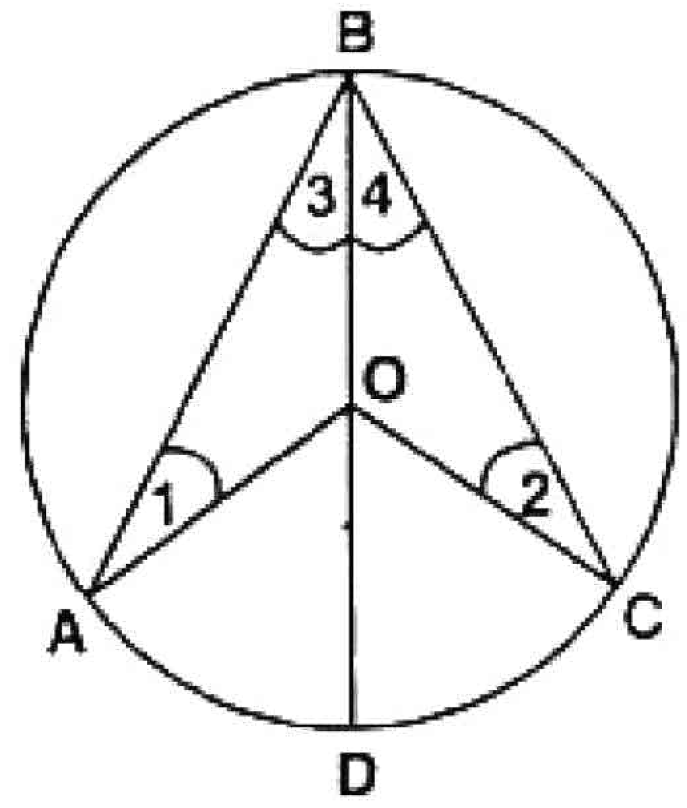
Unit 12 ANGLE IN A SEGMENT OF A CIRCLE

THEOREM 1

The measure of a central angle of a minor arc of a circle, is double that of the angle subtended by the corresponding major arc.



Given: $\overset{\frown}{AC}$ is an arc of a circle with centre O

Whereas $\angle AOC$ is the central angle and $\angle ABC$ is circum angle.

To Prove: $m\angle AOC = 2m\angle ABC$

Construction: Join B with O and produce it to meet the circle at D.

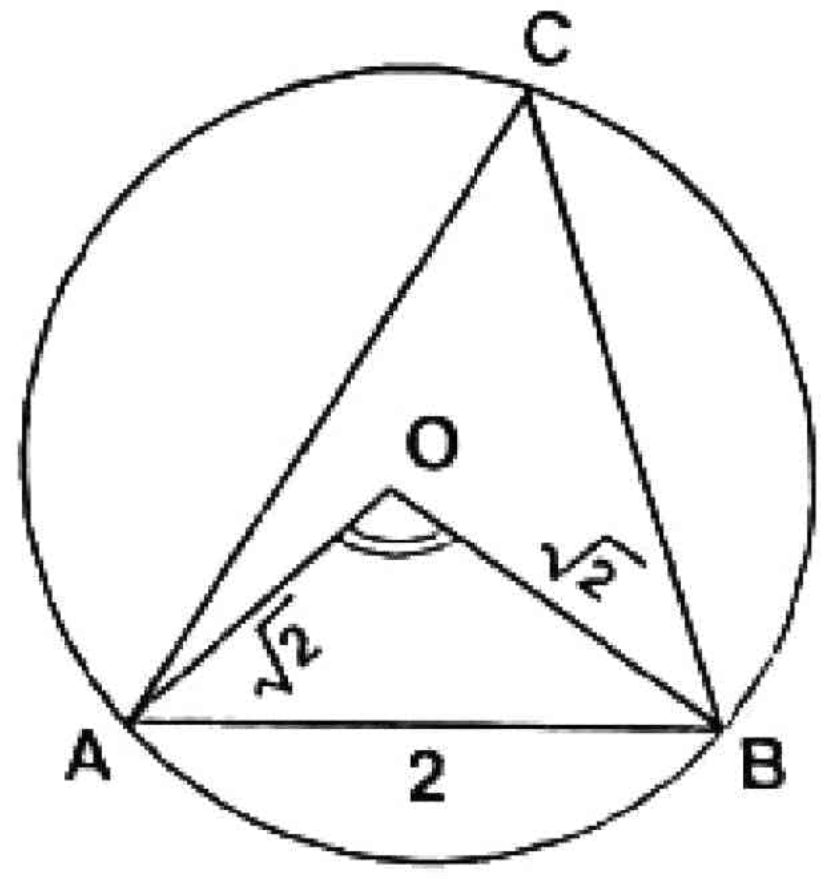
Write angles $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$ and $\angle 6$ as shown in the figure.

Proof:

Statements	Reasons
As $m\angle 1 = m\angle 3$ .... (i)	Angles opposite to equal sides in ΔOAB
and $m\angle 2 = m\angle 4$ .... (ii)	Angles opposite to equal sides in ΔOBC.
Now $m\angle 5 = m\angle 1 + m\angle 3$ ..... (iii)	External angle is the sum of internal opposite angles.
Similarly $m\angle 6 = m\angle 2 + \angle 4$ .... (iv)	
Again $m\angle 5 = m\angle 3 + m\angle 3 = 2m\angle 3$......(v)	Using (i) and (iii)
and $m\angle 6 = m\angle 4 + m\angle 4 = 2m\angle 4$.......(vi)	Using (ii) and (iv)
Then from figure	
$\Rightarrow$ $m\angle 5 + m\angle 6 = 2m\angle 3 + 2m\angle 4$	Adding (v) and (vi)
$\Rightarrow$ $m\angle AOC = 2(m\angle 3 + m\angle 4) = 2\,m\,\angle ABC$	

Example:

The radius of a circle is $\sqrt{2}$ cm. A chord 2 cm in length divides the circle into two segments. Prove that the angle of larger segment is 45°.



Given: In a circle with centre O and radius $m\overline{OA} = m\overline{OB} = \sqrt{2}$ cm.

The length of chord $\overline{AB} = 2$ cm divides the circle into two segments with ACB as larger one.

To Prove: $m\angle ACB = 45°$

Construction: Join O with A and O with B.

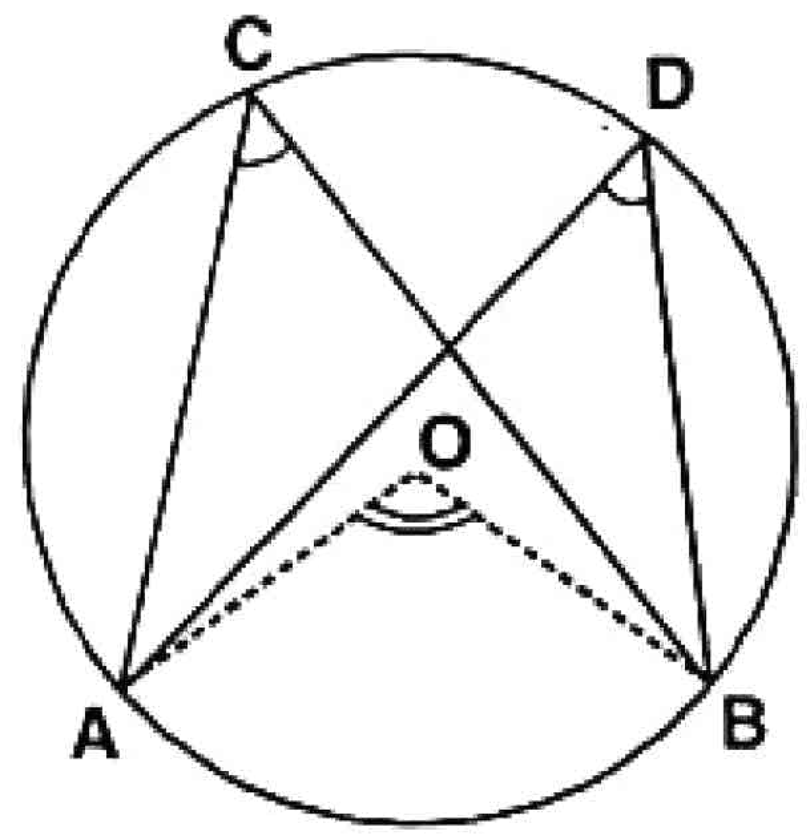
Proof:

Statements	Reasons
In ΔOAB	
$(m\overline{OA})^2 + (m\overline{OB})^2 = (\sqrt{2})^2 + (\sqrt{2})^2$	$m\overline{OA} = m\overline{OB} = \sqrt{2}$ cm
$= 2 + 2 = 4$	
$= (2)^2 = (m\overline{AB})^2$	Given: $m\overline{AB} = 2cm$
$\therefore$ ΔAOB is right angled triangle with $m\angle AOB = 90^\circ$	Which being a central angle standing on an arc AB.
	By theorem 1
Then $m\angle ACB = \frac{1}{2} m\angle AOB$	Circum angle is half of the central angle.
$= \frac{1}{2}(90^\circ) = 45^\circ$	

THEOREM 2

Any two angles in the same segment of a circle are equal.

Given: $\angle ACB$ and $\angle ADB$ are the circum angles in the same segment of a circle with centre O.



To Prove: $m\angle ACB = m\angle ADB$

Construction: Join O with A and O with B.

So that $\angle AOB$ is the central angle.

Proof:

Statements	Reasons
Standing on the same arc AB of a circle.	
$\angle AOB$ is the central angle whereas	Construction
$\angle ACB$ and $\angle ADB$ are circum angles	Given
$\therefore$ $m\angle AOB = 2m\angle ACB$ .... (i)	By theorem 1
and $m\angle AOB = 2m\angle ADB$ .... (ii)	By theorem 1
$\Rightarrow 2m\angle ACB = 2m\angle ADB$	Using (i) and (ii)
Hence, $m\angle ACB = m\angle ADB$	

THEOREM 3

The angle,

- In a semi-circle is a right angle,
- In a segment greater than a semi circle is less than a right angle,
- In a segment less than a semi-circle is greater than a right angle.

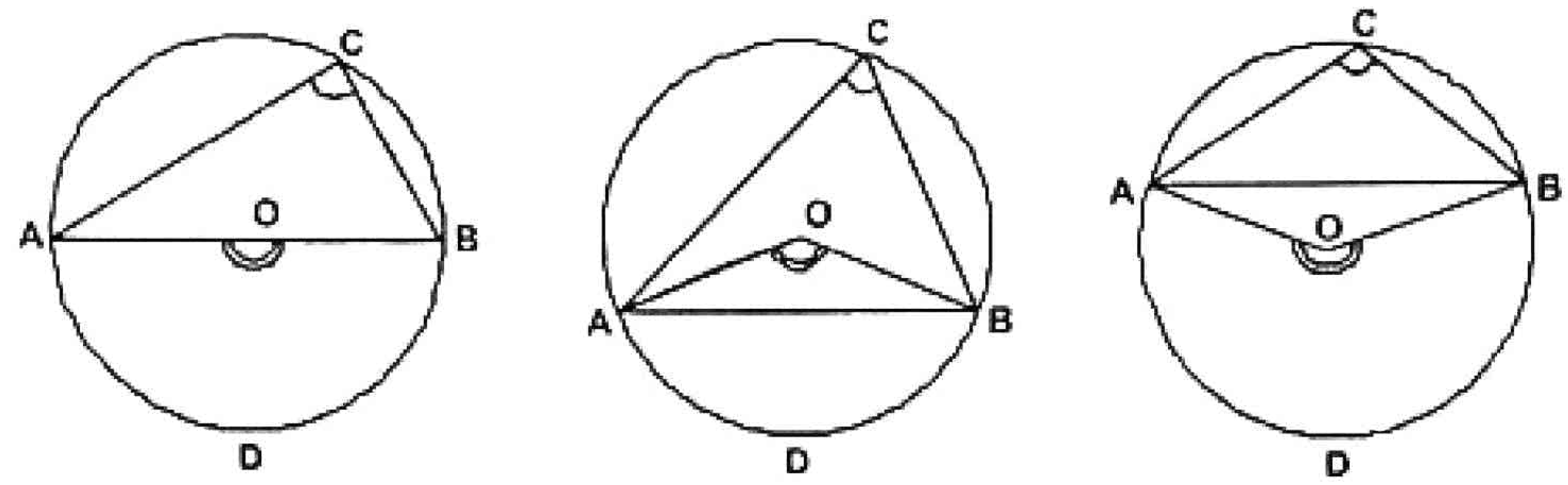


Fig. I Fig. II Fig. III

Given: $\overline{AB}$ is the chord corresponding to an arc ADB

Whereas $\angle AOB$ is a central angle and $\angle$ ACB is a circum angle of a circle with centre O.

To Prove:

In fig (I) If sector ACB is a semi circle then $m\angle ACB = 1\angle rt$

In fig (II) If sector ACB is greater than a semi circle then $m\angle ACB < 1\angle rt$

In fig (III) If sector ACB is less than a semi circle then $m\angle ACB > 1\angle rt$.

Proof:

Statements	Reasons
In each figure, $\overline{AB}$ is the chord of a circle with centre O.	Given
$\angle AOB$ is the central angle standing on an arc ADB.	
Whereas $\angle ACB$ is the circum angle	Given
Such that $m\angle AOB = 2m\angle ACB$ .... (i)	
Now in fig (I) $m\angle AOB = 180^\circ$	By theorem 1
$\therefore$ $m\angle AOB = 2\angle rt$ .... (ii)	A straight angle
$\Rightarrow$ $m\angle ACB = 1\angle rt$	
In fig (II) $m\angle AOB < 180^\circ$	Using (i) and (ii)
$\therefore$ $m\angle AOB < 2\angle rt$ .... (iii)	
$\Rightarrow$ $m\angle ACB < 1\angle rt$	
In fig (III) $m\angle AOB > 180^\circ$	Using (i) and (iii)
$\therefore$ $m\angle AOB > 2\angle rt$ .... (iv)	
$\Rightarrow$ $m\angle ACB > 1\angle rt$	
	Using (i) and (iv)

Corollary 1:

The angles subtended by an arc at the circumference of a circle are equal.

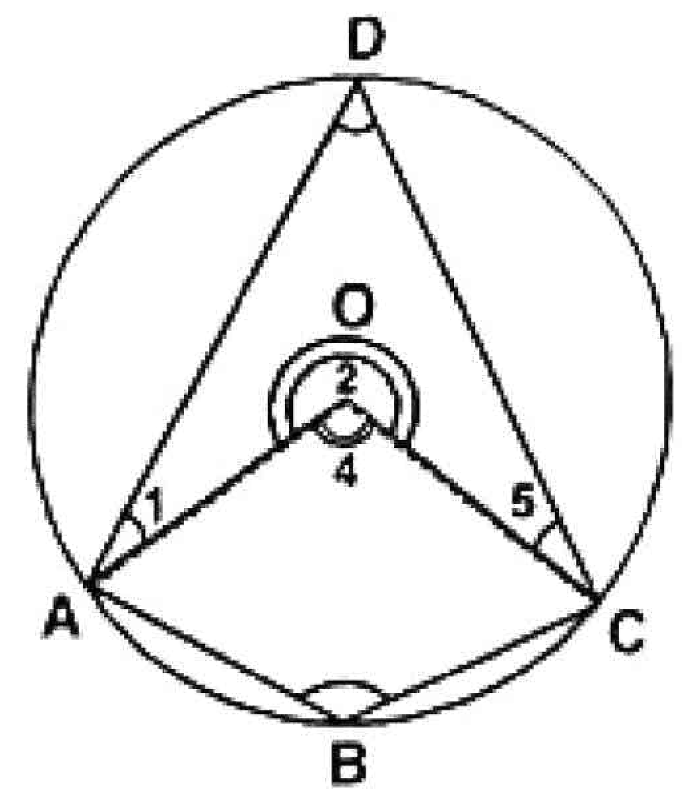
Corollary 2:

The angles in the same segment of a circle are congruent.

THEOREM 4

The opposite angles of any quadrilateral inscribed in a circle are supplementary.

Given: ABCD is a quadrilateral inscribed in a circle with centre O.



To Prove: $\begin{cases} m\angle A + m\angle C = 2\angle rts \\ m\angle B + m\angle D = 2\angle rts \end{cases}$

Construction: Draw $\overline{OA}$ and $\overline{OC}$.

Write ∠1, ∠2, ∠3, ∠4, ∠5 and ∠6 as shown in the figure.

Proof:

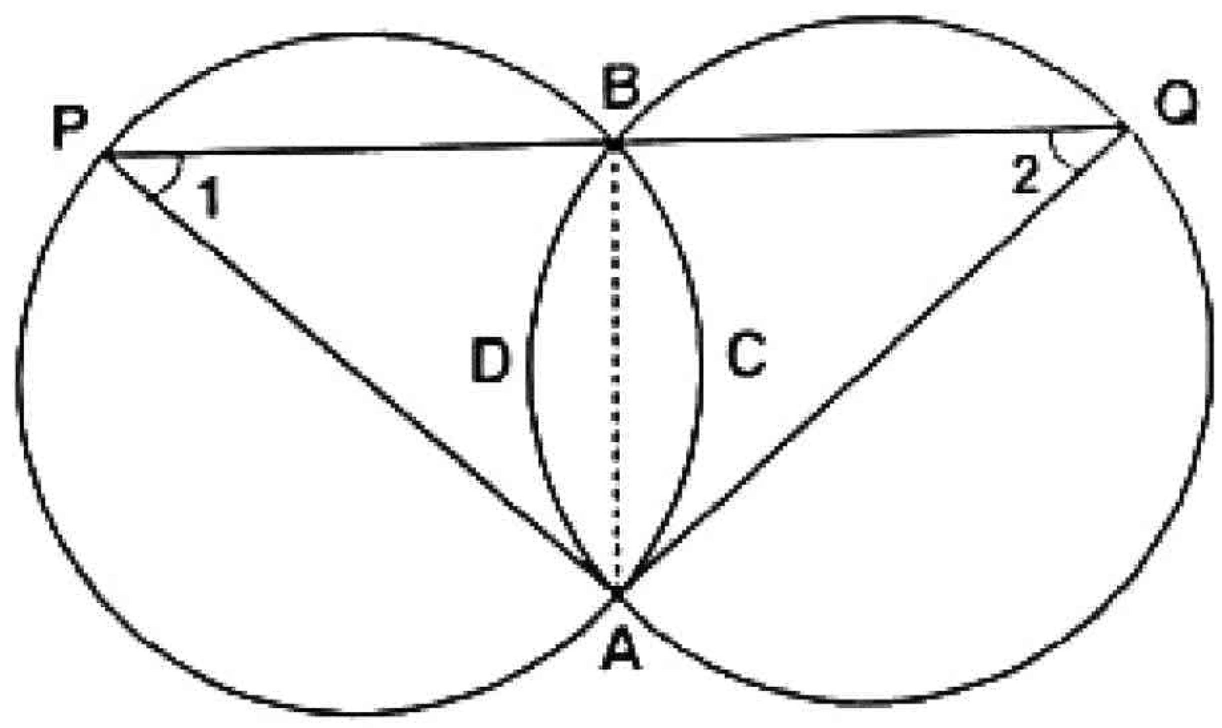
Statements	Reasons
Standing on the same arc ADC, ∠2 is a central angle. Whereas ∠B is the circum angle	Arc ADC of the circle with centre O.
$\therefore \quad m\angle B = \frac{1}{2}(m\angle 2)$ .... (i)	By theorem 1
Standing on the same arc ABC, ∠4 is a central angle whereas ∠D is the circum angle.	Arc ABC of the circle with centre O.
$\therefore \quad m\angle D = \frac{1}{2}(m\angle 4)$ .... (ii)	By theorem 1
$\Rightarrow \quad m\angle B + m\angle D = \frac{1}{2}m\angle 2 + \frac{1}{2}m\angle 4$	Adding (i) and (ii)
$= \frac{1}{2}(m\angle 2 + m\angle 4) = \frac{1}{2}$ (Total central angle)	
i.e., $m\angle B + m\angle D = \frac{1}{2}(4\angle rt) = 2\angle rt$	
Similarly $m\angle A + m\angle C = 2\angle rt$	

Corollary 1: In equal circles or in the same circle if two minor arcs are equal then angles inscribed by their corresponding major arcs are also equal.

Corollary 2: In equal circles or in the same circle, two equal arcs subtend equal angles at the circumference and vice versa.

Example 1: **Two equal circles intersect in A and B. Through B, a straight line is drawn to meet the circumferences at P and Q respectively. Prove that $m\overline{AP} = m\overline{AQ}$.**

Given: Two equal circles cut each other at points A and B. A straight line PBQ drawn through B meets the circles at P and Q respectively.



To Prove: $m\overline{AP} = m\overline{AQ}$

Construction: Join the points A and B. Also draw $\overline{AP}$ and $\overline{AQ}$.

Write ∠1 and ∠2 as shown in the figure.

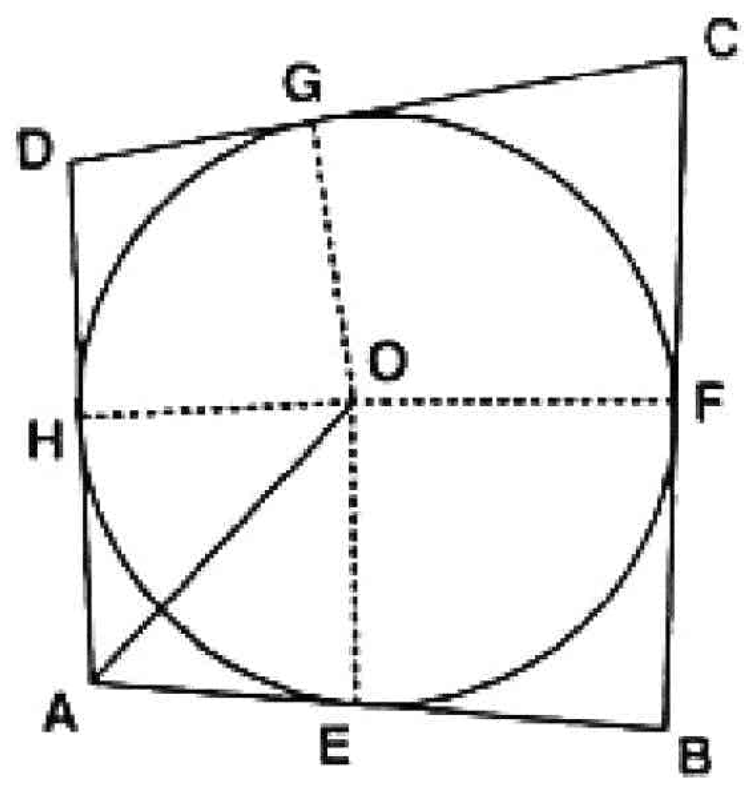
Proof

Statements	Reasons
∵ $m\widehat{ACB} = m\widehat{ADB}$	Arcs about the common chord AB.
∴ $m\angle 1 = m\angle 2$	Corresponding angles made by opposite arcs.
So $m\overline{AQ} = m\overline{AP}$	Sides opposite to equal angles in ΔAPQ.
or $m\overline{AP} = m\overline{AQ}$	

Example 2: **ABCD is a quadrilateral circumscribed about a circle.**

Show that $m\overline{AB} + m\overline{CD} = m\overline{BC} + m\overline{DA}$

Given: ABCD is a quadrilateral circumscribed about a circle with centre O. So that each side becomes tangent to the circle.



To Prove: $m\overline{AB} + m\overline{CD} = m\overline{BC} + m\overline{DA}$

Construction: Drawn $\overline{OE} \perp \overline{AB}$, $\overline{OF} \perp \overline{BC}$

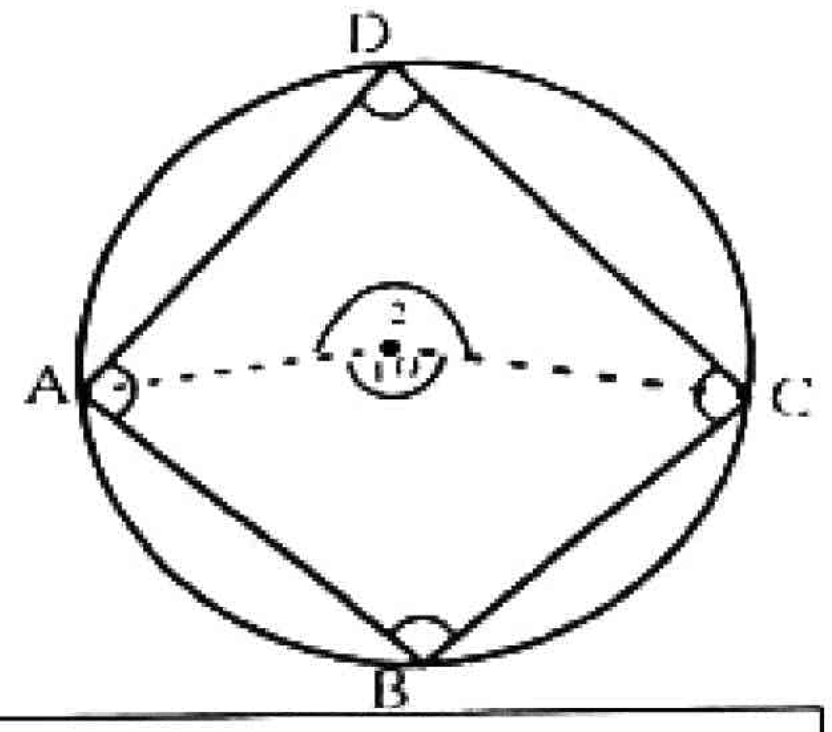
$\overline{OG} \perp \overline{CD}$ and $\overline{OH} \perp \overline{DA}$

Proof

Statements	Reasons
∴ $m\overline{AE} = m\overline{HA}$ and $m\overline{EB} = m\overline{BF}$ ....(i)	Since tangents drawn from a point to the circle are equal in length.
$m\overline{CG} = m\overline{FC}$ and $m\overline{GD} = m\overline{DH}$ ....(ii)	
$(m\overline{AE} + m\overline{EB}) + (m\overline{CG} + m\overline{GD}) = (m\overline{BF} + m\overline{FC}) + (m\overline{DH} + m\overline{HA})$	Adding (i) and (ii).
or $m\overline{AB} + m\overline{CD} = m\overline{BC} + m\overline{DA}$	

EXERCISE 12.1

Q.1 Prove that in a given cyclic quadrilateral, sum of opposite angles is two right angles and conversely.



Given: A circle with centre "O"

ABCD is a cyclic quadrilateral

To Prove: $m\angle B + m\angle D = 180^\circ$

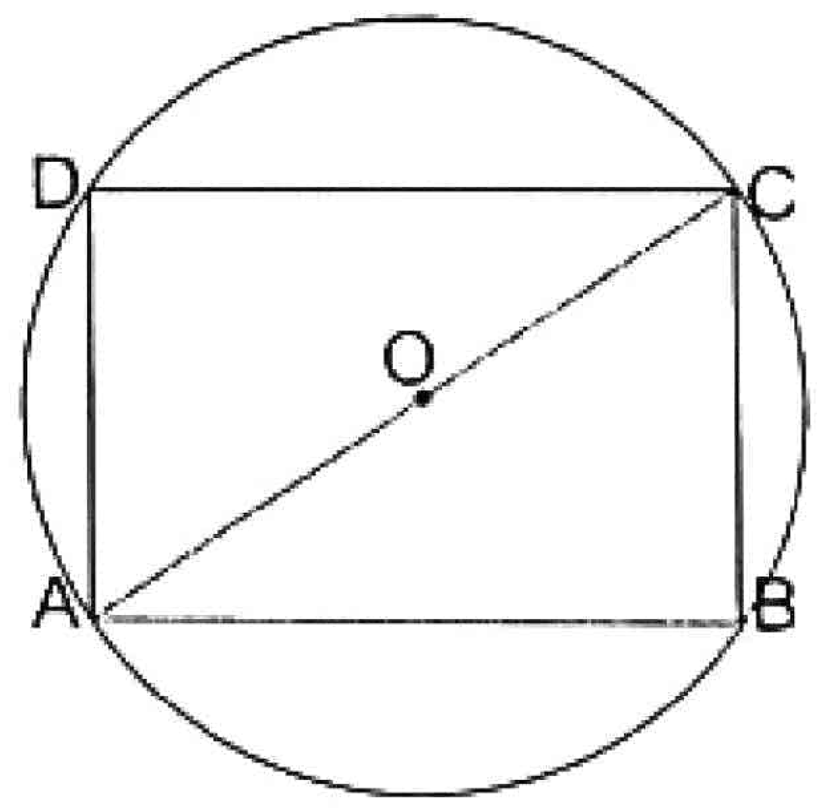
$m\angle BCD + m\angle DAB = 180^\circ$

Construction: Join O with A and C

Proof:

Statements	Reasons
$m\angle 1 = 2m\angle D$..........(i)	$\angle 1$, $\angle 2$ are central angles and $\angle D$, $\angle B$ are
$m\angle 2 = 2m\angle B$..........(ii)	circum angles in Arcs
$m\angle 1 + m\angle 2 = 2m\angle D + 2m\angle B$	Adding (i) and (ii)
$m\angle 1 + m\angle 2 = 2(m\angle D + m\angle B)$	
or $2(m\angle D + m\angle B) = m\angle 1 + m\angle 2$	By symmetric property
$2(m\angle D + m\angle B) = 360^\circ$	Sum of all central angles is 360°
$m\angle D + m\angle B = \frac{360^\circ}{2}$	Dividing by 2
$m\angle D + m\angle B = 180^\circ$	
Similarly $m\angle BCD + m\angle DAB = 180^\circ$	

Q.2 Show that parallelogram inscribed in a circle will be a rectangle.



Given: ABCD is a parallelogram inscribed in the circle with centre "O"

$m\overline{AB} = m\overline{DC}$ and $\overline{AB} \parallel \overline{DC}$

$m\overline{AD} = m\overline{BC}$ and $\overline{AD} \parallel \overline{BC}$

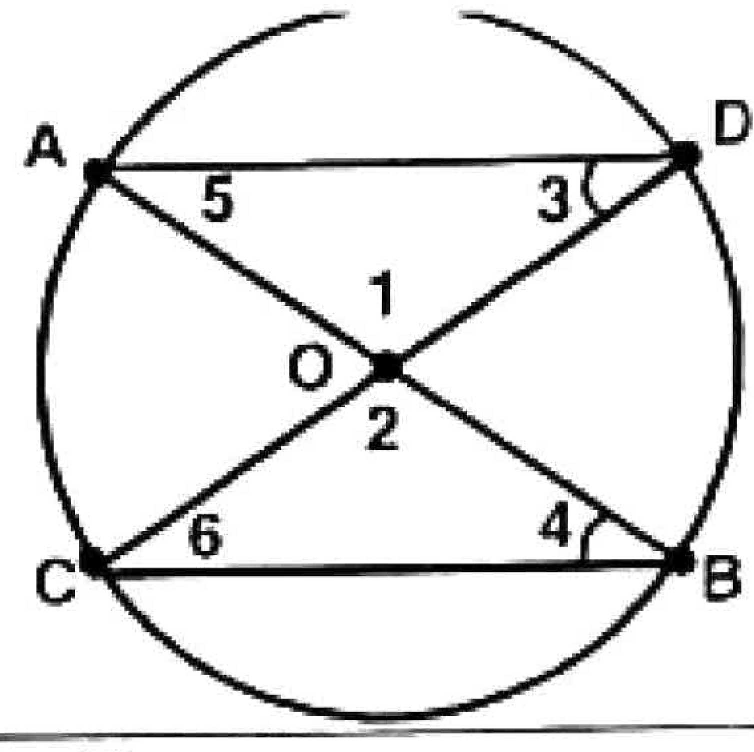
To Prove: ABCD is a rectangle

Construction: Join A with C

Proof:

Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta ADC$	
$m\overline{AC} = m\overline{AC}$	Common
$m\overline{AB} = m\overline{DC}$	Given
$m\overline{BC} = m\overline{AD}$	Given
$\therefore$ $\Delta ABC \cong \Delta ADC$	S.S. S $\cong$ S. S. S
Thus, $m\angle B = m\angle D$........................(i)	Corresponding angles of congruent triangles
$m\angle B + m\angle D = 180^\circ$..........(ii)	Opposite angles of parallelogram
$\Rightarrow$ $m\angle B = m\angle D = 90^\circ$	From (i)
Similarly $m\angle BAD = m\angle BCD = 90^\circ$	From (i) and (ii)
Hence ABCD is rectangle	

Q.3 AOB and COD are two intersecting chords of a circle. Show that ΔAOD and BOC are equiangular.



Given: In a circle $\overline{AOB}$ and $\overline{COD}$ are two intersecting chords at point O.

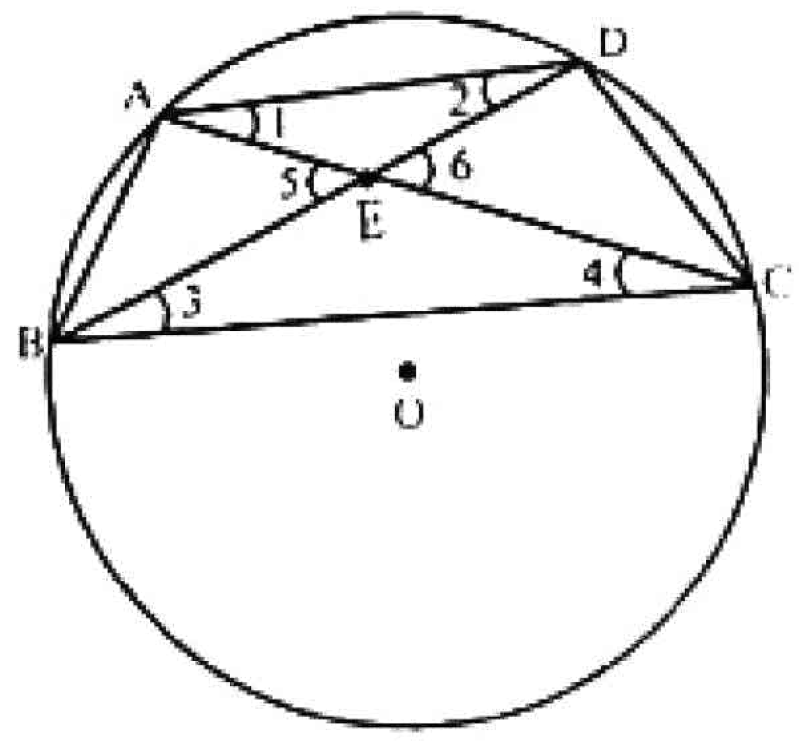
To Prove: ΔAOD and ΔBOC are equiangular

Construction: Join A with C and D. Join B with C and D.

Proof:

Statements	Reasons
$m\angle 1 \cong m\angle 2$........(i)	Vertical angles
$\overline{AC}$ is chord and angles $\angle 3$, $\angle 4$ are in the same segment.	
$\angle 3 \cong \angle 4$............(ii)	
Now $\overline{BD}$ is chord and angles $\angle 5$, $\angle 6$ are in the same segments	
Therefore $\angle 5 \cong \angle 6$............(iii)	
Thus, ΔAOD and ΔBOC are equiangular	From (i), (ii) and (iii)

Q.4 $\overline{AD}$ and $\overline{BC}$ are two parallel chords of a circle prove that arc AB ≅ are CD and arc AC ≅ arc BD.



Given: A circle with centre "O". Two chords $\overline{AD}$ and $\overline{BC}$ are such that $\overline{AD} \parallel \overline{BC}$.

To Prove: arc AB ≅ arc CD and arc AC ≅ arc BD

Construction: Join A to B and C. Join D to B and C. $\overline{AC}$ and $\overline{BD}$ intersect each other at point E. some angles are named as $\angle 1, \angle 2, \angle 3, \angle 4, \angle 5, \angle 6$.

Proof:

Statements	Reasons
$m\angle 1 = m\angle 3$ ....(i)	Angles inscribed by an arc in the same segment are equal.
$m\angle 2 = m\angle 4$ ....(ii)	
$m\angle 1 = m\angle 4$ ....(iii)	Alternate angles are congruent $(\overline{AD} \parallel \overline{BC})$
$m\angle 3 = m\angle 4$ ....(iv)	From (i) and (iii)
$m\angle 1 = m\angle 2$ ....(v)	From (ii) and (iii)
In $\Delta AEB \leftrightarrow \Delta DEC$	
$\overline{AE} \cong \overline{ED}$	Side opposite to equal angles (v)
$m\angle 5 = m\angle 6$	vertical angles
$\overline{BE} \cong \overline{EC}$	Sides opposite to equal angles (iv)
∴ $\Delta AED \cong \Delta DEC$	S.A.S ≅ S.A.S
$\overline{AB} \cong \overline{CD}$	Corresponding sides of congruent.
Thus arc AB ≅ arc CD (Hence Proved)	Arcs corresponding to congruent chords are congruent.
$m\widehat{BC} \cong m\widehat{CB}$	Self congruent
$m\widehat{BA} + m\widehat{AC} = m\widehat{CD} + m\widehat{DB}$	
$\cancel{m\widehat{AB}} + m\widehat{AC} = \cancel{m\widehat{AB}} + m\widehat{BD}$	∵ arc $\widehat{AB}$ ≅ arc CD proved
$m\widehat{AC} = m\widehat{BD}$	
or arc AC ≅ arc BD (Hence proved)	

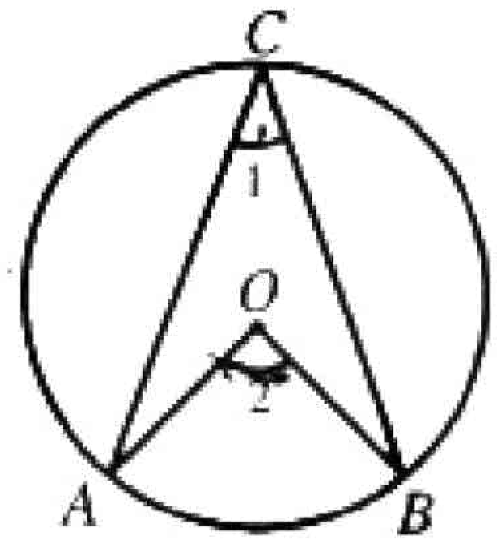
MISCELLANEOUS EXERCISE – 12

Q. 1 Multiple Choice Questions

Four possible answers are given for the following questions.

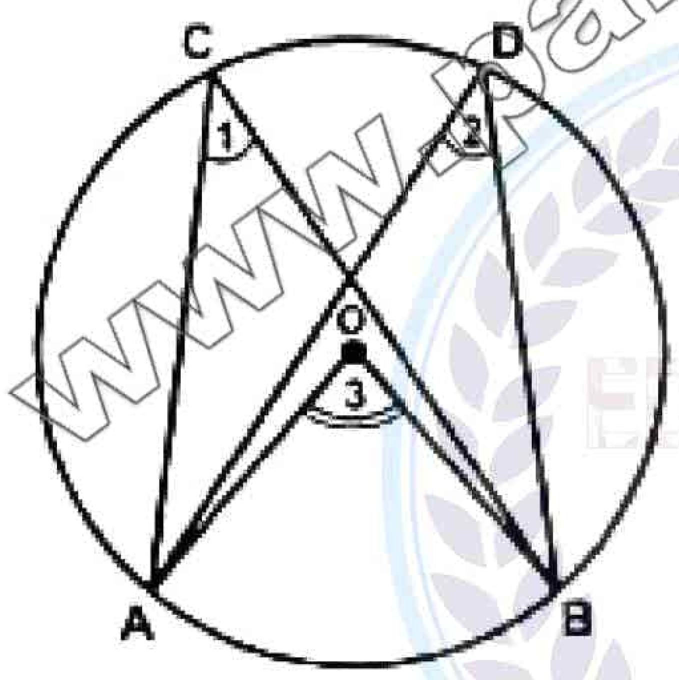
1. A circle passes through the vertices of a right angled ΔABC with $m\overline{AC} = 3$cm and $m\overline{BC} = 4$cm, $m\angle C = 90^\circ$. Radius of the circle is:
   (a) 1.5 cm (b) 2.0 cm
   (c) 2.5 cm (d) 3.5 cm

2. In the adjacent circular figure, central and inscribed angles stand on the same arc AB.



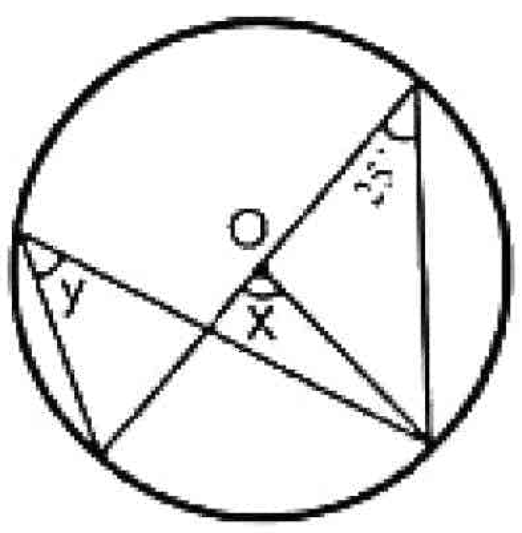
   (a) $m\angle 1 = m\angle 2$
   (b) $m\angle 1 = 2m\angle 2$
   (c) $m\angle 2 = 3m\angle 1$
   (d) $m\angle 2 = 2m\angle 1$

3. In the adjacent figure if $m\angle 3 = 75^\circ$, then find $m\angle 1$ and $m\angle 2$



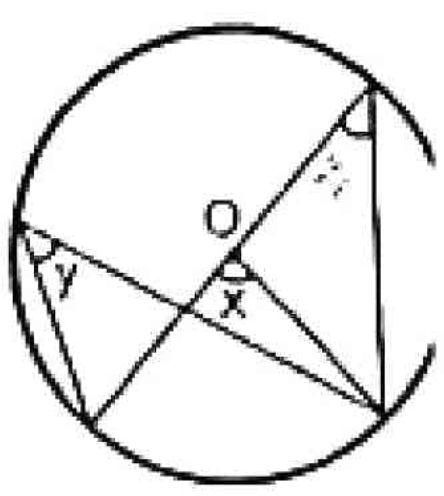
   (a) $37\frac{1^\circ}{2}, 37\frac{1^\circ}{2}$
   (b) $37\frac{1^\circ}{2}, 75^\circ$
   (c) $75^\circ, 37\frac{1^\circ}{2}$
   (d) $75^\circ, 75^\circ$

4. Given that O is the centre of the circle, the angle marked x will be.



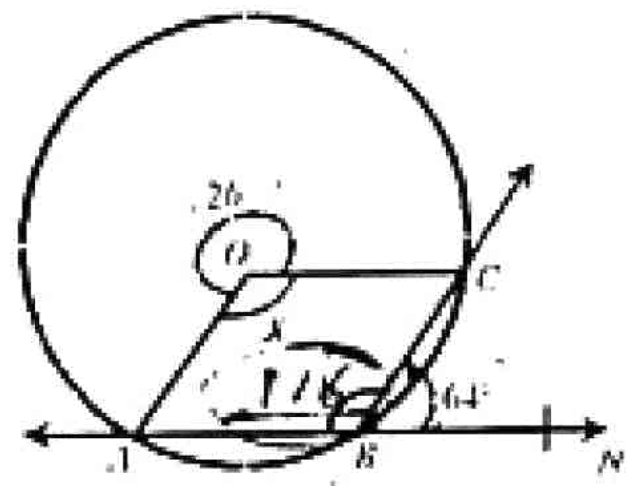
   (a) $12\frac{1^\circ}{2}$ (b) 25°
   (c) 50° (d) 75°

5. Given that O is the centre of the circle the angle marked y will be.



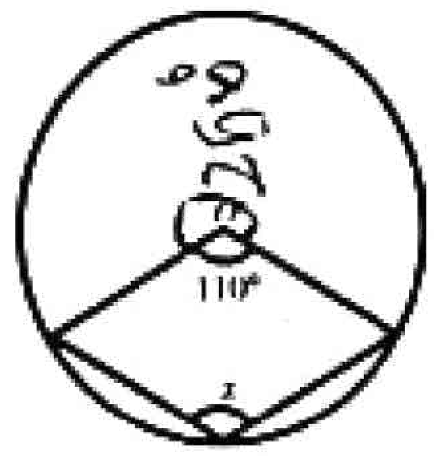
   (a) $12\frac{1^\circ}{2}$ (b) 25°
   (c) 50° (d) 75°

6. In the figure, O is the centre of the circle and $\overleftrightarrow{ABN}$ is a straight line. The obtuse angle AOC = x is.



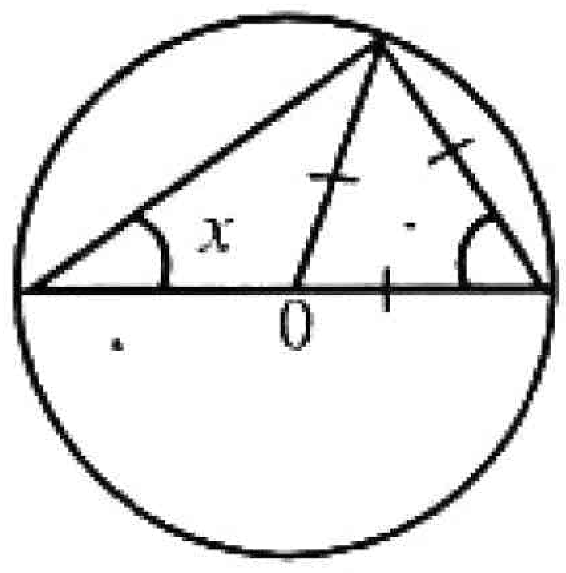
   (a) 32°
   (b) 64°
   (c) 96°
   (d) 116°

7. In the figure, O is the centre of the circle, then the angle x is



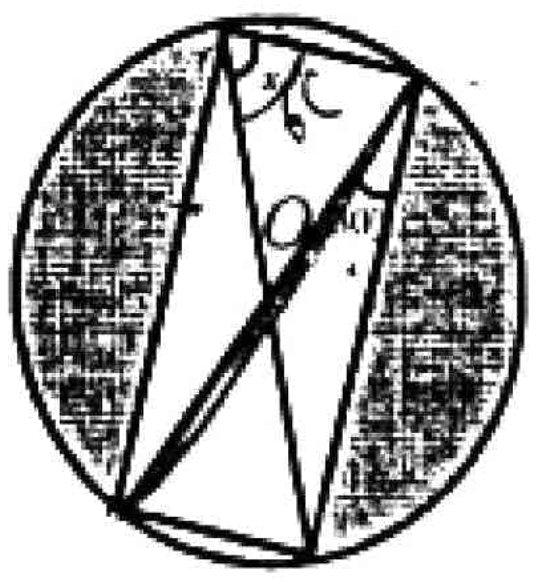
   (a) 55°
   (b) 110°
   (c) 220°
   (d) 125°

8. In the figure, O is the centre of the circle then angle x is.



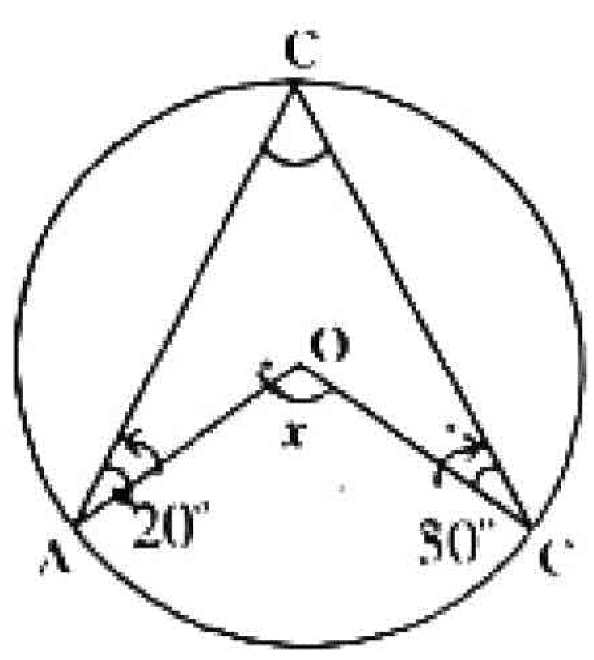
   (a) 15°
   (b) 30°
   (c) 45°
   (d) 60°

9. In the figure ,O is the centre of the circle then the angle x is



   (a) 15°
   (b) 30°
   (c) 45°
   (d) 60°

10. In the figure, O is the centre of the circle then the angle x is.



   (a) 50°
   (b) 75°
   (c) 100°
   (d) 125°

ANSWER KEY

1.	c	2.	d	3.	a	4.	c	5.	b
6.	d	7.	b	8.	b	9.	d	10.	c