MATHEMATICS 2nd YEAR

UNIT # 01

FUNCTIONS AND LIMITS

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Contents

Sherazi Mathematics

اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Function:- A function f from a set X to a set Y is a rule or correspondence that assigns to each element x in X a unique element y in Y. The set X is called domain of f. The set of corresponding elements y in Y is called range of f.

If a variable y depends on a variable x in such a way that the value of x determines exactly one value of y, then we say that "y is function of x".

SWISS mathematician EULER (1707-1783) invented a symbolic way to write a statement "y is a function of x" as **y=f(x)** which is read as "y is equal to f of x"

NOTE:- The term function was recognized by GERMAN mathematician LEBNITZ (1646-1716).

Domain:- The set of all possible inputs of a function is called domain. * The domain of every function $f(x)$ is set of all real numbers for which $f(x)$ is defined. * The values at which $f(x)$ becomes undefined or complex valued will be excluded from real numbers. * Domain is also known as pre-images.

Range:- The set of all possible out puts of a function is called range. * Range is also known as images.

Example 1: Given $f(x) = x^3 - 2x^2 + 4x - 1$ find (i) $f(0)$ (ii) $f(1)$ (iii) $f(-2)$ (iv) $f(1+x)$ (v) $f(1/x)$, $x \neq 0$

Solution: $f(x) = x^3 - 2x^2 + 4x - 1$

(i) $f(0) = 0 - 0 + 0 - 1 = -1$

(ii) $f(1) = (1)^3 - 2(1)^2 + 4(1) - 1 = 1 - 2 + 4 - 1 = 2$

(iii) $f(-2) = (-2)^3 - 2(-2)^2 + 4(-2) - 1 = -8 - 8 - 8 - 1 = -25$

(iv) $f(1+x) = (1+x)^3 - 2(1+x)^2 + 4(1+x) - 1$

$= 1 + x^3 + 3x^2 + 3x - 2(1 + x^2 + 2x) + 4 + 4x - 1$

$= 1 + x^3 + 3x^2 + 3x - 2 - 2x^2 - 4x + 4 + 4x - 1$

$= x^3 + x^2 + 3x + 2$

(v) $f(1/x) = (1/x)^3 - 2(\frac{1}{x})^2 + 4(\frac{1}{x}) - 1$

$= \frac{1}{x^3} - \frac{2}{x^2} + \frac{4}{x} - 1, \quad x \neq 0$

Example 2: Let $f(x) = x^2$. Find the domain and range of f.

Solution: $D_f = (-\infty, +\infty)$

$R_f = [0, +\infty)$

Example 3: Let $f(x) = \frac{x}{x^2-4}$. Find the domain and range of f.

Solution: $f(x) = \frac{x}{x^2-4}$

$\because f(x)$ becomes undefined for $x^2 - 4 = 0$, so $x^2 - 4 \neq 0 \rightarrow x^2 \neq 4$
$\rightarrow x \neq \pm 2$

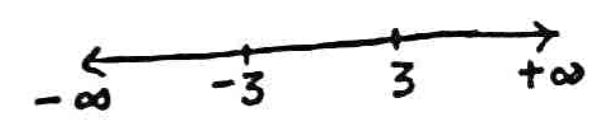
so $D_f = R - \{-2, 2\}$, R_f = set of all real numbers $= R$

or $R_f = (-\infty, +\infty)$

Example 4: Let $f(x) = \sqrt{x^2 - 9}$

Solution: $\because f(x)$ becomes complex for $x^2 - 9 < 0$, so $x^2 - 9 \geqslant 0 \rightarrow x^2 \geqslant 9$
$\rightarrow x \geqslant \pm 3 \quad \rightarrow -3 \geqslant x \geqslant 3$ Thus

$D_f = (-\infty, -3] \cup [3, +\infty)$



$R_f = [0, +\infty)$

Example 5: Find the domain and range of the function $f(x) = x^2 + 1$

Solution: $D_f = (-\infty, +\infty)$, $R_f = [1, +\infty)$

Example 6: Find the domain and range of $f(x) = \begin{cases} x & \text{when } 0 \leq x \leq 1 \\ x-1 & \text{when } 1 < x \leq 2 \end{cases}$

Solution: $D_f = [0,1] \cup (1,2]$ or $D_f = [0,2]$

$R_f = [0,1]$

Types of functions

Algebraic functions:

Any function generated by algebraic operations is known as algebraic function. Algebraic functions are classified as below.

(i) Polynomial function: A function P of the form

$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots\ldots + a_2 x^2 + a_1 x + a_0$

for all x, where the coefficients $a_n, a_{n-1}, a_{n-2} \ldots\ldots a_2, a_1, a_0$ are real numbers and the exponents are non-negative integers, is called a polynomial function.

(ii) Linear function: If the degree of polynomial function is 1, then it is called linear function.

(iii) Quadratic function: If the degree of polynomial function is 2, then it is called a quadratic function.

(iv) Identity function: A function for which $I(x)=x$ or $y=x$ is called identity function. It is denoted by I

(v) Constant function: A function for which $f(x)=b$ or $y=b$ is called constant function.

(vi) Rational function: The quotient of two polynomials such as $f(x)=\frac{P(x)}{Q(x)}$ where $Q(x)\neq 0$ is called rational function

(vii) Exponential function: A function in which the variable appears as exponent (power), is called exponential function. e.g., $y=e^{ax}$, $y=e^{x}$ etc.

Logarithmic function: If $x=a^y$, then $y=\log_a x$, where $a>0$, $a\neq 1$ is called Logarithmic function. * $\log_{10} x$ is known as common logarithm.
* $\log_e x$ is known as natural logarithm.

Trigonometric Functions:- sinx,cosx,tanx,secx, cosecx,cotx are called trigonmetric functions.

Inverse trigonometric Functions:-

$\sin^{-1}x, \cos^{-1}x, \tan^{-1}x, \csc^{-1}x, \sec^{-1}x$, $\cot^{-1}x$ are called Inverse trigonometric functions.

NOTE:- For Domain & Range of Trigonometric functions & Inverse Trigonometric functions , Consult Your Instructor/Mentor or See Unit # 11 (First Year Math) Discussed In Details. Thank You

Hyperbolic functions:

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, $\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$

$\operatorname{csch} x = \frac{2}{e^x - e^{-x}}$, $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Inverse Hyperbolic functions:

$\sinh^{-1}x = \ln(x+\sqrt{x^2+1})$, $\forall x$

$\cosh^{-1}x = \ln(x+\sqrt{x^2-1})$, $x \geq 1$

$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $|x|<1$

$\operatorname{csch}^{-1}x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$, $x\neq 0$

$\operatorname{sech}^{-1}x = \ln\left(\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right)$, $0<x\leq 1$

$\coth^{-1}x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$, $|x|<1$

Explicit function: If y is easily expressed in terms of x, then y is called explicit function. Symbolically $y=f(x)$

Implicit function: If the two variables x and y are so mixed up s.that y cannot be expressed in terms of x, then this type of function is called implicit function. Symbolically $f(x,y)=0$

Parametric functions: If x and y are expressed in terms of third variable (say t) such as $x=f(t)$, $y=g(t)$ then these equations are called parametric equations.

Even function: A function f is said to be an even if $f(-x)=f(x)$, for every x in the domain of f.

Odd function: A function f is said to be an odd if $f(-x)=-f(x)$, for every number x in the domain of f.

Example 1: Show that the parametric equations $x=a\cos t$ and $y=a\sin t$ represent the equation of the circle $x^2+y^2=a^2$

Solution: $x=a\cos t$ —— (i)

$y=a\sin t$ —— (ii)

Squaring and adding (i) and (ii)

$\Rightarrow x^2+y^2 = (a\cos t)^2 + (a\sin t)^2$

$= a^2\cos^2 t + a^2\sin^2 t$

$= a^2(\cos^2 t + \sin^2 t)$

$x^2+y^2 = a^2(1)$

$\Rightarrow x^2+y^2=a^2$ (Eq. of circle) which is

Example 2: prove the identities

(i) $\cosh^2 x - \sinh^2 x = 1$ (ii) $\cosh^2 x + \sinh^2 x = \cosh 2x$

Solution: (i) $\cosh^2 x - \sinh^2 x = 1$

$$L.H.S = \cosh^2 x - \sinh^2 x$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4}$$

$$= \frac{\cancel{e^{2x}} + \cancel{e^{-2x}} + 2 - \cancel{e^{2x}} - \cancel{e^{-2x}} + 2}{4} = \frac{4}{4} = 1$$

$$= R.H.S$$

Hence $\cosh^2 x - \sinh^2 x = 1$

(ii) $\cosh^2 x + \sinh^2 x = \cosh 2x$

$$L.H.S = \cosh^2 x + \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + e^{-2x} + 2}{4} + \frac{e^{2x} + e^{-2x} - 2}{4}$$

$$= \frac{e^{2x} + e^{-2x} + \cancel{2} + e^{2x} + e^{-2x} - \cancel{2}}{4} = \frac{2e^{2x} + 2e^{-2x}}{4}$$

$$= 2\left(\frac{e^{2x} + e^{-2x}}{4}\right) = \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x$$

$$= R.H.S$$

Hence $\cosh^2 x + \sinh^2 x = \cosh 2x$

Example 3: Determine whether the following functions are even or odd.

Solution: (a) $f(x) = 3x^4 - 2x^2 + 7$

$$\Rightarrow f(-x) = 3(-x)^4 - 2(-x)^2 + 7$$

$$= 3x^4 - 2x^2 + 7 = f(x)$$

Thus $f(x)$ is even.

(b) $f(x) = \dfrac{3x}{x^2+1} \Rightarrow f(-x) = \dfrac{3(-x)}{(-x)^2+1}$

$$= -\frac{3x}{x^2+1} = -f(x)$$

Thus $f(-x)$ is odd.

(c) $f(x) = \sin x + \cos x$

$$\Rightarrow f(-x) = \sin(-x) + \cos(-x)$$

$$= -\sin x + \cos x$$

$$\neq \pm f(x)$$

Thus $f(x)$ is neither even nor odd.

Exercise 1.1

Q1. Given that: (a) $f(x) = x^2 - x$

(b) $f(x) = \sqrt{x+4}$ Find (i) $f(-2)$

(ii) $f(0)$ (iii) $f(x-1)$ (iv) $f(x^2+4)$

Solution: (a) $f(x) = x^2 - x$

(i) $f(-2) = (-2)^2 - (-2) = 4 + 2 = 6$

(ii) $f(0) = (0)^2 - 0 = 0$

(iii) $f(x-1) = (x-1)^2 - (x-1) = x^2 + 1 - 2x - x + 1$

$$= x^2 - 3x + 2$$

(iv) $f(x^2+4) = (x^2+4)^2 - (x^2+4) = x^4 + 16 + 8x^2 - x^2 - 4$

$$= x^4 + 7x^2 + 12$$

(b) $f(x) = \sqrt{x+4}$ (i) $f(-2) = \sqrt{-2+4} = \sqrt{2}$

(ii) $f(0) = \sqrt{0+4} = \sqrt{4} = 2$

(iii) $f(x-1) = \sqrt{x-1+4} = \sqrt{x+3}$

(iv) $f(x^2+4) = \sqrt{x^2+4+4} = \sqrt{x^2+8}$

Q2. Find $\dfrac{f(a+h) - f(a)}{h}$ and simplify

where (i) $f(x) = 6x - 9$ (ii) $f(x) = \sin x$

(iii) $f(x) = x^3 + 2x^2 - 1$ (iv) $f(x) = \cos x$

Solution: (i) $f(x) = 6x - 9$

$$\frac{f(a+h) - f(a)}{h} = \frac{\{6(a+h) - 9\} - (6a - 9)}{h}$$

$$= \frac{\cancel{6a} + 6h - \cancel{9} - \cancel{6a} + \cancel{9}}{h} = \frac{6\cancel{h}}{\cancel{h}} = 6$$

(ii) $f(x) = \sin x$

$$\frac{f(a+h) - f(a)}{h} = \frac{\sin(a+h) - \sin a}{h}$$

$$= \frac{1}{h}\left\{2\cos\left(\frac{a+h+a}{2}\right)\sin\left(\frac{a+h-a}{2}\right)\right\}$$

$$= \frac{1}{h}\left\{2\cos\left(\frac{2a+h}{2}\right)\sin\frac{h}{2}\right\}$$

$$= \frac{2}{h}\cos\left(a + \frac{h}{2}\right)\sin\frac{h}{2}$$

(iii) $f(x) = x^3 + 2x^2 - 1$

$$f(a+h) = (a+h)^3 + 2(a+h)^2 - 1$$

$$= a^3 + h^3 + 3a^2h + 3ah^2 + 2a^2 + 2h^2 + 4ah - 1$$

$$f(a) = a^3 + 2a^2 - 1$$

$$\frac{f(a+h) - f(a)}{h} = \frac{\cancel{a^3} + h^3 + 3a^2h + 3ah^2 + \cancel{2a^2} + 2h^2 + 4ah - \cancel{1} - \cancel{a^3} - \cancel{2a^2} + \cancel{1}}{h}$$

$$= \frac{\cancel{h}(h^2 + 3a^2 + 3ah + 2h + 4a)}{\cancel{h}}$$

$$= h^2 + 3a^2 + 3ah + 2h + 4a$$

(iv) $f(x) = \cos x$

$$\frac{f(a+h)-f(a)}{h} = \frac{\cos(a+h)-\cos a}{h}$$

$$= \frac{1}{h}\left(-2\sin\left(\frac{a+h+a}{2}\right)\sin\left(\frac{a+h-a}{2}\right)\right)$$

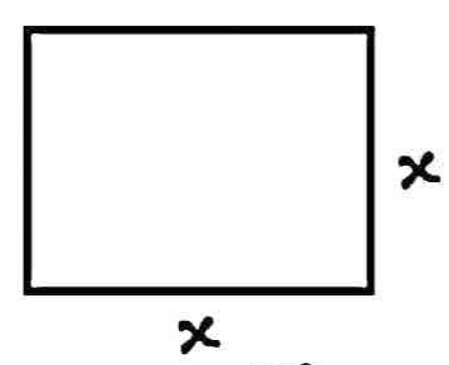
$$= \frac{1}{h}\left(-2\sin\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right)$$

$$= -\frac{2}{h}\sin\left(\frac{a+h}{2}\right)\sin\frac{h}{2}$$

Q3. Express the following

(a) The perimeter P of square as a function of its area A.

Solution: Let each side of square be 'x' then



Perimeter:-

$P = 4x$ —— (i)

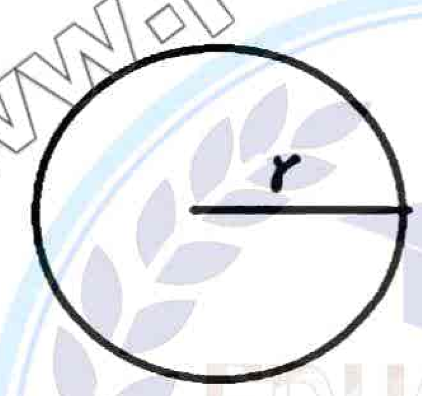
Area:- $A = x \times x = x^2 \rightarrow x = \sqrt{A}$

put value of x in (i)

(i) $\rightarrow P = 4\sqrt{A}$

(b) The area A of a circle as a function of its circumference C.

Solution: Let r be the radius of circle then



Area:-

$A = \pi r^2$ —— (i)

Circumference:-

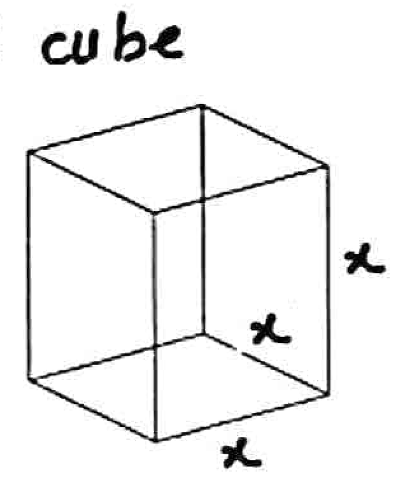
$C = 2\pi r \rightarrow r = \frac{C}{2\pi}$ put in (i)

(i) $\rightarrow A = \pi\cdot\left(\frac{C}{2\pi}\right)^2 = \pi\cdot\frac{C^2}{4\pi^2} = \frac{C^2}{4\pi}$

$\rightarrow A = \frac{C^2}{4\pi}$

(c) The Volume V of a cube as a function of the area A of its base.

Solution: Let each side of cube be x then



Volume:-

$V = x \times x \times x$

$V = x^3$ —— (i)

Area of base:-

$A = x^2 \rightarrow x = \sqrt{A}$ put in (i)

(i) $\rightarrow V = (\sqrt{A})^3 \rightarrow V = A^{3/2}$

Q4. Find the domain and range of the functions g defined below.

(i) $g(x) = 2x - 5$

$D_g = (-\infty, +\infty)$, $R_g = (-\infty, +\infty)$

(ii) $g(x) = \sqrt{x^2-4}$

$g(x)$ becomes complex for $x^2 - 4 < 0$, so

$x^2 - 4 \geqslant 0 \rightarrow x^2 \geqslant 4 \rightarrow x \geqslant \pm 2$

$\rightarrow -2 \geqslant x \geqslant 2$ so $D_g = (-\infty, -2] \cup [2, +\infty)$

$R_g = [0, +\infty)$

(iii) $g(x) = \sqrt{x+1}$

$g(x)$ becomes complex for $x + 1 < 0$

so $x + 1 \geqslant 0 \rightarrow x \geqslant -1$

Thus

$D_g = [-1, +\infty)$, $R_g = [0, +\infty)$

(iv) $g(x) = |x - 3|$

$D_g = (-\infty, +\infty)$, $R_g = [0, +\infty)$

(v) $g(x) = \begin{cases} 6x + 7 & \text{if } x \leq -2 \\ 4 - 3x & \text{if } x > -2 \end{cases}$

$D_g = (-\infty, -2] \cup (-2, +\infty)$

$R_g = (-\infty, -5] \cup (10, +\infty)$

(vi) $g(x) = \begin{cases} x - 1 & \text{if } x < 3 \\ 2x + 1 & \text{if } x \geqslant 3 \end{cases}$

$D_g = (-\infty, 3) \cup [3, +\infty)$

$R_g = (-\infty, 2) \cup [7, +\infty)$

(vii) $g(x) = \frac{x^2 + 3x + 2}{x + 1}$, $x \neq -1$

$D_g = R - \{-1\}$

$R_g = R - \{1\}$

$\because g(x) = \frac{x^2+3x+2}{x+1}$

$= \frac{\cancel{(x+1)}(x+2)}{\cancel{x+1}}$

$g(x) = x + 2$

$g(-1) = -1 + 2 = 1$

(viii) $g(x) = \frac{x^2 - 16}{x - 4}$, $x \neq 4$

$D_g = R - \{4\}$

$R_g = R - \{8\}$

$\because g(x) = \frac{x^2-16}{x-4}$

$= \frac{\cancel{(x-4)}(x+4)}{\cancel{x-4}}$

$g(x) = x + 4$

$g(4) = 4 + 4 = 8$

Q5. Given $f(x) = x^3 - ax^2 + bx + 1$ If $f(2) = -3$, $f(-1) = 0$. Find the values of a and b.

Solution:- $f(x) = x^3 - ax^2 + bx + 1$

$\rightarrow f(2) = (2)^3 - a(2)^2 + b(2) + 1$

$\rightarrow -3 = 8 - 4a + 2b + 1$

$\rightarrow -4a + 2b + 12 = 0 \rightarrow -2a + b + 6 = 0 \longrightarrow (I)$

Also, $f(-1) = (-1)^3 - a(-1)^2 + b(-1) + 1$

$\rightarrow 0 = -\cancel{1} - a - b + \cancel{1}$

$\rightarrow -a - b = 0 \longrightarrow II$

$I + II \rightarrow$

$$\begin{array}{r} -2a + \cancel{b} + 6 = 0 \\ -a - \cancel{b} \quad = 0 \\ \hline -3a + 6 = 0 \end{array} \rightarrow -3a = -6$$

$\Rightarrow \boxed{a = 2}$ put in II $\quad -2 - b = 0$

$\rightarrow \boxed{b = -2}$

Q6. A stone falls from a height of 60m on the ground, the height h after x second is approximately given by $h(x) = 40 - 10x^2$

(i) What is the height of the stone when: (a) $x = 1$ sec? (b) $x = 1.5$ sec (c) $x = 1.7$ sec

(ii) When does the stone strike the ground?

Solution:-

(i) $h(x) = 40 - 10x^2$

(a) $h(1) = 40 - 10(1)^2 = 40 - 10 = 30$

$\rightarrow h(1) = 30m$

(b) $h(1.5) = 40 - 10(1.5)^2 = 40 - 22.5$

$= 17.5m$

(c) $h(1.7) = 40 - 10(1.7)^2 = 40 - 28.9$

$= 11.1m$

(ii) When does stone strikes the ground then $h(x) = 0$

$\rightarrow h(x) = 40 - 10x^2$

$\rightarrow 0 = 40 - 10x^2$

$\rightarrow 10x^2 = 40$

$\rightarrow x^2 = 4$

$\rightarrow x = \pm 2$

$\rightarrow x = 2$, (neglect -2)

Q7. Show that the parametric equation:

(i) $x = at^2$, $y = 2at$ represent the equation of parabola $y^2 = 4ax$

Solution:- $x = at^2$ —— I

$y = 2at \rightarrow t = \frac{y}{2a}$ put in I

$\rightarrow x = a\left(\frac{y}{2a}\right)^2 = \cancel{a} \cdot \frac{y^2}{4a^{\cancel{2}}}$

$\rightarrow x = \frac{y^2}{4a} \rightarrow y^2 = 4ax$

(ii) $x = a\cos\theta$, $y = b\sin\theta$ represent the equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution:- $x = a\cos\theta$ —— I

$y = b\sin\theta$ —— II

From I, $\frac{x}{a} = \cos\theta$ —— III

From II, $\frac{y}{b} = \sin\theta$ —— IV

Squaring and adding III and IV

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = (\cos\theta)^2 + (\sin\theta)^2$

$= \cos^2\theta + \sin^2\theta$

$\rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(iii) $x = a\sec\theta$, $y = b\tan\theta$ represent the equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution:- $x = a\sec\theta$ —— I

$y = b\tan\theta$ —— II

$I \Rightarrow \frac{x}{a} = \sec\theta$, $\quad II \Rightarrow \frac{y}{b} = \tan\theta$

$\Rightarrow \frac{x^2}{a^2} = \sec^2\theta$ —— III, $\quad \rightarrow \frac{y^2}{b^2} = \tan^2\theta$ —— IV

$III - IV \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2\theta - \tan^2\theta$

$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Q8. Prove the identities

(i) $\sinh 2x = 2\sinh x\cosh x$

Solution:- $R.H.S = 2\sinh x\cosh x$

$$= \cancel{2}\cdot\frac{e^{x}-e^{-x}}{\cancel{2}}\cdot\frac{e^{x}+e^{-x}}{2} = \frac{e^{2x}-e^{-2x}}{2}$$

$$= \sinh 2x = L.H.S$$

Hence $\sinh 2x = 2\sinh x\cosh x$

(ii) $\operatorname{sech}^2 x = 1-\tanh^2 x$

Solution:- $R.H.S = 1-\tanh^2 x$

$$= 1-\left(\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right)^2 = 1-\frac{(e^{x}-e^{-x})^2}{(e^{x}+e^{-x})^2}$$

$$= \frac{(e^{2x}+e^{-2x}+2)-(e^{2x}+e^{-2x}-2)}{(e^{x}+e^{-x})^2}$$

$$= \frac{e^{2x}+e^{-2x}+2-e^{+2x}+e^{-2x}+2}{(e^{x}+e^{-x})^2}$$

$$= \frac{4}{(e^{x}+e^{-x})^2} = \left(\frac{2}{e^{x}+e^{-x}}\right)^2$$

$$= \frac{1}{\left(\frac{e^{x}+e^{-x}}{2}\right)^2} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$= L.H.S$$

Hence $\operatorname{sech}^2 x = 1-\tanh^2 x$

(iii) $\operatorname{csch}^2 x = \coth^2 x - 1$

Solution:- $R.H.S = \coth^2 x - 1$

$$= \left(\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}\right)^2 - 1 = \frac{(e^{x}+e^{-x})^2}{(e^{x}-e^{-x})^2} - 1$$

$$= \frac{e^{2x}+e^{-2x}+2-(e^{2x}+e^{-2x}-2)}{(e^{x}-e^{-x})^2}$$

$$= \frac{e^{2x}+e^{-2x}+2-e^{+2x}-e^{-2x}+2}{(e^{x}-e^{-x})^2}$$

$$= \frac{4}{(e^{x}-e^{-x})^2} = \left(\frac{2}{e^{x}-e^{-x}}\right)^2$$

$$= \left(\frac{1}{\frac{e^{x}-e^{-x}}{2}}\right)^2 = \frac{1}{\sinh^2 x} = \operatorname{csch}^2 x = L.H.S$$

Hence $\operatorname{csch}^2 x = \coth^2 x - 1$

Q9. Determine whether the given function f is even or odd.

Solution:- (i) $f(x) = x^3 + x$

$$\Rightarrow f(-x) = (-x)^3 + (-x) = -x^3 - x$$

$$= -(x^3+x) = -f(x)$$

Thus $f(x)$ is odd.

(ii) $f(x) = (x+2)^2$

$$\Rightarrow f(-x) = (-x+2)^2 \neq \pm f(x)$$

Thus $f(x)$ is neither even nor odd.

(iii) $f(x) = x\sqrt{x^2+5}$

$$\Rightarrow f(-x) = -x\sqrt{(-x)^2+5}$$

$$= -x\sqrt{x^2+5} = -f(x)$$

Thus $f(x)$ is odd.

(iv) $f(x) = \dfrac{x-1}{x+1}$

$$\Rightarrow f(-x) = \frac{-x-1}{-x+1} = -\frac{(x+1)}{1-x} \neq \pm f(x)$$

Thus $f(x)$ is neither even nor odd.

(v) $f(x) = x^{2/3} + 6$

$$\Rightarrow f(x) = (-x)^{2/3} + 6$$

$$= [(-x)^2]^{1/3} + 6$$

$$= (x^2)^{1/3} + 6 = x^{2/3} + 6 = f(x)$$

Thus $f(x)$ is even.

(vi) $f(x) = \dfrac{x^3-x}{x^2+1}$

$$\Rightarrow f(-x) = \frac{(-x)^3-(-x)}{(-x)^2+1} = \frac{-x^3+x}{x^2+1}$$

$$= -\frac{(x^3-x)}{x^2+1} = -f(x)$$

Thus $f(x)$ is odd.

Composition of function:- If f is a function from set A to set B and g is a function from set B to set C then composition of f and g is denoted by $(fog)(x) = f(g(x))\ \forall x \in X$

Inverse of a function:- Let f be a bijective (1-1 and onto) function from set A to set B i-e., $f: A \longrightarrow B$ then its inverse is f^{-1} which is surjective (onto) function from B to A i-e., $f^{-1}: B \longrightarrow A$. In this case $D_f = R_{f^{-1}}$ and $R_f = D_{f^{-1}}$

Example 1:- Let the real valued functions f and g be defined by $f(x) = 2x+1$ and $g(x) = x^2 - 1$ Obtain the expression for (i) $fg(x)$ (ii) $gf(x)$ (iii) $f^2(x)$ (iv) $g^2(x)$

Solution:- (i) $fg(x) = f(g(x)) = f(x^2-1)$

$= 2(x^2-1)+1 = 2x^2-2+1 = 2x^2-1 \longrightarrow \mathrm{I}$

(ii) $gf(x) = g(f(x)) = g(2x+1) = (2x+1)^2 - 1$

$= 4x^2+1+4x-1 = 4x^2+4x \longrightarrow \mathrm{II}$

(iii) $f^2(x) = f(f(x)) = f(2x+1) = 2(2x+1)+1$

$= 4x+3$

(iv) $g^2(x) = g(g(x)) = g(x^2-1) = (x^2-1)^2 - 1$

$= x^4+1-2x^2-1 = x^4-2x^2$

Also from I and II, we observe that

$fg(x) \neq gf(x)$

Example 2:- Let $f: R \rightarrow R$ be the function defined by $f(x) = 2x+1$. Find $f^{-1}(x)$

Solution:- $f(x) = 2x+1$

$\because y = f(x)$ so $y = 2x+1$

$\Rightarrow f^{-1}(y) = x$ —(i)

$\Rightarrow y-1 = 2x$

$\Rightarrow x = \dfrac{y-1}{2}$

$\Rightarrow f^{-1}(y) = \dfrac{y-1}{2}$ from (i)

Replace y by x

$\Rightarrow f^{-1}(x) = \dfrac{x-1}{2}$

Verification:

$f(f^{-1}(x)) = f\left(\dfrac{x-1}{2}\right) = \cancel{2}\left(\dfrac{x-1}{\cancel{2}}\right)+1 = x$

$f^{-1}(f(x)) = f^{-1}(2x+1) = \dfrac{2x+1-1}{2} = x$

Hence $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

Example 3:- Without finding the inverse, state the domain and range of f^{-1}, where $f(x) = 2+\sqrt{x-1}$

Solution: $f(x) = 2+\sqrt{x-1}$

$f(x)$ becomes complex valued when $x-1<0$ or $x<1$ so

$D_f = [1, +\infty)$, $R_f = [2, +\infty)$

By def. of inverse function f^{-1}, we have

$D_{f^{-1}} = R_f = [2, +\infty)$

$R_{f^{-1}} = D_f = [1, +\infty)$

Exercise 1.2

Q1. The real valued functions f and g are defined below. Find (a) $fog(x)$ (b) $gof(x)$ (c) $fof(x)$ (d) $gog(x)$ (i) $f(x) = 2x+1$; $g(x) = \dfrac{3}{x-1}$, $x \neq 1$

Solution:- (a) $fog(x) = f(g(x)) = f\left(\dfrac{3}{x-1}\right)$

$= 2\left(\dfrac{3}{x-1}\right)+1 = \dfrac{6}{x-1}+1 = \dfrac{6+x-1}{x-1}$

$= \dfrac{5+x}{x-1}$

(b) $gof(x) = g(f(x)) = g(2x+1)$

$= \dfrac{3}{2x+1-1} = \dfrac{3}{2x}$

(c) $fof(x) = f(f(x)) = f(2x+1)$

$= 2(2x+1)+1 = 4x+3$

(d) $gog(x) = g(g(x)) = g\left(\dfrac{3}{x-1}\right) = \dfrac{3}{\dfrac{3}{x-1}-1}$

$= \dfrac{3}{\dfrac{3-(x-1)}{x-1}} = \dfrac{3(x-1)}{3-x+1} = \dfrac{3(x-1)}{4-x}$

(ii) $f(x) = \sqrt{x+1}$, $g(x) = \frac{1}{x^2}$

Solution:- (a) $fog(x) = f(g(x))$

$= f\left(\frac{1}{x^2}\right) = \sqrt{\frac{1}{x^2}+1} = \sqrt{\frac{1+x^2}{x^2}}$

$= \frac{\sqrt{1+x^2}}{x}$

(b) $gof(x) = g(f(x)) = g(\sqrt{x+1})$

$= \frac{1}{(\sqrt{x+1})^2} = \frac{1}{x+1}$

(c) $fof(x) = f(f(x)) = f(\sqrt{x+1})$

$= \sqrt{\sqrt{x+1}+1}$

(d) $gog(x) = g(g(x)) = g\left(\frac{1}{x^2}\right)$

$= \frac{1}{\left(\frac{1}{x^2}\right)^2} = \frac{1}{\frac{1}{x^4}} = x^4$

(iii) $f(x) = \frac{1}{\sqrt{x-1}}$, $g(x) = (x^2+1)^2$

Solution:- (a) $fog(x) = f(g(x))$

$= f((x^2+1)^2) = \frac{1}{\sqrt{(x^2+1)^2-1}}$

$= \frac{1}{\sqrt{x^4+1+2x^2-1}} = \frac{1}{\sqrt{x^4+2x^2}} = \frac{1}{\sqrt{x^2(x^2+2)}}$

$= \frac{1}{x\sqrt{x^2+2}}$

(b) $gof(x) = g(f(x)) = g\left(\frac{1}{\sqrt{x-1}}\right)$

$= \left[\left(\frac{1}{\sqrt{x-1}}\right)^2+1\right]^2 = \left(\frac{1}{x-1}+1\right)^2$

$= \left(\frac{1+x-1}{x-1}\right)^2 = \left(\frac{x}{x-1}\right)^2$

(c) $fof(x) = f(f(x)) = f\left(\frac{1}{\sqrt{x-1}}\right)$

$= \frac{1}{\sqrt{\frac{1}{\sqrt{x-1}}-1}} = \frac{1}{\left(\frac{1-\sqrt{x-1}}{\sqrt{x-1}}\right)^{1/2}} = \left(\frac{1-\sqrt{x-1}}{\sqrt{x-1}}\right)^{-1/2}$

$= \left(\frac{\sqrt{x-1}}{1-\sqrt{x-1}}\right)^{1/2} = \sqrt{\frac{\sqrt{x-1}}{1-\sqrt{x-1}}}$

(d) $gog(x) = g(g(x))$

$= g((x^2+1)^2)$

$= ((x^2+1)^2+1)^2$

(iv) $f(x) = 3x^4 - 2x^2$, $g(x) = \frac{2}{\sqrt{x}}$

Solution:- (a) $fog(x) = f(g(x))$

$= f\left(\frac{2}{\sqrt{x}}\right) = 3\left(\frac{2}{\sqrt{x}}\right)^4 - 2\left(\frac{2}{\sqrt{x}}\right)^2$

$= 3\left(\frac{16}{x^2}\right) - \frac{8}{x} = \frac{48}{x^2} - \frac{8}{x} = \frac{48-8x}{x^2}$

(b) $gof(x) = g(f(x)) = g(3x^4-2x^2)$

$= \frac{2}{\sqrt{3x^4-2x^2}} = \frac{2}{\sqrt{x^2(3x^2-2)}} = \frac{2}{x\sqrt{3x^2-2}}$

(c) $fof(x) = f(f(x)) = f(3x^4-2x^2)$

$= 3(3x^4-2x^2)^4 - 2(3x^4-2x^2)^2$

(d) $gog(x) = g(g(x)) = g\left(\frac{2}{\sqrt{x}}\right)$

$= \frac{2}{\sqrt{\frac{2}{\sqrt{x}}}} = \frac{2}{\left(\frac{2}{\sqrt{x}}\right)^{1/2}} = 2\left(\frac{2}{\sqrt{x}}\right)^{-1/2}$

$= 2\left(\frac{\sqrt{x}}{2}\right)^{1/2} = 2\sqrt{\frac{\sqrt{x}}{2}} = \frac{\sqrt{2}\cancel{\sqrt{2}}\sqrt{\sqrt{x}}}{\cancel{\sqrt{2}}}$

$= \sqrt{2\sqrt{x}}$

Q2. For the real valued function, f defined below, find (a) $f^{-1}(x)$ (b) $f^{-1}(-1)$ and verify

$f(f^{-1}(x)) = f^{-1}(f(x)) = x$

(i) $f(x) = -2x+8$

Solution:- $f(x) = -2x+8$

Let $y = f(x)$ then

$y = -2x+8 \Rightarrow \frac{y-8}{-2} = x$

$\rightarrow x = \frac{y-8}{-2}$ $\quad \because y = f(x)$

$\rightarrow f^{-1}(y) = \frac{y-8}{-2}$ $\quad \Rightarrow f^{-1}(y) = x$

Replace y by x, we have

$f^{-1}(x) = \frac{x-8}{-2}$

Put $x=-1$, $f^{-1}(-1) = \frac{-1-8}{-2} = \frac{9}{2}$

(ii) $f(x) = 3x^3 + 7$

Solution:- $f(x) = 3x^3 + 7$

Let $y = f(x)$ then $y = 3x^3 + 7$

$\Rightarrow \dfrac{y-7}{3} = x^3 \Rightarrow x = \left(\dfrac{y-7}{3}\right)^{1/3}$

$\because y = f(x) \Rightarrow f^{-1}(y) = x$

so $f^{-1}(y) = \left(\dfrac{y-7}{3}\right)^{1/3}$

Replace y by x, we have

$f^{-1}(x) = \left(\dfrac{x-7}{3}\right)^{1/3}$

Put $x = -1$ $\quad f^{-1}(-1) = \left(\dfrac{-8}{3}\right)^{1/3}$

Verification:-

$f(f^{-1}(x)) = f\left[\left(\dfrac{x-7}{3}\right)^{1/3}\right] = 3\left[\left(\dfrac{x-7}{3}\right)^{1/3}\right]^{3} + 7$

$= \cancel{3}\left(\dfrac{x-7}{\cancel{3}}\right) + 7 = x - 7 + 7 = x$

$f^{-1}(f(x)) = f^{-1}(3x^3 + 7) = \left(\dfrac{3x^3 + 7 - 7}{3}\right)^{1/3}$

$= \left(\dfrac{3x^3}{3}\right)^{1/3} = x$

Hence $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

(iii) $f(x) = (-x+9)^3$

Solution:- $f(x) = (-x+9)^3$

Let $y = f(x)$ then $y = (-x+9)^3$

$\Rightarrow y^{1/3} = -x + 9 \Rightarrow y^{1/3} - 9 = -x$

$\Rightarrow x = 9 - y^{1/3} \Rightarrow f^{-1}(y) = 9 - y^{1/3}$

$(\because y = f(x) \Rightarrow f^{-1}(y) = x)$

Replace y by x, we have

$f^{-1}(x) = 9 - x^{1/3}$

Put $x = -1$, $f^{-1}(-1) = 9 - (-1)^{1/3}$

$= 9 - (-1) = 10$

Verification:-

$f(f^{-1}(x)) = f(9 - x^{1/3}) = [-(9 - x^{1/3}) + 9]^3$

$= (-\cancel{9} + x^{1/3} + \cancel{9})^{\cancel{3}} = x$

$f^{-1}(f(x)) = f^{-1}((-x+9)^3)$

$= 9 - ((-x+9)^3)^{1/3} = 9 - (-x+9)$

$= 9 + x - 9 = x$

Hence $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

(iv) $f(x) = \dfrac{2x+1}{x-1}$

Solution:- $f(x) = \dfrac{2x+1}{x-1}$

Let $y = f(x)$ then $y = \dfrac{2x+1}{x-1}$

$\Rightarrow (x-1)y = 2x+1 \Rightarrow xy - y = 2x + 1$

$\Rightarrow xy - 2x = y + 1 \Rightarrow x(y-2) = 1 + y$

$\Rightarrow x = \dfrac{1+y}{y-2}$ $\quad \because y = f(x) \Rightarrow x = f^{-1}(y)$

$\Rightarrow f^{-1}(y) = \dfrac{1+y}{y-2}$

Replace y by x, we have

$f^{-1}(x) = \dfrac{1+x}{x-2}$

Put $x = -1$, $f^{-1}(-1) = \dfrac{1+(-1)}{-1-2} = 0$

Verification:-

$f(f^{-1}(x)) = f\left(\dfrac{1+x}{x-2}\right) = \dfrac{2\left(\dfrac{1+x}{x-2}\right) + 1}{\dfrac{1+x}{x-2} - 1}$

$= \dfrac{\dfrac{2(1+x) + x - 2}{\cancel{x-2}}}{\dfrac{1 + x - (x-2)}{\cancel{x-2}}} = \dfrac{\cancel{2} + 2x + x - \cancel{2}}{1 + \cancel{x} - \cancel{x} + 2}$

$= \dfrac{3x}{3} = x$

$f^{-1}(f(x)) = f^{-1}\left(\dfrac{2x+1}{x-1}\right) = \dfrac{1 + \dfrac{2x+1}{x-1}}{\dfrac{2x+1}{x-1} - 2}$

$= \dfrac{\dfrac{x - \cancel{1} + 2x + \cancel{1}}{\cancel{x-1}}}{\dfrac{2x + 1 - 2(x-1)}{\cancel{x-1}}} = \dfrac{3x}{\cancel{2x} + 1 - \cancel{2x} + 2} = \dfrac{3x}{3}$

$= x$

Hence $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

Q 3. Without finding the inverse, state the domain and range of f^{-1}. (i) $f(x)=\sqrt{x+2}$ (ii) $f(x)=\frac{x-1}{x-4}, x\neq 4$ (iii) $f(x)=\frac{1}{x+3}, x\neq -3$ (iv) $f(x)=(x-5)^2, x\geqslant 5$

Solution:- (i) $f(x)=\sqrt{x+2}$

$\because f(x)$ becomes complex for $x+2<0$

so $x+2\geqslant 0 \rightarrow x\geqslant -2$

Thus $D_f=[-2,+\infty)$, $R_f=[0,+\infty)$

By def. of Inverse function,

$D_{f^{-1}}=R_f=[0,+\infty)$

$R_{f^{-1}}=D_f=[-2,+\infty)$

(ii) $f(x)=\frac{x-1}{x-4}, x\neq 4$

$f(x)$ becomes undefined for $x-4=0$

so $x-4\neq 0 \rightarrow x\neq 4$

so $D_f=R-\{4\}$

$R_f=R-\{\frac{1}{1}\}=R-\{1\}$

By def. of Inverse function,

$D_{f^{-1}}=R_f=R-\{1\}$

$R_{f^{-1}}=D_f=R-\{4\}$

> Remember,
> $f(x)=\frac{ax+b}{cx+d}$
> $D_f=R-\{-\frac{d}{c}\}$
> $R_f=R-\{\frac{a}{c}\}$

(iii) $f(x)=\frac{1}{x+3}, x\neq -3$

$f(x)$ becomes undefined for $x+3=0$

so $x+3\neq 0 \rightarrow x\neq -3$

Thus $D_f=R-\{-3\}$

$R_f=R-\{0\}$

By def. of Inverse function,

$D_{f^{-1}}=R_f=R-\{0\}$

$R_{f^{-1}}=D_f=R-\{-3\}$

> Remember
> $f(x)=\frac{0x+b}{cx+d}$
> $D_f=R-\{-\frac{d}{c}\}$
> $R_f=R-\{\frac{0}{c}\}$
> $R_f=R-\{0\}$

(iv) $f(x)=(x-5)^2, x\geqslant 5$

$D_f=[5,+\infty)$, $R_f=[0,+\infty)$

By def. of inverse function.

$D_{f^{-1}}=R_f=[0,+\infty)$, $R_{f^{-1}}=D_f=[5,+\infty)$

Limit of a function:-

Let $f(x)$ be a function then a number L is said to be limit of $f(x)$ when x approaches to a (from both left and right hand side of a), Symbolically it is written as; $\lim_{x\to a} f(x)=L$ and read as "Limit of f of x as x approaches to a is equal to L".

Theorems on Limits of functions:-

(i) $\lim_{x\to a}[f(x)+g(x)]$

$=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)=L+M$

(ii) $\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$

$=L-M$

(iii) $\lim_{x\to a}[k\,f(x)]=k\lim_{x\to a} f(x)=kL$

(iv) $\lim_{x\to a}[f(x)g(x)]=\lim_{x\to a} f(x)\lim_{x\to a} g(x)$

$=LM$

(v) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}=\frac{L}{M}$

(vi) $\lim_{x\to a}[f(x)]^n=\left[\lim_{x\to a} f(x)\right]^n=L^n$

Example: If $P(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$ is a polynomial function of degree n then show that $\lim_{x\to c} P(x)=P(c)$
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Solution:-

$P(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0 \longrightarrow 1$

Applying $\lim_{x\to c}$ on eq 1

$\lim_{x\to c} P(x)=\lim_{x\to c}(a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0)$

$=a_nc^n+a_{n-1}c^{n-1}+\dots+a_1c+a_0$

$=P(c)$ Thus $\lim_{x\to c} P(x)=P(c)$

Theorem:- Prove that

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1},$$ where n is an integer and $a>0$

Proof:- Case I: Suppose n is a +ive integer.

$$L.H.S=\lim_{x\to a}\frac{x^n-a^n}{x-a}\quad\left(\frac{0}{0}\right)\text{ form}$$

$$=\lim_{x\to a}\frac{\cancel{(x-a)}(x^{n-1}+x^{n-2}\cdot a+x^{n-3}\cdot a^2+\dots+xa^{n-2}+a^{n-1})}{\cancel{x-a}}$$

$$=\lim_{x\to a}(x^{n-1}+x^{n-2}\cdot a+x^{n-3}\cdot a^2+\dots+xa^{n-2}+a^{n-1})$$

$$=a^{n-1}+a^{n-2}\cdot a+a^{n-3}\cdot a^2+\dots+a\cdot a^{n-2}+a^{n-1}$$

$$=\underbrace{a^{n-1}+a^{n-1}+a^{n-1}+\dots+a^{n-1}+a^{n-1}}_{(n\text{— times})}$$

$$=na^{n-1}$$

Thus $$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$$

Case II: Suppose n is -ive integer. Let $n=-m$ (where m is +ive integer)

then $$\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\frac{x^{-m}-a^{-m}}{x-a}$$

$$=\lim_{x\to a}(x^{-m}-a^{-m})\cdot\frac{1}{x-a}=\lim_{x\to a}\left(\frac{1}{x^m}-\frac{1}{a^m}\right)\frac{1}{x-a}$$

$$=\lim_{x\to a}\left(\frac{a^m-x^m}{a^mx^m}\right)\frac{1}{x-a}$$

$$=\lim_{x\to a}\left(\frac{x^m-a^m}{x^ma^m}\right)\left(\frac{-1}{x-a}\right)$$

$$=\lim_{x\to a}\left(\frac{x^m-a^m}{x-a}\right)\left(\frac{-1}{x^ma^m}\right)$$

$$=\lim_{x\to a}\left(\frac{x^m-a^m}{x-a}\right)\cdot\lim_{x\to a}\left(\frac{-1}{x^ma^m}\right)$$

$$=ma^{m-1}\cdot\left(\frac{-1}{a^{2m}}\right)$$

$$=-ma^{m-1-2m}=-ma^{-m-1}=na^{n-1}\quad(\because n=-m)$$

Thus $$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$$

Theorem:- Prove that

$$\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}=\frac{1}{2\sqrt{a}}$$

Proof:-

$$L.H.S=\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}\quad\left(\frac{0}{0}\right)\text{form}$$

$$=\lim_{x\to 0}\left(\frac{\sqrt{x+a}-\sqrt{a}}{x}\times\frac{\sqrt{x+a}+\sqrt{a}}{\sqrt{x+a}+\sqrt{a}}\right)$$

$$=\lim_{x\to 0}\frac{x+\cancel{a}-\cancel{a}}{x(\sqrt{x+a}+\sqrt{a})}$$

$$=\lim_{x\to 0}\frac{\cancel{x}}{\cancel{x}(\sqrt{x+a}+\sqrt{a})}$$

$$=\lim_{x\to 0}\frac{1}{\sqrt{x+a}+\sqrt{a}}=\frac{1}{\sqrt{a}+\sqrt{a}}$$

$$=\frac{1}{2\sqrt{a}}$$

Thus $$\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}=\frac{1}{2\sqrt{a}}$$

Example 1: Evaluate

(i) $$\lim_{x\to 1}\frac{x^2-1}{x^2-x}$$

Solution:- $$\lim_{x\to 1}\frac{x^2-1}{x^2-x}\quad\left(\frac{0}{0}\right)\text{form}$$

$$=\lim_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{x\cancel{(x-1)}}=\lim_{x\to 1}\frac{x+1}{x}$$

$$=\frac{1+1}{1}=2$$

(ii) $$\lim_{x\to 3}\frac{x-3}{\sqrt{x}-\sqrt{3}}\quad\left(\frac{0}{0}\right)\text{form}$$

$$=\lim_{x\to 3}\left(\frac{x-3}{\sqrt{x}-\sqrt{3}}\times\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}}\right)$$

$$=\lim_{x\to 3}\frac{\cancel{(x-3)}(\sqrt{x}+\sqrt{3})}{\cancel{x-3}}$$

$$=\lim_{x\to 3}(\sqrt{x}+\sqrt{3})=\sqrt{3}+\sqrt{3}$$
$$=2\sqrt{3}$$

Example 2:- Evaluate

$$\lim_{x\to+\infty} \frac{5x^4-10x^2+1}{-3x^3+10x^2+50}$$

Solution:- $\displaystyle\lim_{x\to+\infty} \frac{5x^4-10x^2+1}{-3x^3+10x^2+50}$

Dividing up and down by x^3, we get

$$= \lim_{x\to\infty} \frac{\frac{5x^4}{x^3} - \frac{10x^2}{x^3} + \frac{1}{x^3}}{\frac{-3x^3}{x^3} + \frac{10x^2}{x^3} + \frac{50}{x^3}}$$

$$= \lim_{x\to\infty} \frac{5x - \frac{10}{x} + \frac{1}{x^3}}{-3 + \frac{10}{x} + \frac{50}{x^3}}$$

$$= \frac{5(\infty) - \frac{10}{\infty} + \frac{1}{(\infty)^3}}{-3 + \frac{10}{\infty} + \frac{50}{(\infty)^3}} = \frac{\infty - 0 + 0}{-3+0+0}$$

$$= \infty$$

Example 3:- Evalue $\displaystyle\lim_{x\to-\infty} \frac{4x^4-5x^3}{3x^5+2x^2+1}$

Solution:- $\displaystyle\lim_{x\to-\infty} \frac{4x^4-5x^3}{3x^5+2x^2+1}$

$\because x<0$, dividing up and down by $(-x)^5 = -x^5$

$$= \lim_{x\to-\infty} \frac{-\frac{4}{x} + \frac{5}{x^2}}{-3 - \frac{2}{x^3} - \frac{1}{x^5}} = \frac{-0+0}{-3-0-0} = 0$$

Example 4: Evaluate

(i) $\displaystyle\lim_{x\to-\infty} \frac{2-3x}{\sqrt{3+4x^2}}$ (ii) $\displaystyle\lim_{x\to+\infty} \frac{2-3x}{\sqrt{3+4x^2}}$

Solution:- (i) $\displaystyle\lim_{x\to-\infty} \frac{2-3x}{\sqrt{3+4x^2}}$

$\because \sqrt{x^2} = |x| = -x$ as $x<0$

$\therefore$ Dividing up and down by $-x$,

$$= \lim_{x\to-\infty} \frac{-\frac{2}{x}+3}{\sqrt{\frac{3}{x^2}+4}} = \frac{-0+3}{\sqrt{0+4}} = \frac{3}{2}$$

(ii) $\displaystyle\lim_{x\to\infty} \frac{2-3x}{\sqrt{3+4x^2}}$

Here $\sqrt{x^2} = |x| = x$ as $x>0$

$\therefore$ Dividing up and down by x

$$= \lim_{x\to+\infty} \frac{\frac{2}{x}-3}{\sqrt{\frac{3}{x^2}+4}} = \frac{0-3}{\sqrt{0+4}} = -\frac{3}{2}$$

Theorem:- Prove that

$$\lim_{x\to+\infty}\left(1+\frac{1}{n}\right)^n = e$$

Proof:- using Bionomial theorem, we have

$$\left(1+\frac{1}{n}\right)^n = 1+n\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3+\cdots$$

$$=1+1+\frac{1}{2!}\left(\frac{n-1}{n}\right)+\frac{1}{3!}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)+\cdots$$

$$=2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots$$

when $n\to\infty$, $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}$ ... all tend to zero

Thus,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

$$= 2+0.5+0.16667+\cdots$$

$$= 2.718281$$

$\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ Hence proved.

Deduction:- $\displaystyle\lim_{x\to 0}(1+x)^{1/x} = e$

We know that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \to (i)$$

Put $n=\frac{1}{x} \Rightarrow x=\frac{1}{n}$ in (i)

when $n\to\infty$, $x\to 0$

so (i) $\displaystyle\lim_{x\to 0}(1+x)^{1/x} = e$

Theorem:- Prove that

$$\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$$

Proof:- $L.H.S=\lim_{x\to 0}\frac{a^x-1}{x}$

Put $a^x-1=y \Rightarrow a^x=1+y$

so $x=\log_a(1+y)$

as $x\to 0$, $y\to 0$ so

$$L.H.S=\lim_{y\to 0}\frac{y}{\log_a(1+y)}$$

$$=\lim_{y\to 0}\frac{1}{\frac{1}{y}\log_a(1+y)}=\lim_{y\to 0}\frac{1}{\log_a(1+y)^{1/y}}$$

$$=\frac{1}{\log_a e}=\log_e a \qquad \because \lim_{y\to 0}(1+y)^{1/y}=e$$

$$=R.H.S$$

Thus $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$

Deduction:- $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)=\log_e e=1$

Since we know that

$$\lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\log_e a \rightarrow (I)$$

Put $a=e$ in (I), we get

$$\lim_{x\to 0}\frac{e^x-1}{x}=\log_e e=1$$

Important results to remember

(i) $\lim_{x\to +\infty}(e^x)=\infty$

(ii) $\lim_{x\to -\infty}(e^x)=\lim_{x\to -\infty}\left(\frac{1}{e^{-x}}\right)=0$

(iii) $\lim_{x\to \pm\infty}\left(\frac{a}{x}\right)=0$, where a is any real number.

Example 5:- Express each limit in terms of 'e'

(a) $\lim_{x\to\infty}\left(1+\frac{3}{n}\right)^{2n}$

(b) $\lim_{h\to 0}(1+2h)^{1/h}$

Solution:- (a) $\lim_{x\to\infty}\left(1+\frac{3}{n}\right)^{2n}$

$$=\left[\left(1+\frac{3}{n}\right)^{n/3}\right]^{6}=e^6$$

(b) $\lim_{h\to 0}(1+2h)^{1/h}$

$$=\left[\lim_{h\to 0}(1+2h)^{1/2h}\right]^2$$

$$=e^2$$

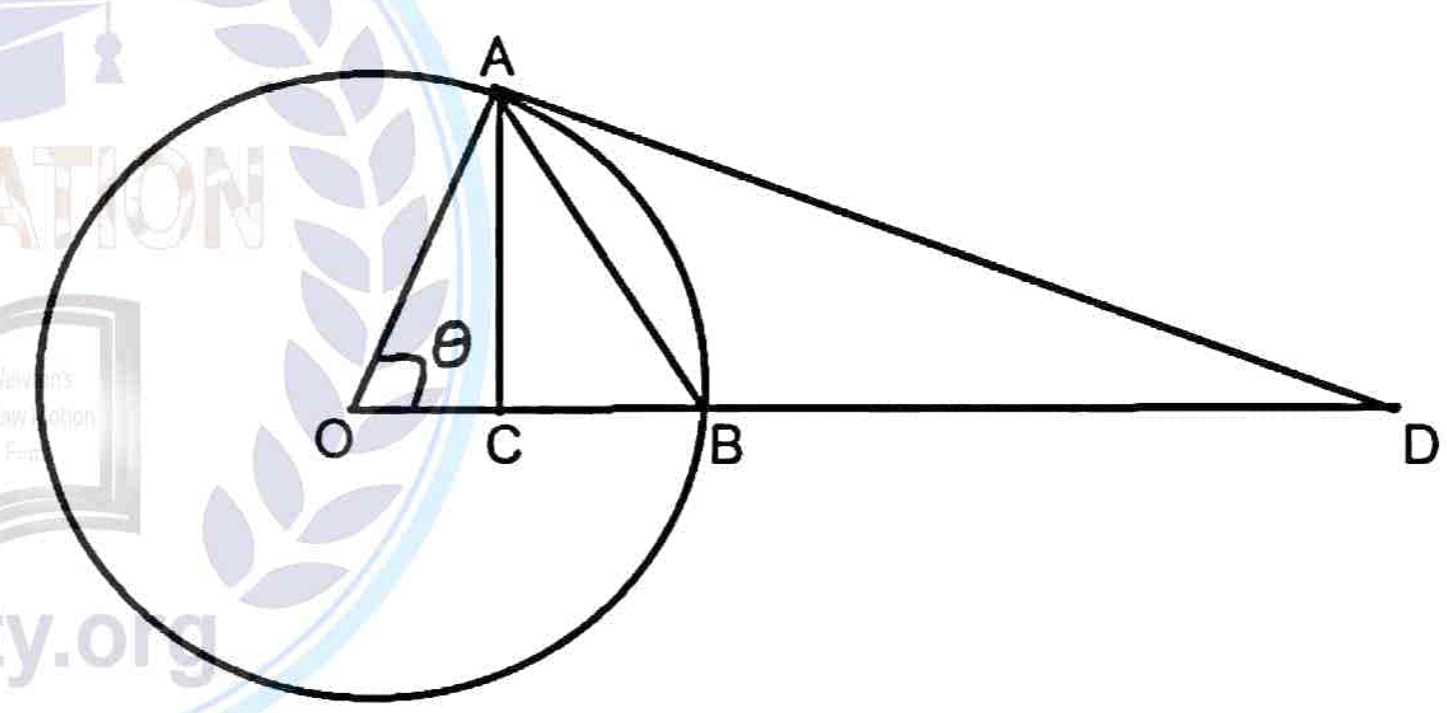
The Sandwitch Theorem

Let f, g and h be functions s.that $f(x)\le g(x)\le h(x)$ for all numbers x in some open interval containing 'c', except possibly at c itself. If $\lim_{x\to c}f(x)=L$ and $\lim_{x\to c}h(x)=L$, then $g(x)$ is sandwitch b/w $f(x)$ and $h(x)$ so that

$$\lim_{x\to c}g(x)=L$$

Theorem:- If θ is measured in radian, then $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$

Proof:- Draw a unit cirle (radius 1) in which



Area of $\Delta OAB <$ Area of sector $OAB <$ Area of $\Delta OAD \longrightarrow (I)$

Now

Area of $\Delta OAB=\frac{1}{2}$(base)(perpendicular)

$$=\frac{1}{2}|OB||AC| \qquad \text{where } \frac{|AC|}{|OA|}=\sin\theta$$

$$=\frac{1}{2}(1)(\sin\theta) \qquad |AC|=|OA|\sin\theta$$

$$=\frac{1}{2}\sin\theta \qquad |AC|=\sin\theta$$

$\because$ Radius $=|OA|=|OB|=1$

Area of sector $OAB = \frac{1}{2} r^2 \theta$

$= \frac{1}{2}(1)^2\theta = \frac{1}{2}\theta$

Area of $\Delta OAD = \frac{1}{2}$(base)(perpendicular)

$= \frac{1}{2}|OA||AD|$ where $\frac{|AD|}{|OA|} = \tan\theta$

$= \frac{1}{2}(1)(\tan\theta)$ $\quad |AD| = |OA|\tan\theta$

$= \frac{1}{2}\tan\theta$ $\quad |AD| = \tan\theta$

Now by (I)

$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$

or $\sin\theta < \theta < \tan\theta$ ('x' by 2)

or $\frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\sin\theta}{\cos\theta} \times \frac{1}{\sin\theta}$ ($\div$ by $\sin\theta$)

or $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$

Take reciprocal and limit $\theta \to 0$

$\lim_{\theta\to 0}(1) > \lim_{\theta\to 0}\frac{\sin\theta}{\theta} > \lim_{\theta\to 0}(\cos\theta)$

$1 > \lim_{\theta\to 0}\frac{\sin\theta}{\theta} > 1$

Applying sandwitch theorem

$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ Hence proved.

Example 6: Evaluate $\lim_{\theta\to 0}\frac{\sin 7\theta}{\theta}$

Solution: $\lim_{\theta\to 0}\frac{\sin 7\theta}{\theta}$

$= 7\left(\lim_{\theta\to 0}\frac{\sin 7\theta}{7\theta}\right)$ ('x' and $\div$ by 7)

$= 7(1) = 7$

Example 7: Evaluate $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$

Solution:- $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} \times \frac{1+\cos\theta}{1+\cos\theta}$

$= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)} = \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)}$

$= \lim_{\theta\to 0}\sin\theta \cdot \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \cdot \lim_{\theta\to 0}\frac{1}{1+\cos\theta}$

$= (0)(1)\left(\frac{1}{1+1}\right) = 0$

Exercise 1.3

Q1. Evaluate each limit by using theorems of limits:

(i) $\lim_{x\to 3}(2x+4)$

Solution:- $\lim_{x\to 3}(2x+4)$

$= \lim_{x\to 3}(2x) + \lim_{x\to 3}(4) = 2(3)+4 = 10$

(ii) $\lim_{x\to 1}(3x^2-2x+4)$

Solution:- $\lim_{x\to 1}(3x^2-2x+4)$

$= 3(1)^2 - 2(1) + 4 = 3-2+4 = 5$

(iii) $\lim_{x\to 3}\sqrt{x^2+x+4}$

Solution:- $\lim_{x\to 3}\sqrt{x^2+x+4}$

$= \sqrt{(3)^2+3+4} = \sqrt{16} = 4$

(iv) $\lim_{x\to 2} x\sqrt{x^2-4}$

Solution:- $\lim_{x\to 2} x\sqrt{x^2-4}$

$= (2)\sqrt{(2)^2-4} = 0$

(v) $\lim_{x\to 2}\left(\sqrt{x^3+1} - \sqrt{x^2+5}\right)$

Solution:- $\lim_{x\to 2}\left(\sqrt{x^3+1} - \sqrt{x^2+5}\right)$

$= \sqrt{(2)^3+1} - \sqrt{(2)^2+5} = 3-3 = 0$

(vi) $\lim_{x\to -2}\frac{2x^3+5x}{3x-2}$

Solution:- $\lim_{x\to -2}\frac{2x^3+5x}{3x-2}$

$= \frac{2(-2)^3+5(-2)}{3(-2)-2} = \frac{-16-10}{-8}$

$= \frac{-26}{-8} = \frac{13}{4}$

Q2. Evaluate each limit by using algebraic techniques.

(i) $\lim\limits_{x\to -1} \dfrac{x^3-x}{x+1}$

Solution:- $\lim\limits_{x\to -1} \dfrac{x^3-x}{x+1}$ $\left(\frac{0}{0}\right)$ form

$= \lim\limits_{x\to -1} \dfrac{x(x^2-1)}{x+1} = \lim\limits_{x\to -1} \dfrac{x(x-1)\cancel{(x+1)}}{\cancel{x+1}}$

$= \lim\limits_{x\to -1} x(x-1) = (-1)(-1-1) = 2$

(ii) $\lim\limits_{x\to 0} \left(\dfrac{3x^3+4x}{x^2+x}\right)$

Solution:- $\lim\limits_{x\to 0} \left(\dfrac{3x^3+4x}{x^2+x}\right)$ $\left(\frac{0}{0}\right)$ form

$= \lim\limits_{x\to 0} \dfrac{x(3x^2+4)}{x(x+1)} = \lim\limits_{x\to 0} \left(\dfrac{3x^2+4}{x+1}\right)$

$= \dfrac{3(0)^2+4}{0+1} = \dfrac{4}{1} = 4$

(iii) $\lim\limits_{x\to 2} \dfrac{x^3-8}{x^2+x-6}$

Solution:- $\lim\limits_{x\to 2} \dfrac{x^3-8}{x^2+x-6}$ $\left(\frac{0}{0}\right)$ form

$= \lim\limits_{x\to 2} \dfrac{(x)^3-(2)^3}{x^2+3x-2x-6}$

$= \lim\limits_{x\to 2} \dfrac{(x-2)(x^2+4+2x)}{x(x+3)-2(x+3)}$

$= \lim\limits_{x\to 2} \dfrac{\cancel{(x-2)}(x^2+4+2x)}{(x+3)\cancel{(x-2)}}$

$= \lim\limits_{x\to 2} \dfrac{x^2+4+2x}{x+3} = \dfrac{(2)^2+4+2(2)}{2+3} = \dfrac{12}{5}$

(iv) $\lim\limits_{x\to 1} \dfrac{x^3-3x^2+3x-1}{x^3-x}$

Solution:- $\lim\limits_{x\to 1} \dfrac{x^3-3x^2+3x-1}{x^3-x}$ $\left(\frac{0}{0}\right)$ form

$= \lim\limits_{x\to 1} \dfrac{(x-1)^3}{x(x^2-1)}$ $\quad \because (x-1)^3 = x^3-3x^2+3x-1$

$= \lim\limits_{x\to 1} \dfrac{(x-1)^3}{x(x-1)(x+1)} = \lim\limits_{x\to 1} \dfrac{(x-1)^2}{x(x+1)}$

$= \dfrac{(1-1)^2}{1(1+1)} = 0$

(v) $\lim\limits_{x\to -1} \left(\dfrac{x^3+x}{x^2-1}\right)$

Solution:- $\lim\limits_{x\to -1} \left(\dfrac{x^3+x^2}{x^2-1}\right)$ $\left(\frac{0}{0}\right)$ forms

$= \lim\limits_{x\to -1} \dfrac{x^2\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} = \lim\limits_{x\to -1} \dfrac{x^2}{x-1}$

$= \dfrac{(-1)^2}{-1-1} = \dfrac{1}{-2}$

(vi) $\lim\limits_{x\to 4} \dfrac{2x^2-32}{x^3-4x^2}$

Solution:- $\lim\limits_{x\to 4} \dfrac{2x^2-32}{x^3-4x^2}$ $\left(\frac{0}{0}\right)$ forms

$= \lim\limits_{x\to 4} \dfrac{2(x^2-16)}{x^2(x-4)} = \lim\limits_{x\to 4} \dfrac{2\cancel{(x-4)}(x+4)}{x^2\cancel{(x-4)}}$

$= \lim\limits_{x\to 4} \dfrac{2(x+4)}{x^2} = \dfrac{2(4+4)}{(4)^2} = 1$

(vii) $\lim\limits_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{x-2}$

Solution:- $\lim\limits_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{x-2}$ $\left(\frac{0}{0}\right)$ form

$= \lim\limits_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{x-2} \times \dfrac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}$

$= \lim\limits_{x\to 2} \dfrac{(\sqrt{x})^2-(\sqrt{2})^2}{x-2(\sqrt{x}+\sqrt{2})}$

$= \lim\limits_{x\to 2} \dfrac{\cancel{x-2}}{\cancel{(x-2)}(\sqrt{x}+\sqrt{2})} = \lim\limits_{x\to 2} \dfrac{1}{\sqrt{x}+\sqrt{2}}$

$= \dfrac{1}{\sqrt{2}+\sqrt{2}} = \dfrac{1}{2\sqrt{2}}$

(viii) $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

Solution:- $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ $\left(\frac{0}{0}\right)$ form

$= \lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h} \times \dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$= \lim\limits_{h\to 0} \dfrac{\cancel{x}+\cancel{h}-\cancel{x}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$

$= \lim\limits_{h\to 0} \dfrac{1}{\sqrt{x+h}+\sqrt{x}}$

$= \dfrac{1}{\sqrt{x}+\sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

(ix) $\lim_{x\to a} \dfrac{x^n - a^n}{x^m - a^m}$

Solution:- $\lim_{x\to a} \dfrac{x^n - a^n}{x^m - a^m}$ $\left(\frac{0}{0}\right)$ form

Dividing up and down by $x-a$

$$= \lim_{x\to a}\left(\frac{\frac{x^n-a^n}{x-a}}{\frac{x^m-a^m}{x-a}}\right) = \frac{\lim_{x\to a}\frac{x^n-a^n}{x-a}}{\lim_{x\to a}\frac{x^m-a^m}{x-a}}$$

$$= \frac{na^{n-1}}{ma^{m-1}} \quad \left(\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right)$$

$$= \frac{n}{m}a^{n-1-m+1} = \frac{n}{m}a^{n-m}$$

Q3. Evaluate the following limits.

(i) $\lim_{x\to 0} \dfrac{\sin 7x}{x}$

Solution:- $\lim_{x\to 0} \dfrac{\sin 7x}{x}$ $\left(\frac{0}{0}\right)$ form

$$= 7\left(\lim_{x\to 0}\frac{\sin 7x}{7x}\right) = 7(1) = 7 \quad \because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

(ii) $\lim_{x\to 0} \dfrac{\sin x^\circ}{x}$

Solution:- $\lim_{x\to 0} \dfrac{\sin x^\circ}{x}$ $\left(\frac{0}{0}\right)$ form

$$= \lim_{x\to 0}\frac{\sin\frac{\pi x}{180}}{x} \quad \because 1^\circ = \frac{\pi}{180}\text{rad} \quad \text{so } x^\circ = \frac{\pi x}{180}\text{rad}$$

$$= \lim_{x\to 0}\frac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}} \times \frac{\pi}{180}$$

$$= 1 \times \frac{\pi}{180} = \frac{\pi}{180}$$

(iii) $\lim_{\theta\to 0} \dfrac{1-\cos\theta}{\sin\theta}$

Solution:- $\lim_{\theta\to 0} \dfrac{1-\cos\theta}{\sin\theta}$ $\left(\frac{0}{0}\right)$ form

$$= \lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta} \times \frac{1+\cos\theta}{1+\cos\theta}$$

$$= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)} = \lim_{\theta\to 0}\frac{\sin^{\cancel{2}}\theta}{\cancel{\sin\theta}(1+\cos\theta)}$$

$$= \lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta} = \frac{\sin 0}{1+\cos 0}$$

$$= \frac{0}{1+1} = 0$$

(iv) $\lim_{x\to \pi} \dfrac{\sin x}{\pi - x}$

Solution:- $\lim_{x\to \pi} \dfrac{\sin x}{\pi - x}$ $\left(\frac{0}{0}\right)$ form

Put $\pi - x = t$

$\Rightarrow x = \pi - t$

when $x\to\pi$ then $t\to 0$ so

$$\lim_{x\to\pi}\frac{\sin x}{\pi - x} = \lim_{t\to 0}\frac{\sin(\pi - t)}{t}$$

$$= \lim_{t\to 0}\frac{\sin t}{t} \quad \because \sin(\pi-\theta) = \sin\theta$$

$$= 1$$

(v) $\lim_{x\to 0} \dfrac{\sin ax}{\sin bx}$

Solution:- $\lim_{x\to 0} \dfrac{\sin ax}{\sin bx}$ $\left(\frac{0}{0}\right)$ form

$$= \lim_{x\to 0}\left(\frac{\frac{\sin ax}{ax}\times ax}{\frac{\sin bx}{bx}\times bx}\right)$$

$$= \frac{\lim_{x\to 0}\frac{\sin ax}{ax}\times ax}{\lim_{x\to 0}\frac{\sin bx}{bx}\times bx} = \frac{1\times ax}{1\times bx} = \frac{a}{b}$$

(vi) $\lim_{x\to 0} \dfrac{x}{\tan x}$

Solution:- $\lim_{x\to 0} \dfrac{x}{\tan x}$ $\left(\frac{0}{0}\right)$ form

$$= \lim_{x\to 0} x\cot x \quad \because \cot x = \frac{1}{\tan x}$$

$$= \lim_{x\to 0} x \cdot \frac{\cos x}{\sin x}$$

$$= \lim_{x\to 0}\frac{x}{\sin x} \cdot \lim_{x\to 0}\cos x$$

$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^{-1} \cdot \lim_{x\to 0}\cos x$$

$$= (1)^{-1}\cdot\cos 0 = 1\cdot 1 = 1$$

(vii) $\lim\limits_{x\to 0} \dfrac{1-\cos 2x}{x^2}$

Solution:- $\lim\limits_{x\to 0} \dfrac{1-\cos 2x}{x^2}$ $\left(\frac{0}{0}\right)$ form

$= \lim\limits_{x\to 0} \dfrac{2\sin^2 x}{x^2}$ $\quad \because 1-\cos\theta = 2\sin^2\frac{\theta}{2}$

$\Rightarrow 1-\cos 2\theta = 2\sin^2\theta$

$= 2\left(\lim\limits_{x\to 0} \dfrac{\sin x}{x}\right)^2 = 2(1)^2 = 2$

(viii) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{\sin^2 x}$

Solution:- $\lim\limits_{x\to 0} \dfrac{1-\cos x}{\sin^2 x}$ $\left(\frac{0}{0}\right)$ form

$= \lim\limits_{x\to 0} \dfrac{1-\cos x}{1-\cos^2 x}$ $\quad \because \sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \sin^2\theta = 1-\cos^2\theta$

$= \lim\limits_{x\to 0} \dfrac{\cancel{1-\cos x}}{\cancel{(1-\cos x)}(1+\cos x)}$ $\quad =(1-\cos\theta)(1+\cos\theta)$

$= \lim\limits_{x\to 0} \dfrac{1}{1+\cos x} = \dfrac{1}{1+\cos 0} = \dfrac{1}{1+1} = \dfrac{1}{2}$

(ix) $\lim\limits_{\theta\to 0} \dfrac{\sin^2\theta}{\theta}$

Solution:- $\lim\limits_{\theta\to 0} \dfrac{\sin^2\theta}{\theta}$ $\left(\frac{0}{0}\right)$ form

$= \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} \lim\limits_{\theta\to 0} \sin\theta = 1 \cdot \sin 0 = 1\cdot 0 = 0$

(x) $\lim\limits_{x\to 0} \dfrac{\sec x - \cos x}{x}$

Solution:- $\lim\limits_{x\to 0} \dfrac{\sec x - \cos x}{x}$ $\left(\frac{0}{0}\right)$ form

$= \lim\limits_{x\to 0} \dfrac{1}{x}\left(\dfrac{1}{\cos x} - \cos x\right) = \lim\limits_{x\to 0} \dfrac{1}{x}\left(\dfrac{1-\cos^2 x}{\cos x}\right)$

$= \lim\limits_{x\to 0} \dfrac{1}{x}\left(\dfrac{\sin^2 x}{\cos x}\right) = \lim\limits_{x\to 0} \dfrac{\sin x}{x} \cdot \lim\limits_{x\to 0} \dfrac{\sin x}{\cos x}$

$= \lim\limits_{x\to} \dfrac{\sin x}{x} \cdot \lim\limits_{x\to 0} \tan x = 1 \cdot \tan 0 = 0$

(xi) $\lim\limits_{\theta\to 0} \dfrac{1-\cos p\theta}{1+\cos q\theta}$

Solution:- $\lim\limits_{\theta\to 0} \dfrac{1-\cos p\theta}{1+\cos q\theta}$ $\left(\frac{0}{0}\right)$ form

$= \lim\limits_{\theta\to 0} \dfrac{\cancel{2}\sin^2\frac{p\theta}{2}}{\cancel{2}\sin^2\frac{q\theta}{2}} = \dfrac{\left(\lim\limits_{\theta\to 0} \sin\frac{p\theta}{2}\right)^2}{\left(\lim\limits_{\theta\to 0} \sin\frac{q\theta}{2}\right)^2}$

$= \dfrac{\left(\lim\limits_{\theta\to 0} \dfrac{\sin\frac{p\theta}{2}}{\frac{p\theta}{2}} \times \frac{p\theta}{2}\right)^2}{\left(\lim\limits_{\theta\to 0} \dfrac{\sin\frac{q\theta}{2}}{\frac{q\theta}{2}} \times \frac{q\theta}{2}\right)^2} = \dfrac{\left(1\times\frac{p\theta}{2}\right)^2}{\left(1\times\frac{q\theta}{2}\right)^2}$

$= \dfrac{\frac{p^2\cancel{\theta^2}}{\cancel{4}}}{\frac{q^2\cancel{\theta^2}}{\cancel{4}}} = \dfrac{p^2}{q^2}$

(xii) $\lim\limits_{\theta\to 0} \dfrac{\tan\theta - \sin\theta}{\sin^3\theta}$

Solution:- $\lim\limits_{\theta\to 0} \dfrac{\tan\theta - \sin\theta}{\sin^3\theta}$ $\left(\frac{0}{0}\right)$ form

$= \lim\limits_{\theta\to 0} \dfrac{1}{\sin^3\theta}\left(\dfrac{\sin\theta}{\cos\theta} - \sin\theta\right)$

$= \lim\limits_{\theta\to 0} \dfrac{1}{\sin^3\theta}\left(\dfrac{\sin\theta - \sin\theta\cos\theta}{\cos\theta}\right)$

$= \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\sin^3\theta}\left(\dfrac{1-\cos\theta}{\cos\theta}\right) = \lim\limits_{\theta\to 0} \dfrac{1}{\sin^2\theta} \dfrac{(1-\cos\theta)}{\cos\theta}$

$= \lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{(1-\cos^2\theta)\cos\theta}$

$= \lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{(1-\cos\theta)(1+\cos\theta)\cos\theta}$

$= \dfrac{1}{(1+\cos 0)\cos 0} = \dfrac{1}{(1+1)1} = \dfrac{1}{2}$

Q4. Express each limit in terms of 'e'.

(i) $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^{2n}$

Solution:- $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^{2n}$

$= \left[\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n\right]^2 = e^2$

(ii) $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^{\frac{n}{2}}$

Solution:- $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^{\frac{n}{2}}$

$= \left[\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n\right]^{1/2} = e^{1/2}$

(iii) $\lim\limits_{n\to\infty} \left(1+\frac{1}{3n}\right)$

Solution:- $\lim\limits_{n\to\infty} \left(1+\frac{1}{3n}\right)^n$

$= \lim\limits_{n\to\infty} \left(1+\frac{1}{3n}\right)^{\frac{3n}{3}} = \left[\lim\limits_{n\to\infty} \left(1+\frac{1}{3n}\right)^{3n}\right]^{\frac{1}{3}}$

$= e^{1/3}$

(iv) $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$

Solution:- $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$

$$= \lim_{n\to\infty}\left(1+\left(-\frac{1}{n}\right)\right)^n = \left[\lim_{n\to\infty}\left(1+\left(-\frac{1}{n}\right)\right)^{-n}\right]^{-1}$$

$$= e^{-1}$$

(v) $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n$

Solution:- $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n$

$$= \lim_{n\to\infty}\left(1+\frac{4}{n}\right)^{\frac{4n}{4}} = \left[\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^{\frac{n}{4}}\right]^4$$

$$= e^4$$

(vi) $\lim_{x\to 0}(1+3x)^{2/x}$

Solution:- $\lim_{x\to 0}(1+3x)^{2/x}$

$$= \lim_{x\to 0}(1+3x)^{\frac{2}{x}\times\frac{3}{3}} = \lim_{x\to 0}(1+3x)^{\frac{6}{3x}}$$

$$= \left[\lim_{x\to 0}(1+3x)^{\frac{1}{3x}}\right]^6 = e^6$$

(vii) $\lim_{x\to 0}(1+2x^2)^{\frac{1}{x^2}}$

Solution:- $\lim_{x\to 0}(1+2x^2)^{\frac{1}{x^2}}$

$$= \lim_{x\to 0}(1+2x^2)^{\frac{2}{2x^2}} = \left[\lim_{x\to 0}(1+2x^2)^{\frac{1}{2x^2}}\right]^2$$

$$= e^2$$

(viii) $\lim_{h\to 0}(1-2h)^{\frac{1}{h}}$

Solution:- $\lim_{h\to 0}(1-2h)^{\frac{1}{h}}$

$$= \lim_{h\to 0}(1+(-2h))^{1/h} = \lim_{h\to 0}(1+(-2h))^{\frac{-2}{-2h}}$$

$$= \left[\lim_{h\to 0}(1+(-2h))^{\frac{1}{-2h}}\right]^{-2} = e^{-2}$$

(ix) $\lim_{x\to 0}\left(\frac{x}{1+x}\right)^x$

Solution:- $\lim_{x\to 0}\left(\frac{x}{1+x}\right)^x$

$$= \lim_{x\to 0}\left(\frac{1+x}{x}\right)^{-x} = \lim_{x\to 0}\left(\frac{1}{x}+1\right)^{x(-1)}$$

$$= \left[\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x\right]^{-1} = e^{-1}$$

(x) $\lim_{x\to 0}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right), \quad x<0$

Solution:-

$$\lim_{x\to 0}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right), \quad x<0$$

Here as $\lim_{x\to 0}\frac{1}{x} = -\infty$ if $x<0$ so

$$= \frac{e^{-\infty}-1}{e^{-\infty}+1} = \frac{\frac{1}{e^{\infty}}-1}{\frac{1}{e^{\infty}}+1}$$

$$= \frac{\frac{1}{\infty}-1}{\frac{1}{\infty}+1} = \frac{0-1}{0+1}$$

$$= -1$$

(xi) $\lim_{x\to 0}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right), \quad x>0$

Solution:-

$$\lim_{x\to 0}\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right), \quad x>0$$

Here as $\lim_{x\to 0}\frac{1}{x} = \infty$ if $x>0$ so

$$= \lim_{x\to 0}\frac{\cancel{e^{1/x}}\left(1-\frac{1}{e^{1/x}}\right)}{\cancel{e^{1/x}}\left(1+\frac{1}{e^{1/x}}\right)}$$

$$= \frac{1-\frac{1}{e^{\infty}}}{1+\frac{1}{e^{\infty}}} = \frac{1-\frac{1}{\infty}}{1+\frac{1}{\infty}}$$

$$= \frac{1-\frac{1}{\infty}}{1+\frac{1}{\infty}} = \frac{1-0}{1+0} = 1$$

The Left Hand Limit:-

If $\lim_{x\to a^-} f(x) = L$. It means $f(x)$ takes value L as x approaches to 'a' from the left side of 'a' (i.e., from $-\infty$ to a) then $\lim_{x\to a^-} f(x)=L$ is called left hand limit.

The Right Hand Limit:-

If $\lim_{x\to a^+} f(x) = L$. It means $f(x)$ takes value L as x approaches to 'a' from the right side of 'a' (i.e., from a to ∞) then $\lim_{x\to a^+} f(x)=L$ is called Right hand limit.

Existence of Limit of function (criteria):-

$\lim_{x\to a} f(x) = L$ if and only if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

i.e., L.H.L = R.H.L

Example 1: Determine whether $\lim_{x\to 2} f(x)$ and $\lim_{x\to 4} f(x)$ exist, when

$$f(x) = \begin{cases} 2x+1 & \text{if } 0 \le x \le 2 \\ 7-x & \text{if } 2 \le x \le 4 \\ x & \text{if } 4 \le x \le 6 \end{cases}$$

Solution:-

(i) $\boxed{\text{L.H.L:-}}$ $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(2x+1)$

$= 2(2)+1 = 5$

$\boxed{\text{R.H.L:-}}$ $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(7-x) = 7-2$

$= 5$

As L.H.L = R.H.L = 5

$\Rightarrow \lim_{x\to 2} f(x)$ exists.

(ii) $\boxed{\text{L.H.L:-}}$ $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(7-x)$

$= 7-4 = 3$

$\boxed{\text{R.H.L:-}}$ $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}(x) = 4$

$\because$ L.H.L $\neq$ R.H.L

so $\lim_{x\to 4} f(x)$ does not exist.

Continuous function:-

A function f is said to be continuous at a number x=a if (i) f(a) is defined (ii) $\lim_{x\to a} f(x)$ exists (iii) $\lim_{x\to a} f(x) = f(a)$

Discontinuous function:-

A function f(x) is said to be discontinuous at x=a if $\lim_{x\to a} f(x) \neq f(a)$

* If f(x) is not defined at x=a then f(x) is called discontinuous.
* Any function which does not satisfy at least one of three conditions of continuity is called discontinuous.

Example 2:- Discuss the conitinuity of the function $f(x) = \frac{x^2-1}{x-1}$ at $x=1$

Solution:- $f(x) = \frac{x^2-1}{x-1}$

At $x=1$, $f(x) = \frac{(1)^2-1}{1-1} = \frac{1-1}{1-1} = \frac{0}{0}$

$\rightarrow$ f(1) is not defined.

Thus f(x) is discontinuous at x=1

Example 3:- For $f(x) = 3x^2-5x+4$, discuss the continuity of f at x=1

Solution:- $f(x) = 3x^2-5x+4$

at x=1, $f(1) = 3(1)^2 - 5(1) + 4 = 2$

Now $\lim_{x\to 1} f(x) = \lim_{x\to 1}(3x^2-5x+4)$

$= 3(1)^2 - 5(1) + 4 = 2$

As $\lim_{x \to 1} f(x) = f(1)$. so $f(x)$ is continuous at $x = 1$

Example 4:- Discuss the continuity of the function $f(x)$ and $g(x)$ at $x = 3$.

(a) $f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ (b) $g(x) = \frac{x^2-3}{x-3}$ if $x \neq 3$

Solution:- (a) $f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$

$\because$ at $x = 3$, $f(3) = 6$

Now $\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2-9}{x-3}$

$= \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3}$

$= \lim_{x \to 3} (x+3) = 3+3 = 6$

As $f(3) = \lim_{x \to 3} f(x) = 6$. so $f(x)$ is continuous at $x = 3$.

(b) $g(x) = \frac{x^2-3}{x-3}$ if $x \neq 3$

It means $g(x)$ is not defined at $x = 3$. So $g(x)$ is discontinuous at $x = 3$.

Example 5:- Discuss the continuity of f at 3, when $f(x) = \begin{cases} x-1, & \text{if } x < 3 \\ 2x+1, & \text{if } 3 \leq x \end{cases}$

Solution:- $f(x) = \begin{cases} x-1, & \text{if } x < 3 \\ 2x+1, & \text{if } 3 \leq x \end{cases}$

L.H.L:- $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x-1) = 3-1$

$= 2$

R.H.L:- $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2x+1)$

$= 2(3)+1 = 7$

At $x = 3$, $\lim_{x \to 3} f(x) = \lim_{x \to 3} (2x+1)$

$= 2(3)+1 = 7$

$\because$ L.H.L $\neq$ R.H.L so $\lim_{x \to 3} f(x)$ does not exist. Hence $f(x)$ is discontinuous at $x = 3$

Exercise 1.4

Q1. Determine the left hand limit and the right hand limit and then, find the limit of the following functions when $x \to c$

(i) $f(x) = 2x^2 + x - 5$, $c = 1$

Solution:- L.H.L:-

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x^2 + x - 5) = 2(1)^2 + 1 - 5$

$= -2$

R.H.L:- $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x^2 + x - 5)$

$= 2(1)^2 + 1 - 5 = -2$

As L.H.L = R.H.L so

$\lim_{x \to 1} f(x) = -2$

(ii) $f(x) = \frac{x^2-9}{x-3}$, $c = -3$

Solution:- L.H.L:-

$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} \frac{x^2-9}{x-3}$

$= \lim_{x \to -3^-} \frac{(x-3)(x+3)}{x-3}$

$= \lim_{x \to -3^-} (x+3) = -3+3 = 0$

R.H.L:- $\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \frac{x^2-9}{x-3}$

$= \lim_{x \to -3^+} \frac{(x-3)(x+3)}{x-3} = \lim_{x \to -3^+} (x+3)$

$= -3+3 = 0$

As L.H.L = R.H.L so $\lim_{x \to -3} f(x) = 0$

(iii) $f(x) = |x-5|$, $c = 5$

Solution:- L.H.L:-

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} |x-5| = 5-5 = 0$

R.H.L:- $\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} |x-5| = 5-5$
$= 0$

As L.H.L = R.H.L so

$\lim_{x \to 5} f(x) = 0$

Q2. Discuss the continuity of $f(x)$ at $x=c$

(i) $f(x)=\begin{cases}2x+5 & \text{if } x\le 2\\ 4x+1 & \text{if } x>2\end{cases}$, $c=2$

Solution:- $\boxed{\text{L.H.L:-}}$

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(2x+5)=2(2)+5=9$

$\boxed{\text{R.H.L:-}}$ $\lim_{x\to 2^+} f(x)=\lim_{x\to 2}(4x+1)=4(2)+1=9$

$\boxed{\text{At } x=2}$, $f(x)=2x+5$

$\rightarrow f(2)=2(2)+5=9$

As L.H.L = R.H.L so $\lim_{x\to 2} f(x)$ exists

$\rightarrow \lim_{x\to 2} f(x)=f(2)$ so $f(x)$ is continuous at $x=2$

(ii) $f(x)=\begin{cases}3x-1 & \text{if } x<1\\ 4 & \text{if } x=1,\\ 2x & \text{if } x>1\end{cases}$ $c=1$

Solution:- $\boxed{\text{L.H.L :-}}$ $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(3x-1)$

$=3(1)-1=2$

$\boxed{\text{R.H.L:-}}$ $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(2x)=2(1)=2$

$\boxed{\text{At } x=1}$, $f(x)=4 \Rightarrow f(1)=4$

As L.H.L = R.H.L so $\lim_{x\to 1} f(x)$ exists.

but $\lim_{x\to 1} f(x)\ne f(1)$ Hence $f(x)$ is discontinuous.

Q3. If $f(x)=\begin{cases}3x & \text{if } x\le -2\\ x^2-1 & \text{if } -2<x<2\\ 3 & \text{if } x\ge 2\end{cases}$

Discuss continuity at $x=2$ and $x=-2$

Solution:- (i) $x=2$

$\boxed{\text{L.H.L:-}}$ $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(x^2-1)$

$=(2)^2-1=3$

$\boxed{\text{R.H.L :-}}$ $\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(3)=3$

$\because$ L.H.L = R.H.L so $\lim_{x\to 2} f(x)$ exists

and $\lim_{x\to 2} f(x)=3$

$\boxed{\text{At } x=2}$ $f(x)=3 \rightarrow f(2)=3$

so $\lim_{x\to 2} f(x)=f(2)$

Hence $f(x)$ is continuous at $x=2$

(ii) $x=-2$

$\boxed{\text{L.H.L:-}}$ $\lim_{x\to -2^-} f(x)=\lim_{x\to -2^-}(3x)=3(-2)$

$=-6$

$\boxed{\text{R.H.L:-}}$ $\lim_{x\to -2^+} f(x)=\lim_{x\to -2^+}(x^2-1)$

$=(-2)^2-1=4-1=3$

$\boxed{\text{At } x=-2}$, $f(x)=3x \Rightarrow f(-2)=3(-2)=-6$

$\because$ L.H.L $\ne$ R.H.L so

$f(x)$ is discontinuous at $x=-2$

Q4. If $f(x)=\begin{cases}x+2, & x\le -1\\ c+2, & x>-1\end{cases}$, find "c" so that $\lim_{x\to -1} f(x)$ exists.

Solution:- $\boxed{\text{L.H.L:-}}$

$\lim_{x\to -1^-} f(x)=\lim_{x\to -1^-}(x+2)=-1+2=1$

$\boxed{\text{R.H.L :-}}$ $\lim_{x\to -1^+} f(x)=\lim_{x\to -1^+}(c+2)$

$=c+2$

Given that $\lim_{x\to -1} f(x)$ exists. so

L.H.L = R.H.L

i.e., $\lim_{x\to -1^-} f(x)=\lim_{x\to -1^+} f(x)$

$\rightarrow 1=c+2$

$\rightarrow 1-2=c \rightarrow c=-1$

Q5. Find the value of m and n, so that given function f is continuous at $x=3$

(i) $f(x)=\begin{cases}mx & \text{if } x<3\\ n & \text{if } x=3\\ -2x+9 & \text{if } x>3\end{cases}$

Solution:- $\boxed{\text{L.H.L:-}}$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(mx) = 3m$

$\boxed{\text{R.H.L:-}}$ $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(-2x+9)$

$= -2(3)+9 = 3$

$\boxed{\text{At } x=3}$ $f(x) = n \Rightarrow f(3) = n$

We know that for a continuous function $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$

$3m = 3 = n$

$\Rightarrow 3m = 3$ and n=3

$\boxed{\text{Hence } m=1,\ n=3}$

(ii) $f(x) = \begin{cases} mx & \text{if } x<3 \\ x^2 & \text{if } x \geq 3 \end{cases}$

Solution:- $\boxed{\text{L.H.L:-}}$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(mx) = 3m$

$\boxed{\text{R.H.L:-}}$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(x^2) = (3)^2 = 9$

$\boxed{\text{At } x=3 :-}$ $f(x) = x^2 \Rightarrow f(3) = (3)^2 = 9$

Given that $f(x)$ is continuous. so

L.H.L = R.H.L = $f(3)$

$\Rightarrow 3m = 9 = 9 \Rightarrow 3m = 9$

$\boxed{\Rightarrow m=3}$

Q6. If $f(x) = \begin{cases} \dfrac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}, & x \neq 2 \\ k, & x=2 \end{cases}$

Find value of k so that f is continuous.

Solution:- $\boxed{\text{At } x=2}$

$f(x) = k \Rightarrow f(2) = k$

Now we find $\lim_{x\to 2} f(x)$

$\lim_{x\to 2} f(x) = \lim_{x\to 2} \dfrac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}$ $(\frac{0}{0})$form

ultiplying & dividing by $\sqrt{2x+5}+\sqrt{x+7}$

$\lim_{x\to 2} \dfrac{(\sqrt{2x+5}-\sqrt{x+7})(\sqrt{2x+5}+\sqrt{x+7})}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})}$

as $(a-b)(a+b) = (a^2 - b^2)$

$= \lim_{x\to 2} \dfrac{2x+5-(x+7)}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})}$

$= \lim_{x\to 2} \dfrac{\cancel{x-2}}{\cancel{(x-2)}(\sqrt{2x+5}+\sqrt{x+7})}$

$\lim_{x\to 2} f(x) = \dfrac{1}{\sqrt{2(2)+5}+\sqrt{2+7}} = \dfrac{1}{6}$

$\because$ given function is continuous at $x=2$. so

$\boxed{\lim_{x\to 2} f(x) = f(2)}$

$\Rightarrow \dfrac{1}{6} = k \Rightarrow \boxed{k = \dfrac{1}{6}}$