MATHEMATICS 1st YEAR

UNIT # 09

FUNDAMENTALS OF TRIGONOMETRY

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Contents

Sherazi Mathematics

اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

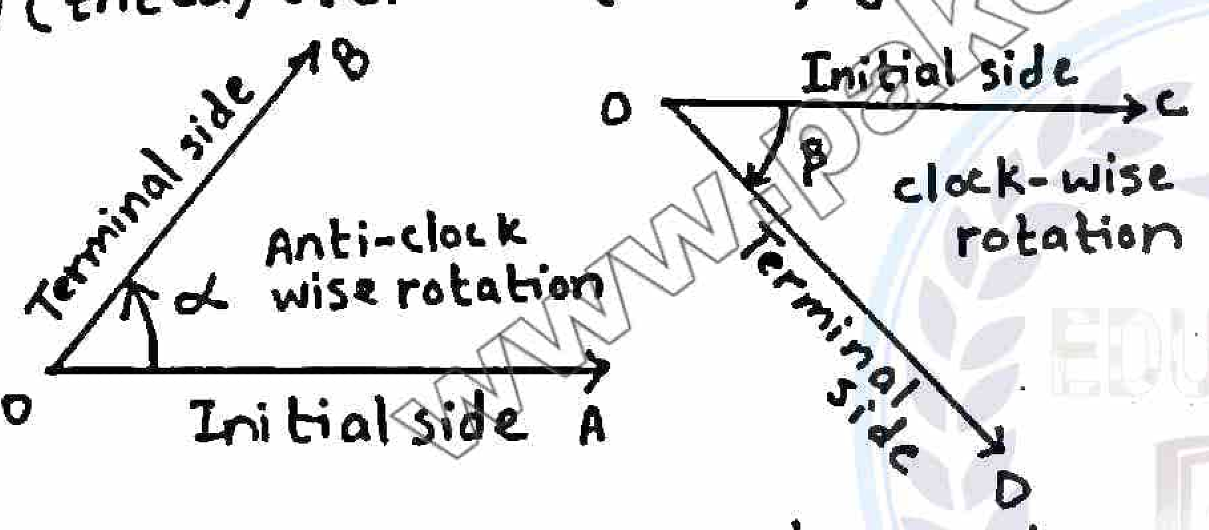
Trigonometry:-

The word Trigonometry has been derived from three Greek words: Trei(three), Goni (angles) and Metron(measurement). So it means measurement of triangle.

Units of Measures of Angles

Angle:- Two rays with a common starting point form an angle.

* One of the rays of angle is called initial side and the other as terminal side.
* An angle is said to be positive if the rotation is anti-clockwise.
* An angle is said to be negative if the rotation is clock-wise.
* Angles are usually denoted by Greek letters α (alpha), β (beta), γ (gamma), θ (theta) etc. (see figures)



* There are two commonly used measurements for angles: Degrees and Radians.

Sexagesimal System

(Degree, Minute and Second) ($D^\circ M' S''$)

The system in which one complete revolution is divided into 360 parts, each part is called degree. Then one degree is divided into 60 parts, each part is called minute. Now one minute is divided into 60 parts, each part is called a second. so

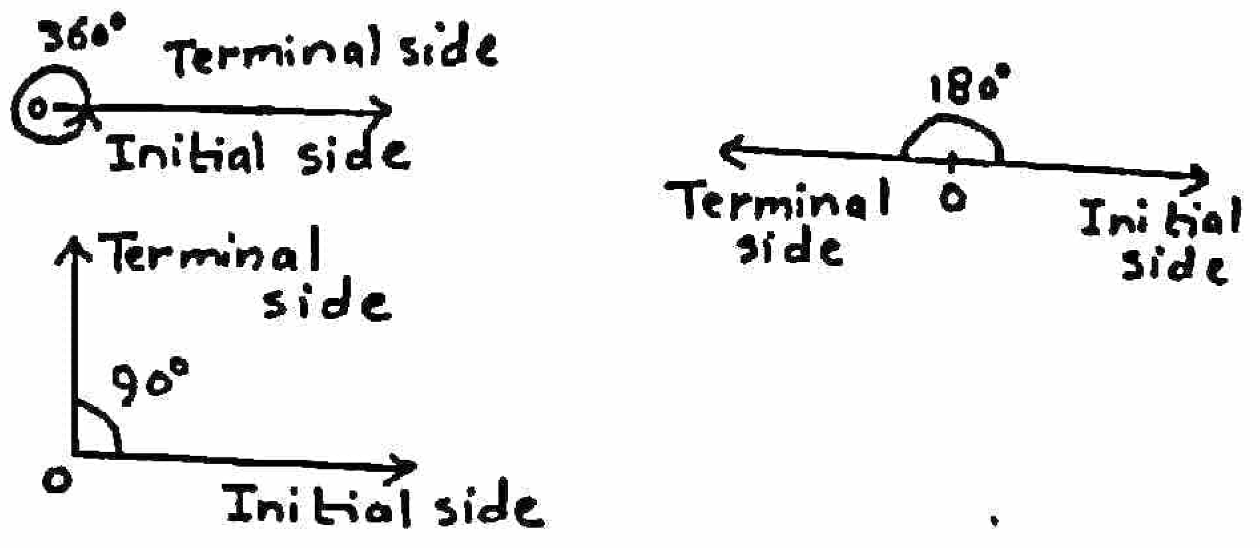
1 rotation (anti-clockwise) = 360°

One degree (1°) = $60'$ (60 minutes)

one minute ($1'$) = $60''$ (60 seconds)

$\frac{1}{2}$ rotation (anti-clockwise) = 180° is called a straight angle.

$\frac{1}{4}$ rotation (anti-clockwise) = 90° is called right angle.



Remember:- Sexagesimal System is also known as "ENGLISH SYSTEM".

Note that:- $1^\circ = 60' \rightarrow 1' = \left(\frac{1}{60}\right)^\circ$
$1' = 60'' \rightarrow 1'' = \left(\frac{1}{60}\right)'$ $\quad 1'' = \left(\frac{1}{60\times 60}\right)^\circ = \left(\frac{1}{3600}\right)^\circ$

Conversion from $D^\circ M' S''$ to a decimal form and vice versa

(i) $16^\circ 30' = \left[16 + \frac{30}{60}\right]^\circ = \left(16 + \frac{1}{2}\right)^\circ$
$= (16 + 0.5)^\circ = 16.5^\circ$ $\quad \because 1' = \left(\frac{1}{60}\right)^\circ \rightarrow 30' = \left(\frac{30}{60}\right)^\circ$

(ii) $45.25^\circ = 45^\circ + 0.25 \times 60'$ $\quad \because 1^\circ = 60' \rightarrow 0.25^\circ = 0.25 \times 60'$
$= 45^\circ 15'$

Example 1. Convert $18^\circ 6' 21''$ to decimal form.

Solution:- $\because 1' = \left(\frac{1}{60}\right)^\circ$, $1'' = \left(\frac{1}{3600}\right)^\circ$

$18^\circ 6' 21'' = \left[18 + \frac{6}{60} + \frac{21}{3600}\right]^\circ$ $\quad \because 6' = \left(\frac{6}{60}\right)^\circ$, $21'' = \left(\frac{21}{3600}\right)^\circ$

$= (18 + 0.1 + 0.005833)^\circ$

$= 18.105833^\circ$

Example 2. Convert 21.256° to the $D^\circ M' S''$ form.

Solution:- $21.256^\circ = 21^\circ + 0.256^\circ$

$= 21^\circ + 0.256(1^\circ)$

$= 21^\circ + 0.256(60')$ $\quad \because 1^\circ = 60'$

$= 21^\circ + 15.36'$

$= 21^\circ + 15' + 0.36'$

$= 21^\circ + 15' + 0.36(1')$

$= 21^\circ + 15' + 0.36(60'')$ $\quad \because 1' = 60''$

$= 21^\circ + 15' + 21.6''$

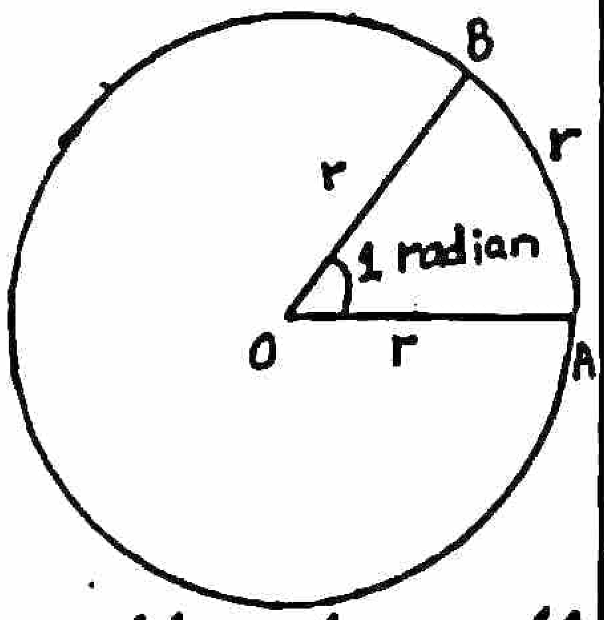
$= 21^\circ 15' 22''$

Circular System (Radians)

Radian:- An angle subtended at the centre of the circle by an arc, whose length is equal to the radius of the circle is called one radian.

In fig; $\widehat{AB} = \overline{OA} = r$

so $m\angle AOB = 1$ radian

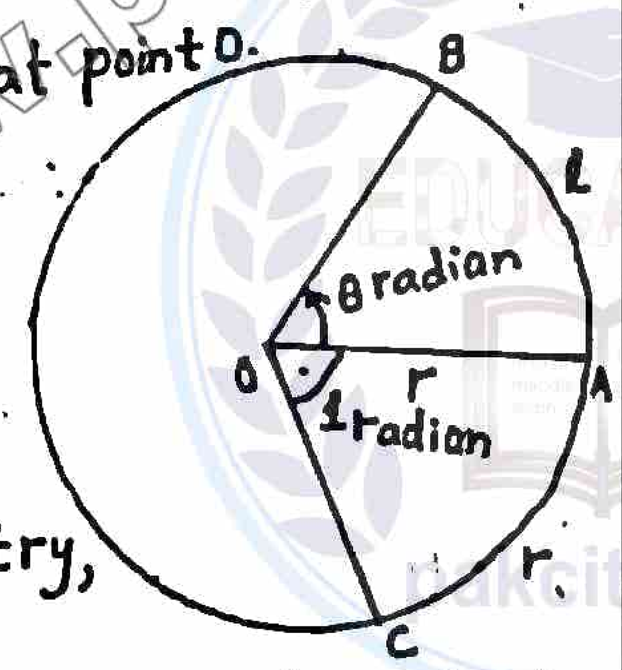


Relation between the length of arc of a circle and the circular measure of its central angle

Prove that $l = r\theta$

where r = radius of cirle, l = arc length
θ = circular measure of central angle.

Proof:- Let r be the radius of a circle having center at point O.
Let $\widehat{AB} = l$, $m\angle AOB = \theta$
Take an arc $\widehat{AC} = r$.
By def., $m\angle AOC = 1$ radian



By elementary geometry,

$$\frac{m\angle AOB}{m\angle AOC} = \frac{m\widehat{AB}}{m\widehat{AC}}$$

(∵ measure of central angles of a circle are proportional to the lengths of their arcs)

$$\rightarrow \frac{\theta \text{ radian}}{1 \text{ radian}} = \frac{l}{r}$$

$$\rightarrow \theta = \frac{l}{r} \rightarrow \boxed{l = r\theta}$$ Hence proved.

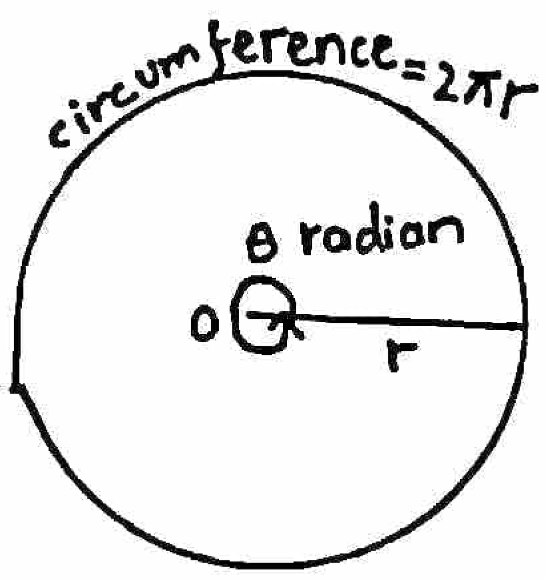
Conversion of Radian into Degree and vice versa

∵ circumference of circle of radius r $= 2\pi r$

We know that

$$l = r\theta$$

$$\rightarrow 2\pi r = r\theta$$



$$\rightarrow \theta = 2\pi \text{ radian}$$

Thus $2\pi = 360°$

$$\rightarrow \pi = 180° \longrightarrow \text{(i)}$$

$$\rightarrow 1 \text{radian} = \frac{180°}{\pi}$$

$$\boxed{\rightarrow 1 \text{rad} = \frac{180°}{3.1416} \approx 57.296°}$$

Further by (i)

$$\pi = 180°$$

$$\rightarrow 1° = \frac{\pi}{180} \text{rad}$$

$$\approx \frac{3.14}{180}$$

$$\boxed{\rightarrow 1° \approx 0.0175 \text{rad}}$$

Example 3. Convert the following angles in degree: (i) $\frac{2\pi}{3}$ radians (ii) 3 radians

Solution:- (i) $\frac{2\pi}{3}$ radians

$= \frac{2}{3}(180°)$ ∵ π rad $= 180°$

$= 2(60°) = 120°$

(ii) 3 radians $= 3(1 \text{radian})$

$\approx 3(57.296°) \approx 171.888°$

Example 4. Convert $54°45'$ into radians.

Solution:- $54°45' = 54° + 45'$

$= \left[54 + \frac{45}{60}\right]°$ ∵ $1' = \left(\frac{1}{60}\right)°$ $\rightarrow 45' = \left(\frac{45}{60}\right)°$

$= \left(\frac{3240 + 45}{60}\right)° = \left(\frac{3285}{60}\right)°$

$= \left(\frac{219}{4}\right)° = \left(\frac{219}{4}\right)(1°)$

$\approx \left(\frac{219}{4}\right)(0.0175)$ (∵ $1° = 0.0175$ rad)

≈ 0.958 radians.

Example 5. An angle subtends an angle of 70° at the center of a circle and its length is 132 m.m. Find the radius of the circle.

Solution:- Here $\theta = 70° = 70 \times 0.01745$

$\theta = 1.2215$ rad, $l = 132$ mm

$r = ?$

∵ $l = r\theta \rightarrow r = \frac{l}{\theta} = \frac{132}{1.2215}$

$\rightarrow r = 108.06$ mm or $r = 108$ mm

Example 6. Find the length of the equatorial arc subtending an angle of 1° at the centre of the earth, taking the radius of the earth as 6400km.

Solution:- Here $\theta = 1^\circ = 0.01745$ rad

Radius of earth $= r = 6400$km

length of arc $= l = ?$

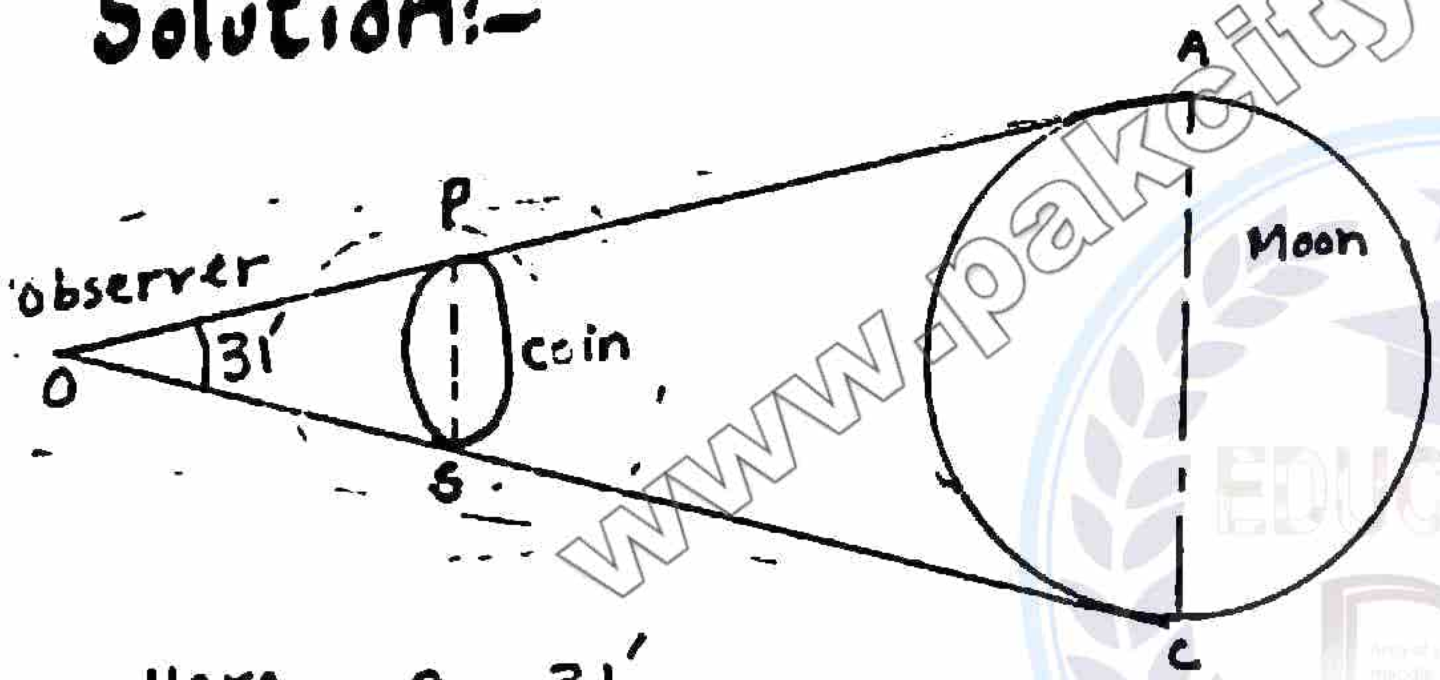
we know that

$l = r\theta$

$\rightarrow l = (6400)(0.01745) = 111.7$ Km

Example 7. Find correct to the nearest centimeter, the distance at which a coin of diameter '1'cm should be held so as to conceal the full moon whose diameter subtends an angle of $31'$ at the eye of the observer on the earth.

Solution:-



Here $\theta = 31'$

$\rightarrow \theta = \left(\frac{31}{60}\right)^\circ = 0.51^\circ$

$\theta = 0.51 \times 0.01745 = 0.00889$ rad

$\rightarrow \theta = 0.009$ radian

Diameter of coin $= m\overline{PS} = 1$cm

Diameter of coin from observer's eye $= m\overline{OP} = r = ?$

$\because l = r\theta$

$\rightarrow r = \frac{l}{\theta} = \frac{1}{0.009} = 111.11$ cm

Exercise 9.1

Q1. Express the following sexagesimal measures of angles in radians:

Solution:- (i) 30°

$= 30 \times \frac{\pi}{180} = \frac{\pi}{6}$ rad

(ii) 45°

$= 45 \times \frac{\pi}{180} = \frac{\pi}{4}$ rad

$\because 1^\circ = \frac{\pi}{180}$ rad

(iii) 60°

$= 60^\circ \times \frac{\pi}{180} = \frac{\pi}{3}$ rad

(iv) 75°

$= 75 \times \frac{\pi}{180} = \frac{5\pi}{12}$ rad

(v) 90°

$= 90 \times \frac{\pi}{180} = \frac{\pi}{2}$ rad

$\because 1^\circ = \frac{\pi}{180}$ rad

(vi) 105°

$= 105 \times \frac{\pi}{180} = \frac{7\pi}{12}$ rad

(vii) 120°

$= 120 \times \frac{\pi}{180} = \frac{2\pi}{3}$ rad

(viii) 135°

$= 135 \times \frac{\pi}{180} = \frac{3\pi}{4}$ rad

(ix) 150°

$= 150 \times \frac{\pi}{180} = \frac{5\pi}{6}$ rad

(x) $10^\circ 15'$

$= \left[10 + \frac{15}{60}\right]^\circ = \left[10 + \frac{1}{4}\right]^\circ$

$\because 1' = \left(\frac{1}{60}\right)^\circ$

$\rightarrow 15' = \left(\frac{15}{60}\right)^\circ$

$= \left(\frac{40+1}{4}\right)^\circ = \left(\frac{41}{4}\right)^\circ = \frac{41}{4} \times \frac{\pi}{180} = \frac{41\pi}{720}$ rad

(xi) $35^\circ 20'$

$= \left(35 + \frac{20}{60}\right)^\circ = \left(35 + \frac{1}{3}\right)^\circ = \left(\frac{105+1}{3}\right)^\circ$

$= \left(\frac{106}{3}\right)^\circ = \frac{106}{3} \times \frac{\pi}{180} = \frac{53\pi}{270}$ rad

(xii) $75^\circ 6' 30''$

$= \left[75 + \frac{6}{60} + \frac{30}{60 \times 60}\right]^\circ = \left(75 + \frac{1}{10} + \frac{1}{120}\right)^\circ$

$= \left(\frac{900+12+1}{120}\right)^\circ = \left(\frac{9013}{120}\right)^\circ = \frac{9013}{120} \times \frac{\pi}{180}$

$= \frac{9013}{21600}$ rad.

(xiii) $120'40''$

$= \left(\frac{120}{60} + \frac{40}{60\times60}\right)^\circ = \left(2 + \frac{1}{90}\right)^\circ = \left(\frac{180+1}{90}\right)^\circ$

$= \left(\frac{181}{90}\right)^\circ = \frac{181}{90} \times \frac{\pi}{180} = \frac{181\pi}{16200}\text{ rad}$

(xiv) $154^\circ 20''$

$= \left[154 + \frac{20}{60\times60}\right]^\circ = \left(154 + \frac{1}{180}\right)^\circ$

$= \left(\frac{27720+1}{180}\right)^\circ = \left(\frac{27721}{180}\right)^\circ = \frac{27721}{180} \times \frac{\pi}{180}$

$= \frac{27721\pi}{32400}\text{ rad}$

(xv) 0°

$= 0 \times \frac{\pi}{180} = 0\text{ rad}$

(xvi) $3''$

$= \left(\frac{3}{60\times60}\right)^\circ = \left(\frac{1}{1200}\right)^\circ = \frac{1}{1200} \times \frac{\pi}{180}$

$= \frac{\pi}{21600}\text{ rad}$

Q2. Convert the following radian measures of angles into the measures of sexagesimal system:

Solution:- i) $\frac{\pi}{8}$

$= \frac{\pi}{8} \times \frac{180^\circ}{\pi} = 22.5^\circ = 22^\circ + 0.5^\circ$

$= 22^\circ + (0.5\times60)' = 22^\circ 30'$ $\quad \because 1^\circ = 60'$

(ii) $\frac{\pi}{6}$ $\quad \because 1\text{ rad} = \frac{180^\circ}{\pi}$

$= \frac{\pi}{6} \times \frac{180^\circ}{\pi} = 30^\circ$

(iii) $\frac{\pi}{4}$

$= \frac{\pi}{4} \times \frac{180^\circ}{\pi} = 45^\circ$

(iv) $\frac{\pi}{3}$

$= \frac{\pi}{3} \times \frac{180^\circ}{\pi} = 60^\circ$

(v) $\frac{\pi}{2}$

$= \frac{\pi}{2} \times \frac{180^\circ}{\pi} = 90^\circ$

(vi) $\frac{2\pi}{3}$

$= \frac{2\pi}{3} \times \frac{180^\circ}{\pi} = 120^\circ$

(vii) $\frac{3\pi}{4}$

$= \frac{3\pi}{4} \times \frac{180^\circ}{\pi} = 135^\circ$

(viii) $\frac{5\pi}{6}$

$= \frac{5\pi}{6} \times \frac{180^\circ}{\pi} = 150^\circ$

(ix) $\frac{7\pi}{12}$

$= \frac{7\pi}{12} \times \frac{180^\circ}{\pi} = 105^\circ$

(x) $\frac{9\pi}{5}$

$= \frac{9\pi}{5} \times \frac{180^\circ}{\pi} = 324^\circ$

(xi) $\frac{11\pi}{27}$

$= \frac{11\pi}{27} \times \frac{180^\circ}{\pi} = \frac{220^\circ}{3} = 73.33^\circ = 73^\circ + 0.33^\circ$

$= 73^\circ + (0.33\times60)' = 73^\circ 20'$

(xii) $\frac{13\pi}{16}$

$= \frac{13\pi}{16} \times \frac{180^\circ}{\pi} = \left(\frac{585}{4}\right)^\circ = 146.25^\circ$

$= 146^\circ + 0.25^\circ = 146^\circ + 0.25\times60' = 146^\circ 15'$

(xiii) $\frac{17\pi}{24}$

$= \frac{17\pi}{24} \times \frac{180^\circ}{\pi} = \left(\frac{255}{2}\right)^\circ = 127.5^\circ$

$= 127 + (0.5\times60)' = 127^\circ 30'$

(xiv) $\frac{25\pi}{36}$

$= \frac{25\pi}{36} \times \frac{180^\circ}{\pi} = 125^\circ$

(xv) $\frac{19\pi}{32}$

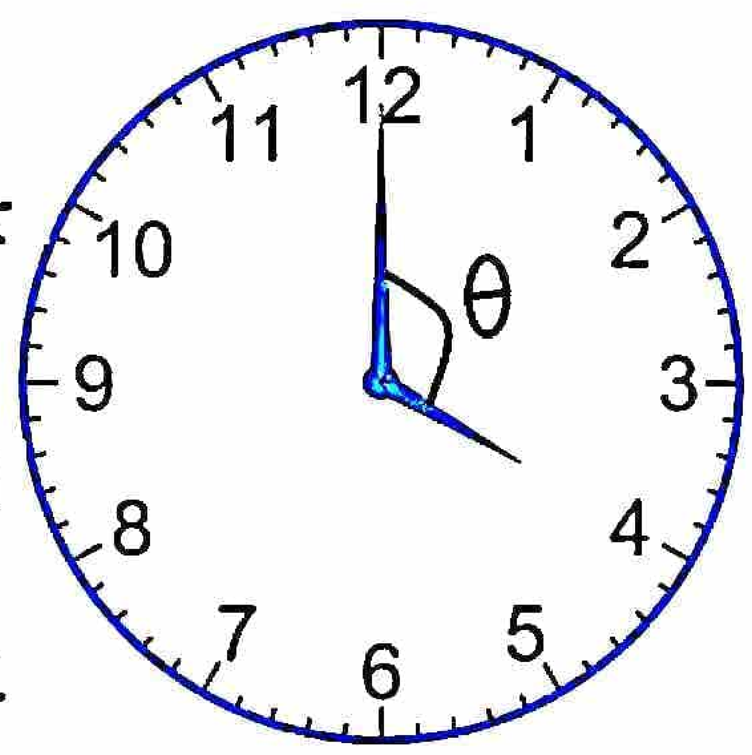
$= \frac{19\pi}{32} \times \frac{180^\circ}{\pi} = \left(\frac{855}{8}\right)^\circ = 106.875^\circ$

$= 106^\circ + (0.875\times60)' = 106^\circ 52.5'$

$= 106^\circ 52' + (0.5\times60'') = 106^\circ 52' 30''$

Q3. What is the circular measure of the angle between the hands of a watch at 4'o clock?

Solution:-



Angle traced in 12 hours $= 2\pi$

Angle traced in 1 hour $= \frac{2\pi}{12}$

$= \frac{\pi}{6}\text{ rad}$

Ang le b/w hands of watch at 4'O Clock $= 4 \times \frac{\pi}{6}$

$= \frac{2\pi}{3}\text{ rad}$

Q4. Find θ, when:
i) $l = 1.5cm$, $r = 2.5cm$
ii) $l = 3.2\ m$, $r = 2m$

Solution:- i) $l = 1.5cm$, $r = 2.5cm$

$\because l = r\theta \rightarrow \theta = \frac{l}{r} = \frac{1.25}{2.5} = 0.6\ rad$

ii) $l = 3.2\ m$, $r = 2m$

$\because l = r\theta \rightarrow \theta = \frac{l}{r} = \frac{3.2}{2} = 1.6\ rad$

Q5. Find l, when:
i) $\theta = \pi$ radians, $r = 6cm$
ii) $\theta = 65°20'$, $r = 18mm$

Solution:- i) $\theta = \pi$ radians, $r = 6cm$

$\because l = r\theta \rightarrow l = 6\pi = 6(3.14)$

$\rightarrow l = 18.86cm$

ii) $\theta = 65°20'$, $r = 18mm$

$\rightarrow \theta = \left(65 + \frac{20}{60}\right)° = \left(65 + \frac{1}{3}\right)°$

$= \left(\frac{195+1}{3}\right)° = \left(\frac{196}{3}\right)°$

$\theta = \frac{196}{3} \times 0.01745 = 1.14\ rad$

$\because l = r\theta = 18(1.14) = 20.57mm$

Q6. Find r, when:
i) $l = 5cm$, $\theta = \frac{1}{2}$ radian
ii) $l = 56cm$, $\theta = 45°$

Solution:- i) $l = 5cm$, $\theta = \frac{1}{2}rad$

$\because l = r\theta \rightarrow r = \frac{l}{\theta} = \frac{5}{\frac{1}{2}} = 10\ cm$

ii) $l = 56cm$, $\theta = 45°$

$\theta = 45 \times 0.01745 = 0.7852\ rad$

$\because l = r\theta \rightarrow r = \frac{l}{\theta} = \frac{56}{0.7852}$

$\rightarrow r = 71.31cm$

Q7. What is the length of the arc intercepted on a circle of radius 14cms by the arms of a central angle of 45°?

Solution:- Here $\theta = 45° = 45 \times 0.01745$
$= 0.7852\ rad$

$r = 14cms$, $l = ?$

$\because l = r\theta \rightarrow l = 14 \times 0.7852$

$l = 10.99cm$

Q8. Find the radius of the circle, in which the arms of a central angle of measure 1 radian cut off an arc of length 35 cm

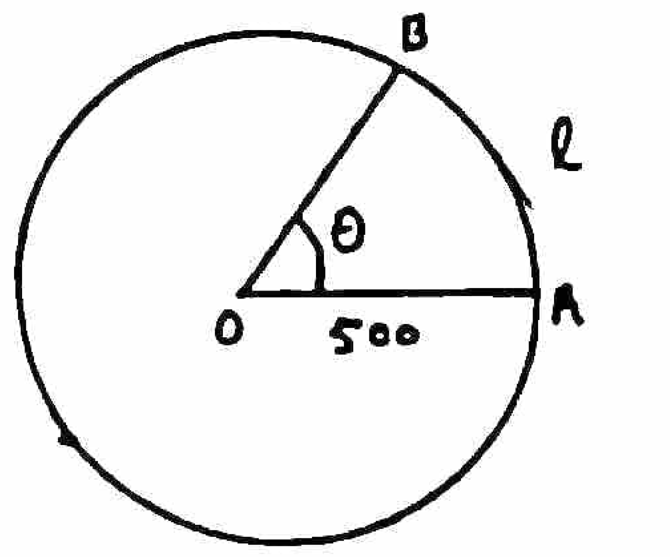
Solution:- $l = 35cm$, $\theta = 1\ rad$
$r = ?$

$\because l = r\theta \rightarrow r = \frac{l}{\theta} = \frac{35}{1}$

$\rightarrow r = 35cm$

Q9. A railway train is running on a circular track of radius 500 meters at the rate of 30km per hour. Through what angle will it turn in 10 sec?

Solution:- Here



$V = 30\ km/h$, $t = 10sec$

$v = \frac{30 \times 1000}{60 \times 60} = \frac{25}{3} m/s$

$r = 500m$, $\theta = ?$

$\because s = l = v \times t$

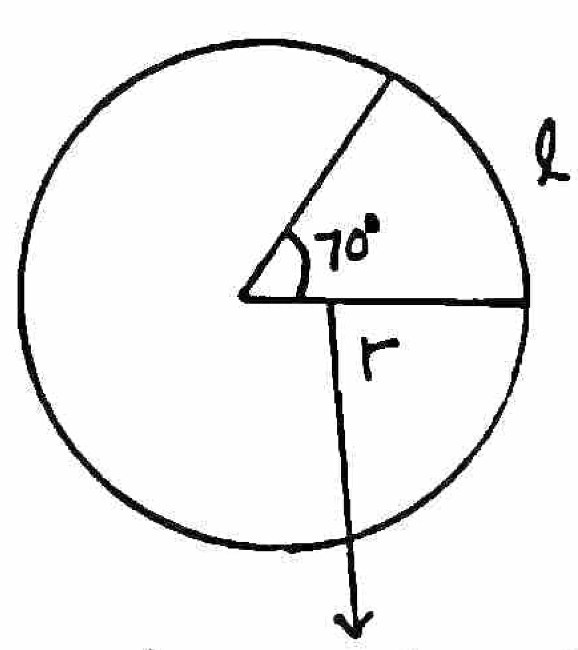
$\rightarrow l = \frac{25}{3} \times 10 = \frac{250}{3}$

$\because l = r\theta \rightarrow \theta = \frac{l}{r} = \frac{\frac{250}{3}}{500} = \frac{250}{3} \times \frac{1}{500}$

$\rightarrow \theta = \frac{1}{6}$ radian

Q10. A hourse is tethered to a peg by a rope of 9 meters length and it can move in a circle with the peg as centre. If the horse moves along the circumference of the circle, keeping the rope tight, how far will it have gone when the rope has turned through an angle of 70°?

Solution:- Here



(length of rope become radius)

$r = 9m$, $\theta = 70°$, $l = ?$

$\theta = 70 \times 0.01745$

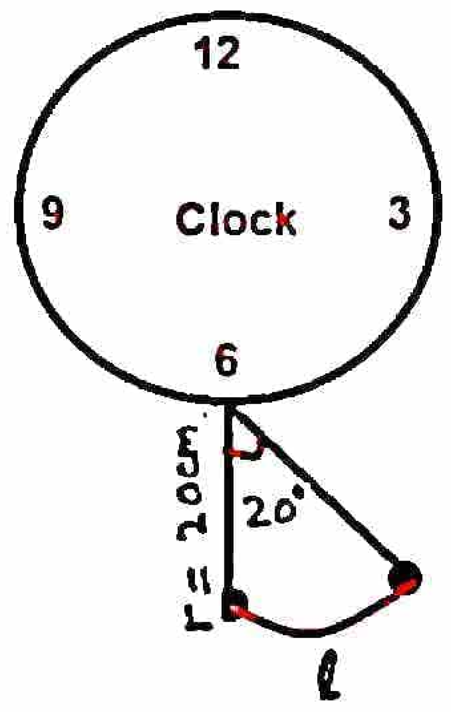
$\theta = 1.22\ rad$

$\because l = r\theta$

$\rightarrow l = 8 \times 1.22 = 10.99m$

Q11. The pendulum of a clock is 20cm long and it swings through an angle of 20° each second. How far does the tip of the pendulum move in 1 second?

Solution:- Here



$\theta = 20° = 20 \times \frac{\pi}{180}$

$\rightarrow \theta = \frac{\pi}{9}$ rad

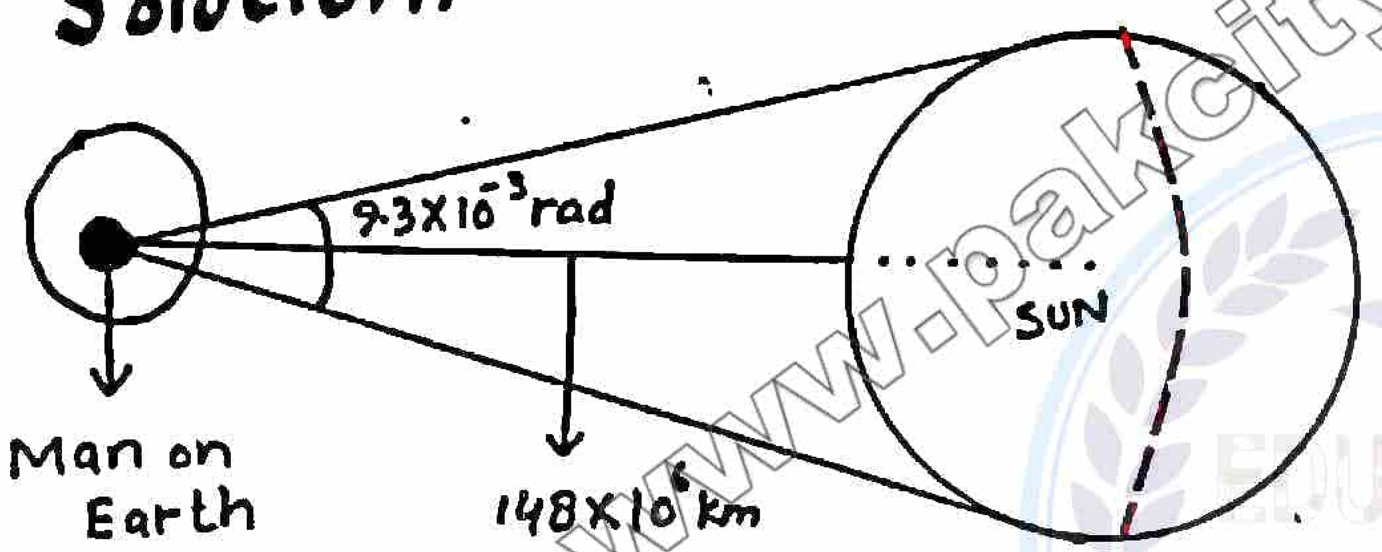
$r = 20cm, \quad l = ?$

$\because l = r\theta$

$\rightarrow l = 20 \times \frac{\pi}{9} = 6.98cm$

Q12. Assuming the average distance of the earth from the sun to be 148×10^6 km and the angle subtended by the sun at the eye of a person on the earth of measure 9.3×10^{-3} rad. Find the diameter of the sun.

Solution:-



Here

$r = 148 \times 10^6$ km, $\quad \theta = 9.3 \times 10^{-3}$ rad

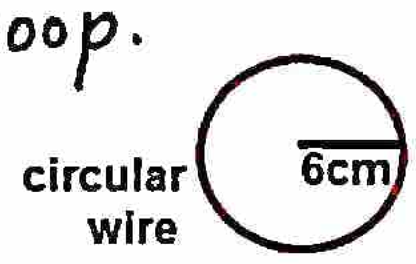
$l = ?$

$\because l = r\theta \rightarrow l = (148 \times 10^6)(9.3 \times 10^{-3})$

$\rightarrow l = 1376400$ km

Q13. A circular wire of radius 6cm is cut straightened and then bent so as to lie along the circumference of a hoop of radius 24cm. Find the measure of angle which it subtends at the centre of the hoop.

Solution:- Here



$r = 24cm$ (of Hoop)

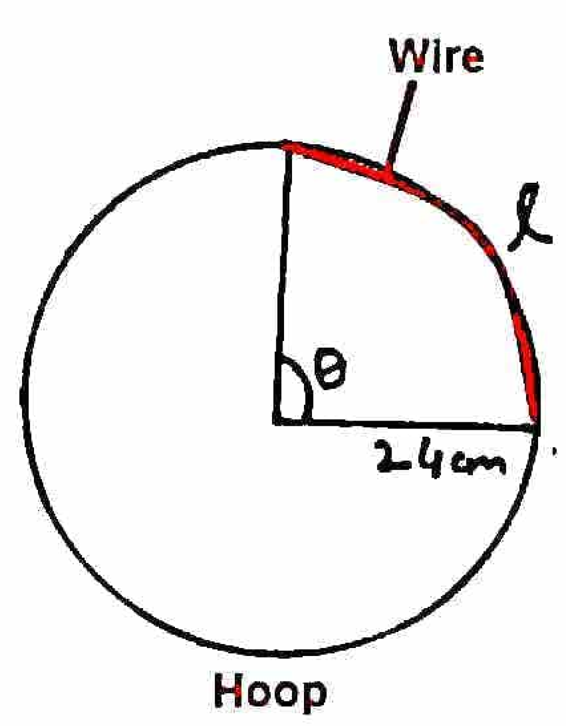
$r = 6cm$ (of circle)

$\theta = ?$

$\because l = 2\pi r$

$= 2\pi(6)$

$l = 12\pi$



$\because l = r\theta$

$\rightarrow \theta = \frac{l}{r} = \frac{12\pi}{24} = \frac{\pi}{2}$ rad

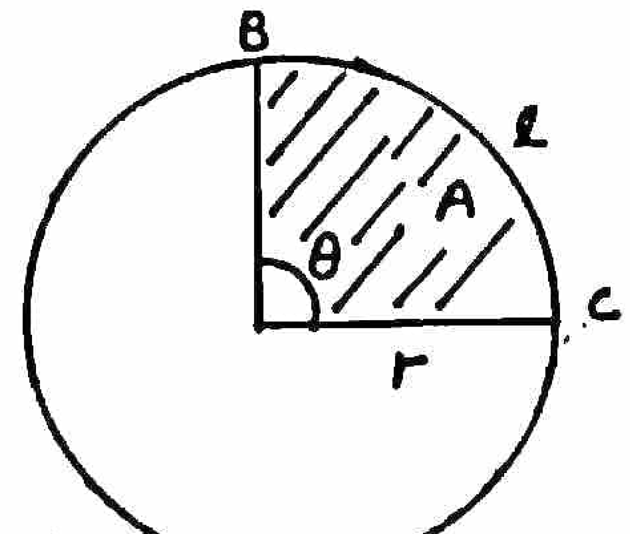
Q14. Show that the area of a sector of a circular region of radius r is $\frac{1}{2}r^2\theta$, where θ is circular measure of the central angle of the sector.

Solution:-

Let A = Area of sector

θ = central angle

r = radius



We know by elementary geometry that

Area of sector : Area of circle = θ : 2π

$\rightarrow \frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\theta}{2\pi}$

$\rightarrow \frac{A}{\pi r^2} = \frac{\theta}{2\pi} \quad \rightarrow A = \frac{\theta}{2\pi} \times \pi r^2$

$\rightarrow A = \frac{1}{2} r^2 \theta \quad$ Hence proved

Q15. Two cities A and B lie on the equator such that their longitudes are 45°E and 25°W respectively. Find the distance between the two cities, taking radius of the earth as 6400 kms.

Solution:- Here

$\theta = 45° + 25° = 70°$

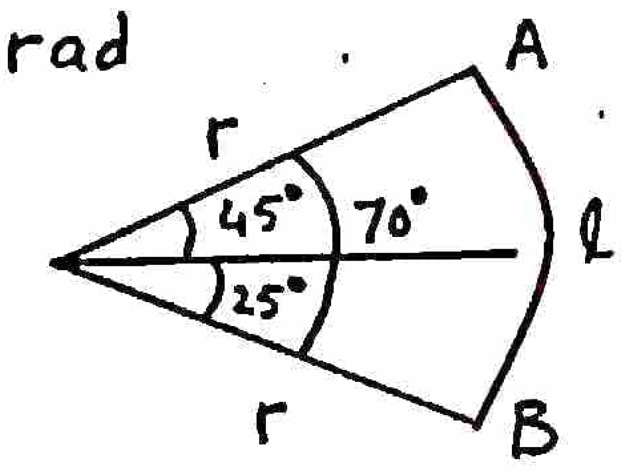
$= 70 \times 0.01745 = 1.2217$ rad

$r = 6400$ km, $l = ?$

$\because l = r\theta$

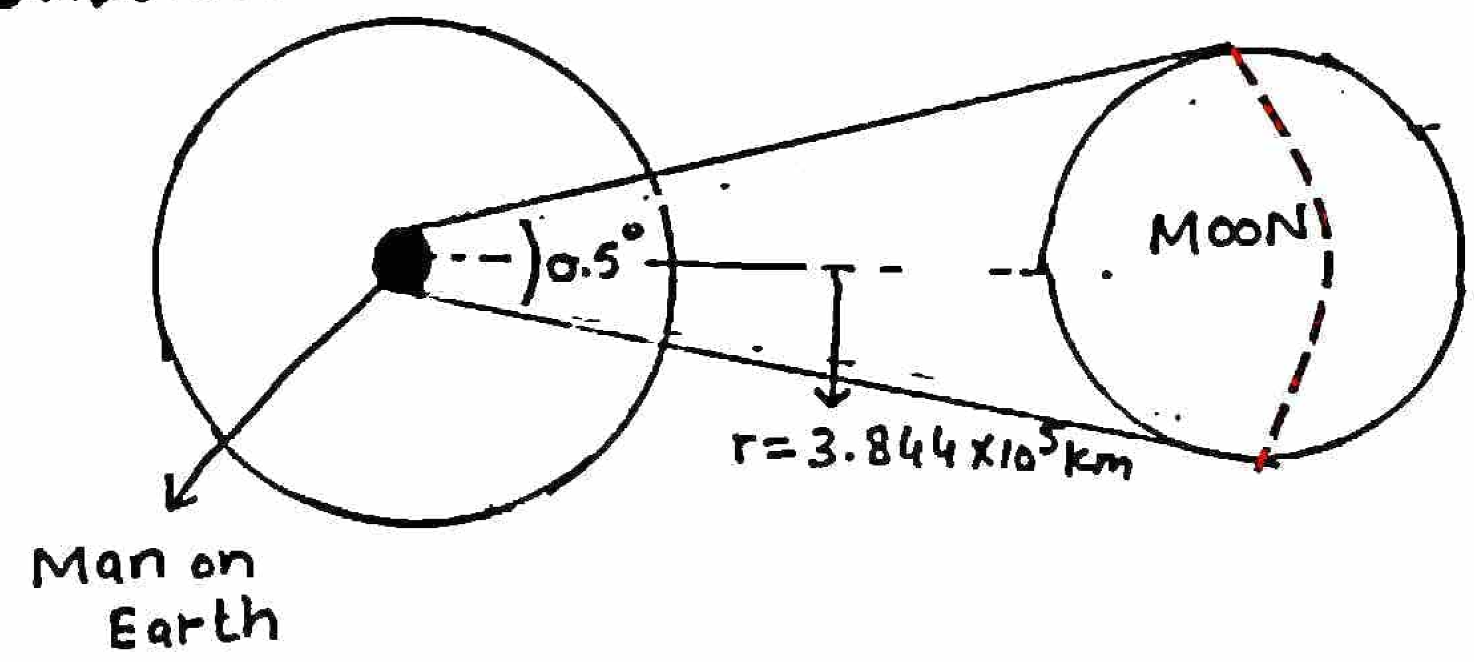
$= (6400)(1.2217)$

$l = 7819$ kms



Q16. The moon subtends an angle of 0.5° at the eye of an observer on earth. The distance of the moon from the earth is 3.844×10^5 km approx. What is the length of the diameter of the moon?

Solution:-



Here $\theta = 0.5^\circ = 0.5 \times 0.01745$
$= 0.008726\,rad$

$r = 3.844 \times 10^5\,km$, $l = ?$

$\because l = r\theta \rightarrow l = (3.844 \times 10^5)(0.008726)$

$\rightarrow l = 3554\,km$

Q17. The angle subtended by the earth at the eye of a spaceman, landed on the moon, is $1^\circ 54'$. The radius of the earth is 6400 km. Find the approximate distance between the moon and the earth.

Solution:- Here

$$\theta = 1^\circ 54' = \left(1 + \frac{54}{60}\right)^\circ = \left(\frac{60+54}{60}\right)^\circ$$

$$= \left(\frac{114}{60}\right)^\circ = \frac{114}{60} \times 0.01745$$

$\theta = 0.033\,rad$

Diameter of earth $= l = 2r$

$\rightarrow l = 2 \times 6400 = 12800\,km$

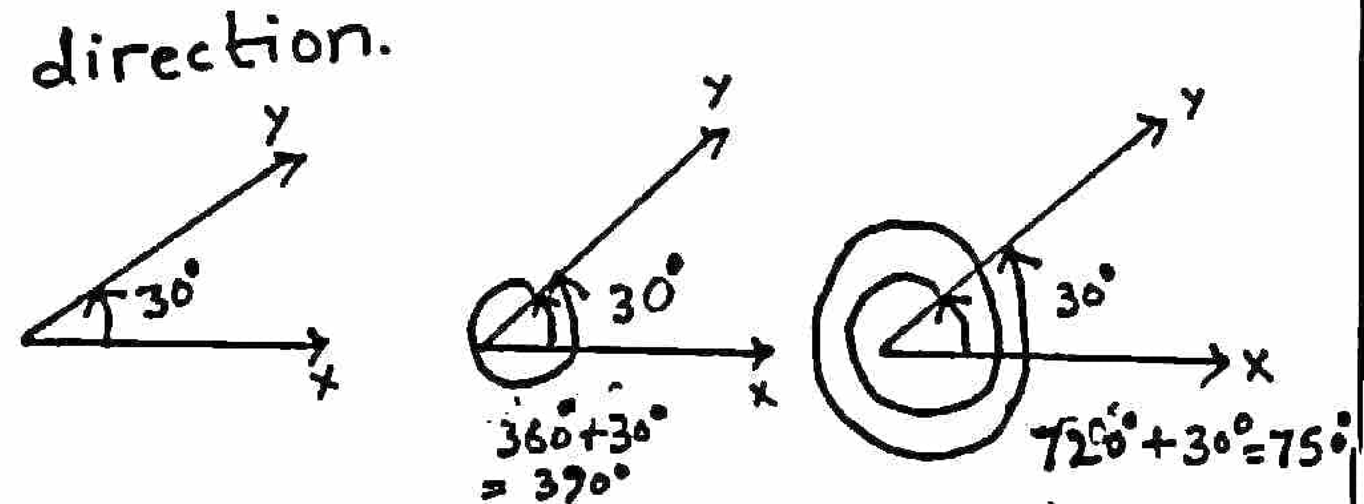
Distance between moon and earth $= r = ?$

$\because l = r\theta$

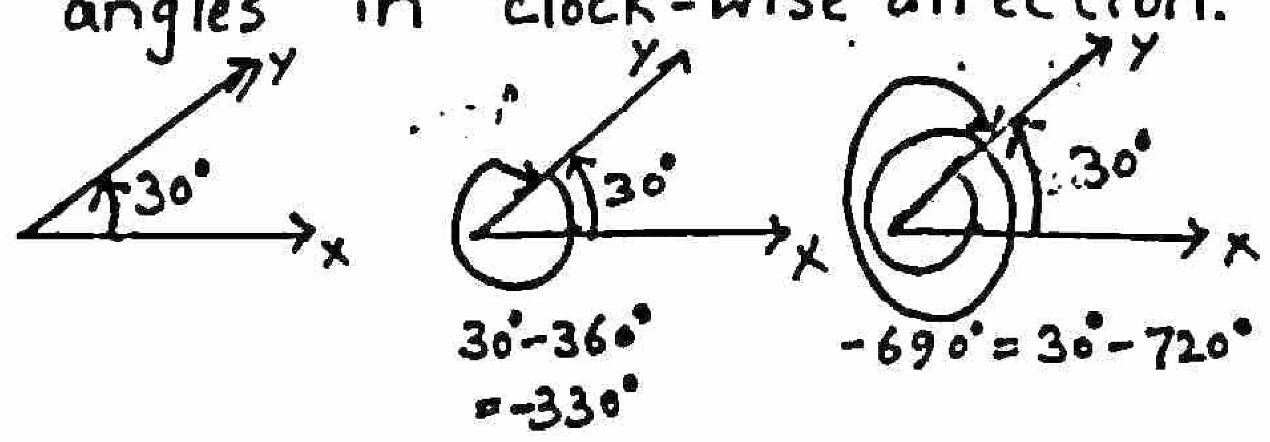
$\rightarrow \theta = \frac{l}{r} = \frac{12800}{0.033} = 385992.6\,km$

General Angle
(Coterminal angles)

The angles having same initial and terminal sides are called coterminal angles. e.g., 30°, 390°, 750° are coterminal angles in anti-clockwise direction.



Also 30°, -330°, -690° are coterminal angles in clock-wise direction.



Note:- In general, if angle θ is in degrees, then $\theta + 360k$, $k \in Z$ is an angle coterminal with θ

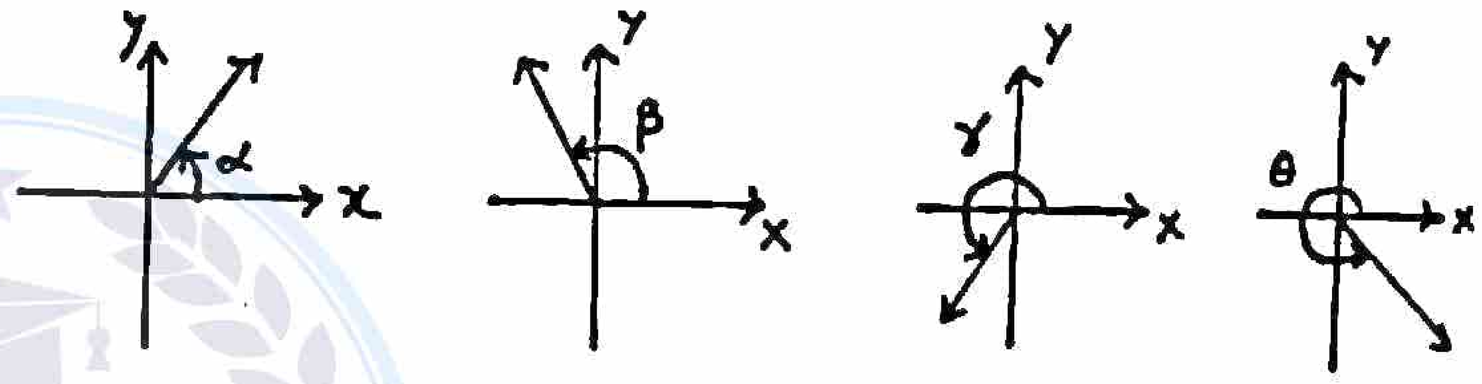
* If angle θ is in radians, then $\theta + 2k\pi$, $k \in Z$, is an angle coterminal with θ

$$\boxed{\rightarrow \text{General angle is } \theta + 2k\pi,\ k \in Z}$$

* Trignometric functions or trignometric ratios of coterminal angles are same.

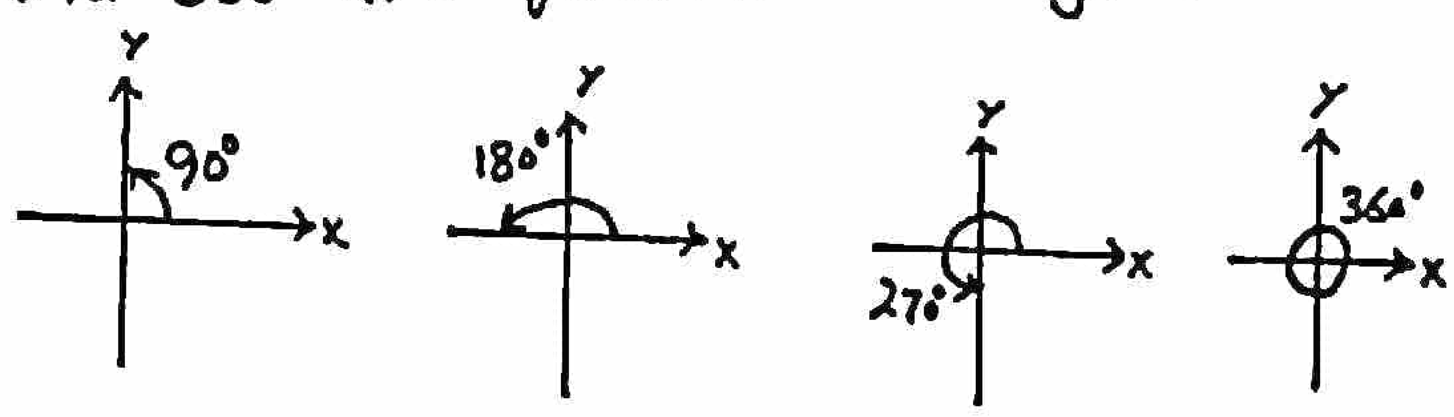
Angle In the Standard Position

An angle is said to be in standard position if its vertex lies at the origin of a rectangular coordinate system and its initial side along the positive x-axis.



Angles α, β, γ and θ are in standard position.

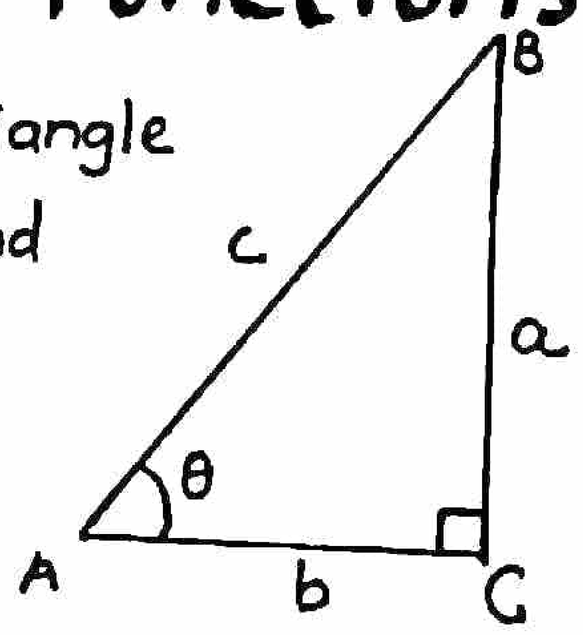
Quadrantal angles:- If the terminal side of an angle falls on x-axis or y-axis, it is called a quadrantal angle. i.e., 90°, 180°, 270° and 360° are quadrantal angles.



Trigonometric Functions

Consider a right triangle ABC with $\angle C = 90^\circ$ and sides a, b, c.

Let $m\angle A = \theta$ radian



The side AB opposite to 90° is called hypotenuse (hyp)

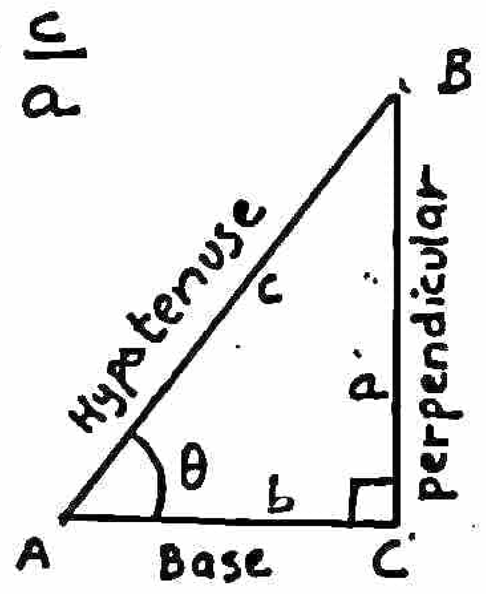
The side BC opposite to θ is called perpendicular (per) and the side AC related to angle θ is called adjacent (adj) or Base.

- Trigonometric functions of angle θ are defined as below:

$\sin\theta = \frac{a}{c}$; $\operatorname{cosec}\theta = \frac{c}{a}$

$\cos\theta = \frac{b}{c}$; $\sec\theta = \frac{c}{b}$

$\tan\theta = \frac{a}{b}$; $\cot\theta = \frac{b}{a}$



We observe that

$\csc\theta = \frac{1}{\sin\theta}$ or $\sin\theta = \frac{1}{\csc\theta}$

$\sec\theta = \frac{1}{\cos\theta}$ or $\cos\theta = \frac{1}{\sec\theta}$

$\tan\theta = \frac{1}{\cot\theta}$ or $\cot\theta = \frac{1}{\tan\theta}$

Also $\tan\theta = \frac{\sin\theta}{\cos\theta}$ or $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Fundamental Identities

For any real number θ,

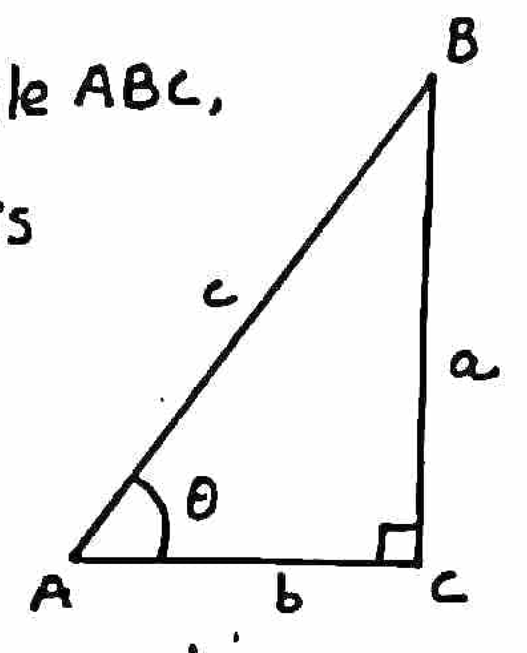
i) $\sin^2\theta + \cos^2\theta = 1$

ii) $1 + \tan^2\theta = \sec^2\theta$

iii) $1 + \cot^2\theta = \csc^2\theta$

Proof:- (i)

For any right triangle ABC, Pythagora's therem is



$(Base)^2 + (perp)^2 = (Hyp)^2$

$\rightarrow b^2 + a^2 = c^2 \longrightarrow (i)$

Dividing (i) by c^2, we get

$\rightarrow \frac{b^2}{c^2} + \frac{a^2}{c^2} = \frac{c^2}{c^2}$

$\rightarrow \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = 1$

$\rightarrow (\cos\theta)^2 + (\sin\theta)^2 = 1$

$\rightarrow \boxed{\sin^2\theta + \cos^2\theta = 1}$

(ii) Dividing (i) by b^2, we get

$\frac{b^2}{b^2} + \frac{a^2}{b^2} = \frac{c^2}{b^2}$

$\rightarrow 1 + \left(\frac{a}{b}\right)^2 = \left(\frac{c}{b}\right)^2$

$\rightarrow 1 + (\tan\theta)^2 = (\sec\theta)^2$

$\rightarrow \boxed{1 + \tan^2\theta = \sec^2\theta}$

(iii) Dividing (i) by a^2, we get

$\frac{b^2}{a^2} + \frac{a^2}{a^2} = \frac{c^2}{a^2}$

$\rightarrow \left(\frac{b}{a}\right)^2 + 1 = \left(\frac{c}{a}\right)^2$

$\rightarrow (\cot\theta)^2 + 1 = (\csc\theta)^2$

$\rightarrow \boxed{1 + \cot^2\theta = \csc^2\theta}$

Important note:-

(i) $\sin^2\theta + \cos^2\theta = 1 \rightarrow \sin^2\theta = 1 - \cos^2\theta$

$\rightarrow \sin\theta = \pm\sqrt{1-\cos^2\theta}$

or $\cos^2\theta = 1 - \sin^2\theta \rightarrow \cos\theta = \pm\sqrt{1-\sin^2\theta}$

(ii) $1 + \tan^2\theta = \sec^2\theta \rightarrow 1 = \sec^2\theta - \tan^2\theta$

$\rightarrow \tan^2\theta = \sec^2\theta - 1 \rightarrow \tan\theta = \pm\sqrt{\sec^2\theta - 1}$

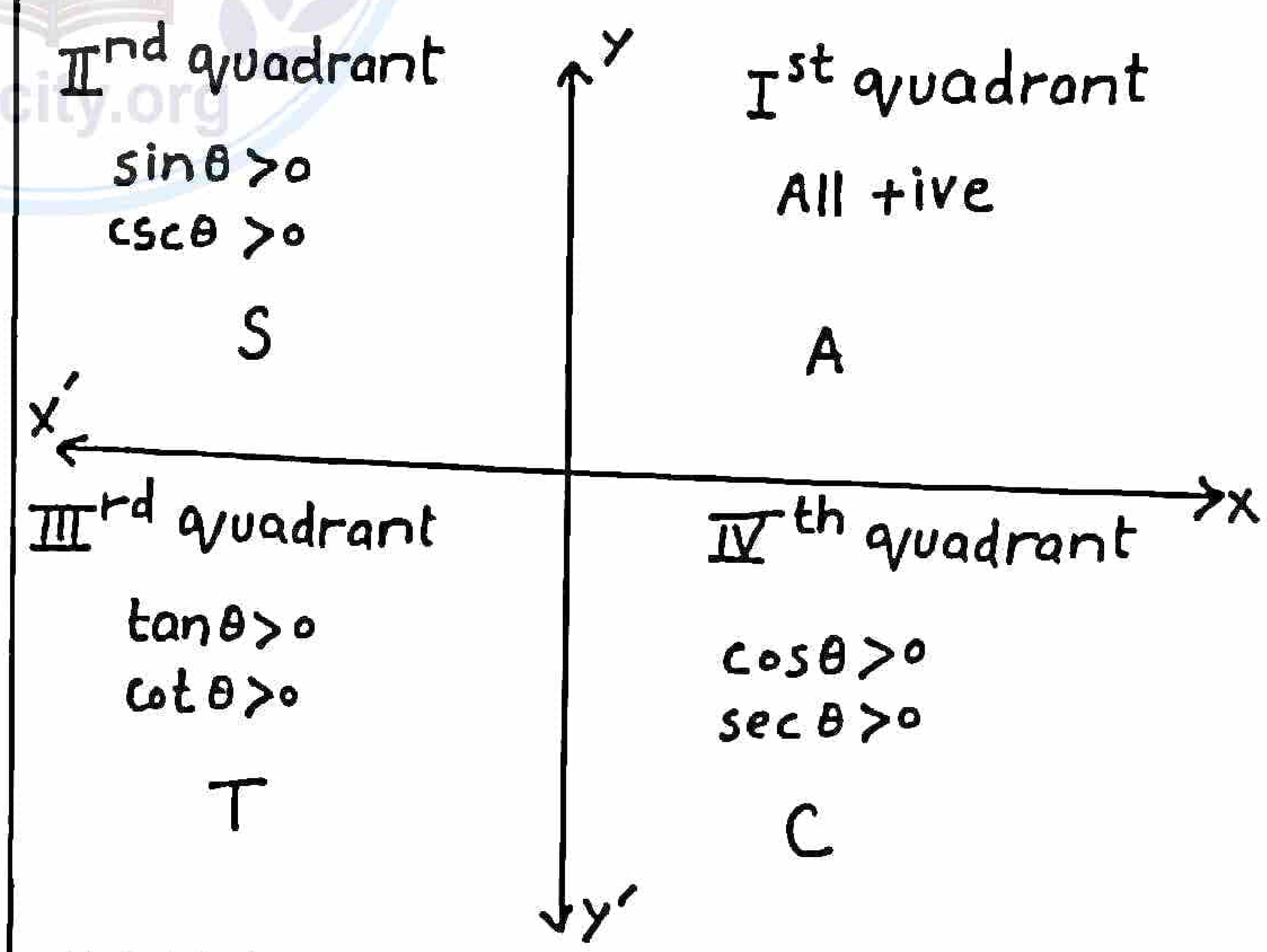
or $\sec\theta = \pm\sqrt{1+\tan^2\theta}$

(iii) $1 + \cot^2\theta = \csc^2\theta \rightarrow 1 = \csc^2\theta - \cot^2\theta$

$\rightarrow \cot^2\theta = \csc^2\theta - 1 \rightarrow \cot\theta = \pm\sqrt{\csc^2\theta - 1}$

or $\csc\theta = \pm\sqrt{1-\cot^2\theta}$

Signs of the Trigonometric functions



CAST

C for $\cos\theta$ and its reciprocal $\sec\theta$

A for all trigonometric functions

S for $\sin\theta$ and its reciprocal $\csc\theta$

T for $\tan\theta$ and its reciprocal $\cot\theta$

It is clear from the above figure that $\sin(-\theta)=-\sin\theta$; $\csc(-\theta)=-\csc\theta$

$\cos(-\theta)=\cos\theta$; $\sec(-\theta)=\sec\theta$

$\tan(-\theta)=-\tan\theta$; $\cot(-\theta)=-\cot\theta$

Example 1. If $\tan\theta=\frac{8}{15}$ and the terminal arm of the angle is in the III quadrant, find the values of the other trigonometric functions of θ.

Solution:- $\tan\theta=\frac{8}{15}$

$\Rightarrow \cot\theta=\frac{15}{8}$ $\left(\because \cot\theta=\frac{1}{\tan\theta}\right)$

$\because \sec^2\theta = 1+\tan^2\theta$

$\Rightarrow \sec\theta=\pm\sqrt{1+\tan^2\theta}=\pm\sqrt{1+\left(\frac{8}{15}\right)^2}$

$\sec\theta=\pm\sqrt{1+\frac{64}{225}}=\pm\sqrt{\frac{225+64}{225}}$

$\sec\theta=\pm\sqrt{\frac{289}{225}}=\pm\frac{17}{15}$

$\Rightarrow \sec\theta=-\frac{17}{15}$ $(\because \theta \text{ in III quad})$

Now $\cos\theta=-\frac{15}{17}$ $\left(\because \cos\theta=\frac{1}{\sec\theta}\right)$

$\because \sin^2\theta+\cos^2\theta=1$

$\Rightarrow \sin^2\theta=1-\cos^2\theta$

$\Rightarrow \sin\theta=\pm\sqrt{1-\cos^2\theta}$

$=\pm\sqrt{1-\left(-\frac{15}{17}\right)^2}=\pm\sqrt{1-\frac{225}{289}}$

$=\pm\sqrt{\frac{289-225}{289}}=\pm\sqrt{\frac{64}{289}}$

$\sin\theta=\pm\frac{8}{17}$ $\Rightarrow \sin\theta=-\frac{8}{17}$ $(\theta \text{ in III quad})$

so $\csc\theta=-\frac{17}{8}$ $\left(\because \csc\theta=\frac{1}{\sin\theta}\right)$

Example 2. Find the value of other five trigonometric functions of θ, $\cos\theta=\frac{12}{13}$ and the terminal side of the angle is not in the I quadrant.

Solution:- $\cos\theta=\frac{12}{13}$ $(\theta \text{ not in I quad})$

$\because$ $\cos\theta$ is +ive in I and IV quadrant. but given that θ not in I quad. so it means θ is in IV quad.

Now $\cos\theta=\frac{12}{13}$

$\Rightarrow \sec\theta=\frac{13}{12}$ $\left(\because \sec\theta=\frac{1}{\cos\theta}\right)$

$\because \sin^2\theta+\cos^2\theta=1$

$\Rightarrow \sin\theta=\pm\sqrt{1-\cos^2\theta}$

$=\pm\sqrt{1-\left(\frac{12}{13}\right)^2}=\pm\sqrt{1-\frac{144}{169}}$

$=\pm\sqrt{\frac{169-144}{169}}=\pm\sqrt{\frac{25}{169}}$

$\Rightarrow \sin\theta=\pm\frac{5}{13}$

$\Rightarrow \sin\theta=-\frac{5}{13}$ $(\because \theta \text{ is in IV quad})$

$\csc\theta=-\frac{13}{5}$ $\left(\because \csc\theta=\frac{1}{\sin\theta}\right)$

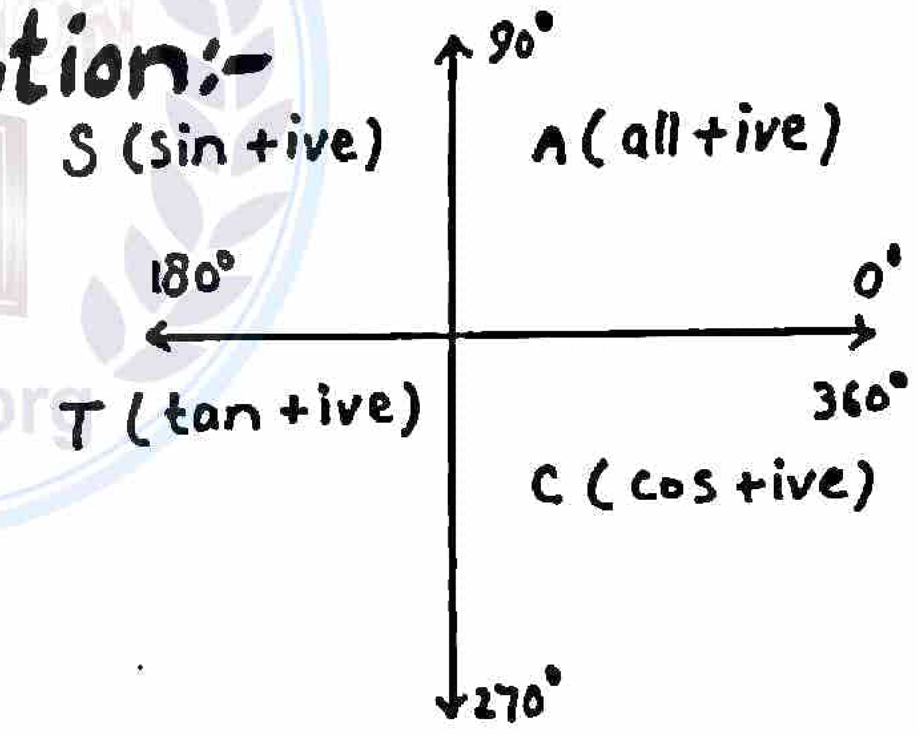
$\because \tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{-5/13}{12/13}=-\frac{5}{12}$

and $\cot\theta=-\frac{12}{5}$ $\left(\because \cot\theta=\frac{1}{\tan\theta}\right)$

Exercise 9.2

Q1. Find the signs of the following (i) $\sin 160^\circ$ (ii) $\cos 190^\circ$ (iii) $\tan 115^\circ$ (iv) $\sec 245^\circ$ (v) $\cot 80^\circ$ (vi) $\operatorname{cosec} 297^\circ$

Solution:-



(i) $\sin 160^\circ$ + ive (iv) $\sec 245^\circ$ − ive

(ii) $\cos 190^\circ$ − ive (v) $\cot 80^\circ$ + ive

(iii) $\tan 115^\circ$ − ive (vi) $\operatorname{cosec} 297^\circ$ − ive

Q2. Fill in the blanks:

Solution:-

(i) $\sin(-310^\circ)=\ldots - \ldots \sin 310^\circ$

(ii) $\cos(-75^\circ)=\ldots + \ldots \cos 75^\circ$

(iii) $\tan(-182^\circ)=\ldots - \ldots \tan 182^\circ$

(iv) $\cot(-137^\circ)=\ldots - \ldots \cot 137^\circ$

(v) $\sec(-216^\circ)=\ldots + \ldots \sec 216^\circ$

(vi) $\operatorname{cosec}(-15^\circ)=\ldots - \ldots \operatorname{cosec} 15^\circ$

Q3. In which quadrant are the terminal arms of the angle lie when

Solution:- (i) $\sin\theta < 0$ and $\cos\theta > 0$

$\rightarrow$ terminal arm lies in IV quad.

(ii) $\cot\theta > 0$ and $\operatorname{cosec}\theta > 0$

$\rightarrow$ Terminal arm lies in I quad.

(iii) $\tan\theta < 0$ and $\cos\theta > 0$

$\rightarrow$ Terminal arm lies in IV quad.

(iv) $\sec\theta < 0$ and $\sin\theta < 0$

$\rightarrow$ Terminal arm lies in III quad.

(v) $\cot\theta > 0$ and $\sin\theta < 0$

$\rightarrow$ Terminal arm lies in III quad.

(vi) $\cos\theta < 0$ and $\tan\theta < 0$

$\rightarrow$ Terminal arm lies in II quad.

Q4. Find the values of the remaining trigonometric functions:

(i) $\sin\theta = \frac{12}{13}$ and the terminal arm of the angle is in quad. I.

Solution:- $\sin\theta = \frac{12}{13}$ (θ is in I quad)

$\rightarrow \csc\theta = \frac{13}{12}$

$\because \cos\theta = \pm\sqrt{1-\sin^2\theta}$

$= \pm\sqrt{1-\left(\frac{12}{13}\right)^2}$

$= \pm\sqrt{1-\frac{144}{169}} = \pm\sqrt{\frac{169-144}{169}}$

$\cos\theta = \pm\sqrt{\frac{25}{169}} = \pm\frac{5}{13}$

$\rightarrow \cos\theta = \frac{5}{13}$ (θ is in I quad)

$\sec\theta = \frac{13}{5}$

$\because \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{12/13}{5/13} = \frac{12}{5}$

$\rightarrow \cot\theta = \frac{5}{12}$

(ii) $\cos\theta = \frac{9}{41}$ and the terminal arm of the angle is in quad IV.

Solution:- $\cos\theta = \frac{9}{41}$ (θ is in IV quad)

$\sec\theta = \frac{41}{9}$

$\because \sin\theta = \pm\sqrt{1-\cos^2\theta}$

$= \pm\sqrt{1-\left(\frac{9}{41}\right)^2} = \pm\sqrt{1-\frac{81}{1681}}$

$= \pm\sqrt{\frac{1681-81}{1681}} = \pm\sqrt{\frac{1600}{1681}}$

$\sin\theta = \pm\frac{40}{41} \rightarrow \sin\theta = -\frac{40}{41}$ (θ is in IV quad)

$\rightarrow \csc\theta = \frac{-41}{40}$

$\because \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-40/41}{9/41} = \frac{-40}{9}$

and $\cot\theta = -\frac{9}{40}$ $\quad \because \cot\theta = \frac{1}{\tan\theta}$

(iii) $\cos\theta = -\frac{\sqrt{3}}{2}$ and the terminal arm of the angle is in quad III.

Solution:- $\cos\theta = -\frac{\sqrt{3}}{2}$ (θ is in III quad)

$\rightarrow \sec\theta = -\frac{2}{\sqrt{3}}$

$\because \sin\theta = \pm\sqrt{1-\cos^2\theta}$

$= \pm\sqrt{1-\left(\frac{-\sqrt{3}}{2}\right)^2} = \pm\sqrt{1-\frac{3}{4}}$

$\sin\theta = \pm\sqrt{\frac{4-3}{4}} = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

$\rightarrow \sin\theta = -\frac{1}{2}$ ($\because \theta$ is in III quad)

$\rightarrow \csc\theta = -\frac{2}{1} = -2$ ($\because \csc\theta = \frac{1}{\sin\theta}$)

$\because \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}}$

$\cot\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$

(iv) $\tan\theta = -\frac{1}{3}$ and the terminal arm of the angle is in quad II.

Solution:- $\tan\theta = \frac{-1}{3}$ (θ is in II quad)

$\rightarrow \cot\theta = -\frac{3}{1} = -3$

$\because \sec\theta = \pm\sqrt{1+\tan^2\theta}$

$= \pm\sqrt{1+\left(-\frac{1}{3}\right)^2}$

$\Rightarrow \sec\theta = \pm\sqrt{1+\frac{1}{9}} = \pm\sqrt{\frac{9+1}{9}}$

$\sec\theta = \pm\frac{\sqrt{10}}{3}$

$\Rightarrow \sec\theta = -\frac{\sqrt{10}}{3}$ ($\because \theta$ is in II quad)

$\Rightarrow \cos\theta = -\frac{3}{\sqrt{10}}$ $\quad \because \cos\theta = \frac{1}{\sec\theta}$

$\because \tan\theta = \frac{\sin\theta}{\cos\theta}$

$\Rightarrow \sin\theta = \tan\theta \cdot \cos\theta$

$\sin\theta = (-\frac{1}{3})(-\frac{3}{\sqrt{10}}) = \frac{1}{\sqrt{10}}$

$\therefore \csc\theta = \frac{\sqrt{10}}{1} = \sqrt{10}$

(v) $\sin\theta = -\frac{1}{\sqrt{2}}$ and the terminal arm of the angle is not in quad III.

Solution:- $\sin\theta = \frac{-1}{\sqrt{2}}$ (θ is not in III quad)

$\because \sin\theta$ is $-$ive in III and IV quad but given that θ is not in quad III so θ is in IV quad.

Now $\csc\theta = -\frac{\sqrt{2}}{1} = -\sqrt{2}$

$\because \cos\theta = \pm\sqrt{1-\sin^2\theta}$

$= \pm\sqrt{1-(\frac{-1}{\sqrt{2}})^2} = \pm\sqrt{1-\frac{1}{2}}$

$= \pm\sqrt{\frac{2-1}{2}} = \pm\sqrt{\frac{1}{2}}$

$\cos\theta = \pm\frac{1}{\sqrt{2}}$ $\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$ (θ is in IV quad)

$\Rightarrow \sec\theta = \frac{\sqrt{2}}{1} = \sqrt{2}$

$\because \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \frac{-1}{1} = -1$

and $\cot\theta = -1$ $\quad (\because \cot\theta = \frac{1}{\tan\theta})$

Q5. If $\cot\theta = \frac{15}{8}$ and the terminal arm of the angle is not in quad. I, find the values of $\cos\theta$ and $\operatorname{cosec}\theta$.

Solution:- $\because \cot\theta = \frac{15}{8}$ (θ is not in I quad)

$\cot\theta$ is $+$ive in I and III quad but given that θ is not in I so θ is in III. quad.

Now $\tan\theta = \frac{8}{15}$ $\quad (\because \tan\theta = \frac{1}{\cot\theta})$

$\because \sec^2\theta = 1+\tan^2\theta$

$= 1+(\frac{8}{15})^2$

$\sec^2\theta = 1+\frac{64}{225} = \frac{225+64}{225}$

$\Rightarrow \sec^2\theta = \frac{289}{225}$

$\Rightarrow \sec\theta = \pm\sqrt{\frac{289}{225}} = \pm\frac{17}{15}$

$\sec\theta = -\frac{17}{15}$ ($\because \theta$ is in III quad)

$\Rightarrow \cos\theta = -\frac{15}{17}$

$\because \sin\theta = \pm\sqrt{1-\cos^2\theta}$

$= \pm\sqrt{1-(\frac{-15}{17})^2} = \pm\sqrt{1-\frac{225}{289}}$

$= \pm\sqrt{\frac{289-225}{289}} = \pm\sqrt{\frac{64}{289}}$

$\sin\theta = \pm\frac{8}{17}$ $\Rightarrow \sin\theta = -\frac{8}{17}$ (θ is in III quad)

$\Rightarrow \operatorname{cosec}\theta = -\frac{17}{8}$

Q6. If $\operatorname{cosec}\theta = \frac{m^2+1}{2m}$ and $m>0$, $(0<\theta<\frac{\pi}{2})$, find the values of the remaining trignometric ratios.

Solution:- $\operatorname{cosec}\theta = \frac{m^2+1}{2m}$

$\because 0<\theta<\frac{\pi}{2} \Rightarrow \theta$ is in I quad.

$\Rightarrow \sin\theta = \frac{2m}{m^2+1}$ $\quad (\because \sin\theta = \frac{1}{\csc\theta})$

$\because \cos\theta = \pm\sqrt{1-\sin^2\theta}$

$= \pm\sqrt{1-(\frac{2m}{m^2+1})^2}$

$$\cos\theta = \pm\sqrt{1-\frac{4m^2}{(m^2+1)^2}}$$

$$= \pm\sqrt{\frac{(m^2+1)^2-4m^2}{(m^2+1)^2}}$$

$$= \pm\sqrt{\frac{m^4+1+2m^2-4m^2}{(m^2+1)^2}}$$

$$= \pm\sqrt{\frac{m^4+1-2m^2}{(m^2+1)^2}}$$

$$\cos\theta = \pm\sqrt{\frac{(m^2-1)^2}{(m^2+1)^2}}$$

$$\rightarrow \cos\theta = \pm\frac{m^2-1}{m^2+1}$$

$$\rightarrow \cos\theta = \frac{m^2-1}{m^2+1} \quad (\because \theta \text{ is in I quad})$$

Also $\sec\theta = \dfrac{m^2+1}{m^2-1}$

$$\because \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{2m}{m^2+1}}{\frac{m^2-1}{m^2+1}} = \frac{2m}{m^2-1}$$

$$\rightarrow \tan\theta = \frac{2m}{m^2-1} \text{ and } \cot\theta = \frac{m^2-1}{2m}$$

Q7. If $\tan\theta = \frac{1}{\sqrt{7}}$ and the terminal arm of the angle is not in the III quad., find values of $\dfrac{\csc^2\theta-\sec^2\theta}{\csc^2\theta+\sec^2\theta}$.

Solution:- $\tan\theta = \frac{1}{\sqrt{7}}$ (θ is not in III quad)

$$\because \sec^2\theta = 1+\tan^2\theta$$

$$= 1+\left(\frac{1}{\sqrt{7}}\right)^2 = 1+\frac{1}{7} = \frac{7+1}{7}$$

$$\rightarrow \sec^2\theta = \frac{8}{7} \quad \rightarrow \cos^2\theta = \frac{7}{8}$$

$$\because \sin^2\theta = 1-\cos^2\theta$$

$$\rightarrow \sin^2\theta = 1-\frac{7}{8} = \frac{8-7}{8} = \frac{1}{8}$$

Now,

$$\frac{\csc^2\theta-\sec^2\theta}{\csc^2\theta+\sec^2\theta} = \frac{8-8/7}{8+8/7} = \frac{\frac{56-8}{7}}{\frac{56+8}{7}}$$

$$= \frac{48}{64} = 3/4$$

Q8. If $\cot\theta = \frac{5}{2}$ and the terminal arm of the angle is in the I quad., find the value of $\dfrac{3\sin\theta+4\cos\theta}{\cos\theta-\sin\theta}$

Solution:- $\cot\theta = \frac{5}{2}$ (θ is in I quad)

$$\rightarrow \tan\theta = \frac{2}{5}$$

$$\because \sec^2\theta = 1+\tan^2\theta$$

$$= 1+\left(\frac{2}{5}\right)^2 = 1+\frac{4}{25} = \frac{25+4}{25}$$

$$\sec^2\theta = \frac{29}{25} \rightarrow \sec\theta = \pm\sqrt{\frac{29}{25}}$$

$$\rightarrow \sec\theta = \pm\frac{\sqrt{29}}{5}$$

$$\sec\theta = \frac{\sqrt{29}}{5} \quad (\because \theta \text{ is in I quad})$$

$$\rightarrow \cos\theta = \frac{5}{\sqrt{29}}$$

$$\because \sin\theta = \pm\sqrt{1-\cos^2\theta}$$

$$= \pm\sqrt{1-\left(\frac{5}{\sqrt{29}}\right)^2} = \pm\sqrt{1-\frac{25}{29}}$$

$$\rightarrow \sin\theta = \pm\sqrt{\frac{29-25}{29}} = \pm\sqrt{\frac{4}{29}} = \pm\frac{2}{\sqrt{29}}$$

$$\sin\theta = \frac{2}{\sqrt{29}} \quad (\theta \text{ is in I quad})$$

Now

$$\frac{3\sin\theta+4\cos\theta}{\cos\theta-\sin\theta} = \frac{3\left(\frac{2}{\sqrt{29}}\right)+4\left(\frac{5}{\sqrt{29}}\right)}{\frac{5}{\sqrt{29}}-\frac{2}{\sqrt{29}}}$$

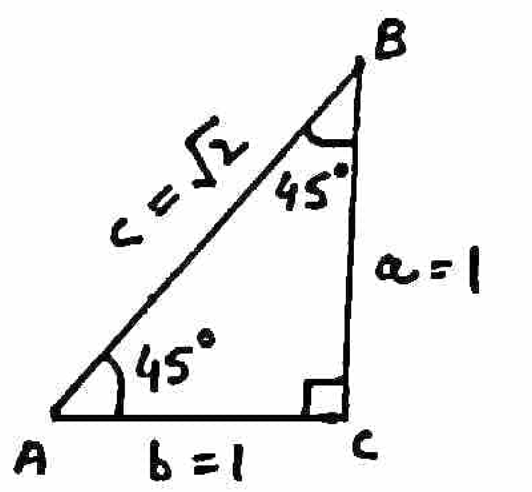
$$= \frac{\frac{6}{\sqrt{29}}+\frac{20}{\sqrt{29}}}{\frac{5-2}{\sqrt{29}}} = \frac{\frac{26}{\sqrt{29}}}{\frac{3}{\sqrt{29}}} = \frac{26}{3}$$

The values of Trigonometric Functions of acute angles 45°, 30° and 60°

(a) Trigonometric functions of 45°

we take $a=b=1$

By Pathagora's theorem,



$$c^2 = a^2+b^2$$

$$\rightarrow c^2 = (1)^2+(1)^2 = 1+1$$

$$c^2 = 2 \rightarrow c = \sqrt{2}$$

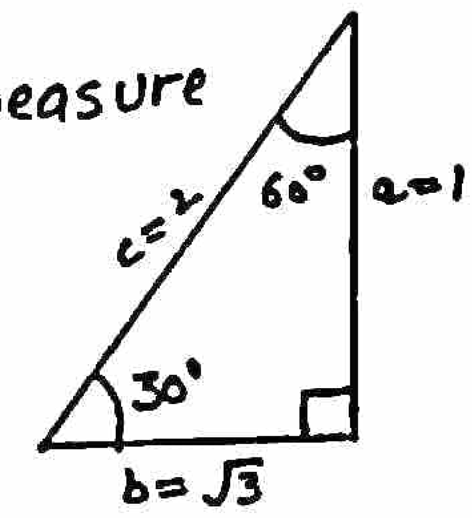
Now $\sin 45^\circ = \frac{a}{c} = \frac{1}{\sqrt{2}}$; $\operatorname{cosec} 45^\circ = \frac{c}{a} = \frac{\sqrt{2}}{1}$

$\cos 45^\circ = \frac{b}{c} = \frac{1}{\sqrt{2}}$; $\sec 45^\circ = \frac{c}{b} = \frac{\sqrt{2}}{1}$

$\tan 45^\circ = \frac{a}{b} = \frac{1}{1} = 1$; $\cot 45^\circ = \frac{b}{a} = \frac{1}{1} = 1$

(b) Trigonometric functions of 30°

By elementry geometry, in a right triangle the measure of side opposite to 30° is half of the hypotenuse.



So Take $c = 2$ then $a = 1$

By pathagoras theorem,

$$c^2 = a^2 + b^2 \rightarrow b^2 = c^2 - a^2$$

$$\rightarrow b^2 = (2)^2 - (1)^2 = 4 - 1 = 3$$

$$\rightarrow b = \sqrt{3}$$ Now

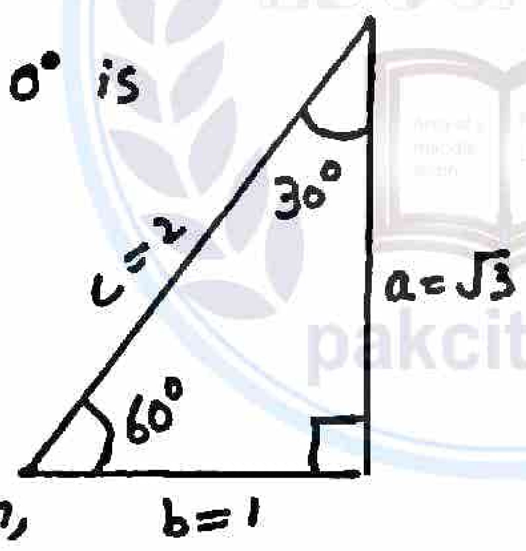
$\sin 30^\circ = \frac{a}{c} = \frac{1}{2}$; $\csc 30^\circ = \frac{c}{a} = \frac{2}{1} = 2$

$\cos 30^\circ = \frac{b}{c} = \frac{\sqrt{3}}{2}$; $\sec 30^\circ = \frac{c}{b} = \frac{2}{\sqrt{3}}$

$\tan 30^\circ = \frac{a}{b} = \frac{1}{\sqrt{3}}$; $\cot 30^\circ = \frac{b}{a} = \frac{\sqrt{3}}{1} = \sqrt{3}$

(c) Trigonometric functions of 60°

By elementry geometry, in a right triangle the measure of the side opposite to 30° is half the hypotenuse.



so take $c=2$ then $b=1$

By pathagoras theorem,

$$c^2 = a^2 + b^2$$

$$\rightarrow a^2 = c^2 - b^2 = (2)^2 - (1)^2 = 4 - 1 = 3$$

$$\rightarrow a = \sqrt{3}$$

Now $\sin 60^\circ = \frac{a}{c} = \frac{\sqrt{3}}{2}$; $\operatorname{cosec} 60^\circ = \frac{c}{a} = \frac{2}{\sqrt{3}}$

$\cos 60^\circ = \frac{b}{c} = \frac{1}{2}$; $\sec 60^\circ = \frac{c}{b} = \frac{2}{1} = 2$

$\tan 60^\circ = \frac{a}{b} = \frac{\sqrt{3}}{1} = \sqrt{3}$; $\cot 60^\circ = \frac{b}{a} = \frac{1}{\sqrt{3}}$

Example 3. Find the values of all the trigonometric functions of

(i) 420° (ii) $-\frac{7\pi}{4}$ (iii) $\frac{19\pi}{3}$

Solution:- (i) 420°

we know that $\theta = 2k\pi + \theta$, $k \in Z$

$\rightarrow 420^\circ = 360^\circ(1) + 60^\circ$, $k = 1$

Thus values of trigonometric functions at 420° and 60° are same. so

$\sin 420^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}$; $\csc 420^\circ = \csc 60^\circ = \frac{2}{\sqrt{3}}$

$\cos 420^\circ = \cos 60^\circ = \frac{1}{2}$; $\sec 420^\circ = \sec 60^\circ = 2$

$\tan 420^\circ = \tan 60^\circ = \sqrt{3}$; $\cot 420^\circ = \cot 60^\circ = \frac{1}{\sqrt{3}}$

(ii) $-\frac{7\pi}{4}$

$\because \theta = 2k\pi + \theta$, $k \in Z$

$$-\frac{7\pi}{4} = \frac{-8\pi + \pi}{4} = -\frac{8\pi}{4} + \frac{\pi}{4}$$

$$\rightarrow -\frac{7\pi}{4} = -2\pi + \frac{\pi}{4} = (-1)2\pi + \frac{\pi}{4}, \; k = -1$$

Thus values of trigonometric functions at $-\frac{7\pi}{4}$ and $\frac{\pi}{4}$ are same. so

$\sin(-\frac{7\pi}{4}) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$; $\csc(-\frac{7\pi}{4}) = \csc\frac{\pi}{4} = \sqrt{2}$

$\cos(-\frac{7\pi}{4}) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$; $\sec(-\frac{7\pi}{4}) = \sec\frac{\pi}{4} = \sqrt{2}$

$\tan(-\frac{7\pi}{4}) = \tan\frac{\pi}{4} = 1$; $\cot(-\frac{7\pi}{4}) = \cot\frac{\pi}{4} = 1$

(iii) $\frac{19\pi}{3}$

$$\frac{19\pi}{3} = \frac{18\pi + \pi}{3} = \frac{18\pi}{3} + \frac{\pi}{3} = 6\pi + \frac{\pi}{3}$$

$$= 3(2\pi) + \frac{\pi}{3} \quad , \; k = 3$$

Thus values of trigonometric functions at $\frac{19\pi}{3}$ and $\frac{\pi}{3}$ are same. so

$\sin\frac{19\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$; $\csc\frac{19\pi}{3} = \csc\frac{\pi}{3} = \frac{2}{\sqrt{3}}$

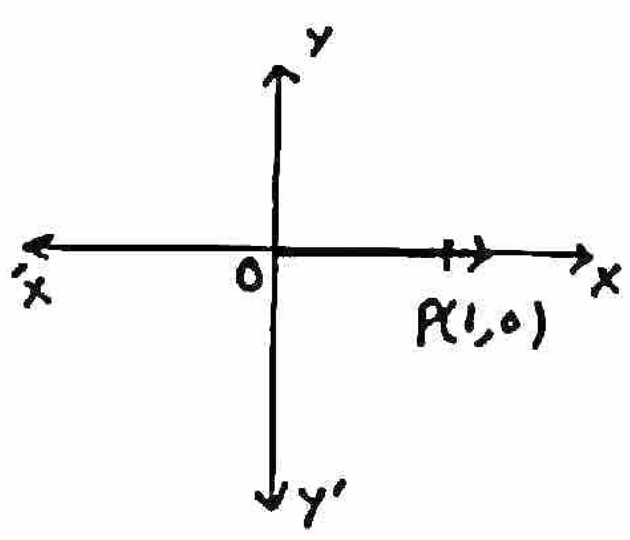
$\cos\frac{19\pi}{3} = \cos\frac{\pi}{3} = \frac{1}{2}$; $\sec\frac{19\pi}{3} = \sec\frac{\pi}{3} = 2$

$\tan\frac{19\pi}{3} = \tan\frac{\pi}{3} = \sqrt{3}$; $\cot\frac{19\pi}{3} = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$

The values of the Trigonometric Functions of angles 0°, 90°, 180°, 270°, 360°

(a) When $\theta = 0^\circ$

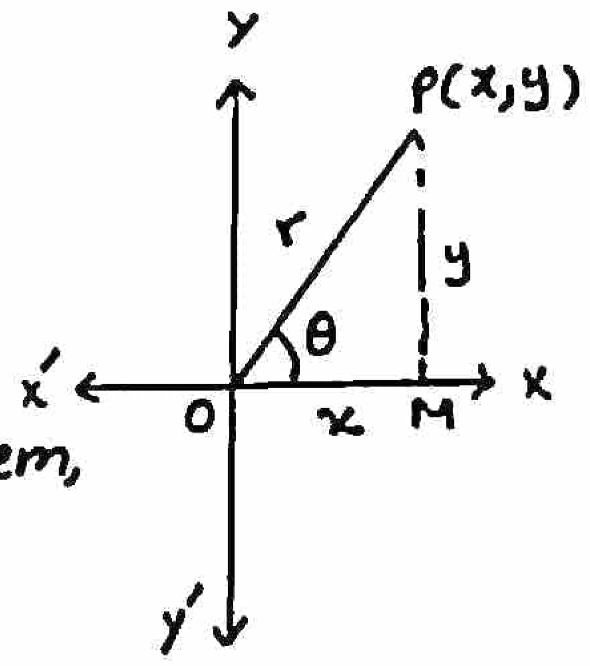
The point (1,0) lies on the terminal



side of 0°

$\Rightarrow x = 1$ and $y = 0$

Note:-



In ΔMOP

$OM = x,\ MP = y,\ OP = r$

By Pathagora's theorem,

$r^2 = x^2 + y^2$

$\Rightarrow r = \sqrt{x^2+y^2}$

$\sin\theta = \frac{y}{r}$; $\csc\theta = \frac{r}{y}$; $\cos\theta = \frac{x}{r}$

$\sec\theta = \frac{r}{x}$; $\tan\theta = \frac{y}{x}$; $\cot\theta = \frac{x}{y}$

$\because r = \sqrt{x^2+y^2} = \sqrt{(1)^2+(0)^2} = \sqrt{1+0} = 1$

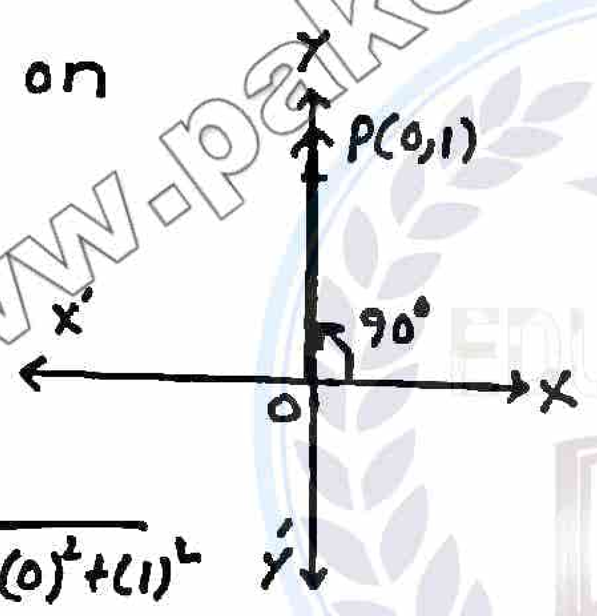
$\therefore \sin 0^\circ = \frac{y}{r} = \frac{0}{1} = 0$; $\csc 0^\circ = \frac{r}{y} = \frac{1}{0} = \infty$

$\cos 0^\circ = \frac{x}{r} = \frac{1}{1} = 1$; $\sec 0^\circ = \frac{r}{x} = \frac{1}{1} = 1$

$\tan 0^\circ = \frac{y}{x} = \frac{0}{1} = 0$; $\cot 0^\circ = \frac{x}{y} = \frac{1}{0} = \infty$

(b) When $\theta = 90^\circ$

The point $(0,1)$ lies on the terminal side of angle 90°.



$\Rightarrow x = 0,\ y = 1$

so $r = \sqrt{x^2+y^2} = \sqrt{(0)^2+(1)^2}$

$\Rightarrow r = 1$

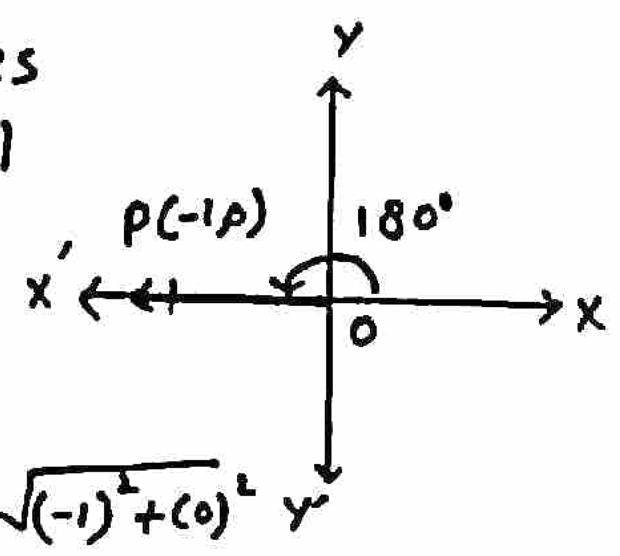
$\therefore \sin 90^\circ = \frac{y}{r} = \frac{1}{1} = 1$; $\csc 90^\circ = \frac{r}{y} = \frac{1}{1} = 1$

$\cos 90^\circ = \frac{x}{r} = \frac{0}{1} = 0$; $\sec 90^\circ = \frac{r}{x} = \frac{1}{0} = \infty$

$\tan 90^\circ = \frac{y}{x} = \frac{1}{0} = \infty$; $\cot 90^\circ = \frac{x}{y} = \frac{0}{1} = 0$

(c) When $\theta = 180^\circ$

The point $(-1,0)$ lies on the terminal side of angle 180°



$\Rightarrow x = -1,\ y = 0$

so $r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(0)^2}$

$r = \sqrt{1+0} = 1$

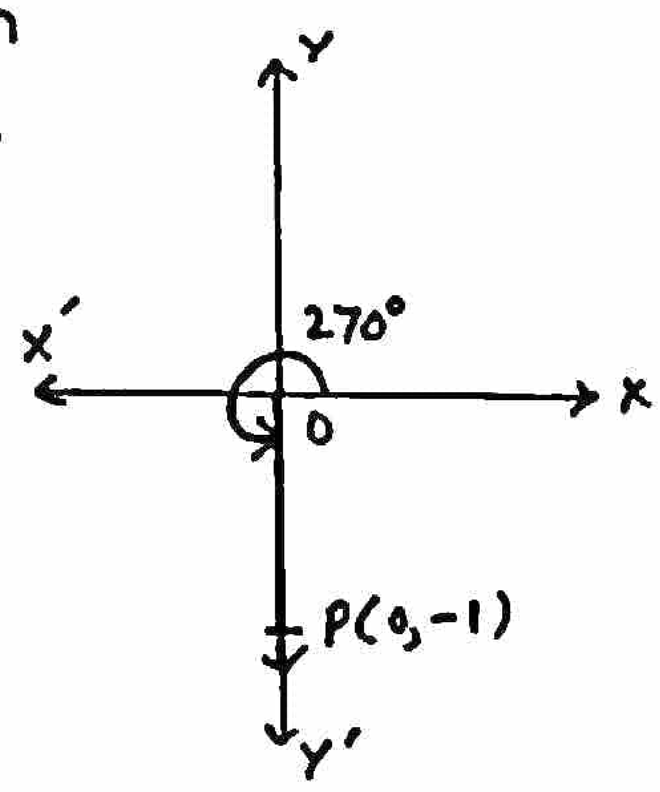
$\therefore \sin 180^\circ = \frac{y}{r} = \frac{0}{1} = 0$; $\csc 180^\circ = \frac{r}{y} = \frac{1}{0} = \infty$

$\cos 180^\circ = \frac{x}{r} = \frac{-1}{1} = -1$; $\sec 180^\circ = \frac{r}{x} = \frac{1}{-1} = -1$

$\tan 180^\circ = \frac{y}{x} = \frac{0}{-1} = 0$; $\cot 180^\circ = \frac{x}{y} = \frac{-1}{0} = \infty$

(d) When $\theta = 270^\circ$

The point $(0,-1)$ lies on the terminal side of angle 270°



$\Rightarrow x = 0,\ y = -1$

so $r = \sqrt{x^2+y^2}$

$r = \sqrt{(0)^2+(-1)^2} = 1$

$\therefore \sin 270^\circ = \frac{y}{r} = \frac{-1}{1} = -1$

$\csc 270^\circ = \frac{r}{y} = \frac{1}{-1} = -1$; $\cos 270^\circ = \frac{x}{r} = \frac{0}{1} = 0$

$\sec 270^\circ = \frac{r}{x} = \frac{1}{0} = \infty$

$\tan 270^\circ = \frac{y}{x} = \frac{-1}{0} = \infty$; $\cot 270^\circ = \frac{x}{y} = \frac{0}{-1} = 0$

Example 4. Find the values of all trigonometric functions of (i) 360° (ii) $-\frac{\pi}{2}$ (iii) 5π

Solution:- We know that

General angle: $\theta = \theta + 2k\pi$, $k \in \mathbb{Z}$

(i) $360^\circ = 0^\circ + 1(360^\circ)$, $k = 1$

Thus values of trigonometric functions at 360° and 0° are same. so

$\sin 360^\circ = \sin 0^\circ = 0$; $\csc 360^\circ = \csc 0^\circ = \frac{1}{0} = \infty$

$\cos 360^\circ = \cos 0^\circ = 1$; $\sec 360^\circ = \sec 0^\circ = 1$

$\tan 360^\circ = \tan 0^\circ = 0$; $\cot 360^\circ = \cot 0^\circ = \frac{1}{0} = \infty$

(ii) $-\frac{\pi}{2}$

$-\frac{\pi}{2} = -\frac{4\pi+3\pi}{2} = -\frac{4\pi}{2} + \frac{3\pi}{2} = -2\pi + \frac{3\pi}{2}$

$= (-1)2\pi + \frac{3\pi}{2}$

Thus values of trigonometric functions at $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$ are same. so

$\sin(-\frac{\pi}{2}) = \sin\frac{3\pi}{2} = -1$; $\csc(-\frac{\pi}{2}) = \csc\frac{3\pi}{2} = -1$

$\cos(-\frac{\pi}{2}) = \cos\frac{3\pi}{2} = 0$; $\sec(-\frac{\pi}{2}) = \sec\frac{3\pi}{2} = \frac{1}{0} = \infty$

$\tan(-\frac{\pi}{2}) = \tan\frac{3\pi}{2} = \infty$; $\cot(-\frac{\pi}{2}) = \cot\frac{3\pi}{2} = 0$

(iii) 5π

$5\pi = 4\pi + \pi = 2(2\pi) + \pi$, $k = 2$

Thus values of trigonometric functions at 5π and π are same. so

$\sin 5\pi = \sin\pi = 0$; $\csc 5\pi = \csc\pi = \frac{1}{0} = \infty$

$\cos 5\pi = \cos\pi = -1$; $\sec 5\pi = \sec\pi = -1$

$\tan 5\pi = \tan\pi = 0$; $\cot 5\pi = \cot\pi = \frac{1}{0} = \infty$

Dear Students , Remember these values on your finger tips :-)

θ	0°	30°	45°	60°	90°	180°	270°	360°
Sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0

Exercise 9.3

Q1. Verify the following:

i) $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ = \sin 30^\circ$

Solution:-

$L.H.S = \sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

$= \frac{3}{4} - \frac{1}{4} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$

$= \frac{1}{2} = \sin 30^\circ = R.H.S$

Hence proved.

(ii) $\sin^2\frac{\pi}{6} + \sin^2\frac{\pi}{3} + \tan^2\frac{\pi}{4} = 2$

$L.H.S = \sin^2\frac{\pi}{6} + \sin^2\frac{\pi}{3} + \tan^2\frac{\pi}{4}$

$= \sin^2 30^\circ + \sin^2 60^\circ + \tan^2 45^\circ$

$= \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + (1)^2$

$= \frac{1}{4} + \frac{3}{4} + 1 = \frac{1+3+4}{4} = \frac{8}{4}$

$= 2 = R.H.S$

Hence proved.

(iii) $2\sin 45^\circ + \frac{1}{2}\operatorname{cosec} 45^\circ = \frac{3}{\sqrt{2}}$

$L.H.S = 2\sin 45^\circ + \frac{1}{2}\operatorname{cosec} 45^\circ$

$= 2\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}(\sqrt{2})$

$= \frac{2}{\sqrt{2}} + \frac{\sqrt{2}}{2} = \frac{4+(\sqrt{2})(\sqrt{2})}{2\sqrt{2}}$

$= \frac{4+2}{2\sqrt{2}} = \frac{6}{2\sqrt{2}} = \frac{3}{\sqrt{2}} = R.H.S$

Hence proved.

(iv) $\sin^2\frac{\pi}{6} : \sin^2\frac{\pi}{4} : \sin^2\frac{\pi}{3} : \sin^2\frac{\pi}{2} = 1:2:3:4$

$L.H.S = \sin^2\frac{\pi}{6} : \sin^2\frac{\pi}{4} : \sin^2\frac{\pi}{3} : \sin^2\frac{\pi}{2}$

$= \sin^2 30^\circ : \sin^2 45^\circ : \sin^2 60^\circ : \sin^2 90^\circ$

$= \left(\frac{1}{2}\right)^2 : \left(\frac{1}{\sqrt{2}}\right)^2 : \left(\frac{\sqrt{3}}{2}\right)^2 : (1)^2$

$= \frac{1}{4} : \frac{1}{2} : \frac{3}{4} : 1$

Multiplying by 4

$= 1 : 2 : 3 : 4 = R.H.S$

Hence proved

Q2. Evaluate the following

i) $\dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{6}}$ ii) $\dfrac{1-\tan^2\frac{\pi}{3}}{1+\tan^2\frac{\pi}{3}}$

Solution:- i) $\dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{6}}$

$= \dfrac{\sqrt{3} - \frac{1}{\sqrt{3}}}{1+(\sqrt{3})\left(\frac{1}{\sqrt{3}}\right)} = \dfrac{\frac{3-1}{\sqrt{3}}}{1+1} = \dfrac{\frac{2}{\sqrt{3}}}{2} = \frac{1}{\sqrt{3}}$

ii) $\dfrac{1-\tan^2\frac{\pi}{3}}{1+\tan^2\frac{\pi}{3}}$

$= \dfrac{1-(\sqrt{3})^2}{1+(\sqrt{3})^2} = \dfrac{1-3}{1+3} = \dfrac{-2}{4} = -\frac{1}{2}$

Q3. Verify the following when $\theta = 30^\circ, 45^\circ$

i) $\sin 2\theta = 2\sin\theta\cos\theta$

Solution:- when $\theta = 30^\circ$

$L.H.S = \sin 2\theta = \sin 2(30^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$

$R.H.S = 2\sin\theta\cos\theta = 2\sin 30^\circ \cos 30^\circ$

$= 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$

Hence $L.H.S = R.H.S$

when $\theta = 45^\circ$

$L.H.S = \sin 2\theta = \sin 2(45^\circ)$
$= \sin 90^\circ = 1$

$R.H.S = 2\sin\theta\cos\theta$
$= 2\sin 45^\circ \cos 45^\circ = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$
$= 2\left(\frac{1}{2}\right) = 1$

Hence $L.H.S = R.H.S$

ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

Solution:- when $\theta = 30^\circ$

$L.H.S = \cos 2\theta = \cos 2(30^\circ) = \cos 60^\circ$
$= \frac{1}{2}$

$R.H.S = \cos^2\theta - \sin^2\theta$
$= \cos^2 30^\circ - \sin^2 30^\circ$
$= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4} = \frac{3-1}{4}$
$= \frac{2}{4} = \frac{1}{2}$

Hence $L.H.S = R.H.S$

when $\theta = 45^\circ$

$L.H.S = \cos 2\theta$
$= \cos 2(45^\circ) = \cos 90^\circ = 0$

$R.H.S = \cos^2\theta - \sin^2\theta$
$= \cos^2 45^\circ - \sin^2 45^\circ$
$= \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 = 0$

Hence $L.H.S = R.H.S$

iii) $\cos 2\theta = 2\cos^2\theta - 1$

Solution:- when $\theta = 30^\circ$

$L.H.S = \cos 2\theta$
$= \cos 2(30^\circ) = \cos 60^\circ = \frac{1}{2}$

$R.H.S = 2\cos^2\theta - 1$
$= 2\cos^2 30^\circ - 1$
$= 2\left(\frac{\sqrt{3}}{2}\right)^2 - 1 = 2\left(\frac{3}{4}\right) - 1$
$= \frac{3}{2} - 1 = \frac{3-2}{2} = \frac{1}{2}$

Hence $L.H.S = R.H.S$

when $\theta = 45^\circ$

$L.H.S = \cos 2\theta = \cos 2(45^\circ) = \cos 90^\circ$
$= 0$

$R.H.S = 2\cos^2\theta - 1$
$= 2\cos^2 45^\circ - 1 = 2\left(\frac{1}{\sqrt{2}}\right)^2 - 1$
$= 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$

Hence $L.H.S = R.H.S$

(iv) $\cos 2\theta = 1 - 2\sin^2\theta$

Solution:- when $\theta = 30^\circ$

$L.H.S = \cos 2\theta$
$= \cos 2(30^\circ) = \cos 60^\circ = \frac{1}{2}$

$R.H.S = 1 - 2\sin^2\theta$
$= 1 - 2\sin^2 30^\circ = 1 - 2\left(\frac{1}{2}\right)^2$
$= 1 - 2\left(\frac{1}{4}\right) = 1 - \frac{1}{2} = \frac{2-1}{2} = \frac{1}{2}$

Hence $L.H.S = R.H.S$

when $\theta = 45^\circ$

$L.H.S = \cos 2\theta$
$= \cos 2(45^\circ) = \cos 90^\circ = 0$

$R.H.S = 1 - 2\sin^2\theta$
$= 1 - 2\sin^2 45^\circ = 1 - 2\left(\frac{1}{\sqrt{2}}\right)^2$
$= 1 - 2\left(\frac{1}{2}\right) = 1 - 1 = 0$

Hence $L.H.S = R.H.S$

v) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

Solution:- when $\theta = 30^\circ$

$L.H.S = \tan 2\theta$
$= \tan 2(30^\circ) = \tan 60^\circ = \sqrt{3}$

$R.H.S = \frac{2\tan\theta}{1-\tan^2\theta}$
$= \frac{2\tan 30^\circ}{1-\tan^2 30^\circ} = \frac{2\left(\frac{1}{\sqrt{3}}\right)}{1-\left(\frac{1}{\sqrt{3}}\right)^2}$
$= \frac{\frac{2}{\sqrt{3}}}{1-\frac{1}{3}} = \frac{2}{\sqrt{3}} \Big/ \frac{3-1}{3} = \frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}}$
$= \frac{2}{\sqrt{3}} \times \frac{3}{2} = \frac{3}{\sqrt{3}} = \frac{\sqrt{3}\sqrt{3}}{\sqrt{3}} = \sqrt{3}$

Hence $L.H.S = R.H.S$

when $\theta = 45°$

$L.H.S = \tan 2\theta = \tan 2(45°)$

$= \tan 90° = \infty$ (undefined)

$R.H.S = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\tan 45°}{1-\tan^2 45°}$

$= \frac{2(1)}{1-(1)^2} = \frac{2}{1-1} = \frac{2}{0} = \infty$

Hence $L.H.S = R.H.S$

Q4. Find x, if $\tan^2 45° - \cos^2 60° = x \sin 45° \cos 45° \tan 60°$

Solution:-

$\tan^2 45° - \cos^2 60° = x\sin 45° \cos 45° \tan 60°$

$\Rightarrow (1)^2 - (\frac{1}{2})^2 = x(\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}})(\sqrt{3})$

$\Rightarrow 1 - \frac{1}{4} = x(\frac{\sqrt{3}}{2})$

$\Rightarrow \frac{4-1}{4} = x\frac{\sqrt{3}}{2}$

$\Rightarrow \frac{3}{4} = x\frac{\sqrt{3}}{2}$

$\Rightarrow \frac{3}{4} \times \frac{2}{\sqrt{3}} = x \Rightarrow x = \frac{3}{2\sqrt{3}}$

$\Rightarrow x = \frac{\sqrt{3}\sqrt{3}}{2\sqrt{3}} \Rightarrow x = \frac{\sqrt{3}}{2}$

Q5. Find the values of the trigonometric functions of the following quadrantal angles: i) $-\pi$

Solution:- we know that

General angle: $\theta = \theta + 2K\pi$, $K \in Z$

$-\pi = -2\pi + \pi = (-1)2\pi + \pi$, $K = -1$

Thus values of trigonometric functions at $-\pi$ and π are same.

$\therefore \sin(-\pi) = \sin\pi = 0$

$\csc(-\pi) = \csc\pi = \infty$

$\cos(-\pi) = \cos\pi = -1$

$\sec(-\pi) = \sec\pi = -1$

$\tan(-\pi) = \tan\pi = 0$

$\cot(-\pi) = \cot\pi = \infty$

ii) -3π

Solution:-

$-3\pi = -4\pi + \pi = -2(2\pi) + \pi$

Thus values of trigonometric functions at -3π and π are same. so

$\sin(-3\pi) = \sin\pi = 0$; $\csc(-3\pi) = \csc\pi = \infty$

$\cos(-3\pi) = \cos\pi = -1$; $\sec(-3\pi) = \sec\pi = -1$

$\tan(-3\pi) = \tan\pi = 0$; $\cot(-3\pi) = \cot\pi = \infty$

iii) $\frac{5}{2}\pi$

Solution:-

$\frac{5}{2}\pi = \frac{4\pi+\pi}{2} = \frac{4\pi}{2} + \frac{\pi}{2} = 2\pi + \frac{\pi}{2}$

$= (1)2\pi + \frac{\pi}{2}$, $K = 1$

Thus values of trigonometric functions at $\frac{5\pi}{2}$ and $\frac{\pi}{2}$ are same. so

$\sin(\frac{5\pi}{2}) = \sin\frac{\pi}{2} = 1$; $\csc(\frac{5\pi}{2}) = \csc(\frac{\pi}{2}) = 1$

$\cos(\frac{5\pi}{2}) = \cos\frac{\pi}{2} = 0$; $\sec(\frac{5\pi}{2}) = \sec\frac{\pi}{2} = 0$

$\tan(\frac{5\pi}{2}) = \tan\frac{\pi}{2} = \infty$; $\cot\frac{5\pi}{2} = \cot\frac{\pi}{2} = 0$

iv) $-\frac{9}{2}\pi$

Solution:-

$-\frac{9}{2}\pi = \frac{-12\pi + 3\pi}{2} = \frac{-12\pi}{2} + \frac{3\pi}{2}$

$= -6\pi + \frac{3\pi}{2} = (-3)(2\pi) + \frac{3\pi}{2}$, $K = -3$

$(\because \theta = K(2\pi) + \theta,\ K \in Z)$

Thus values of trigonometric functions at $-\frac{9}{2}\pi$ and $\frac{3\pi}{2}$ are same. so

$\sin(-\frac{9}{2}\pi) = \sin(\frac{3}{2}\pi) = -1$; $\csc(-\frac{9}{2}\pi) = \csc(\frac{3}{2}\pi) = -1$

$\cos(-\frac{9}{2}\pi) = \cos(\frac{3\pi}{2}) = 0$; $\sec(-\frac{9}{2}\pi) = \sec(\frac{3}{2}\pi) = \infty$

$\tan(-\frac{9}{2}\pi) = \tan\frac{3\pi}{2} = \infty$; $\cot(-\frac{9}{2}\pi) = \cot(\frac{3}{2}\pi) = 0$

v) -15π

Solution:-

$-15\pi = -16\pi + \pi = (-8)(2\pi) + \pi$, $K = -8$

Thus values of trigonometric values at -15π and π are same. so

$\sin(-15\pi) = \sin\pi = 0$; $\csc(-15\pi) = \csc\pi = \infty$

$\cos(-15\pi) = \cos\pi = -1$; $\sec(-15\pi) = \sec\pi = -1$

$\tan(-15\pi) = \tan\pi = 0$; $\cot(-15\pi) = \cot\pi = \infty$

vi) 1530°

Solution:- $\because \theta = k(2\pi) + \theta$, $k \in Z$

$1530^\circ = 1440^\circ + 90^\circ$

$= 4(360^\circ) + 90^\circ$, $k = 4$

Thus values of trigonometric functions at 1530° and 90° are same. so

$\sin 1530^\circ = \sin 90^\circ = 1$; $\csc 1530^\circ = \csc 90^\circ = 1$

$\cos 1530^\circ = \cos 90^\circ = 0$; $\sec 1530^\circ = \sec 90^\circ = \infty$

$\tan 1530^\circ = \tan 90^\circ = \frac{1}{0} = \infty$; $\cot 1530^\circ = \cot 90^\circ = 0$

vii) -2430°

Solution:-

$-2430^\circ = -2520^\circ + 90^\circ$

$= -7(360^\circ) + 90^\circ$, $k = -7$

Thus values of trigonometric functions at -2430° and 90° are same. so

$\sin(-2430^\circ) = \sin 90^\circ = 1$; $\csc(-2430^\circ) = \csc 90^\circ = 1$

$\cos(-2430^\circ) = \cos 90^\circ = 0$; $\sec(-2430^\circ) = \sec 90^\circ = \infty$

$\tan(-2430^\circ) = \tan 90^\circ = \infty$; $\cot(-2430^\circ) = \cot 90^\circ = 0$

viii) $\frac{235\pi}{2}$

Solution:-

$\frac{235\pi}{2} = \frac{232\pi + 3\pi}{2} = \frac{232\pi}{2} + \frac{3\pi}{2}$

$= 116\pi + \frac{3\pi}{2} = 58(2\pi) + \frac{3\pi}{2}$, $k = 58$

Thus trigonometric function of $\frac{235\pi}{2}$ and $\frac{3\pi}{2}$ are same. so

$\sin\left(\frac{235\pi}{2}\right) = \sin\frac{3\pi}{2} = -1$; $\csc\left(\frac{235\pi}{2}\right) = \csc\frac{3\pi}{2} = -1$

$\cos\left(\frac{235\pi}{2}\right) = \cos\frac{3\pi}{2} = 0$

$\sec\left(\frac{235\pi}{2}\right) = \sec\frac{3\pi}{2} = \infty$

$\tan\left(\frac{235\pi}{2}\right) = \tan\frac{3\pi}{2} = \infty$

$\cot\left(\frac{235\pi}{2}\right) = \cot\frac{3\pi}{2} = 0$

ix) $\frac{407\pi}{2}$

Solution:-

$\frac{407\pi}{2} = \frac{404\pi + 3\pi}{2} = \frac{404\pi}{2} + \frac{3\pi}{2}$

$= 202\pi + \frac{3\pi}{2} = 101(2\pi) + \frac{3\pi}{2}$

$k = 101$

Thus trigonometric functions of $\frac{407\pi}{2}$ and $\frac{3\pi}{2}$ are same. so

$\sin\left(\frac{407\pi}{2}\right) = \sin\frac{3\pi}{2} = -1$

$\csc\left(\frac{407\pi}{2}\right) = \csc\frac{3\pi}{2} = -1$

$\cos\left(\frac{407\pi}{2}\right) = \cos\frac{3\pi}{2} = 0$

$\sec\left(\frac{407\pi}{2}\right) = \sec\frac{3\pi}{2} = \infty$

$\tan\left(\frac{407\pi}{2}\right) = \tan\frac{3\pi}{2} = \infty$

$\cot\left(\frac{407\pi}{2}\right) = \cot\frac{3\pi}{2} = 0$

Q6. Find the values of the trigonometric functions of the following angles:

i) 390°

Solution:- we know that

General angle : $\theta = k(2\pi) + \theta$, $k \in Z$

$390^\circ = 360^\circ + 30^\circ = (1)\,360^\circ + 30^\circ$, $k = 1$

Thus trigonometric function of 390° and 30° are same. so

$\sin 390^\circ = \sin 30^\circ = \frac{1}{2}$

$\csc 390^\circ = \csc 30^\circ = 2$

$\cos 390^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$

$\sec 390^\circ = \sec 30^\circ = \frac{2}{\sqrt{3}}$

$\tan 390^\circ = \tan 30^\circ = \frac{1}{\sqrt{3}}$

$\cot 390^\circ = \cot 30^\circ = \sqrt{3}$

ii) -330°

Solution:-

$-330^\circ = -360^\circ + 30^\circ = (-1)\,360^\circ + 30^\circ$, $k = -1$

Trigonometric functions of -330° and 30° are same. so

$\sin(-330^\circ) = \sin 30^\circ = \frac{1}{2}$

$\csc(-330^\circ) = \csc 30^\circ = 2$

$\cos(-330°) = \cos 30° = \frac{\sqrt{3}}{2}$

$\sec(-330°) = \sec 30° = \frac{2}{\sqrt{3}}$

$\tan(-330°) = \tan 30° = \frac{1}{\sqrt{3}}$

$\cot(-330°) = \cot 30° = \sqrt{3}$

iii) $765°$

Solution:-

$765° = 720° + 45° = 2(360°) + 45°$, $K = 2$

Trigonometric functions of $765°$ and $45°$ are same. so

$\sin 765° = \sin 45° = \frac{1}{\sqrt{2}}$

$\csc 765° = \csc 45° = \sqrt{2}$

$\cos 765° = \cos 45° = \frac{1}{\sqrt{2}}$

$\sec 765° = \sec 45° = \sqrt{2}$

$\tan 765° = \tan 45° = 1$

$\cot 765° = \cot 45° = 1$

iv) $-675°$

Solution:-

$-675° = -720° + 45°$,

$= (-2)(360°) + 45°$, $K = -2$

Trigonometric functions of $-675°$ and $45°$ are same. so

$\sin(-675°) = \sin 45° = \frac{1}{\sqrt{2}}$

$\csc(-675°) = \csc 45° = \sqrt{2}$

$\cos(-675°) = \cos 45° = \frac{1}{\sqrt{2}}$

$\sec(-675°) = \sec 45° = \sqrt{2}$

$\tan(-675°) = \tan 45° = 1$

$\cot(-675°) = \cot 45° = 1$

v) $-\frac{17}{3}\pi$

Solution:-

$-\frac{17}{3}\pi = -\frac{18\pi + \pi}{3} = -\frac{18\pi}{3} + \frac{\pi}{3}$

$= -6\pi + \frac{\pi}{3} = -3(2\pi) + \frac{\pi}{3}$

Trigonometric functions of $-\frac{17\pi}{3}$ and $\frac{\pi}{3}$ are same. so

$\sin(-\frac{17}{3}\pi) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\csc(-\frac{17\pi}{3}) = \csc\frac{\pi}{3} = \frac{2}{\sqrt{3}}$

$\cos(-\frac{17\pi}{3}) = \cos\frac{\pi}{3} = \frac{1}{2}$

$\sec(-\frac{17\pi}{3}) = \sec\frac{\pi}{3} = 2$

$\tan(-\frac{17\pi}{3}) = \tan\frac{\pi}{3} = \sqrt{3}$

$\cot(-\frac{17\pi}{3}) = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$

vi) $\frac{13}{3}\pi$

Solution:-

$\frac{13}{3}\pi = \frac{12\pi + \pi}{3} = \frac{12\pi}{3} + \frac{\pi}{3}$

$= 4\pi + \frac{\pi}{3}$, $K = 4$

Trigonometric functions of $\frac{13}{3}\pi$ and $\frac{\pi}{3}$ are same.

$\sin\frac{13\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\csc\frac{13\pi}{3} = \csc\frac{\pi}{3} = \frac{2}{\sqrt{3}}$

$\cos\frac{13\pi}{3} = \cos\frac{\pi}{3} = \frac{1}{2}$

$\sec\frac{13\pi}{3} = \sec\frac{\pi}{3} = 2$

$\tan\frac{13\pi}{3} = \tan\frac{\pi}{3} = \sqrt{3}$

$\cot\frac{13\pi}{3} = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$

vii) $\frac{25}{6}\pi$

Solution:-

$\frac{25\pi}{6} = \frac{24\pi + \pi}{6} = \frac{24\pi}{6} + \frac{\pi}{6}$

$= 4\pi + \frac{\pi}{6} = 2(2\pi) + \frac{\pi}{6}$, $K = 2$

Trigonometric functions of $\frac{25\pi}{6}$ and $\frac{\pi}{6}$ are same. so

$\sin\frac{25\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2}$

$\csc\frac{25\pi}{6} = \csc\frac{\pi}{6} = 2$

$\cos\frac{25\pi}{6} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\sec\frac{25\pi}{6} = \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$

$\tan\frac{25\pi}{6} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$\cot\frac{25\pi}{6} = \cot\frac{\pi}{6} = \sqrt{3}$

viii) $-\frac{71\pi}{6}$

Solution:-

$$-\frac{71\pi}{6} = \frac{-72\pi+\pi}{6} = -\frac{72\pi}{6} + \frac{\pi}{6}$$

$$= -12\pi + \frac{\pi}{6} = -6(2\pi) + \frac{\pi}{6}, \quad k=-6$$

Trigonometric functions of $-\frac{71\pi}{6}$ and $\frac{\pi}{6}$ are same. so

$\sin(-\frac{71\pi}{6}) = \sin\frac{\pi}{6} = \frac{1}{2}$

$\csc(-\frac{71\pi}{6}) = \csc\frac{\pi}{6} = 2$

$\cos(-\frac{71\pi}{6}) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\sec(-\frac{71\pi}{6}) = \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$

$\tan(-\frac{71\pi}{6}) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$\cot(-\frac{71\pi}{6}) = \cot\frac{\pi}{6} = \sqrt{3}$

ix) -1035°

Solution:-

$$-1035^\circ = -1080^\circ + 45^\circ$$

$$= (-3)(360^\circ) + 45^\circ, \quad k=-3$$

Trigonometric functions of -1035° and 45° are same. so

$\sin(-1035^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}$

$\csc(-1035^\circ) = \csc 45^\circ = \sqrt{2}$

$\cos(-1035^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}$

$\sec(-1035^\circ) = \sec 45^\circ = \sqrt{2}$

$\tan(-1035^\circ) = \tan 45^\circ = 1$

$\cot(-1035^\circ) = \cot 45^\circ = 1$

Domains of Trigonometric functions and Fundamental Identities

i) $\sin\theta$, for all $\theta \in R$

ii) $\cos\theta$, for all $\theta \in R$

iii) $\csc\theta = \frac{1}{\sin\theta}$, for all $\theta \in R$ but $\theta \neq n\pi$, $n \in Z$

iv) $\sec\theta = \frac{1}{\cos\theta}$, $\forall\ \theta \in R$, $\theta \neq (\frac{2n+1}{2})\pi$, $n \in Z$

v) $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\forall\ \theta \in R$, $\theta \neq (2n+1)\frac{\pi}{2}$, $n \in Z$

vi) $\cot\theta = \frac{\cos\theta}{\sin\theta}$, $\forall\ \theta \in R$, $\theta \neq n\pi$, $n \in Z$

vii) $\sin^2\theta + \cos^2\theta = 1$, $\forall\ \theta \in R$

viii) $1 + \tan^2\theta = \sec^2\theta$, $\forall\ \theta \in R$, $\theta \neq (2n+1)\frac{\pi}{2}$, $n \in Z$

ix) $1 + \cot^2\theta = \csc^2\theta$, $\forall\ \theta \in R$ but $\theta \neq n\pi$

Example 1. Prove that $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$, $\forall\ \theta \in R$

Solution:- $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$

L.H.S $= \cos^4\theta - \sin^4\theta$

$= (\cos^2\theta)^2 - (\sin^2\theta)^2$

$= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$

$= (1)(\cos^2\theta - \sin^2\theta) \quad \because \cos^2\theta + \sin^2\theta = 1$

$= \cos^2\theta - \sin^2\theta = R.H.S$

Hence proved.

Example 2. Prove that:

$\sec^2 A + \csc^2 A = \sec^2 A \csc^2 A$

(where $A \neq \frac{n\pi}{2}$, $n \in Z$)

Solution:- $\sec^2 A + \csc^2 A = \sec^2 A \csc^2 A$

L.H.S $= \sec^2 A + \csc^2 A$

$= \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} = \frac{\sin^2 A + \cos^2 A}{\cos^2 A \sin^2 A}$

$= \frac{1}{\cos^2 A \sin^2 A} \quad (\because \sin^2 A + \cos^2 A = 1)$

$= \frac{1}{\cos^2 A} \cdot \frac{1}{\sin^2 A} = \sec^2 A \csc^2 A = R.H.S$

Hence proved.

Example 3. Prove that $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$, where θ is not an odd multiple of $\frac{\pi}{2}$.

Solution:- $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

L.H.S $= \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$

$= \sqrt{\frac{1-\sin\theta}{1+\sin\theta} \times \frac{1-\sin\theta}{1-\sin\theta}}$

$= \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}} = \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}$

$$= \sqrt{\left(\frac{1-\sin\theta}{\cos\theta}\right)^2} = \frac{1-\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta = R.H.S$$

Hence proved.

Example 4. Show that

$\cot^4\theta + \cot^2\theta = \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta$, where θ is not an integral multiple of $\frac{\pi}{2}$.

Solution:– $\cot^4\theta + \cot^2\theta = \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta$

L.H.S $= \cot^4\theta + \cot^2\theta$

$$= (\cot^2\theta)^2 + \cot^2\theta$$

$$= \cot^2\theta(\cot^2\theta + 1)$$

$$= (\operatorname{cosec}^2\theta - 1)(\operatorname{cosec}^2\theta) \quad \because 1+\cot^2\theta = \operatorname{cosec}^2\theta \Rightarrow \cot^2\theta = \csc^2\theta - 1$$

$$= \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta = R.H.S$$

Hence proved.

Exercise 9.4

Prove the following identities, state the domain of θ in each case:

Q1. $\tan\theta + \cot\theta = \operatorname{cosec}\theta\sec\theta$

Solution:– L.H.S $= \tan\theta + \cot\theta$

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta} \quad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$= \operatorname{cosec}\theta\sec\theta = R.H.S$$

Hence proved

Domain:– $\theta \in R$ but $\theta \neq \frac{n\pi}{2}$

Q2. $\sec\theta\operatorname{cosec}\theta\sin\theta\cos\theta = 1$

Solution:–

L.H.S $= \sec\theta\operatorname{cosec}\theta\sin\theta\cos\theta$

$$= \frac{1}{\cancel{\cos\theta}} \cdot \frac{1}{\cancel{\sin\theta}} \cancel{\sin\theta}\cancel{\cos\theta}$$

$$= 1 = R.H.S$$

Hence proved.

Domain:– $\theta \in R$ but $\theta \neq \frac{n\pi}{2}$

Q3. $\cos\theta + \tan\theta\sin\theta = \sec\theta$

Solution:– L.H.S $= \cos\theta + \tan\theta\sin\theta$

$$= \cos\theta + \frac{\sin\theta}{\cos\theta}\cdot\sin\theta = \cos\theta + \frac{\sin^2\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos\theta} = \frac{1}{\cos\theta} = \sec\theta = R.H.S$$

Hence proved.

Domain:– $\theta \in R$ but $\theta \neq \frac{(2n+1)\pi}{2}$

Q4. $\operatorname{cosec}\theta + \tan\theta\sec\theta = \operatorname{cosec}\theta\sec^2\theta$

Solution:– L.H.S $= \operatorname{cosec}\theta + \tan\theta\sec\theta$

$$= \frac{1}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta} = \frac{1}{\sin\theta} + \frac{\sin\theta}{\cos^2\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos^2\theta} = \frac{1}{\sin\theta\cos^2\theta} = \frac{1}{\sin\theta}\cdot\frac{1}{\cos^2\theta}$$

$$= \operatorname{cosec}\theta\sec^2\theta = R.H.S$$

Hence proved

Domain:– $\theta \in R$ but $\theta \neq \frac{n\pi}{2}$

Q5. $\sec^2\theta - \operatorname{cosec}^2\theta = \tan^2\theta - \cot^2\theta$

Solution:– L.H.S $= \sec^2\theta - \operatorname{cosec}^2\theta$

$$= 1 + \tan^2\theta - (1 + \cot^2\theta) \quad \left(\begin{matrix}\because 1+\tan^2\theta = \sec^2\theta \\ 1+\cot^2\theta = \operatorname{cosec}^2\theta\end{matrix}\right)$$

$$= 1 + \tan^2\theta - 1 - \cot^2\theta$$

$$= \tan^2\theta - \cot^2\theta = R.H.S$$

Hence proved

Domain:– $\theta \in R$ but $\theta \neq \frac{(2n+1)\pi}{2}$

Q6. $\cot^2\theta - \cos^2\theta = \cot^2\theta\cos^2\theta$

Solution:– L.H.S $= \cot^2\theta - \cos^2\theta$

$$= \frac{\cos^2\theta}{\sin^2\theta} - \cos^2\theta = \frac{\cos^2\theta - \cos^2\theta\sin^2\theta}{\sin^2\theta}$$

$$= \frac{\cos^2\theta(1-\sin^2\theta)}{\sin^2\theta} = \cot^2\theta\cdot\cos^2\theta = R.H.S$$

Hence proved

Domain:– $\theta \in R$ but $\theta \neq n\pi$

Q7. $(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)=1$

Solution:-

L.H.S $=(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)$

$=\sec^2\theta-\tan^2\theta$

$=1+\tan^2\theta-\tan^2\theta$

$=1=$ R.H.S $\quad \because 1+\tan^2\theta=\sec^2\theta$

Hence proved

Domain:- $\theta\in R$ but $\theta\neq\frac{(2n+1)\pi}{2}$

Q8. $2\cos^2\theta-1=1-2\sin^2\theta$

Solution:- L.H.S $=2\cos^2\theta-1$

$=2(1-\sin^2\theta)-1 \quad \because \cos^2\theta+\sin^2\theta=1 \Rightarrow \cos^2\theta=1-\sin^2\theta$

$=2-2\sin^2\theta-1$

$=1-2\sin^2\theta=$ R.H.S

Hence proved

Domain:- $\theta\in R$

Q9. $\cos^2\theta-\sin^2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}$

Solution:- R.H.S $=\frac{1-\tan^2\theta}{1+\tan^2\theta}$

$=\frac{1-\tan^2\theta}{\sec^2\theta}=\frac{1}{\sec^2\theta}-\frac{\tan^2\theta}{\sec^2\theta}$

$=\cos^2\theta-\frac{\sin^2\theta}{\cos^2\theta}.\cos^2\theta \quad (\because 1+\tan^2\theta=\sec^2\theta)$

$=\cos^2\theta-\sin^2\theta=$ L.H.S

Hence proved.

Domain:- $\theta\in R$ but $\theta\neq\frac{(2n+1)\pi}{2}$

Q10. $\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\frac{\cot\theta-1}{\cot\theta+1}$

Solution:- R.H.S $=\frac{\cot\theta-1}{\cot\theta+1}$

$=\frac{\frac{\cos\theta}{\sin\theta}-1}{\frac{\cos\theta}{\sin\theta}+1} \quad \because \cot\theta=\frac{\cos\theta}{\sin\theta}$

$=\frac{\cos\theta-\sin\theta}{\sin\theta}\times\frac{\sin\theta}{\cos\theta+\sin\theta}$

$=\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=$ L.H.S

Hence proved.

Domain:- $\theta\in R$ but $\theta\neq n\pi$

Q11. $\frac{\sin\theta}{1+\cos\theta}+\cot\theta=\operatorname{cosec}\theta$

Solution:- L.H.S $=\frac{\sin\theta}{1+\cos\theta}+\cot\theta$

$=\frac{\sin\theta}{1+\cos\theta}+\frac{\cos\theta}{\sin\theta}$

$=\frac{\sin^2\theta+\cos^2\theta+\cos\theta}{\sin\theta(1+\cos\theta)}$

$=\frac{1+\cancel{\cos\theta}}{\sin\theta(1+\cancel{\cos\theta})}=\frac{1}{\sin\theta}=\operatorname{cosec}\theta$

$=$ R.H.S

Hence proved.

Domain:- $\theta\in R$ but $\theta\neq n\pi$

Q12. $\frac{\cot^2\theta-1}{1+\cot^2\theta}=2\cos^2\theta-1$

Solution:- L.H.S $=\frac{\cot^2\theta-1}{1+\cot^2\theta}$

$=\frac{\cot^2\theta-1}{\operatorname{cosec}^2\theta} \quad \because 1+\cot^2\theta=\operatorname{cosec}^2\theta$

$=\frac{\cot^2\theta}{\operatorname{cosec}^2\theta}-\frac{1}{\operatorname{cosec}^2\theta}$

$=\frac{\cos^2\theta}{\cancel{\sin^2\theta}}\cancel{\sin^2\theta}-\sin^2\theta$

$=\cos^2\theta-\sin^2\theta=\cos^2\theta-(1-\cos^2\theta)$

$=\cos^2\theta-1+\cos^2\theta \quad (\because \sin^2\theta+\cos^2\theta=1 \Rightarrow \sin^2\theta=1-\cos^2\theta)$

$=2\cos^2\theta-1=$ R.H.S

Hence proved.

Domain:- $\theta\in R$ but $\theta\neq n\pi$

Q13. $\dfrac{1+\cos\theta}{1-\cos\theta} = (\operatorname{cosec}\theta+\cot\theta)^2$

Solution:- R.H.S $= (\operatorname{cosec}\theta+\cot\theta)^2$

$= \left(\dfrac{1}{\sin\theta} + \dfrac{\cos\theta}{\sin\theta}\right)^2 = \left(\dfrac{1+\cos\theta}{\sin\theta}\right)^2$

$= \dfrac{(1+\cos\theta)^2}{\sin^2\theta}$ $\quad \because \sin^2\theta = 1-\cos^2\theta$

$= \dfrac{(1+\cos\theta)^2}{1-\cos^2\theta} = \dfrac{(1+\cos\theta)^{\cancel{2}}}{(1-\cos\theta)\cancel{(1+\cos\theta)}}$

$= \dfrac{1+\cos\theta}{1-\cos\theta} = $ L.H.S

Hence proved

Domain:- $\theta \in R$ but $\theta \neq n\pi$

Q14. $(\sec\theta-\tan\theta)^2 = \dfrac{1-\sin\theta}{1+\sin\theta}$

Solution:- L.H.S $= (\sec\theta-\tan\theta)^2$

$= \left(\dfrac{1}{\cos\theta} - \dfrac{\sin\theta}{\cos\theta}\right)^2 = \left(\dfrac{1-\sin\theta}{\cos\theta}\right)^2$

$= \dfrac{(1-\sin\theta)^2}{\cos^2\theta} = \dfrac{(1-\sin\theta)^2}{1-\sin^2\theta}$

$= \dfrac{(1-\sin\theta)^{\cancel{2}}}{\cancel{(1-\sin\theta)}(1+\sin\theta)}$ $\quad (\because \cos^2\theta = 1-\sin^2\theta)$

$= \dfrac{1-\sin\theta}{1+\sin\theta} = $ R.H.S

Hence proved

Domain:- $\theta \in R$ but $\theta \neq \dfrac{(2n+1)\pi}{2}$

Q15. $\dfrac{2\tan\theta}{1+\tan^2\theta} = 2\sin\theta\cos\theta$

Solution:- L.H.S $= \dfrac{2\tan\theta}{1+\tan^2\theta}$

$= \dfrac{2\tan\theta}{\sec^2\theta} = 2\tan\theta\cos^2\theta$

$= 2\dfrac{\sin\theta}{\cos\theta}\cos^2\theta = 2\sin\theta\cos\theta = $ R.H.S

Hence proved

Domain:- $\theta \in R$ but $\theta \neq \dfrac{(2n+1)\pi}{2}$

Q16. $\dfrac{1-\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1+\sin\theta}$

Solution:- L.H.S $= \dfrac{1-\sin\theta}{\cos\theta}$

$= \dfrac{1-\sin\theta}{\cos\theta} \times \dfrac{1+\sin\theta}{1+\sin\theta}$

$= \dfrac{1-\sin^2\theta}{\cos\theta(1+\sin\theta)} = \dfrac{\cos^2\theta}{\cos\theta(1+\sin\theta)}$

$= \dfrac{\cos\theta}{1+\sin\theta} = $ R.H.S $\quad (\because \cos^2\theta = 1-\sin^2\theta)$

Hence proved

Domain:- $\theta \in R$ but $\theta \neq \dfrac{(2n+1)\pi}{2}$

Q17. $(\tan\theta+\cot\theta)^2 = \sec^2\theta\operatorname{cosec}^2\theta$

Solution:- L.H.S $= (\tan\theta+\cot\theta)^2$

$= \left(\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}\right)^2 = \left(\dfrac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\right)^2$

$= \dfrac{1}{\cos^2\theta\ \sin^2\theta} = \dfrac{1}{\cos^2\theta}\cdot\dfrac{1}{\sin^2\theta}$

$= \sec^2\theta\operatorname{cosec}^2\theta = $ R.H.S

Hence proved.

Domain:- $\theta \in R$ but $\theta \neq \dfrac{n\pi}{2}$

Q18. $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \tan\theta+\sec\theta$

Solution:- $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}$ $\quad \because 1+\tan^2\theta = \sec^2\theta$

$= \dfrac{(\tan\theta+\sec\theta)-(\sec^2\theta-\tan^2\theta)}{\tan\theta-\sec\theta+1}$ $\quad 1 = \sec^2\theta-\tan^2\theta$

$= \dfrac{(\tan\theta+\sec\theta)-(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}{\tan\theta-\sec\theta+1}$

$= \dfrac{(\tan\theta+\sec\theta)[1-(\sec\theta-\tan\theta)]}{\tan\theta-\sec\theta+1}$

$= \dfrac{(\tan\theta+\sec\theta)\cancel{(1-\sec\theta+\tan\theta)}}{\cancel{\tan\theta-\sec\theta+1}}$

$= \tan\theta+\sec\theta = $ R.H.S

Hence proved

Domain:- $\theta \in R$ but $\theta \neq \dfrac{(2n+1)\pi}{2}$

Q19. $\dfrac{1}{\operatorname{cosec}\theta-\cot\theta}-\dfrac{1}{\sin\theta}=\dfrac{1}{\sin\theta}-\dfrac{1}{\operatorname{cosec}\theta+\cot\theta}$

Solution:-

$$L.H.S=\frac{1}{\operatorname{cosec}\theta-\cot\theta}-\frac{1}{\sin\theta}$$

$$=\frac{1}{\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}}-\frac{1}{\sin\theta}$$

$$=\frac{1}{\frac{1-\cos\theta}{\sin\theta}}-\frac{1}{\sin\theta}$$

$$=\frac{\sin\theta}{1-\cos\theta}-\frac{1}{\sin\theta}=\frac{\sin^2\theta-1+\cos\theta}{\sin\theta(1-\cos\theta)}$$

$$=\frac{1-\cos^2\theta-1+\cos\theta}{\sin\theta(1-\cos\theta)}=\frac{\cos\theta-\cos^2\theta}{\sin\theta(1-\cos\theta)}$$

$$=\frac{\cos\theta\cancel{(1-\cos\theta)}}{\sin\theta\cancel{(1-\cos\theta)}}=\frac{\cos\theta}{\sin\theta}=\cot\theta$$

$$R.H.S=\frac{1}{\sin\theta}-\frac{1}{\operatorname{cosec}\theta+\cot\theta}$$

$$=\frac{1}{\sin\theta}-\frac{1}{\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}}$$

$$=\frac{1}{\sin\theta}-\frac{1}{\frac{1+\cos\theta}{\sin\theta}}$$

$$=\frac{1}{\sin\theta}-\frac{\sin\theta}{1+\cos\theta}$$

$$=\frac{1+\cos\theta-\sin^2\theta}{\sin\theta(1+\cos\theta)}$$

$$=\frac{1+\cos\theta-(1-\cos^2\theta)}{\sin\theta(1+\cos\theta)}$$

$$=\frac{1+\cos\theta-1+\cos^2\theta}{\sin\theta(1+\cos\theta)}$$

$$=\frac{\cos\theta+\cos^2\theta}{\sin\theta(1+\cos\theta)}=\frac{\cos\theta\cancel{(1+\cos\theta)}}{\sin\theta\cancel{(1+\cos\theta)}}$$

$$=\frac{\cos\theta}{\sin\theta}=\cot\theta$$

Hence proved

Domain:- $\theta\in R$ but $\theta\neq\frac{(2n+1)\pi}{2}$ and $\theta\neq n\pi$

Q20. $\sin^3\theta-\cos^3\theta=(\sin\theta-\cos\theta)(1+\sin\theta\cos\theta)$

Solution:- $L.H.S=\sin^3\theta-\cos^3\theta$

$$=(\sin\theta-\cos\theta)(\sin^2\theta+\cos^2\theta+\sin\theta\cos\theta)$$

$$[\because a^3-b^3=(a-b)(a^2+b^2-ab)]$$

$$=(\sin\theta-\cos\theta)(1+\sin\theta\cos\theta)$$

$$=R.H.S$$

Hence proved

Domain:- $\theta\in R$

Q21.

$\sin^6\theta-\cos^6\theta=(\sin^2\theta-\cos^2\theta)(1-\sin^2\theta\cos^2\theta)$

Solution:-

$$L.H.S=\sin^6\theta-\cos^6\theta$$

$$=(\sin^2\theta)^3-(\cos^2\theta)^3$$

$$=(\sin^2\theta-\cos^2\theta)((\sin^2\theta)^2+(\cos^2\theta)^2+\sin^2\theta\cos^2\theta)$$

$$=(\sin^2\theta-\cos^2\theta)[(\sin^2\theta)^2+(\cos^2\theta)^2+2\sin^2\theta\cos^2\theta-\sin^2\theta\cos^2\theta]$$

$$=(\sin^2\theta-\cos^2\theta)((\sin^2\theta+\cos^2\theta)^2-\sin^2\theta\cos^2\theta]$$

$$=(\sin^2\theta-\cos^2\theta)((1)^2-\sin^2\theta\cos^2\theta)$$

$$=(\sin^2\theta-\cos^2\theta)(1-\sin^2\theta\cos^2\theta)$$

$$=R.H.S$$

Hence proved **Domain:-** $\theta\in R$

Q22. $\sin^6\theta+\cos^6\theta=1-3\sin^2\theta\cos^2\theta$

Solution:- $L.H.S=\sin^6\theta+\cos^6\theta$

$$=(\sin^2\theta)^3+(\cos^2\theta)^3$$

$$=(\sin^2\theta+\cos^2\theta)[(\sin^2\theta)^2+(\cos^2\theta)^2-\sin^2\theta\cos^2\theta]$$

$$=(1)[(\sin^2\theta)^2+(\cos^2\theta)^2+2\sin^2\theta\cos^2\theta-3\sin^2\theta\cos^2\theta]$$

$$=((\sin^2\theta+\cos^2\theta)^2-3\sin^2\theta\cos^2\theta)$$

$$=[(1)^2-3\sin^2\theta\cos^2\theta]=1-3\sin^2\theta\cos^2\theta$$

$$=R.H.S$$ Hence proved

Domain:- $\theta\in R$

Q23. $\dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta} = 2\sec^2\theta$

Solution:- $L.H.S = \dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta}$

$$= \frac{1-\cancel{\sin\theta} + 1 + \cancel{\sin\theta})}{(1+\sin\theta)(1-\sin\theta)}$$

$$= \frac{2}{1-\sin^2\theta}$$

$$= \frac{2}{\cos^2\theta} = 2\sec^2\theta = R.H.S$$

Hence proved. Domain:- $\theta \in R$ but $\theta \neq \dfrac{(2n+1)\pi}{2}$

Q24. $\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta} + \dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} = \dfrac{2}{1-2\sin^2\theta}$

Solution:- $L.H.S = \dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta} + \dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$

$$= \frac{(\cos\theta+\sin\theta)^2 + (\cos\theta-\sin\theta)^2}{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$$

$$= \frac{\cos^2\theta+\sin^2\theta+\cancel{2\cos\theta\sin\theta}+\cos^2\theta+\sin^2\theta-\cancel{2\cos\theta\sin\theta}}{\cos^2\theta-\sin^2\theta}$$

$$= \frac{2\cos^2\theta+2\sin^2\theta}{1-\sin^2\theta-\sin^2\theta} = \frac{2(\cos^2\theta+\sin^2\theta)}{1-2\sin^2\theta}$$

$$= \frac{2}{1-2\sin^2\theta} = R.H.S \qquad \left(\because \cos^2\theta+\sin^2\theta = 1 \rightarrow \cos^2\theta = 1-\sin^2\theta\right)$$

Hence proved.

Domain:- $\theta \in R$ but $\theta \neq \dfrac{(2n+1)\pi}{2}$