Question # 1

Find the definition, the derivatives w.r.t 'x' of the following functions defined as:

(i) $2x^2+1$ (ii) $2-\sqrt{x}$ (iii) $\dfrac{1}{\sqrt{x}}$ (iv) $\dfrac{1}{x^3}$

(v) $\dfrac{1}{x-a}$ (vi) $x(x-3)$ (vii) $\dfrac{2}{x^4}$ (viii) $(x+4)^{\frac{1}{3}}$

(ix) $x^{\frac{3}{2}}$ (x) $x^{5/2}$ (xi) $x^m, m\in N$ (xii) $\dfrac{1}{x^m}, m\in N$

(xiii) x^{40} (xiv) x^{-100}

Solution

(i) Let $y=2x^2+1$

$$\Rightarrow\ y+\delta y=2(x+\delta x)^2+1 \qquad \Rightarrow\ \delta y=2(x+\delta x)^2+1-y$$

$$\Rightarrow\ \delta y=2\left(x^2+2x\delta x+\delta x^2\right)+1-2x^2-1 \qquad \because\ y=2x^2+1$$

$$\Rightarrow\ \delta y=2x^2+4x\delta x+2\delta x^2-2x^2 \quad \Rightarrow\ \delta y=2x^2+4x\delta x+2\delta x^2-2x^2$$

$$\Rightarrow\ \delta y=4x\delta x+2\delta x^2$$

$$=\delta x(4x+2\delta x)$$

Dividing by δx

$$\frac{\delta y}{\delta x}=4x+2\delta x$$

Taking limit when $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}(4x+2\delta x)$$

$$\Rightarrow\ \frac{dy}{dx}=4x+2(0)$$

$$\Rightarrow\ \frac{dy}{dx}=4x \quad \text{i.e.} \quad \boxed{\frac{d}{dx}\left(2x^2+1\right)=4x}$$

(ii) Let $y=2-\sqrt{x}$

$$\Rightarrow\ y+\delta y=2-\sqrt{x+\delta x} \qquad \Rightarrow\ \delta y=2-\sqrt{x+\delta x}-y$$

$$\Rightarrow\ \delta y=2-\sqrt{x+\delta x}-2+\sqrt{x} \qquad \Rightarrow\ \delta y=x^{\frac{1}{2}}-(x+\delta x)^{\frac{1}{2}}$$

$$\Rightarrow\ \delta y=x^{\frac{1}{2}}-x^{\frac{1}{2}}\left(1+\frac{\delta x}{x}\right)^{\frac{1}{2}}$$

$$\Rightarrow\ \delta y=x^{\frac{1}{2}}-x^{\frac{1}{2}}\left(1+\frac{1}{2}\cdot\frac{\delta x}{x}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+\ldots\right)$$

$$= x^{\frac{1}{2}} - x^{\frac{1}{2}} - x^{\frac{1}{2}}\left(\frac{\delta x}{2x} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\frac{\delta x^2}{x^2} + \ldots\right)$$

$$= -x^{\frac{1}{2}}\delta x\left(\frac{1}{2x} - \frac{1}{8}\frac{\delta x}{x^2} + \ldots\right)$$

Dividing by δx, we have

$$\frac{\delta y}{\delta x} = -x^{\frac{1}{2}}\left(\frac{1}{2x} - \frac{1}{8}\frac{\delta x}{x^2} + \ldots\right)$$

Taking limit as

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = -x^{\frac{1}{2}}\lim_{\delta x \to 0}\left(\frac{1}{2x} - \frac{1}{8}\frac{\delta x}{x^2} + \ldots\right)$$

$$\Rightarrow \frac{dy}{dx} = -x^{\frac{1}{2}}\left(\frac{1}{2x} - 0 + 0 - \ldots\right)$$

$$= -x^{\frac{1}{2}} \cdot \frac{1}{2x} = -\frac{1}{2}x^{\frac{1}{2}-1} \Rightarrow \boxed{\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}}}$$

(iii) Let $y = \frac{1}{\sqrt{x}} \Rightarrow y = x^{-\frac{1}{2}}$

Now do yourself as above

(iv) Let $y = \frac{1}{x^3} \Rightarrow y = x^{-3}$

$$\Rightarrow y + \delta y = (x + \delta x)^{-3}$$

$$\Rightarrow \delta y = (x + \delta x)^{-3} - x^{-3}$$

$$\Rightarrow \delta y = x^{-3}\left[\left(1 + \frac{\delta x}{x}\right)^{-3} - 1\right]$$

$$= x^{-3}\left[\left(1 - \frac{3\delta x}{x} + \frac{-3(-3-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \ldots\right) - 1\right]$$

$$= x^{-3}\left[1 - \frac{3\delta x}{x} + \frac{-3(-4)}{2}\left(\frac{\delta x}{x}\right)^2 + \ldots - 1\right]$$

$$= x^{-3}\left[-\frac{3\delta x}{x} + \frac{-3(-4)}{2}\left(\frac{\delta x}{x}\right)^2 + \ldots\right]$$

$$= x^{-3} \cdot \frac{\delta x}{x}\left[-3 + 6\left(\frac{\delta x}{x}\right) - \ldots\right]$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x}=x^{-3-1}\left[-3+6\left(\frac{\delta x}{x}\right)-.....\right]$$

Taking limit on both sided, we get

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}x^{-4}\left[-3+6\left(\frac{\delta x}{x}\right)-.....\right]$$

$$\Rightarrow \frac{dy}{dx}=x^{-4}[-3+0-0+.....]$$

$$\Rightarrow \frac{dy}{dx}=-3x^{-4} \quad \text{or} \quad \boxed{\frac{dy}{dx}=-\frac{3}{x^4}}$$

(v) Let $y=\frac{1}{x-a}$

$$\Rightarrow y=(x-a)^{-1}$$

$$\Rightarrow y+\delta y=(x+\delta x-a)^{-1}$$

$$\Rightarrow \delta y=(x-a+\delta x)^{-1}-y$$

$$\Rightarrow \delta y=(x-a+\delta x)^{-1}-(x-a)^{-1}$$

$$=(x-a)^{-1}\left[\left(1+\frac{\delta x}{x-a}\right)^{-1}-1\right]$$

$$=(x-a)^{-1}\left[\left(1-\frac{\delta x}{x-a}+\frac{-1(-1-1)}{2!}\left(\frac{\delta x}{x-a}\right)^2+....\right)-1\right]$$

$$\Rightarrow \delta y=(x-a)^{-1}\left[1-\frac{\delta x}{x-a}+\frac{-1(-1-1)}{2!}\left(\frac{\delta x}{x-a}\right)^2+....-1\right]$$

$$=(x-a)^{-1}\left[-\frac{\delta x}{x-a}+\frac{-1(-2)}{2}\left(\frac{\delta x}{x-a}\right)^2+....\right]$$

$$=(x-a)^{-1}\cdot\frac{\delta x}{x-a}\left[-1+\left(\frac{\delta x}{x-a}\right)-....\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x}=(x-a)^{-1-1}\left[-1+\left(\frac{\delta x}{x-a}\right)-....\right]$$

Taking limit when $\delta x\to 0$, we have

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}(x-a)^{-1-1}\left[-1+\left(\frac{\delta x}{x-a}\right)-....\right]$$

$$\Rightarrow \frac{dy}{dx}=(x-a)^{-2}[-1+0-0+....] \quad \Rightarrow \boxed{\frac{dy}{dx}=\frac{-1}{(x-a)^2}}$$

(vi) Let $y = x(x-3)$

$= x^2 - 3x$

Do yourself

(vii) Let $y = \dfrac{2}{x^4} = 2x^{-4}$

$\Rightarrow y+\delta y = 2(x+\delta x)^{-4}$

Do yourself

(viii) Let $y = (x+4)^{\frac{1}{3}}$

$$\Rightarrow y+\delta y=(x+\delta x+4)^{\frac{1}{3}}$$

$$\Rightarrow \delta y=(x+\delta x+4)^{\frac{1}{3}} - y$$

$$=(x+4+\delta x)^{\frac{1}{3}}-(x+4)^{\frac{1}{3}}$$

$$=(x+4)^{\frac{1}{3}}\left[\left(1+\frac{\delta x}{x+4}\right)^{\frac{1}{3}}-1\right]$$

$$=(x+4)^{\frac{1}{3}}\left[\left(1+\frac{1}{3}\frac{\delta x}{x+4}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{\delta x}{x+4}\right)^2+....\right)-1\right]$$

$$=(x+4)^{\frac{1}{3}}\left[\frac{\delta x}{3(x+4)}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}\left(\frac{\delta x}{x+4}\right)^2+....\right]$$

$$=(x+4)^{\frac{1}{3}}\cdot\frac{\delta x}{x+4}\left[\frac{1}{3}-\frac{1}{9}\left(\frac{\delta x}{x+4}\right)+....\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x}=(x+4)^{\frac{1}{3}-1}\left[\frac{1}{3}-\frac{1}{9}\left(\frac{\delta x}{x+4}\right)+....\right]$$

Taking limit when $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}(x+4)^{-\frac{2}{3}}\left[\frac{1}{3}-\frac{1}{9}\left(\frac{\delta x}{x+4}\right)+....\right]$$

$$\Rightarrow \frac{dy}{dx}=(x+4)^{-\frac{2}{3}}\left[\frac{1}{3}-0+0-....\right] \qquad \Rightarrow \boxed{\frac{dy}{dx}=\frac{1}{3}(x+4)^{-\frac{2}{3}}}$$

(ix) Let $y = x^{\frac{3}{2}}$

$$\Rightarrow y+\delta y=(x+\delta x)^{\frac{3}{2}}$$

$$\Rightarrow \delta y=(x+\delta x)^{\frac{3}{2}}-x^{\frac{3}{2}}$$

$$=x^{\frac{3}{2}}\left[\left(1+\frac{\delta x}{x}\right)^{\frac{3}{2}}-1\right]$$

$$=x^{\frac{3}{2}}\left[\left(1+\frac{3}{2}\frac{\delta x}{x}+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^{2}+....\right)-1\right]$$

$$=x^{\frac{3}{2}}\left[\frac{3\delta x}{2x}+\frac{\frac{3}{2}\left(\frac{1}{2}\right)}{2}\left(\frac{\delta x}{x}\right)^{2}+....\right]$$

$$=x^{\frac{3}{2}}\cdot\frac{\delta x}{x}\left[\frac{3}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+....\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x}=x^{\frac{3}{2}-1}\left[\frac{3}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+....\right]$$

Taking limit when $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}x^{\frac{1}{2}}\left[\frac{3}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+....\right]$$

$$\Rightarrow \frac{dy}{dx}=x^{\frac{1}{2}}\left[\frac{3}{2}-0+0-....\right] \qquad \Rightarrow \boxed{\frac{dy}{dx}=\frac{3}{2}x^{\frac{1}{2}}}$$

(x) Let $y = x^{5/2}$

Do yourself as above.

(xi) Let $y = x^m$

$$\Rightarrow y+\delta y = (x+\delta x)^m$$

$$\Rightarrow \delta y = (x+\delta x)^m - x^m$$

$$= x^m\left[\left(1+\frac{\delta x}{x}\right)^m-1\right]$$

$$= x^m\left[\left(1+m\cdot\frac{\delta x}{x}+\frac{m(m-1)}{2!}\left(\frac{\delta x}{x}\right)^2+....\right)-1\right]$$

$$= x^m\left[\frac{m\delta x}{x}+\frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right)^2+....\right]$$

$$= x^m\cdot\frac{\delta x}{x}\left[m+\frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right)+....\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x} = x^{m-1}\left[m + \frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right) + \dots\right]$$

Taking limit when $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} x^{m-1}\left[m + \frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right) + \dots\right]$$

$$\Rightarrow \frac{dy}{dx} = x^{m-1}\left[m + 0 + 0\dots\right] \Rightarrow \boxed{\frac{dy}{dx} = mx^{m-1}}$$

(xii) Let $y = \dfrac{1}{x^m} = x^{-m}$

Do yourself as above, just change the m by $-m$ in above question.

(xiii) Let $y = x^{40}$

$$\Rightarrow y + \delta y = (x+\delta x)^{40}$$

$$\Rightarrow \delta y = (x+\delta x)^{40} - x^{40}$$

$$= \left[\binom{40}{0}x^{40} + \binom{40}{1}x^{39}\delta x + \binom{40}{2}x^{38}\delta x^2 + \dots + \binom{40}{40}\delta x^{40}\right] - x^{40}$$

$$= (1)x^{40} + \binom{40}{1}x^{39}\delta x + \binom{40}{2}x^{38}\delta x^2 + \dots + \binom{40}{40}\delta x^{40} - x^{40}$$

$$= \binom{40}{1}x^{39}\delta x + \binom{40}{2}x^{38}\delta x^2 + \dots + \binom{40}{40}\delta x^{40}$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \binom{40}{1}x^{39} + \binom{40}{2}x^{38}\delta x + \dots + \binom{40}{40}\delta x^{39}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[\binom{40}{1}x^{39} + \binom{40}{2}x^{38}\delta x + \dots + \binom{40}{40}\delta x^{39}\right]$$

$$\frac{dy}{dx} = \left[\binom{40}{1}x^{39} + 0 + 0 + \dots + 0\right]$$

$$\Rightarrow \frac{dy}{dx} = \binom{40}{1}x^{39} \quad \text{or} \quad \boxed{\frac{dy}{dx} = 40x^{39}}$$

(xiv) Let $y = x^{-100}$

Do yourself Question # 1(xii), Replace m by -100.

Question # 2

Find $\frac{dy}{dx}$ from the first principles if

(i) $\sqrt{x+2}$ (ii) $\frac{1}{\sqrt{x+a}}$

Solution

(i) Let $y = \sqrt{x+2} = (x+2)^{\frac{1}{2}}$

Now do yourself as Question # 1(ix)

(ii) Let $y = \frac{1}{\sqrt{x+a}} = (x+a)^{-\frac{1}{2}}$

Now do yourself as Question # 1 (ix)

Question # 1

Find from first principles, the derivatives of the following expensions w.r.t. their respective independent variables:

(i) $(ax+b)^3$ (ii) $(2x+3)^5$ (iii) $(3t+2)^{-2}$

(iv) $(ax+b)^{-5}$ (v) $\dfrac{1}{(az-b)^7}$

Solution

(i) Let $y = (ax+b)^3$

$$\Rightarrow y+\delta y = (a(x+\delta x)+b)^3$$

$$\Rightarrow \delta y = (ax+b+a\delta x)^3 - y$$

$$= ((ax+b)+a\delta x)^3 - (ax+b)^3$$

$$= \left[(ax+b)^3 + 3(ax+b)^2(a\delta x) + 3(ax+b)(a\delta x)^2 + (a\delta x)^3\right] - (ax+b)^3$$

$$= 3a(ax+b)^2\delta x + 3a^2(ax+b)\delta x^2 + a^3\delta x^3$$

$$= \delta x\left(3a(ax+b)^2 + 3a^2(ax+b)\delta x + a^3\delta x^2\right)$$

Dividing by δx

$$\frac{\delta y}{\delta x} = 3a(ax+b)^2 + 3a^2(ax+b)\delta x + a^3\delta x^2$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[3a(ax+b)^2 + 3a^2(ax+b)\delta x + a^3\delta x^2\right]$$

$$\Rightarrow \frac{dy}{dx} = 3a(ax+b)^2 + 3a^2(ax+b)(0) + a^3(0)^2$$

$$\Rightarrow \frac{dy}{dx} = 3a(ax+b)^2 + 0 + 0 \quad \Rightarrow \boxed{\frac{dy}{dx} = 3a(ax+b)^2}$$

(ii) Let $y = (2x+3)^5$

$$\Rightarrow y+\delta y = (2(x+\delta x)+3)^5$$

$$\Rightarrow \delta y = (2x+2\delta x+3)^5 - y$$

$$= ((2x+3)+2\delta x)^5 - (2x+3)^5$$

$$= \left[\binom{5}{0}(2x+3)^5 + \binom{5}{1}(2x+3)^4(2\delta x) + \binom{5}{2}(2x+3)^3(2\delta x)^2 + \dots\right.$$

$$\left.\dots + \binom{5}{5}(2\delta x)^5\right] - (2x+3)^5$$

$$= \left[(1)(2x+3)^5 + 2\binom{5}{1}(2x+3)^4\delta x + 4\binom{5}{2}(2x+3)^3\delta x^2 + \ldots \right.$$

$$\left. \ldots + 32\binom{5}{5}\delta x^5\right] - (2x+3)^5$$

$$= 2\binom{5}{1}(2x+3)^4\delta x + 4\binom{5}{2}(2x+3)^3\delta x^2 + \ldots + 32\binom{5}{5}\delta x^5$$

Dividing by δx

$$\frac{\delta y}{\delta x} = 2\binom{5}{1}(2x+3)^4 + 4\binom{5}{2}(2x+3)^3\delta x + \ldots + 32\binom{5}{5}\delta x^4$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[2\binom{5}{1}(2x+3)^4 + 4\binom{5}{2}(2x+3)^3\delta x + \ldots + 32\binom{5}{5}\delta x^4\right]$$

$$\Rightarrow \frac{dy}{dx} = \left[2\binom{5}{1}(2x+3)^4 + 0 + 0 + \ldots + 0\right]$$

$$\Rightarrow \frac{dy}{dx} = 2(5)(2x+3)^4 \quad \text{or} \quad \boxed{\frac{dy}{dx} = 10(2x+3)^4}$$

(iii) Let $y = (3t+2)^{-2}$

$$\Rightarrow y + \delta y = (3(t+\delta t)+2)^{-2}$$

$$\Rightarrow \delta y = (3t+3\delta t+2)^{-2} - y$$

$$\Rightarrow \delta y = ((3t+2)+3\delta t)^{-2} - (3t+2)^{-2}$$

$$= (3t+2)^{-2}\left(1+\frac{3\delta t}{3t+2}\right)^{-2} - (3t+2)^{-2} = (3t+2)^{-2}\left[\left(1+\frac{3\delta t}{3t+2}\right)^{-2} - 1\right]$$

$$= (3t+2)^{-2}\left[\left(1+(-2)\frac{3\delta t}{3t+2} + \frac{-2(-2-1)}{2!}\left(\frac{3\delta t}{3t+2}\right)^2 + \ldots\right) - 1\right]$$

$$\Rightarrow \delta y = (3t+2)^{-2}\left[1 - \frac{6\delta t}{3t+2} + \frac{-2(-3)}{2}\left(\frac{\delta t}{3t+2}\right)^2 + \ldots - 1\right]$$

$$= (3t+2)^{-2}\left[-\frac{6\delta t}{3t+2} + 3\left(\frac{3\delta t}{3t+2}\right)^2 + \ldots\right]$$

$$= (3t+2)^{-1}\cdot\frac{3\delta t}{3t+2}\left[-2 + 3\left(\frac{3\delta t}{3t+2}\right) + \ldots\right]$$

Dividing by δt

$$\frac{\delta y}{\delta t} = 3(3t+2)^{-2-1}\left[-2 + \left(\frac{3\delta t}{3t+2}\right) + \ldots\right]$$

Taking limit when $\delta t \to 0$, we have

$$\lim_{\delta t\to 0}\frac{\delta y}{\delta t}=\lim_{\delta t\to 0}3(3t+2)^{-3}\left[-2+\left(\frac{3\delta t}{3t+2}\right)+....\right]$$

$$\Rightarrow \frac{dy}{dx}=3(3t+2)^{-3}[-2+0-0+....] \Rightarrow \boxed{\frac{dy}{dx}=-6(3t+2)^{-3}}$$

(iv) Let $y = (ax+b)^{-5}$

Do yourself

(v) Let $y = \dfrac{1}{(az-b)^7} = (az-b)^{-7}$

$$\Rightarrow y+\delta y = \left(a(z+\delta z)-b\right)^{-7}$$

$$\Rightarrow \delta y = \left((az-b)+a\delta z\right)^{-7}-\left(az-b\right)^{-7}$$

$$\Rightarrow \delta y = \left(az-b\right)^{-7}\left[\left(1+\frac{a\delta z}{(az-b)}\right)^{-7}-1\right]$$

Differentiate w.r.t. 'x'

Question # 1

$x^4+2x^3+x^2$

Solution Let $y = x^4+2x^3+x^2$

Differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^4+2x^3+x^2\right)$$
$$= \frac{d}{dx}x^4+2\frac{d}{dx}x^3+\frac{d}{dx}x^2$$
$$= 4x^{4-1}+2\left(3x^{3-1}\right)+2x^{2-1}$$
$$= 4x^3+6x^2+2x$$

Question # 2

$x^{-3}+2x^{-\frac{3}{2}}+3$

Solution Let $y=x^{-3}+2x^{-\frac{3}{2}}+3$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{-3}+2x^{-\frac{3}{2}}+3\right)$$
$$=\frac{d}{dx}x^{-3}+2\frac{d}{dx}x^{-\frac{3}{2}}+\frac{d}{dx}(3)$$
$$=-3x^{-3-1}+2\left(-\frac{3}{2}x^{-\frac{3}{2}-1}\right)+0$$
$$\Rightarrow \frac{dy}{dx}=-3x^{-4}-3x^{-\frac{5}{2}}$$
$$\text{or} \quad \frac{dy}{dx}=-3\left(\frac{1}{x^4}+\frac{1}{x^{5/2}}\right)$$

Question # 3

$\dfrac{a+x}{a-x}$

Solution Let $y=\dfrac{a+x}{a-x}$

$$\text{Now} \quad \frac{dy}{dx}=\frac{d}{dx}\left(\frac{a+x}{a-x}\right) = \frac{(a-x)\frac{d}{dx}(a+x)-(a+x)\frac{d}{dx}(a-x)}{(a-x)^2}$$
$$=\frac{(a-x)(0+1)-(a+x)(0-1)}{(a-x)^2}$$

$$=\frac{(a-x)(1)-(a+x)(-1)}{(a-x)^2}$$

$$=\frac{a-x+a+x}{(a-x)^2} \quad =\frac{2a}{(a-x)^2} \qquad \textit{Answer}$$

Question # 4

$$\frac{2x-3}{2x+1}$$

Solution Let $y=\dfrac{2x-3}{2x+1}$

Now $\dfrac{dy}{dx}=\dfrac{d}{dx}\left(\dfrac{2x-3}{2x+1}\right)$

$$=\frac{(2x+1)\frac{d}{dx}(2x-3)-(2x-3)\frac{d}{dx}(2x+1)}{(2x+1)^2}$$

$$=\frac{(2x+1)(2-0)-(2x-3)(2+0)}{(2x+1)^2}$$

$$=\frac{(2x+1)(2)-(2x-3)(2)}{(2x+1)^2}$$

$$=\frac{2(2x+1-2x+3)}{(2x+1)^2}$$

$$=\frac{2(4)}{(2x+1)^2} \quad =\frac{8}{(2x+1)^2} \qquad \textit{Answer}$$

Question # 5

$(x-5)(3-x)$

Solution Let $y=(x-5)(3-x)$

$$=3x-x^2-15+5x$$

$$=-x^2+8x-15$$

Now

$$\frac{dy}{dx}=\frac{dy}{dx}\left(-x^2+8x-15\right)$$

$$=\frac{dy}{dx}\left(-x^2\right)+8\frac{d}{dx}(x)-\frac{d}{dx}(15)$$

$$=-2x^{2-1}+8(1)-0 \quad =-2x+8 \qquad \textit{Answer}$$

Question # 6

$$\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$$

Solution Let $y=\left(\sqrt{x}-\dfrac{1}{\sqrt{x}}\right)^2$

$$=\left(\sqrt{x}\right)^2+\left(\frac{1}{\sqrt{x}}\right)^2-2\left(\sqrt{x}\right)\left(\frac{1}{\sqrt{x}}\right)$$

$$=x+\frac{1}{x}-2 \quad =x+x^{-1}-2$$

Now diff. w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\left(x+x^{-1}-2\right) \quad =\frac{d}{dx}(x)+\frac{d}{dx}(x^{-1})-\frac{d}{dx}(2)$$

$$=1+(-1\cdot x^{-1-1})-0= 1-x^{-2}$$

$$=1-\frac{1}{x^2} \quad = \frac{x^2-1}{x^2} \qquad \textit{Answer}$$

Question # 7

$$\frac{\left(1+\sqrt{x}\right)\left(x-x^{3/2}\right)}{\sqrt{x}}$$

Solution Consider $y = \dfrac{\left(1+\sqrt{x}\right)\left(x-x^{3/2}\right)}{\sqrt{x}}$

$$=\frac{\left(1+\sqrt{x}\right)\; x\left(1-x^{\frac{1}{2}}\right)}{\sqrt{x}}$$

$$=\frac{x\left(1+\sqrt{x}\right)\left(1-\sqrt{x}\right)}{\sqrt{x}} \qquad \text{Since } x^{\frac{3}{2}}=x^{1+\frac{1}{2}}$$

$$=\frac{\left(\sqrt{x}\right)^2\left(1-\left(\sqrt{x}\right)^2\right)}{\sqrt{x}}$$

$$=\sqrt{x}(1-x) \quad = x^{\frac{1}{2}}(1-x) \quad = x^{\frac{1}{2}}-x^{\frac{3}{2}}$$

Now

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{\frac{1}{2}}-x^{\frac{3}{2}}\right)$$

$$=\frac{1}{2}x^{\frac{1}{2}-1}-\frac{3}{2}x^{\frac{3}{2}-1}$$

$$=\frac{1}{2}x^{-\frac{1}{2}}-\frac{3}{2}x^{\frac{1}{2}}$$

$$=\frac{1}{2}\left(\frac{1}{\sqrt{x}}-3\sqrt{x}\right) \qquad \textit{Answer}$$

Question # 8

$$\frac{\left(x^2+1\right)^2}{x^2-1}$$

Solution Let $y = \dfrac{\left(x^2+1\right)^2}{x^2-1}$

Differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\left(x^2+1\right)^2}{x^2-1}\right)$$

$$= \frac{\left(x^2-1\right)\frac{d}{dx}\left(x^2+1\right)^2-\left(x^2+1\right)^2\frac{d}{dx}\left(x^2-1\right)}{\left(x^2-1\right)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\left(x^2-1\right)\,2\left(x^2+1\right)^{2-1}\frac{d}{dx}\left(x^2+1\right)-\left(x^2+1\right)^2(2x)}{\left(x^2-1\right)^2}$$

$$= \frac{\left(x^2-1\right)\,2\left(x^2+1\right)(2x)-\left(x^2+1\right)^2(2x)}{\left(x^2-1\right)^2}$$

$$= \frac{2x\left(x^2+1\right)\left[2\left(x^2-1\right)-\left(x^2+1\right)\right]}{\left(x^2-1\right)^2}$$

$$= \frac{2x\left(x^2+1\right)\left[2x^2-2-x^2-1\right]}{\left(x^2-1\right)^2}$$

$$= \frac{2x\left(x^2+1\right)\left(x^2-3\right)}{\left(x^2-1\right)^2} \quad \textit{Answer}$$

Question # 9

$$\frac{x^2+1}{x^2-3}$$

Solution Let $y = \dfrac{x^2+1}{x^2-3}$

Differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-3}\right)$$

$$= \frac{\left(x^2-3\right)\frac{d}{dx}\left(x^2+1\right)-\left(x^2+1\right)\frac{d}{dx}\left(x^2-3\right)}{\left(x^2-3\right)^2}$$

$$= \frac{(x^2-3)\ (2x)-(x^2+1)(2x)}{(x^2-3)^2} = \frac{2x(x^2-3-x^2-1)}{(x^2-3)^2}$$

$$= \frac{2x(-4)}{(x^2-3)^2} = \frac{-8x}{(x^2-3)^2} \qquad \textit{Answer}$$

Question # 10

$$\frac{\sqrt{1+x}}{\sqrt{1-x}}$$

Solution Let $y = \frac{\sqrt{1+x}}{\sqrt{1-x}} = \left(\frac{1+x}{1-x}\right)^{1/2}$

Now $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1+x}{1-x}\right)^{1/2}$

$$= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\left(\frac{(1-x)\frac{d}{dx}(1+x)-(1+x)\frac{d}{dx}(1-x)}{(1-x)^2}\right)$$

$$= \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}\left(\frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}\right)$$

$$= \frac{1}{2}\frac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}}\left(\frac{1-x+1+x}{(1-x)^2}\right) = \frac{(1-x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}}\left(\frac{2}{(1-x)^2}\right)$$

$$= \frac{1}{(1+x)^{\frac{1}{2}}(1-x)^{2-\frac{1}{2}}} = \frac{1}{\sqrt{1+x}\ (1-x)^{\frac{3}{2}}} \qquad \textit{Answer}$$

Question # 11

$$\frac{2x-1}{\sqrt{x^2+1}}$$

Solution Let $y = \frac{2x-1}{\sqrt{x^2+1}}$

Differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x-1}{(x^2+1)^{1/2}}\right)$$

$$= \frac{(x^2+1)^{1/2}\ \frac{d}{dx}(2x-1)-(2x-1)\frac{d}{dx}(x^2+1)^{1/2}}{\left((x^2+1)^{1/2}\right)^2}$$

$$= \frac{\left(x^2+1\right)^{1/2}(2)-(2x-1)\,\frac{1}{2}\left(x^2+1\right)^{-1/2}\frac{d}{dx}(x^2+1)}{\left(x^2+1\right)}$$

$$= \frac{2\left(x^2+1\right)^{1/2}-(2x-1)\dfrac{1}{2\left(x^2+1\right)^{1/2}}(2x)}{\left(x^2+1\right)}$$

$$= \frac{1}{\left(x^2+1\right)}\left(2\left(x^2+1\right)^{1/2}-\frac{2x^2-x}{\left(x^2+1\right)^{1/2}}\right)$$

$$= \frac{1}{\left(x^2+1\right)}\left(\frac{2x^2+2-2x^2+x}{\left(x^2+1\right)^{1/2}}\right)$$

$$= \frac{x+2}{\left(x^2+1\right)\sqrt{x^2+1}} \quad \text{or} \quad \frac{x+2}{\left(x^2+1\right)^{3/2}} \quad \textit{Answer}$$

Question # 12

$$\frac{\sqrt{a-x}}{\sqrt{a+x}}$$

Solution

Do yourself as Question # 10

Question # 13

$$\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$$

Solution Let $y = \dfrac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$

$$= \left(\frac{x^2+1}{x^2-1}\right)^{\frac{1}{2}}$$

Differentiating w.r.t x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$=\frac{1}{2}\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}}\left(\frac{(x^2-1)(2x)-(x^2+1)(2x)}{\left(x^2-1\right)^2}\right)$$

$$=\frac{1}{2}\frac{\sqrt{x^2-1}}{\sqrt{x^2+1}}\left(\frac{2x^3-2x-2x^3-2x}{\left(x^2-1\right)^2}\right)$$

$$=\frac{1}{2}\frac{\sqrt{x^2-1}}{\sqrt{x^2+1}}\left(\frac{-4x}{\left(x^2-1\right)^2}\right)$$

$$=\frac{-2\sqrt{x^2-1}}{\left(x^2-1\right)^2\sqrt{x^2+1}}=\frac{-2}{\left(x^2-1\right)^{2-\frac{1}{2}}\sqrt{x^2+1}}$$

$$=\frac{-2}{\left(x^2-1\right)^{\frac{3}{2}}\sqrt{x^2+1}}\quad \textit{Answer}$$

Question # 14

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$$

Solution Assume $y=\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

$$=\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\cdot\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\qquad \textit{Rationalizing}$$

$$=\frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)^2}{\left(\sqrt{1+x}\right)^2-\left(\sqrt{1-x}\right)^2}$$

$$=\frac{\left(\sqrt{1+x}\right)^2+\left(\sqrt{1-x}\right)^2-2\left(\sqrt{1+x}\right)\left(\sqrt{1-x}\right)}{1+x-1+x}$$

$$=\frac{1+x+1-x-2\sqrt{(1+x)(1-x)}}{2x}$$

$$=\frac{2-2\sqrt{1-x^2}}{2x}=\frac{2\left(1-\left(1-x^2\right)^{\frac{1}{2}}\right)}{2x}$$

$$=\frac{1-\left(1-x^2\right)^{\frac{1}{2}}}{x}$$

Now differentiation w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1-\left(1-x^2\right)^{\frac{1}{2}}}{x}\right)$$

$$= \frac{x\,\frac{d}{dx}\left(1-\left(1-x^2\right)^{\frac{1}{2}}\right)-\left(1-\left(1-x^2\right)^{\frac{1}{2}}\right)\frac{d}{dx}x}{x^2}$$

$$= \frac{1}{x^2}\cdot\left[x\left(0-\frac{1}{2}\left(1-x^2\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(1-x^2\right)\right)-\left(1-\left(1-x^2\right)^{\frac{1}{2}}\right)(1)\right]$$

$$= \frac{1}{x^2}\cdot\left[x\left(-\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x)\right)-1+\left(1-x^2\right)^{\frac{1}{2}}\right]$$

$$= \frac{1}{x^2}\cdot\left[\frac{x^2}{\left(1-x^2\right)^{\frac{1}{2}}}-1+\left(1-x^2\right)^{\frac{1}{2}}\right] = \frac{1}{x^2}\cdot\left[\frac{x^2-\left(1-x^2\right)^{\frac{1}{2}}+1-x^2}{\left(1-x^2\right)^{\frac{1}{2}}}\right]$$

$$= \frac{1}{x^2}\cdot\left[\frac{1-\left(1-x^2\right)^{\frac{1}{2}}}{\left(1-x^2\right)^{\frac{1}{2}}}\right] = \frac{1-\sqrt{1-x^2}}{x^2\sqrt{1-x^2}} \quad \textit{Answer}$$

Question # 15

$$\frac{x\sqrt{a+x}}{\sqrt{a-x}}$$

Solution Let $y = \dfrac{x\sqrt{a+x}}{\sqrt{a-x}} = x\left(\dfrac{a+x}{a-x}\right)^{1/2}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}x\left(\frac{a+x}{a-x}\right)^{1/2}$$

$$= x\frac{d}{dx}\left(\frac{a+x}{a-x}\right)^{1/2} + \left(\frac{a+x}{a-x}\right)^{1/2}\frac{d}{dx}x \quad \ldots\ldots\ldots\ldots \text{(i)}$$

Now $\dfrac{d}{dx}\left(\dfrac{a+x}{a-x}\right)^{1/2} = \dfrac{1}{2}\left(\dfrac{a+x}{a-x}\right)^{1/2-1}\dfrac{d}{dx}\left(\dfrac{a+x}{a-x}\right)$

$$= \frac{1}{2}\left(\frac{a+x}{a-x}\right)^{-1/2}\left(\frac{(a-x)\,\frac{d}{dx}(a+x)-(a+x)\frac{d}{dx}(a-x)}{(a-x)^2}\right)$$

$$= \frac{1}{2}\left(\frac{a-x}{a+x}\right)^{1/2}\left(\frac{(a-x)\,(1)-(a+x)(-1)}{(a-x)^2}\right)$$

$$= \frac{1}{2}\frac{(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}}\left(\frac{a-x+a+x}{(a-x)^{2}}\right) = \frac{1}{2}\frac{1}{(a+x)^{\frac{1}{2}}(a-x)^{-\frac{1}{2}}}\cdot\left(\frac{2a}{(a-x)^{2}}\right)$$

$$= \frac{a}{(a+x)^{\frac{1}{2}}(a-x)^{2-\frac{1}{2}}} = \frac{a}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}}$$

Using in eq. (i)

$$\frac{dy}{dx} = x\cdot\frac{a}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}} + \left(\frac{a+x}{a-x}\right)^{1/2}(1)$$

$$= \frac{ax}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}} + \frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$= \frac{ax+(a+x)(a-x)}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}} = \frac{ax+a^2-x^2}{\sqrt{a+x}(a-x)^{\frac{3}{2}}} \quad \textit{Answer}$$

Question # 16

If $y=\sqrt{x}-\frac{1}{\sqrt{x}}$, show that $2x\frac{dy}{dx}+y= 2\sqrt{x}$

Solution Since $y = \sqrt{x}-\frac{1}{\sqrt{x}}$

$$= x^{\frac{1}{2}}-x^{-\frac{1}{2}}$$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)$$

$$= \frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{-\frac{3}{2}}$$

Multiplying by $2x$

$$2x\frac{dy}{dx} = 2x\left(\frac{1}{2}x^{-\frac{1}{2}}\right)+2x\left(\frac{1}{2}x^{-\frac{3}{2}}\right) \Rightarrow 2x\frac{dy}{dx} = x^{-\frac{1}{2}+1}+x^{-\frac{3}{2}+1}$$

$$2x\frac{dy}{dx} = x^{\frac{1}{2}}+x^{-\frac{1}{2}}$$

Adding y on both sides

$$2x\frac{dy}{dx}+y= x^{\frac{1}{2}}+x^{-\frac{1}{2}}+y$$

$$\Rightarrow 2x\frac{dy}{dx}+y= x^{\frac{1}{2}}+x^{-\frac{1}{2}}+x^{\frac{1}{2}}-x^{-\frac{1}{2}} \qquad \because\ y=x^{\frac{1}{2}}-x^{-\frac{1}{2}}$$

$$\Rightarrow 2x\frac{dy}{dx}+y= 2x^{\frac{1}{2}} \qquad \Rightarrow 2x\frac{dy}{dx}+y= 2\sqrt{x} \qquad \textit{Proved}$$

Question # 17

If $y = x^4+2x^2+2$, prove that $\frac{dy}{dx} = 4x\sqrt{y-1}$

Solution Since $y = x^4 + 2x^2 + 2$

Now $\frac{dy}{dx} = \frac{d}{dx}\left(x^4 + 2x^2 + 2\right)$

$\Rightarrow \frac{dy}{dx} = 4x^{4-1} + 2\left(2x^{2-1}\right) + 0$

$= 4x^3 + 4x$

$\Rightarrow \frac{dy}{dx} = 4x\left(x^2 + 1\right)$ ……… (i)

Now $y = x^4 + 2x^2 + 2$

$\Rightarrow y - 1 = x^4 + 2x^2 + 2 - 1$

$= x^4 + 2x^2 + 1 = \left(x^2 + 1\right)^2$

$\Rightarrow \sqrt{y-1} = \left(x^2 + 1\right)$ i.e. $\left(x^2 + 1\right) = \sqrt{y-1}$

Using it in eq. (i), we have

$\Rightarrow \frac{dy}{dx} = 4x\sqrt{y-1}$ as required.

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Question # 1

Find by making suitable substitution in the following functions defined as:

(i) $y = \sqrt{\dfrac{1-x}{1+x}}$

(ii) $y = \sqrt{x+\sqrt{x}}$

(iii) $y = x\sqrt{\dfrac{a+x}{a-x}}$

(iv) $y=\left(3x^2-2x+7\right)^6$

(v) $\dfrac{\sqrt{a^2+x^2}}{\sqrt{a^2-x^2}}$

Solution

(i)

$$y = \sqrt{\frac{1-x}{1+x}}$$

Put $u = \dfrac{1-x}{1+x}$

So $y = \sqrt{u} \quad \Rightarrow y = u^{\frac{1}{2}}$

Now diff. u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$= \frac{(1+x)\frac{d}{dx}(1-x)-(1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}$$

$$= \frac{-1-x-1+x}{(1+x)^2}$$

$$\Rightarrow \frac{du}{dx} = \frac{-2}{(1+x)^2}$$

Now diff. y w.r.t. u

$$\frac{dy}{du} = \frac{d}{du}u^{\frac{1}{2}}$$

$$= \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{du} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\cdot\frac{-2}{(1+x)^2}$$

$$= \frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}\cdot\frac{-1}{(1+x)^2}$$

$$= \frac{-1}{(1-x)^{\frac{1}{2}}(1+x)^{2-\frac{1}{2}}}$$

$$= \frac{-1}{\sqrt{1-x}\,(1+x)^{\frac{3}{2}}} \quad \textit{Answer}$$

(ii)

$$y = \sqrt{x+\sqrt{x}}$$

Let $u = x+\sqrt{x} = x+x^{\frac{1}{2}}$

$$\Rightarrow y = \sqrt{u} = u^{\frac{1}{2}}$$

Diff. u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx}\left(x+x^{\frac{1}{2}}\right)$$

$$= 1+\frac{1}{2}x^{-\frac{1}{2}} = 1+\frac{1}{2\sqrt{x}}$$

$$= \frac{2\sqrt{x}+1}{2\sqrt{x}}$$

Now diff. y w.r.t. x

$$\frac{dy}{du} = \frac{d}{du}u^{\frac{1}{2}}$$

$$= \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2u^{\frac{1}{2}}} = \frac{1}{2\left(x+\sqrt{x}\right)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{du} = \frac{1}{2\sqrt{x+\sqrt{x}}}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{2\sqrt{x+\sqrt{x}}}\cdot\frac{2\sqrt{x}+1}{2\sqrt{x}}$$

$$= \frac{2\sqrt{x}+1}{4\sqrt{x}\cdot\sqrt{x+\sqrt{x}}} \quad \textit{Answer}$$

(iii)

$$y = x\sqrt{\frac{a+x}{a-x}}$$

Put $u = \frac{a+x}{a-x}$

So $y = x\sqrt{u} = x(u)^{\frac{1}{2}}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}x(u)^{\frac{1}{2}}$$

$$= x\frac{d}{dx}(u)^{\frac{1}{2}} + (u)^{\frac{1}{2}}\frac{d}{dx}x$$

$$= x\frac{1}{2}(u)^{-\frac{1}{2}}\frac{du}{dx} + (u)^{\frac{1}{2}}(1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{2}(u)^{-\frac{1}{2}}\frac{du}{dx} + (u)^{\frac{1}{2}} \quad \ldots\ldots. (i)$$

Now diff. u w.r.t. x

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{a+x}{a-x}\right)$$

$$= \frac{(a-x)\frac{d}{dx}(a+x) - (a+x)\frac{d}{dx}(a-x)}{(a-x)^2}$$

$$= \frac{(a-x)(0+1) - (a+x)(0-1)}{(a-x)^2}$$

$$= \frac{(a-x)(1) - (a+x)(-1)}{(a-x)^2}$$

$$= \frac{a-x+a+x}{(a-x)^2} = \frac{2a}{(a-x)^2}$$

Using value of u and $\frac{du}{dx}$ in eq. (i)

$$\frac{dy}{dx} = \frac{x}{2}\left(\frac{a+x}{a-x}\right)^{-\frac{1}{2}}\frac{2a}{(a-x)^2} + \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}$$

$$= \frac{(a+x)^{-\frac{1}{2}}}{(a-x)^{-\frac{1}{2}}}\cdot\frac{ax}{(a-x)^2} + \frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$= \frac{ax}{(a+x)^{\frac{1}{2}}(a-x)^{2-\frac{1}{2}}} + \frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$= \frac{ax}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}} + \frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}$$

$$= \frac{ax + (a+x)(a-x)}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{ax + a^2 - x^2}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}}}$$

(iv)

Do yourself as above

(v)

Do yourself as above

Question # 2

Find $\frac{dy}{dx}$ if:

(i) $3x+4y+7 = 0$

(ii) $xy + y^2 = 2$

(iii) $x^2 - 4xy - 5y = 0$

(iv) $4x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

(v) $x\sqrt{1+y} + y\sqrt{1+x} = 0$

(vi) $y\left(x^2-1\right) = x\sqrt{x^2+4}$

Solution

(i)

$$3x+4y+7 = 0$$

Diff. w.r.t. x.

$$\frac{d}{dx}(3x+4y+7) = \frac{d}{dx}(0)$$

$$\Rightarrow 3(1)+4\frac{dy}{dx}+0 = 0 \quad \Rightarrow 4\frac{dy}{dx} = -3$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{3}{4}}$$

(ii) $xy+y^2 = 2$

Differentiating w.r.t. x

$$\frac{d}{dx}\left(xy+y^2\right) = \frac{d}{dx}(2)$$

$$\Rightarrow \frac{d}{dx}(xy)+\frac{d}{dx}y^2 = 0$$

$$\Rightarrow x\frac{dy}{dx}+y\frac{dx}{dx}+2y\frac{dy}{dx} = 0$$

$$\Rightarrow (x+2y)\frac{dy}{dx}+y(1) = 0$$

$$\Rightarrow (x+2y)\frac{dy}{dx} = -y$$

$$\Rightarrow (x+2y)\frac{dy}{dx} = -y$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-y}{x+2y}}$$

(iii)

Do yourself

(iv)

$$4x^2+2hxy+by^2+2gx+2fy+c = 0$$

Differentiating w.r.t. x

$$\frac{d}{dx}\left(4x^2+2hxy+by^2+2gx+2fy+c\right) = \frac{d}{dx}(0)$$

$$\Rightarrow 4\frac{d}{dx}(x^2)+2h\frac{d}{dx}(xy)+b\frac{d}{dx}(y^2)$$
$$+2g\frac{d}{dx}(x)+2f\frac{d}{dx}(y)+\frac{d}{dx}(c) = 0$$

$$\Rightarrow 4(2x)+2h\left(x\frac{dy}{dx}+y(1)\right)+b\cdot 2y\frac{dy}{dx}$$
$$+2g(1)+2f\frac{dy}{dx}+0 = 0$$

$$\Rightarrow 8x+2hx\frac{dy}{dx}+2hy+2by\frac{dy}{dx}$$
$$+2g+2f\frac{dy}{dx} = 0$$

$$\Rightarrow 2(hx+by+f)\frac{dy}{dx}+2(4x+hy++g)=0$$

$$\Rightarrow 2(hx+by+f)\frac{dy}{dx}=-2(4x+hy++g)$$

$$\Rightarrow (hx+by+f)\frac{dy}{dx}=-(4x+hy++g)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{4x+hy++g}{hx+by+f}}$$

(v)

$$x\sqrt{1+y}+y\sqrt{1+x} = 0 \quad \Rightarrow x(1+y)^{\frac{1}{2}}+y(1+x)^{\frac{1}{2}} = 0$$

Differentiating w.r.t. x

$$\Rightarrow \frac{d}{dx}\left[x(1+y)^{\frac{1}{2}}\right]+\frac{d}{dx}\left[y(1+x)^{\frac{1}{2}}\right] = \frac{d}{dx}(0)$$

$$\Rightarrow x\frac{d}{dx}(1+y)^{\frac{1}{2}}+(1+y)^{\frac{1}{2}}\frac{dx}{dx}+y\frac{d}{dx}(1+x)^{\frac{1}{2}}+(1+x)^{\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\Rightarrow\ x\cdot\frac{1}{2}(1+y)^{-\frac{1}{2}}\frac{dy}{dx}+(1+y)^{\frac{1}{2}}(1)\ +y\cdot\frac{1}{2}(1+x)^{-\frac{1}{2}}(1)+(1+x)^{\frac{1}{2}}\frac{dy}{dx}=0$$

$$\Rightarrow\ \frac{x}{2(1+y)^{\frac{1}{2}}}\frac{dy}{dx}+(1+y)^{\frac{1}{2}}+\frac{y}{2(1+x)^{\frac{1}{2}}}+(1+x)^{\frac{1}{2}}\frac{dy}{dx}=0$$

$$\Rightarrow\ \left[\frac{x}{2(1+y)^{\frac{1}{2}}}+(1+x)^{\frac{1}{2}}\right]\frac{dy}{dx}=-\left[(1+y)^{\frac{1}{2}}+\frac{y}{2(1+x)^{\frac{1}{2}}}\right]$$

$$\Rightarrow\ \left[\frac{x+2(1+x)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}}{2(1+y)^{\frac{1}{2}}}\right]\frac{dy}{dx}=-\left[\frac{2(1+x)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}+y}{2(1+x)^{\frac{1}{2}}}\right]$$

$$\Rightarrow\ \left[\frac{x+2\sqrt{(1+x)(1+y)}}{2\sqrt{1+y}}\right]\frac{dy}{dx}=-\left[\frac{2\sqrt{(1+x)(1+y)}+y}{2\sqrt{1+x}}\right]$$

$$\Rightarrow\ \frac{dy}{dx}=-\frac{2\sqrt{(1+x)(1+y)}+y}{2\sqrt{1+x}}\cdot\frac{2\sqrt{1+y}}{x+2\sqrt{(1+x)(1+y)}}$$

$$\Rightarrow\ \frac{dy}{dx}=-\frac{\sqrt{1+y}\left(2\sqrt{(1+x)(1+y)}+y\right)}{\sqrt{1+x}\left(x+2\sqrt{(1+x)(1+y)}\right)}\qquad \textit{Answer}$$

(vi)

$$y\left(x^2-1\right)=x\sqrt{x^2+4}$$

Differentiating w.r.t x

$$\frac{d}{dx}y\left(x^2-1\right)=\frac{d}{dx}x\left(x^2+4\right)^{\frac{1}{2}}$$

$$\Rightarrow\ y\frac{d}{dx}\left(x^2-1\right)+\left(x^2-1\right)\frac{dy}{dx}=x\frac{d}{dx}\left(x^2+4\right)^{\frac{1}{2}}+\left(x^2+4\right)^{\frac{1}{2}}\frac{dx}{dx}$$

$$\Rightarrow\ y(2x)+\left(x^2-1\right)\frac{dy}{dx}=x\frac{1}{2}\left(x^2+4\right)^{-\frac{1}{2}}(2x)+\left(x^2+4\right)^{\frac{1}{2}}(1)$$

$$\Rightarrow\ 2xy+\left(x^2-1\right)\frac{dy}{dx}=\frac{x^2}{\left(x^2+4\right)^{\frac{1}{2}}}+\left(x^2+4\right)^{\frac{1}{2}}$$

$$\Rightarrow\ \left(x^2-1\right)\frac{dy}{dx}=\frac{x^2}{\left(x^2+4\right)^{\frac{1}{2}}}+\left(x^2+4\right)^{\frac{1}{2}}-2xy$$

$$\Rightarrow\ \left(x^2-1\right)\frac{dy}{dx}=\frac{x^2}{\left(x^2+4\right)^{\frac{1}{2}}}+\left(x^2+4\right)^{\frac{1}{2}}-2xy$$

$$\Rightarrow\ \left(x^2-1\right)\frac{dy}{dx}=\frac{x^2+x^2+4-2xy\left(x^2+4\right)^{\frac{1}{2}}}{\left(x^2+4\right)^{\frac{1}{2}}}\qquad\Rightarrow\ \boxed{\frac{dy}{dx}=\frac{2x^2+4-2xy\sqrt{x^2+4}}{\left(x^2-1\right)\sqrt{x^2+4}}}$$

Question # 3

Find $\dfrac{dy}{dx}$ of the following parametric functions:

(i) $x = \theta + \dfrac{1}{\theta}$ and $y = \theta + 1$

(ii) $x = \dfrac{a(1-t^2)}{1+t^2}$, $y = \dfrac{2bt}{1+t^2}$

Solution

(i) Since $x = \theta + \dfrac{1}{\theta}$

$\Rightarrow x = \theta + \theta^{-1}$

Differentiating x w.r.t. θ

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(\theta + \theta^{-1}\right)$$

$$= 1 - \theta^{-2} = 1 - \frac{1}{\theta^2} = \frac{\theta^2 - 1}{\theta^2}$$

$$\Rightarrow \frac{d\theta}{dx} = \frac{\theta^2}{\theta^2 - 1}$$

Now $y = \theta + 1$

Diff. w.r.t. θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\theta + 1) \Rightarrow \frac{dy}{d\theta} = 1$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = 1 \cdot \frac{\theta^2}{\theta^2 - 1}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{\theta^2}{\theta^2 - 1}}$$

(ii) Since $x = \dfrac{a(1-t^2)}{1+t^2}$

Diff. w.r.t. t

$$\frac{dx}{dt} = a\frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$$

$$= a\frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= a\frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}$$

$$= a\frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2}$$

$$\Rightarrow \frac{dx}{dt} = \frac{-4at}{(1+t^2)^2} \Rightarrow \frac{dt}{dx} = \frac{(1+t^2)^2}{-4at}$$

Now $y = \dfrac{2bt}{1+t^2}$y

Diff. w.r.t. t

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{2bt}{1+t^2}\right)$$

$$= \frac{(1+t^2)\frac{d}{dt}2bt - 2bt\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)2b(1) - 2bt(2t)}{(1+t^2)^2}$$

$$= \frac{2b + 2bt^2 - 4bt^2}{(1+t^2)^2} = \frac{2b - 2bt^2}{(1+t^2)^2}$$

$$= \frac{2b(1-t^2)}{(1+t^2)^2}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{2b(1-t^2)}{(1+t^2)^2} \cdot \frac{(1+t^2)^2}{-4at}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{b(1-t^2)}{2at}}$$

Question # 4

Prove that $y\dfrac{dy}{dx}+x=0$ if $x = \dfrac{1-t^2}{1+t^2}$, $y = \dfrac{2t}{1+t^2}$

Solution Since $x = \dfrac{1-t^2}{1+t^2}$

Differentiating w.r.t. t, we get (*solve yourself as above*)

$$\frac{dx}{dt} = \frac{-4t}{\left(1+t^2\right)^2} \Rightarrow \frac{dt}{dx} = \frac{\left(1+t^2\right)^2}{-4t}$$

Now $y = \dfrac{2t}{1+t^2}$

Differentiating w.r.t. t, we get (*solve yourself as above*)

$$\frac{dy}{dt} = \frac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \frac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}\cdot\frac{\left(1+t^2\right)^2}{-4t}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1-t^2}{2t}$$

Multiplying both sides by y

$$\Rightarrow y\frac{dy}{dx} = -y\cdot\frac{1-t^2}{2t}$$

$$= -\frac{2t}{1+t^2}\cdot\frac{1-t^2}{2t}$$

$$\Rightarrow y\frac{dy}{dx} = -\frac{1-t^2}{1+t^2}$$

$$\Rightarrow y\frac{dy}{dx} = -x \qquad \because x = \frac{1-t^2}{1+t^2}$$

$$\Rightarrow y\frac{dy}{dx} + x = 0$$ *Proved.*

Question # 5

Differentiate

(i) $x^2-\dfrac{1}{x^2}$ w.r.t. x^4

(ii) $\left(1+x^2\right)^n$ w.r.t. x^2

(iii) $\dfrac{x^2+1}{x^2-1}$ w.r.t. $\dfrac{x-1}{x+1}$

(iv) $\dfrac{ax+b}{cx+d}$ w.r.t. $\dfrac{ax^2+b}{ax^2+d}$

(v) $\dfrac{x^2+1}{x^2-1}$ w.r.t. x^3

Solution

(i) Suppose $y = x^2-\dfrac{1}{x^2}$ and $u=x^4$

Diff. y w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2-\frac{1}{x^2}\right)$$

$$= \frac{d}{dx}\left(x^2-x^{-2}\right) = 2x+2x^{-3}$$

$$= 2\left(x+\frac{1}{x^3}\right)$$

$$\Rightarrow \frac{dy}{dx} = 2\left(\frac{x^4+1}{x^3}\right)$$

Now diff. u w.r.t x

$$\frac{du}{dx} = \frac{d}{dx}\left(x^4\right)$$

$$\Rightarrow \frac{du}{dx} = 4x^3$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$

$$= \frac{dy}{dx}\cdot\frac{1}{\frac{du}{dx}}$$

$$\Rightarrow \frac{dy}{du} = 2\left(\frac{x^4+1}{x^3}\right)\cdot\frac{1}{4x^3}$$

$$\Rightarrow \boxed{\frac{dy}{du} = \frac{x^4+1}{2x^6}}$$

(ii) Let $y=(1+x^2)^n$ and $u = x^2$

Differentiation y w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(1+x^2)^n$$
$$= n(1+x^2)^{n-1}\frac{d}{dx}(1+x^2)$$
$$= n(1+x^2)^{n-1}(2x)$$
$$= 2nx(1+x^2)^{n-1}$$

Now differentiating u w.r.t x

$$\frac{du}{dx} = \frac{d}{dx}x^2$$
$$= 2x \quad \Rightarrow \frac{dx}{du}=\frac{1}{2x}$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$
$$\Rightarrow \frac{dy}{du} = 2nx(1+x^2)^{n-1}\cdot\frac{1}{2x}$$
$$\Rightarrow \boxed{\frac{dy}{du} = n(1+x^2)^{n-1}}$$

(iii) Let $y=\frac{x^2+1}{x^2-1}$ and $u = \frac{x-1}{x+1}$

Diff. y w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$
$$= \text{Solve yourself} = \frac{-4x}{(x^2-1)^2}$$

Now diff. u w.r.t x

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{x-1}{x+1}\right)$$
$$= \text{Solve yourself} = \frac{2}{(x+1)^2}.$$
$$\Rightarrow \frac{dx}{du} = \frac{(x+1)^2}{2}.$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$
$$= \frac{-4x}{(x^2-1)^2}\cdot\frac{(x+1)^2}{2}$$
$$= \frac{-2x}{(x-1)^2(x+1)^2}\cdot(x+1)^2$$
$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-2x}{(x-1)^2}}$$

(iv) Let $y=\frac{ax+b}{cx+d}$ and $u=\frac{ax^2+b}{ax^2+d}$

Diff. y w.r.t. x

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{ax+b}{cx+d}\right)$$
$$=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$
$$= \frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}$$
$$= \frac{acx+ad-acx-bc}{(cx+d)^2}$$
$$\Rightarrow \frac{dy}{dx}= \frac{ad-bc}{(cx+d)^2}$$

Now diff. u w.r.t x

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{ax^2+b}{ax^2+d}\right)$$
$$=\frac{(ax^2+d)\frac{d}{dx}(ax^2+b)-(ax^2+b)\frac{d}{dx}(ax^2+d)}{(ax^2+d)^2}$$
$$= \frac{(ax^2+d)(2ax)-(ax^2+b)(2ax)}{(ax^2+d)^2}$$
$$= \frac{2ax(ax^2+d-ax^2-b)}{(ax^2+d)^2}$$
$$= \frac{2ax(d-b)}{(ax^2+d)^2}$$

$$\Rightarrow \frac{dx}{du} = \frac{(ax^2+d)^2}{2ax(d-b)}$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$

$$= \frac{ad-bc}{(cx+d)^2}\cdot\frac{(ax^2+d)^2}{2ax(d-b)}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{(ad-bc)(ax^2+d)^2}{2ax(cx+d)^2(d-b)}}$$

(v) Let $y=\frac{x^2+1}{x^2-1}$ and $u=x^3$

Diff. y w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$= \textit{Solve yourself}$

$$=\frac{-4x}{(x^2-1)^2}$$

Now diff. u w.r.t x

$$\frac{du}{dx}=\frac{d}{dx}x^3$$

$$=3x^2$$

$$\Rightarrow \frac{dx}{du}=\frac{1}{3x^2}$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$

$$=\frac{-4x}{(x^2-1)^2}\cdot\frac{1}{3x^2}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-4}{3x(x^2-1)^2}}$$

Some Important Derivative Formulas

- $\frac{d}{dx}c = 0$ where c is constant
- $\frac{d}{dx}x^n = nx^{n-1}$
- $\frac{d}{dx}\sin x = \cos x$
- $\frac{d}{dx}\tan x = \sec^2 x$
- $\frac{d}{dx}\csc x = -\csc x \cot x$
- $\frac{d}{dx}\cos x = -\sin x$
- $\frac{d}{dx}\cot x = -\csc^2 x$
- $\frac{d}{dx}\sec x = \sec x \tan x$
- $\frac{d}{dx}Sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}Tan^{-1}x = \frac{1}{1+x^2}$
- $\frac{d}{dx}Sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}Cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}Cot^{-1}x = \frac{-1}{1+x^2}$
- $\frac{d}{dx}Csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$

Question # 1

Difference the following trigonometric functions from the first principles.

(i) $\sin 2x$ (ii) $\tan 3x$ (iii) $\sin 2x + \cos 2x$ (iv) $\cos x^2$

(v) $\tan^2 x$ (vi) $\sqrt{\tan x}$ (vii) $\cos\sqrt{x}$

Solution

(i) Suppose $y = \sin 2x$

$$\Rightarrow y + \delta y = \sin 2(x+\delta x)$$

$$\Rightarrow \delta y = \sin 2(x+\delta x) - y$$

$$= \sin 2(x+\delta x) - \sin 2x$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = \frac{\sin(2x+2\delta x) - \sin 2x}{\delta x}$$

$$= \frac{2\cos\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{2x+2\delta x-2x}{2}\right)}{\delta x}$$

$$= \frac{2\cos(2x+\delta x)\sin(\delta x)}{\delta x}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{2\cos(2x+\delta x)\sin(\delta x)}{\delta x}$$

$$\frac{dy}{dx} = 2\lim_{\delta x\to 0}\cos(2x+\delta x)\cdot\frac{\sin(\delta x)}{\delta x}$$

$$= 2\lim_{\delta x\to 0}\cos(2x+\delta x)\cdot\lim_{\delta x\to 0}\frac{\sin(\delta x)}{\delta x}$$

$$= 2\cos(2x+0)\cdot(1) \qquad \because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 2\cos 2x}$$

(ii) Let $y = \tan 3x$

$$\Rightarrow y+\delta y = \tan 3(x+\delta x)$$

$$\Rightarrow \delta y = \tan(3x+3\delta x)-\tan 3x$$

$$= \frac{\sin(3x+3\delta x)}{\cos(3x+3\delta x)}-\frac{\sin 3x}{\cos 3x} = \frac{\sin(3x+3\delta x)\cos 3x-\cos(3x+3\delta x)\sin 3x}{\cos(3x+3\delta x)\cos 3x}$$

$$= \frac{\sin(3x+3\delta x-3x)}{\cos(3x+3\delta x)\cos 3x} = \frac{\sin(3\delta x)}{\cos(3x+3\delta x)\cos 3x}$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\cdot\frac{\sin(3\delta x)}{\cos(3x+3\delta x)\cos 3x}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\sin(3\delta x)}{\delta x\ \cos(3x+3\delta x)\cos 3x}$$

$$\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\sin(3\delta x)}{\delta x}\cdot\frac{1}{\cos(3x+3\delta x)\cos 3x}\cdot\frac{3}{3} \qquad \times\text{ing and } \div\text{ing 3 on R.H.S}$$

$$= 3\lim_{\delta x\to 0}\frac{\sin(3\delta x)}{3\delta x}\cdot\lim_{\delta x\to 0}\frac{1}{\cos(3x+3\delta x)\cos 3x}$$

$$= 3(1)\cdot\frac{1}{\cos(3x+3(0))\cos 3x}$$

$$= \frac{3}{\cos 3x\cos 3x} = \frac{3}{\cos^2 3x}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 3\sec^2 3x}$$

(iii) Let $y = \sin 2x+\cos 2x$

$$\Rightarrow y+\delta y = \sin 2(x+\delta x)+\cos 2(x+\delta x)$$

$$\Rightarrow \delta y = \sin 2(x+\delta x)+\cos 2(x+\delta x)-y$$

$$= \sin 2(x+\delta x)+\cos 2(x+\delta x)-\sin 2x-\cos 2x$$

$$= \left[\sin(2x+2\delta x)-\sin 2x\right]+\left[\cos(2x+2\delta x)-\cos 2x\right]$$

$$= \left[2\cos\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{2x+2\delta x-2x}{2}\right)\right]$$

$$+\left[-2\sin\left(\frac{2x+2\delta x-2x}{2}\right)\sin\left(\frac{2x+2\delta x-2x}{2}\right)\right]$$

$$= 2\cos(2x+\delta x)\sin(\delta x) - 2\sin(2x+\delta x)\sin(\delta x)$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\left[2\cos(2x+\delta x)\sin(\delta x) - 2\sin(2x+\delta x)\sin(\delta x)\right]$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{\delta x}\left[2\cos(2x+\delta x)\sin(\delta x) - 2\sin(2x+\delta x)\sin(\delta x)\right]$$

$$\frac{dy}{dx} = 2\lim_{\delta x\to 0}\cos(2x+\delta x)\lim_{\delta x\to 0}\frac{\sin(\delta x)}{\delta x} - 2\lim_{\delta x\to 0}\sin(2x+\delta x)\lim_{\delta x\to 0}\frac{\sin(\delta x)}{\delta x}$$

$$= 2\cos(2x+0)\cdot(1) - 2\sin(2x+0)\cdot(1) \qquad \text{Since } \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 2\cos 2x - 2\sin 2x}$$

(iv) Let $y = \cos x^2$

$$\Rightarrow y+\delta y = \cos(x+\delta x)^2$$

$$\Rightarrow \delta y = \cos(x+\delta x)^2 - \cos x^2$$

$$= -2\sin\left(\frac{(x+\delta x)^2 + x^2}{2}\right)\sin\left(\frac{(x+\delta x)^2 - x^2}{2}\right)$$

$$= -2\sin\left(\frac{x^2+2x\delta x+\delta x^2+x^2}{2}\right)\sin\left(\frac{x^2+2x\delta x+\delta x^2-x^2}{2}\right)$$

$$= -2\sin\left(\frac{2x^2+2x\delta x+\delta x^2}{2}\right)\cdot\sin\left(\frac{2x\delta x+\delta x^2}{2}\right)$$

$$= -2\sin\left(x^2+x\delta x+\frac{\delta x^2}{2}\right)\cdot\sin\left(x+\frac{\delta x}{2}\right)\delta x$$

Dividing by δx

$$\frac{\delta y}{\delta x} = -\frac{1}{\delta x}\cdot 2\sin\left(x^2+x\delta x+\frac{\delta x^2}{2}\right)\cdot\sin\left(x+\frac{\delta x}{2}\right)\delta x$$

$\times$ing and $\div$ing $\left(x+\frac{\delta x}{2}\right)$ on R.H.S

$$\Rightarrow \frac{\delta y}{\delta x} = -\left[\frac{2}{\delta x}\sin\left(x^2+x\delta x+\frac{\delta x^2}{2}\right)\cdot\sin\left(x+\frac{\delta x}{2}\right)\delta x\right]\cdot\frac{\left(x+\frac{\delta x}{2}\right)}{\left(x+\frac{\delta x}{2}\right)}$$

$$= -\left[2\sin\left(x^2+x\delta x+\frac{\delta x^2}{2}\right)\cdot\frac{\sin\left(x+\frac{\delta x}{2}\right)\delta x}{\left(x+\frac{\delta x}{2}\right)\delta x}\right]\cdot\left(x+\frac{\delta x}{2}\right)$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = -\lim_{\delta x\to 0}\left[2\sin\left(x^2+x\delta x+\frac{\delta x^2}{2}\right)\cdot\frac{\sin\left(x+\frac{\delta x}{2}\right)\delta x}{\left(x+\frac{\delta x}{2}\right)\delta x}\right]\cdot\left(x+\frac{\delta x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = -2\lim_{\delta x\to 0}\sin\left(x^2+x\delta x+\frac{\delta x^2}{2}\right)\cdot\lim_{\delta x\to 0}\frac{\sin\left(x+\frac{\delta x}{2}\right)\delta x}{\left(x+\frac{\delta x}{2}\right)\delta x}\cdot\lim_{\delta x\to 0}\left(x+\frac{\delta x}{2}\right)$$

$$= -2\sin\left(x^2+(0)+(0)\right)\cdot(1)\cdot\left(x+(0)\right)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -2x\sin x^2}$$

(v) Let $y = \tan^2 x$

$$\Rightarrow y+\delta y = \tan^2(x+\delta x)$$

$$\Rightarrow \delta y = \tan^2(x+\delta x)-\tan^2 x$$

$$= \left(\tan(x+\delta x)+\tan x\right)\cdot\left(\tan(x+\delta x)-\tan x\right)$$

$$= \left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin(x+\delta x)}{\cos(x+\delta x)}-\frac{\sin x}{\cos x}\right)$$

$$= \left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin(x+\delta x)\cos x-\sin x\cos(x+\delta x)}{\cos(x+\delta x)\cos x}\right)$$

$$= \left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin(x+\delta x-x)}{\cos(x+\delta x)\cos x}\right)$$

$$= \left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin\delta x}{\cos(x+\delta x)\cos x}\right)$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin\delta x}{\cos(x+\delta x)\cos x}\right)$$

Taking limit when $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{\delta x}\left(\tan(x+\delta x)+\tan x\right)\cdot\left(\frac{\sin\delta x}{\cos(x+\delta x)\cos x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x\to 0}\left(\frac{\tan(x+\delta x)+\tan x}{\cos(x+\delta x)\cos x}\right)\cdot\lim_{\delta x\to 0}\left(\frac{\sin\delta x}{\delta x}\right)$$

$$= \left(\frac{\tan(x+0)+\tan x}{\cos(x+0)\cos x}\right)\cdot(1) = \frac{\tan x+\tan x}{\cos x\cdot\cos x} = \frac{2\tan x}{\cos^2 x}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 2\tan x\ \sec^2 x}$$

(vi) Let $y = \sqrt{\tan x}$

$$\Rightarrow y+\delta y = \sqrt{\tan(x+\delta x)}$$

$$\Rightarrow \delta y = \sqrt{\tan(x+\delta x)} - \sqrt{\tan x}$$

$$= \left(\sqrt{\tan(x+\delta x)} - \sqrt{\tan x}\right)\cdot\left(\frac{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}\right)$$

$$= \frac{\tan(x+\delta x) - \tan x}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}$$

$$= \frac{1}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}\cdot\left(\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}\right)$$

Now do yourself as above.

(vii) Let $y = \cos\sqrt{x}$

$$\Rightarrow y+\delta y = \cos\sqrt{x+\delta x}$$

$$\Rightarrow \delta y = \cos\sqrt{x+\delta x} - \cos\sqrt{x}$$

$$= -2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)$$

Dividing by δx

$$\frac{\delta y}{\delta x} = -\frac{2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = -2\lim_{\delta x\to 0}\frac{\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x}$$

As $\delta x = \left(\sqrt{x+\delta x}+\sqrt{x}\right)\left(\sqrt{x+\delta x}-\sqrt{x}\right)$, putting in above

$$\Rightarrow \frac{dy}{dx} = -2\lim_{\delta x\to 0}\frac{\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\sqrt{x+\delta x}+\sqrt{x}\right)\left(\sqrt{x+\delta x}-\sqrt{x}\right)}$$

$$= -\lim_{\delta x\to 0}\frac{\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)}{\left(\sqrt{x+\delta x}+\sqrt{x}\right)}\cdot\lim_{\delta x\to 0}\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}$$

$$= -\frac{\sin\left(\frac{\sqrt{x+0}+\sqrt{x}}{2}\right)}{\left(\sqrt{x+0}+\sqrt{x}\right)}\cdot(1) \qquad \Rightarrow \boxed{\frac{dy}{dx} = -\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}}$$

Question # 2

Differentiate the following w.r.t. the variable involved.

(i) $x^2\sec 4x$ (ii) $\tan^3\theta\ \sec^2\theta$ (iii) $\left(\sin 2\theta-\cos 3\theta\right)^2$ (iv) $\cos\sqrt{x}+\sqrt{\sin x}$

Solution

(i) Assume $y = x^2\sec 4x$

Differentiating w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}x^2\sec 4x$$

$$= x^2\frac{d}{dx}\sec 4x+\sec 4x\frac{d}{dx}x^2$$

$$= x^2\sec 4x\tan 4x\frac{d}{dx}(4x)+\sec 4x\ (2x)$$

$$= x^2\sec 4x\tan 4x(4)+2x\sec 4x$$

$$= 2x\sec 4x\left(2x\tan 4x+1\right)$$

(ii) Let $y = \tan^3\theta\ \sec^2\theta$

Diff. w.r.t θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\tan^3\theta\ \sec^2\theta$$

$$= \tan^3\theta\frac{d}{d\theta}\sec^2\theta+\sec^2\theta\frac{d}{d\theta}\tan^3\theta$$

$$= \tan^3\theta\left(2\sec\theta\frac{d}{d\theta}\sec\theta\right)+\sec^2\theta\left(3\tan^2\theta\frac{d}{d\theta}\tan\theta\right)$$

$$= \tan^3\theta\left(2\sec\theta\cdot\sec\theta\tan\theta\right)+\sec^2\theta\left(3\tan^2\theta\cdot\sec^2\theta\right)$$

$$= \sec^2\theta\tan^2\theta\left(2\tan^2\theta+3\sec^2\theta\right)$$

(iii) Let $y = \left(\sin 2\theta-\cos 3\theta\right)^2$

Diff. w.r.t θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(\sin 2\theta-\cos 3\theta\right)^2$$

$$= 2\left(\sin 2\theta-\cos 3\theta\right)\frac{d}{d\theta}\left(\sin 2\theta-\cos 3\theta\right)$$

$$= 2\left(\sin 2\theta-\cos 3\theta\right)\left(\cos 2\theta\cdot\frac{d}{d\theta}(2\theta)+\sin 3\theta\cdot\frac{d}{d\theta}(3\theta)\right)$$

$$= 2\left(\sin 2\theta-\cos 3\theta\right)\left(\cos 2\theta\cdot(2)+\sin 3\theta\cdot(3)\right)$$

$$= 2(\sin 2\theta - \cos 3\theta)(2\cos 2\theta + 3\sin 3\theta)$$

(iv) Let $y = \cos\sqrt{x} + \sqrt{\sin x}$

$$= \cos(x)^{\frac{1}{2}} + (\sin x)^{\frac{1}{2}}$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos(x)^{\frac{1}{2}} + (\sin x)^{\frac{1}{2}}\right)$$

$$= -\sin(x)^{\frac{1}{2}}\frac{d}{dx}x^{\frac{1}{2}} + \frac{1}{2}(\sin x)^{-\frac{1}{2}}\frac{d}{dx}(\sin x)$$

$$= -\sin(x)^{\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + \frac{1}{2}(\sin x)^{-\frac{1}{2}}(\cos x)$$

$$= \frac{1}{2}\left(\frac{\cos x}{\sqrt{\sin x}} - \frac{\sin\sqrt{x}}{\sqrt{x}}\right)$$

Question # 3

Find $\frac{dy}{dx}$ if

(i) $y = x\cos y$ (ii) $x = y\cos y$

Solution

(i) Since $y = x\cos y$

$$\frac{dy}{dx} = \frac{d}{dx}x\cos y$$

$$= x\frac{d}{dx}\cos y + \cos y\frac{dx}{dx}$$

$$= x(-\sin y)\frac{dy}{dx} + \cos y(1)$$

$$\Rightarrow \frac{dy}{dx} + x\sin y\frac{dy}{dx} = \cos y \quad \Rightarrow (1 + x\sin y)\frac{dy}{dx} = \cos y$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos y}{1 + x\sin y}$$

(ii) *Do yourself as above*

Question # 4

Find the derivative w.r.t. "x"

(i) $\cos\sqrt{\frac{1+x}{1+2x}}$ (ii) $\sin\sqrt{\frac{1+2x}{1+x}}$

Solution

(i) Since $y = \cos\sqrt{\frac{1+x}{1+2x}}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\cos\sqrt{\frac{1+x}{1+2x}}$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\frac{d}{dx}\left(\sqrt{\frac{1+x}{1+2x}}\right) = -\sin\sqrt{\frac{1+x}{1+2x}}\frac{d}{dx}\left(\frac{1+x}{1+2x}\right)^{\frac{1}{2}}$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{1}{2}\left(\frac{1+x}{1+2x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{1+x}{1+2x}\right)$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{1}{2}\left(\frac{1+2x}{1+x}\right)^{\frac{1}{2}}\left(\frac{(1+2x)\frac{d}{dx}(1+x)-(1+x)\frac{d}{dx}(1+2x)}{(1+2x)^2}\right)$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}}\left(\frac{(1+2x)(1)-(1+x)(2)}{(1+2x)^2}\right)$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}}\left(\frac{1+2x-2-2x}{(1+2x)^2}\right)$$

$$= -\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}}\left(\frac{-1}{(1+2x)^2}\right)$$

$$= \frac{1}{2}\sin\sqrt{\frac{1+x}{1+2x}}\cdot\frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}(1+2x)^{2-\frac{1}{2}}}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{2\sqrt{1+x}(1+2x)^{\frac{3}{2}}}\sin\sqrt{\frac{1+x}{1+2x}}}$$

(ii) *Do yourself as above.*

Question # 5

Differentiate

(i) $\sin x$ w.r.t. $\cot x$ (ii) $\sin^2 x$ w.r.t. $\cos^4 x$

Solution

(i) Let $y = \sin x$ and $u = \cot x$

Diff. y w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\sin x$$
$$= \cos x$$

Now diff. u w.r.t x

$$\frac{du}{dx} = \frac{d}{dx}\cot x$$
$$= -\csc^2 x$$

$$\Rightarrow \frac{dx}{du} = -\frac{1}{\csc^2 x}$$
$$= -\sin^2 x$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$
$$= (\cos x)(-\sin^2 x) = -\sin^2 x \cos x$$

(ii) Let $y = \sin^2 x$ and $u = \cos^4 x$

Diff. y w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\sin^2 x$$
$$= 2\sin x\frac{d}{dx}(\sin x) = 2\sin x \cos x$$

Now diff. u w.r.t x

$$\frac{du}{dx} = \frac{d}{dx}\cos^4 x$$
$$= 4\cos^3 x\frac{d}{dx}(\cos x) = 4\cos^3 x(-\sin x)$$
$$= -4\sin x\cos^3 x$$
$$\Rightarrow \frac{dx}{du} = -\frac{1}{4\sin x\cos^3 x}$$

Now by chain rule

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$
$$= (2\sin x \cos x)\left(-\frac{1}{4\sin x\cos^3 x}\right)$$
$$= -\frac{1}{2}\sec^2 x$$

Question # 6

If $\tan y(1+\tan x) = 1-\tan x$, show that $\frac{dy}{dx} = -1$

Solution

Since $\tan y(1+\tan x) = 1-\tan x$

$$\Rightarrow \tan y = \frac{1-\tan x}{1+\tan x}$$
$$= \frac{1-\tan x}{1+1\cdot\tan x} = \frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\cdot\tan x} = \tan\left(\frac{\pi}{4}-x\right)$$
$$\Rightarrow y = \frac{\pi}{4}-x$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - x\right)$$

$$= 0 - 1 \Rightarrow \frac{dy}{dx} = -1$$

Question # 7

If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\infty}}}$, Prove that $(2y-1)\frac{dy}{dx} = \sec^2 x$.

Solution

Since $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\infty}}}$

Taking square on both sides

$$y^2 = \tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\infty}}$$

$$= \tan x + \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\infty}}}$$

$$\Rightarrow y^2 = \tan x + y$$

Diff. w.r.t x

$$\frac{d}{dx} y^2 = \frac{d}{dx}(\tan x + y)$$

$$\Rightarrow 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \quad \Rightarrow 2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

$$\Rightarrow (2y-1)\frac{dy}{dx} = \sec^2 x$$

Question # 8

If $x = a\cos^3\theta$, $y = b\sin^3\theta$, Show that $a\frac{dy}{dx} + b\tan\theta = 0$

Solution

$x = a\cos^3\theta$, $y = b\sin^3\theta$

Diff. x w.r.t θ

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\cos^3\theta\right)$$

$$= a\cdot 3\cos^2\theta\frac{d}{d\theta}(\cos\theta) = 3a\cos^2\theta(-\sin\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = -3a\sin\theta\cos^2\theta \quad \Rightarrow \frac{d\theta}{dx} = \frac{-1}{3a\sin\theta\cos^2\theta}$$

Now diff. y w.r.t θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(b\sin^3\theta\right)$$

$$= b\cdot 3\sin^2\theta\frac{d}{d\theta}(\sin\theta) = 3b\sin^2\theta\cos\theta$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= 3b\sin^2\theta\cos\theta\cdot-\frac{1}{3a\sin\theta\cos^2\theta}$$

$$= -\frac{b}{a}\tan\theta$$

$$\Rightarrow a\frac{dy}{dx} = -b\tan\theta \qquad \Rightarrow a\frac{dy}{dx} + b\tan\theta = 0$$

Question # 9

Find $\frac{dy}{dx}$ if $x = a(\cos t + \sin t)$ and $y = a(\sin t - t\cos t)$

Solution

$x = a(\cos t + \sin t)$ and $y = a(\sin t - t\cos t)$

Do yourself

Derivative of inverse trigonometric formulas

(i) $\frac{d}{dx}Sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

See proof on book page 76

(ii) $\frac{d}{dx}Cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

Proof

Let $y = \cos^{-1}x$ where $x \in [0,\pi]$

$\Rightarrow \cos y = x$

Diff. w.r.t x

$$\frac{d}{dx}\cos y = \frac{dx}{dx} \qquad \Rightarrow -\sin y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

$$= \frac{-1}{\sqrt{1-\cos^2 y}}$$ Since $\sin y$ is positive for $x \in [0,\pi]$

$$= \frac{-1}{\sqrt{1-x^2}}$$

(iii) $\frac{d}{dx}Tan^{-1}x = \frac{1}{1+x^2}$

See proof on book at page 77

(iv) $\frac{d}{dx}Cot^{-1}x = \frac{-1}{1+x^2}$

Proof

Let $y = \cot^{-1}x$

$\Rightarrow \cot y = x$

Diff. w.r.t x

$$\frac{d}{dx}\cot y = \frac{d}{dx}x \quad \Rightarrow -\csc^2 y\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\csc^2 y}$$

$$= \frac{-1}{1+\cot^2 y} \qquad \because 1+\cot^2 y = \csc^2 y$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{1+x^2}$$

(v) $\frac{d}{dx}Sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$

Proof

Let $y = \sec^{-1}x \qquad \Rightarrow \sec y = x$

Diff. w.r.t x

$$\frac{d}{dx}\sec y = \frac{d}{dx}x \quad \Rightarrow \sec y\tan y\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y\tan y}$$

$$= \frac{1}{\sec y\sqrt{\sec^2 y-1}} \qquad \because 1+\tan^2 y = \sec^2 y$$

$$\Rightarrow \frac{d}{dx}Sec^{-1}x = \frac{1}{x\sqrt{x^2-1}} \qquad \because \sec y = x$$

(vi) $\frac{d}{dx}Csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$

See on book at page 77

Question # 10

Differentiate w.r.t. "x"

(i) $Cos^{-1}\frac{x}{a}$ (ii) $\cot^{-1}\frac{x}{a}$ (iii) $\frac{1}{a}Sin^{-1}\frac{a}{x}$

(iv) $Sin^{-1}\sqrt{1-x^2}$ (v) $Sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$ (vi) $Cot^{-1}\left(\frac{2x}{1-x^2}\right)$

(vii) $\text{Cos}^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution

(i) Let $y = Cos^{-1}\frac{x}{a}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}Cos^{-1}\frac{x}{a}$$

$$= \frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\frac{d}{dx}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}\frac{d}{dx}x$$

$$= \frac{-1}{\sqrt{\frac{a^2-x^2}{a^2}}}\cdot\frac{1}{a}(1) = \frac{-a}{\sqrt{a^2-x^2}}\cdot\frac{1}{a} = \frac{-1}{\sqrt{a^2-x^2}} \quad Ans$$

(ii) Let $y = \cot^{-1}\frac{x}{a}$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\cot^{-1}\frac{x}{a}$$

$$= \frac{-1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{d}{dx}\left(\frac{x}{a}\right) = \frac{-1}{\frac{a^2+x^2}{a^2}}\cdot\frac{1}{a}\frac{d}{dx}(x)$$

$$= \frac{-a^2}{a^2+x^2}\cdot\frac{1}{a}(1) = \frac{-a}{a^2+x^2}.$$

(iii) Let $y = \frac{1}{a}Sin^{-1}\frac{a}{x}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{1}{a}\frac{d}{dx}Sin^{-1}\frac{a}{x}$$

$$= \frac{1}{a}\frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}}\frac{d}{dx}\left(\frac{a}{x}\right) = \frac{1}{a}\frac{1}{\sqrt{\frac{x^2-a^2}{x^2}}}\cdot a\frac{d}{dx}\left(x^{-1}\right)$$

$$= \frac{x}{\sqrt{x^2-a^2}}\left(-x^{-2}\right) = \frac{x}{\sqrt{x^2-a^2}}\left(-\frac{1}{x^2}\right) = -\frac{1}{x\sqrt{x^2-a^2}} \quad Ans$$

(iv) Let $y = Sin^{-1}\sqrt{1-x^2}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}Sin^{-1}\sqrt{1-x^2}$$

$$= \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\cdot\frac{d}{dx}\sqrt{1-x^2} = \frac{1}{\sqrt{1-1+x^2}}\cdot\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}\frac{d}{dx}\left(1-x^2\right)$$

$$= \frac{1}{\sqrt{x^2}}\cdot\frac{1}{2}\frac{1}{\left(1-x^2\right)^{\frac{1}{2}}}(-2x) = -\frac{1}{x}\cdot\frac{x}{\sqrt{1-x^2}} = -\frac{1}{\sqrt{1-x^2}}$$

(v) Let $y = Sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}Sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2-1}}\cdot\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\frac{\left(x^2+1\right)^2-\left(x^2-1\right)^2}{(x^2-1)^2}}}\cdot\left(\frac{\left(x^2-1\right)\frac{d}{dx}\left(x^2+1\right)-\left(x^2+1\right)\frac{d}{dx}\left(x^2-1\right)}{\left(x^2-1\right)^2}\right)$$

$$= \frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\cdot\frac{\sqrt{\left(x^4+2x^2+1\right)-\left(x^4+2x^2+1\right)}}{(x^2-1)}}\cdot\left(\frac{\left(x^2-1\right)(2x)-\left(x^2+1\right)(2x)}{\left(x^2-1\right)^2}\right)$$

$$= \frac{\left(x^2-1\right)^2}{\left(x^2+1\right)\cdot\sqrt{x^4+2x^2+1-x^4+2x^2-1}}\cdot\left(\frac{2x\left(x^2-1-x^2-1\right)}{\left(x^2-1\right)^2}\right)$$

$$= \frac{1}{\left(x^2+1\right)\cdot\sqrt{4x^2}}\cdot\left(2x(-2)\right) = \frac{-4x}{\left(x^2+1\right)\cdot 2x} = \frac{-2}{\left(x^2+1\right)} \quad Ans$$

(vi) *Do yourself as above.*

(vii) *Do yourself as above.*

Question # 11

Show that $\frac{dy}{dx} = \frac{y}{x}$ if $\frac{y}{x} = \tan^{-1}\frac{x}{y}$

Solution

Since $\frac{y}{x} = Tan^{-1}\frac{x}{y} \quad \Rightarrow \quad y = x\,Tan^{-1}\frac{x}{y}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(x\,Tan^{-1}\frac{x}{y}\right)$$

$$= x\frac{d}{dx}\left(Tan^{-1}\frac{x}{y}\right) + Tan^{-1}\frac{x}{y}\cdot\frac{d}{dx}(x)$$

$$= x\left(\frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{d}{dx}\left(\frac{x}{y}\right)\right) + Tan^{-1}\frac{x}{y}\cdot(1)$$

$$= x\left(\frac{1}{\frac{y^2+x^2}{y^2}}\left(\frac{y(1)-x\frac{dy}{dx}}{y^2}\right)\right) + Tan^{-1}\frac{x}{y} \quad = \quad \frac{x}{y^2+x^2}\left(y - x\frac{dy}{dx}\right) + \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{y^2+x^2} - \frac{x^2}{y^2+x^2}\cdot\frac{dy}{dx} + \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{x^2}{y^2+x^2}\cdot\frac{dy}{dx} = \frac{xy}{y^2+x^2} + \frac{y}{x} \qquad \Rightarrow \left(1+\frac{x^2}{y^2+x^2}\right)\cdot\frac{dy}{dx} = \frac{y}{x}\left(\frac{x^2}{y^2+x^2}+1\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \quad \textit{Proved}$$

Question # 12

If $y = \tan\left(pTan^{-1}x\right)$, show that $\left(1+x^2\right)y_1 - p\left(1+y^2\right) = 0$

Solution

Since $y = \tan\left(pTan^{-1}x\right) \quad \Rightarrow Tan^{-1}y = pTan^{-1}x$

Differentiating w.r.t x

$$\frac{d}{dx}Tan^{-1}y = p\frac{d}{dx}Tan^{-1}x$$

$$\Rightarrow \frac{1}{1+y^2}\frac{dy}{dx} = p\cdot\frac{1}{1+x^2} \qquad \Rightarrow \left(1+x^2\right)\frac{dy}{dx} = p\left(1+y^2\right)$$

$$\Rightarrow \left(1+x^2\right)y_1 - p\left(1+y^2\right) = 0 \qquad \text{Since } \frac{dy}{dx} = y_1$$

2.10 Derivative of General Exponential Function (Page 80)

A function define by

$$f(x)=a^x \text{ where } a>0,\ a\neq 1$$

is called general exponential function.

Suppose $\quad y=a^x$

$\Rightarrow \quad y+\delta y=a^{x+\delta x} \quad \Rightarrow \delta y=a^{x+\delta x}-y$

$\Rightarrow \quad \delta y=a^{x+\delta x}-a^x \qquad$ Since $y=a^x$

$\Rightarrow \quad \delta y=a^x(a^{\delta x}-1)$

Dividing by δx

$$\frac{\delta y}{\delta x}=\frac{a^x(a^{\delta x}-1)}{\delta x}$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{a^x(a^{\delta x}-1)}{\delta x}$$

$\Rightarrow \quad \frac{dy}{dx}=\lim_{\delta x\to 0}a^x\left(\frac{a^{\delta x}-1}{\delta x}\right) \quad \Rightarrow \quad \frac{dy}{dx}=a^x\lim_{\delta x\to 0}\left(\frac{a^{\delta x}-1}{\delta x}\right)$

$\Rightarrow \quad \boxed{\frac{d}{dx}(a^x)=a^x.\ln a} \qquad$ Since $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$

Derivative of Natural Exponential Function

The exponential function $f(x)=e^x$, where $e=2.71828...$, is called Natural Exponential Function.

Suppose $\quad y = e^x$

Do yourself … Just Change *a* by *e* in above article. You'll get

$$\boxed{\frac{d}{dx}e^x = e^x}$$

2.11 Derivative of General Logarithmic Function (page 81)

If $a>0,\ a\neq 1$ and $x=a^y$, then the function defined by $y=\log_a x \quad (x>0)$ is called General Logarithmic Function.

Suppose $\quad y=\log_a x$

$\Rightarrow \quad y+\delta y=\log_a(x+\delta x) \quad \Rightarrow \delta y=\log_a(x+\delta x)-y$

$\Rightarrow \quad \delta y=\log_a(x+\delta x)-\log_a x$

$\quad = \log_a\left(\frac{x+\delta x}{x}\right) \qquad$ Since $\log_a m-\log_a n=\log_a\frac{m}{n}$

Dividing both sides by δx

$$\frac{\delta y}{\delta x}=\frac{1}{\delta x}\log_a\left(\frac{x+\delta x}{x}\right)$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{\delta x}\log_a\left(\frac{x+\delta x}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{1}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right)$$

$$= \lim_{\delta x\to 0}\frac{x}{x}\cdot\frac{1}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right) \qquad \div\text{ing and }\times\text{ing by } x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}\lim_{\delta x\to 0}\frac{x}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}\lim_{\delta x\to 0}\log_a\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} \qquad \text{Since } m\log_a x = \log_a x^m$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}\log_a\left[\lim_{\delta x\to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}\log_a e \qquad \text{Since } \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

$$\Rightarrow \quad \frac{d}{dx}(\log_a x) = \frac{1}{x}\frac{1}{\log_e a} \qquad \text{Since } \log_a b = \frac{1}{\log_b a}$$

$$\Rightarrow \quad \frac{d}{dx}(\log_a x) = \frac{1}{x\ln a} \qquad \text{Since } \log_e a = \ln a$$

Derivative of Natural Logarithmic Function

The logarithmic function $f(x) = \log_e x$ where $e = 2.71828...$ is called Natural Logarithmic Function. And we write $\ln x$ instead of $\log_e x$ for our ease.

Suppose $\qquad y = \ln x$

$$\Rightarrow \quad y + \delta y = \ln(x+\delta x) \quad \Rightarrow \quad \delta y = \ln(x+\delta x) - y$$

$$\Rightarrow \quad \delta y = \ln(x+\delta x) - \ln x$$

$$\Rightarrow \quad \delta y = \ln\left(\frac{x+\delta x}{x}\right) \qquad \text{Since} \Rightarrow \ \ln m - \ln n = \ln\frac{m}{n}$$

$$= \ln\left(1+\frac{\delta x}{x}\right)$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x} = \ln\left(1+\frac{\delta x}{x}\right)$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{\delta x}\ln\left(1+\frac{\delta x}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{x}{x} \cdot \frac{1}{\delta x} \ln\left(1 + \frac{\delta x}{x}\right) \qquad \div\text{ing and} \times\text{ing by } x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} \lim_{\delta x \to 0} \frac{x}{\delta x} \ln\left(1 + \frac{\delta x}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} \lim_{\delta x \to 0} \ln\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}} \qquad \text{Since } m\ln x = \ln x^m$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} \ln\left[\lim_{\delta x \to 0}\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} \ln e \qquad \text{Since } \lim_{x \to 0}(1+x)^{\frac{1}{x}} = e$$

$$\Rightarrow \quad \frac{d}{dx}(\ln x) = \frac{1}{x} \cdot 1 \qquad \text{Since } \ln e = \log_e e = 1$$

$$\Rightarrow \quad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Exercise 2.6 (Questions)

Question # 1

Find $f'(x)$ if

(i) $f(x) = e^{\sqrt{x}-1}$ (ii) $f(x) = x^3 e^{\frac{1}{x}}, (x \neq 0)$ (iii) $f(x) = e^x(1 + \ln x)$

(iv) $f(x) = \dfrac{e^x}{e^{-x}+1}$ (v) $f(x) = \ln\left(e^x + e^{-x}\right)$ (vi) $f(x) = \dfrac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$

(vii) $f(x) = \sqrt{\ln\left(e^{2x} + e^{-2x}\right)}$ (viii) $f(x) = \ln\sqrt{\left(e^{2x} + e^{-2x}\right)}$

Solution

(i) $f(x) = e^{\sqrt{x}-1}$

Diff. w.r.t x

$$\frac{d}{dx} f(x) = \frac{d}{dx} e^{\sqrt{x}-1}$$

$$\Rightarrow f'(x) = e^{\sqrt{x}-1} \frac{d}{dx}\left(\sqrt{x} - 1\right)$$

$$= e^{\sqrt{x}-1}\left(\frac{1}{2} x^{-\frac{1}{2}} - 0\right) = \frac{e^{\sqrt{x}-1}}{2\sqrt{x}} \quad Ans.$$

(ii) $f(x) = x^3 e^{\frac{1}{x}}$

Diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{d}{dx}x^3e^{\frac{1}{x}}$$

$$\Rightarrow f'(x) = x^3\frac{d}{dx}e^{\frac{1}{x}} + e^{\frac{1}{x}}\frac{d}{dx}x^3$$

$$= x^3e^{\frac{1}{x}}\frac{d}{dx}\left(\frac{1}{x}\right) + e^{\frac{1}{x}}\left(3x^2\right)$$

$$= x^3e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) + e^{\frac{1}{x}}\left(3x^2\right) \qquad \because \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = -x^{-2} = -\frac{1}{x^2}$$

$$= -xe^{\frac{1}{x}} + 3x^2e^{\frac{1}{x}} = xe^{\frac{1}{x}}(3x-1) \quad \textit{Ans.}$$

(iii) $f(x) = e^x(1+\ln x)$

Diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{d}{dx}e^x(1+\ln x)$$

$$\Rightarrow f'(x) = e^x\frac{d}{dx}(1+\ln x) + (1+\ln x)\frac{d}{dx}e^x$$

$$= e^x\left(0+\frac{1}{x}\right) + (1+\ln x)e^x$$

$$\Rightarrow f'(x) = e^x\left(\frac{1}{x}+1+\ln x\right) \quad \text{or} \quad f'(x) = e^x\left(\frac{1+x(1+\ln x)}{x}\right)$$

(iv) $f(x) = \dfrac{e^x}{e^{-x}+1}$

Diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{e^x}{e^{-x}+1}\right)$$

$$\Rightarrow f'(x) = \frac{\left(e^{-x}+1\right)\frac{d}{dx}e^x - e^x\frac{d}{dx}\left(e^{-x}+1\right)}{\left(e^{-x}+1\right)^2}$$

$$= \frac{\left(e^{-x}+1\right)e^x - e^x\left(e^{-x}(-1)+0\right)}{\left(e^{-x}+1\right)^2} = \frac{e^x\left(e^{-x}+1+e^{-x}\right)}{\left(e^{-x}+1\right)^2}$$

$$\Rightarrow f'(x) = \frac{e^x\left(2e^{-x}+1\right)}{\left(e^{-x}+1\right)^2} \quad \textit{Ans.}$$

(v) $f(x) = \ln\left(e^x + e^{-x}\right)$

Diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{d}{dx}\ln\left(e^x + e^{-x}\right)$$

$$\Rightarrow f'(x) = \frac{1}{\left(e^x + e^{-x}\right)}\frac{d}{dx}\left(e^x + e^{-x}\right)$$

$$= \frac{1}{\left(e^x + e^{-x}\right)}\left(e^x + e^{-x}(-1)\right)$$

$$\Rightarrow f'(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \text{or} \quad f'(x) = \tanh x \qquad \because \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

(vi) $f(x) = \dfrac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$

Diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}\right)$$

$$= \frac{\left(e^{ax} + e^{-ax}\right)\frac{d}{dx}\left(e^{ax} - e^{-ax}\right) - \left(e^{ax} - e^{-ax}\right)\frac{d}{dx}\left(e^{ax} + e^{-ax}\right)}{\left(e^{ax} + e^{-ax}\right)^2}$$

$$= \frac{\left(e^{ax} + e^{-ax}\right)\left(e^{ax}(a) - e^{-ax}(-a)\right) - \left(e^{ax} - e^{-ax}\right)\left(e^{ax}(a) + e^{-ax}(-a)\right)}{\left(e^{ax} + e^{-ax}\right)^2}$$

$$= \frac{a\left(e^{ax} + e^{-ax}\right)\left(e^{ax} + e^{-ax}\right) - a\left(e^{ax} - e^{-ax}\right)\left(e^{ax} - e^{-ax}\right)}{\left(e^{ax} + e^{-ax}\right)^2}$$

$$= \frac{a\left[\left(e^{ax} + e^{-ax}\right)^2 - \left(e^{ax} - e^{-ax}\right)^2\right]}{\left(e^{ax} + e^{-ax}\right)^2}$$

$$= \frac{a\left[(e^{2ax} + e^{-2ax} + 2e^{ax}e^{-ax}) - (e^{2ax} + e^{-2ax} - 2e^{ax}e^{-ax})\right]}{\left(e^{ax} + e^{-ax}\right)^2}$$

$$= \frac{a\left[e^{2ax} + e^{-2ax} + 2 - e^{2ax} - e^{-2ax} + 2\right]}{\left(e^{ax} + e^{-ax}\right)^2} \qquad \because e^{ax}e^{-ax} = e^0 = 1$$

$$\Rightarrow f'(x) = \frac{4a}{\left(e^{ax} + e^{-ax}\right)^2} \quad \textit{Ans.}$$

(vii) $f(x) = \sqrt{\ln\left(e^{2x}+e^{-2x}\right)}$

$$\Rightarrow \frac{d}{dx}f(x) = \frac{d}{dx}\left[\ln\left(e^{2x}+e^{-2x}\right)\right]^{\frac{1}{2}}$$

$$\Rightarrow f'(x) = \frac{1}{2}\left[\ln\left(e^{2x}+e^{-2x}\right)\right]^{-\frac{1}{2}}\frac{d}{dx}\ln\left(e^{2x}+e^{-2x}\right)$$

$$= \frac{1}{2\left[\ln\left(e^{2x}+e^{-2x}\right)\right]^{\frac{1}{2}}}\cdot\frac{1}{\left(e^{2x}+e^{-2x}\right)}\frac{d}{dx}\left(e^{2x}+e^{-2x}\right)$$

$$= \frac{1}{2\sqrt{\ln\left(e^{2x}+e^{-2x}\right)}}\cdot\frac{1}{\left(e^{2x}+e^{-2x}\right)}\left(e^{2x}(2)+e^{-2x}(-2)\right)$$

$$= \frac{1}{2\sqrt{\ln\left(e^{2x}+e^{-2x}\right)}}\cdot\frac{2\left(e^{2x}-e^{-2x}\right)}{\left(e^{2x}+e^{-2x}\right)} = \frac{\left(e^{2x}-e^{-2x}\right)}{\left(e^{2x}+e^{-2x}\right)\sqrt{\ln\left(e^{2x}+e^{-2x}\right)}} \qquad \textit{Ans.}$$

(viii) $f(x) = \ln\sqrt{\left(e^{2x}+e^{-2x}\right)}$

$$= \ln\left(e^{2x}+e^{-2x}\right)^{\frac{1}{2}} \quad \Rightarrow f(x) = \frac{1}{2}\ln\left(e^{2x}+e^{-2x}\right) \qquad \because \ln x^m = m\ln x$$

Now diff. w.r.t x

$$\frac{d}{dx}f(x) = \frac{1}{2}\frac{d}{dx}\ln\left(e^{2x}+e^{-2x}\right)$$

Now do yourself

Question # 2

Find $\frac{dy}{dx}$ if

(i) $y = x^2\ln\sqrt{x}$ (ii) $y = x\sqrt{\ln x}$ (iii) $y = \frac{x}{\ln x}$

(iv) $y = x^2\ln\frac{1}{x}$ (v) $y = \ln\sqrt{\frac{x^2-1}{x^2+1}}$ (vi) $y = \ln\left(x+\sqrt{x^2+1}\right)$

(vii) $y = \ln\left(9-x^2\right)$ (viii) $y = e^{-2x}\sin 2x$ (ix) $y = e^{-x}\left(x^3+2x^2+1\right)$

(x) $y = xe^{\sin x}$ (xi) $y = 5e^{3x-4}$ (xii) $y = (x+1)^x$

(xiii) $y = (\ln x)^{\ln x}$ (xiv) $y = \frac{\sqrt{x^2-1}\,(x+1)}{\left(x^3+1\right)^{3/2}}$

Solution

(i) $y = x^2\ln\sqrt{x}$

$$\Rightarrow y = x^2 \ln(x)^{\frac{1}{2}} \qquad \Rightarrow y = \frac{1}{2}x^2 \ln x \qquad \because \ln x^m = m\ln x$$

Now diff. w.r.t x

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}x^2 \ln x$$

$$= \frac{1}{2}\left(x^2 \frac{d}{dx}\ln x + \ln x \frac{d}{dx}x^2\right)$$

$$= \frac{1}{2}\left(x^2 \cdot \frac{1}{x} + \ln x(2x)\right) = \frac{1}{2}x + x\ln x \text{ or } \frac{1}{2}x + 2x\ln\sqrt{x} \quad \textit{Ans.}$$

(ii) $y = x\sqrt{\ln x}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}x(\ln x)^{\frac{1}{2}}$$

$$= x\frac{d}{dx}(\ln x)^{\frac{1}{2}} + (\ln x)^{\frac{1}{2}}\frac{d}{dx}(x)$$

$$= x \cdot \frac{1}{2}(\ln x)^{-\frac{1}{2}}\frac{d}{dx}(\ln x) + (\ln x)^{\frac{1}{2}}(1) = \frac{x}{2(\ln x)^{\frac{1}{2}}}\left(\frac{1}{x}\right) + (\ln x)^{\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1 + 2\ln x}{2\sqrt{\ln x}} \quad \textit{Answer}$$

(iii) $y = \dfrac{x}{\ln x}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{\ln x}\right)$$

$$= \frac{\ln x\frac{dx}{dx} - x\frac{d}{dx}\ln x}{(\ln x)^2} = \frac{\ln x(1) - x \cdot \frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} \qquad \textit{Answer}$$

(iv) $y = x^2 \ln\dfrac{1}{x}$

$$\Rightarrow y = x^2 \ln x^{-1} \qquad \Rightarrow y = -x^2 \ln x$$

Now do yourself.

(v) $y = \ln\sqrt{\dfrac{x^2-1}{x^2+1}}$

$$\Rightarrow \; y = \ln\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}} \quad \Rightarrow \; y = \frac{1}{2}\ln\left(\frac{x^2-1}{x^2+1}\right)$$

Now diff. w.r.t x

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\ln\left(\frac{x^2-1}{x^2+1}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{\left(\frac{x^2-1}{x^2+1}\right)}\cdot\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)$$

$$= \frac{x^2+1}{2\left(x^2-1\right)}\cdot\left(\frac{\left(x^2+1\right)\frac{d}{dx}\left(x^2-1\right)-\left(x^2-1\right)\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}\right)$$

$$= \frac{1}{2\left(x^2-1\right)}\cdot\left(\frac{\left(x^2+1\right)(2x)-\left(x^2-1\right)(2x)}{\left(x^2+1\right)}\right)$$

$$= \frac{1}{2\left(x^2-1\right)}\cdot\left(\frac{2x\left(x^2+1-x^2+1\right)}{\left(x^2+1\right)}\right) = \frac{1}{\left(x^2-1\right)}\cdot\left(\frac{x(2)}{\left(x^2+1\right)}\right) = \frac{2x}{\left(x^4-1\right)} \quad \textit{Ans.}$$

(vi) $\quad y = \ln\left(x+\sqrt{x^2+1}\right)$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\ln\left(x+\sqrt{x^2+1}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}\left(x+\sqrt{x^2+1}\right) = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}\frac{d}{dx}\left(x^2+1\right)\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{2\left(x^2+1\right)^{\frac{1}{2}}}\cdot(2x)\right) = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}} \quad \textit{Answer}$$

(vii) $\quad y = \ln\left(9-x^2\right)$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\ln\left(9-x^2\right)$$

$$= \frac{1}{9-x^2} \cdot \frac{d}{dx}\left(9-x^2\right) \quad = \frac{1}{9-x^2} \cdot (-2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{9-x^2}$$

(viii) $\quad y = e^{-2x} \sin 2x$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} e^{-2x} \sin 2x$$

$$= e^{-2x} \frac{d}{dx} \sin 2x + \sin 2x \frac{d}{dx} e^{-2x}$$

$$= e^{-2x} \cos 2x \,(2) + \sin 2x \; e^{-2x} (-2) \quad = 2e^{-2x}\left(\cos 2x - \sin 2x\right) \quad \textit{Answer}$$

(ix) $\quad y = e^{-x}\left(x^3 + 2x^2 + 1\right)$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} e^{-x}\left(x^3 + 2x^2 + 1\right)$$

$$= e^{-x} \frac{d}{dx}\left(x^3 + 2x^2 + 1\right) + \left(x^3 + 2x^2 + 1\right) \frac{d}{dx} e^{-x}$$

$$= e^{-x}\left(3x^2 + 4x + 0\right) + \left(x^3 + 2x^2 + 1\right) \cdot e^{-x} \,(-1)$$

$$= e^{-x}\left(3x^2 + 4x\right) - \left(x^3 + 2x^2 + 1\right) \cdot e^{-x} \quad = e^{-x}\left(3x^2 + 4x - x^3 - 2x^2 - 1\right)$$

$$= e^{-x}\left(-x^3 + x^2 + 4x - 1\right) \qquad \textit{Answer}$$

(x) $\quad y = xe^{\sin x}$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} xe^{\sin x}$$

$$= x \frac{d}{dx} e^{\sin x} + e^{\sin x} \frac{d}{dx} x$$

$$= x \cdot e^{\sin x} \frac{d}{dx} \sin x + e^{\sin x} (1) \quad = x \cdot e^{\sin x} \cos x + e^{\sin x}$$

$$= e^{\sin x}\left(x \cos x + 1\right) \qquad \textit{Answer}$$

(xi) *Do yourself*

(xii) $\quad y = \left(x+1\right)^x$

Taking log on both sides

$$\ln y = \ln(x+1)^x \quad \Rightarrow \ln y = x\ln(x+1)$$

Diff w.r.t x

$$\frac{d}{dx}\ln y = \frac{d}{dx}x\ln(x+1)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}\ln(x+1) + \ln(x+1)\frac{dx}{dx}$$

$$= x\cdot\frac{1}{x+1}\frac{d}{dx}(x+1) + \ln(x+1)(1)$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{x}{x+1}(1) + \ln(x+1)\right)$$

$$= (x+1)^x\left(\frac{x}{x+1} + \ln(x+1)\right) \qquad \textit{Answer}$$

(xiii) $y = (\ln x)^{\ln x}$

Taking log on both sides

$$\ln y = \ln(\ln x)^{\ln x} \quad \Rightarrow \ln y = (\ln x)\cdot\ln(\ln x)$$

Diff w.r.t x

$$\frac{d}{dx}\ln y = \frac{d}{dx}(\ln x)\cdot\ln(\ln x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (\ln x)\frac{d}{dx}\ln(\ln x) + \ln(\ln x)\frac{d}{dx}(\ln x)$$

$$= (\ln x)\cdot\frac{1}{\ln x}\frac{d}{dx}(\ln x) + \ln(\ln x)\cdot\frac{1}{x}$$

$$= \frac{1}{x} + \frac{\ln(\ln x)}{x} = \frac{1+\ln(\ln x)}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1+\ln(\ln x)}{x}\right) \qquad \Rightarrow \frac{dy}{dx} = (\ln x)^{\ln x}\left(\frac{1+\ln(\ln x)}{x}\right)$$

(xiv) $$y = \frac{\sqrt{x^2-1}\,(x+1)}{(x^3+1)^{3/2}} \Rightarrow y = \frac{((x+1)(x-1))^{\frac{1}{2}}(x+1)}{\left[(x+1)(x^2-x+1)\right]^{\frac{3}{2}}}$$

$$\Rightarrow y = \frac{(x+1)^{\frac{1}{2}}(x-1)^{\frac{1}{2}}(x+1)}{(x+1)^{\frac{3}{2}}(x^2-x+1)^{\frac{3}{2}}} \quad \Rightarrow y = \frac{(x+1)^{\frac{3}{2}}(x-1)^{\frac{1}{2}}}{(x+1)^{\frac{3}{2}}(x^2-x+1)^{\frac{3}{2}}}$$

$$\Rightarrow\ y = \frac{(x-1)^{\frac{1}{2}}}{\left(x^2-x+1\right)^{\frac{3}{2}}}$$

Taking log on both sides

$$\ln y = \ln\frac{(x-1)^{\frac{1}{2}}}{\left(x^2-x+1\right)^{\frac{3}{2}}}$$

$$= \ln(x-1)^{\frac{1}{2}} - \ln\left(x^2-x+1\right)^{\frac{3}{2}}$$

$$\Rightarrow\ \ln y = \frac{1}{2}\ln(x-1) - \frac{3}{2}\ln\left(x^2-x+1\right)$$

Now diff. w.r.t x

$$\frac{d}{dx}\ln y = \frac{1}{2}\frac{d}{dx}\ln(x-1) - \frac{3}{2}\frac{d}{dx}\ln\left(x^2-x+1\right)$$

$$\Rightarrow\ \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\frac{1}{x-1}\frac{d}{dx}(x-1) - \frac{3}{2}\frac{1}{\left(x^2-x+1\right)}\frac{d}{dx}\left(x^2-x+1\right)$$

$$= \frac{1}{2(x-1)}(1) - \frac{3}{2\left(x^2-x+1\right)}(2x-1) \quad = \frac{1}{2(x-1)} - \frac{3(2x-1)}{2\left(x^2-x+1\right)}$$

$$\Rightarrow\ \frac{dy}{dx} = y\left[\frac{x^2-x+1-3(2x-1)(x-1)}{2(x-1)\left(x^2-x+1\right)}\right]$$

$$= \frac{(x-1)^{\frac{1}{2}}}{\left(x^2-x+1\right)^{\frac{3}{2}}}\cdot\left[\frac{x^2-x+1-3\left(2x^2-x-2x+1\right)}{2(x-1)\left(x^2-x+1\right)}\right]$$

$$= \left[\frac{x^2-x+1-6x^2+3x+6x-3}{2(x-1)^{1-\frac{1}{2}}\left(x^2-x+1\right)^{\frac{3}{2}+1}}\right] \quad = \frac{-5x^2+8x-2}{2(x-1)^{\frac{1}{2}}\left(x^2-x+1\right)^{\frac{5}{2}}}$$

$$\Rightarrow\ \frac{dy}{dx} = -\frac{5x^2-8x+2}{2\sqrt{x-1}\left(x^2-x+1\right)^{\frac{5}{2}}}\ \textit{Ans.}$$

(xv) $y = \dfrac{(x+2)^2\cdot\sqrt{x-1}}{\sqrt{x^2+x-2}}$

$$\Rightarrow\ y = \frac{(x+2)^2\cdot\sqrt{x-1}}{\sqrt{x^2+2x-x-2}} \quad \Rightarrow\ y = \frac{(x+2)^2\cdot\sqrt{x-1}}{\sqrt{x(x+2)-1(x+2)}}$$

$$\Rightarrow y = \frac{(x+2)^2 \cdot \sqrt{x-1}}{\sqrt{(x+2)(x-1)}} \quad \Rightarrow y = (x+2)^{2-\frac{1}{2}} \Rightarrow y = (x+2)^{\frac{3}{2}}$$

Now diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(x+2)^{\frac{3}{2}}$$

Do yourself

2.1.3 Derivative of Hyperbolic Function (page 85)

The hyperbolic functions are define by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad , x \in R \; ; \qquad \cosh x = \frac{e^x + e^{-x}}{2} \quad , x \in R$$

$$\text{and} \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad , x \in R$$

The reciprocal of these functions are defined as;

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \quad , x \in R - \{0\}; \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \quad , x \in R$$

$$\text{and} \qquad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad , x \in R - \{0\}$$

and there derivatives are

(i) $\frac{d}{dx}(\sinh x) = \cosh x$ (ii) $\frac{d}{dx}(\cosh x) = \sinh x$

(iii) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$ (iv) $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

(v) $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$ (vi) $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Proof:

(i) $$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{d}{dx}\left(\frac{1}{2}\left(e^x - e^{-x}\right)\right) = \frac{1}{2}\frac{d}{dx}\left(e^x - e^{-x}\right)$$

$$= \frac{1}{2}\left(\frac{d}{dx}e^x - \frac{d}{dx}e^{-x}\right) = \frac{1}{2}\left(e^x - e^{-x} \quad e^{-x}\right)$$

$$= \left(\frac{e^x + e^{-x}}{2}\right) = \cosh x$$

(ii) *Similar as above.*

(iii) *See the below (iv) proof.*

(iv) $$\frac{d}{dx}\coth x = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)$$

$$=\frac{\left(e^{x}-e^{-x}\right)\frac{d}{dx}\left(e^{x}+e^{-x}\right)-\left(e^{x}+e^{-x}\right)\frac{d}{dx}\left(e^{x}-e^{-x}\right)}{\left(e^{x}-e^{-x}\right)^{2}}$$

$$=\frac{\left(e^{x}-e^{-x}\right)\left(e^{x}+e^{-x}(-1)\right)-\left(e^{x}+e^{-x}\right)\left(e^{x}-e^{-x}(-1)\right)}{\left(e^{x}-e^{-x}\right)^{2}}$$

$$=\frac{\left(e^{x}-e^{-x}\right)\left(e^{x}-e^{-x}\right)-\left(e^{x}+e^{-x}\right)\left(e^{x}+e^{-x}\right)}{\left(e^{x}-e^{-x}\right)^{2}}$$

$$=\frac{\left(e^{x}-e^{-x}\right)^{2}-\left(e^{x}+e^{-x}\right)^{2}}{\left(e^{x}-e^{-x}\right)^{2}}$$

$$=\frac{(e^{2x}+e^{-2x}-2e^{x}e^{-x})-(e^{2x}+e^{-2x}+2e^{x}e^{-x})}{\left(e^{x}-e^{-x}\right)^{2}}$$

$$=\frac{e^{2x}+e^{-2x}-2-e^{2x}-e^{-2x}-2}{\left(e^{x}-e^{-x}\right)^{2}} \qquad \because\ e^{x}e^{-x}=e^{0}=1$$

$$=\frac{-4}{\left(e^{x}-e^{-x}\right)^{2}}=-\left(\frac{2}{e^{x}-e^{-x}}\right)^{2}=-\operatorname{csch}^{2}x$$

(v) $$\frac{d}{dx}(\operatorname{sech}x)=\frac{d}{dx}\left(\frac{2}{e^{x}+e^{-x}}\right)=\frac{d}{dx}2\left(e^{x}+e^{-x}\right)^{-1}=2\frac{d}{dx}\left(e^{x}+e^{-x}\right)^{-1}$$

$$=2\left[(-1)\left(e^{x}+e^{-x}\right)^{-1-1}\frac{d}{dx}\left(e^{x}+e^{-x}\right)\right]$$

$$=-2\left(e^{x}+e^{-x}\right)^{-2}\left(e^{x}+e^{-x}(-1)\right)=\frac{-2}{\left(e^{x}+e^{-x}\right)^{2}}\left(e^{x}-e^{-x}\right)$$

$$=\frac{-2\left(e^{x}-e^{-x}\right)}{\left(e^{x}+e^{-x}\right)\left(e^{x}+e^{-x}\right)}=-\frac{2}{\left(e^{x}+e^{-x}\right)}\frac{\left(e^{x}-e^{-x}\right)}{\left(e^{x}+e^{-x}\right)}$$

$$=-\operatorname{sech}x\tanh x$$

(vi) *Do yourself as above (v).*

2.14 Derivative of Inverse Hyperbolic Function (page 86)

(i) $\frac{d}{dx}\sinh^{-1}x=\frac{1}{\sqrt{1+x^{2}}}$ (ii) $\frac{d}{dx}\cosh^{-1}x=\frac{1}{\sqrt{x^{2}-1}}$

(iii) $\frac{d}{dx}\tanh^{-1}x=\frac{1}{1-x^{2}}$ (iv) $\frac{d}{dx}\coth^{-1}x=\frac{1}{1-x^{2}}$

(v) $\frac{d}{dx}\text{sech}^{-1}x = \frac{-1}{x\sqrt{1-x^2}}$ (vi) $\frac{d}{dx}\text{csch}^{-1}x = \frac{-1}{x\sqrt{1+x^2}}$

Proof:

(i) Let $y = \sinh^{-1}x \Rightarrow \sinh y = x$

differentiate w.r.t. x.

$$\frac{d}{dx}\sinh y = \frac{d}{dx}x \quad \Rightarrow \cosh y\frac{dy}{dx} = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+\sinh^2 y}} \qquad \because \cosh^2 x - \sinh^2 x = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} \qquad \because \sinh y = x$$

(ii) Do yourself as above.

(iii) Do yourself as (iv) below *or* see book at page 88.

(iv) Let $y = \coth^{-1}x \Rightarrow \coth y = x$

differentiate w.r.t. x

$$\frac{d}{dx}\coth y = \frac{d}{dx}x \quad \Rightarrow -\text{csch}^2 y\frac{dy}{dx} = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{1}{-\text{csch}^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{-(\coth^2 y - 1)} \qquad \because \coth^2 y - 1 = \text{csch}^2 y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{-\coth^2 y + 1} = \frac{1}{1-\coth^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-x^2} \qquad \because \coth y = x$$

(v) Suppose $y = \text{sech}^{-1}x \Rightarrow \text{sech}\, y = x$

differentiate w.r.t. x

$$\frac{d}{dx}\text{sech}\, y = \frac{d}{dx}x \quad \Rightarrow -\text{sech}\, y\tanh y\frac{dy}{dx} = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{1}{-\text{sech}\, y\tanh y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\text{sech}\, y\sqrt{1-\tanh^2 y}} \qquad \because 1-\tanh^2 y = \text{sech}^2 y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}} \qquad \because \text{sech}\, y = x$$

(vi) *Do yourself as above*

Question # 3

Find $\frac{dy}{dx}$ if

(i) $y = \cosh 2x$ (ii) $y = \sinh 3x$ (iii) $y = \tanh^{-1}(\sin x), -\frac{\pi}{2} < x < \frac{\pi}{2}$

(iv) $y = \sinh^{-1}\left(x^3\right)$ (v) $y = \left(\ln \tanh x\right)$ (vi) $y = \sinh^{-1}\left(\frac{x}{2}\right)$

Solution

(i) $y = \cosh 2x$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\cosh 2x \Rightarrow \frac{dy}{dx} = \sinh 2x \frac{d}{dx}(2x) \Rightarrow \frac{dy}{dx} = 2\sinh 2x$$

(ii) *Do yourself*

(iii) $y = \tanh^{-1}(\sin x) \Rightarrow \tanh y = \sin x$

Diff. w.r.t x

$$\frac{d}{dx}\tanh y = \frac{d}{dx}(\sin x)$$

$$\Rightarrow \operatorname{sech}^2 y \frac{dy}{dx} = \cos x \Rightarrow \frac{dy}{dx} = \frac{\cos x}{\operatorname{sech}^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{1 - \tanh^2 y}$$

$\because \cosh^2\theta - \sinh^2\theta = 1$

$\therefore 1 - \tanh^2\theta = \operatorname{sech}^2\theta$

$\because \sin x = \tanh y$

$$= \frac{\cos x}{1 - \sin^2 x}$$

$$= \frac{\cos x}{\cos^2 x} \Rightarrow \frac{dy}{dx} = \sec x$$

(iv) $y = \sinh^{-1}\left(x^3\right) \Rightarrow \sinh y = x^3$

$$\Rightarrow \frac{d}{dx}\sinh y = \frac{d}{dx}x^3 \Rightarrow \cosh y \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{\cosh y}$$

$$= \frac{3x^2}{\sqrt{1 + \sinh^2 y}}$$

$\because \cosh^2 y - \sinh^2 y = 1$

$$= \frac{3x^2}{\sqrt{1+\left(x^3\right)^2}} = \frac{3x^2}{\sqrt{1+x^6}}. \quad \textit{Answer}$$

(v) *Do yourself*

(vi) $y = \sinh^{-1}\left(\frac{x}{2}\right) \Rightarrow \sinh y = \frac{x}{2}$

Now diff w.r.t x

$$\frac{d}{dx}\sinh y = \frac{d}{dx}\left(\frac{x}{2}\right) \Rightarrow \cosh y\frac{dy}{dx} = \frac{1}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\cosh y}$$

$\because \cosh^2 y - \sinh^2 y = 1$

$\therefore \cosh^2 y = 1 + \sinh^2 y$

$$= \frac{1}{2\sqrt{1+\sinh^2 y}}$$

$$= \frac{1}{2\sqrt{1+(x/2)^2}} = \frac{1}{2\sqrt{(4+x^2)/2}} = \frac{1}{\sqrt{4+x^2}} \quad \textit{Answer.}$$

Question # 1

Find y_2 if

(i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$ (ii) $y = (2x+5)^{\frac{3}{2}}$ (iii) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

Solution

(i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(2x^5 - 3x^4 + 4x^3 + x - 2\right)$$

$$\Rightarrow y_1 = 2\left(5x^4\right) - 3\left(4x^3\right) + 4\left(3x^2\right) + 1 - 0$$

$$= 10x^4 - 12x^3 + 12x^2 + 1$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = \frac{d}{dx}\left(10x^4 - 12x^3 + 12x^2 + 1\right)$$

$$\Rightarrow y_2 = 10\left(4x^3\right) - 12\left(3x^2\right) + 12(2x) + 0$$

$$= 40x^3 - 36x^2 + 24x \quad Ans.$$

(ii) $y = (2x+5)^{\frac{3}{2}}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(2x+5)^{\frac{3}{2}}$$

$$\Rightarrow y_1 = \frac{3}{2}(2x+5)^{\frac{3}{2}-1}\frac{d}{dx}(2x+5)$$

$$= \frac{3}{2}(2x+5)^{\frac{1}{2}}(2) = 3(2x+5)^{\frac{1}{2}}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = 3\frac{d}{dx}(2x+5)^{\frac{1}{2}}$$

$$\Rightarrow y_2 = 3\cdot\frac{1}{2}(2x+5)^{-\frac{1}{2}}(2) \quad \Rightarrow y_2 = \frac{3}{\sqrt{2x+5}}$$

(iii) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

$$\Rightarrow y = (x)^{\frac{1}{2}} + (x)^{-\frac{1}{2}}$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left[(x)^{\frac{1}{2}}+(x)^{-\frac{1}{2}}\right] \quad \Rightarrow y_1 = \frac{1}{2}(x)^{-\frac{1}{2}}-\frac{1}{2}(x)^{-\frac{3}{2}}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = \frac{1}{2}\frac{d}{dx}\left[(x)^{-\frac{1}{2}}-(x)^{-\frac{3}{2}}\right]$$

$$\Rightarrow y_2 = \frac{1}{2}\left[-\frac{1}{2}(x)^{-\frac{3}{2}}+\frac{3}{2}(x)^{-\frac{5}{2}}\right]$$

$$= \frac{1}{4}\left[-\frac{1}{x^{\frac{3}{2}}}+\frac{3}{x^{\frac{5}{2}}}\right] = \frac{1}{4}\left[\frac{-x+3}{x^{\frac{5}{2}}}\right] \quad \text{or} \quad y_2 = \frac{3-x}{4x^{\frac{5}{2}}}$$

Question # 2

Find y_2 if

(i) $y = x^2e^{-x}$ (ii) $y = \ln\left(\frac{2x+3}{3x+2}\right)$

Solution

(i) $y = x^2e^{-x}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}x^2e^{-x}$$

$$\Rightarrow y_1 = x^2\frac{d}{dx}e^{-x}+e^{-x}\frac{d}{dx}x^2$$

$$= x^2e^{-x}(-1)+e^{-x}(2x)$$

$$= e^{-x}\left(-x^2+2x\right)$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = \frac{d}{dx}e^{-x}\left(-x^2+2x\right)$$

$$y_2 = e^{-x}\frac{d}{dx}\left(-x^2+2x\right)+\left(-x^2+2x\right)\frac{d}{dx}e^{-x}$$

$$= e^{-x}(-2x+2)+\left(-x^2+2x\right)e^{-x}(-1)$$

$$= e^{-x}\left(-2x+2+x^2-2x\right)$$

$$= e^{-x}\left(x^2-4x+2\right)$$

(ii) $y = \ln\left(\frac{2x+3}{3x+2}\right)$

$$\Rightarrow y = \ln(2x+3)-\ln(3x+2)$$

Diff. w.r.t x

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\ln(2x+3)-\frac{d}{dx}\ln(3x+2)$$

$$\Rightarrow y_1 = \frac{1}{2x+3}(2)-\frac{1}{3x+2}(3)$$

$$= 2(2x+3)^{-1} - 3(3x+2)^{-1}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = 2\frac{d}{dx}(2x+3)^{-1} - 3\frac{d}{dx}(3x+2)^{-1}$$

$$\Rightarrow y_2 = 2\left[-(2x+3)^{-2}(2)\right] - 3\left[-(3x+2)^{-2}(3)\right]$$

$$\Rightarrow y_2 = -\frac{4}{(2x+3)^2} + \frac{9}{(3x+2)^2} \quad \textit{Ans.}$$

$$\text{OR} \quad y_2 = \frac{-4(3x+2)^2 + 9(3x+2)^2}{(2x+3)^2(3x+2)^2}$$

$$= \frac{-4(9x^2+12x+4) + 9(4x^2+12x+9)^2}{(2x+3)^2(3x+2)^2}$$

$$= \frac{-36x^2 - 48x - 16 + 36x^2 + 108x + 81}{(2x+3)^2(3x+2)^2} = \frac{60x+65}{(2x+3)^2(3x+2)^2} \quad \textit{Ans.}$$

(iii) $y = \sqrt{\frac{1-x}{1+x}}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$$

By solving, you will get (differentiate here)

$$\Rightarrow y_1 = \frac{-1}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{2}}} = -(1-x)^{-\frac{1}{2}}(1+x)^{-\frac{3}{2}}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = -\frac{d}{dx}\left[(1-x)^{-\frac{1}{2}}(1+x)^{-\frac{3}{2}}\right]$$

$$\Rightarrow y_2 = -(1-x)^{-\frac{1}{2}}\frac{d}{dx}(1+x)^{-\frac{3}{2}} - (1+x)^{-\frac{3}{2}}\frac{d}{dx}(1-x)^{-\frac{1}{2}}$$

$$= -(1-x)^{-\frac{1}{2}}\left(-\frac{3}{2}(1+x)^{-\frac{5}{2}}(1)\right) - (1+x)^{-\frac{3}{2}}\left(-\frac{1}{2}(1-x)^{-\frac{3}{2}}(-1)\right)$$

$$= \frac{3}{2(1-x)^{\frac{1}{2}}(1+x)^{\frac{5}{2}}} - \frac{1}{2(1+x)^{\frac{3}{2}}(1-x)^{\frac{3}{2}}}$$

$$= \frac{3(1-x)-(1+x)}{2(1-x)^{\frac{3}{2}}(1+x)^{\frac{5}{2}}} = \frac{3-3x-1-x}{2(1-x)^{\frac{3}{2}}(1+x)^{\frac{5}{2}}}$$

$$= \frac{2-4x}{2(1-x)^{\frac{3}{2}}(1+x)^{\frac{5}{2}}} = \frac{1-2x}{(1-x)^{\frac{3}{2}}(1+x)^{\frac{5}{2}}} \quad \textit{Ans.}$$

Question # 3

Find y_2 if

(i) $x^2+y^2 = a^2$ (ii) $x^3-y^3 = a^3$ (iii) $x = a\cos\theta,\ y = a\sin\theta$

(iv) $x = at^2,\ y = bt^4$ (v) $x^2+y^2+2gx+2fy+c=0$

Solution

(i) $x^2+y^2 = a^2$

Diff. w.r.t x

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}a^2 \Rightarrow 2x+2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2y\,y_1 = -2x \Rightarrow y_1 = -\frac{x}{y}$$

Again diff. w.r.t x

$$\Rightarrow \frac{dy_1}{dx} = -\frac{d}{dx}\left(\frac{x}{y}\right) \Rightarrow y_2 = -\left(\frac{y\frac{dx}{dx}-x\frac{dy}{dx}}{y^2}\right)$$

$$\Rightarrow y_2 = -\left(\frac{y(1)-x\left(-\frac{x}{y}\right)}{y^2}\right) \qquad \because \frac{dy}{dx} = -\frac{x}{y}$$

$$= -\left(\frac{y+\frac{x^2}{y}}{y^2}\right) = -\left(\frac{\frac{y^2+x^2}{y}}{y^2}\right)$$

$$= -\left(\frac{x^2+y^2}{y^3}\right) \quad \textit{Ans.}$$

OR $$y_2 = -\frac{a^2}{y^3} \qquad \because x^2+y^2 = a^2$$

(ii) $x^3-y^3 = a^3$

Diff. w.r.t x

$$\frac{d}{dx}\left(x^3-y^3\right) = \frac{d}{dx}a^3$$

$$3x^2-3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow -3y^2y_1 = -3x^2 \Rightarrow y_1 = \frac{x^2}{y^2}$$

Again diff. w.r.t x

$$\Rightarrow \frac{dy_1}{dx} = \frac{d}{dx}\left(\frac{x^2}{y^2}\right)$$

$$\Rightarrow y_2 = \frac{y^2\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(y^2)}{\left(y^2\right)^2}$$

$$= \frac{y^2(2x) - x^2\left(2y\frac{dy}{dx}\right)}{y^4}$$

$$= \frac{2xy^2 - 2x^2y\left(\frac{x^2}{y^2}\right)}{y^4} \qquad \because \frac{dy}{dx} = \frac{x^2}{y^2}$$

$$= \frac{2xy^2 - \frac{2x^4}{y}}{y^4} = \frac{\frac{2xy^3 - 2x^4}{y}}{y^4}$$

$$= \frac{-2x\left(x^3 - y^3\right)}{y^5} \quad \textit{Ans.}$$

OR $$y_2 = \frac{-2x\left(a^3\right)}{y^5} \qquad \because x^3 - y^3 = a^3$$

$$\Rightarrow y_2 = -\frac{2a^3x}{y^5}$$

(iii) $x = a\cos\theta \quad , \quad y = a\sin\theta$

Diff. x w.r.t θ

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}\cos\theta$$

$$= -a\sin\theta$$

$$\Rightarrow \frac{d\theta}{dx} = -\frac{1}{a\sin\theta}$$

Diff y w.r.t θ

$$\frac{dy}{d\theta} = a\frac{d}{d\theta}\sin\theta$$

$$= a\cos\theta$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= a\cos\theta\cdot\frac{-1}{a\sin\theta} \qquad \Rightarrow y_1 = -\cot\theta$$

Now diff. y_1 w.r.t θ

$$\frac{dy_1}{dx} = -\frac{d}{dx}\cot\theta$$

$$\Rightarrow y_2 = +\operatorname{cosec}^2\theta\frac{d\theta}{dx}$$

$$= \operatorname{cosec}^2\theta.\left(-\frac{1}{a\sin\theta}\right)$$

$$\Rightarrow y_2 = \frac{-1}{a\sin^3\theta}$$

(iv) $x = at^2 \quad , \quad y = bt^4$

Diff. x w.r.t t

$$\frac{dx}{dt} = a\frac{d}{dt}t^2$$
$$= 2at$$
$$\Rightarrow \frac{dt}{dx} = \frac{1}{2at}$$

Diff. y w.r.t t

$$\frac{dy}{dt} = b\frac{d}{dt}(t^4)$$
$$= 4bt^3$$

Now by chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$
$$= 4bt^3\cdot\frac{1}{2at} \qquad \Rightarrow y_1 = \frac{2b}{a}t^2$$

Now diff. y_1 w.r.t x

$$\frac{dy_1}{dx} = \frac{2b}{a}\frac{d}{dx}\left(t^2\right) = \frac{2b}{a}\frac{d}{dt}\left(t^2\right)\cdot\frac{dt}{dx}$$
$$\Rightarrow y_2 = \frac{2b}{a}(2t)\cdot\frac{1}{2at} \qquad \Rightarrow y_2 = \frac{2b}{a^2}$$

(v) $x^2 + y^2 + 2gx + 2fy + c = 0$

$$\Rightarrow \frac{d}{dx}\left(x^2 + y^2 + 2gx + 2fy + c\right) = \frac{d}{dx}(0)$$
$$\Rightarrow 2x + 2y\frac{dy}{dx} + 2g(1) + 2f\frac{dy}{dx} + 0 = 0$$
$$\Rightarrow (2y+2f)\frac{dy}{dx} + (2x+2g) = 0$$
$$\Rightarrow (2y+2f)\frac{dy}{dx} = -(2x+2g)$$
$$\Rightarrow \frac{dy}{dx} = -\frac{(2x+2g)}{(2y+2f)} \Rightarrow y_1 = -\frac{x+g}{y+f}$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = -\frac{d}{dx}\left(\frac{x+g}{y+f}\right)$$
$$\Rightarrow y_2 = -\left(\frac{(y+f)\frac{d}{dx}(x+g) - (x+g)\frac{d}{dx}(y+f)}{(y+f)^2}\right)$$

$$= -\frac{(y+f)(1)-(x+g)\frac{dy}{dx}}{(y+f)^2} = -\frac{(y+f)-(x+g)\left(-\frac{x+g}{y+f}\right)}{(y+f)^2}$$

$$= -\frac{\frac{(y+f)^2+(x+g)^2}{y+f}}{(y+f)^2} = -\frac{(y+f)^2+(x+g)^2}{(y+f)^3} \qquad \textit{Ans.}$$

OR $$y_2 = -\frac{y^2+2yf+f^2+x^2+2xg+g^2}{(y+f)^3}$$

$$= -\frac{(x^2+y^2+2gx+2fy+c)-c+f^2+g^2}{(y+f)^3}$$

$$= -\frac{0-c+f^2+g^2}{(y+f)^3} \qquad \because x^2+y^2+2gx+2fy+c=0$$

$$\Rightarrow y_2 = \frac{c-f^2-g^2}{(y+f)^3} \qquad \textit{Ans.}$$

Question # 4

Find y_4 if

(i) $y = \sin 3x$ (ii) $y = \cos^3 x$ (iii) $y = \ln(x^2-9)$

Solution

(i) $y = \sin 3x$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(\sin 3x)$$

$$\Rightarrow y_1 = \cos 3x\,(3) \qquad \Rightarrow y_1 = 3\cos 3x$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = 3\frac{d}{dx}\cos 3x \qquad \Rightarrow y_2 = 3(-\sin 3x\,(3)) \qquad \Rightarrow y_2 = -9\sin 3x$$

Again diff. w.r.t x

$$\frac{dy_2}{dx} = -9\frac{d}{dx}\sin 3x$$

$$\Rightarrow y_3 = -9\cos 3x\,(3) \qquad \Rightarrow y_3 = -27\cos 3x$$

Again diff. w.r.t x

$$\frac{dy_3}{dx} = -27\frac{d}{dx}\cos 3x \qquad \Rightarrow y_4 = -27(-\sin 3x\,(3))$$

$$\Rightarrow \boxed{y_4 = 81\sin 3x}$$

(ii) $y = \cos^3 x$

Diff w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos^3 x\right)$$

$$\Rightarrow y_1 = 3\left(\cos^2 x\right)\frac{d}{dx}\cos x$$

$$\Rightarrow y_1 = 3\left(\cos^2 x\right)\left(-\sin x\right)$$

$$\Rightarrow y_1 = 3\left(1-\sin^2 x\right)\left(-\sin x\right) \qquad \Rightarrow y_1 = -3\sin x + 3\sin^3 x$$

Again diff. w.r.t x

$$\frac{dy_1}{dx} = -3\frac{d}{dx}\sin x + 3\frac{d}{dx}\sin^3 x$$

$$\Rightarrow y_2 = -3\cos x + 9\sin^2 x\frac{d}{dx}\sin x$$

$$\Rightarrow y_2 = -3\cos x + 9\left(1-\cos^2 x\right)\cos x$$

$$= -3\cos x + 9\cos x - 9\cos^3 x \quad = 6\cos x - 9\cos^3 x$$

Again diff. w.r.t x

$$\frac{dy_2}{dx} = 6\frac{d}{dx}\cos x - 9\frac{d}{dx}\cos^3 x$$

$$\Rightarrow y_3 = 6\left(-\sin x\right) - 9\left(-3\sin x + 3\sin^3 x\right) \qquad \because \frac{d}{dx}(\cos^3 x) = -3\sin x + 3\sin^3 x$$

$$= -6\sin x + 27\sin x - 27\sin^3 x \quad = 21\sin x - 27\sin^3 x$$

Again diff. w.r.t x

$$\frac{dy_3}{dx} = 21\frac{d}{dx}\sin x - 27\frac{d}{dx}\sin^3 x$$

$$\Rightarrow y_4 = 21(\cos x) - 27\left(3\sin^2 x\right)\frac{d}{dx}\sin x$$

$$= 21\cos x - 81\sin^2 x(\cos x) \quad = 21\cos x - 81\left(1-\cos^2 x\right)(\cos x)$$

$$= 21\cos x - 81\cos x + 81\cos^3 x \quad = -60\cos x + 54\cos^3 x$$

Alternative:

$$y = \cos^3 x$$

Since $\cos 3x = 4\cos^3 x - 3\cos x$

$$\Rightarrow \cos 3x - 3\cos x = 4\cos^3 x \quad \Rightarrow \cos^3 x = \frac{1}{4}(\cos 3x - 3\cos x)$$

Therefore

$$y = \frac{1}{4}(\cos 3x - 3\cos x)$$

Now diff. w.r.t x

$$\Rightarrow \frac{dy}{dx} = \frac{1}{4}\left(\frac{d}{dx}\cos 3x - 3\frac{d}{dx}\cos x\right)$$

Do yourself

(iii) $y = \ln\left(x^2-9\right)$

$$= \ln\left[(x+3)(x-3)\right] = \ln(x+3)+\ln(x-3)$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\ln(x+3)+\frac{d}{dx}\ln(x-3)$$

$$\Rightarrow y_1 = \frac{1}{x+3}+\frac{1}{x-3}$$

$$= (x+3)^{-1}+(x-3)^{-1}$$

Again diff w.r.t x

$$\frac{dy_1}{dx} = \frac{d}{dx}(x+3)^{-1}+\frac{d}{dx}(x-3)^{-1}$$

$$\Rightarrow y_2 = -(x+3)^{-2}-(x-3)^{-2}$$

Again diff. w.r.t x

$$\frac{dy_2}{dx} = -\frac{d}{dx}(x+3)^{-2}-\frac{d}{dx}(x-3)^{-2} \quad \Rightarrow y_3 = 2(x+3)^{-3}+2(x-3)^{-3}$$

Again diff. w.r.t x

$$\frac{dy_3}{dx} = 2\frac{d}{dx}(x+3)^{-3}+2\frac{d}{dx}(x-3)^{-3}$$

$$\Rightarrow y_4 = 2\left(-3(x+3)^{-4}\right)+2\left(-3(x-3)^{-4}\right)$$

$$= \frac{-6}{(x+3)^4}+\frac{-6}{(x+3)^4} = -6\left[\frac{1}{(x+3)^4}+\frac{1}{(x+3)^4}\right] \quad \textit{Ans.}$$

Question # 5

If $x=\sin\theta$, $y=\sin m\theta$, Show that $\left(1-x^2\right)y_2 - xy_1 + m^2y = 0$

Solution $x = \sin\theta$ …… (i), $y = \sin m\theta$ ……..(ii)

From (i) $\theta=\sin^{-1}x$, putting in (ii)

$$y = \sin\left(m\sin^{-1}x\right)$$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\sin m\left(\sin^{-1}x\right)$$

$$\Rightarrow y_1 = \cos\left(m\sin^{-1}x\right)\frac{d}{dx}m\sin^{-1}x$$

$$= \cos\left(m\sin^{-1}x\right)\cdot m\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow y_1\sqrt{1-x^2} = m\cos\left(m\sin^{-1}x\right)$$

Taking square on both sides.

$$y_1^2\left(1-x^2\right) = m^2\cos^2\left(m\sin^{-1}x\right)$$

$$\Rightarrow y_1^2\left(1-x^2\right) = m^2\left(1-\sin^2(m\sin^{-1}x)\right) \qquad \because \cos^2 x = 1-\sin^2 x$$

$$\Rightarrow y_1^2\left(1-x^2\right)=m^2\left(1-y^2\right) \qquad \text{From (ii)}$$

Now again diff. w.r.t x

$$\frac{d}{dx}y_1^2\left(1-x^2\right)=m^2\frac{d}{dx}\left(1-y^2\right)$$

$$\Rightarrow y_1^2\frac{d}{dx}\left(1-x^2\right)+\left(1-x^2\right)\frac{d}{dx}y_1^2=m^2\left(0-2y\frac{dy}{dx}\right)$$

$$\Rightarrow y_1^2(-2x)+\left(1-x^2\right)2y_1\frac{dy_1}{dx}=-2m^2y\frac{dy}{dx}$$

$$\Rightarrow -2xy_1^2+\left(1-x^2\right)2y_1y_2=-2m^2y\,y_1$$

$$\Rightarrow 2y_1\left(-xy_1+\left(1-x^2\right)y_2\right)=2y_1\left(-m^2y\right)$$

$$\Rightarrow -xy_1+\left(1-x^2\right)y_2=-m^2y$$

$$\Rightarrow \left(1-x^2\right)y_2-xy_1+m^2y=0 \qquad \textit{Proved}$$

Question # 6

If $y=e^x\sin x$, show that $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$

Solution $\quad y=e^x\sin x$

Diff. w.r.t x

$$\frac{dy}{dx}=\frac{d}{dx}e^x\sin x$$

$$=e^x\frac{d}{dx}\sin x+\sin x\frac{d}{dx}e^x$$

$$=e^x\cos x+\sin x\,e^x \quad = e^x(\cos x+\sin x)$$

Again diff. w.r.t x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}e^x(\cos x+\sin x)$$

$$\Rightarrow \frac{d^2y}{dx^2}=e^x\frac{d}{dx}(\cos x+\sin x)+(\cos x+\sin x)\frac{d}{dx}e^x$$

$$=e^x(-\sin x+\cos x)+(\cos x+\sin x)e^x \quad = e^x(-\sin x+\cos x+\cos x+\sin x)$$

$$=e^x(2\cos x) \;=\; 2e^x\cos x$$

Now

$$\text{L.H.S}=\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y$$

$$=2e^x\cos x-2e^x(\cos x+\sin x)+2e^x\sin x$$

$$=2e^x(\cos x-\cos x-\sin x+\sin x)$$

$$=0$$

i.e. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0 \qquad \textit{Proved}$

Question # 7

If $y = e^{ax}\sin bx$, show that $\dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx} + \left(a^2+b^2\right)y = 0$

Solution $y = e^{ax}\sin bx$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}e^{ax}\sin bx$$

$$= e^{ax}\frac{d}{dx}\sin bx + \sin bx\frac{d}{dx}e^{ax} = e^{ax}\cos bx(b) + \sin bx\, e^{ax}(a)$$

$$= e^{ax}\left(b\cos bx + a\sin bx\right)$$

Again diff. w.r.t x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}e^{ax}\left(b\cos bx + a\sin bx\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^{ax}\frac{d}{dx}\left(b\cos bx + a\sin bx\right) + \left(b\cos bx + a\sin bx\right)\frac{d}{dx}e^{ax}$$

$$= e^{ax}\left(-b\sin bx(b) + a\cos bx(b)\right) + \left(b\cos bx + a\sin bx\right)e^{ax}(a)$$

$$= e^{ax}\left(-b^2\sin bx + ab\cos bx + ab\cos bx + a^2\sin bx\right)$$

$$= e^{ax}\left(2ab\cos bx + a^2\sin bx - b^2\sin bx\right)$$

$$= e^{ax}\left(2ab\cos bx + 2a^2\sin bx - a^2\sin bx - b^2\sin bx\right)$$

$$= e^{ax}\left[2a\left(b\cos bx + a\sin bx\right) - \left(a^2+b^2\right)\sin bx\right]$$

$$= 2ae^{ax}\left(b\cos bx + a\sin bx\right) - \left(a^2+b^2\right)e^{ax}\sin bx$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2a\frac{dy}{dx} - \left(a^2+b^2\right)y \quad \Rightarrow \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + \left(a^2+b^2\right)y = 0$$

Question # 8

If $y = \left(Cos^{-1}x\right)^2$, prove that $\left(1-x^2\right)y_2 - xy_1 - 2 = 0$

Solution $y = \left(Cos^{-1}x\right)^2$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(Cos^{-1}x\right)^2 \quad \Rightarrow y_1 = 2\left(Cos^{-1}x\right)\frac{d}{dx}Cos^{-1}x$$

$$\Rightarrow y_1 = 2\left(Cos^{-1}x\right)\cdot\frac{-1}{\sqrt{1-x^2}} \quad \Rightarrow y_1\sqrt{1-x^2} = -2\left(Cos^{-1}x\right)$$

On squaring both sides

$$y_1^2\left(1-x^2\right) = 4\left(Cos^{-1}x\right)^2$$

$$\Rightarrow y_1^2\left(1-x^2\right) = 4y \qquad \because y = \left(Cos^{-1}x\right)^2$$

Again diff. w.r.t x

$$\frac{d}{dx}y_1^2\left(1-x^2\right) = 4\frac{dy}{dx}$$

$$\Rightarrow \left(1-x^2\right)\frac{d}{dx}y_1^2 + y_1^2\frac{d}{dx}\left(1-x^2\right) = 4y_1$$

$$\Rightarrow \left(1-x^2\right)\cdot 2y_1\frac{dy_1}{dx} + y_1^2(-2x) = 4y_1 \quad \Rightarrow 2y_1\left[\left(1-x^2\right)y_2 - xy_1\right] = 4y_1$$

$$\Rightarrow \left(1-x^2\right)y_2 - xy_1 - 2 = 0$$

Question # 9

If $y = a\cos(\ln x) + b\sin(\ln x)$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Solution $y = a\cos(\ln x) + b\sin(\ln x)$

Diff. w.r.t x

$$\frac{dy}{dx} = a\frac{d}{dx}\cos(\ln x) + b\frac{d}{dx}\sin(\ln x)$$

$$= a\left[-\sin(\ln x)\right]\frac{d}{dx}(\ln x) + b\cos(\ln x)\frac{d}{dx}(\ln x)$$

$$= -a\sin(\ln x)\frac{1}{x} + b\cos(\ln x)\frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx} = -a\sin(\ln x) + b\cos(\ln x)$$

Again diff. w.r.t x

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] = -a\frac{d}{dx}\sin(\ln x) + b\frac{d}{dx}\cos(\ln x)$$

$$\Rightarrow x\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\left(\frac{dx}{dx}\right) = -a\cos(\ln x)\frac{d}{dx}(\ln x) + b(-\sin(\ln x))\frac{d}{dx}(\ln x)$$

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot(1) = -a\cos(\ln x)\cdot\frac{1}{x} - b\sin(\ln x)\cdot\frac{1}{x}$$

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x}\left(a\cos(\ln x) + b\sin(\ln x)\right) \quad \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \quad \textit{Proved}$$

Taylor Series Expansion of Function

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+....$$

Maclaurin Series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+....$$

Question # 1

Apply the Maclaurin series expansion to prove that:

(i) $\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+......$

(ii) $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+......$

(iii) $\sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}+.....$

(iv) $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+.....$

(v) $e^{2x} = 1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+.....$

Solution

(i) Let $f(x) = \ln(1+x)$

$\Rightarrow f(0) = \ln(1+0) = 0$

$f'(x) = \frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$

$\Rightarrow f'(0) = \frac{1}{1+0} = 1$

$f''(x) = \frac{d}{dx}(1+x)^{-1} = -(1+x)^{-2}$

$\Rightarrow f''(0) = -(1+0)^{-2} = -1$

$f'''(x) = \frac{d}{dx}\left[-(1+x)^{-2}\right] = +2(1+x)^{-3}$

$\Rightarrow f'''(0) = 2(1+0)^{-3} = 2$

$f^{(iv)}(x) = \frac{d}{dx}2(1+x)^{-3} = -6(1+x)^{-4}$

$\Rightarrow f^{(iv)}(0) = -6(1+0)^{-4} = -6$

By Maclaurin series

$$f(x)= f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+......$$

$$\Rightarrow \ln(1+x) = 0+x(1)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(2)+\frac{x^4}{4!}(-6)+......$$

$$= x-\frac{x^2}{2\cdot 1}+\frac{x^3}{3\cdot 2\cdot 1}(2)-\frac{x^4}{4\cdot 3\cdot 2\cdot 1}(6)+......$$

$$= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+......$$

(ii) Let $f(x) = \cos x \Rightarrow f(0) = \cos(0) = 1$

$$f'(x) = \frac{d}{dx}\cos x = -\sin x \quad \Rightarrow f'(0) = -\sin(0) = 0$$

$$f''(x) = \frac{d}{dx}(-\sin x) = -\cos x \quad \Rightarrow f''(0) = -\cos(0) = -1$$

$$f'''(x) = \frac{d}{dx}(-\cos x) = +\sin x \quad \Rightarrow f'''(0) = \sin(0) = 0$$

$$f^{(iv)}(x) = \frac{d}{dx}\sin x = \cos x \quad \Rightarrow f^{(iv)}(x) = \cos(0) = 1$$

$$f^{(v)}(x) = \frac{d}{dx}\cos x = -\sin x \quad \Rightarrow f^{(v)}(x) = -\sin(0) = 0$$

$$f^{(vi)}(0) = \frac{d}{dx}(-\sin x) = -\cos x \quad \Rightarrow f^{(vi)}(0) = -\cos(0) = -1$$

Now by Maclaurin series

$$f(x)= f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+......$$

$$\Rightarrow \cos x = 1+x(0)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(0)+\frac{x^4}{4!}(1)+\frac{x^5}{5!}(0)+\frac{x^6}{6!}(-1)+......$$

$$= 1+0-\frac{x^2}{2!}+0+\frac{x^4}{4!}+0-\frac{x^6}{6!}+......$$

$$= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+......$$

(iii) Let $f(x) = \sqrt{1+x}$

$$= (1+x)^{\frac{1}{2}} \quad \Rightarrow f(0) = (1+0)^{\frac{1}{2}} = 1$$

$$f'(x) = \frac{d}{dx}(1+x)^{\frac{1}{2}}$$

$$= \frac{1}{2}(1+x)^{-\frac{1}{2}}(1) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$$

$$\Rightarrow f'(0) = \frac{1}{2}(1+0)^{-\frac{1}{2}} = \frac{1}{2}$$

$$f''(x) = \frac{d}{dx}\left[\frac{1}{2}(1+x)^{-\frac{1}{2}}\right] = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$$

$$\Rightarrow f''(0) = -\frac{1}{4}(1+0)^{-\frac{3}{2}} = -\frac{1}{4}$$

$$f'''(x) = -\frac{1}{4}\frac{d}{dx}\left[(1+x)^{-\frac{3}{2}}\right]$$

$$= -\frac{1}{4}\left[-\frac{3}{2}(1+x)^{-\frac{5}{2}}\right] = \frac{3}{8}(1+x)^{-\frac{5}{2}}$$

$$\Rightarrow f'''(0) = \frac{3}{8}(1+0)^{-\frac{5}{2}} = \frac{3}{8}$$

Now by Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots$$

$$\Rightarrow \sqrt{1+x} = 1 + x\cdot\frac{1}{2} + \frac{x^2}{2!}\cdot\left(-\frac{1}{4}\right) + \frac{x^3}{3!}\cdot\frac{3}{8} + \ldots..$$

$$= 1 + x\cdot\frac{1}{2} + \frac{x^2}{2}\cdot\left(-\frac{1}{4}\right) + \frac{x^3}{6}\cdot\frac{3}{8} + \ldots..$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \ldots..$$

(iv) Let $f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$$f'(x) = \frac{d}{dx}\left(e^x\right) = e^x \Rightarrow f'(0) = e^0 = 1$$

$$f''(x) = \frac{d}{dx}\left(e^x\right) = e^x \Rightarrow f''(0) = e^0 = 1$$

$$f'''(x) = \frac{d}{dx}\left(e^x\right) = e^x \Rightarrow f'''(0) = e^0 = 1$$

By Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots$$

$$\Rightarrow e^x = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \ldots..$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots..$$

(v) Let $f(x) = e^{2x} \Rightarrow f(0) = e^{2(0)} = e^0 = 1$

$$f'(x) = \frac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$$

$$\Rightarrow f'(0) = 2e^{2(0)} = 2(1) = 2$$

$$f''(x) = 2\frac{d}{dx}\left(e^{2x}\right) = 2\left(2e^{2x}\right) = 4e^{2x}$$

$$\Rightarrow f''(0) = 4e^{2(0)} = 4(1) = 4$$

$$f'''(x) = 4\frac{d}{dx}\left(e^{2x}\right) = 4\left(2e^{2x}\right) = 8e^{2x}$$

$$\Rightarrow f'''(0) = 8e^{2(0)} = 8$$

By Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots$$

$$\Rightarrow e^{2x} = 1 + x(2) + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(8) + \ldots\ldots$$

$$= 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \ldots\ldots$$

Question # 2
Show that

$$\cos(x+h) = \cos x - h\sin x - \frac{h^2}{\lfloor 2}\cos x + \frac{h^3}{\lfloor 3}\sin x + \ldots\ldots$$

and evaluate $\cos 61^\circ$.

Solution Let $f(x) = \cos x$

$$f'(x) = \frac{d}{dx}\cos x = -\sin x$$

$$f''(x) = -\frac{d}{dx}\sin x = -\cos x$$

$$f'''(x) = -\frac{d}{dx}\cos x = -(-\sin x) = \sin x$$

By Taylor series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots.$$

$$\Rightarrow \cos(x+h) = \cos x + h(-\sin x) + \frac{h^2}{2!}(-\cos x) + \frac{h^3}{3!}(\sin x) + \ldots\ldots$$

$$\Rightarrow \cos(x+h) = \cos x - h\sin x - \frac{h^2}{\lfloor 2}\cos x + \frac{h^3}{\lfloor 3}\sin x + \ldots\ldots$$

Put $x = 60^\circ$ and $h = 1^\circ = \frac{\pi}{180} = 0.01745$ rad

$$\cos(60+1) = \cos 60 - (0.01745)\sin 60 - \frac{(0.01745)^2}{\lfloor 2}\cos 60 + \frac{(0.01745)^3}{\lfloor 3}\sin 60 + \ldots\ldots$$

Increasing and Decreasing Function (Page 104)

Let f be defined on an interval (a,b) and let $x_1, x_2 \in (a,b)$. Then

1. f is increasing on the interval (a,b) if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$
2. f is decreasing on the interval (a,b) if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$

Theorem (Page 105)

Let f be differentiable on the open interval (a,b).

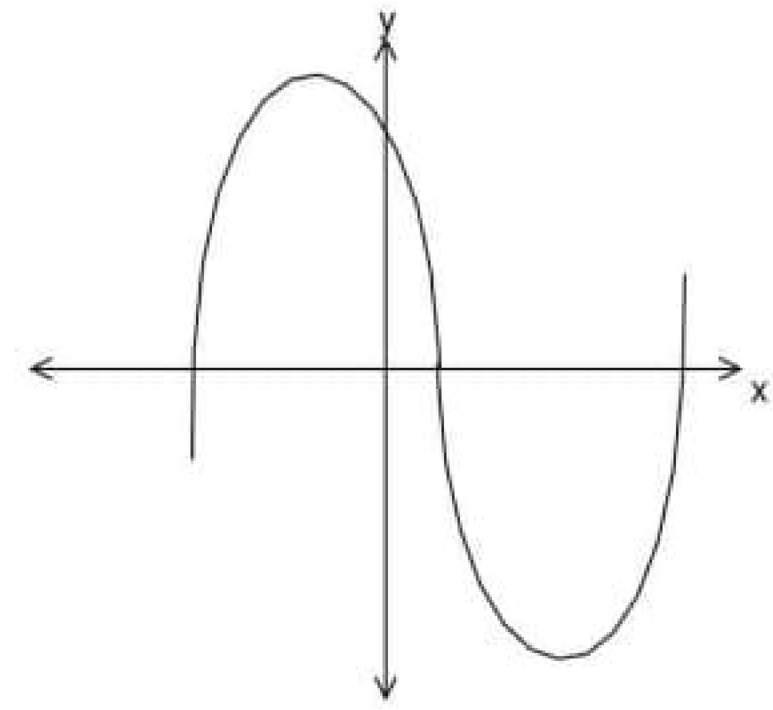
1- f is increasing on (a,b) if $f'(x) > 0$ for each $x \in (a,b)$.

2- f is decreasing on (a,b) if $f'(x) < 0$ for each $x \in (a,b)$.

First Derivative Test (Page 109)

Let f be differentiable in neighbourhood of c, where $f'(c) = 0$.

1. The function has relative maxima at $x = c$ if $f'(x) > 0$ before $x = c$ and $f'(x) < 0$ after $x = c$.
2. The function has relative minima at $x = c$ if $f'(x) < 0$ before $x = c$ and $f'(x) > 0$ after $x = c$.



Second Derivative Test (Page 111)

Let f be differential function in a neighbourhood of c, where $f'(c) = 0$. Then

1- f has relative maxima at c if $f''(c) < 0$.

2- f has relative minima at c if $f''(c) > 0$.

Question # 1

Determine the intervals in which f is increasing or decreasing for the domain mentioned in each case.

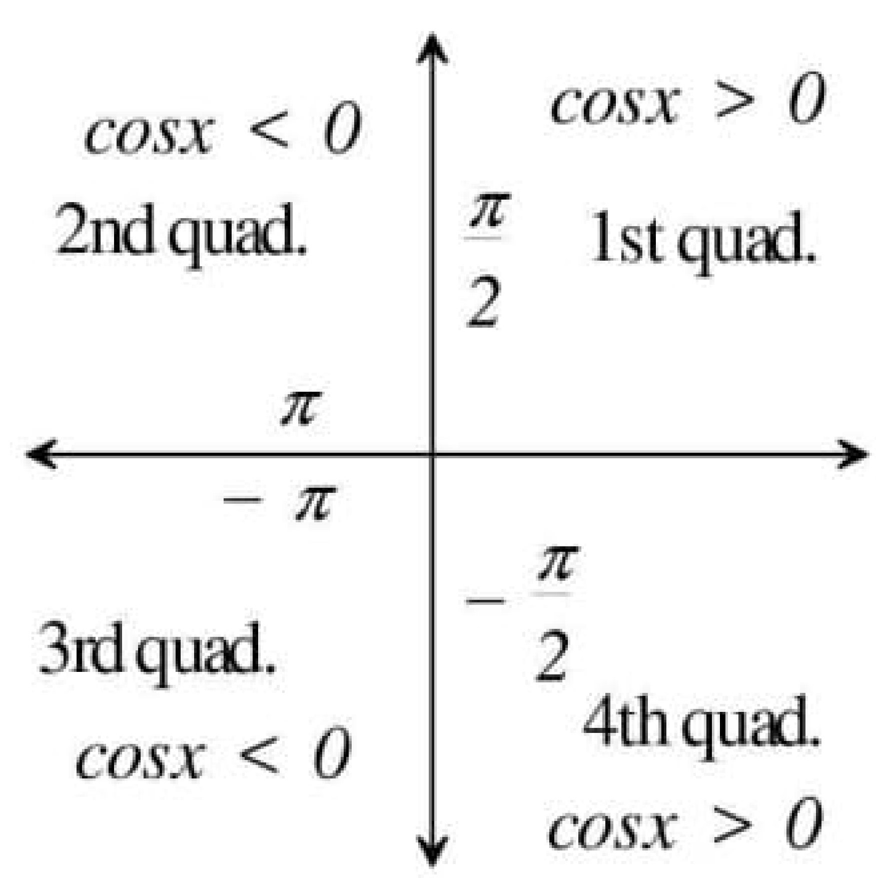
(i) $f(x) = \sin x$; $x \in [-\pi, \pi]$

(ii) $f(x) = \cos x$; $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(iii) $f(x) = 4 - x^2$; $x \in [-2, 2]$

(iv) $f(x) = x^2 + 3x + 2$; $x \in [-4, 1]$

Solution



(i) $f(x) = \sin x$; $x \in [-\pi, \pi]$

$\Rightarrow f'(x) = \cos x$

Put $f'(x) = 0 \Rightarrow \cos x = 0$

$\Rightarrow x = -\dfrac{\pi}{2}, \dfrac{\pi}{2}$

So we have sub-intervals $\left(-\pi,-\dfrac{\pi}{2}\right)$, $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$, $\left(\dfrac{\pi}{2},\pi\right)$

$f'(x)=\cos x<0$ whenever $x\in\left(-\pi,-\dfrac{\pi}{2}\right)$

So f is decreasing on the interval $\left(-\pi,-\dfrac{\pi}{2}\right)$.

$f'(x)=\cos x>0$ whenever $x\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

So f is increasing on the interval $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$.

$f'(x)=\cos x>0$ whenever $x\in\left(\dfrac{\pi}{2},\pi\right)$

So f is decreasing on the interval $\left(\dfrac{\pi}{2},\pi\right)$.

(ii) $f(x)=\cos x$; $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

$\Rightarrow f'(x)=-\sin x$

Put $f'(x)=0 \Rightarrow -\sin x=0 \Rightarrow \sin x=0 \Rightarrow x = 0$

So we have sub-intervals $\left(-\dfrac{\pi}{2},0\right)$ and $\left(0,\dfrac{\pi}{2}\right)$.

Now $f'(x)=-\sin x>0$ whenever $x\in\left(-\dfrac{\pi}{2},0\right)$

So f is increasing on $\left(-\dfrac{\pi}{2},0\right)$

$f'(x)=-\sin x<0$ whenever $x\in\left(0,\dfrac{\pi}{2}\right)$

So f is decreasing on $\left(0,\dfrac{\pi}{2}\right)$.

(iii) $f(x)=4-x^2$; $x\in[-2,2]$

$\Rightarrow f'(x) = -2x$

Put $f'(x)=0 \Rightarrow -2x=0 \Rightarrow x=0$

So we have subintervals $(-2,0)$ and $(0,2)$

$\because f'(x)=-2x>0$ whenever $x\in(-2,0)$

$\therefore f$ is increasing on the interval $(-2,0)$

Also $f'(x)=-2x<0$ whenever $x\in(0,2)$

$\therefore f$ is decreasing on $(0,2)$

(iv) $f(x) = x^2+3x+2$; $x\in[-4,1]$

$\Rightarrow f'(x) = 2x+3$

Put $f'(x)=0 \Rightarrow 2x+3=0 \Rightarrow x=-\frac{3}{2}$

So we have sub-intervals $\left(-4,-\frac{3}{2}\right)$ and $\left(-\frac{3}{2},1\right)$

Now $f'(x)=2x+3<0$ whenever $x\in\left(-4,-\frac{3}{2}\right)$

So f is decreasing on $\left(-4,-\frac{3}{2}\right)$

Also $f'(x)>0$ whenever $x\in\left(-\frac{3}{2},1\right)$

Therefore f is increasing on $\left(-\frac{3}{2},1\right)$.

Question # 2

Ind the extreme values of the following functions defined as:

(i) $f(x)=1-x^3$

(ii) $f(x) = x^2-x-2$

(iii) $f(x) = 5x^2-6x+2$

(iv) $f(x) = 3x^2$

(v) $f(x) = 3x^2-4x+5$

(vi) $f(x) = 2x^3-2x^2-36x+3$

(vii) $f(x) = x^4-4x^2$

(viii) $f(x) = (x-2)^2(x-1)$

(ix) $f(x) = 5+3x-x^3$

Solution

(i) $f(x)=1-x^3$

Diff. w.r.t x

$f'(x) = -3x^2$ ……. (i)

For stationary points, put $f'(x) = 0$

$\Rightarrow -3x^2 = 0 \Rightarrow x=0$

Diff (i) w.r.t x

$f''(x)=-6x$ ………. (ii)

Now put $x=0$ in (ii)

$f''(0)=-6(0)=0$

So second derivative test fails to determinate the extreme points.

Put $x=0-\varepsilon=-\varepsilon$ in (i)

$f'(x)=-3(-\varepsilon)^2=-3\varepsilon^2<0$

Put $x=0+\varepsilon=\varepsilon$ in (i)

$f'(x)=-3(\varepsilon)^2 = -3\varepsilon^2 <0$

As $f'(x)$ doest not change its sign before and after $x=0$.

Since at $x=0$, $f(x)=1$ therefore $(0,1)$ is the point of inflexion.

(ii) $f(x) = x^2 - x - 2$

Diff. w.r.t. x

$f'(x) = 2x - 1$ .... (i)

For stationary points, put $f'(x) = 0$

$$\Rightarrow 2x-1=0 \quad \Rightarrow 2x=1 \quad \Rightarrow x=\frac{1}{2}$$

Diff (i) w.r.t x

$$f''(x) = \frac{d}{dx}(2x-1) = 2$$

As $f''\left(\frac{1}{2}\right) = 2 > 0$

Thus $f(x)$ is minimum at $x=\frac{1}{2}$

Now $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 2 = \frac{1}{4} - \frac{1}{2} - 2 = -\frac{9}{4}$

(iii) $f(x) = 5x^2 - 6x + 2$

Diff. w.r.t. x

$f'(x) = 10x - 6$ .... (i)

For stationary points, put $f'(x) = 0$

$$\Rightarrow 10x-6=0 \quad \Rightarrow 10x=6 \quad \Rightarrow x=\frac{6}{10} \quad \Rightarrow x=\frac{3}{5}$$

Diff (i) w.r.t x

$$f''(x) = \frac{d}{dx}(10x-6) = 10$$

As $f''\left(\frac{3}{5}\right) = 10 > 0$

Thus $f(x)$ is minimum at $x=\frac{3}{5}$

And $f\left(\frac{3}{5}\right) = 5\left(\frac{3}{5}\right)^2 - 6\left(\frac{3}{5}\right) + 2 = \frac{9}{5} - \frac{18}{5} + 2 = \frac{1}{5}$

(iv) $f(x) = 3x^2$

Diff. w.r.t x

$f'(x) = 6x$ .... (i)

For stationary points, put $f'(x)=0$

$$\Rightarrow 6x=0 \quad \Rightarrow x=0$$

Diff. (i) w.r.t x

$f''(x) = 6$

At $x = 0$

$$f''(0) = 6 > 0$$

$\Rightarrow$ f has minimum value at $x = 0$

And $f(0) = 3(0)^2 = 0$

(v) *Do yourself*

(vi) $f(x) = 2x^3 - 2x^2 - 36x + 3$

Diff. w.r.t x

$$f'(x) = \frac{d}{dx}\left(2x^3 - 2x^2 - 36x + 3\right) = 6x^2 - 4x - 36 \text{ ....(i)}$$

For stationary points, put $f'(x) = 0$

$$\Rightarrow 6x^2 - 4x - 36 = 0$$

$$\Rightarrow 3x^2 - 2x - 12 = 0 \qquad \div\text{ing by 2}$$

$$\Rightarrow x = \frac{2 \pm \sqrt{4 - 4(3)(-18)}}{2(3)}$$

$$= \frac{2 \pm \sqrt{4 + 216}}{6} = \frac{2 \pm \sqrt{220}}{6} = \frac{2 \pm 2\sqrt{55}}{6} = \frac{1 \pm \sqrt{55}}{3}$$

Diff. (i) w.r.t x

$$f''(x) = \frac{d}{dx}\left(6x^2 - 4x - 36\right) = 12x - 4$$

Now $f''\left(\frac{1+\sqrt{55}}{3}\right) = 12\left(\frac{1+\sqrt{55}}{3}\right) - 4$

$$= 4\left(1 + \sqrt{55}\right) - 4 = 4 + 4\sqrt{55} - 4 = 4\sqrt{55} > 0$$

$\Rightarrow$ $f(x)$ has relative minima at $x = \frac{1+\sqrt{55}}{3}$.

And $f\left(\frac{1+\sqrt{55}}{3}\right) = 2\left(\frac{1+\sqrt{55}}{3}\right)^3 - 2\left(\frac{1+\sqrt{55}}{3}\right)^2 - 36\left(\frac{1+\sqrt{55}}{3}\right) + 3$

$$= \frac{2}{27}\left(1+\sqrt{55}\right)^3 - \frac{2}{9}\left(1+\sqrt{55}\right)^2 - 12\left(1+\sqrt{55}\right) + 3$$

$$= \frac{2}{27}\left(1 + 3\sqrt{55} + 3\cdot 55 + 55\sqrt{55}\right) - \frac{2}{9}\left(1 + 2\sqrt{55} + 55\right) - 12\left(1 + \sqrt{55}\right) + 3$$

$$= \frac{2}{27}\left(166 + 58\sqrt{55}\right) - \frac{2}{9}\left(56 + 2\sqrt{55}\right) - 12\left(1 + \sqrt{55}\right) + 3$$

$$= \frac{332}{27} + \frac{116}{27}\sqrt{55} - \frac{112}{9} - \frac{4}{9}\sqrt{55} - 12 - 12\sqrt{55} + 3$$

$$= -\frac{247}{27} - \frac{220}{27}\sqrt{55} = -\frac{1}{27}\left(247 + 220\sqrt{55}\right)$$

Also $f''\left(\frac{1-\sqrt{55}}{3}\right) = 12\left(\frac{1-\sqrt{55}}{3}\right)-4$

$$= 4\left(1-\sqrt{55}\right)-4 = 4-4\sqrt{55}-4 = -4\sqrt{55} < 0$$

$\Rightarrow f(x)$ has relative maxima at $x = \frac{1+\sqrt{55}}{3}$.

And Since $f\left(\frac{1+\sqrt{55}}{3}\right) = -\frac{1}{27}\left(247+220\sqrt{55}\right)$

Therefore by replacing $\sqrt{55}$ by $-\sqrt{55}$, we have

$$f\left(\frac{1-\sqrt{55}}{3}\right) = -\frac{1}{27}\left(247-220\sqrt{55}\right)$$

(vii) $f(x) = x^4-4x^2$

Diff. w.r.t. x

$$f'(x) = 4x^3-8x \text{ …….. (i)}$$

For critical points put $f'(x)=0$

$$\Rightarrow 4x^3-8x=0 \quad \Rightarrow 4x\left(x^2-2\right)=0$$

$$\Rightarrow 4x=0 \quad \text{or} \quad x^2-2=0$$

$$\Rightarrow x=0 \quad \text{or} \quad x^2=2 \quad \Rightarrow x=\pm\sqrt{2}$$

Now diff. (i) w.r.t x

$$f''(x) = 12x^2-8$$

For $x=-\sqrt{2}$

$$f''(-\sqrt{2}) = 12(-\sqrt{2})^2-8 = 24-8 = 16 > 0$$

$\Rightarrow f$ has relative minima at $x=-\sqrt{2}$

And $f\left(-\sqrt{2}\right) = \left(-\sqrt{2}\right)^4-4\left(-\sqrt{2}\right)^2 = 4-8 = -4$

For $x=0$

$$f''(0) = 12(0)-8 = -8 < 0$$

$\Rightarrow f$ has relative maxima at $x=0$

And $f(0) = (0)^4-4(0)^2 = 0$

For $x=\sqrt{2}$

$$f''(\sqrt{2}) = 12(\sqrt{2})^2-8 = 24-8 = 16 > 0$$

$\Rightarrow f$ has relative minima at $x=\sqrt{2}$

And $f\left(\sqrt{2}\right) = \left(\sqrt{2}\right)^4-4\left(\sqrt{2}\right)^2 = 4-8 = -4$

(viii) $f(x) = (x-2)^2(x-1)$

$= (x^2-4x+4)(x-1) = x^3-4x^2+4x-x^2+4x-4$

$= x^3-5x^2+8x-4$

Diff. w.r.t. x

$f'(x) = 3x^2-10x+8$

For critical (stationary) points, put $f'(x) = 0$

$\Rightarrow 3x^2-10x+8=0 \Rightarrow 3x^2-6x-4x+8=0$

$\Rightarrow 3x(x-2)-4(x-2)=0 \Rightarrow (x-2)(3x-4)=0$

$\Rightarrow (x-2)=0$ or $(3x-4)=0$

$\Rightarrow x=2$ or $x = \dfrac{4}{3}$

Now diff. (i) w.r.t x

$f''(x) = 6x-10$

For $x = 2$

$f''(2) = 6(2)-10 = 2 > 0$

$\Rightarrow f$ has relative minima at $x=2$

And $f(2) = (2-2)^2(2-1) = 0$

For $x=\dfrac{4}{3}$

$$f''\left(\frac{4}{3}\right) = 6\left(\frac{4}{3}\right)-10 = 8-10 = -2 < 0$$

$\Rightarrow f$ has relative maxima at $x=\dfrac{4}{3}$

And $f\left(\dfrac{4}{3}\right) = \left(\dfrac{4}{3}-2\right)^2\left(\dfrac{4}{3}-1\right) = \left(-\dfrac{2}{3}\right)^2\left(\dfrac{1}{3}\right) = \left(\dfrac{4}{9}\right)\left(\dfrac{1}{3}\right) = \dfrac{4}{27}$

(ix) $f(x) = 5+3x-x^3$

Diff. w.r.t x

$f'(x) = 3-3x^2$ .... (i)

For stationary points, put $f'(x)=0$

$\Rightarrow 3-3x^2=0 \Rightarrow 3x^2=3 \Rightarrow x^2=1 \Rightarrow x=\pm 1$

Diff. (i) w.r.t x

$f''(x) = -6x$

For $x=1$

$f''(1) = -6(1) = -6 < 0$

$\Rightarrow f$ has relative maxima at $x=1$

And $f(1) = 5+3(1)-(1)^3 = 5+3-1 = 7$

For $x=-1$

$f''(-1) = -6(-1) = 6 > 0$

$\Rightarrow f$ has relative minima at $x = -1$, and

$f(-1) = 5 + 3(-1) - (-1)^3 = 5 - 3 + 1 = 3$

Question # 3

Find the maximum and minimum values of the function defined by the following equation occurring in the interval $[0, 2\pi]$

$$f(x) = \sin x + \cos x$$

Solution $f(x) = \sin x + \cos x$ where $x \in [0, 2\pi]$

Diff. w.r.t x

$$f'(x) = \cos x - \sin x \ \ldots\ldots\ldots \text{(i)}$$

For stationary points, put $f'(x) = 0$

$$\cos x - \sin x = 0$$

$$\Rightarrow -\sin x = -\cos x \quad \Rightarrow \frac{\sin x}{\cos x} = 1 \quad \Rightarrow \tan x = 1$$

$$\Rightarrow x = \tan^{-1}(1) \quad \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4} \quad \text{when } x \in [0, 2\pi]$$

Now diff. (i) w.r.t x

$$f''(x) = -\sin x - \cos x$$

For $x = \frac{\pi}{4}$

$$f''\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -2\left(\frac{1}{\sqrt{2}}\right) < 0$$

$\Rightarrow f$ has relative maxima at $x = \frac{\pi}{4}$

And $f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2\left(\frac{1}{\sqrt{2}}\right) = \left(\sqrt{2}\right)^2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

For $x = \frac{5\pi}{4}$

$$f''\left(\frac{5\pi}{4}\right) = -\sin\left(\frac{5\pi}{4}\right) - \cos\left(\frac{5\pi}{4}\right)$$

$$= -\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2\left(\frac{1}{\sqrt{2}}\right) > 0$$

$\Rightarrow f$ has relative minima at $x = \frac{5\pi}{4}$

And $f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) + \cos\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -2\left(\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$

Question # 4

Show that $y = \frac{\ln x}{x}$ has maximum value at $x = e$

Solution $y = \dfrac{\ln x}{x}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{x\cdot\frac{1}{x} - \ln x\cdot(1)}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-\ln x}{x^2} \quad \ldots\ldots\ldots \text{(i)}$$

For critical points, put $\dfrac{dy}{dx}=0$

$$\Rightarrow \frac{1-\ln x}{x^2} = 0 \Rightarrow 1-\ln x = 0 \Rightarrow \ln x = 1$$

$$\Rightarrow \ln x = \ln e \quad \Rightarrow x = e \qquad \because \ln e = 1$$

Diff. (i) w.r.t x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1-\ln x}{x^2}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{x^2\cdot\left(-\frac{1}{x}\right)-(1-\ln x)\cdot(2x)}{(x^2)^2} = \frac{-x-2x+2x\ln x}{x^4} = \frac{-3x+2x\ln x}{x^4}$$

At $x=e$

$$\left.\frac{d^2y}{dx^2}\right|_{x=e} = \frac{-3e+2e\cdot\ln e}{e^4}$$

$$= \frac{-3e+2e\cdot(1)}{e^4} = \frac{-e}{e^4} = -\frac{1}{e^3} < 0$$

$\Rightarrow$ y has a maximum value at $x=e$.

Question # 5

Show that $y = x^x$ has maximum value at $x=\dfrac{1}{e}$.

Solution $y = x^x$

Taking log on both sides

$$\ln y = \ln x^x \Rightarrow \ln y = x\ln x$$

Diff. w.r.t x

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}x\ln x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\cdot\frac{d}{dx}\ln x + \ln x\cdot\frac{dx}{dx}$$

$$= x\cdot\frac{1}{x} + \ln x\cdot(1)$$

$$\Rightarrow \frac{dy}{dx} = y(1+\ln x) \qquad \Rightarrow \frac{dy}{dx} = x^x(1+\ln x) \quad \ldots\ldots\ldots\ldots \text{(i)}$$

For critical point, put $\dfrac{dy}{dx}=0$

$$\Rightarrow\ x^x(1+\ln x)=0 \quad \Rightarrow\ 1+\ln x=0 \ \text{ as } x^x\neq 0$$

$$\Rightarrow\ \ln x=-1 \quad \Rightarrow\ \ln x=-\ln e \qquad \because\ \ln e=1$$

$$\Rightarrow\ \ln x = \ln e^{-1} \quad \Rightarrow\ x = e^{-1} \quad \Rightarrow\ x = \frac{1}{e}$$

Diff. (i) w.r.t x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}x^x(1+\ln x)$$

$$\Rightarrow\ \frac{d^2y}{dx^2} = x^x\frac{d}{dx}(1+\ln x)+(1+\ln x)\frac{d}{dx}x^x$$

$$= x^x\cdot\frac{1}{x}+(1+\ln x)\cdot x^x(1+\ln x) \qquad \text{from (i)}$$

$$= x^x\left(\frac{1}{x}+(1+\ln x)^2\right)$$

At $x=\dfrac{1}{e}$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1/e} = \left(\frac{1}{e}\right)^{\frac{1}{e}}\left(\frac{1}{1/e}+\left(1+\ln\frac{1}{e}\right)^2\right)$$

$$= \left(\frac{1}{e}\right)^{\frac{1}{e}}\left(e+\left(1+\ln e^{-1}\right)^2\right) = \left(\frac{1}{e}\right)^{\frac{1}{e}}\left(e+(1-\ln e)^2\right)$$

$$= \left(\frac{1}{e}\right)^{\frac{1}{e}}\left(e+(1-1)^2\right) = \left(\frac{1}{e}\right)^{\frac{1}{e}}\cdot e > 0$$

$\Rightarrow$ y has a minimum value at $x=\dfrac{1}{e}$

Question # 1
Find two positive integers whose sum is 30 and their product will be maximum.
Solution
Let x and $30-x$ be two positive integers and P denotes product integers then

$$P = x(30-x)$$
$$= 30x - x^2$$

Diff. w.r.t. x

$$\frac{dP}{dx} = 30-2x \;\ldots\ldots\; \text{(i)}$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = -2 \;\ldots\ldots\; \text{(ii)}$$

For critical points, put $\frac{dP}{dx}=0$

$$\Rightarrow 30-2x=0$$
$$\Rightarrow -2x=-30 \qquad \Rightarrow x=15$$

Putting value of x in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=2} = -2 < 0$$

$\Rightarrow P$ is maximum at $x=15$
Other +tive integer $= 30-x$
$= 30-15 = 15$
Hence 15 and 15 are the required positive numbers.

Question # 2
Divide 20 into two parts so that the sum of their squares will be minimum.
Solution
Let x be the part of 20 then other is $20-x$.
Let S denotes sum of squares then

$$S = x^2 + (20-x)^2$$
$$= x^2 + 400 - 40x + x^2$$
$$= 2x^2 - 40x + 400$$

Diff. w.r.t x

$$\frac{dS}{dx} = 4x-40 \;\ldots\ldots\; \text{(i)}$$

Again diff. w.r.t x

$$\frac{d^2S}{dx^2} = 4 \;\ldots\ldots\; \text{(ii)}$$

For stationary points put $\frac{dS}{dx}=0$

$$\Rightarrow 4x-40=0 \qquad \Rightarrow 4x=40$$
$$\Rightarrow x=10$$

Putting value of x in (ii)

$$\left.\frac{d^2S}{dx^2}\right|_{x=10} = 4 > 0$$

$\Rightarrow S$ is minimum at $x=10$
Other integer $= 20-x = 20-10 = 10$
Hence 10, 10 are the two parts of 20.

Question # 3
Find two positive integers whose sum is 12 and the product of one with the square of the other will be maximum.
Solution
Let x and $12-x$ be two +tive integers and P denotes product of one with square of the other then

$$P = x(12-x)^2$$
$$\Rightarrow P = x\left(144-24x+x^2\right)$$
$$= x^3 - 24x^2 + 144x$$

Diff. w.r.t x

$$\frac{dP}{dx} = 3x^2 - 48x + 144 \;\ldots\; \text{(i)}$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = 6x-48 \;\ldots\; \text{(ii)}$$

For critical points put $\frac{dP}{dx}=0$

$$3x^2 - 48x + 144 = 0$$
$$\Rightarrow x^2 - 16x + 48 = 0$$
$$\Rightarrow x^2 - 4x - 12x + 48 = 0$$
$$\Rightarrow x(x-4) - 12(x-4) = 0$$
$$\Rightarrow (x-4)(x-12) = 0$$
$$\Rightarrow x=4 \quad \text{or} \quad x=12$$

We can not take $x=12$ as sum of integers is 12. So put $x=4$ in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=4} = 6(4)-48$$
$$= 24-48 = -24 < 0$$

$\Rightarrow$ P is maximum at $x=4$.

So the other integer $= 12-4 = 8$

Hence 4, 8 are the required integers.

Alternative Method: (by Irfan Mehmood: Fazaia Degree College Risalpur)

Let x and $12-x$ be two positive integers and P denotes product of one with square of the other then

$$P = x^2(12-x)$$
$$\Rightarrow P = 12x^2-x^3$$

Diff. w.r.t x

$$\frac{dP}{dx} = 24x-3x^2 \;\ldots\ldots\; \text{(i)}$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = 24-6x \;\ldots\ldots\; \text{(ii)}$$

For critical points put $\frac{dP}{dx}=0$

$$24x-3x^2 = 0$$
$$\Rightarrow 3x(x-8) = 0$$
$$\Rightarrow x=0 \text{ or } x=8$$

We cannot take $x=0$ as given integers are positive. So put $x=8$ in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=8} = 24-6(8)$$
$$= 24-48 = -24 < 0$$

$\Rightarrow$ P is maximum at $x=8$.

So the other integer $= 12-8 = 4$

Hence 4, 8 are the required integers.

Question # 4

The perimeter of a triangle is $16cm$. If one side is of length $6cm$, What are length of the other sides for maximum area of the triangle.

Solution

Let the remaining sides of the triangles are x and y

$$\text{Perimeter} = 16$$
$$\Rightarrow 6+x+y = 16$$
$$\Rightarrow x+y = 16-6 \quad \Rightarrow x+y = 10$$
$$\Rightarrow y = 10-x \;\ldots\ldots\; \text{(i)}$$

Now suppose A denotes the square of the area of triangle then

$$A = s(s-a)(s-b)(s-c)$$

Where $s = \frac{a+b+c}{2} = \frac{6+x+y}{2}$

$$= \frac{6+x+10-x}{2} \quad \text{from (i)}$$
$$= \frac{16}{2} = 8$$

So $A = 8(8-6)(8-x)(8-y)$

$$= 8(2)(8-x)(8-10+x)$$
$$= 16(8-x)(-2+x)$$
$$= 16(-16+2x+8x-x^2)$$
$$\Rightarrow A = 16(-16+10x-x^2)$$

Diff. w.r.t x

$$\frac{dA}{dx} = 16(10-2x) \;\ldots\ldots\ldots\; \text{(i)}$$

Again diff. w.r.t x

$$\frac{d^2A}{dx^2} = 16(-2) = -32$$

For critical points put $\frac{dA}{dx}=0$

$$16(10-2x) = 0$$
$$\Rightarrow (10-2x) = 0 \quad \Rightarrow -2x = -10$$
$$\Rightarrow x = 5$$

Putting value of x in (*ii*)

$$\left.\frac{d^2A}{dx^2}\right|_{x=5} = -32 < 0$$

$\Rightarrow$ A is maximum at $x=5$

Putting value of x in (i)

$$y = 10-5 = 5$$

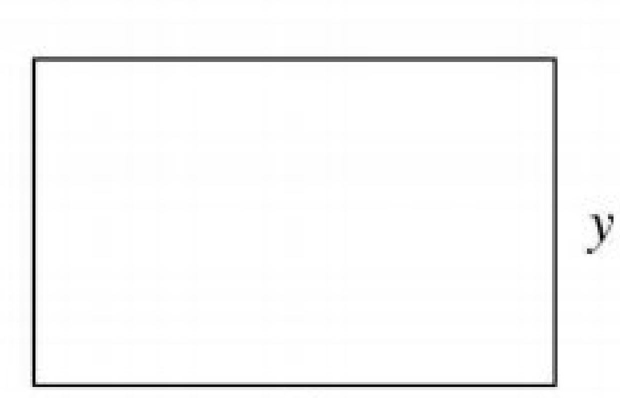
Hence length of remaining sides of triangles are $5cm$ and $5cm$.

Question # 5

Find the dimensions of a rectangle of largest area having perimeter $120cm$.

Solution

Let x and y be the length and breadth of rectangle, then



Area $= A = xy$ ….. (i)

Perimeter $= 120$

$\Rightarrow\ x+x+y+y = 120$

$\Rightarrow\ 2x+2y = 120$

$\Rightarrow\ x+y = 60$

$\Rightarrow\ y = 60-x$ ……. (ii)

Putting in (i)

$$A = x(60-x)$$

$$\Rightarrow A = 60x-x^2$$

Diff. w.r.t x

$$\frac{dA}{dx} = 60-2x \text{ ……….. (iii)}$$

Again diff. w.r.t x

$$\frac{d^2A}{dx^2} = -2 \text{ ………… (iv)}$$

For critical points put $\frac{dA}{dx}=0$

$$60-2x = 0 \quad \Rightarrow -2x = -60$$

$$\Rightarrow x = 30$$

Putting value of x in (iv)

$$\left.\frac{d^2A}{dx^2}\right|_{x=30} = -2 < 0$$

$\Rightarrow$ A is maximum at $x=30$

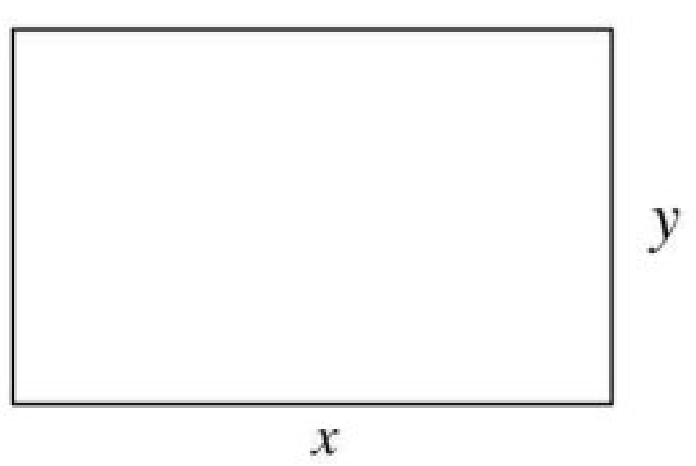
Putting value of x in (ii)

$$y = 60-30 = 30$$

Hence dimension of rectangle is $30cm$, $30cm$.

Question # 6

Find the lengths of the sides of a variable rectangle having area $36cm^2$ when its perimeter is minimum.



Solution

Let x and y be the length and breadth of the rectangle then

Area $= xy$

$\Rightarrow 36 = xy$

$\Rightarrow y = 36/x$ … (i)

Now perimeter $= 2x+2y$

$$\Rightarrow P = 2x+2\left(36/x\right)$$

$$= 2\left(x+36x^{-1}\right)$$

Diff. P w.r.t x

$$\frac{dP}{dx} = 2\left(1-36x^{-2}\right) \text{ … (ii)}$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = 2\left(0-36\left(-2x^{-3}\right)\right)$$

$$= 2\left(72x^{-3}\right) = \frac{144}{x^3}$$

For critical points put $\frac{dP}{dx}=0$

$$2\left(1-36x^{-2}\right)=0 \quad \Rightarrow 1-\frac{36}{x^2}=0$$

$$\Rightarrow 1=\frac{36}{x^2} \quad \Rightarrow x^2 = 36 \quad \Rightarrow x = \pm 6$$

Since length can not be negative therefore

$$x=6$$

Putting value of x in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=6} = \frac{144}{(6)^3} > 0$$

Hence P is minimum at $x=6$.

Putting in eq. (i)

$$y = \frac{36}{6} = 6$$

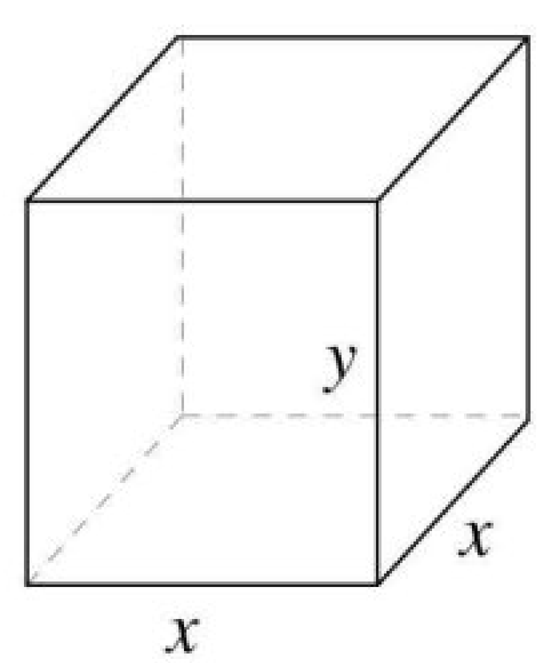
Hence $6cm$ and $6cm$ are the lengths of the sides of the rectangle.

Question # 7

A box with a square base and open top is to have a volume of 4 cubic dm. Find the dimensions of the box which will require the least material.

Solution

Let x be the lengths of the sides of the base and y be the height of the box.



Then Volume

$= x\cdot x\cdot y$

$\Rightarrow 4 = x^2y$

$$\Rightarrow y = \frac{4}{x^2} \text{ … (i)}$$

Suppose S denotes the surface area of the box, then

$$S = x^2+4xy$$

$$\Rightarrow S = x^2+4x\left(\frac{4}{x^2}\right)$$

$$\Rightarrow S = x^2 + 16x^{-1}$$

Diff. S w.r.t x

$$\frac{dS}{dx} = 2x - 16x^{-2} \dots \text{ (ii)}$$

Again diff. w.r.t x

$$\frac{d^2S}{dx^2} = 2 - 16\left(-2x^{-3}\right)$$

$$= 2 + \frac{32}{x^3} \dots \text{ (iii)}$$

For critical points, put $\frac{dS}{dx} = 0$

$$2x - 16x^{-2} = 0 \quad \Rightarrow 2x - \frac{16}{x^2} = 0$$

$$\Rightarrow \frac{2x^3 - 16}{x^2} = 0$$

$$\Rightarrow 2x^3 - 16 = 0 \quad \Rightarrow 2x^3 = 16$$

$$\Rightarrow x^3 = 8 \quad \Rightarrow x = 2$$

Putting in (ii)

$$\left.\frac{d^2S}{dx^2}\right|_{x=2} = 2 + \frac{32}{(2)^3} > 0$$

$\Rightarrow S$ is min. when $x = 2$

Putting value of x in (i)

$$y = \frac{4}{(2)^2} = 1$$

Hence $2dm$, $2dm$ and $1dm$ are the dimensions of the box.

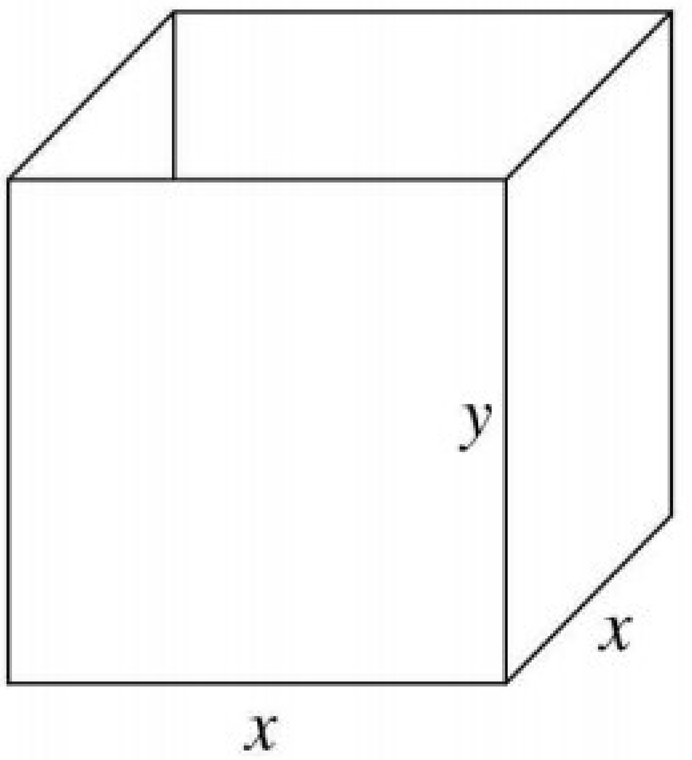
Question # 8

Find the dimensions of a rectangular garden having perimeter 80 meters if its area is to be maximum.

Solution

Do yourself as question # 5.

Question # 9



An open tank of square base of side x and vertical sides is to be constructed to contain a given quantity of water. Find the depth in terms of x if the expense of lining the inside of the tank with lead will be least.

Solution

Let y be the height of the open tank.

Then Volume $= x \cdot x \cdot y$

$$\Rightarrow V = x^2 y$$

$$\Rightarrow y = \frac{V}{x^2} \dots\dots\dots\dots \text{ (i)}$$

If S denotes the surface area the open tank, then

$$S = x^2 + 4xy$$

$$= x^2 + 4x\left(\frac{V}{x^2}\right)$$

$$\Rightarrow S = x^2 + 4Vx^{-1}$$

Diff. w.r.t x

$$\frac{dS}{dx} = 2x - 4Vx^{-2} \dots\dots\dots \text{ (ii)}$$

Again diff. w.r.t x

$$\frac{d^2S}{dx^2} = 2 - 4V\left(-2x^{-3}\right)$$

$$= 2 + \frac{8V}{x^3} \dots\dots\dots\dots\dots \text{ (iii)}$$

For critical points, put $\frac{dS}{dx} = 0$

$$2x - 4Vx^{-2} = 0 \quad \Rightarrow 2x - \frac{4V}{x^2} = 0$$

$$\Rightarrow \frac{2x^3 - 4V}{x^2} = 0 \quad \Rightarrow 2x^3 - 4V = 0$$

$$\Rightarrow 2x^3 = 4V \quad \Rightarrow x^3 = 2V$$

$$\Rightarrow x = (2V)^{\frac{1}{3}}$$

Putting in (ii)

$$\left.\frac{d^2S}{dx^2}\right|_{x=(2V)^{\frac{1}{3}}} = 2 + \frac{8V}{\left((2V)^{\frac{1}{3}}\right)^3}$$

$$= 2 + \frac{8V}{2V} = 2 + 4 = 6 > 0$$

$\Rightarrow S$ is minimum when $x = (2V)^{\frac{1}{3}}$

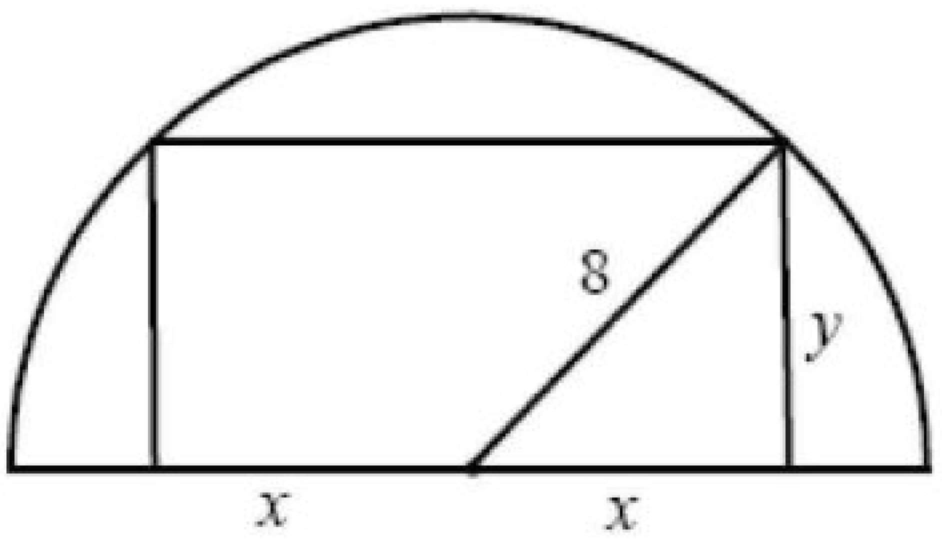
i.e. $x^3 = 2V \quad \Rightarrow V = \frac{x^3}{2}$

Putting in (i)

$$y = \frac{x^3/2}{x^2} = \frac{x}{2}$$

Hence height of the open tank is $\frac{x}{2}$.

Question # 10
Find the dimensions of the rectangular of maximum area which fits inside the semi-circle of radius $8cm$



Solution

Let $2x$ & y be dimension of rectangle.
Then from figure, using Pythagoras theorem
$$x^2 + y^2 = 8^2$$
$$\Rightarrow y^2 = 64 - x^2 \text{ ……….. (i)}$$
Now Area of the rectangle is given by
$$A = 2x \cdot y$$
Squaring both sides
$$A^2 = 4x^2y^2$$
$$= 4x^2\left(64 - x^2\right)$$
$$= 256x^2 - 4x^4$$
Now suppose
$$f = A^2 = 256x^2 - 4x^4 \text{ ……….. (ii)}$$
Diff. w.r.t x
$$\frac{df}{dx} = 512x - 16x^3 \text{ ……….. (iii)}$$
Again diff. w.r.t x
$$\frac{d^2f}{dx^2} = 512 - 48x^2 \text{ ………. (iv)}$$
For critical points, put $\frac{df}{dx} = 0$
$$\Rightarrow 512x - 16x^3 = 0$$
$$\Rightarrow 16x\left(32 - x^2\right) = 0$$
$$\Rightarrow 16x = 0 \quad \text{or} \quad 32 - x^2 = 0$$
$$\Rightarrow x = 0 \quad \text{or} \quad x^2 = 32$$
$$\Rightarrow x = \pm 4\sqrt{2}$$
Since x can not be zero or $-$ive, therefore
$$x = 4\sqrt{2}$$
Putting in (iv)
$$\left.\frac{d^2f}{dx^2}\right|_{x=4\sqrt{2}} = 512 - 48\left(4\sqrt{2}\right)^2$$
$$= 512 - 48(32) = 512 - 1536$$
$$= -1024 < 0$$
$\Rightarrow$ Area is max. for $x = 4\sqrt{2}$

Hence length $= 2x = 2\left(4\sqrt{2}\right)$

Breadth $= y = \sqrt{64 - \left(4\sqrt{2}\right)^2}$
$$= \sqrt{64 - 32} = \sqrt{32} = 4\sqrt{2}$$
Hence dimension is $8\sqrt{2}$ cm and $4\sqrt{2}$ cm.

Question # 11
Find the point on the curve $y = x^2 - 1$ that is closest to the point $(3,-1)$

Solution

Let $P(x, y)$ be point and let $A(3,-1)$.

Then $d = |AP| = \sqrt{(x-3)^2 + (y+1)^2}$
$$\Rightarrow d^2 = (x-3)^2 + (y+1)^2$$
$$= (x-3)^2 + \left(x^2 - 1 + 1\right)^2$$
$\because$ $y = x^2 - 1$ (given)
$$\Rightarrow d^2 = (x-3)^2 + x^4$$
Let $\quad f = d^2 = (x-3)^2 + x^4$.
Diff. w.r.t x
$$\frac{df}{dx} = 2(x-3) + 4x^3 \text{ ………… (i)}$$
Again diff. w.r.t x
$$\frac{d^2f}{dx^2} = 2 + 12x^2 \text{ ……… (ii)}$$
For stationary points, put $\frac{df}{dx} = 0$
$$2(x-3) + 4x^3 = 0$$

$$\Rightarrow 2x-6+4x^3 = 0$$

$$\Rightarrow 4x^3+2x-6 = 0$$

$$\Rightarrow 2x^3+x-3 = 0 \qquad \div\text{ing by 2}$$

By synthetic division

1	2	0	1	-3
	↓	2	2	3
	2	2	3	0

$$\Rightarrow x = 1 \quad \text{or} \quad 2x^2+2x+3 = 0$$

$$\Rightarrow x = \frac{-2\pm\sqrt{4-4(2)(3)}}{4}$$

$$= \frac{-2\pm\sqrt{-20}}{4}$$

This is complex and not acceptable.
Now put $x=1$ in (*ii*)

$$\left.\frac{d^2 f}{dx^2}\right|_{x=1} = 2+12(1)^2 = 14>0$$

$\Rightarrow d$ is minimum at $x=1$.

Also $y = 1^2-1 = 0$.

Hence $(1,0)$ is the required point.

Question # 12

Find the point on the curve $y=x^2+1$ that is closest to the point $(18,1)$

Solution

Do yourself as Q # 11