1}{2}c$$

(iii) Let $I = \int \sqrt{4-5x^2}\ dx$

$= \int \sqrt{4-5x^2} \cdot 1\ dx$

Integrating by parts

$$I = \sqrt{4-5x^2} \cdot x - \int x \cdot \frac{1}{2}(4-5x^2)^{-\frac{1}{2}} \cdot (-10x)\ dx$$

$$= \sqrt{4-5x^2} \cdot x - \int \frac{-5x^2}{(4-5x^2)}\, dx$$

$$= \sqrt{4-5x^2} \cdot x - \int \frac{4-5x^2-4}{(4-5x^2)}\, dx$$

$$= \sqrt{4-5x^2} \cdot x - \int \left(\frac{4-5x^2}{(4-5x^2)^{\frac{1}{2}}} - \frac{4}{(4-5x^2)^{\frac{1}{2}}} \right) dx$$

$$= \sqrt{4-5x^2} \cdot x - \int \left((4-5x^2)^{\frac{1}{2}} - \frac{4}{(4-5x^2)^{\frac{1}{2}}} \right) dx$$

$$= \sqrt{4-5x^2} \cdot x - \int \sqrt{4-5x^2}\, dx + 4\int \frac{1}{\sqrt{4-5x^2}}\ dx$$

$$\Rightarrow\ I = \sqrt{4-5x^2} \cdot x - I + 4\int \frac{1}{\sqrt{5\left(\frac{4}{5} - x^2\right)}}\ dx$$

$$\Rightarrow\ I + I = \sqrt{4-5x^2} \cdot x + 4\int \frac{1}{\sqrt{5}\sqrt{\frac{4}{5} - x^2}}\ dx$$

$$\Rightarrow\ 2I = \sqrt{4-5x^2} \cdot x + \frac{4}{\sqrt{5}} \int \frac{1}{\sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 - x^2}}\ dx$$

$$= \sqrt{4-5x^2} \cdot x + \frac{4}{\sqrt{5}} Sin^{-1}\left(\frac{x}{2/\sqrt{5}} \right) + c_1$$

$$\because\ \int \frac{dx}{\sqrt{a^2-x^2}} = Sin^{-1}\frac{x}{a}$$

$$\Rightarrow\ I = \frac{x}{2}\sqrt{4-5x^2} + \frac{4}{2\sqrt{5}} Sin^{-1}\left(\frac{\sqrt{5}\,x}{2} \right) + \frac{1}{2}c_1$$

$$=\frac{x}{2}\sqrt{4-5x^2}+\frac{2}{\sqrt{5}}Sin^{-1}\left(\frac{\sqrt{5}\,x}{2}\right)+c$$

Where $c=\frac{1}{2}c_1$

(iv) *Same as above.*

(v) *Same as Q # 4(ii)*

Use $\int\frac{dx}{\sqrt{x^2+4}}=\ln\left|x+\sqrt{x^2+4}\right|+c$

(vi) *Do yourself as Question # 2(v)*

Important Formula

Since $\frac{d}{dx}\left(e^{ax}f(x)\right)=e^{ax}\frac{d}{dx}f(x)+f(x)\frac{d}{dx}e^{ax}$

$$=e^{ax}f'(x)+f(x)\cdot e^{ax}(a)$$

$$=e^{ax}\left[a\,f(x)+f'(x)\right]$$

On integrating

$$\int\frac{d}{dx}\left(e^{ax}f(x)\right)dx=\int e^{ax}\left[a\,f(x)+f'(x)\right]dx$$

$$\Rightarrow\ e^{ax}f(x)=\int e^{ax}\left[a\,f(x)+f'(x)\right]dx$$

$$\Rightarrow\ \boxed{\int e^{ax}\left[a\,f(x)+f'(x)\right]dx=e^{ax}f(x)+c}$$

Question # 5

Evaluate the following integrals.

(i) $\int e^x\left(\frac{1}{x}+\ln x\right)dx$ (ii) $\int e^x\left(\cos x+\sin x\right)dx$

(iii) $\int e^{ax}\left[a\sec^{-1}x+\frac{1}{x\sqrt{x^2-1}}\right]dx$

(iv) $\int e^{3x}\left(\frac{3\sin x-\cos x}{\sin^2 x}\right)dx$

(v) $\int\frac{xe^x}{(1+x)^2}dx$ (vi) $\int\frac{xe^x}{(1+x)^2}dx$

(vii) $\int e^{-x}\left(\cos x-\sin x\right)dx$

(viii) $\int\frac{e^{m\tan^{-1}x}}{1+x^2}dx$ (ix) $\int\frac{2x}{1-\sin x}dx$

(x) $\int\frac{e^x(1+x)}{(2+x)^2}dx$ (xi) $\int\left(\frac{1-\sin x}{1-\cos x}\right)e^x\,dx$

Solution

(i) Let $I=\int e^x\left(\frac{1}{x}+\ln x\right)dx$

$$=\int e^x\left(\ln x+\frac{1}{x}\right)dx$$

Put $f(x)=\ln x\ \Rightarrow\ f'(x)=\frac{1}{x}$

So $I=\int e^x\left(f(x)+f'(x)\right)dx$

$$=e^xf(x)+c\quad=e^x\ln x+c$$

(ii) Let $I=\int e^x\left(\cos x+\sin x\right)dx$

$$=\int e^x\left(\sin x+\cos x\right)dx$$

Put $f(x)=\sin x\ \Rightarrow\ f'(x)=\cos x$

So $I=\int e^x\left(f(x)+f'(x)\right)dx$

$$=e^xf(x)+c$$

$$=e^x\sin x+c$$

(iii) Let $I=\int e^{ax}\left[a\sec^{-1}x+\frac{1}{x\sqrt{x^2-1}}\right]dx$

Put $f(x)=\sec^{-1}x\ \Rightarrow\ f'(x)=\frac{1}{x\sqrt{x^2-1}}$

So $I=\int e^{ax}\left[a\,f(x)+f'(x)\right]dx$

$$=e^{ax}f(x)+c$$

$$=e^{ax}\sec^{-1}x+c$$

(iv) Let $I=\int e^{3x}\left(\frac{3\sin x-\cos x}{\sin^2 x}\right)dx$

$$=\int e^{3x}\left(\frac{3\sin x}{\sin^2 x}-\frac{\cos x}{\sin^2 x}\right)dx$$

$$=\int e^{3x}\left(3\frac{1}{\sin x}-\frac{\cos x}{\sin x\cdot\sin x}\right)dx$$

$$=\int e^{3x}\left(3\csc x-\csc x\cot x\right)dx$$

Put $f(x)=\csc x\ \Rightarrow\ f'(x)=-\csc x\cot x$

$\Rightarrow\ I=\int e^{3x}\left(3f(x)+f'(x)\right)dx$

$$=e^{3x}f(x)+c$$

$$=e^{3x}\csc x+c$$

(v) Let $I=\int e^{2x}\left(-\sin x+2\cos x\right)dx$

$$=\int e^{2x}\left(2\cos x-\sin x\right)dx$$

Put $f(x)=\cos x\ \Rightarrow\ f'(x)=-\sin x$

So $I=\int e^{2x}\left(2f(x)+f'(x)\right)dx$

$$= e^{2x} f(x) + c$$
$$= e^{2x} \cos x + c$$

(vi) Let $I = \int \frac{xe^x}{(1+x)^2} dx$

$$= \int \frac{(1+x-1)e^x}{(1+x)^2} dx$$
$$= \int e^x \left[\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2} \right] dx$$
$$= \int e^x \left[\frac{1}{(1+x)} - \frac{1}{(1+x)^2} \right] dx$$

Put $f(x) = \frac{1}{1+x} = (1+x)^{-1}$

$\Rightarrow f'(x) = -(1+x)^{-2} = -\frac{1}{(1+x)^2}$

So $I = \int e^x \big(f(x) + f'(x) \big) dx$

$$= e^x f(x) + c$$
$$= e^x \left(\frac{1}{1+x} \right) + c$$

(vii) Let $I = \int e^{-x} (\cos x - \sin x) dx$

$$= \int e^{-x} \big((-1)\sin x + \cos x \big) dx$$

Put $f(x) = \sin x \quad \Rightarrow \quad f'(x) = \cos x$

So $I = \int e^{-x} \big((-1) f(x) + f'(x) \big) dx$

$$= e^{-x} f(x) + c$$
$$= e^{-x} \sin x + c$$

(viii) Let $I = \int \frac{e^{m\tan^{-1}x}}{1+x^2} dx$

$$= \int e^{m\tan^{-1}x} \cdot \frac{1}{1+x^2} dx$$

Put $t = \tan^{-1} x \quad \Rightarrow \quad dt = \frac{1}{1+x^2} dx$

So $I = \int e^{mt} dt$

$$= \frac{e^{mt}}{m} + c$$
$$= \frac{1}{m} e^{m\tan^{-1}x} + c$$

Important Integral

Let $I = \int \tan x \, dx$

$$= \int \frac{\sin x}{\cos x} dx$$

Put $t = \cos x \quad \Rightarrow \quad dt = -\sin x \, dx$

$\Rightarrow \quad -dt = \sin x \, dx$

So $I = \int \frac{-dt}{t} = -\int \frac{dt}{t}$

$$= -\ln|t| + c$$
$$= -\ln|\cos x| + c$$
$$= \ln|\cos x|^{-1} + c \qquad \because \; m\ln x = \ln x^m$$
$$= \ln\left| \frac{1}{\cos x} \right| + c \quad = \ln|\sec x| + c$$
$$\Rightarrow \quad \boxed{\int \tan x \, dx = \ln|\sec x| + c}$$

Similarly, we have

$$\boxed{\int \cot x \, dx = \ln|\sin x| + c}$$

(ix) Let $I = \int \frac{2x}{1-\sin x} dx$

$$= \int \frac{2x}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x} dx$$
$$= \int \frac{2x(1+\sin x)}{1-\sin^2 x} dx$$
$$= \int \frac{2x + 2x\sin x)}{\cos^2 x} dx$$
$$= \int \left(\frac{2x}{\cos^2 x} + \frac{2x\sin x}{\cos^2 x} \right) dx$$
$$= \int \frac{2x}{\cos^2 x} dx + \int \frac{2x\sin x}{\cos x \cdot \cos x} dx$$
$$= 2\int x\sec^2 x \, dx + 2\int x\sec x \tan x \, dx$$

Integrating by parts

$$I = 2\left[x\cdot\tan x - \int \tan x \cdot 1 \, dx \right] + 2\left[x\cdot\sec x - \int \sec x(1) \, dx \right]$$
$$= 2\left[x\cdot\tan x - \ln|\sec x| \right] + 2\left[x\cdot\sec x - \ln|\sec x + \tan x| \right] + c$$
$$= 2x\tan x - 2\ln|\sec x| + 2x\sec x - 2\ln|\sec x + \tan x| + c$$

(x) Let $I = \int \frac{e^x(1+x)}{(2+x)^2} dx$

$$=\int\frac{e^x(2+x-1)}{(2+x)^2}dx$$

$$=\int e^x\left(\frac{2+x}{(2+x)^2}-\frac{1}{(2+x)^2}\right)dx$$

$$=\int e^x\left((2+x)^{-1}-(2+x)^{-2}\right)dx$$

Put $f(x)=(2+x)^{-1} \quad\Rightarrow\quad f'(x)=-(2+x)^{-2}$

So $I=\int e^x\left(f(x)+f'(x)\right)dx$

$$=e^x f(x)+c$$

$$=e^x(2+x)^{-1}+c$$

$$=\frac{e^x}{2+x}+c$$

(xi) Let $I=\int\left(\frac{1-\sin x}{1-\cos x}\right)e^x\,dx$

$$=\int\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)e^x\,dx$$

$$=\int\left(\frac{1}{2\sin^2\frac{x}{2}}-\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)e^x\,dx$$

$$=\int\left(\frac{1}{2}\text{cosec}^2\frac{x}{2}-\cot\frac{x}{2}\right)e^x\,dx$$

$$=\int e^x\left(-\cot\frac{x}{2}+\frac{1}{2}\text{cosec}^2\frac{x}{2}\right)dx$$

Put $f(x)=-\cot\frac{x}{2} \quad\Rightarrow\quad f'(x)=\text{cosec}^2\frac{x}{2}\cdot\frac{1}{2}$

$$\Rightarrow\quad f'(x)=\frac{1}{2}\text{cosec}^2\frac{x}{2}$$

So $I=\int e^x\left(f(x)+f'(x)\right)$

$$=e^x f(x)+c$$

$$=e^x\left(-\cot\frac{x}{2}\right)+c$$

$$=-e^x\cot\frac{x}{2}+c.$$

EXERCISE 3.5

Q.1 $\int \frac{3x+1}{x^2-x-6}\,dx$

Solution:

$$\int \frac{3x+1}{x^2-x-6}\,dx$$

$$= \int \frac{3x+1 \qquad dx}{x^2-3x+2x-6}$$

$$= \int \frac{3x+1}{x(x-3)+2(x-3)}\,dx$$

$$= \int \frac{3x+1}{(x+2)(x-3)}\,dx$$

Let

$$\frac{3x+1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3} \quad \text{—— (1)}$$

Where A & B are constant of partial fractions which are to be determined.

Multiplying $(x+2)(x-3)$ on both sides in (1)

$$3x+1 = A(x-3) + B(x+2) \quad \text{———— (2)}$$

<u>To find A</u>

Put $x+2=0$

$x=-2$ in (2)

$$3(-2)+1 = A(-2-3)$$

$$-6+1 = A(-5)$$

$$-5 = -5A$$

$$A = \frac{-5}{-5}$$

$$\boxed{A = 1}$$

<u>To find B</u>

Put $x-3=0$

$$x = 3 \text{ in } (2)$$

$$\begin{aligned} 3(3) + 1 &= B(3 + 2) \\ 9 + 1 &= 5B \\ 5B &= 10 \\ B &= \frac{10}{5} \end{aligned}$$

∴ From eq. (1)

$$\frac{3x + 1}{(x + 2)(x - 3)} = \frac{1}{x + 2} + \frac{2}{x - 3}$$

$$\int \frac{3x + 1}{(x + 2)(x - 3)}\,dx = \int \frac{dx}{x + 2} + 2\int \frac{dx}{x - 3}$$

$$= \boxed{ln|x + 2| + 2ln|x - 3| + c}$$ Ans.

Q.2 $\displaystyle \int \frac{\mathbf{5x + 8}}{\mathbf{(x + 3)(2x - 1)}}\,\mathbf{dx}$

Solution:

$$\int \frac{5x + 8}{(x + 3)(2x - 1)}\,dx$$

Let

$$\frac{5x + 8}{(x + 3)(2x - 1)} = \frac{A}{x + 3} + \frac{B}{2x - 1} \quad \text{———} \quad (1)$$

Where A & B are constant of partial fractions which are to be determined.

Multiplying (x + 3) (2x – 1) on both sides in (1)

$$5x + 8 = A(2x - 1) + B(x + 3) \quad \text{———} \quad (2)$$

To find A

Put $x + 3 = 0$

$$x = -3 \text{ in } (2)$$

$$\begin{aligned} 5(-3) + 8 &= A[2(-3) - 1] \\ -15 + 8 &= A(-6 - 1) \\ -7 &= -7A \\ A &= \frac{-7}{-7} \end{aligned}$$

$$\boxed{A = 1}$$

To find B

Put $2x - 1 = 0$

$$2x = 1$$

$$x = \frac{1}{2} \text{ in } (2)$$

$$5\left(\frac{1}{2}\right) + 8 = B\left(\frac{1}{2} + 3\right)$$

$$\frac{5}{2} + 8 = B\left(\frac{1+6}{2}\right)$$

$$\frac{5+16}{2} = \left(\frac{7}{2}\right) B$$

$$\frac{21}{2} = \frac{7}{2} B$$

$$\frac{21}{2} \times \frac{2}{7} = B$$

$$\boxed{B = 3}$$

$\therefore$ From eq. (1)

$$\frac{5x+8}{(x+3)(2x-1)} = \frac{1}{x+3} + \frac{3}{2x-1}$$

Integrate

$$\int \frac{5x+8 \quad dx}{(x+3)(2x-1)} = \int \frac{dx}{x+3} + \frac{3}{2}\int \frac{2dx}{2x-1}$$

$$= \boxed{ln\,|x+3| + \frac{3}{2}\, ln\,|2x-1| + c} \qquad \text{Ans.}$$

Q.3 $\displaystyle \int \frac{\mathbf{x^2 + 3x - 34}}{\mathbf{x^2 + 2x - 15}}\ \mathbf{dx}$ **(Guj. Board 2006)**

Solution:

$$\int \frac{x^2 + 3x - 34}{x^2 + 2x - 15}\ dx$$

$$= \int \left(1 + \frac{x - 19}{x^2 + 2x - 15}\right) dx$$

$$\begin{array}{r@{\,}l} & \quad 1 \\ x^2 + 2x - 15 & \overline{\big)\, x^2 + 3x - 34} \\ & \underline{\pm x^2 \pm 2x \mp 15} \\ & \qquad x - 19 \end{array}$$

$$= \int dx + \int \frac{x - 19}{x^2 + 5x - 3x - 15}\ dx$$

$$= x + \int \frac{x - 19}{x(x+5) - 3(x+5)}\ dx$$

$$= x + \int \frac{x - 19}{(x-3)(x+5)}\ dx$$

$$= \quad x + I \quad \text{———} \quad (1)$$

$$I = \int \frac{x-19}{(x-3)(x+5)}\,dx$$

Let

$$\frac{x-19}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5} \quad \text{———} \quad (2)$$

Where A & B are constant of potential fractions which are to be determined.

Multiplying $(x-3)(x+5)$ on both sides in (2)

$$X - 19 = A(x+5) + B(x-3) \quad \text{———} \quad (3)$$

To find A

Put $x - 3 = 0$

$x = 3$ in (3)

$$3 - 19 = A(3+5)$$

$$-16 = 8A$$

$$A = \frac{-16}{8}$$

$$\boxed{A = -2}$$

To find B

Put $x + 5 = 0$

$x = -5$ in (3)

$$-5 - 19 = B(-5-3)$$

$$-24 = -8B$$

$$B = \frac{-24}{-8}$$

$$\boxed{B = 3}$$

$\therefore$ From eq. (2)

$$\frac{x-19}{(x-3)(x+5)} = \frac{-2}{x-3} + \frac{3}{x+5}$$

Integrate

$$\int \frac{x-19}{(x-3)(x+5)}\,dx = 2\int \frac{dx}{x-3} + 3\int \frac{dx}{x+5}$$

$$I = -2\,ln\,|x-3| + 3\,ln\,|x+5| + c$$

$\therefore$ From eq. (1)

$$= \boxed{x - 2ln\,|x-3| + 3ln\,|x+5| + c} \quad \text{Ans.}$$

Q.4 $\displaystyle\int \frac{(a-b)\,x}{(x-a)\,(x-b)}\,dx$

Solution:

$$\int \frac{(a-b)\,x}{(x-a)\,(x-b)}\,dx$$

Let

$$\frac{(a-b)x}{(x-a)\,(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \quad \text{———} \quad (1)$$

Where A & B are constant of partial fractions which are to be determined.

Multiplying (x – a) (x – b) on both sides in (1)

$$(a-b)\,x = A\,(x-b) + B\,(x-a) \quad \text{———} \quad (2)$$

To find A

Put $x - a = 0$

$x = a$ in (2)

$$(a-b)\,a = A\,(a-b)$$

$$\frac{(a-b)a}{a-b} = A$$

$$\boxed{A = a}$$

To find B

Put $x - b = 0$

$x = b$ in eq. (2)

$$(a-b)\,b = B\,(b-a)$$

$$\frac{(a-b)\,b}{b-a} = B$$

$$B = \frac{-(b-a)\,b}{b-a}$$

$$\boxed{B = -b}$$

From eq. (1)

$$\frac{(a-b)\,x}{(x-a)\,(x-b)} = \frac{a}{x-a} + \frac{-b}{x-b}$$

Integrate

$$\int \frac{(a-b)\,x}{(x-a)\,(x-b)}\,dx = a\int\frac{dx}{x-a} - b\int\frac{dx}{x-b}$$

$$= \boxed{a\,ln\,|x-a| - b\,ln\,|x-b| + c}$$ Ans.

Q.5 $\int \frac{3-x}{1-x-6x^2}\,dx$

Solution:

$$\int \frac{3-x}{1-x-6x^2}\,dx$$

$$= \int \frac{3-x}{1-3x+2x-6x^2}\,dx$$

$$= \int \frac{3-x}{(1-3x)+2x\,(1-3x)}\,dx$$

$$= \int \frac{3-x}{(1+2x)\,(1-3x)}\,dx$$

Let

$$\frac{3-x}{(1+2x)\,(1-3x)} = \frac{A}{1+2x} + \frac{B}{1-3x} \quad \text{———} \quad (1)$$

Where A and B are constant of partial fractions which are to be determined

Multiplying $(1+2x)\,(1-3x)$ on both sides in (1)

$3 - x \;=\; A\,(1-3x) + B\,(1+2x)$ ——— (2)

<u>To find A</u>

Put $1 + 2x = 0$

$2x = -1$

$x = \frac{-1}{2}$ in (2)

$$3 - \frac{-1}{2} = A\left[1 - 3\left(\frac{-1}{2}\right)\right]$$

$$3 + \frac{1}{2} = A\left(1 + \frac{3}{2}\right)$$

$$\frac{6+1}{2} = A\left(\frac{2+3}{2}\right)$$

$$\frac{7}{2} = \left(\frac{5}{2}\right)A$$

$$\frac{7\times 2}{2\times 5} = A$$

$$\boxed{A = \frac{7}{5}}$$

To find B

Put $1 - 3x = 0$

$3x = 1$

$x = \frac{1}{3}$ in (2)

$$3 - \frac{1}{3} = B\left(1 + 2\left(\frac{1}{3}\right)\right)$$

$$\frac{9-1}{3} = B\left(1 + \frac{2}{3}\right)$$

$$\frac{8}{3} = B\left(\frac{3+2}{3}\right)$$

$$\frac{8}{3} = \left(\frac{5}{3}\right)B$$

$$\frac{8}{3} \times \frac{3}{5} = B$$

$$\boxed{B = \frac{8}{5}}$$

∴ From eq. (1)

$$\frac{3-x}{(1+2x)(1-3x)} = \frac{\frac{7}{5}}{1+2x} + \frac{\frac{8}{5}}{1-3x}$$

Integrate

$$\int \frac{3-x}{(1+2x)(1-3x)}\,dx = \frac{7}{5}\int \frac{dx}{1+2x} + \frac{8}{5}\int \frac{dx}{1-3x}$$

$$= \frac{7}{5.2}\int \frac{2dx}{1+2x} - \frac{8}{5\times 3}\int \frac{-3}{1-3x}\,dx$$

$$= \boxed{\frac{7}{10}\ \ln|1+2x| - \frac{8}{5}\ \ln|1-3x| + c}$$ Ans.

Q.6 $\int \frac{2x}{x^2 - a^2}\,dx$

Solution:

$$\int \frac{2x}{x^2 - a^2}\,dx$$

$$= \int \frac{2x}{(x+a)(x-a)}\,dx$$

Let

$$\frac{2x}{(x+a)(x-a)} = \frac{A}{x+a} + \frac{B}{x-a} \quad \text{———} \quad (1)$$

Where A and B are constant of partial fractions which are to be determined

Multiplying (x + a) (x – a) on both sides in (1)

$$2x \quad = \quad A(x-a) + B(x+a) \text{———} \quad (2)$$

To find A

$$\text{Put} \quad x + a = 0$$

$$x = -a \quad \text{in } (2)$$

$$2(-a) = A(-a-a)$$

$$-2a = 2a\,A$$

$$A = \frac{-2a}{-2a}$$

$$\boxed{A = 1}$$

To find B

$$\text{Put} \quad x - a = 0$$

$$x = a \quad \text{in } (2)$$

$$2a = B(a+a)$$

$$2a = 2a\,B$$

$$B = \frac{2a}{2a}$$

$$\boxed{B = 1}$$

$\therefore$ From eq. (1)

$$\frac{2x}{(x+a)(x-a)} = \frac{1}{x+a} + \frac{1}{x-a}$$

Integrate

$$\int \frac{2x}{(x+a)(x-a)}\,dx = \int \frac{dx}{x+a} + \int \frac{dx}{x-a}$$

$$= \boxed{\ln|x+a| + \ln|x-a| + c} \qquad \text{Ans.}$$

Q.7 $\int \frac{1}{6x^2 - 5x - 4}\,dx$

Solution:

$$\int \frac{1}{6x^2 - 5x - 4}\,dx$$

$$= \int \frac{1}{x^2 + 8x - 3x - 4}\, dx$$

$$= \int \frac{1}{2x\,(3x + 4) - 1\,(3x + 4)}\, dx$$

$$= \int \frac{1}{(2x - 1)\,(3x + 4\,)}\, dx$$

Let

$$\frac{1}{(2x - 1)\,(3x + 4)} = \frac{A}{2x - 1} + \frac{B}{3x + 4} \text{ ———— } (1)$$

Where A and B are constant of partial fractions which are to be determined.

Multiplying $(2x - 1)\,(3x + 4)$ on both sides in (1)

$$1 = A\,(3x + 4) + B\,(2x - 1) \text{ ———— } (2)$$

<u>To find A</u>

Put $2x - 1 = 0$

$2x = 1$

$x = \frac{1}{2}$ in eq. (2)

$$1 = A\left(\frac{3}{2} + 4\right)$$

$$1 = A\left(\frac{3 + 8}{2}\right)$$

$$2 = A\,(11)$$

$$A = \frac{2}{11}$$

<u>To find B</u>

Put $3x + 4 = 0$

$3x = -4$

$x = \frac{-4}{3}$ in eq. (2)

$$1 = B\left[\,2\left(\frac{-4}{3}\right) - 1\,\right]$$

$$1 = B\left(\frac{-8}{3} - 1\right)$$

$$1 = B\left(\frac{-8-3}{3}\right)$$

$$3 = -11B$$

$$\boxed{B = \frac{-3}{11}}$$

$\therefore$ From eq. (1)

$$\frac{1}{(2x - 1)(3x + 4)} = \frac{\frac{2}{11}}{2x - 1} + \frac{\frac{-3}{11}}{3x + 4}$$

$$\int \frac{dx}{(2x - 1)(3x + 4)} = \frac{1}{11}\int \frac{2dx}{2x - 1} - \frac{1}{11}\int \frac{3}{3x + 4}\,dx$$

$$= \boxed{\frac{1}{11}\,ln\,|2x - 1| - \frac{1}{11}\,ln\,|3x + 4| + c} \quad \text{Ans.}$$

Q.8 $\displaystyle \int \frac{2x^3 - 3x^2 - x - 7}{2x^2 - 3x - 2}\,dx$

Solution:

$$\int \frac{2x^3 - 3x^2 - x - 7}{2x^2 - 3x - 2}\,dx$$

$$\because \begin{array}{r|l} & x \\ 2x^2 - 3x - 2 & 2x^3 - 3x^2 - x - 7 \\ & \underline{\underset{-}{2x^3} \underset{+}{\mp} 3x^2 \underset{+}{\mp} 2x} \\ & x - 7 \end{array}$$

$$= \int \left(x + \frac{x - 7}{2x^2 - 3x - 2}\right) dx$$

$$= \int xdx + \int \frac{x - 7}{2x^2 - 4x + x - 2}\,dx$$

$$= \frac{x^2}{2} + \int \frac{x - 7}{2x(x - 2) + 1(x - 2)}\,dx$$

$$= \frac{x^2}{2} + \int \frac{x - 7}{(2x + 1)(x - 2)}\,dx$$

$$= \frac{x^2}{2} + I \quad \text{———} \quad (1)$$

Where

$$I = \int \frac{x - 7}{(2x + 1)(x - 2)}\,dx$$

$$\frac{x - 7}{(2x + 1)(x - 2)} = \frac{A}{2x + 1} + \frac{B}{x - 2} \quad \text{———} \quad (2)$$

Where A & B are constant of partial fractions which are to be determined.

Multiplying (2x + 1) (x – 2) on both sides in (2)

$$x-7 = A(x-2)+B(2x+1) \quad \text{———} \quad (3)$$

To find A

$$\text{Put} \quad 2x+1 = 0$$
$$2x = -1$$
$$x = \frac{-1}{2} \text{ in } (3)$$
$$\frac{-1}{2}-7 = A\left(\frac{-1}{2}-2\right)$$
$$\frac{-1-14}{2} = A\left(\frac{-1-4}{2}\right)$$
$$\frac{-15}{2} = A\left(\frac{-5}{2}\right)$$
$$\frac{-15}{2} \times \frac{-2}{2} = A$$
$$\boxed{A = 3}$$

To find B

$$\text{Put} \quad x-2 = 0$$
$$x = 2 \text{ in } (3)$$
$$2-7 = B[2(2)+1]$$
$$-5 = B(4+1)$$
$$-5 = 5B$$
$$B = \frac{-5}{5}$$
$$\boxed{B = -1}$$

∴ From eq. (2)

$$\frac{x-7}{(2x+1)(x-2)} = \frac{3}{2x+1} + \frac{-1}{x-2}$$
$$\int \frac{x-7 \quad dx}{(2x+1)(x-2)} = \frac{3}{2}\int \frac{2dx}{2x+1} - \int \frac{dx}{x-2}$$
$$= \boxed{\frac{3}{2} \ln|2x+1| - \ln|x-2| + c} \quad \text{Ans.}$$

Q.9 $\displaystyle \int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} dx$

Solution:

$$\int \frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)}\,dx$$

Let $$\frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \quad \text{———} \quad (1)$$

Where A, B and C are constant of partial fractions which are to be determined.

Multiplying $(x-1)(x-2)(x-3)$ on both sides in (1)

$$3x^2 - 12x + 11 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \text{———}(2)$$

To find A

Put $x - 1 = 0$

$x = 1$ in (2)

$$3(1)^2 - 12(1) + 11 = A(1-2)(1-3)$$

$$3 - 12 + 11 = A(-1)(-2)$$

$$2 = 2A$$

$$A = \frac{2}{2}$$

$$\boxed{A = 1}$$

To find B

Put $x - 2 = 0$

$x = 2$ in (2)

$$3(2)^2 - 12(2) + 11 = B(2-1)(2-3)$$

$$3(4) - 24 + 11 = B(1)(-1)$$

$$12 - 24 + 11 = -B$$

$$-1 = -B$$

$$\boxed{B = 1}$$

To find C

∴ From eq. (1)

$$\frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}$$

Integrate

$$\int \frac{3x^2 - 12x + 11\quad dx}{(x-1)(x-2)(x-3)} = \int \frac{dx}{x-1} + \int \frac{dx}{x-2} + \int \frac{dx}{x-3}$$

$$= \boxed{\ln|x-1| + \ln|x-2| + \ln|x-3| + c} \quad \text{Ans.}$$

Q.10 $\displaystyle\int \frac{2x-1}{x(x-1)(x-3)}\,dx$

Solution:

$$\int \frac{2x-1}{x(x-1)(x-3)}\,dx$$

Let

$$\frac{2x-1}{x(x-1)(x-3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-3} \text{ ——— } (1)$$

Where A, B and C are constant of partial fractions which are to be determined.

Multiplying $x(x-1)(x-3)$ on both sides in (1)

$$2x - 1 = A(x-1)(x-3) + Bx(x-3) + Cx(x-1) \text{ —— } (2)$$

To find A

Put $x = 0$ in eq (2)

$$2(0) - 1 = A(0-1)(0-3)$$

$$-1 = A(-1)(-3)$$

$$-1 = 3A$$

$$\boxed{A = \frac{-1}{3}}$$

To find B

Put $x - 1 = 0$

$x = 1$ in (2)

$$2(1) - 1 = B(1)(1-3)$$

$$2 - 1 = B(-2)$$

$$1 = -2B$$

$$\boxed{B = \frac{-1}{2}}$$

To find C

Put $x - 3 = 0$

$x = 3$ in (2)

$$2(3) - 1 = C(3)(3-1)$$

$$6 - 1 = C(3)(2)$$

$$5 = 6C$$

$$\boxed{C = \frac{5}{6}}$$

∴ From eq. (1)

$$\frac{2x-1}{x(x-1)(x-3)} = \frac{\frac{-1}{3}}{x} + \frac{\frac{-1}{2}}{x-1} + \frac{\frac{5}{6}}{x-3}$$

Integrate $\int \frac{2x-1}{x(x-1)(x-3)}\,dx = \frac{-1}{3}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x-1} + \frac{5}{6}\int \frac{dx}{x-3}$

$$= \boxed{\frac{-1}{3}\ln|x| - \frac{1}{2}\ln|x-1| + \frac{5}{6}\ln|x-3| + c} \quad \text{Ans.}$$

Q.11 $\int \frac{5x^2+9x+6}{(x^2-1)(2x+3)}\,dx$

Solution:

$$\int \frac{5x^2+9x+6}{(x^2-1)(2x+3)}\,dx$$

$$= \int \frac{5x^2+9x+6}{(x+1)(x-1)(2x+3)}\,dx$$

Let

$$\frac{5x^2+9x+6}{(x+1)(x-1)(2x+3)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{2x+3} \quad \text{——— (1)}$$

Where A, B and C are constant of partial fractions which are to be determined.

Multiplying (x + 1) (x – 1) (2x + 3) on both sides in (1)

$$5x^2+9x+6 = A(x-1)(2x+3) + B(x+1)(2x+3) + C(x-1)(x-1) \text{— (2)}$$

To find A

Put $x + 1 = 0$

$x = 1$ in (2)

$$5(-1)^2 + 9(-1) + 6 = A(-1-1)(-2+3)$$

$$5 - 9 + 6 = A(-2)(1)$$

$$2 = -2A$$

$$A = \frac{2}{-2}$$

$$\boxed{A = -1}$$

To find B

Put $x - 1 = 0$

$x = 1$ in (2)

$$
\begin{aligned}
5(1)^2 + 9(1) + 6 &= B\,(1 + 1)\,(2 + 3) \\
5 + 9 + 6 &= B\,(2\,)\,(5) \\
20 &= 10\,B \\
B &= \frac{20}{10} \\
&\boxed{B = 2}
\end{aligned}
$$

To find C

$$
\begin{aligned}
\text{Put} \quad 2x + 3 &= 0 \\
2x &= -3 \\
x &= \frac{-3}{2} \text{ in (2)}
\end{aligned}
$$

$$
\begin{aligned}
5\left(\frac{-3}{2}\right)^2 + 9\left(\frac{-3}{2}\right) + 6 &= C\left(\frac{-3}{2} + 1\right)\left(\frac{-3}{2} - 1\right) \\
5\left(\frac{9}{4}\right) - \frac{27}{2} + 6 &= C\left(\frac{-3+2}{2}\right)\left(\frac{-3-2}{2}\right) \\
\frac{45}{4} - \frac{27}{2} + 6 &= C\left(\frac{-1}{2}\right)\left(\frac{-5}{2}\right) \\
\frac{45 - 54 + 24}{4} &= \frac{5}{4}\,C \\
\frac{15}{4} \times \frac{4}{5} &= C \\
&\boxed{C = 3}
\end{aligned}
$$

∴ From eq. (1)

$$\frac{5x^2 + 9x + 6}{(x + 1)\,(x - 1)\,(2x + 3)} = \frac{-1}{x + 1} + \frac{2}{x - 1} + \frac{3}{2x + 3}$$

Integrate

$$\int \frac{5x^2 + 9x + 6}{(x + 1)\,(x - 1)\,(2x + 3)}\,dx = -\int \frac{dx}{x + 1} + 1\int \frac{dx}{x - 1} + \frac{3}{2}\int \frac{2}{2x + 3}\,dx$$

$$= \boxed{ln\,|x+1| + 2ln\,|x - 1| + \frac{3}{2}\,ln\,|2x + 3| \; + \; c} \quad \text{Ans.}$$

Q.12 $\displaystyle\int \frac{\mathbf{4 + 7x}}{\mathbf{(1 + x)^2\,(2 + 3x)}}\,\mathbf{dx}$

Solution:

$$\int \frac{4+7x}{(1+x)^2\,(2+3x)}\,dx$$

Let

$$\frac{4+7x}{(2+3x)(1+x)^2} = \frac{A}{2+3x} + \frac{B}{1+x} + \frac{C}{(1+x)^2} \quad \text{———} \quad (1)$$

Where A, B and C are constant of partial fractions which are to be determined.

Multiplying $(2+3x)\,(1+x)^2$ on both sides in (1)

$$4+7x = A\,(1+x)^2 + B(2+3x)\,(1+x) + C\,(2+3x) \text{ ——}(2)$$

$$4+7x = A\,(1+x^2+2x) + B(2+5x+3x^2) + 2C + 3Cx$$

$$4+7x = (A+3B)\,x^2 + (2A+5B+3C)\,x + (A+2B+2C) \text{ ——} \quad (3)$$

<u>To find A</u>

Put $2 + 3x = 0$

$3x = -2$

$x = \frac{-2}{3}$ in (3)

$$4+7\left(\frac{-2}{3}\right) = A\,(1-\frac{2}{3})^2$$

$$4-\frac{14}{3} = A\left(\frac{3-2}{3}\right)^2$$

$$\frac{12-14}{3} = A\left(\frac{1}{3}\right)^2$$

$$\frac{-2}{3} = \frac{1}{9}A$$

$$\frac{-2\times 9}{3} = A$$

$$\boxed{A = -6}$$

<u>To find C</u>

Put $1 + x = 0$

$x = -1$ in (2)

$$4+7\,(-1) = C\,(2-3)$$

$$4-7 = C\,(-1)$$

$$-3 = -C$$

$$\boxed{C = 3}$$

To find B comparing the coefficient of x^2 in (3)

$$A + 3B = 0$$

Definite Integral

Let $\int f(x)dx = \varphi(x)+c$

Then $\int_a^b f(x)\,dx = |\varphi(x)|_a^b$ or $[\varphi(x)]_a^b$

$$= \varphi(b)-\varphi(a)$$

Also

- $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
- $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$

where $a<c<b$

Evaluate the following definite integrals:

Question # 1

$$\int_1^2 \left(x^2+1\right) dx$$

Solution

$$\int_1^2 \left(x^2+1\right) dx$$

$$= \int_1^2 x^2\ dx + \int_1^2 dx$$

$$= \left|\frac{x^3}{3}\right|_1^2 + |x|_1^2 = \left(\frac{2^3}{3}-\frac{1^3}{3}\right)+(2-1)$$

$$= \frac{8}{3}-\frac{1}{3}+1 \quad = \frac{10}{3}$$

Question # 2

$$\int_{-1}^1 \left(x^{\frac{1}{3}}+1\right) dx$$

Solution

$$\int_{-1}^1 \left(x^{\frac{1}{3}}+1\right) dx$$

$$= \int_{-1}^1 x^{\frac{1}{3}} dx + \int_{-1}^1 dx$$

$$= \left|\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right|_{-1}^1 + |x|_{-1}^1$$

$$= \left|\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right|_{-1}^1 + (1-(-1))$$

$$= \frac{3}{4}\left((1)^{\frac{4}{3}} - (-1)^{\frac{4}{3}}\right) + (1+1)$$

$$= \frac{3}{4}(1-1)+2 \quad = 2$$

Question # 3

$$\int_{-2}^0 \frac{1}{(2x-1)^2} dx$$

Solution

$$\int_{-2}^0 \frac{1}{(2x-1)^2} dx$$

$$= \int_{-2}^0 (2x-1)^{-2}\ dx$$

$$= \left|\frac{(2x-1)^{-2+1}}{(-2+1)\cdot 2}\right|_{-2}^0$$

$$= \left|\frac{(2x-1)^{-1}}{(-1)\ 2}\right|_{-2}^0$$

$$= \frac{(2(0)-1)^{-1}}{-2} - \frac{(2(-2)-1)^{-1}}{-2}$$

$$= \frac{(0-1)^{-1}}{-2} - \frac{(-4-1)^{-1}}{-2}$$

$$= \frac{(-1)^{-1}}{-2} - \frac{(-5)^{-1}}{-2}$$

$$= \frac{1}{(-2)(-1)} - \frac{1}{(-2)(-5)}$$

$$= \frac{1}{2} - \frac{1}{10} \quad = \frac{2}{5}$$

Question # 4

$$\int_{-6}^2 \sqrt{3-x}\,dx$$

Solution

$$\int_{-6}^{2}\sqrt{3-x}\,dx$$

$$=\int_{-6}^{2}(3-x)^{\frac{1}{2}}\,dx$$

$$=\left|\frac{(3-x)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)(-1)}\right|_{-6}^{2} \quad =\left|\frac{(3-x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(-1)}\right|_{-6}^{2}$$

$$=-\frac{2}{3}\left|(3-x)^{\frac{3}{2}}\right|_{-6}^{2}$$

$$=-\frac{2}{3}\left((3-2)^{\frac{3}{2}}-(3+6)^{\frac{3}{2}}\right)$$

$$=-\frac{2}{3}\left((1)^{\frac{3}{2}}-(9)^{\frac{3}{2}}\right) \quad =-\frac{2}{3}(1-27) \quad =\frac{52}{3}.$$

Question # 5

$$\int_{1}^{\sqrt{5}}\sqrt{(2t-1)^{3}}\,dt$$

Solution

$$\int_{1}^{\sqrt{5}}\sqrt{(2t-1)^{3}}\,dt$$

$$=\int_{1}^{\sqrt{5}}(2t-1)^{\frac{3}{2}}\,dt$$

$$=\left|\frac{(2t-1)^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right)\cdot 2}\right|_{1}^{\sqrt{5}} \quad =\left|\frac{(2t-1)^{\frac{5}{2}}}{\left(\frac{5}{2}\right)\cdot 2}\right|_{1}^{\sqrt{5}}$$

$$=\left|\frac{(2t-1)^{\frac{5}{2}}}{5}\right|_{1}^{\sqrt{5}} =\frac{(2\sqrt{5}-1)^{\frac{5}{2}}}{5}-\frac{(2(1)-1)^{\frac{5}{2}}}{5}$$

$$=\frac{(2\sqrt{5}-1)^{\frac{5}{2}}}{5}-\frac{1}{5}$$

$$=\frac{\sqrt{(2\sqrt{5}-1)^{5}}}{5}-\frac{1}{5} \qquad \textit{Ans.}$$

Question # 6

$$\int_{2}^{\sqrt{5}}x\sqrt{x^{2}-1}\,dx$$

Solution

$$\int_{2}^{\sqrt{5}}x\sqrt{x^{2}-1}\,dx$$

$$=\int_{2}^{\sqrt{5}}\left(x^{2}-1\right)^{\frac{1}{2}}\cdot x\,dx$$

$$=\frac{1}{2}\int_{2}^{\sqrt{5}}\left(x^{2}-1\right)^{\frac{1}{2}}\cdot 2x\,dx$$

$$=\frac{1}{2}\int_{2}^{\sqrt{5}}\left(x^{2}-1\right)^{\frac{1}{2}}\cdot\frac{d}{dx}\left(x^{2}-1\right)\,dx$$

$$=\frac{1}{2}\left|\frac{\left(x^{2}-1\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{2}^{\sqrt{5}} \quad =\frac{1}{2}\left|\frac{\left(x^{2}-1\right)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{2}^{\sqrt{5}}$$

$$=\frac{1}{2}\cdot\frac{2}{3}\left[\left(\left(\sqrt{5}\right)^{2}-1\right)^{\frac{3}{2}}-\left((2)^{2}-1\right)^{\frac{3}{2}}\right]$$

$$=\frac{1}{3}\left[(5-1)^{\frac{3}{2}}-(4-1)^{\frac{3}{2}}\right] \quad =\frac{1}{3}\left[(4)^{\frac{3}{2}}-(3)^{\frac{3}{2}}\right]$$

$$=\frac{1}{3}\left[\left(2^{2}\right)^{\frac{3}{2}}-(3)^{1+\frac{1}{2}}\right] \quad =\frac{1}{3}\left[(2)^{3}-3(3)^{\frac{1}{2}}\right]$$

$$=\frac{1}{3}\left[8-3\sqrt{3}\right]$$

Question # 7

$$\int_{1}^{2}\frac{x}{x^{2}+2}\,dx$$

Solution

$$\int_{1}^{2}\frac{x}{x^{2}+2}\,dx$$

$$=\frac{1}{2}\int_{1}^{2}\frac{2x}{x^{2}+2}\,dx$$

$$=\frac{1}{2}\int_{1}^{2}\frac{\frac{d}{dx}\left(x^{2}+2\right)}{x^{2}+2}\,dx \quad =\frac{1}{2}\left|\ln\left|x^{2}+2\right|\right|_{1}^{2}$$

$$=\frac{1}{2}\left(\ln\left|2^{2}+2\right|-\ln\left|1^{2}+2\right|\right)$$

$$=\frac{1}{2}\left(\ln 6-\ln 3\right)$$

$$=\frac{1}{2}\ln\left(\frac{6}{3}\right) \quad =\frac{1}{2}\ln 2$$

Question # 8

$$\int_2^3 \left(x-\frac{1}{x}\right)^2 \, dx$$

Solution

$$\int_2^3 \left(x-\frac{1}{x}\right)^2 \, dx \;= \int_2^3 \left(x^2+\frac{1}{x^2}-2\right) dx$$

$$= \int_2^3 x^2 \, dx + \int_2^3 x^{-2} \, dx - 2\int_2^3 dx$$

Now do yourself

Question # 9

$$\int_{-1}^{1} \left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} \; dx$$

Solution

$$\int_{-1}^{1} \left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} \; dx$$

$$= \int_{-1}^{1} \left(\frac{2x+1}{2}\right)\left(x^2+x+1\right)^{\frac{1}{2}} \; dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(x^2+x+1\right)^{\frac{1}{2}}(2x+1) \; dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(x^2+x+1\right)^{\frac{1}{2}}\frac{d}{dx}(2x+1) \; dx$$

$$= \frac{1}{2}\left|\frac{\left(x^2+x+1\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{-1}^{1}$$

NOTE

$$(3)^{\frac{3}{2}} = (3)^{1+\frac{1}{2}}$$

$$= 3^1 \cdot 3^{\frac{1}{2}} = 3\sqrt{3}$$

$$= \frac{1}{2}\left|\frac{\left(x^2+x+1\right)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{-1}^{1}$$

$$= \frac{1}{3}\left|\left(x^2+x+1\right)^{\frac{3}{2}}\right|_{-1}^{1}$$

$$= \frac{1}{3}\left[\left((1)^2+(1)+1\right)^{\frac{3}{2}} - \left((-1)^2+(-1)+1\right)^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}\left[(1+1+1)^{\frac{3}{2}} - (1-1+1)^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}\left[(3)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] \;= \frac{1}{3}\left[3\sqrt{3}-1\right]$$

$$= \sqrt{3} - \frac{1}{3}$$

Question # 10

$$\int_0^3 \frac{dx}{x^2+9}$$

Solution

$$\int_0^3 \frac{dx}{x^2+9} \;= \int_0^3 \frac{dx}{x^2+3^2}$$

$$= \left|\frac{1}{3}Tan^{-1}\frac{x}{3}\right|_0^3$$

$$= \frac{1}{3}Tan^{-1}\left(\frac{3}{3}\right) - \frac{1}{3}Tan^{-1}\left(\frac{0}{3}\right)$$

$$= \frac{1}{3}Tan^{-1}(1) - \frac{1}{3}Tan^{-1}(0)$$

$$= \frac{1}{3}\left(\frac{\pi}{4}\right) - \frac{1}{3}(0) \;= \frac{\pi}{12}$$

Question # 11

$$\int_{\pi/6}^{\pi/3} \cos t dt$$

Solution *Do yourself*

Question # 12

$$\int_1^2 \left(x+\frac{1}{x}\right)^{\frac{1}{2}}\left(1-\frac{1}{x^2}\right) dx$$

Solution *Do yourself*

Question # 13

$$\int_1^2 \ln x \; dx$$

Solution

Let $I = \int_1^2 \ln x \; dx \;= \int_1^2 \ln x \cdot 1 \; dx$

Integrating by parts

$$I = \left|\ln x \cdot x\right|_1^2 - \int_1^2 x \cdot \frac{1}{x} dx$$

$$= \left|x \ln x\right|_1^2 - \int_1^2 dx$$

$$= \left(2\cdot\ln 2 - 1\cdot\ln 1\right) - \left|x\right|_1^2$$

$$= \left(2\cdot\ln 2 - 1\cdot(0)\right) - (2-1)$$

$$=(2\cdot\ln 2-0)-1=2\ln 2-1$$

Question # 14

$$\int_0^2\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)\,dx$$

Solution

$$\int_0^2\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)\,dx$$

$$=\int_0^2 e^{\frac{x}{2}}\,dx-\int_0^2 e^{-\frac{x}{2}}\,dx$$

$$=\left|\frac{e^{\frac{x}{2}}}{1/2}\right|_0^2-\left|\frac{e^{-\frac{x}{2}}}{-1/2}\right|_0^2\quad=2\left|e^{\frac{x}{2}}\right|_0^2+2\left|e^{-\frac{x}{2}}\right|_0^2$$

$$=2\left(e^{\frac{2}{2}}-e^{\frac{0}{2}}\right)+2\left(e^{-\frac{2}{2}}-e^{-\frac{0}{2}}\right)$$

$$=2\left(e^{1}-e^{0}\right)+2\left(e^{-1}-e^{0}\right)$$

$$=2\left(e-1+\frac{1}{e}-1\right)\quad=2\left(e+\frac{1}{e}-2\right)$$

$$=2\left(\frac{e^2+1-2e}{e}\right)\quad=2\frac{(e-1)^2}{e}$$

Question # 15

$$\int_0^{\pi/4}\frac{\cos\theta+\sin\theta}{\cos 2\theta+1}\,d\theta$$

Solution

$$\text{Let } I=\int_0^{\pi/4}\frac{\cos\theta+\sin\theta}{\cos 2\theta+1}\,d\theta$$

$$=\int_0^{\pi/4}\frac{\cos\theta+\sin\theta}{2\cos^2\theta}\,d\theta$$

$$\because\ \cos^2\theta=\frac{1+\cos 2\theta}{2}$$

$$=\int_0^{\pi/4}\left(\frac{\cos\theta}{2\cos^2\theta}++\frac{\sin\theta}{2\cos^2\theta}\right)d\theta$$

$$=\int_0^{\pi/4}\frac{1}{2\cos\theta}\,d\theta+\int_0^{\pi/4}\frac{\sin\theta}{2\cos\theta\cdot\cos\theta}\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/4}\sec\theta\,d\theta+\frac{1}{2}\int_0^{\pi/4}\sec\theta\tan\theta\,d\theta$$

$$=\frac{1}{2}\Big|\ln|\sec\theta+\tan\theta|\Big|_0^{\pi/4}+\frac{1}{2}\Big|\sec\theta\Big|_0^{\pi/4}$$

$$=\frac{1}{2}\left(\ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|-\ln|\sec(0)+\tan(0)|\right)+\frac{1}{2}\left(\sec\frac{\pi}{4}-\sec(0)\right)$$

$$=\frac{1}{2}\left(\ln\left|\sqrt{2}+1\right|-\ln|1+0|\right)+\frac{1}{2}\left(\sqrt{2}-1\right)$$

$$=\frac{1}{2}\left(\ln\left|\sqrt{2}+1\right|-0\right)+\frac{1}{2}\left(\sqrt{2}-1\right)$$

$$=\frac{1}{2}\left(\ln\left|\sqrt{2}+1\right|+\sqrt{2}-1\right)\qquad \textit{Ans.}$$

Question # 16

$$\int_0^{\pi/6}\cos^3\theta\,d\theta$$

Solution

$$\int_0^{\pi/6}\cos^3\theta\,d\theta\ =\int_0^{\pi/6}\cos^2\theta\cdot\cos\theta\,d\theta$$

$$=\int_0^{\pi/6}\left(1-\sin^2\theta\right)\cos\theta\,d\theta$$

$$=\int_0^{\pi/6}\cos\theta\,d\theta-\int_0^{\pi/6}\sin^2\theta\cos\theta\,d\theta$$

$$=\Big|\sin\theta\Big|_0^{\pi/6}-\int_0^{\pi/6}\sin^2\theta\frac{d}{d\theta}\sin\theta\,d\theta$$

$$=\left(\sin\frac{\pi}{6}-\sin(0)\right)-\left|\frac{\sin^3\theta}{3}\right|_0^{\pi/6}$$

$$=\left(\frac{1}{2}-0\right)-\frac{1}{3}\left(\sin^3\frac{\pi}{6}-\sin^3(0)\right)$$

$$=\frac{1}{2}-\frac{1}{3}\left(\left(\frac{1}{2}\right)^2-(0)^3\right)$$

$$=\frac{1}{2}-\frac{1}{3}\left(\frac{1}{8}\right)\quad=\frac{1}{2}-\frac{1}{24}\quad=\frac{11}{24}$$

Question # 17

$$\int_{\pi/6}^{\pi/4}\cos^2\theta\cdot\cot^2\theta\,d\theta$$

Solution

$$\int_{\pi/6}^{\pi/4} \cos^2\theta\cdot\cot^2\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/4} \cos^2\theta\left(\operatorname{cosec}^2\theta-1\right) d\theta$$

$$= \int_{\pi/6}^{\pi/4} \left(\cos^2\theta\operatorname{cosec}^2\theta-\cos^2\theta\right) d\theta$$

$$= \int_{\pi/6}^{\pi/4} \left(\cos^2\theta\frac{1}{\sin^2\theta}-\cos^2\theta\right) d\theta$$

$$= \int_{\pi/6}^{\pi/4} \cot^2\theta\, d\theta-\int_{\pi/6}^{\pi/4} \cos^2\theta\, d\theta$$

$$= \int_{\pi/6}^{\pi/4} \left(\operatorname{cosec}^2\theta-1\right) d\theta-\int_{\pi/6}^{\pi/4} \left(\frac{1+\cos\theta}{2}\right) d\theta$$

$$= \int_{\pi/6}^{\pi/4} \csc^2\theta\, d\theta-\int_{\pi/6}^{\pi/4} d\theta-\frac{1}{2}\int_{\pi/6}^{\pi/4} d\theta-\frac{1}{2}\int_{\pi/6}^{\pi/4} \cos 2\theta d\theta$$

$$= \left|-\cot\theta\right|_{\pi/6}^{\pi/4}-\frac{3}{2}\int_{\pi/6}^{\pi/4} d\theta-\frac{1}{2}\left|\frac{\sin 2\theta}{2}\right|_{\pi/6}^{\pi/4}$$

$$= \left(-\cot\frac{\pi}{4}+\cot\frac{\pi}{6}\right)-\frac{3}{2}\left|\theta\right|_{\pi/6}^{\pi/4} -\frac{1}{2}\left(\frac{\sin 2\left(\pi/4\right)}{2}-\frac{\sin 2\left(\pi/6\right)}{2}\right)$$

$$= \left(-1+\sqrt{3}\right)-\frac{3}{2}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{\sqrt{3}/2}{2}\right)$$

$$= \left(-1+\sqrt{3}\right)-\frac{3}{2}\left(\frac{\pi}{12}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{\sqrt{3}}{4}\right)$$

$$= -1+\sqrt{3}-\frac{\pi}{8}-\frac{1}{4}+\frac{\sqrt{3}}{8} \qquad = -\frac{5}{4}+\frac{9}{8}\sqrt{3}-\frac{\pi}{8}$$

$$= \frac{9\sqrt{3}-10-\pi}{8}$$

Question # 18

$$\int_{0}^{\pi/4} \cos^4 t\, dt$$

Solution

$$\int_{0}^{\pi/4} \cos^4 t\, dt \quad = \int_{0}^{\pi/4} \left(\cos^2 t\right)^2 dt$$

$$= \int_{0}^{\pi/4} \left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$$= \int_{0}^{\pi/4} \left(\frac{1+2\cos 2t+\cos^2 2t}{4}\right) dt$$

$$= \frac{1}{4}\int_{0}^{\pi/4} \left(1+2\cos 2t+\cos^2 2t\right) dt$$

$$= \frac{1}{4}\int_{0}^{\pi/4} \left(1+2\cos 2t+\frac{1+\cos 4t}{2}\right) dt$$

$$= \frac{1}{4}\int_{0}^{\pi/4} \left(\frac{2+4\cos 2t+1+\cos 4t}{2}\right) dt$$

$$= \frac{1}{8}\int_{0}^{\pi/4} \left(3+4\cos 2t+\cos 4t\right) dt$$

$$= \frac{1}{8}\left|3t+4\frac{\sin 2t}{2}+\frac{\sin 4t}{4}\right|_{0}^{\pi/4}$$

$$= \frac{1}{8}\Bigg(3\left(\frac{\pi}{4}\right)+2\sin 2\left(\pi/4\right)+\frac{\sin 4\left(\pi/4\right)}{4} -3(0)-2\sin 2(0)-\frac{\sin 4(0)}{4}\Bigg)$$

$$= \frac{1}{8}\left(\frac{3\pi}{4}+2+\frac{0}{4}-0-0-\frac{0}{4}\right) \quad = \frac{1}{8}\left(\frac{3\pi}{4}+2\right)$$

$$= \frac{1}{8}\left(\frac{3\pi+8}{4}\right) \quad = \frac{3\pi+8}{32}$$

Question # 19

$$\int_{0}^{\pi/3} \cos^2\theta\sin\theta\, d\theta$$

Solution

Let $I = \int_{0}^{\pi/3} \cos^2\theta\sin\theta\, d\theta$

Put $t=\cos\theta \quad \Rightarrow \quad dt=-\sin\theta\, d\theta$

$\Rightarrow \; -dt=\sin\theta\, d\theta$

When $\theta=0$ then $t=1$

And when $\theta = \pi/3$ then $t = 1/2$

So $I = \int_{1}^{1/2} t^2(-dt)$

$$= -\int_{1}^{1/2} t^2\,dt \quad = -\left|\frac{t^3}{3}\right|_{1}^{1/2}$$

$$= -\left(\frac{\left(1/2\right)^3}{3} - \frac{(1)^3}{3}\right) \quad = -\left(\frac{1/8}{3} - \frac{1}{3}\right)$$

$$= -\left(\frac{1}{24} - \frac{1}{3}\right) \quad = -\left(-\frac{7}{24}\right) \quad = \frac{7}{24}$$

Question # 20

$$\int_{0}^{\pi/4} \left(1+\cos^2\theta\right)\tan^2\theta\,d\theta$$

Solution

$$\int_{0}^{\pi/4} \left(1+\cos^2\theta\right)\tan^2\theta\,d\theta$$

$$= \int_{0}^{\pi/4} \left(1+\cos^2\theta\right)\frac{\sin^2\theta}{\cos^2\theta}\,d\theta$$

$$= \int_{0}^{\pi/4} \left(\frac{\sin^2\theta}{\cos^2\theta} + \sin^2\theta\right)d\theta$$

$$= \int_{0}^{\pi/4} \left(\tan^2\theta + \sin^2\theta\right)d\theta$$

$$= \int_{0}^{\pi/4} \left(\sec^2\theta - 1 + \frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \int_{0}^{\pi/4} \left(\frac{2\sec^2\theta - 2 + 1 - \cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/4} \left(2\sec^2\theta - 1 - \cos 2\theta\right)d\theta$$

$$= \frac{1}{2}\left|2\tan\theta - \theta - \frac{\sin 2\theta}{2}\right|_{0}^{\pi/4}$$

$$= \frac{1}{2}\left(2\tan\frac{\pi}{4} - \frac{\pi}{4} - \frac{\sin 2\left(\pi/4\right)}{2} - 2\tan(0) + 0 + \frac{\sin 2(0)}{2}\right)$$

$$= \frac{1}{2}\left(2(1) - \frac{\pi}{4} - \frac{1}{2} - 2(0) + 0 + 0\right)$$

$$= \frac{1}{2}\left(\frac{3}{2} - \frac{\pi}{4}\right) \quad = \frac{1}{2}\left(\frac{6-\pi}{4}\right) \quad = \frac{6-\pi}{8}$$

Question # 21

$$\int_{0}^{\pi/4} \frac{\sec\theta}{\sin\theta + \cos\theta}\,d\theta$$

Solution

Let $I = \int_{0}^{\pi/4} \frac{\sec\theta}{\sin\theta + \cos\theta}\,d\theta$

$$= \int_{0}^{\pi/4} \frac{\sec\theta}{\cos\theta\left(\frac{\sin\theta}{\cos\theta} + 1\right)}\,d\theta$$

$$= \int_{0}^{\pi/4} \frac{\sec^2\theta}{\left(\tan\theta + 1\right)}\,d\theta$$

Put $t = \tan\theta + 1 \quad \Rightarrow \quad dt = \sec^2\theta\,d\theta$

When $x = 0$ then $t = 1$

Also when $x = \frac{\pi}{4}$ then $t = 2$

So $I = \int_{1}^{2} \frac{dt}{t}$

$$= \left|\ln t\right|_{1}^{2}$$

$$= \ln 2 - \ln 1 \quad = \ln 2 - 0 \quad = \ln 2$$

Review

If $f(x) = \begin{cases} g(x) & : \quad a \le x \le b \\ h(x) & : \quad b \le x \le c \end{cases}$

Then

$$\int_{a}^{c} f(x)\,dx = \int_{a}^{b} g(x) + \int_{b}^{c} h(x)$$

Question # 22

$$\int_{-1}^{5} \left|x-3\right|dx$$

Solution

Let $I=\int_{-1}^{5}|x-3|\,dx$

Since

$$|x-3|=\begin{cases} x-3 & \text{if } x-3\geq 0 \Rightarrow x\geq 3\\ -(x-3) & \text{if } x-3<0 \Rightarrow x<3\end{cases}$$

So $\int_{-1}^{5}|x-3|\,dx=\int_{-1}^{3}[-(x-3)]\,dx+\int_{3}^{5}(x-3)\,dx$

$$=-\int_{-1}^{3}(x-3)\,dx+\int_{3}^{5}(x-3)\,dx$$

$$=-\left|\frac{(x-3)^2}{2}\right|_{-1}^{3}+\left|\frac{(x-3)^2}{2}\right|_{3}^{5}$$

$$=-\left(\frac{(3-3)^2}{2}-\frac{(-1-3)^2}{2}\right)+\left(\frac{(5-3)^2}{2}-\frac{(3-3)^2}{2}\right)$$

$$=-\left(\frac{0}{2}-\frac{16}{2}\right)+\left(\frac{4}{2}-\frac{0}{2}\right)\quad=8+2\quad=10$$

Question # 23

$$\int_{1/8}^{1}\frac{\left(x^{1/3}+2\right)^2}{x^{2/3}}\,dx$$

Solution

Let $I=\int_{1/8}^{1}\frac{\left(x^{1/3}+2\right)^2}{x^{2/3}}\,dx$

$$=\int_{1/8}^{1}\left(x^{1/3}+2\right)^2x^{-2/3}\,dx$$

Put $t=x^{\frac{1}{3}}+2$

$\Rightarrow dt=\frac{1}{3}x^{-\frac{2}{3}}dx \quad\Rightarrow 3dt=x^{-\frac{2}{3}}dx$

When $x=\frac{1}{8}$ then $t=\frac{5}{2}$

And when $x=1$ then $t=3$

So $I=\int_{5/2}^{3}(t)^2\,3dt \quad =3\left|\frac{t^3}{3}\right|_{5/2}^{3}$

$$=3\left(\frac{3^3}{3}-\frac{\left(5/2\right)^3}{3}\right)\quad=3\left(\frac{27}{3}-\frac{125/8}{3}\right)$$

$$=3\left(\frac{27}{3}-\frac{125}{24}\right)\quad=3\left(\frac{91}{24}\right)\quad=\frac{91}{8}$$

Question # 24

$$\int_{1}^{3}\frac{x^2-2}{x+1}\,dx$$

Solution

$$\begin{array}{r|l} & \quad x-1\\ x+1 & \overline{)\,x^2-2}\\ & \;\;\underset{-}{x^2}\underset{-}{+x}\\ \hline & \quad -x-2\\ & \quad \underset{+}{-x}\underset{+}{-1}\\ \hline & \qquad -1\end{array}$$

$$\int_{1}^{3}\frac{x^2-2}{x+1}\,dx$$

$$=\int_{1}^{3}\left(x-1-\frac{1}{x+1}\right)dx$$

$$=\int_{1}^{3}x\,dx-\int_{1}^{3}dx-\int_{1}^{3}\frac{dx}{x+1}$$

$$=\left|\frac{x^2}{2}\right|_{1}^{3}-|x|_{1}^{3}-\left|\ln|x+1|\right|_{1}^{3}$$

$$=\left(\frac{3^2}{2}-\frac{1^2}{2}\right)-(3-1)-\left(\ln|3+1|-\ln|1+1|\right)$$

$$=\left(\frac{9}{2}-\frac{1}{2}\right)-(2)-(\ln 4-\ln 2)$$

$$=4-2-\ln\frac{4}{2}\quad=2-\ln 2$$

Question # 25

$$\int_{2}^{3}\frac{3x^2-2x+1}{(x-1)\left(x^2+1\right)}\,dx$$

Solution

$$\int_{2}^{3}\frac{3x^2-2x+1}{(x-1)\left(x^2+1\right)}\,dx$$

$$=\int_{2}^{3}\frac{3x^2-2x+1}{x^3-x^2+x-1}\,dx$$

$$=\int_{2}^{3}\frac{\frac{d}{dx}\left(x^3-x^2+x-1\right)}{x^3-x^2+x-1}\,dx$$

$$=\left|\ln\left|x^3-x^2+x-1\right|\right|_{2}^{3}$$

$$=\ln\left|3^3-3^2+3-1\right|-\ln\left|2^3-2^2+2-1\right|$$

$$=\ln|27-9+3-1|-\ln|8-4+2-1|$$

$$=\ln 20-\ln 5\quad=\ln\frac{20}{5}\quad=\ln 4$$

Question # 26

$$\int_0^{\pi/4} \frac{\sin x-1}{\cos^2 x}\,dx$$

Solution

$$\int_0^{\pi/4} \frac{\sin x-1}{\cos^2 x}\,dx \;=\int_0^{\pi/4}\left(\frac{\sin x}{\cos^2 x}-\frac{1}{\cos^2 x}\right)dx$$

$$=\int_0^{\pi/4}\left(\frac{\sin x}{\cos x\cdot\cos x}-\frac{1}{\cos^2 x}\right)dx$$

$$=\int_0^{\pi/4}\left(\sec x\tan x-\sec^2 x\right)dx$$

$$=\left|\sec x-\tan x\right|_0^{\pi/4}$$

$$=\left(\sec\frac{\pi}{4}-\tan\frac{\pi}{4}\right)-\left(\sec(0)-\tan(0)\right)$$

$$=\sqrt{2}-1-1+0 \quad =\sqrt{2}-2$$

Question # 27

$$\int_0^{\pi/4} \frac{1}{1+\sin x}\,dx$$

Solution

$$\text{Let } I=\int_0^{\pi/4} \frac{1}{1+\sin x}\,dx$$

$$=\int_0^{\pi/4} \frac{1}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x}\,dx$$

$$=\int_0^{\pi/4} \frac{1-\sin x}{1-\sin^2 x}\,dx \quad =\int_0^{\pi/4} \frac{1-\sin x}{\cos^2 x}\,dx$$

Now same as Question # 24

Question # 28

$$\int_0^{1} \frac{3x}{\sqrt{4-3x}}\,dx$$

Solution

$$\text{Let } I=\int_0^{1} \frac{3x}{\sqrt{4-3x}}\,dx$$

Put $t=4-3x \quad\Rightarrow\quad 3x=4-t$

Also $dt=-3dx \quad\Rightarrow\quad -\frac{1}{3}dt=dx$

When $x=0$ then $t=4$

And when $x=1$ then $t=1$

$$\text{So } I=\int_4^{1}\frac{4-t}{\sqrt{t}}\left(-\frac{1}{3}dt\right)$$

$$=-\frac{1}{3}\int_4^{1}\left(\frac{4}{t^{1/2}}-\frac{t}{t^{1/2}}\right)dt$$

$$=+\frac{1}{3}\int_1^{4}\left(4t^{-\frac{1}{2}}-t^{\frac{1}{2}}\right)dt$$

Now do yourself

Question # 29

$$\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x\left(2+\sin x\right)}\,dx$$

Solution

$$\text{Let } I=\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x\left(2+\sin x\right)}\,dx$$

Put $t=\sin x \quad\Rightarrow\quad dt=\cos x\,dx$

When $x=\frac{\pi}{6}$ then $t=\frac{1}{2}$

When $x=\frac{\pi}{2}$ then $t=1$

$$\text{So } I=\int_{1/2}^{1}\frac{dt}{t\left(2+t\right)}$$

Now consider

$$\frac{1}{t\left(2+t\right)}=\frac{A}{t}+\frac{B}{2+t}$$

$$\Rightarrow\; 1=A\left(2+t\right)+Bt \;\ldots\ldots\text{ (i)}$$

Put $t=0$ in (i)

$$1=A(2+0)+B(0) \quad\Rightarrow\quad 1=2A \quad\Rightarrow\quad A=\frac{1}{2}$$

Put $2+t=0\Rightarrow\; t=-2$ in eq. (i)

$$1=0+B(-2) \quad\Rightarrow\quad 1=-2B \quad\Rightarrow\quad B=-\frac{1}{2}$$

$$\text{So}\qquad \frac{1}{t\left(2+t\right)}=\frac{1/2}{t}+\frac{-1/2}{2+t}$$

$$\Rightarrow\; \int_{1/2}^{1}\frac{1}{t(2+t)}dt=\int_{1/2}^{1}\frac{1/2}{t}\,dt+\int_{1/2}^{1}\frac{-1/2}{2+t}\,dt$$

$$=\frac{1}{2}\int_{1/2}^{1}\frac{1}{t}\,dt-\frac{1}{2}\int_{1/2}^{1}\frac{1}{2+t}\,dt$$

$$=\frac{1}{2}\left|\ln\left|t\right|\right|_{1/2}^{1}-\frac{1}{2}\left|\ln\left|2+t\right|\right|_{1/2}^{1}$$

$$=\frac{1}{2}\left[\ln|1|-\ln\left|\frac{1}{2}\right|\right]$$
$$-\frac{1}{2}\left[\ln|2+1|-\ln\left|2+\frac{1}{2}\right|\right]$$
$$=\frac{1}{2}\left[0-\ln\frac{1}{2}\right]-\frac{1}{2}\left[\ln 3-\ln\frac{5}{2}\right]$$
$$=\frac{1}{2}\left[-\ln\frac{1}{2}-\ln 3+\ln\frac{5}{2}\right]$$
$$=\frac{1}{2}\ln\left(\frac{5/2}{1/2\times 3}\right)\quad=\frac{1}{2}\ln\left(\frac{5}{3}\right)$$

Question # 30

$$I=\int_0^{\pi/2}\frac{\sin x\,dx}{(1+\cos x)(2+\cos x)}$$

Solution

Let $I=\int_0^{\pi/2}\frac{\sin x\,dx}{(1+\cos x)(2+\cos x)}$

Put $t=\cos x\quad\Rightarrow\quad dt=-\sin x\,dx$

$\Rightarrow\ -dt=\sin x\,dx$

When $x=0$ then $t=1$

And when $x=\frac{\pi}{2}$ then $t=0$

So $I=\int_1^0\frac{-dt}{(1+t)(2+t)}$

$$=-\int_1^0\frac{dt}{(1+t)(2+t)}\quad=\int_0^1\frac{dt}{(1+t)(2+t)}$$

Now consider

$$\frac{1}{(1+t)(2+t)}=\frac{A}{1+t}+\frac{B}{2+t}$$

$\Rightarrow\ 1=A(2+t)+B(1+t)\ \ ....\ (i)$

Put $1+t=0\quad\Rightarrow\quad t=-1$ in (i)

$1=A(2-1)+0\quad\Rightarrow\quad A=1$

Put $2+t=0\quad\Rightarrow\quad t=-2$ in (i)

$1=0+B(1-2)\quad\Rightarrow\quad 1=-B\quad$ i.e. $B=-1$

So

$$\frac{1}{(1+t)(2+t)}=\frac{1}{1+t}+\frac{-1}{2+t}$$

$$\int_0^1\frac{1}{(1+t)(2+t)}\,dt=\int_0^1\frac{1}{1+t}\,dt-\int_0^1\frac{1}{2+t}\,dt$$
$$=\left|\ln|1+t|\right|_0^1-\left|\ln|2+t|\right|_0^1$$
$$=\left(\ln|1+1|-\ln|1+0|\right)$$
$$-\left(\ln|2+1|-\ln|2+0|\right)$$
$$=\ln 2-0-\ln 3+\ln 2$$
$$=\ln\left(\frac{2\times 2}{3}\right)\quad=\ln\left(\frac{4}{3}\right)$$

Example 4

Find the area bounded by the curve

$$f(x) = x^3 - 2x^2 + 1$$

and the x-axis in the first quadrant.

Solution

Put $f(x) = 0$

$$\Rightarrow x^3 - 2x + 1 = 0$$

By synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -2 & 0 & 1 \\ & \downarrow & 1 & -1 & -1 \\ \hline & 1 & -1 & -1 & \boxed{0} \end{array}$$

$$\Rightarrow (x-1)(x^2 - x - 1) = 0$$

$$\Rightarrow x - 1 = 0 \quad \text{or} \quad x^2 - x - 1 = 0$$

$$\Rightarrow x = 1 \quad \text{or} \quad x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Thus the curve cuts the x-axis at $x = 1$, $\frac{1 \pm \sqrt{5}}{2}$

Since we are taking area in the first quad. only

$\therefore x = 1, \frac{1+\sqrt{5}}{2}$ ignoring $\frac{1-\sqrt{5}}{2}$ as it is $-$ive.

Intervals in 1st quad. are $[0,1]$ & $\left[1, \frac{1+\sqrt{5}}{2}\right]$

Since $f(x) \geq 0$ whenever $x \in [0,1]$

and $f(x) \leq 0$ whenever $x \in \left[1, \frac{1+\sqrt{5}}{2}\right]$

$$\therefore \text{Area in } 1^{\text{st}} \text{ quad.} = \int_0^1 \left(x^3 - 2x^2 + 1\right) dx$$

$$= \left| \frac{x^4}{4} - 2\frac{x^3}{3} + x \right|_0^1$$

$$= \left(\frac{1}{2} - \frac{2}{3} + 1\right) - 0$$

$$= \frac{7}{12} \text{ sq. unit}$$

Question # 1

Find the area between the x-asis and the curve $y = x^2 + 1$ from $x = 1$ to $x = 2$.

Solution

$$y = x^2 + 1 \quad ; \quad x = 1 \text{ to } x = 2$$

$$\therefore \ y \geq 0 \text{ whenever } x \in [1, 2]$$

$$\therefore \ \text{Area} = \int_1^2 (x^2 + 1) dx$$

$$= \int_1^2 x^2 dx + \int_1^2 dx$$

$$= \left| \frac{x^3}{3} \right|_1^2 + \left| x \right|_1^2$$

$$= \left(\frac{(2)^3}{3} - \frac{(1)^3}{3} \right) + (2 - 1)$$

$$= \left(\frac{8}{3} - \frac{1}{3} \right) + 1$$

$$= \frac{7}{3} + 1 = \frac{10}{3} \text{ sq. unit.}$$

Question # 2

Find the area above the x-asis and under the curve $y = 5 - x^2$ from $x = -1$ to $x = 2$.

Solution

$$y = 5 - x^2 \quad ; \quad x = -1 \text{ to } x = 2$$

$$\therefore \ y > 0 \text{ whenever } x \in (-1, 2)$$

$$\therefore \ \text{Area} = \int_{-1}^2 (5 - x^2) dx$$

$$= \left| 5x - \frac{x^3}{3} \right|_{-1}^2$$

$$= \left(5(2) - \frac{(2)^3}{3} \right) - \left(5(-1) - \frac{(-1)^3}{3} \right)$$

$$= \left(10 - \frac{8}{3} \right) - \left(-5 + \frac{1}{3} \right)$$

$$= \frac{22}{3} - \left(-\frac{14}{3} \right) = \frac{22}{3} + \frac{14}{3}$$

$$= \frac{36}{3} = 12 \text{ sq. unit}$$

Question # 3

Find the area below the curve $y = 3\sqrt{x}$ and above the x-axis between $x = 1$ to $x = 4$.

Solution

$$y = 3\sqrt{x} \quad ; \qquad x=1 \text{ to } x=4$$

Since $y \geq 0$ when $x \in [1,4]$

$$\therefore \text{ Area } = \int_1^4 3\sqrt{x}\, dx$$

$$= \int_1^4 3x^{\frac{1}{2}}\, dx = 3\int_1^4 x^{\frac{1}{2}}\, dx$$

$$= 3\left|\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_1^4 = 3\left|\frac{x^{\frac{3}{2}}}{3/2}\right|_1^4$$

$$= 3\times\frac{2}{3}\left|x^{\frac{3}{2}}\right|_1^4 = 2\left((4)^{\frac{3}{2}}-(1)^{\frac{3}{2}}\right)$$

$$= \frac{3}{4}\left((4)^{\frac{4}{3}}-(1)^{\frac{4}{3}}\right) = 2\left((2^2)^{\frac{3}{2}}-1\right)$$

$$= 2(8-1) = 14 \text{ sq. unit}$$

Question # 4

Find the area bounded by cos function from $x=-\dfrac{\pi}{2}$ to $x=\dfrac{\pi}{2}$

Solution

$$y = \cos x \quad ; \qquad x=-\frac{\pi}{2} \text{ to } x=\frac{\pi}{2}$$

$$\because \quad y>0 \quad \text{whenever } x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$\therefore \text{ Area } = \int_{-\pi/2}^{\pi/2} \cos x\, dx$$

$$= \left|\sin x\right|_{-\pi/2}^{\pi/2}$$

$$= \sin\left(\frac{\pi}{2}\right)-\sin\left(-\frac{\pi}{2}\right)$$

$$= 1+1 = 2 \text{ sq. unit}$$

Question # 5

Find the area between the x-asis and the curve $y = 4x-x^2$

Solution

$$y = 4x-x^2$$

Putting $y=0$, we have

$$4x-x^2 = 0$$

$$\Rightarrow \quad x(4-x) = 0$$

$$\Rightarrow \quad x = 0 \text{ or } x = 4$$

Now $y>0$ when $x\in(0,4)$

$$\therefore \text{ Area } = \int_0^4 \left(4x-x^2\right)dx$$

$$= \left|\frac{4x^2}{2}-\frac{x^3}{3}\right|_0^4 = \left|2x^2-\frac{x^3}{3}\right|_0^4$$

$$= \left(2(4)^2-\frac{(4)^3}{3}\right)-\left(2(0)^2-\frac{(0)^3}{3}\right)$$

$$= \left(32-\frac{64}{3}\right)-(0-0)$$

$$= \frac{32}{2} \text{ sq. unit.}$$

Question # 6

Determine the area bounded by the parabola $y = x^2+2x-3$ and the x-axis.

Solution

$$y = x^2+2x-3$$

Putting $y = 0$, we have

$$x^2+2x-3 = 0$$

$$\Rightarrow \quad x^2+3x-x-2 = 0$$

$$\Rightarrow \quad x(x+3)-1(x+3) = 0$$

$$\Rightarrow \quad (x+3)(x-1) = 0$$

$$\Rightarrow \quad x = -3 \quad \text{or} \quad x = 1$$

Now $y \leq 0$ whenever $x\in[-3,1]$

$$\therefore \text{ Area } = -\int_{-3}^{1}\left(x^2+2x-3\right)dx$$

$$= -\left|\frac{x^3}{3}+\frac{2x^2}{2}-3x\right|_{-3}^{1}$$

$$= -\left|\frac{x^3}{3}+x^2-3x\right|_{-3}^{1}$$

$$= -\left(\frac{(1)^3}{3}+(1)^2-3(1)\right) + \left(\frac{(-3)^3}{3}+(-3)^2-3(-3)\right)$$

$$= -\left(\frac{1}{3}+1-3\right)+\left(\frac{-27}{3}+9+9\right)$$

$$= -\left(-\frac{5}{3}\right)+(-9+18)$$

$$= \frac{5}{3}+9 = \frac{32}{3} \text{ sq. unit}$$

Question # 7

Find the area bounded by the curve $y = x^3+1$, the x-axis and line $x=2$.

Solution

$y = x^3+1$

Putting $y=0$, we have

$$x^3+1 = 0$$

$$\Rightarrow (x+1)(x^2-x+1) = 0$$

$$\Rightarrow x+1=0 \quad \text{or} \quad x^2-x+1=0$$

$$\Rightarrow x=-1 \quad \text{or} \quad x=\frac{1\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}$$

$$=\frac{1\pm\sqrt{1-4}}{2}$$

$$\Rightarrow x=\frac{1\pm\sqrt{-3}}{2}$$

Which is not possible.

Now $y\geq 0$ when $x\in[-1,2]$

$$\therefore \text{ Area } = \int_{-1}^{2}(x^3+1)dx$$

$$= \left|\frac{x^4}{4}+x\right|_{-1}^{2}$$

$$= \left(\frac{(2)^4}{4}+2\right)-\left(\frac{(-1)^4}{4}-1\right)$$

$$= \left(\frac{16}{4}+2\right)-\left(\frac{1}{4}-1\right)$$

$$= 6-\frac{3}{4} = \frac{27}{4} \text{ sq. unit}$$

Question # 8

Find the area bounded by the curve $y = x^3-2x+4$ and the x-axis.

Solution

$y = x^3-2x+4 \quad ; \quad x=1$

Putting $y=0$, we have

$$x^3-2x+4 = 0$$

By synthetic division

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -2 & 4 \\ & \downarrow & -2 & 4 & -4 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$$\Rightarrow (x+2)(x^2-2x+2) = 0$$

$$\Rightarrow x+2=0 \quad \text{or} \quad x^2-2x+2 = 0$$

$$\Rightarrow x=-2 \quad \text{or} \quad x=\frac{2\pm\sqrt{(-2)^2-4(1)(2)}}{2}$$

$$= \frac{2\pm\sqrt{4-8}}{2}$$

$$= \frac{2\pm\sqrt{-4}}{2}$$

This is imaginary.

Now $y\geq 0$ when $x\in[-2,1]$

$$\therefore \text{ Area } = \int_{-2}^{1}(x^3-2x+4)dx$$

$$= \int_{-2}^{1}x^3dx-2\int_{-2}^{1}x\,dx+4\int_{-2}^{1}dx$$

$$= \left|\frac{x^4}{4}\right|_{-2}^{1}-2\left|\frac{x^2}{2}\right|_{-2}^{1}+4|x|_{-2}^{1}$$

$$= \left(\frac{(1)^4}{4}-\frac{(-2)^4}{4}\right)-2\left(\frac{(1)^2}{2}-\frac{(-2)^2}{2}\right)+4(1-(-2))$$

$$= \left(\frac{1}{4}-\frac{16}{4}\right)-2\left(\frac{1}{2}-\frac{4}{2}\right)+4(1+2)$$

$$= \left(\frac{1}{4}-4\right)-2\left(\frac{1}{2}-2\right)+4(3)$$

$$= \left(-\frac{15}{4}\right)-2\left(-\frac{3}{2}\right)+12$$

$$= -\frac{15}{4}+3+12 = \frac{45}{4} \text{ sq. unit}$$

Question # 9

Find the area between the curve

Solution

$y = x^3-4x$

Putting $y=0$, we have

$$x^3-4x = 0$$

$$\Rightarrow x(x^2-4) = 0$$

$$\Rightarrow x(x+2)(x-2) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x=-2 \quad \text{or} \quad x=2$$

Now $y\geq 0$ whenever $x\in[-2,0]$

And $y\leq 0$ whenever $x\in[0,2]$

$$\therefore \text{ Area } = \int_{-2}^{0}y\,dx-\int_{0}^{2}y\,dx$$

$$= \int_{-2}^{0}\left(x^3-4x\right)dx-\int_{0}^{2}\left(x^3-4x\right)dx$$

$$= \left|\frac{x^4}{4}-4\frac{x^2}{2}\right|_{-2}^{0}-\left|\frac{x^4}{4}-4\frac{x^2}{2}\right|_{0}^{2}$$

$$= \left|\frac{x^4}{4}-2x^2\right|_{-2}^{0}-\left|\frac{x^4}{4}-2x^2\right|_{0}^{2}$$

$$= \left(\frac{(0)^4}{4}-2(0)^2\right)-\left(\frac{(-2)^4}{4}-2(-2)^2\right)$$
$$-\left(\frac{(2)^4}{4}-2(2)^2\right)+\left(\frac{(0)^4}{4}-2(0)^2\right)$$

$$= (0-0)-\left(\frac{16}{4}-8\right)$$
$$-\left(\frac{16}{4}-8\right)+(0-0)$$

$$= -(4-8)-(4-8) \quad = -(-4)-(-4)$$

$$= 4+4 \quad = 8 \text{ sq. unit.}$$

Question # 9

Find the area between the curve $y = x(x-1)(x+1)$ and the x-axis.

Solution

$$y = x(x-1)(x+1)$$

Putting $y = 0$, we have

$$x(x-1)(x+1) = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = -1$$

Now $y \geq 0$ whenever $x \in [-1,0]$

And $y \leq 0$ whenever $x \in [0,1]$

$$\therefore \text{ Area} = \int_{-1}^{0} y\,dx - \int_{0}^{1} y\,dx$$

$$= \int_{-1}^{0} x(x-1)(x+1)\,dx$$
$$-\int_{0}^{1} x(x-1)(x+1)\,dx$$

$$= \int_{-1}^{0}\left(x^3-x\right)dx-\int_{0}^{1}\left(x^3-x\right)dx$$

$$= \left|\frac{x^4}{4}-\frac{x^2}{2}\right|_{-1}^{0}-\left|\frac{x^4}{4}-\frac{x^2}{2}\right|_{0}^{1}$$

$$= \left(\frac{(0)^4}{4}-\frac{(0)^2}{2}\right)-\left(\frac{(-1)^4}{4}-\frac{(-1)^2}{2}\right)$$
$$-\left(\frac{(1)^4}{4}-\frac{(1)^2}{2}\right)+\left(\frac{(0)^4}{4}-\frac{(0)^2}{2}\right)$$

$$= (0-0)-\left(\frac{1}{4}-\frac{1}{2}\right)$$
$$-\left(\frac{1}{4}-\frac{1}{2}\right)+(0-0)$$

$$= 0-\left(-\frac{1}{4}\right)-\left(-\frac{1}{4}\right)+0$$

$$= \frac{1}{4}+\frac{1}{4} \quad = \frac{1}{2} \text{ sq. unit}$$

Question # 11

Find the area between the x-asis and the curve $y = \cos\frac{1}{2}x$ from $x=-\pi$ to $x=\pi$

Solution

$$g(x) = \cos\frac{1}{2}x \quad ; \quad x=-\pi \text{ to } x=\pi$$

$$\because \; g(x) \geq 0 \quad \text{when} \quad x \in [-\pi,\pi]$$

$$\therefore \text{ Area} = \int_{-\pi}^{\pi} \cos\frac{1}{2}x\,dx$$

$$= \left|\frac{\sin\frac{x}{2}}{1/2}\right|_{-\pi}^{\pi} \quad = 2\left|\sin\frac{x}{2}\right|_{-\pi}^{\pi}$$

$$= 2\left(\sin\left(\frac{\pi}{2}\right)-\sin\left(\frac{-\pi}{2}\right)\right)$$

$$= 2\left(1-(-1)\right) \quad = 2(1+1)$$

$$= 2(2) \quad = 4 \text{ sq. unit.}$$

Question # 12

Find the area between the x-asis and the curve $y = \sin 2x$ from $x=0$ to $x=\frac{\pi}{3}$

Solution

$$y = \sin 2x \quad ; \quad x=0 \text{ to } x=\frac{\pi}{3}$$

$$\because \quad y \geq 0 \quad \text{when} \quad x \in \left[0,\frac{\pi}{3}\right]$$

$$\therefore \text{ Area} = \int_{0}^{\pi/3} \sin 2x\,dx$$

$$= \left|-\frac{\cos 2x}{2}\right|_{0}^{\pi/3} \quad = -\frac{1}{2}\left(\cos\frac{2\pi}{3}-\cos(0)\right)$$

$$= -\frac{1}{2}\left(-\frac{1}{2}-1\right) = -\frac{1}{2}\left(-\frac{3}{2}\right) = \frac{3}{4} \text{ sq. unit.}$$

Question # 13

Find the area between the x-asis and the curve

$y = \sqrt{2ax-x^2}$ when $a>0$

Solution

$$y = \sqrt{2ax-x^2}$$

Putting $y=0$, we have

$$\sqrt{2ax-x^2} = 0$$

On squaring

$$2ax-x^2 = 0$$

$$\Rightarrow \quad x(2a-x) = 0$$

$$\Rightarrow \quad x=0 \quad \text{or} \quad 2a-x = 0 \Rightarrow \quad x=2a$$

$$\because \quad y\geq 0 \quad \text{when} \quad x\in[0,2a]$$

$$\therefore \quad \text{Area} = \int_0^{2a}\sqrt{2ax-x^2}\,dx$$

$$= \int_0^{2a}\sqrt{a^2-a^2+2ax-x^2}\,dx$$

$$= \int_0^{2a}\sqrt{a^2-\left(a^2-2ax+x^2\right)}\,dx$$

$$= \int_0^{2a}\sqrt{a^2-(a-x)^2}\,dx$$

Put $a-x = a\sin\theta$

$\Rightarrow \quad -dx = a\cos\theta\,d\theta$

$\Rightarrow \quad dx = -a\cos\theta\,d\theta$

When $x=0$

$$a-0 = a\sin\theta \quad \Rightarrow \quad a\sin\theta = a$$

$$\Rightarrow \quad \sin\theta = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{2}$$

When $x=2a$

$$a-2a = a\sin\theta \quad \Rightarrow \quad -a = a\sin\theta$$

$$\Rightarrow \quad -1 = \sin\theta \quad \Rightarrow \quad \theta = -\frac{\pi}{2}$$

$$\text{So area} = \int_{\pi/2}^{-\pi/2}\sqrt{a^2-a^2\sin^2\theta}\,(-a\cos\theta d\theta)$$

$$= -a\int_{\pi/2}^{-\pi/2}\sqrt{a^2\left(1-\sin^2\theta\right)}\,\cos\theta\,d\theta$$

$$= -a\int_{\pi/2}^{-\pi/2}\sqrt{a^2\cos^2\theta}\,\cos\theta\,d\theta$$

$$= -a\int_{\pi/2}^{-\pi/2} a\cos\theta\cdot\cos\theta\,d\theta$$

$$= -a^2\int_{\pi/2}^{-\pi/2}\cos^2\theta\,d\theta$$

$$= -a^2\int_{\pi/2}^{-\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= -\frac{a^2}{2}\int_{\pi/2}^{-\pi/2}(1+\cos 2\theta)\,d\theta$$

$$= -\frac{a^2}{2}\left|\theta+\frac{\sin 2\theta}{2}\right|_{\pi/2}^{-\pi/2}$$

$$= -\frac{a^2}{2}\left(-\frac{\pi}{2}+\sin(-\pi)-\frac{\pi}{2}-\sin\pi\right)$$

$$= -\frac{a^2}{2}(-\pi-0-0)$$

$$= -\frac{a^2}{2}(-\pi) = \frac{a^2\pi}{2} \text{ sq. unit}$$

Question # 1

Check each of the following equations written against the differential equation is its solution.

(i) $x\dfrac{dy}{dx}=1+y\ ,\quad y=cx-1$

(ii) $x^2(2y+1)\dfrac{dy}{dx}-1=0\ ,\quad y^2+y=c-\dfrac{1}{x}$

(iii) $y\dfrac{dy}{dx}-e^{2x}=1,\quad y^2 = 2x+e^{2x}+c$

(iv) $\dfrac{1}{x}\dfrac{dy}{dx}-2y = 0,\quad y = ce^{x^2}$

(v) $\dfrac{dy}{dx} = \dfrac{y^2+1}{e^{-x}},\quad y = Tan\left(e^x+c\right)$

Solution

(i) $x\dfrac{dy}{dx}=1+y$

$$\Rightarrow\ x\,dy=(1+y)\,dx\ \Rightarrow\ \frac{dy}{1+y}=\frac{dx}{x}$$

Integrating both sides

$$\int\frac{dy}{1+y}=\int\frac{dx}{x}$$

$$\Rightarrow\ \ln(1+y)=\ln x+\ln c$$

$$=\ln cx$$

$$\Rightarrow\ 1+y=cx$$

$$\Rightarrow\ y=cx-1\qquad \textit{Proved}$$

(ii) $x^2(2y+1)\dfrac{dy}{dx}-1=0$

$$\Rightarrow\ x^2(2y+1)\frac{dy}{dx}=1\ \Rightarrow\ x^2(2y+1)\,dy=dx$$

$$\Rightarrow\ (2y+1)\,dy=\frac{1}{x^2}dx$$

On integrating

$$\int(2y+1)\,dy=\int\frac{1}{x^2}dx$$

$$\Rightarrow\ 2\int ydy+\int dy=\int x^{-2}\,dx$$

$$\Rightarrow\ 2\cdot\frac{y^2}{2}+y=\frac{x^{-2+1}}{-2+1}+c$$

$$\Rightarrow\ y^2+y=\frac{x^{-1}}{-1}+c$$

$$\Rightarrow\ y^2+y=c-\frac{1}{x}\qquad \textit{Proved}$$

(iii) $y\dfrac{dy}{dx}-e^{2x}=1$

$$\Rightarrow\ y\frac{dy}{dx} = 1+e^{2x}\quad \Rightarrow\ ydy = \left(1+e^{2x}\right)dx$$

On integrating

$$\int ydy = \int\left(1+e^{2x}\right)dx$$

$$\Rightarrow\ \frac{y^2}{2} = x+\frac{e^{2x}}{2}+\frac{c}{2}\quad \Rightarrow\ y^2 = 2x+e^{2x}+c$$

$$\Rightarrow\ y^2 = 2x+e^{2x}+c$$

(iv) $\dfrac{1}{x}\dfrac{dy}{dx}-2y = 0$

$$\Rightarrow\ \frac{1}{x}\frac{dy}{dx} = 2y\quad \Rightarrow\ \frac{dy}{dx} = 2xy$$

$$\Rightarrow\ \frac{dy}{y} = 2xdx$$

On integrating

$$\int\frac{dy}{y} = 2\int xdx$$

$$\Rightarrow\ \ln y = 2\cdot\frac{x^2}{2}+\ln c$$

$$= x^2+\ln c$$

$$= x^2\ln e+\ln c\qquad \because\ \ln e=1$$

$$= \ln e^{x^2}+\ln c$$

$$\Rightarrow\ \ln y = \ln ce^{x^2}$$

$$\Rightarrow\ y = ce^{x^2}\qquad \textit{Proved}$$

(v) $\dfrac{dy}{dx} = \dfrac{y^2+1}{e^{-x}}\quad \Rightarrow\ \dfrac{dy}{y^2+1} = e^x dx$

Integrating both sides

$$\Rightarrow\ \int\frac{dy}{y^2+1} = \int e^x dx$$

$$\Rightarrow\ Tan^{-1}y = e^x+c$$

$$\Rightarrow\ y = Tan\left(e^x+c\right)$$

Solve the following differential equations:

Question # 2

$$\frac{dy}{dx} = -y$$

Solution

$$\frac{dy}{dx} = -y\quad \Rightarrow\ \frac{dy}{y} = -dx$$

On integrating

$$\int\frac{dy}{y} = -\int dx$$

$$\ln y = -x + \ln c$$
$$= -x\ln e + \ln c \qquad \because \ln e = 1$$
$$= \ln e^{-x} + \ln c$$
$$\Rightarrow \ln y = \ln ce^{-x} \qquad \Rightarrow y = ce^{-x}$$

Question # 3

$$ydx + xdy = 0$$

Solution

$$ydx + xdy = 0 \quad \Rightarrow ydx = -xdy$$
$$\Rightarrow \frac{dx}{x} = -\frac{dy}{y}$$

On integrating

$$\ln x = -\ln y + \ln c$$
$$\Rightarrow \ln x = \ln\frac{c}{y}$$
$$\Rightarrow x = \frac{c}{y} \qquad \Rightarrow xy = c$$

Question # 4

$$\frac{dy}{dx} = \frac{1-x}{y}$$

Solution *Do yourself*

Question # 5

$$\frac{dy}{dx} = \frac{y}{x^2}, (y > 0)$$

Solution

$$\frac{dy}{dx} = \frac{y}{x^2} \qquad \Rightarrow \frac{dy}{y} = x^{-2}dx$$

Integrating

$$\int\frac{dy}{y} = \int x^{-2}dx$$
$$\Rightarrow \ln y = \frac{x^{-2+1}}{-2+1} + \ln c$$
$$\Rightarrow \ln y = \frac{x^{-1}}{-1} + \ln c$$
$$\Rightarrow \ln y = -\frac{1}{x} + \ln c$$
$$\Rightarrow \ln y = -\frac{1}{x}\ln e + \ln c$$
$$= \ln e^{-\frac{1}{x}} + \ln c$$
$$\Rightarrow \ln y = \ln ce^{-\frac{1}{x}} \qquad \Rightarrow y = ce^{-\frac{1}{x}}$$

Question # 6

$$\sin y \operatorname{cosec} x\frac{dy}{dx} = 1$$

Solution

$$\sin y \operatorname{cosec} x\frac{dy}{dx} = 1$$
$$\Rightarrow \sin y\, dy = \frac{dx}{\operatorname{cosec} x}$$
$$\Rightarrow \sin y\, dy = \sin x\, dx$$

Integrating

$$\int \sin y\, dy = \int \sin x\, dx$$
$$\Rightarrow -\cos y = -\cos x - c$$
$$\Rightarrow \cos y = \cos x + c$$

Question # 7

$$xdy + y(x-1)dx = 0$$

Solution

$$xdy + y(x-1)dx = 0$$
$$\Rightarrow xdy = -y(x-1)dx$$
$$\Rightarrow \frac{dy}{y} = -\frac{x-1}{x}dx$$
$$\Rightarrow \frac{dy}{y} = -\left(\frac{x}{x} - \frac{1}{x}\right)dx$$
$$\Rightarrow \frac{dy}{y} = -\left(1 - \frac{1}{x}\right)dx$$

On integrating

$$\int\frac{dy}{y} = -\int\left(1 - \frac{1}{x}\right)dx$$
$$\Rightarrow \ln y = -x + \ln x + \ln c$$
$$= -x\ln e + \ln x + \ln c$$
$$= \ln e^{-x} + \ln x + \ln c$$
$$\Rightarrow \ln y = \ln cxe^{-x} \Rightarrow y = cxe^{-x}$$

Question # 8

$$\frac{x^2+1}{y+1} = \frac{x}{y}\frac{dy}{dx}, (x, y > 0)$$

Solution

$$\frac{x^2+1}{y+1} = \frac{x}{y}\frac{dy}{dx}$$
$$\Rightarrow \frac{x^2+1}{x}dx = \frac{y+1}{y}dy$$

On integrating

$$\int\frac{x^2+1}{x}dx = \int\frac{y+1}{y}dy$$
$$\Rightarrow \int\left(\frac{x^2}{x} + \frac{1}{x}\right)dx = \int\left(\frac{y}{y} + \frac{1}{y}\right)dy$$

$$\Rightarrow \int\left(x+\frac{1}{x}\right)dx = \int\left(1+\frac{1}{y}\right)dy$$

$$\Rightarrow \int x\,dx+\int\frac{1}{x}dx = \int dy+\int\frac{1}{y}dy$$

$$\Rightarrow \frac{x^2}{2}+\ln x = y+\ln y-\ln c$$

$$\Rightarrow \frac{x^2}{2}\ln e+\ln x+\ln c = y\ln e+\ln y$$

$$\Rightarrow \ln e^{\frac{x^2}{2}}+\ln x+\ln c = \ln e^y+\ln y$$

$$\Rightarrow \ln cxe^{\frac{x^2}{2}} = \ln ye^y$$

$$\Rightarrow cxe^{\frac{x^2}{2}} = ye^y \quad \text{i.e. } ye^y = cxe^{\frac{x^2}{2}}$$

Question # 9

$$\frac{1}{x}\frac{dy}{dx}=\frac{1}{2}\left(1+y^2\right)$$

Solution *Do yourself*

Question # 10

$$2x^2y\frac{dy}{dx}=x^2-1$$

Solution *Do yourself*

Question # 11

$$\frac{dy}{dx}+\frac{2xy}{2y+1} = x$$

Solution

$$\frac{dy}{dx}+\frac{2xy}{2y+1} = x$$

$$\Rightarrow \frac{dy}{dx} = x-\frac{2xy}{2y+1}$$

$$= x\left(1-\frac{2y}{2y+1}\right)$$

$$= x\left(\frac{2y+1-2y}{2y+1}\right)$$

$$\Rightarrow \frac{dy}{dx} = x\left(\frac{1}{2y+1}\right)\Rightarrow (2y+1)dy = x\,dx$$

Now do yourself

Question # 12

$$\left(x^2-yx^2\right)\frac{dy}{dx} + y^2+xy^2 = 0$$

Solution

$$\left(x^2-yx^2\right)\frac{dy}{dx} + y^2+xy^2 = 0$$

$$\Rightarrow \left(x^2-yx^2\right)\frac{dy}{dx} = -y^2-xy^2$$

$$\Rightarrow x^2(1-y)\frac{dy}{dx} = -y^2(1+x)$$

$$\Rightarrow \frac{1-y}{y^2}dy = -\frac{1+x}{x^2}dx$$

Now do yourself

Question # 13

$$\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy = 0$$

Solution

$$\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy = 0$$

$$\Rightarrow \sec^2 x\tan y\,dx = -\sec^2 y\tan x\,dy$$

$$\Rightarrow \frac{\sec^2 x}{\tan x}dx = -\frac{\sec^2 y}{\tan y}dy$$

On integrating

$$\int\frac{\sec^2 x}{\tan x}dx = -\int\frac{\sec^2 y}{\tan y}dy$$

$$\Rightarrow \int\frac{\frac{d}{dx}(\tan x)}{\tan x}dx = -\int\frac{\frac{d}{dy}(\tan y)}{\tan y}dy$$

$$\Rightarrow \ln\tan x = -\ln\tan y+\ln c$$

$$\Rightarrow \ln\tan x+\ln\tan y = \ln c$$

$$\Rightarrow \ln(\tan x\tan y) = \ln c$$

$$\Rightarrow \tan x\tan y = c$$

Question # 14

$$\left(y-x\frac{dy}{dx}\right) = 2\left(y^2+\frac{dy}{dx}\right)$$

Solution

$$\left(y-x\frac{dy}{dx}\right) = 2\left(y^2+\frac{dy}{dx}\right)$$

$$\Rightarrow y-x\frac{dy}{dx} = 2y^2+2\frac{dy}{dx}$$

$$\Rightarrow y-2y^2 = 2\frac{dy}{dx}+x\frac{dy}{dx}$$

$$\Rightarrow y(1-2y) = (2+x)\frac{dy}{dx}$$

$$\Rightarrow \frac{dx}{2+x} = \frac{dy}{y(1-2y)}$$

On integrating

$$\int\frac{dx}{2+x} = \int\frac{dy}{y(1-2y)} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

Now consider

$$\frac{1}{y(1-2y)} = \frac{A}{y}+\frac{B}{1-2y}$$

$$\Rightarrow\ 1 = A(1-2y)+By\ \ldots\ldots\ldots\ \text{(ii)}$$

Put $y=0$ in (ii)

$$1 = A(1-2(0))+0 \Rightarrow\ A=1$$

Put $1-2y=0 \ \Rightarrow\ 2y=1 \ \Rightarrow\ y=\frac{1}{2}$ in (ii)

$$1 = 0+B\left(\frac{1}{2}\right) \quad \Rightarrow\ B=2$$

So $\dfrac{1}{y(1-2y)} = \dfrac{1}{y}+\dfrac{2}{1-2y}$

Using in (i) ←*

$$\int\frac{dx}{2+x} = \int\left(\frac{1}{y}+\frac{2}{1-2y}\right)dy$$

$$= \int\frac{1}{y}dy+\int\frac{2}{1-2y}dy$$

$$= \int\frac{1}{y}dy-\int\frac{-2}{1-2y}dy$$

$$\Rightarrow\ \int\frac{dx}{2+x} = \int\frac{1}{y}dy-\int\frac{\frac{d}{dx}(1-2y)}{1-2y}dy$$

$$\Rightarrow\ \ln(2+x) = \ln y-\ln(1-2y)-\ln c$$

$$\Rightarrow\ \ln(2+x)+\ln c = \ln y-\ln(1-2y)$$

$$\Rightarrow\ \ln c(2+x) = \ln\frac{y}{(1-2y)}$$

$$\Rightarrow\ c(2+x) = \frac{y}{(1-2y)}$$

$$\Rightarrow\ y = c(2+x)(1-2y)$$

Alternative (←*)

$$\int\frac{dx}{2+x} = \int\left(\frac{1}{y}+\frac{2}{1-2y}\right)dx$$

$$= \int\frac{1}{y}dy+\int\frac{2}{1-2y}dy$$

$$= \int\frac{1}{y}dy-\int\frac{2}{2y-1}dy$$

$$\Rightarrow\ \int\frac{dx}{2+x} = \int\frac{1}{y}dy-\int\frac{\frac{d}{dx}(2y-1)}{2y-1}dy$$

$$\Rightarrow\ \ln(2+x) = \ln y-\ln(2y-1)-\ln c$$

$$\Rightarrow\ \ln(2+x)+\ln c = \ln y-\ln(2y-1)$$

$$\Rightarrow\ \ln c(2+x) = \ln\frac{y}{(2y-1)}$$

$$\Rightarrow\ c(2+x) = \frac{y}{(2y-1)}$$

i.e. $\dfrac{y}{(2y-1)} = c(2+x)$

Review

- ○ $\int\tan x\,dx = \ln|\sec x| = -\ln|\cos x|$
- ○ $\int\cot x\,dx = \ln|\sin x| = -\ln|\csc x|$
- ○ $\int\sec x\,dx = \ln|\sec x+\tan x|$
- ○ $\int\csc x\,dx = \ln|\csc x-\cot x|$

Question # 15

$$1+\cos x\tan y\frac{dy}{dx} = 0$$

Solution

$$1+\cos x\tan y\frac{dy}{dx} = 0$$

$$\Rightarrow\ \cos x\tan y\frac{dy}{dx} = -1$$

$$\Rightarrow\ \tan y\,dy = -\frac{1}{\cos x}dx$$

$$\Rightarrow\ \tan y\,dy = -\sec x\,dx$$

On integrating

$$\int\tan y\,dy = -\int\sec x\,dx$$

$$\Rightarrow\ -\ln|\cos y| = -\ln|\sec x+\tan x|-\ln c$$

$$\Rightarrow\ \ln|\cos y| = +\ln|\sec x+\tan x|+\ln c$$

$$\Rightarrow\ \ln|\cos y| = \ln|c(\sec x+\tan x)|$$

$$\Rightarrow\ \cos y = c(\sec x+\tan x)$$

Question # 16

$$y-x\frac{dy}{dx} = 3\left(1+x\frac{dy}{dx}\right)$$

Solution

$$y-x\frac{dy}{dx} = 3\left(1+x\frac{dy}{dx}\right)$$

$$\Rightarrow\ y-x\frac{dy}{dx} = 3+3x\frac{dy}{dx}$$

$$\Rightarrow\ y-3 = 3x\frac{dy}{dx}+x\frac{dy}{dx}$$

$$= (3x+x)\frac{dy}{dx}$$

$$\Rightarrow\ y-3 = 4x\frac{dy}{dx} \quad \Rightarrow\ \frac{dx}{x} = 4\frac{dy}{y-3}$$

Now do yourself

Question # 17

$$\sec x+\tan y\frac{dy}{dx} = 0$$

Solution

$$\sec x+\tan y\frac{dy}{dx} = 0$$

$$\Rightarrow \tan y\frac{dy}{dx} = -\sec x$$

$$\Rightarrow \tan y\,dy = -\sec x\,dx$$

Now do yourself as Question # 15

Question # 18

$$\left(e^{x}+e^{-x}\right)\frac{dy}{dx} = e^{x}-e^{-x}$$

Solution

$$\left(e^{x}+e^{-x}\right)\frac{dy}{dx} = e^{x}-e^{-x}$$

$$\Rightarrow dy = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}dx$$

On integrating

$$\int dy = \int\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}dx$$

$$\Rightarrow y = \int\frac{\frac{d}{dx}\left(e^{x}+e^{-x}\right)}{e^{x}+e^{-x}}dx$$

$$\Rightarrow y = \ln\left(e^{x}+e^{-x}\right)+c$$

Question # 19

Find the general solution of the equation $\frac{dy}{dx}-x = xy^{2}$. Also find the perpendicular solution if $y=1$ when $x=0$.

Solution

$$\frac{dy}{dx}-x = xy^{2} \quad\Rightarrow\quad \frac{dy}{dx} = x+xy^{2}$$

$$\Rightarrow \frac{dy}{dx} = x\left(1+y^{2}\right) \quad\Rightarrow\quad \frac{dy}{1+y^{2}} = x\,dx$$

$$\Rightarrow \int\frac{dy}{1+y^{2}} = \int x\,dx$$

$$\Rightarrow Tan^{-1}y = \frac{x^{2}}{2}+c$$

$$\Rightarrow y = Tan\left(\frac{x^{2}}{2}+c\right)$$

Question # 20

Solve the differential equation $\frac{dx}{dt} = 2x$ given that $x=4$ when $t=0$

Solution

$$\frac{dx}{dt} = 2x \quad\Rightarrow\quad \frac{dx}{x} = 2dt$$

$$\Rightarrow \int\frac{dx}{x} = 2\int dt$$

$$\Rightarrow \ln x = 2t+\ln c$$

$$= \ln e^{2t}+\ln c \qquad \because\ \ln e^{x}=x$$

$$\Rightarrow \ln x = \ln ce^{2t}$$

$$\Rightarrow x = ce^{2t} \ ......\ \text{(i)}$$

When $t=0$ then $x=4$, putting in (i)

$$4 = ce^{2(0)} \quad\Rightarrow\quad 4 = c\,e^{0}$$

$$\Rightarrow 4 = c(1) \quad\Rightarrow\quad c = 4$$

Putting in (i)

$$\Rightarrow x = 4e^{2t}$$

Question # 21

Solve the differential equation $\frac{ds}{dt}+2st = 0$. Also find the perpendicular solution if $s=4e$, when $t=0$

Solution

$$\frac{ds}{dt}+2st = 0$$

$$\Rightarrow \frac{ds}{dt} = -2st \quad\Rightarrow\quad \frac{ds}{s} = -2t\,dt$$

On integrating

$$\int\frac{ds}{s} = -2\int t\,dt$$

$$\Rightarrow \ln s = -2\frac{t^{2}}{2}+\ln c$$

$$= -t^{2}+\ln c$$

$$= \ln e^{-t^{2}}+\ln c \qquad \because\ \ln e^{x}=x$$

$$\Rightarrow \ln s = \ln ce^{-t^{2}}$$

$$\Rightarrow s = ce^{-t^{2}} \ ....\ \text{(i)}$$

When $t=0$ then $s=4e$, using in (i)

$$4e = ce^{-(0)^{2}} \quad\Rightarrow\quad 4e = c(1)$$

$$\Rightarrow c=4e$$

Putting in (i)

$$s = 4e\cdot e^{-t^{2}}$$

$$\Rightarrow s = 4e^{1-t^{2}}$$

Question # 22

In a culture, bacteria increases at the rate proportional to the number of bacteria present. If bacteria are 200 initially and are doubled in 2 hours, find the number of bacteria present four hours later.

Solution

Number of bacteria initially $= 200$

No. of bacteria after two hours $= 2(200)$

$= 400$

No. of bacteria after four hours $= 2(400)$

$= 800$ *Ans.*

Question # 23

A ball is thrown vertically upward with a velocity of $2450 cm/\sec$. Neglecting air resistance, find

- (i) velocity of ball at any time t
- (ii) distance travelled in any time t
- (iii) maximum height attained by the ball.

Solution

i) When a body is projected upward its acceleration is $-g$. (where $g = 980\text{ cm/sec}^2$)

i.e. acceleration $= \dfrac{dv}{dt} = -g$,

where v is velocity of ball.

$$\Rightarrow \frac{dv}{dt} = -980$$

$$\Rightarrow dv = -980\,dt$$

On integrating

$$\int dv = -980\int dt$$

$$\Rightarrow v = -980t + c_1 \;\ldots\ldots\ldots\; \text{(i)}$$

Initially, when $t = 0$ then $v = 2450\text{ cm/sec}$

$$2450 = -980(0) + c_1$$

$$\Rightarrow c_1 = 2450$$

Putting in (i)

$$\boxed{v = -980t + 2450}$$

ii) Since velocity $= v = \dfrac{dx}{dt}$

where x is height of ball.

$$\Rightarrow \frac{dx}{dt} = -980t + 2450$$

$$\Rightarrow dx = (-980t + 2450)\,dt$$

Integrating

$$\int dx = \int(-980t + 2450)\,dt$$

$$\Rightarrow x = -980\frac{t^2}{2} + 2450t + c_2$$

$$\Rightarrow x = -490t^2 + 2450t + c_2 \;\ldots\ldots\ldots\; \text{(ii)}$$

Initially, when $t = 0$ then $x = 0$

$$0 = -490\,(0) + 2450(0) + c_2$$

$$\Rightarrow c_2 = 0$$

Putting value of c_2 in (ii)

$$\Rightarrow x = -490t^2 + 2450t + 0$$

$$\Rightarrow \boxed{x = 2450t - 490t^2}$$

iii)

$$\because \; v = -980t + 2450$$

When body is at max. height then $v = 0$

$$\Rightarrow -980t + 2450 = 0$$

$$\Rightarrow 980t = 2450 \quad \Rightarrow t = \frac{2450}{980}$$

$$\Rightarrow t = 2.5\sec$$

Since $x = 2450t - 4980t^2$

When $t = 2.5\sec$

$$x = 2450(2.5) - 490(2.5)^2$$

$$= 6125 - 3062.5$$

$$= 3062.5$$

Hence ball attains max. height of 3062.5 cm.
