MATHEMATICS 2nd YEAR

UNIT # 06

CONIC SECTIONS
(6.7-6.9)

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Contents

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اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Tangents and Normals

Remember,

Given curve and point (x_1, y_1) (say)

slope of tangent at $P(x_1,y_1) = m = \left.\frac{dy}{dx}\right|_{(x_1,y_1)}$

slope of Normal $= -\frac{1}{m}$

Equation of tangent is

$$y - y_1 = m(x - x_1)$$

Equation of Normal is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

> The normal to a curve at a Point on the curve is $\perp$ar to the tangent through the point of tangency.

Example 1. Find equation of tangent and normal to

i) $y^2 = 4ax$

ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

iii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; at the point (x_1,y_1)

Solution:- i) $y^2 = 4ax \longrightarrow (1)$

Diff (1) w.r.t x, we get

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{2a}{y_1} \quad \left(\text{slope of tangent at } (x_1,y_1)\right)$$

Now Equation of tangent to (1) at (x_1, y_1) is

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

$$\Rightarrow yy_1 - y_1^2 = 2ax - 2ax_1$$

$$\Rightarrow yy_1 - 2ax = y_1^2 - 2ax_1$$

Adding $-2ax_1$ to both sides

$$\Rightarrow yy_1 - 2ax - 2ax_1 = y_1^2 - 2ax_1 - 2ax_1$$

$$\Rightarrow yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

$$\Rightarrow yy_1 = 2a(x + x_1) + y_1^2 - 4ax_1 \longrightarrow (2)$$

$\because (x_1,y_1)$ lies on $y^2 = 4ax$

$$\Rightarrow y_1^2 = 4ax_1$$

or $y_1^2 - 4ax_1 = 0$ put in (2)

$$\Rightarrow yy_1 = 2a(x + x_1) \quad (\text{Req. tangent eq.})$$

Now slope of normal $= -\frac{y_1}{2a}$

($\because$ slope of normal = -ive reciprocal of slope of tangent)

so equation of normal is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow (1)$

Diff (1) w.r.t 'x', we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{\cancel{2}y}{b^2}\frac{dy}{dx} = -\frac{\cancel{2}x}{a^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = -\frac{b^2}{a^2}\frac{x_1}{y_1}$$

Now equation of tangent to (1) at (x_1,y_1) is

$$y - y_1 = -\frac{b^2 x_1}{a^2 y_1}(x - x_1)$$

$$\Rightarrow \frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = \frac{-xx_1}{a^2} + \frac{x_1^2}{a^2}$$

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \longrightarrow (2)$$

Since (x_1, y_1) lies on (1) so

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \quad \text{put in (2)}$$

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \left(\text{Req. tangent eq.}\right)$$

Now slope of normal at (x_1, y_1) is $\frac{a^2 y_1}{b^2 x_1}$

so eq. of normal at (x_1, y_1) is

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

$$\Rightarrow b^2 x_1 y - b^2 x_1 y_1 = a^2 y_1 x - a^2 x_1 y_1$$

or $a^2 y_1 x - b^2 x_1 y = x_1 y_1 (a^2 - b^2)$

$\div$ing both sides by $x_1 y_1$

$$\Rightarrow \frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

iii) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \longrightarrow (1)$$

Diff (1) w.r.t x, we get

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \cancel{-}\frac{\cancel{2}y}{b^2}\frac{dy}{dx} = \cancel{-}\frac{\cancel{2}x}{a^2}$$

or $$\frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \frac{b^2}{a^2}\frac{x_1}{y_1}$$

Now Eq. of tangent to (1) is

$$y - y_1 = \frac{b^2 x_1}{a^2 y_1}(x - x_1)$$

$$\Rightarrow \frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = \frac{xx_1}{a^2} - \frac{x_1^2}{a^2}$$

$$\Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} \longrightarrow (2)$$

since (x_1, y_1) lies on (1) so

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \quad \text{put in (2)}$$

$$\Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad \left(\text{Req. tangent eq.}\right)$$

Now slope of Normal at (x_1, y_1) is $-\frac{a^2 y_1}{b^2 x_1}$

so eq. of normal at (x_1, y_1) is

$$y - y_1 = -\frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

$$b^2 x_1 y - b^2 x_1 y_1 = -a^2 y_1 x + a^2 x_1 y_1$$

$$a^2 y_1 x + b^2 x_1 y = (a^2 + b^2) x_1 y_1$$

$\div$ing both sides by $x_1 y_1$

$$\Rightarrow \frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$$

Remarks:-

An equation of the tangent at the point (x_1, y_1) of any conic can be written by making replacements in the equation of the conic as under:

Replace x^2 by xx_1

y^2 by yy_1

x by $\frac{1}{2}(x + x_1)$

y by $\frac{1}{2}(y + y_1)$

Example 1. Write equations of the tangent and normal to the parabola $x^2 = 16y$ at the point whose abscissa is 8.

Solution:-

$$x^2 = 16y \longrightarrow (1)$$

$\because$ x(abscissa) $= 8$ lies on (1)

so $64 = 16y \rightarrow y = 4$

Thus we have to find eq. of tangent and normal at $(8,4)$

Diff (1) w.r.t. x, we get

$$\cancel{16}^{\,8}\frac{dy}{dx} = \cancel{2}x$$

$$\rightarrow \frac{dy}{dx} = \frac{x}{8}$$

$$\rightarrow \left.\frac{dy}{dx}\right|_{(8,4)} = \frac{8}{8} = 1$$

Now eq. of tangent at $(8,4)$ is $y - 4 = 1(x-8)$

$$\rightarrow y - 4 = x - 8$$

$$\rightarrow x - y = 4$$

or $x - y - 4 = 0$

Also eq. of normal at $(8,4)$ is $y - 4 = -1(x-8)$

$$\rightarrow y - 4 = -x + 8$$

$$\rightarrow x + y - 12 = 0$$

Example 2. Write equations of normal and tangent to conic $\frac{x^2}{8} + \frac{y^2}{9} = 1$ at point $\left(\frac{8}{3}, 1\right)$

Solution:-

Given equation is

$$\frac{x^2}{8} + \frac{y^2}{9} = 1$$

$$\rightarrow 9x^2 + 8y^2 = 72 \longrightarrow (1)$$

Diff (1) w.r.t x, we get

$$18x + 16y\frac{dy}{dx} = 0$$

$$\rightarrow 16y\frac{dy}{dx} = -18x$$

$$\rightarrow \frac{dy}{dx} = -\frac{18}{16}\frac{x}{y} = -\frac{9}{8}\frac{x}{y}$$

$$\rightarrow \left.\frac{dy}{dx}\right|_{\left(\frac{8}{3},1\right)} = \frac{-9}{8}\cdot\frac{\frac{8}{3}}{1} = -3$$

Now eq. of tangent at $\left(\frac{8}{3},1\right)$ is $y - 1 = -3\left(x - \frac{8}{3}\right)$

$$\rightarrow y - 1 = -3x + 8$$

$$\rightarrow 3x + y - 9 = 0$$

Also eq. of normal at $\left(\frac{8}{3},1\right)$ is

$$y - 1 = \frac{1}{3}\left(x - \frac{8}{3}\right)$$

$$\rightarrow y - 1 = \frac{x}{3} - \frac{8}{9}$$

$$\rightarrow \frac{x}{3} - y = -\frac{1}{9}$$

$$\rightarrow 3x - 9y = -1 \rightarrow 3x - 9y + 1 = 0$$

Theorem:

To show that a straight line cuts a conic, in general, in two points and to find the condition that line be a tangent to the conic

Let $y = mx + c$ cut the conics

i) $y^2 = 4ax$ ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

iii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(i) $y^2 = 4ax$ (Parabola).

We find point of intersection of

$y = mx + c \longrightarrow (1)$

and $y^2 = 4ax \longrightarrow (2)$

put value of y from (1) in equation (2)

$\Rightarrow (mx+c)^2 = 4ax$

$\Rightarrow m^2x^2 + c^2 + 2mcx - 4ax = 0$

$m^2x^2 + (2mc-4a)x + c^2 = 0 \longrightarrow (3)$

which is quadratic in x gives two values of x. These values are the x-coordinates of the common points of (1) and (2). Setting these values in (1), we obtain the corresponding ordinates of the points of intersection. Thus line (1) cuts parabola (2) in two points. In order that (1) is tangent to (2), the points of intersection of a line and parabola must be coincident. In this case, the roots of (3) should be real and equal. For this

Disc $= 0$

$\Rightarrow 4(mc-2a)^2 - 4m^2c^2 = 0$

$\Rightarrow m^2c^2 + 4a^2 - 4mca - m^2c^2 = 0$

$\Rightarrow -4mca + 4a^2 = 0$

or $c = \frac{a}{m}$ is the req. condition for (1) to be tangent to (2)

Hence $y = mx + \frac{a}{m}$ is tangent to $y^2 = 4ax$ for all nonzero values of m.

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ellipse)

We find point of intersection of $y = mx + c \longrightarrow (1)$

and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow (2)$

Put value of y from (1) in equation (2)

$\Rightarrow \frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$

$\Rightarrow a^2(mx+c)^2 + bx^2 = a^2b^2$

$\Rightarrow a^2m^2x^2 + a^2c^2 + 2a^2mxc + b^2x^2 - a^2b^2 = 0$

or $(a^2m^2+b^2)x^2 + 2mca^2x + a^2c^2 - a^2b^2 = 0 \longrightarrow (3)$

which is quadratic in x and it gives abscissa of the two points where (1) and (2) intersect. Setting these values in (1), we obtain corresponding ordinates of the points of intersection. Thus line (1) cuts ellipse (2) in two points.

Now (1) is tangent to (2), the points of intersection of a line and ellipse must be coincident. In this case, the roots of (3) should be real and equal. For this

Disc $= 0$

$\Rightarrow (2mca^2)^2 - 4(a^2m^2+b^2)(a^2c^2-a^2b^2) = 0$

$\Rightarrow m^2c^2a^2 - (a^2m^2+b^2)(c^2-b^2) = 0$

$\rightarrow m^2c^2a^2 - a^2m^2c^2 + a^2m^2b^2 - b^2c^2 + b^4 = 0$

$\rightarrow c^2 = a^2m^2 + b^2$

or $c = \pm\sqrt{a^2m^2 + b^2}$

so

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

which are tangent to (2) for all non-zero values of m.

iii) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ (hyperbola)

We replace b^2 by $-b^2$ in (ii) and the line $y = mx + c$ is tangent to $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

if $c = mx \pm \sqrt{a^2m^2 - b^2}$

Thus $y = mx \pm \sqrt{a^2m^2 - b^2}$ are tangents to hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ for all non-zero values of m.

Example 4. Find equation of tangent to the parabola $y^2 = -6x$ which is parallel to the line $2x + y + 1 = 0$. Also find the point of tangency.

Solution:– Given line $2x + y + 1 = 0$

$\rightarrow y = -2x - 1$

Here slope of req. tangent line is $= m = -2$

In the given parabola

$y^2 = -6x \longrightarrow (1)$ (compare with $y^2 = 4ax$)

Here $4a = -6$

$\rightarrow a = -\dfrac{3}{2}$

Equation of tangent is

$y = mx + \dfrac{a}{m}$

$y = -2x + \dfrac{-3/2}{-2}$

$y = -2x + \dfrac{3}{4}$

$\rightarrow 4y = -8x + 3 \longrightarrow (2)$

or $8x + 4y - 3 = 0$

Now we find point of tangency

from (2) $\rightarrow y = -\dfrac{8x+3}{4}$ put in (1)

$\rightarrow \left[-\dfrac{8x+3}{4}\right]^2 = -6x$

$\dfrac{64x^2 + 9 - 48x}{16} = -6x$

$64x^2 + 9 - 48x = -96x$

or $64x^2 + 48x + 9 = 0$

$\rightarrow (8x+3)^2 = 0$

$\rightarrow 8x + 3 = 0 \rightarrow x = -\dfrac{3}{8}$ put in (2)

$4y = -8\left(-\dfrac{3}{8}\right) + 3$

$\rightarrow y = \dfrac{6}{4}$ or $y = \dfrac{3}{2}$

So point of tangency $\left(-\dfrac{3}{8}, \dfrac{3}{2}\right)$

Example 5. Find equations of tangents to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ which are parallel to the line $3x + 8y + 1 = 0$. Also find the points of contact.

Solution:-

Given line $3x+8y+1=0$

$\rightarrow 8y = -3x-1$

$\rightarrow y = \frac{-3}{8}x - \frac{1}{8} \longrightarrow (1)$

Here slope of req. tangent line $= m = \frac{-3}{8}$

In the given ellipse;

$\frac{x^2}{128} + \frac{y^2}{18} = 1 \longrightarrow (2)$

Here $a^2 = 128$, $b^2 = 18$

Now eqs. of tangents are

$y = \frac{-3}{8}x \pm \sqrt{128\left(\frac{-3}{8}\right)^2 + 18}$

$\left(\because y = mx \pm \sqrt{a^2m^2 + b^2}\right)$

$\rightarrow y = \frac{-3}{8}x \pm \sqrt{128\left(\frac{9}{64}\right)+18}$

$\rightarrow y = \frac{-3}{8}x \pm \sqrt{18+18}$

$\rightarrow y = \frac{-3}{8}x \pm 6$

so $y = \frac{-3}{8}x + 6$ and $y = \frac{-3}{8}x - 6$

$\rightarrow 8y = -3x+48$ and $8y = -3x-48$

$\rightarrow 3x+8y-48=0 \longrightarrow (3)$ and $3x+8y+48=0 \longrightarrow (4)$

Now we find points of contanct; Solving (2) and (3)

Put value of y from (3) in eq (2)

From (3)
$3x+8y=48$
$8y=-3x+48$
$y=-\frac{3}{8}x+6$

$\frac{x^2}{128} + \frac{\left(-\frac{3x}{8}+6\right)^2}{18} = 1$

$\rightarrow \frac{x^2}{128} + \frac{\frac{9x^2}{64}+36-\frac{9x}{2}}{18} = 1$

$\rightarrow \frac{x^2}{128} + \frac{9\left[\frac{x^2}{64}+4-\frac{x}{2}\right]}{18} = 1$

$\rightarrow \frac{x^2}{128} + \frac{x^2}{128} + 2 - \frac{x}{4} = 1$

$\rightarrow \frac{2x^2}{128} - \frac{x}{4} + 2 - 1 = 0$

$\rightarrow \frac{x^2}{64} - \frac{x}{4} + 1 = 0$

$\rightarrow \left(\frac{x}{8} - 1\right)^2 = 0$

$\rightarrow \frac{x}{8} - 1 = 0 \rightarrow \frac{x}{8} = 1$

$\rightarrow x = 8$

so $y = \frac{-3}{8}x + 6 \rightarrow y = \frac{-3}{8}(8)+6$

$\rightarrow y = 3$

Thus $(8, 3)$ is point of contact of (3).

Now solving (2) and (4) in similar mannar we get point of contact of (2) is $(-8, -3)$.

Example 6. Show that the product of the distances from the foci to any tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is constant.

Solution:- $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \longrightarrow (1)$

$\because$ The line $y = mx \pm \sqrt{a^2m^2 - b^2} \longrightarrow (2)$

is tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Foci of (1) are $F(-c, 0)$ and $F'(c, 0)$

Distance of $F(-c, 0)$ from (2)

is $d_1 = \frac{\left|-cm + \sqrt{a^2m^2 - b^2}\right|}{\sqrt{1+m^2}}$

Distance of $F(c,0)$ from (2) is

$$d_2 = \frac{|cm + \sqrt{a^2m^2-b^2}|}{\sqrt{1+m^2}}$$

Now

$$d_1 \times d_2 = \frac{|a^2m^2 - b^2 - c^2m^2|}{1+m^2}$$

$$= \frac{|a^2m^2 - c^2 + a^2 - c^2m^2|}{1+m^2} \quad (\because b^2 = c^2 - a^2)$$

$$= \frac{|a^2(1+m^2) - c^2(1+m^2)|}{1+m^2}$$

$$= \frac{|(1+m^2)(a^2-c^2)|}{1+m^2}$$

$$= \frac{(1+m^2)|a^2-c^2|}{1+m^2}$$

$$= |a^2-c^2|$$

$$= c^2 - a^2 \quad \because c > a$$

$d_1 \times d_2 = c^2$ which is constant.

Hence proved.

Intersection of two conics

Suppose we are given two conics $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow (1)$

and $y^2 = 4ax \longrightarrow (2)$

To find the points common to both (1) and (2) we need to solve simultaneously.

Note:- Simultaneous solution set of two equations of second degree consists of four points. Thus two conics will always intersect in four points. These points may be real and distinct, two real and two imaginary, or all imaginary. Two or more points may also coincide.

*Two conics are said to be touch each other if they intersect two or more coincident points.

Example 7. Find the points of intersection of the ellipse

$$\frac{x^2}{43/3} + \frac{y^2}{43/4} = 1 \quad \text{and}$$

hyperbola $\frac{x^2}{7} - \frac{y^2}{14} = 1$

Also sketch the graph of the two conics.

Solution:- Given equations can be written as

$3x^2 + 4y^2 = 43 \longrightarrow (1)$ and $2x^2 - y^2 = 14 \longrightarrow (2)$

By $(1) + 4(2) \rightarrow$

$$\begin{aligned} 3x^2 + 4y^2 &= 43 \\ 8x^2 - 4y^2 &= 56 \\ \hline 11x^2 &= 99 \end{aligned}$$

$\rightarrow x^2 = \frac{99}{11} \rightarrow x^2 = 9$

$\rightarrow x = \pm 3$

for $x = 3$ in (2), $18 - y^2 = 14$

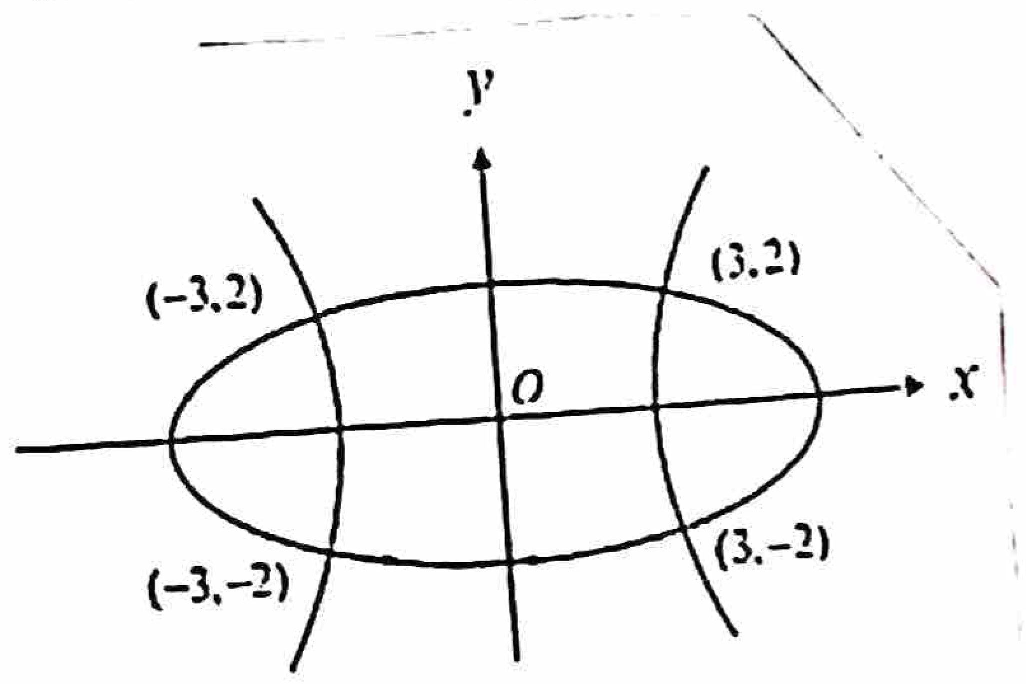
$\rightarrow y^2 = 4 \rightarrow y = \pm 2$

Thus $(3,2)$ and $(3,-2)$ are two points of intersection of two conics.

for $x=-3$ in (2), we get

$\rightarrow y = \pm 2$

So $(-3,2)$ and $(-3,-2)$ are also points of intersection of (1) & (2). The four points of intersection are shown in the fig.



Example 8. Find the points of the intersection of the conics $y = 1+x^2$ and
$$y = 1+4x+x^2$$
Also draw the graph of the conics.

Solution:- Given conics are
$$y = 1+x^2 \longrightarrow (1) \text{ and}$$
$$y = 1+4x-x^2 \longrightarrow (2)$$

$(1) \rightarrow \quad x = \pm\sqrt{y-1}$ Put in (2)

$\rightarrow y = 1 \pm 4\sqrt{y-1} - (y-1)$

$\rightarrow 2y-2 = \pm 4\sqrt{y-1}$

or $y-1 = \pm 2\sqrt{y-1}$

or $(y-1)^2 = 4(y-1)$

$(y-1)^2 - 4(y-1) = 0$

$(y-1)[y-1-4] = 0$

$(y-1)(y-5) = 0$

$\rightarrow y = 1, \; y = 5$

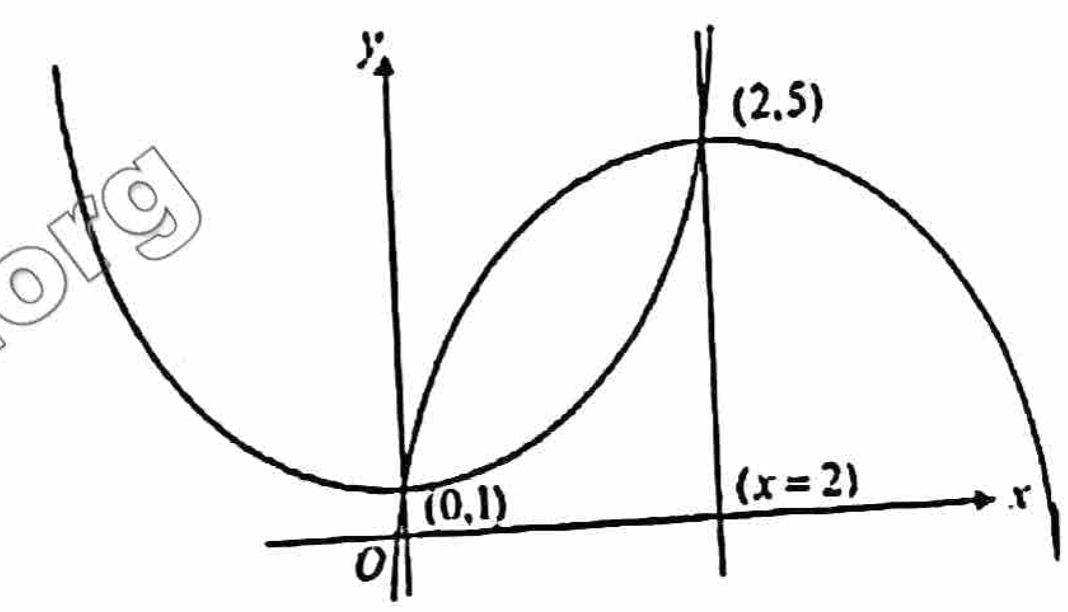
when $y=1$, $x=0$ see (1)

when $y=5$, $x=\pm 2$ see (1)

But $(-2,5)$ does not satisfy equation (2).

Thus $(0,1)$ & $(2,5)$ are the points of intersections of (1) and (2).

(1) is parabola having vertex $(0,1)$ and opening upward and
$y = 1+4x+x^2$ can be written as $y-5 = -(x-2)^2$ which is parabola having vertex $(2,5)$ and opening down ward.



Example 9. Find the equations of common tangents to the two conics $\frac{x^2}{16}+\frac{y^2}{25}=1$ and
$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

Solution:- The tangents with slope m, to the two conics are respectively given

$y = mx \pm \sqrt{16m^2+25}$ and

$y = mx \pm \sqrt{25m^2+9}$

For tangents to be common, we must have

$16m^2+25 = 25m^2+9$

$\rightarrow 9m^2 = 16 \rightarrow m = \pm 4/3$

Using these values of m, equations of the four common tangents are;

$$y = \pm\frac{4}{3}x \pm \sqrt{481}$$

Exercise 6.7

Q1. Find equations of the tangent and normal to each of the following at the indicated point:

(i) $y^2 = 4ax$ at $(at^2, 2at)$

Solution:-

$$y^2 = 4ax \longrightarrow (1)$$

Diff (1) w.r.t 'x' we have

$$2y\frac{dy}{dx} = 4a$$

$$\rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\rightarrow \frac{dy}{dx}\Big|_{(at^2, 2at)} = \frac{\cancel{2a}}{\cancel{2a}t} = \frac{1}{t}$$

Now eq. of tangent at $(at^2, 2at)$

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$\rightarrow yt - 2at^2 = x - at^2$$

$$\rightarrow yt = x + at^2$$

Eq. of normal at $(at^2, 2at)$

$$y - 2at = -t(x - at^2)$$

$$\rightarrow y - 2at = -tx + at^3$$

or $y = 2at - tx + at^3$

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a\cos\theta, b\sin\theta)$

Solution:-

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow (1)$$

Diff (1) w.r.t 'x' we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\rightarrow \frac{2y}{b^2}\frac{dy}{dx} = \frac{-2x}{a^2}$$

$$\rightarrow \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$$

$$\frac{dy}{dx}\Big|_{(a\cos\theta, b\sin\theta)} = -\frac{b^2}{a^2}\cdot\frac{a\cos\theta}{b\sin\theta}$$

$$= -\frac{b\cos\theta}{a\sin\theta}$$

Now eq. of tangent at $(a\cos\theta, b\sin\theta)$

$$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$ya\sin\theta - ab\sin^2\theta = -xb\cos\theta + ab\cos^2\theta$$

$$\rightarrow xb\cos\theta + ya\sin\theta = ab\sin^2\theta + ab\cos^2\theta$$

$$xb\cos\theta + ya\sin\theta = ab(\sin^2\theta + \cos^2\theta)$$

$$\rightarrow \frac{xb}{ab}\cos\theta + \frac{ya}{ab}\sin\theta = \frac{ab(1)}{ab}$$

or $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

Eq. of normal at $(a\cos\theta, b\sin\theta)$

$$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$$

$$\frac{y - b\sin\theta}{a\sin\theta} = \frac{x - a\cos\theta}{b\cos\theta}$$

$$\frac{y}{a\sin\theta} - \frac{b}{a} = \frac{x}{b\cos\theta} - \frac{a}{b}$$

$$\frac{y}{a}\operatorname{cosec}\theta - \frac{x}{b}\sec\theta = \frac{b}{a} - \frac{a}{b}$$

$$\rightarrow \frac{y}{a}\csc\theta - \frac{x}{b}\sec\theta = \frac{b^2 - a^2}{ab}$$

$$-\left[\frac{x}{b}\sec\theta - \frac{y}{a}\csc\theta\right] = \frac{-(a^2 - b^2)}{ab}$$

or $\frac{x}{b}\sec\theta - \frac{y}{a}\csc\theta = \frac{a^2 - b^2}{ab}$

or $ax\sec\theta - by\csc\theta = a^2 - b^2$

(iii) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ at $(a\sec\theta, b\tan\theta)$

Solution:– $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1 \longrightarrow (1)$

Diff (1) w.r.t x, we get

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\Rightarrow -\frac{2y}{b^2}\frac{dy}{dx}=-\frac{2x}{a^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{b^2}{a^2}\frac{x}{y}$$

$$\left.\frac{dy}{dx}\right|_{(a\sec\theta, b\tan\theta)}=\frac{b^2}{a^2}\frac{a\sec\theta}{b\tan\theta}$$

$$=\frac{b\sec\theta}{a\tan\theta}$$

Now eq. of tangent at $(a\sec\theta, b\tan\theta)$

$$y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)$$

$$ay\tan\theta-ab\tan^2\theta=bx\sec\theta-ab\sec^2\theta$$

$$bx\sec\theta-ay\tan\theta=ab\sec^2\theta-ab\tan^2\theta$$

$$bx\sec\theta-ay\tan\theta=ab(\sec^2\theta-\tan^2\theta)$$

$$\frac{bx}{ab}\sec\theta-\frac{ay}{ab}\tan\theta=\frac{ab}{ab}(1)$$

or $\dfrac{x}{a}\sec\theta-\dfrac{y}{b}\tan\theta=1$

Eq. of normal at $(a\sec\theta, b\tan\theta)$

$$y-b\tan\theta=\frac{-a\tan\theta}{b\sec\theta}(x-a\sec\theta)$$

$$\frac{y-b\tan\theta}{a\tan\theta}=-\left(\frac{x-a\sec\theta}{b\sec\theta}\right)$$

$$\frac{y}{a\tan\theta}-\frac{b}{a}=\frac{-x}{a\sec\theta}+\frac{a}{b}$$

$$\Rightarrow \frac{y}{a}\cot\theta+\frac{x}{b}\cos\theta=\frac{b}{a}+\frac{a}{b}$$

$$\frac{y}{a}\cot\theta+\frac{x}{b}\cos\theta=\frac{b^2+a^2}{ab}$$

or $by\cot\theta+xa\cos\theta=a^2+b^2$

Q2. Write equation of the tangent to the given conic at the indicated point.

i) $3x^2=-16y$ at the point whose ordinate is -3

Solution:–

$$3x^2=-16y \longrightarrow (1)$$

and $y(\text{ordinate})=-3$

so (1) $\Rightarrow 3x^2=-16(-3)$

$$\Rightarrow 3x^2=48 \Rightarrow x^2=16$$

$$\Rightarrow x=\pm 4$$

Thus we have to find eqs. of tangents at $(4,-3)$ & $(-4,-3)$

Diff (1) w.r.t x, we get

$$6x=-16\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{6x}{16}=-\frac{3x}{8}$$

At $(4,-3)$:–

$$\left.\frac{dy}{dx}\right|_{(4,-3)}=\frac{-3(4)}{8}=\frac{-3}{2}$$

Now eq. of tangent at $(4,-3)$ with slope $-3/2$

$\therefore\ y+3=-\dfrac{3}{2}(x-4)$

$$\Rightarrow 2y+6=-3x+12$$

$$\Rightarrow 3x+2y-6=0$$

At $(-4,-3)$:–

$$\left.\frac{dy}{dx}\right|_{(-4,-3)}=\frac{-3(-4)}{8}=\frac{3}{2}$$

Now eq. of tangent at $(-4,-3)$ with slope $3/2$

$$y+3=\frac{3}{2}(x+4)$$

$$2y+6=3x+12$$

$$\rightarrow 3x-2y+6=0$$

(ii) $3x^2-7y^2=20$ at the points where $y=-1$

Solution:-

$$3x^2-7y^2=20 \longrightarrow (1)$$

Put $y=-1$ in (1)

So $3x^2-7(-1)^2=20$

$$\rightarrow 3x^2-7=20$$

$$\rightarrow 3x^2=27 \rightarrow x^2=9 \rightarrow x=\pm 3$$

Thus we have to find eqs. of tangents at the points $(3,-1)$ and $(-3,-1)$.

Diff (1) w.r.t x, we get

$$6x-14y\frac{dy}{dx}=0$$

$$\rightarrow -14y\frac{dy}{dx}=-6x$$

$$\rightarrow \frac{dy}{dx}=\frac{3x}{7y}$$

At $(3,-1)$:-

$$\left.\frac{dy}{dx}\right|_{(3,-1)}=\frac{3(3)}{7(-1)}=-\frac{9}{7}$$

So eq. of tangent at $(3,-1)$ with slope $-9/7$

$$y+1=-\frac{9}{7}(x-3)$$

$$7y+7=-9x+27$$

$$\rightarrow 9x+7y-20=0$$

At $(-3,-1)$:-

$$\left.\frac{dy}{dx}\right|_{(-3,-1)}=\frac{3(-3)}{7(-1)}=9/7$$

So eq. of tangent at $(-3,-1)$ with slope $9/7$

$$y+1=9/7(x+3)$$

$$\rightarrow 7y+7=9x+27$$

$$\rightarrow 9x-7y+20=0$$

(iii) $3x^2-7y^2+2x-y-48=0$ at the point where $x=4$

Solution:-

$$3x^2-7y^2+2x-y-48=0 \rightarrow (1)$$

Put $x=4$ in (1)

$$\cancel{48}-7y^2+8-y-\cancel{48}=0$$

$$\rightarrow -7y^2-y+8=0$$

or $7y^2+y-8=0$

$$7y^2+8y-7y-8=0$$

$$y(7y+8)-1(7y+8)=0$$

$$\rightarrow (7y+8)(y-1)=0$$

$$7y+8=0,\quad y-1=0$$

$$y=-\frac{8}{7},\quad y=1$$

Thus we have to find eqs. of tangents at the points $(4,-8/7)$ and $(4,1)$.

Diff (1) w.r.t 'x' we get

$$6x-14y\frac{dy}{dx}+2-\frac{dy}{dx}=0$$

$$6x+2=(14y+1)\frac{dy}{dx}$$

or $$\frac{dy}{dx}=\frac{6x+2}{14y+1}$$

At $(4,1)$:-

$$\left.\frac{dy}{dx}\right|_{(4,1)}=\frac{24+2}{14+1}=\frac{26}{15}$$

Thus eq. of tangent at $(4,1)$ with slope $\frac{26}{15}$

$$y-1=\frac{26}{15}(x-4)$$

$\Rightarrow 15y - 15 = 26x - 104$

or $26x - 15y - 104 + 15 = 0$

or $26x - 15y - 89 = 0$

At $(4, -\frac{8}{7})$:-

$$\frac{dy}{dx}\Big|_{(4,-\frac{8}{7})} = \frac{6(4)+2}{14(-\frac{8}{7})+1} = \frac{26}{-16+1}$$

$$= -\frac{26}{15}$$

Thus eq. of tangent at $(4, -\frac{8}{7})$ with slope $-26/15$.

$$y + \frac{8}{7} = -\frac{26}{15}(x - 4)$$

$$\Rightarrow 15y + \frac{120}{7} = -26x + 104$$

or $26x + 15y = 104 - \frac{120}{7}$

$$= \frac{728 - 120}{7} = \frac{608}{7}$$

$$\Rightarrow 26x + 15y - \frac{608}{7} = 0$$

or $13x + \frac{15}{2}y - \frac{304}{7} = 0$

Q3. Find equations of tangents to each of the following through the given point:

(i) $x^2 + y^2 = 25$ through $(7,-1)$

Solution:-

$$x^2 + y^2 = 25$$

$$\Rightarrow x^2 + y^2 = 5^2 \longrightarrow (1)$$

Here $a = 5$. In this case Eqs. of tangents from any point are of the form

$$y = mx \pm a\sqrt{1+m^2} \quad \forall m \in \mathbb{R}$$

$$\Rightarrow y = mx \pm 5\sqrt{1+m^2} \quad (\because a = 5) \longrightarrow (2)$$

As (2) passes through $(7,-1)$

(2) $\Rightarrow$ $-1 = 7m \pm 5\sqrt{1+m^2}$

$\Rightarrow -1 - 7m = \pm 5\sqrt{1+m^2}$

squaring both sides

$$(-1-7m)^2 = [\pm 5\sqrt{1+m^2}]^2$$

$$1 + 49m^2 + 14m = 25(1+m^2)$$

$$1 + 49m^2 + 14m = 25 + 25m^2$$

$$\Rightarrow 24m^2 + 14m - 24 = 0$$

or $12m^2 + 7m - 12 = 0$

$$12m^2 + 16m - 9m - 12 = 0$$

$$4m(3m+4) - 3(3m+4) = 0$$

$$(3m+4)(4m-3) = 0$$

$$3m + 4 = 0, \quad 4m - 3 = 0$$

$$m = -\frac{4}{3}, \quad m = \frac{3}{4}$$

For $m = -\frac{4}{3}$:-

(2) $\Rightarrow$ $y = -\frac{4}{3}x + 5\sqrt{1+(-\frac{4}{3})^2}$

$$y = -\frac{4}{3}x + 5\sqrt{1+\frac{16}{9}}$$

$$y = -\frac{4}{3}x + 5(5/3)$$

$$y = -\frac{4}{3}x + \frac{25}{3}$$

$\Rightarrow 4x + 3y - 25 = 0$

For $m = 3/4$:-

(2) $\Rightarrow$ $y = \frac{3}{4}x - 5\sqrt{1+9/16}$

$$y = \frac{3}{4}x - 5(5/4)$$

$$y = \frac{3}{4}x - \frac{25}{4}$$

$$\Rightarrow 3x - 4y - 25 = 0$$

Note:-

$m = -\frac{4}{3}$ satisfies the equation $-1 = 7m + 5\sqrt{1+m^2}$

and $m = \frac{3}{4}$ satisfies the equation $-1 = 7m - 5\sqrt{1+m^2}$

(ii) $y^2 = 12x$ through $(1,4)$

Solution:- $y^2 = 12x \longrightarrow (1)$

Here $4a = 12 \rightarrow a = 3$

Eqs. of tangents are of the form in this case

$y = mx + \frac{a}{m} \quad \forall m \in R$

$\rightarrow y = mx + \frac{3}{m} \longrightarrow (2)$

since (2) passes through $(1,4)$

so $4 = m + \frac{3}{m} \rightarrow 4m = m^2 + 3$

or $m^2 - 4m + 3 = 0$

$m^2 - 3m - m + 3 = 0$

$m(m-3) - 1(m-3) = 0$

or $(m-3)(m-1) = 0$

$m = 3, \quad m = 1$

For $m = 1$:-

$(2) \rightarrow y = x + 3$

or $x - y + 3 = 0$

For $m = 3$:-

$(2) \rightarrow y = 3x + \frac{3}{3}$

$\rightarrow 3x - y + 1 = 0$

(iii) $x^2 - 2y^2 = 2$ through $(1,-2)$

Solution:-

$x^2 - 2y^2 = 2$

$\rightarrow \frac{x^2}{2} - y^2 = 1 \quad (\div \text{ by } 2)$

or $\frac{x^2}{2} - \frac{y^2}{1} = 1 \longrightarrow (1)$

Here $a^2 = 2, \quad b^2 = 1$

so eqs. of tangents are of the form in this case

$y = mx \pm \sqrt{a^2m^2 - b^2}$

$\rightarrow y = mx \pm \sqrt{2m^2 - 1} \longrightarrow (2)$

since (2) passes through $(1,-2)$

so $(2) \rightarrow -2 = m \pm \sqrt{2m^2 - 1}$

$\rightarrow -2 - m = \pm \sqrt{2m^2 - 1}$

squaring both sides

$(-2-m)^2 = \left(\pm \sqrt{2m^2 - 1}\right)^2$

$\rightarrow 4 + m^2 + 4m = 2m^2 - 1$

or $2m^2 - m^2 - 4m - 4 - 1 = 0$

or $m^2 - 4m - 5 = 0$

$m^2 - 5m + m - 5 = 0$

$m(m-5) + 1(m-5) = 0$

$(m-5)(m+1) = 0$

$\rightarrow m = 5, \quad m = -1$

For $m = -1$:-

$(2) \rightarrow y = -x - \sqrt{2(-1)^2 - 1}$

$y = -x - 1$

or $x + y + 1 = 0$

For $m = 5$:-

$(2) \rightarrow y = 5x - \sqrt{2(5)^2 - 1}$

$y = 5x - \sqrt{50 - 1}$

$y = 5x - 7$

$\rightarrow 5x - y - 7 = 0$

Note:-

$m = -1$ satisfies the eq. $-2 - m = -\sqrt{2m^2 - 1}$

and $m = 5$ satisfies the eq $-2 - m = -\sqrt{2m^2 - 1}$

Q4. Find equations of the normals to the Parabola $y^2=8x$ which are parallel to the line $2x+3y=10$

Solution:-

$$y^2 = 8x \longrightarrow (1)$$

Diff (1) w.r.t 'x'

$$2y\frac{dy}{dx} = 8 \Rightarrow \frac{dy}{dx} = \frac{4}{y}$$

$\therefore$ slope of tangent to (1) $= \frac{4}{y}$

and slope of normal to (1) $= -\frac{y}{4}$

Given line is

$$2x+3y=10 \longrightarrow (2)$$

$$\Rightarrow 3y = -2x+10$$

$$\text{or} \quad y = -\frac{2}{3}x + \frac{10}{3}$$

slope of given line $= -\frac{2}{3}$

Since normal to (1) is parallel to (2) (Given)

$$\therefore -\frac{y}{4} = -\frac{2}{3} \Rightarrow y = \frac{8}{3}$$

$$\text{so } (1) \Rightarrow \frac{64}{9} = 8x$$

$$\Rightarrow x = 8/9$$

Thus slope of normal to (1) at $(8/9, \frac{8}{3})$ is

$$= -\frac{8/3}{4} = -\frac{8}{3}\times\frac{1}{4} = -\frac{2}{3}$$

Now req. eq of normal is

$$y-\frac{8}{3} = -\frac{2}{3}\left(x-\frac{8}{9}\right)$$

$$3y-8 = -2x+\frac{16}{9}$$

$$\Rightarrow 27y-72 = -18x+16$$

$$\text{or} \quad 18x+27y-88=0$$

Q5. Find equations of tangents to the ellipse $\frac{x^2}{4}+y^2=1$ which are parallel to the line $2x-4y+5=0$

Solution:-

$$\frac{x^2}{4}+\frac{y^2}{1} = 1 \longrightarrow (1)$$

Here $a^2=4$, $b^2=1$

Given line is

$$2x-4y+5=0$$

$$\Rightarrow 2x+5 = 4y \longrightarrow (2)$$

$$\text{or} \quad y = \frac{1}{2}x+\frac{5}{4}$$

$\therefore$ slope of given line $= \frac{1}{2}$

As slope of tangent is $\parallel$ to (2)

so slope of tangent to (1) parallel to (2) $= m = \frac{1}{2}$

Now req. eqs. of tangents are

$$y = mx \pm \sqrt{a^2m^2+b^2}$$

$$\Rightarrow y = \frac{1}{2}x \pm \sqrt{4\left(\frac{1}{2}\right)^2+1}$$

$$y = \frac{1}{2}x \pm \sqrt{1+1}$$

$$y = \frac{1}{2}x \pm \sqrt{2}$$

$$\text{or} \quad 2y = x \pm 2\sqrt{2}$$

$$x-2y+2\sqrt{2} = 0$$

$$\text{and} \quad x-2y-2\sqrt{2} = 0$$

Q6. Find equations of the tangents to the conic $9x^2-4y^2=36$ parallel to $5x-2y+7=0$

Solution:-

$9x^2-4y^2=36$

$\rightarrow \frac{9x^2}{36}-\frac{4y^2}{36}=\frac{36}{36}$ $(\div \text{ by } 36)$

$\rightarrow \frac{x^2}{4}-\frac{y^2}{9}=1 \longrightarrow (1)$

Here $a^2=4$, $b^2=9$

Given line; $5x-2y+7=0$

$\rightarrow 5x+7=2y \longrightarrow (2)$

$\rightarrow \frac{5}{2}x+\frac{7}{2}=y$

$\therefore$ slope of given line $=\frac{5}{2}$

Now slope of tangent $\parallel$ to given line $=\frac{5}{2}$

Thus req. eqs. of tangents to (1) and $\parallel$ to (2) are

$y=\frac{5}{2}x \pm \sqrt{4\left(\frac{25}{4}\right)-9}$

$y=\frac{5}{2}x \pm 4 \rightarrow 2y=5x\pm 8$

So $5x-2y+8=0$

and $5x-2y-8=0$

Q7. Find equations of the common tangents to the given conics

i) $x^2=80y$ and $x^2+y^2=81$

Solution:-

$x^2=80y \longrightarrow (1)$

$x^2+y^2=81 \longrightarrow (2)$

Let $y=mx+c \longrightarrow (3)$ be common tangent to (1) and (2)

using (3) in (1) we have

$x^2=80(mx+c)$

$\rightarrow x^2-80mx-80c=0 \longrightarrow (4)$

If (3) is tangent to (1) then (4) has equal roots.

Disc of (4) $=0$

$\rightarrow (-80m)^2-4(1)(-80c)=0$

$\rightarrow 6400m^2+320c=0$

$\rightarrow c=-\frac{6400m^2}{320}$

$c=-20m^2 \longrightarrow (5)$

using (3) in (2) we have

$x^2+(mx+c)^2=81$

$\rightarrow x^2+m^2x^2+c^2+2mcx=81$

$x^2(1+m^2)+(2mc)x+c^2-81=0 \longrightarrow (6)$

If (3) is tangent to (2) then (6) has equal roots

$\rightarrow$ Disc of (6) $=0$

$\rightarrow 4m^2c^2-4(1+m^2)(c^2-81)=0$

$\rightarrow m^2c^2-(1+m^2)(c^2-81)=0$

$m^2c^2-c^2+81-m^2c^2+81m^2=0$

$-c^2+81(1+m^2)=0$

or $c^2=81(1+m^2) \longrightarrow (7)$

using (5) in (7)

$400m^4-81m^2-81=0$

$\rightarrow 400m^4-225m^2+144m^2-81=0$

$25m^2(16m^2-9)+9(16m^2-9)=0$

$(16m^2-9)(25m^2+9)=0$

$16m^2=9$, $\quad 25m^2=-9$

$m^2=9/16$, $\quad m^2=-9/25$

$\rightarrow m = \pm \frac{3}{4}$, For $m^2 = -\frac{9}{25}$

(Neglacting -ive value $\therefore m^2$ is not -ive)

so $m = \pm \frac{3}{5}$

using $m = \pm \frac{3}{4}$ in (5)

$c = -20\left(\frac{9}{16}\right) = -\frac{45}{4}$

so req. eqs. of common tangent becomes

$y = \pm \frac{3}{4}x - \frac{45}{4}$

$\rightarrow 4y = \pm 3x - 45$

or $\pm 3x - 4y - 45 = 0$

(ii) $y^2 = 16x$ and $x^2 = 2y$

Solution:-

$y^2 = 16x \longrightarrow (1)$

Here $4a = 16 \rightarrow a = 4$

$x^2 = 2y \longrightarrow (2)$

Let $y = mx + c \longrightarrow (3)$

be the equation of common tangents to (1) and (2)

Since $c = \frac{a}{m} \rightarrow c = \frac{4}{m} \rightarrow (4)$

using (4) in (3)

$y = mx + \frac{4}{m}$

Put (3) in (2)

$x^2 = 2(mx + c)$

$x^2 = 2mx + 2c$

$x^2 - 2mx - 2c = 0 \longrightarrow (5)$

$\rightarrow x^2 - 2mc - \frac{8}{m} = 0 \quad (\because c = \frac{4}{m})$

If (3) is tangent to (2) then (6) has equal roots.

so Disc of ( ) = 0

$\rightarrow (-2m)^2 - 4(1)(-\frac{8}{m}) = 0$

$4m^2 + \frac{32}{m} = 0$

or $m^2 + \frac{8}{m} = 0$

$m^3 + 8 = 0 \rightarrow m^3 + 2^3 = 0$

$\rightarrow (m+2)(m^2 - 2m + 4) = 0$

$m + 2 = 0, \quad m^2 - 2m + 4 = 0$

$m = -2 \quad , \quad m = \frac{2 \pm \sqrt{4-16}}{2}$

(Neglecting complex roots)

so put $m = -2$ in eq (4)

so $c = \frac{4}{-2} \rightarrow c = -2$

Thus req. eq of common tangent is

$y = -2x - 2$

or $2x + y + 2 = 0$

Q8. Find the points of intersection of the given conics

i) $\frac{x^2}{18} + \frac{y^2}{8} = 1$ and $\frac{x^2}{3} - \frac{y^2}{3} = 1$

Solution:-

$\frac{x^2}{18} + \frac{y^2}{8} = 1 \longrightarrow (1)$

$\frac{x^2}{3} - \frac{y^2}{3} = 1 \longrightarrow (2)$

Multiplying (2) by $\frac{1}{6}$

(2) $\rightarrow \frac{x^2}{18} - \frac{y^2}{18} = \frac{1}{6} \longrightarrow (3)$

By (1) − (3)

$$\cancel{\frac{x^2}{18}} + \frac{y^2}{8} = 1$$
$$\underset{-}{\cancel{\frac{x^2}{18}}} \underset{+}{-} \frac{y^2}{18} = \underset{-}{\frac{1}{6}}$$

$$\frac{y^2}{8} + \frac{y^2}{18} = 1 - \frac{1}{6}$$
$$\rightarrow \frac{9y^2+4y^2}{72} = \frac{5}{6}$$
$$\rightarrow \frac{13y^2}{72} = \frac{5}{6}$$
$$\rightarrow y^2 = \frac{60}{13} \rightarrow y = \pm\sqrt{\frac{60}{13}}$$

Put $y^2 = \frac{60}{13}$ in (2)

$$\frac{x^2}{3} - \frac{\frac{60}{13}}{3} = 1 \rightarrow \frac{x^2}{3} = 1 + \frac{60}{13}\times\frac{1}{3}$$
$$\rightarrow \frac{x^2}{3} = 1 + \frac{20}{13} = \frac{33}{13}$$
$$\rightarrow x^2 = \frac{99}{13}$$
$$\text{or } x = \pm\sqrt{\frac{99}{13}}$$

So req. points of intersection of given conics

$$\left(\pm\sqrt{\frac{99}{13}}, \pm\sqrt{\frac{60}{13}}\right)$$

(ii) $x^2 + y^2 = 8$ and $x^2 - y^2 = 1$

Solution:-

$$x^2 + y^2 = 8 \longrightarrow (1)$$
$$x^2 - y^2 = 1 \longrightarrow (2)$$

By (1)+(2) $\rightarrow 2x^2 = 9$

$$\rightarrow x^2 = \frac{9}{2} \quad \text{or} \quad x = \pm\frac{3}{\sqrt{2}}$$

Put $x^2 = \frac{9}{2}$ in (1), we get

$$\frac{9}{2} + y^2 = 8$$
$$\rightarrow y^2 = 8 - \frac{9}{2}$$
$$\rightarrow y^2 = \frac{7}{2} \quad \text{or} \quad y = \pm\sqrt{\frac{7}{2}}$$

Thus required points are

$$\left(\pm\sqrt{\frac{9}{2}}, \pm\sqrt{\frac{7}{2}}\right)$$

(iii) $3x^2 - 4y^2 = 12$ and $-2x^2 + 3y^2 = 7$

Solution:-

$$3x^2 - 4y^2 = 12 \longrightarrow (1)$$
$$-2x^2 + 3y^2 = 7 \longrightarrow (2)$$

Multiplying (1) by 2 and (2) by 3 then adding

$$6x^2 - 8y^2 = 24$$
$$-6x^2 + 9y^2 = 21$$

$$y^2 = 45$$
$$\text{or } y = \pm\sqrt{45}$$

Put $y^2 = 45$ in (1) we get

$$3x^2 - 4(45) = 12$$
$$\rightarrow 3x^2 - 180 = 12$$
$$\text{or } x^2 - 60 = 4$$
$$x^2 = 64$$
$$x = \pm\sqrt{64}$$
$$\rightarrow x = \pm 8$$

So req. points are $(\pm 8, \pm\sqrt{45})$

(iv) $3x^2 + 5y^2 = 60$ and $9x^2 + y^2 = 124$

Solution:-

$$3x^2 + 5y^2 = 60 \longrightarrow (1)$$
$$9x^2 + y^2 = 124 \longrightarrow (2)$$

multiplying (1) by 3 and subtract (2) from

$$9x^2 + 15y^2 = 180$$
$$\underset{-}{9x^2} \underset{-}{\pm} y^2 = \underset{-}{124}$$

$$14y^2 = 56 \rightarrow y^2 = 4$$

$\Rightarrow y = \pm 2$

Put $y^2 = 4$ in (1) we get

$3x^2 + 20 = 60$

$\Rightarrow 3x^2 = 40$

or $x^2 = \frac{40}{3}$ or $x = \pm\sqrt{\frac{40}{3}}$

so req. points of intersection of given conics are

$\left(\pm\sqrt{\frac{40}{3}}, \pm 2\right)$

(v) $4x^2 + y^2 = 16$ and

$x^2 + y^2 + y + 8 = 0$

Solution:-

$4x^2 + y^2 = 16 \longrightarrow (1)$

$x^2 + y^2 + y + 8 = 0 \longrightarrow (2)$

Multiplying (2) by 4 and subtract it from (1)

$\cancel{4x^2} + 4y^2 + 4y + 32 = 0$

$\underline{\cancel{4x^2} + y^2 \quad = 16}$

$3y^2 + 4y + 32 = -16$

or $3y^2 + 4y + 48 = 0$

$\Rightarrow y = \frac{-4 \pm \sqrt{16 - 4(3)(48)}}{6}$

$y = \frac{-4 \pm \sqrt{16 - 576}}{6}$

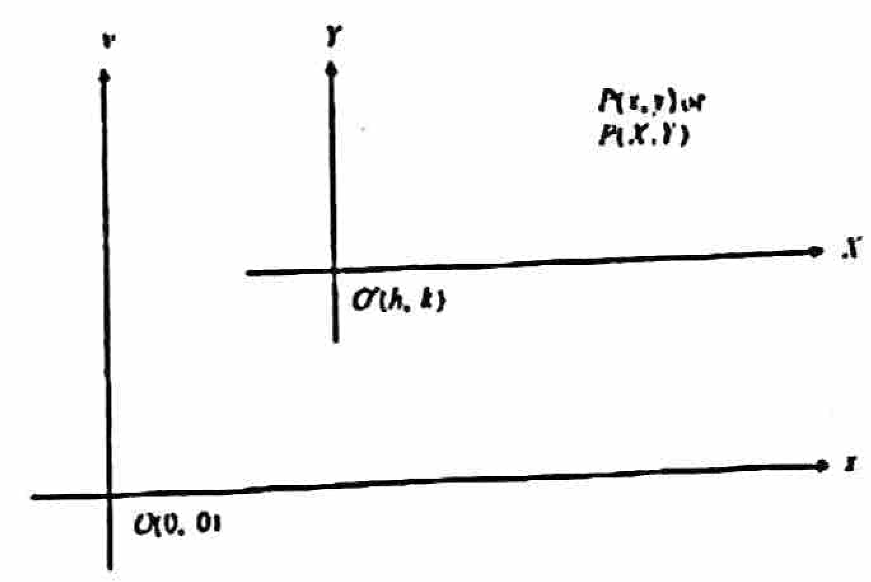
$y = \frac{-4 \pm \sqrt{-560}}{6}$

$\Rightarrow y = \frac{-4 \pm \sqrt{560}\,i}{6}$

At the value of y are complex (imaginary) so no real points of intersection of given conics exist.

Translation and Rotation of Axes

Translation of Axes :-



If a point P has coordinates (x,y) reffered to xy-system and has coordinates (X,Y) reffered to the translated axes O'X, O'Y through O'(h,k) then

$$\left.\begin{aligned} x = X + h, \quad & y = Y + k \\ X = x - h, \quad & Y = y - k \end{aligned}\right\}$$

Example 1. Transform the equation $x^2 + 6x - 8y + 17 = 0$ reffered to $O'(-3,1)$ as origin, axes remaining parallel to the old axes.

Solution:- $x^2 + 6x - 8y + 17 = 0 \longrightarrow (1)$

Eqs. of transformation are $x = X - 3$

$y = Y + 1$

Put values of x and y in (1)

$(X-3)^2 + 6(X-3) - 8(Y+1) + 17 = 0$

$X^2 + 9 - \cancel{6X} + \cancel{6X} - 18 - 8Y - 8 + 17 = 0$

$X^2 - 8Y = 0$ Req. transformed equation.

Example 2. By transforming the equation $x^2+4y^2-4x+8y+4=0$ referred to a new origin and axes remaining parallel to the orignal axes, the first degree terms are removed. Find the coordinates of the new origin and the transformed equation.

Solution:-

$$x^2+4y^2-4x+8y+4=0 \longrightarrow (1)$$

Let coordinates of the new origin be $O'(h,k)$. then eqs. of transformation are

$$x = X+h, \quad y = Y+k$$

Put values of x & y in (1)

$$(X+h)^2+4(Y+K)^2-4(X+h)+8(Y+K)+4=0$$

$$X^2+h^2+2Xh+4Y^2+4Y^2+8YK$$
$$-4X-4h+8Y+8K+4=0$$

$$\Rightarrow X^2+4Y^2+X(2h-4)+Y(8K+8)$$
$$+h^2+4K^2-4h+8K+4=0 \longrightarrow (2)$$

For the removal of first degree terms we put

$$2h-4=0, \quad 8K+8=0$$
$$\Rightarrow h=2, \quad K=-1$$

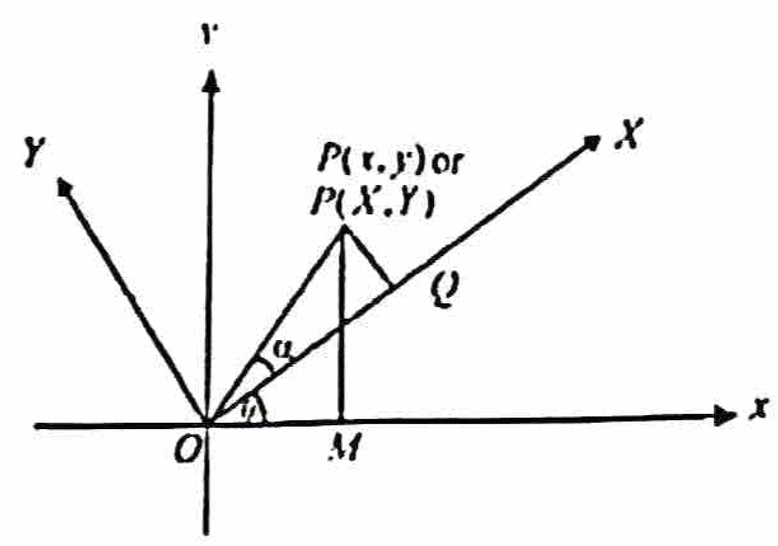
New origin is $O'(2,-1)$

put $h=2$, $K=-1$ in (2)

the transformed eq. is

$$X^2+4Y^2+4+4(-1)^2-8-8+4=0$$
$$\Rightarrow X^2+4Y^2-4=0$$

Rotation of Axes:-



Let xy-coordinate system be given. We rotate Ox, Oy about the origin through an angle $\theta\ (0<\theta<90°)$ so that the new axes are OX and OY as shown in the fig. Let a point $P(x,y)$ referred coordinates, $P(X,Y)$ referred to XY coordinate system. We have to find XY-coordinates in terms of the given coordinates (x,y). Let α be the measure of inclination of OP.

From P, draw PM $\perp$ar to Ox and PQ $\perp$ar to OX. Let $|OP|=r$

From $\triangle OPM$, we have

$$\left.\begin{aligned} OM &= x = r\cos\alpha \\ MP &= y = y\sin\alpha \end{aligned}\right\} \longrightarrow (1)$$

From $\triangle OPQ$,

$$OQ = X = r\cos(\alpha-\theta)$$
$$\Rightarrow X = x\cos\theta + y\sin\theta \quad \text{By (1)}$$

Also

$$MQ = Y = r\sin(\alpha-\theta)$$
$$\Rightarrow Y = y\cos\theta - x\sin\theta \quad \text{By (1)}$$

Thus,

$(X,Y) = (x\cos\theta+y\sin\theta,\ y\cos\theta-x\sin\theta)$ are coordinates of P referred to the new axes OX and OY.

Example 3. Find an equation of $5x^2 - 6xy + 5y^2 - 8 = 0$ with respect to new axes obtained by rotation of axes about the origin through an angle $135°$.

Solution:- Here $\theta = 135°$

Eqs. of transformation are

$$x = X\cos 135° - Y\sin 135°$$

$$= X\left(-\frac{1}{\sqrt{2}}\right) - Y\left(\frac{1}{\sqrt{2}}\right)$$

$$x = -\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}} = -\frac{1}{\sqrt{2}}(X+Y)$$

$$y = X\sin 135° + Y\cos 135°$$

$$= X\left(\frac{1}{\sqrt{2}}\right) + Y\left(-\frac{1}{\sqrt{2}}\right)$$

$$y = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}} = \frac{1}{\sqrt{2}}(X-Y)$$

Put value of x and y in given equation

$$5\left[-\frac{X+Y}{\sqrt{2}}\right]^2 - 6\left(-\frac{X+Y}{\sqrt{2}}\cdot\frac{X-Y}{\sqrt{2}}\right) + 5\left(\frac{X-Y}{\sqrt{2}}\right)^2 - 8 = 0$$

$$\frac{5}{2}(X^2+Y^2+2XY) + 3(X^2-Y^2) + \frac{5}{2}(X^2+Y^2-2XY) - 8 = 0$$

$$\Rightarrow 5X^2+5Y^2+\cancel{10XY}+6X^2-6Y^2+5X^2+5Y^2-\cancel{10XY}-16=0$$

$$\Rightarrow 16X^2+4Y^2-16=0$$

$$\Rightarrow 4X^2+Y^2-4=0 \quad \text{(req. eq.)}$$

Example 4. Find the angle through which the axes be rotated about the origin so that the product term XY is removed from the transformed equation of $5x^2 + 2\sqrt{3}xy + 7x^2 - 16 = 0$. Also find the transformed equation.

Solution:-

Eqs. of transformation are

$x = X\cos\theta - Y\sin\theta$ and

$y = X\sin\theta + Y\cos\theta$

Put values of x and y in given equation

$$5(X\cos\theta - Y\sin\theta)^2 + 2\sqrt{3}(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + 7(X\sin\theta+Y\cos\theta)^2 - 16 = 0 \longrightarrow (1)$$

$$5X^2\cos^2\theta + 5Y^2\sin^2\theta - 10XY\sin\theta\cos\theta + 2\sqrt{3}(X^2\cos\theta\sin\theta + XY\cos^2\theta - XY\sin^2\theta - Y^2\sin\theta\cos\theta) + 7X^2\sin^2\theta + 7Y^2\cos^2\theta + 14XY\sin\theta\cos\theta - 16 = 0 \longrightarrow (2)$$

The eq. (2) will be free from product term XY if

$$-10\sin\theta\cos\theta + 2\sqrt{3}(\cos^2\theta - \sin^2\theta) + 14\sin\theta\cos\theta = 0 \longrightarrow (a)$$

or $4\sin\theta\cos\theta + 2\sqrt{3}\cos 2\theta = 0$

$$\Rightarrow 2(2\sin\theta\cos\theta) + 2\sqrt{3}\cos 2\theta = 0$$

$$\Rightarrow 2\sin 2\theta + 2\sqrt{3}\cos 2\theta = 0$$

$$\Rightarrow 2\sin 2\theta = -2\sqrt{3}\cos 2\theta$$

$$\tan 2\theta = -\frac{2\sqrt{3}}{2}$$

$$\tan 2\theta = -\sqrt{3}$$

$$\tan 2\theta = \tan 120°$$

$$\Rightarrow 2\theta = 120° \text{ or } \theta = 60°$$

Put $\theta = 60°$ (without XY-term) $\because$ see (a) in (2)

$$5X^2\left(\frac{1}{4}\right) + 5Y^2\left(\frac{3}{4}\right) + 2\sqrt{3}\left(X^2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - Y^2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)\right) + 7X^2\left(\frac{3}{2}\right) + 7Y^2\left(\frac{1}{4}\right) - 16 = 0$$

$$\frac{5}{4}X^2 + \frac{15Y^2}{4} + 2\sqrt{3}\cdot\frac{\sqrt{3}}{4}(X^2 - Y^2) + \frac{21X^2}{2} + \frac{7}{4}Y^2 - 16 = 0$$

$\frac{5}{4}X^2+\frac{15Y^2}{4}+\frac{3}{2}X^2-\frac{3}{2}Y^2+\frac{21}{4}X^2+\frac{7}{4}Y^2-16=0$

Multiplying by 4

$5X^2+15Y^2+6X^2-6Y^2+21X^2+7Y^2-64=0$

$\rightarrow 32X^2+16Y^2-64=0$

or $4X^2+2Y^2-8=0$ ($\div$ by 8)

or $2X^2+Y^2-4=0$ ($\div$ by 2)

Exercise 6.8

Q1. Find the equation of each of the following with respect to new axes obtained by shifting the origin to the indicated point:

(i) $x^2+16y-16=0$, $O'(0,1)$

Solution:-

$x^2+16y-16=0 \longrightarrow (1)$

Eqs. of transformation are

$x=X+0$, $y=Y+1$

Put values of x & y in (1)

$X^2+16(Y+1)-16=0$

$X^2+16Y+16-16=0$

or $X^2+16Y=0$ (Req. eq)

(ii) $4x^2+y^2+16x-10y+37=0$

$O'(-2,5)$

Solution:-

$4x^2+y^2+16x-10y+37=0 \longrightarrow (1)$

Eqs. of transformation are

$x=X-2$, $y=Y+5$

Put values of x & y in (1)

$4(X-2)^2+(Y+5)^2+16(X-2)-10(Y+5)+37=0$

$4(X^2+4-4X)+Y^2+25+10Y+16X-32-10Y-50+37=0$

$4X^2+16-16X+Y^2+25+10Y+16X-32-10Y-50+37=0$

$4X^2+Y^2-4=0$ (Req. eq)

(iii) $9x^2+4y^2+18x-16y-11=0$

$O'(-1,2)$

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Solution:-

$9x^2+4y^2+18x-16y-11=0 \longrightarrow (1)$

Eqs. of transformation are

$x=X-1$, $y=Y+2$

Put values of x and y in (1)

$9(X-1)^2+4(Y+2)^2+18(X-1)-16(Y+2)-11=0$

$9(X^2+1-2X)+4(Y^2+4+4Y)+18X-18-16Y-32-11=0$

$9X^2+9-18X+4Y^2+16+16Y+18X-18-16Y-32-11=0$

$9X^2+4Y^2-36=0$ (Req. Eq)

(iv) $x^2-y^2+4x+8y-11=0$

$O'(-2,4)$

Solution:-

$x^2-y^2+4x+8y-11=0 \longrightarrow (1)$

Eqs. of transformation are

$x=X-2$, $y=Y+4$

Put values of x & y in (1)

$(X-2)^2-(Y+4)^2+4(X-2)+8(Y+4)-11=0$

$X^2+4-4X-Y^2-16-8Y+4X-8+8Y+32-11=0$

$X^2-Y^2+1=0$ (Req. Eq)

(v) $9x^2-4y^2+36x+8y-4=0$

$O'(-2,1)$

Solution:-

$9x^2-4y^2+36x+8y-4=0 \longrightarrow (1)$

Eqs. of transformation are

$x=X-2 \quad , \quad y=Y+1$

Put values of x and y in (1)

$9(X-2)^2-4(Y+1)^2+36(X-2)+8(Y+1)-4=0$

$9(X^2+4-4X)-4(Y^2+1+2Y)+36X-72+8Y+8-4=0$

$9X^2+36-36X-4Y^2-4-8Y+36X-72+8Y+8-4=0$

$9X^2-4Y^2-36=0$ (Req. Eq)

Q2. Find coordinates of the new origin (axes remaining parallel) so that first degree terms are removed from the transformed equation of each of the following. Also find the transformed equation:

(i) $3x^2-2y^2+24x+12y+24=0$

Solution:-

$3x^2-2y^2+24x+12y+24=0 \longrightarrow (1)$

Let coordinates of new origin be (h,k). then equations of transformation are

$x=X+h, \quad y=Y+k$

Put values of x & y in (1)

$3(X+h)^2-2(Y+k)^2+24(X+h)+12(Y+k)+24=0$

$3(X^2+h^2+2Xh)-2(Y^2+K^2+2YK)+24X+24h+12Y+12K+24=0$

$3X^2+3h^2+6Xh-2Y^2-2K^2-4YK+24X+24h+12Y+12K+24=0$

$3X^2-2Y^2+6(h+4)X-4(K-3)Y+3h^2-2K^2+24h+12K+24=0 \longrightarrow (2)$

Remove first degree term

put $h+4=0$

$\rightarrow h=-4$

and $K-3=0 \rightarrow K=3$ in (2)

$3X^2-2Y^2+3(-4)^2-2(3)^2+24(-4)+12(3)+24=0$

$3X^2-2Y^2+48-18-96+36+24=0$

$3X^2-2Y^2-6=0$

which is new transformed eq and new origin $(-4,3)$

(ii) $25x^2+9y^2+50x-36y-164=0$

Solution:-

$25x^2+9y^2+50x-36y-164=0 \longrightarrow (1)$

Let coordinates of new origin be (h,K) then eqs. of transformation are

$x=X+h \quad , \quad y=Y+K$

Put values of x & y in (1)

$25(X+h)^2+9(Y+K)^2+50(X+h)-36(Y+K)-164=0$

$25X^2+25h^2+50Xh+9Y^2+9K^2+18YK+50X+50h-36Y-36K-164=0$

$25X^2+9Y^2+50(h+1)X+18(K-2)Y+25h^2+9K^2+50h-36K-164=0 \longrightarrow (2)$

To remove first degree term, we put $h+1=0 \quad , \quad K-2=0$

$h=-1 \quad , \quad K=2$

so the new origin is $(-1, 2)$

Put $h=-1$, $k=2$ in (2)

$$25X^2+9Y^2+(0)X+(0)Y+25+36-50-72-164=0$$

$$\rightarrow 25X^2+9Y^2-225=0$$ (Req. eq)

(iii) $x^2-y^2-6x+2y+7=0$

Solution:-

$$x^2-y^2-6x+2y+7=0 \longrightarrow (1)$$

Suppose (h,k) be the coordinates of new origin then eqs. of transformation are

$$x=X+h \quad , \quad y=Y+k$$

Put values of x and y in (1)

$$(X+h)^2-(Y+k)^2-6(X+h)+2(Y+k)+7=0$$

$$X^2+h^2+2Xh-Y^2-k^2-2Yk-6X-6h+2Y+2k+7=0$$

$$X^2-Y^2+(2h-6)X-2(k-1)Y+h^2-k^2-6h+2k+7=0 \longrightarrow (2)$$

To remove first degree term,

Put $2h-6=0 \rightarrow h=3$

and $k-1=0 \rightarrow k=1$

so new origin is $(3, 1)$

Put $h=3$, $k=1$ in (2)

$$X^2-Y^2+9-1-18+2+7=0$$

or $X^2-Y^2-1=0$ (Req. Eq)

Q3. In each of the following, find an equation referred to the new axes obtained by rotation of axes about the origin through the given angle;

(i) $xy=1 \quad , \quad \theta=45^\circ$

Solution:- $xy=1 \longrightarrow (1)$

Eqs. of transformation are

$$x = X\cos45^\circ - Y\sin45^\circ$$

$$= X\left(\frac{1}{\sqrt2}\right)-Y\left(\frac{1}{\sqrt2}\right)$$

$$x = \frac{X-Y}{\sqrt2} \longrightarrow (i)$$

and

$$y = Y\cos45^\circ + X\sin45^\circ$$

$$= Y\left(\frac{1}{\sqrt2}\right)+X\left(\frac{1}{\sqrt2}\right)$$

$$y = \frac{Y+X}{\sqrt2} \longrightarrow (ii)$$

using (i) & (ii) in (1)

$$\left(\frac{X-Y}{\sqrt2}\right)\left(\frac{X+Y}{\sqrt2}\right) = 1$$

or $\dfrac{X^2-Y^2}{2} = 1$

$$\rightarrow X^2-Y^2 = 2 \text{ (Req. eq)}$$

(ii) $7x^2-8xy+y^2-9=0$

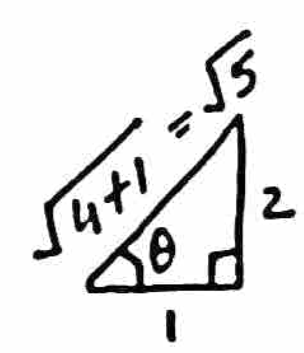
$\theta = \tan^{-1}(2)$

Solution:-

$$7x^2-8xy+y^2-9=0 \longrightarrow (1)$$

$\because \theta = \tan^{-1} 2$

$\rightarrow \tan\theta = \dfrac{2}{1}$



so $\sin\theta = \dfrac{2}{\sqrt5}$

$\cos\theta = \dfrac{1}{\sqrt5}$

Now Eqs. of transformation are

$$x = X\cos\theta - Y\sin\theta$$

$$x = X\left(\frac{1}{\sqrt5}\right)-Y\left(\frac{2}{\sqrt5}\right) = \frac{X-2Y}{\sqrt5} \longrightarrow (i)$$

$$y = Y\cos\theta + X\sin\theta$$

$$= Y\left(\frac{1}{\sqrt5}\right)+X\left(\frac{2}{\sqrt5}\right) = \frac{Y+2X}{\sqrt5} \longrightarrow (ii)$$

using (i) and (ii) in (1)

$$7\left(\frac{X-2Y}{\sqrt{5}}\right)^2-8\left(\frac{X-2Y}{\sqrt{5}}\right)\left(\frac{2X+Y}{\sqrt{5}}\right)+\left(\frac{2X+Y}{\sqrt{5}}\right)^2=0$$

$$7\left(\frac{X^2+4Y^2-4XY}{5}\right)-8\left(\frac{2X^2+XY-4XY-2Y^2}{5}\right)+\frac{4X^2+Y^2+4XY}{5}=0$$

$$7X^2+28Y^2-28XY-16X^2+24XY+16Y^2+4X^2+Y^2+4XY=0$$

$$-5X^2+45Y^2-45=0$$

or $X^2-9Y^2+9=0$ (Req. eq)

(iii) $9x^2+12xy+4y^2-x-y=0$

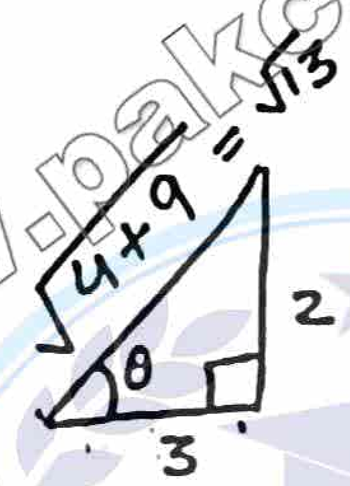
$\theta=\tan^{-1}\frac{2}{3}$

Solution:-

$$9x^2+12xy+4y^2-x-y=0 \longrightarrow (1)$$

$\because \theta=\tan^{-1}\frac{2}{3}$

$\Rightarrow \tan\theta=\frac{2}{3}$



so $\sin\theta=\frac{2}{\sqrt{13}}$

$\cos\theta=\frac{3}{\sqrt{13}}$

Now eqs. of transformation are

$x=X\cos\theta-Y\sin\theta$

$=X\left(\frac{3}{\sqrt{13}}\right)-Y\left(\frac{2}{\sqrt{13}}\right)$

$x=\frac{3X}{\sqrt{13}}-\frac{2Y}{\sqrt{13}}=\frac{3X-2Y}{\sqrt{13}} \longrightarrow (i)$

$y=Y\cos\theta+X\sin\theta$

$=Y\left(\frac{3}{\sqrt{13}}\right)+X\left(\frac{2}{\sqrt{13}}\right)$

$y=\frac{3Y}{\sqrt{13}}+\frac{2X}{\sqrt{13}}=\frac{2X+3Y}{\sqrt{13}} \longrightarrow (ii)$

using (i) & (ii) in (1) we have

$$9\left(\frac{3X-2Y}{\sqrt{13}}\right)^2+12\left(\frac{3X-2Y}{\sqrt{13}}\right)\left(\frac{2X+3Y}{\sqrt{13}}\right)+4\left(\frac{2X+3Y}{\sqrt{13}}\right)^2-\left(\frac{3X-2Y}{\sqrt{13}}\right)-\left(\frac{2X+3Y}{\sqrt{13}}\right)=0$$

$$9\left(\frac{9X^2+4Y^2-12XY}{13}\right)+12\left(\frac{6X^2+9XY-4XY-6Y^2}{13}\right)+4\left(\frac{4X^2+9Y^2+12XY}{13}\right)-\left(\frac{3X-2Y}{\sqrt{13}}\right)-\left(\frac{2X+3Y}{\sqrt{13}}\right)=0$$

$$81X^2+36Y^2-108XY+72X^2+60XY-72Y^2+16X^2+36Y^2+48XY-\sqrt{13}(3X)+\sqrt{13}(2Y)-\sqrt{13}(2X)-\sqrt{13}(3Y)=0$$

$$169X^2-5\sqrt{13}X-\sqrt{13}Y=0$$

$$13(\sqrt{13})^2X^2-5\sqrt{13}X-\sqrt{13}Y=0$$

$\sqrt{13}\left(13X^2-5X-Y\right)=0$ (Req. Eq)

(iv) $x^2-2xy+y^2-2\sqrt{2}x-2\sqrt{2}y+2=0$

$\theta=45^\circ$

Solution:-

$$x^2-2xy+y^2-2\sqrt{2}x-2\sqrt{2}y+2=0 \longrightarrow (1)$$

Eqs. of transformation are

$x=X\cos\theta-Y\sin\theta$

$=X\cos45^\circ-Y\sin45^\circ$

$=X\left(\frac{1}{\sqrt{2}}\right)-Y\left(\frac{1}{\sqrt{2}}\right)$

$x=\frac{X-Y}{\sqrt{2}} \longrightarrow (i)$

$y=X\sin\theta+Y\cos\theta$

$=X\sin45^\circ+Y\cos45^\circ$

$=X\left(\frac{1}{\sqrt{2}}\right)+Y\left(\frac{1}{\sqrt{2}}\right)$

$y=\frac{X+Y}{\sqrt{2}} \longrightarrow (ii)$

using (i) & (ii) in (1)

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2-2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right)+\left(\frac{X+Y}{\sqrt{2}}\right)^2$$

$$-2\sqrt{2}\left(\frac{X-Y}{\sqrt{2}}\right)-2\sqrt{2}\left(\frac{X+Y}{\sqrt{2}}\right)+2=0$$

$$\frac{X^2+Y^2-2XY}{2}-2\left(\frac{X^2-Y^2}{2}\right)+\frac{X^2+Y^2+2XY}{2}$$

$$-2X+\cancel{2Y}-2X-\cancel{2Y}+2=0$$

$$\rightarrow X^2+Y^2-2XY-2X^2+2Y^2+X^2+Y^2+2XY$$

$$-4X\qquad -4X\qquad +4=0$$

$$4Y^2-8X\qquad +4=0$$

or $Y^2-2X+\underline{1}=0$ (Req. Eq)

Q4. Find measure of angle through which the axes be rotated so that the product term XY is removed from the transformed equation. Also find the transformed equation:

(i) $2x^2+6xy+10y^2-11=0$

Solution:-

$$2x^2+6xy+10y^2-11=0 \longrightarrow (1)$$

Let the axes be rotated through an angle θ.

then eqs. of transformation are

$$x = X\cos\theta - Y\sin\theta \longrightarrow (i)$$

$$y = X\sin\theta + Y\cos\theta \longrightarrow (ii)$$

using (i) and (ii) in (1)

$$2(X\cos\theta-Y\sin\theta)^2+6(X\cos\theta-Y\sin\theta)(X\sin\theta+Y\cos\theta)+10(X\sin\theta+Y\cos\theta)^2-11=0$$

$$2(X^2\cos^2\theta+Y^2\sin^2\theta-2XY\cos\theta\sin\theta)+6(X^2\cos\theta\sin\theta+XY\cos^2\theta-YX\sin^2\theta$$

$$-Y^2\cos\theta\sin\theta)+10(X^2\sin^2\theta+Y^2\cos^2\theta-2XY\cos\theta\sin\theta)\quad -11=0$$

$$2X^2\cos^2\theta+2Y^2\sin^2\theta-4XY\cos\theta\sin\theta+6X^2\cos\theta\sin\theta+6XY\cos^2\theta-6YX\sin^2\theta$$

$$-6Y^2\cos\theta\sin\theta+10X^2\sin^2\theta+10Y^2\cos^2\theta-20XY\cos\theta\sin\theta-11=0$$

$$X^2(2\cos^2\theta + 6\cos\theta\sin\theta + 10\sin^2\theta) + XY(-4\cos\theta\sin\theta + 6\cos^2\theta - 6\sin^2\theta + 20\sin\theta\cos\theta) + Y^2(2\sin^2\theta - 6\sin\theta\cos\theta + 10\cos^2\theta) - 11 = 0 \longrightarrow (2)$$

The equation (2) will be free from product term XY if

$$6\cos^2\theta - 6\sin^2\theta + 16\sin\theta\cos\theta = 0 \longrightarrow (a)$$

$$\Rightarrow 3\cos^2\theta - 3\sin^2\theta + 8\sin\theta\cos\theta = 0$$

$$\Rightarrow 3 - 3\tan^2\theta + 8\tan\theta = 0 \qquad (\div \text{ by } \cos^2\theta)$$

or $3\tan^2\theta - 8\tan\theta - 3 = 0$

$$3\tan^2\theta - 9\tan\theta + \tan\theta - 3 = 0$$

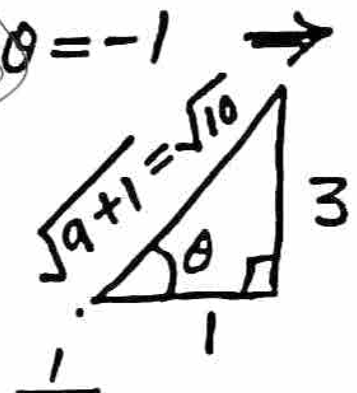
$$3\tan\theta(\tan\theta - 3) + 1(\tan\theta - 3) = 0$$

$$(\tan\theta - 3)(3\tan\theta + 1) = 0$$

$\tan\theta - 3 = 0$, $3\tan\theta + 1 = 0$

$\Rightarrow \tan\theta = 3$, $3\tan\theta = -1 \Rightarrow \tan\theta = -\frac{1}{3}$ (Ignoring)

so we take $\tan\theta = \frac{3}{1}$



$$\Rightarrow \sin\theta = \frac{3}{\sqrt{10}}, \quad \cos\theta = \frac{1}{\sqrt{10}}$$

Put values of $\sin\theta$, $\cos\theta$ in eq (2) (without XY-term) ($\because$ see (a))

$$X^2\left[2\left(\tfrac{1}{10}\right) + 6\left(\tfrac{1}{\sqrt{10}}\right)\left(\tfrac{3}{\sqrt{10}}\right) + 10\left(\tfrac{9}{10}\right)\right] + XY(0) + Y^2\left[2\left(\tfrac{9}{10}\right) - 6\left(\tfrac{3}{\sqrt{10}}\right)\left(\tfrac{1}{\sqrt{10}}\right) + 10\left(\tfrac{1}{10}\right)\right] - 11 = 0$$

$$\Rightarrow X^2\left(\frac{2}{10} + \frac{18}{10} + \frac{90}{10}\right) + Y^2\left(\frac{18}{10} - \frac{18}{10} + \frac{10}{10}\right) - 11 = 0$$

$$\Rightarrow 11X^2 + Y^2 - 11 = 0 \quad (\text{Req. Eq})$$

(ii) $xy + 4x - 3y - 10 = 0$

Solution:–

$$xy + 4x - 3y - 10 = 0 \longrightarrow (1)$$

Let the axes be rotated through an angle θ. then eqs. of transformation are

$$x = X\cos\theta - Y\sin\theta \longrightarrow (i)$$
$$y = X\sin\theta + Y\cos\theta \longrightarrow (ii)$$

using (i) & (ii) in (1) we have

$$\Rightarrow (X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + 4(X\cos\theta - Y\sin\theta) - 3(X\sin\theta + Y\cos\theta) - 10 = 0$$

$$\Rightarrow X^2\sin\theta\cos\theta + XY\cos^2\theta - XY\sin^2\theta - Y^2\cos\theta\sin\theta + 4X\cos\theta - 4Y\sin\theta - 3X\sin\theta - 3Y\cos\theta - 10 = 0$$

$$X^2\sin\theta\cos\theta - Y^2\cos\theta\sin\theta + XY(\cos^2\theta - \sin^2\theta) + X(4\cos\theta - 3\sin\theta) - Y(4\sin\theta + 3\cos\theta) - 10 = 0 \longrightarrow (2)$$

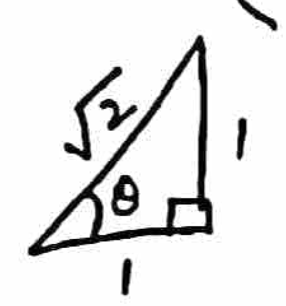
Now the equation (2) will be free from product term XY if

$$\cos^2\theta - \sin^2\theta = 0 \longrightarrow (a)$$

$$\Rightarrow \cos^2\theta = \sin^2\theta \quad \text{or} \quad \tan^2\theta = 1 \Rightarrow \tan\theta = \pm 1$$

we take $\tan\theta = \frac{1}{1}$ ($\tan\theta = -1$ ignoring)

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}, \quad \cos\theta = \frac{1}{\sqrt{2}}$$



so putting values of $\sin\theta, \cos\theta$ in (2) (without XY-term $\because$ see (a))

$$X^2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - Y^2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + XY(0) + X\left(4\left(\frac{1}{\sqrt{2}}\right) - 3\left(\frac{1}{\sqrt{2}}\right)\right) - Y\left(4\left(\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right)\right) - 10 = 0$$

$$\Rightarrow \frac{X^2}{2} - \frac{Y^2}{2} + \frac{X}{\sqrt{2}} - \frac{7Y}{\sqrt{2}} - 10 = 0$$

$$\Rightarrow X^2 - Y^2 + \sqrt{2}X - 7\sqrt{2}Y - 20 = 0 \quad \text{(Req. eq)}$$

(iii) $5x^2 - 6xy + 5y^2 - 8 = 0$

Solution:-

$$5x^2 - 6xy + 5y^2 - 8 = 0 \longrightarrow (1)$$

Let the axes be rotated through an angle θ then eqs. of transformation are

$$x = X\cos\theta - Y\sin\theta \longrightarrow (i)$$

$$y = X\sin\theta + Y\cos\theta \longrightarrow (ii)$$

using (i) & (ii) in (1) we get

$$5(X\cos\theta - Y\sin\theta)^2 - 6(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + 5(X\sin\theta + Y\cos\theta)^2 - 8 = 0$$

$$5(X^2\cos^2\theta + Y^2\sin^2\theta - 2XY\sin\theta\cos\theta) - 6(X^2\sin\theta\cos\theta + XY\cos^2\theta - XY\sin^2\theta - Y^2\sin\theta\cos\theta) + 5(X^2\sin^2\theta + Y^2\cos^2\theta + 2XY\sin\theta\cos\theta) - 8 = 0$$

$$\Rightarrow X^2(5\cos^2\theta - 6\cos\theta\sin\theta + 5\sin^2\theta) + XY(-10\cos\theta\sin\theta - 6\cos^2\theta + 6\sin^2\theta + 10\cos\theta\sin\theta) + Y^2(5\sin^2\theta + 6\sin\theta\cos\theta + 5\cos^2\theta) - 8 = 0 \longrightarrow (2)$$

Now eq (2) will be free from product term XY if

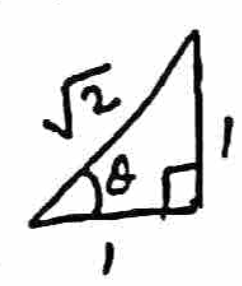
$$\cancel{-10\cos\theta\sin\theta} - 6\cos^2\theta + 6\sin^2\theta \cancel{+10\cos\theta\sin\theta} = 0 \longrightarrow (a)$$

$$\Rightarrow \cancel{6}\sin^2\theta = \cancel{6}\cos^2\theta$$

or $\tan^2\theta = 1 \Rightarrow \tan\theta = \pm 1$

we take $\tan\theta = 1$, $\tan\theta = -1$ (ignoring)

$\tan\theta = \frac{1}{1}$



so $\sin\theta = \frac{1}{\sqrt{2}}$, $\cos\theta = \frac{1}{\sqrt{2}}$

Putting values of $\sin\theta, \cos\theta$ in (2) (without XY-term ∵ see (a))

$$\Rightarrow X^2\left[5\left(\frac{1}{\sqrt{2}}\right)^2 - 6\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 5\left(\frac{1}{\sqrt{2}}\right)^2\right] + XY(0) + Y^2\left(5\left(\frac{1}{\sqrt{2}}\right)^2 + 6\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 5\left(\frac{1}{\sqrt{2}}\right)^2\right) - 8 = 0$$

$$\Rightarrow X^2\left(\frac{5}{2} - \frac{6}{2} + \frac{5}{2}\right) + Y^2\left(\frac{5}{2} + \frac{6}{2} + \frac{5}{2}\right) - 8 = 0$$

$$\Rightarrow 2X^2 + 8Y^2 - 8 = 0$$

or $X^2 + 4Y^2 - 4 = 0$ (Req. Eq)

The general equation of Second degree

$Ax^2 + By^2 + Gx + Fy + C = 0$

The most general eq. of the second degree

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \longrightarrow (1)$

represents a conic

Here is called the discriminant (1) represents

(i) An ellipse or circle if $h^2 - ab < 0$

(ii) A parabola if $h^2 - ab = 0$

(iii) A hyperbola if $h^2 - ab < 0$

*If the axes are rotated about the origin through an angle θ ($0 < \theta < 90°$) where θ is given by $\tan 2\theta = \frac{2h}{a-b}$

If $a = b$ or $a = 0 = b$ then the axes are rotated through an angle $45°$

Equations of transformation are

$x = X\cos\theta - Y\sin\theta \longrightarrow (i)$

$y = X\sin\theta + Y\cos\theta \longrightarrow (ii)$

Solving (i) & (ii) for X, Y we find

$X = x\cos\theta + y\sin\theta$

$Y = -x\sin\theta + y\cos\theta$

Under certain conditions equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ may not represent any conic. In such a case we say (1) represent a degenerate conic, one such degenerate conic is a pair of straight lines represented by (1) if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Example 1. Discuss the conic $7x^2-6\sqrt{3}xy+13y^2-16=0$ and find its elements.

Solution:-

$$7x^2-6\sqrt{3}xy+13y^2-16=0 \longrightarrow (1)$$

In order to remove xy term the angle through which axes be rotated is given by

$$\tan 2\theta = \frac{-6\sqrt{3}}{7-13} \qquad \begin{pmatrix} a=7 \\ b=13 \\ 2h=-6\sqrt{3} \end{pmatrix}$$

$$\tan 2\theta = \frac{-6\sqrt{3}}{-6} = \sqrt{3}$$

$$\Rightarrow 2\theta = \tan^{-1}(\sqrt{3})$$

$$2\theta = 60^\circ \Rightarrow \theta = 30^\circ$$

Eqs. of transformation are

$$x = X\cos 30^\circ - Y\sin 30^\circ$$

$$= X\left(\frac{\sqrt{3}}{2}\right) - Y\left(\frac{1}{2}\right)$$

$$x = \frac{\sqrt{3}X - Y}{2} \longrightarrow (i)$$

$$y = X\sin 30^\circ + Y\cos 30^\circ$$

$$= X\left(\frac{1}{2}\right) + Y\left(\frac{\sqrt{3}}{2}\right)$$

$$y = \frac{X+\sqrt{3}Y}{2} \longrightarrow (ii)$$

using (i) & (ii) in (1)

$$7\left(\frac{\sqrt{3}X-Y}{2}\right)^2 - 6\sqrt{3}\left(\frac{\sqrt{3}X-Y}{2}\right)\left(\frac{X+\sqrt{3}Y}{2}\right) + 13\left(\frac{X+\sqrt{3}Y}{2}\right)^2 = 16$$

$$7\left(\frac{3X^2+Y^2-2\sqrt{3}XY}{4}\right) - 6\sqrt{3}\left(\frac{\sqrt{3}X^2+3XY-XY-\sqrt{3}Y^2}{4}\right) + 13\left(\frac{X^2+3Y^2+2\sqrt{3}XY}{4}\right) = 16$$

$$21X^2+7Y^2-\cancel{14\sqrt{3}XY}-18X^2-\cancel{12\sqrt{3}XY}+18Y^2+13X^2+39Y^2+\cancel{26\sqrt{3}XY}=64$$

$$16X^2+64Y^2=64$$

$$\Rightarrow X^2+4Y^2=4 \quad (\div \text{ by } 16)$$

$$\text{or } \frac{X^2}{4}+\frac{Y^2}{1}=1 \quad (\text{Ellipse}) \longrightarrow (2)$$

Now solving (i) and (ii) for X and Y

$$\text{i)} \Rightarrow x = \frac{\sqrt{3}X-Y}{2}$$

$$\Rightarrow 2x = \sqrt{3}X - Y$$

$$\text{or } Y = \sqrt{3}X - 2x \longrightarrow (a)$$

Put in (ii)

$$\Rightarrow y = \frac{X+\sqrt{3}(\sqrt{3}X-2x)}{2}$$

$$y = \frac{X+3X-2\sqrt{3}x}{2}$$

$$y = \frac{4X-2\sqrt{3}x}{2}$$

$$y = 2X - \sqrt{3}x$$

$$\text{or } 2X = y+\sqrt{3}x$$

$$\Rightarrow X = \frac{\sqrt{3}x+y}{2} \quad \text{put in (a)}$$

$$Y = \sqrt{3}\left(\frac{\sqrt{3}x+y}{2}\right) - 2x$$

$$Y = \frac{3x+\sqrt{3}y-4x}{2}$$

$Y = \frac{-x+\sqrt{3}y}{2}$

Elements of ellipse

Centre:- centre of ellipse (2) is $X = 0$, $Y = 0$

$\Rightarrow \frac{\sqrt{3}x+y}{2} = 0$, $\frac{-x+\sqrt{3}y}{2} = 0$

$\Rightarrow \sqrt{3}x+y=0$, $-x+\sqrt{3}y=0$

$\therefore$ giving $x=0, y=0$

Thus centre is $(0,0)$ of (1)

Length of major axis $= 4$

Length of minor axis $= 2$

Vertices:- Vertices of (,) are: $X = \pm 2$, $Y = 0$

$\frac{\sqrt{3}x+y}{2} = \pm 2$, $\frac{-x+\sqrt{3}y}{2} = 0$

$\Rightarrow \sqrt{3}x+y = \pm 4$, $-x+\sqrt{3}y=0$

$\sqrt{3}x+y = 4$

$-x+\sqrt{3}y = 0 \longrightarrow (a)$

By $\sqrt{3}$ (a) then add

$-\sqrt{3}x+3y = 0$

$\sqrt{3}x + y = 4$

$4y = 4$

$y = 1$ put in (a)

$-x+\sqrt{3} = 0$

$\Rightarrow -x = -\sqrt{3}$

or $x = \sqrt{3}$

$\sqrt{3}x+y = -4$

$-x+\sqrt{3}y = 0 \longrightarrow (b)$

By $\sqrt{3}$ (b) and add

$\sqrt{3}x+y = -4$

$-\sqrt{3}x+3y = 0$

$4y = -4$

$y = -1$ put in (b)

$-x-\sqrt{3} = 0$

$\Rightarrow x = -\sqrt{3}$

so vertices of (1) are $(\sqrt{3}, 1)$ and $(-\sqrt{3}, -1)$

Ends:-

ends of minor axis $= X = 0$

ends of Major axis $= Y = 1$

$\Rightarrow \frac{\sqrt{3}x+y}{2} = 0$, $\frac{-x+\sqrt{3}y}{2} = \pm 1$

$\sqrt{3}x+y = 0$, $-x+\sqrt{3}y = \pm 1$

$\sqrt{3}x+y = 0$

$-x+\sqrt{3}y = 1 \longrightarrow (c)$

By $\sqrt{3}$ (c) and add

$\sqrt{3}x+y = 0$

$-\sqrt{3}x+3y = \sqrt{3}$

$4y = \sqrt{3}$

$y = \frac{\sqrt{3}}{4}$ put in (c)

$-x + \frac{3}{4} = 1$

$-x = 1 - \frac{3}{4}$

$-x = \frac{1}{4}$

$x = -\frac{1}{4}$

$\sqrt{3}x+y = 0$

$-x+\sqrt{3}y = -1 \longrightarrow (d)$

By $\sqrt{3}$ (d) and add

$\sqrt{3}x+y = 0$

$-\sqrt{3}x+3y = -\sqrt{3}$

$4y = -\sqrt{3}$

$y = -\frac{\sqrt{3}}{4}$ put in (d)

$-x - \frac{3}{4} = -1$

$-x = -1 + \frac{3}{4}$

$-x = -\frac{1}{4}$

$x = \frac{1}{4}$

so ends are $\left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$ & $\left(\frac{1}{4}, -\frac{\sqrt{3}}{4}\right)$

Equation:-

equation of major axis is $Y = 0$ i.e., $-x+\sqrt{3}y = 0$

equation of minor axis is $X = 0$ i.e., $\sqrt{3}x+y = 0$

Example 2. Analyze the conic $xy = 4$ and write its elements

Solution:-

$xy = 4 \longrightarrow (1)$

Here $a = 0 = b$, so we rotate the axes through an angle $45°$. Eqs. of transformation

$x = X\cos 45° - Y\sin 45° = \frac{X-Y}{\sqrt{2}} \rightarrow (i)$

$y = X\sin 45° + Y\cos 45° = \frac{X+Y}{\sqrt{2}} \rightarrow (ii)$

using (i) and (ii) in(1)

$$\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right)-4=0$$

$$\frac{X^2-Y^2}{2}=4$$

$$\Rightarrow \frac{X^2}{8}-\frac{Y^2}{8}=1 \text{ (Hyperbola)} \longrightarrow (2)$$

Solving (i) & (ii) for X, Y

$$x=\frac{X-Y}{\sqrt{2}}, \quad y=\frac{X+Y}{\sqrt{2}}$$

$$\Rightarrow \sqrt{2}x = X-Y, \quad \sqrt{2}y = X+Y$$

By adding

$$X+Y=\sqrt{2}y$$
$$X-Y=\sqrt{2}x$$
$$2X=\sqrt{2}(x+y)$$
$$X=\frac{x+y}{\sqrt{2}}$$

and $\sqrt{2}y=\frac{x+y}{\sqrt{2}}+Y$

$$\sqrt{2}y-\frac{x+y}{\sqrt{2}}=Y$$

$$\frac{2y-x-y}{\sqrt{2}}=Y \text{ or } Y=\frac{-x+y}{\sqrt{2}}$$

Elements of hyperbola

Centre:- centre of hyperbola (2) is $X=0, \quad Y=0$

$$\Rightarrow \frac{x+y}{\sqrt{2}}=0, \quad -\frac{x+y}{\sqrt{2}}=0$$

$\Rightarrow x=0, \quad y=0$ is centre of (1)

Equation:-

Eq. of focal axis: $Y=0$

$$\Rightarrow -\frac{x+y}{\sqrt{2}}=0$$

$$\Rightarrow y=x$$

Eq. of conjugate axis: $X=0$

$$\Rightarrow \frac{x+y}{\sqrt{2}}=0$$

or $y=-x$

Eccentricity:- $\sqrt{2}$

Foci:-

Foci of (2) are

$$X=\pm 2\sqrt{2}\cdot\sqrt{2}, \quad Y=0$$

$$\frac{x+y}{\sqrt{2}}=\pm 2\sqrt{2}\cdot\sqrt{2}, \quad \frac{-x+y}{\sqrt{2}}=0$$

$$x+y=\pm 4\sqrt{2}, \quad -x+y=0$$

$x+y=4\sqrt{2}$
$-x+y=0$
$2y=4\sqrt{2}$
$y=2\sqrt{2}$
so $-x+2\sqrt{2}=0$
$\Rightarrow x=2\sqrt{2}$

$-x+y=0$
$x+y=-4\sqrt{2}$
$2y=-4\sqrt{2}$
$y=-2\sqrt{2}$
so $-x-2\sqrt{2}=0$
$x=-2\sqrt{2}$

so Foci of (1) are $(2\sqrt{2}, 2\sqrt{2})$ & $(-2\sqrt{2}, -2\sqrt{2})$

Vertices:-

Vertices of (2) are

$$X=\pm 2\sqrt{2}, \quad Y=0$$

$$\frac{x+y}{\sqrt{2}}=\pm 2\sqrt{2}, \quad \frac{-x+y}{\sqrt{2}}=0$$

$$x+y=\pm 4, \quad -x+y=0$$

$x+y=4$
$-x+y=0$
$2y=4$
$y=2$
so $x=2$

$x+y=-4$
$-x+y=0$
$2y=-4$
$y=-2$
so $x=-2$

so $(2,2)$ & $(-2,-2)$ are vertices of (1)

Asymptotes:-

Asymptotes of (2) are

$$X^2-Y^2=0$$

or $X-Y=0, \quad X+Y=0$

i-e., $\frac{x+y}{\sqrt{2}}=0$ & $\frac{-x+y}{\sqrt{2}}=0$

so $x=0$, $y=0$
are eqs. of asymptotes of (1)

Example 3. By a rotation of axes, eliminate the xy-term in the equation
$$9x^2+12xy+4y^2+2x-3y=0$$
Identify the conic and find its elements.

Solution:-

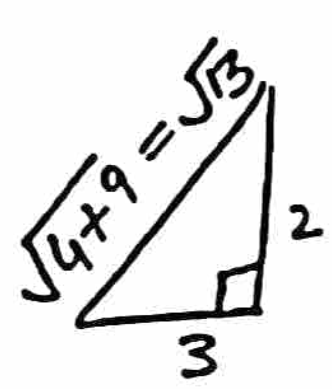
$$9x^2+12xy+4y^2+2x-3y=0 \longrightarrow (1)$$

Here $a=9$, $b=4$,
$2h=12$

The angle through which axes be rotated be given by
$$\tan 2\theta = \frac{12}{9-4} = \frac{12}{5}$$
$$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \frac{12}{5}$$
$$\Rightarrow 5\tan\theta = 6-6\tan^2\theta$$
or $6\tan^2\theta+5\tan\theta-6=0$
$$6\tan^2\theta+9\tan\theta-4\tan\theta-6=0$$
$$3\tan\theta(2\tan\theta+3)-2(2\tan\theta+3)=0$$
$$(2\tan\theta+3)(3\tan\theta-2)=0$$
$2\tan\theta+3=0$, $3\tan\theta-2=0$

$\tan\theta=-\frac{3}{2}$ (Ignoring) $\quad \tan\theta=\frac{2}{3}$

$\tan\theta=\frac{2}{3}$



so $\sin\theta=\frac{2}{\sqrt{13}}$, $\cos\theta=\frac{3}{\sqrt{13}}$

Eqs. of transformation are
$$x = X\cos\theta - Y\sin\theta = \frac{3}{\sqrt{13}}X - \frac{2}{\sqrt{13}}Y \text{ --- (i)}$$
$$y = X\sin\theta + Y\cos\theta = \frac{2}{\sqrt{13}}X + \frac{3}{\sqrt{13}}Y \text{ --- (ii)}$$

using (i) & (ii) in (1)
$$9\left(\frac{3X-2Y}{\sqrt{13}}\right)^2+12\left(\frac{3X-2Y}{\sqrt{13}}\right)\left(\frac{2X+3Y}{\sqrt{13}}\right)$$
$$+4\left(\frac{2X+3Y}{\sqrt{13}}\right)^2+2\left(\frac{3X-2Y}{\sqrt{13}}\right)-3\left(\frac{2X+3Y}{\sqrt{13}}\right)=0$$
$$\frac{9}{13}(9X^2+4Y^2-12XY)+\frac{12}{13}(6X^2-9XY-4XY-6Y^2)$$
$$+\frac{4}{13}(4X^2+9Y^2+12XY)\,.\; -\sqrt{13}Y=0$$
$$X^2\left(\frac{81}{13}+\frac{72}{13}+\frac{16}{13}\right)+\left(-\frac{108}{13}+\frac{60}{13}+\frac{48}{13}\right)XY$$
$$+\left(\frac{36}{13}-\frac{72}{13}+\frac{36}{13}\right)Y^2-\sqrt{13}Y=0$$
$$\Rightarrow 13X^2-\sqrt{13}Y=0$$
$$\Rightarrow X^2=\frac{Y}{\sqrt{13}} \longrightarrow (2) \quad \text{(Parabola)}$$

Solving (i) and (ii) for X, Y
$$x=\frac{3X-2Y}{\sqrt{13}}, \quad y=\frac{2X+3Y}{\sqrt{13}}$$
$$\Rightarrow 3X-2Y=\sqrt{13}x \longrightarrow (a), \quad 2X+3Y=\sqrt{13}y \longrightarrow (b)$$
$3\times(a) \Rightarrow \quad 9X-6Y=3\sqrt{13}x$
$2\times(b) \Rightarrow \quad 4X+6Y=2\sqrt{13}y$

Adding $\quad 13X = \sqrt{13}(3x+2y)$
$$\Rightarrow X=\frac{3x+2y}{\sqrt{13}} \text{ put in (a)}$$
$$3\left[\frac{3x+2y}{\sqrt{13}}\right]-2Y=\sqrt{13}x$$
$$9x+6y-2\sqrt{13}Y=13x$$
$$-2\sqrt{13}Y=13x-9x-6y$$

$$-2\sqrt{13}\,y = 4x - 6y$$

$$y = \frac{2(2x-3y)}{-2\sqrt{13}}$$

$$y = \frac{-2x+3y}{\sqrt{13}}$$

Elements of Parabola

Focus:- $x = 0, \quad y = \frac{1}{4\sqrt{13}}$

$$\Rightarrow \frac{3x+2y}{\sqrt{13}} = 0, \quad \frac{-2x+3y}{\sqrt{13}} = \frac{1}{4\sqrt{13}}$$

$$3x + 2y = 0, \quad -2x+3y = \frac{1}{4}$$

$$3x+2y = 0 \rightarrow (i) \qquad -8x + 12y = 1 \rightarrow (ii)$$

$$8 \times (i) \Rightarrow 24x + 16y = 0$$

$$3 \times (ii) \Rightarrow -24x + 36y = 3$$

Adding $\Rightarrow \quad 52y = 3$

$$\Rightarrow y = \frac{3}{52} \text{ put in } (i)$$

$$(i) \Rightarrow 3x + 2\left(\frac{3}{52}\right) = 0$$

$$3x = -\frac{6}{52} \Rightarrow x = -\frac{2}{52}$$

$$x = -\frac{1}{26}$$

so focus of (1) is $\left(-\frac{1}{26}, \frac{3}{52}\right)$

Vertex:- $x = 0, \quad y = 0$

i-e., $\frac{3x+2y}{\sqrt{13}} = 0, \quad \frac{-2x+3y}{\sqrt{13}} = 0$

$$\Rightarrow 3x+2y = 0, \quad -2x+3y = 0$$

i-e., $x = 0, \quad y = 0$

so vertices of 1 $(0,0)$

Axis:- $x = 0$

$$\Rightarrow 3x + 2y = 0$$

x-intercept $= -\frac{2}{9}$

y-intercept $= \frac{3}{4}$

Example 4. Show that $2x^2 - xy + 5x - 2y + 2 = 0$ represents a pair of lines. Also find an equation of each line.

Solution:-

$$2x^2 - xy + 5x - 2y + 2 = 0.$$

$a = 2, \quad b = 0, \quad 2h = -1 \Rightarrow h = -\frac{1}{2}, \quad c = 2$

$g = \frac{5}{2}, \quad f = 1$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -\frac{1}{2} & \frac{5}{2} \\ -\frac{1}{2} & 0 & 1 \\ \frac{5}{2} & -1 & 2 \end{vmatrix}$$

$$= \frac{1}{2}\left(-1 + \frac{5}{2}\right) + 1\left(-2 + \frac{5}{4}\right)$$

$$= \frac{3}{4} - \frac{3}{4} = 0$$

The given equation represents a degenerate conic which is a pair of lines. So given eq. is

$$2x^2 + (5-y)x + (-2y+2) = 0$$

$$x = \frac{y - 5 \pm \sqrt{[-(y-5)]^2 - 4(2)(-2y+2)}}{4}$$

$$x = \frac{y-5 \pm \sqrt{y^2 + 25 - 10y + 16y - 16}}{4}$$

$$x = \frac{y-5 \pm \sqrt{y^2 - 6y + 9}}{4}$$

$$x = \frac{y-5 \pm y-3}{4}$$

$x = \frac{y-5+y+3}{4}$, $x = \frac{y-5-y-3}{4}$

$x = \frac{2y-2}{4}$ and $x = \frac{-8}{4}$

or $x = \frac{y-1}{2}$, $x = -2$

$2x - y + 1 = 0$, $x + 2 = 0$

Tangent

Find an equation of the tangent to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

at the point (x_1, y_1)

Solution:-

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \longrightarrow (1)$$

Diff (1) w.r.t x, we get

$$2ax + 2hy + 2hx\frac{dy}{dx} + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$\rightarrow (2hx + 2by + 2f)\frac{dy}{dx} = -2ax - 2hy - 2g$$

$$\rightarrow \frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}$$

$$\rightarrow \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}$$

Eq. of tangent at (x_1, y_1)

$$y - y_1 = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}(x - x_1)$$

$$\rightarrow (x - x_1)(ax_1 + hy_1 + g) + (y - y_1)(hx_1 + by_1 + f) = 0$$

or $ax x_1 + hxy_1 + gx + hyx_1 + byy_1 + fy = ax_1^2 + 2hx_1y_1 + gx_1 + by_1^2 + fy_1$

Adding $gx_1 + fy_1 + c$ to both sides and regrouping the terms

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c$$

$$= ax_1^2 + 2hx_1y_1 + 2gx_1 + 2fy_1 + c$$

$$= 0$$

$\because (x_1, y_1)$ lies on (1)

so eq. of tangent to (1) at (x_1, y_1) is

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Note:-

An eq. of the tangent to the general equation of the second degree at the point (x_1, y_1) may be obtained by replacing

x^2 by xx_1

y^2 by yy_1

$2xy$ by $xy_1 + yx_1$

$2x$ by $x + x_1$

$2y$ by $y + y_1$

in the eq. of conic

Example 5. Find the eq. of tangent to the conic $x^2 - xy + y^2 - 2 = 0$ at the point whose ordinate is $\sqrt{2}$

Solution:-

$$x^2 - xy + y^2 - 2 = 0 \longrightarrow (1)$$

Put $y = \sqrt{2}$ in (1)

$$x^2 - \sqrt{2}x + 2 - 2 = 0$$

$$x^2 - \sqrt{2}x = 0$$

$\Rightarrow x(x-\sqrt{2}) = 0$

$\Rightarrow x = 0, \quad x = \sqrt{2}$

The points on the conic are $(0, \sqrt{2})$ & $(\sqrt{2}, \sqrt{2})$

Tangent at $(0, \sqrt{2})$

Now replacing x^2 by xx_1, y^2 by yy_1 and xy by $\frac{1}{2}(xy_1 + yx_1)$ in (1) so

$$xx_1 - (xy_1 + yx_1) + yy_1 - 2 = 0$$

At $(0, \sqrt{2})$ eq. of tangent is

$$x(0) - \frac{1}{2}(x\cdot\sqrt{2} + y(0)) + y\cdot\sqrt{2} - 2 = 0$$

$$\Rightarrow -\frac{x}{\sqrt{2}} + \sqrt{2}\,y - 2 = 0$$

$$\Rightarrow x - 2y + 2\sqrt{2} = 0$$

At $(\sqrt{2}; \sqrt{2})$ Eq. of tangent is

$$\sqrt{2}\,x - \frac{1}{2}(\sqrt{2}\,x + \sqrt{2}\,y) + \sqrt{2}\,y - 2 = 0$$

$$2\sqrt{2}\,x - \sqrt{2}\,x - \sqrt{2}\,y + 2\sqrt{2}\,y - 4 = 0$$

$$\Rightarrow \sqrt{2}\,x + \sqrt{2}\,y - 4 = 0$$

Exercise 6.9

Q1. By a rotation of axes the xy-term in each of the following equations. Identify the conic and find its elements.

(i) $4x^2 - 4xy + y^2 - 6 = 0$

Solution:—

$$4x^2 - 4xy + y^2 - 6 = 0$$

$$\Rightarrow 4x^2 + 2(-2)xy + y^2 - 6 = 0 \longrightarrow (1)$$

Here $a = 4, \quad h = -2, \quad b = 2$

In order to remove the term involving xy, the angle through which axes be rotated is

$$\tan 2\theta = \frac{2h}{a-b} = \frac{2(-2)}{4-1}$$

$$\tan 2\theta = -\frac{4}{3}$$

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{-4}{3}$$

$$\Rightarrow 3\tan\theta = -2 + 2\tan^2\theta$$

$$\Rightarrow 2\tan^2\theta - 3\tan\theta - 2 = 0$$

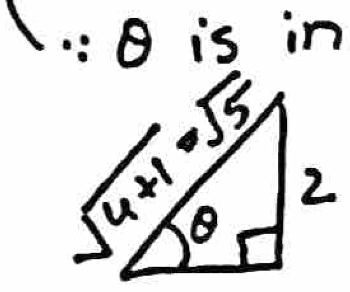
$$2\tan^2\theta - 4\tan\theta + \tan\theta - 2 = 0$$

$$2\tan\theta(\tan\theta - 2) + 1(\tan\theta - 2) = 0$$

$$(\tan\theta - 2)(2\tan\theta + 1) = 0$$

$\tan\theta = 2$, $\quad \tan\theta = -\frac{1}{2}$ (Not admissible $\because \theta$ is in I)

so $\tan\theta = \frac{2}{1}$



$$\Rightarrow \sin\theta = \frac{2}{\sqrt{5}} \quad \& \quad \cos\theta = \frac{1}{\sqrt{5}}$$

Now eqs. of transformation are

$$x = X\cos\theta - Y\sin\theta = \frac{X-2Y}{\sqrt5} \longrightarrow (i)$$

$$y = X\sin\theta + Y\cos\theta = \frac{2X+Y}{\sqrt5} \longrightarrow (ii)$$

using (i) & (ii) in (1)

$$4\left(\frac{X-2Y}{\sqrt5}\right)^2 - 4\left(\frac{X-2Y}{\sqrt5}\right)\left(\frac{2X+Y}{\sqrt5}\right) + \left(\frac{2X+Y}{\sqrt5}\right)^2 - 6 = 0$$

$$\frac{4}{5}(X^2+4Y^2-4XY) - \frac{4}{5}(2X^2+XY-4XY-2Y^2) + \frac{1}{5}(4X^2+Y^2+4XY) - 6 = 0$$

$$4X^2+16Y^2-16XY-8X^2+12XY+8Y^2 + 4X^2+Y^2+4XY-30 = 0$$

$$\Rightarrow 25Y^2 - 30 = 0$$

$$\Rightarrow Y^2 = \frac{30}{25} \quad \text{or} \quad Y^2 = \frac{6}{5}$$

or $Y = \pm\sqrt{\frac{6}{5}}$ (Pair of lines)

From (i) & (ii)

$$X - 2Y = \sqrt5\, x \longrightarrow (iii)$$

$$2X + Y = \sqrt5\, y \longrightarrow (iv)$$

multipling ( ) by 2 and subtract (iv) from it, we get

$$2X - 4Y = 2\sqrt5\, x$$
$$\underline{\underset{-}{}2X \underset{-}{+} Y = \underset{-}{}\sqrt5\, y}$$
$$-5Y = \sqrt5(2x - y)$$

or $Y = \dfrac{\sqrt5(y-2x)}{5}$

$$Y = \frac{y-2x}{\sqrt5}$$

$$\pm\sqrt{\frac{6}{5}} = \frac{y-2x}{\sqrt5} \qquad \because Y = \pm\sqrt{\frac{6}{5}}$$

or $y - 2x = \pm\sqrt6$

Pair of lines are

$$2x - y + \sqrt6 = 0 \quad \&$$
$$2x - y - \sqrt6 = 0$$

(ii) $x^2 - 2xy + y^2 - 8x - 8y = 0$

Solution:-

$$x^2 + 2(-1)xy + y^2 - 8x - 8y = 0 \longrightarrow (1)$$

$a = 1,\quad b = 1,\quad h = -1$

If θ is measure of rotation to remove xy term then

$$\tan 2\theta = \frac{2h}{a-b} = \frac{2(-1)}{1-1}$$

$$2\theta = \tan^{-1}\left(-\frac{2}{0}\right)$$

$$2\theta = 90^\circ$$

$$\Rightarrow \theta = 45^\circ$$

Eqs. of transformation are

$$x = X\cos45^\circ - Y\sin45^\circ = \frac{X-Y}{\sqrt2} \longrightarrow (i)$$

$$y = X\sin45^\circ + Y\cos45^\circ = \frac{X+Y}{\sqrt2} \longrightarrow (ii)$$

using (i) & (ii) in (1)

$$\left(\frac{X-Y}{\sqrt2}\right)^2 - 2\left(\frac{X-Y}{\sqrt2}\right)\left(\frac{X+Y}{\sqrt2}\right) + \left(\frac{X+Y}{\sqrt2}\right)^2 - 8\left(\frac{X-Y}{\sqrt2}\right) - 8\left(\frac{X+Y}{\sqrt2}\right) = 0$$

$$\left[\frac{X-Y}{\sqrt2} - \frac{X+Y}{\sqrt2}\right]^2 - 8\left[\frac{X-Y}{\sqrt2} + \frac{X+Y}{\sqrt2}\right] = 0$$

$$\left(\frac{X-Y-X-Y}{\sqrt2}\right)^2 - 8\left(\frac{X-Y+X+Y}{\sqrt2}\right) = 0$$

$$\frac{4Y^2}{2} - 8\left(\frac{-2X}{\sqrt2}\right) = 0$$

$$\Rightarrow 2Y^2 - \frac{8}{\sqrt2}(2X) = 0$$

$$\text{or} \quad Y^2 - \frac{8}{\sqrt2}X = 0$$

$$\Rightarrow Y^2 = 4\sqrt2\, X \longrightarrow (2) \quad \text{(Parabola)}$$

from (i) & (ii), we have

$X - Y = \sqrt{2}\, x$

$X + Y = \sqrt{2}\, y$

Adding $\Rightarrow 2X = \sqrt{2}(x+y)$

or $X = \dfrac{x+y}{\sqrt{2}}$

Now $\dfrac{x+y}{\sqrt{2}} + Y = \sqrt{2}\, y$

$\Rightarrow Y = \sqrt{2}\, y - \dfrac{x+y}{\sqrt{2}}$

$Y = \dfrac{2y - x - y}{\sqrt{2}}$

$Y = \dfrac{-x+y}{\sqrt{2}}$

Elements of Parabola

Foci:- $X = \sqrt{2}$, $Y = 0$

$\Rightarrow \dfrac{x+y}{\sqrt{2}} = \sqrt{2}$, $\dfrac{-x+y}{\sqrt{2}} = 0$

$\Rightarrow x + y = 2 \rightarrow (a)$, $-x+y=0 \rightarrow (b)$

By (a) + (b) $\Rightarrow 2y = 2 \Rightarrow y = 1$

and $x + 1 = 2 \Rightarrow x = 1$

so foci of (1) is $(1, 1)$

Vertex:- $X = 0$, $Y = 0$

$\Rightarrow \dfrac{x+y}{\sqrt{2}} = 0$, $\dfrac{-x+y}{\sqrt{2}} = 0$

$\Rightarrow x + y = 0$, $-x + y = 0$

Here $x = 0$, $y = 0$

Thus vertex of 1 $(0,0)$

Axis of parabola:- $Y = 0$

Directrix:-

$X = -\sqrt{2}$

$\Rightarrow \dfrac{1}{\sqrt{2}}(x+y) = -\sqrt{2}$

or $x + y = -2$

or $x + y + 2 = 0$

(iii) $x^2 + 2xy + y^2 + 2\sqrt{2}x - 2\sqrt{2}y + 2 = 0$

Solution:-

$x^2 + 2xy + y^2 + 2\sqrt{2}x - 2\sqrt{2}y + 2 = 0 \rightarrow (1)$

$a = 1$, $b = 1$, $h = 1$

If θ is angle of rotation to remove xy-term

$\tan 2\theta = \dfrac{h}{a-b} = \dfrac{1}{1-1} = \dfrac{1}{0}$

$\Rightarrow 2\theta = \tan^{-1}\left(\dfrac{1}{0}\right) = 90^\circ$

or $\theta = 45^\circ$

Eqs. of transformation are

$x = X\cos 45^\circ - Y\sin 45^\circ = \dfrac{X-Y}{\sqrt{2}} \rightarrow (i)$

$y = X\sin 45^\circ + Y\cos 45^\circ = \dfrac{X+Y}{\sqrt{2}} \rightarrow (ii)$

using (i) & (ii) in (1)

$\left(\dfrac{X-Y}{\sqrt{2}}\right)^2 + 2\left(\dfrac{X-Y}{\sqrt{2}}\right)\left(\dfrac{X+Y}{\sqrt{2}}\right) + \left(\dfrac{X+Y}{\sqrt{2}}\right)^2$

$+ 2\sqrt{2}\left(\dfrac{X-Y}{\sqrt{2}}\right) - 2\sqrt{2}\left(\dfrac{X+Y}{\sqrt{2}}\right) + 2 = 0$

$\left(\dfrac{X-Y}{\sqrt{2}} + \dfrac{X+Y}{\sqrt{2}}\right)^2 + 2X - 2Y - 2X - 2Y + 2 = 0$

$\dfrac{4X^2}{2} - 4Y + 2 = 0$

or $2X^2 - 4Y + 2 = 0$

$\Rightarrow X^2 - 2Y + 1 = 0$

$X^2 = 2Y - 1$ (Parabola)

$X^2 = 2\left(Y - \dfrac{1}{2}\right) \rightarrow (2)$

Now from (i) & (ii)

$$x-y=\sqrt{2}x$$
$$x+y=\sqrt{2}y$$
$$2x=\sqrt{2}(x+y) \Rightarrow x=\frac{x+y}{\sqrt{2}}$$

Also $\frac{x+y}{\sqrt{2}}-y=\sqrt{2}x$

or $\frac{x+y}{\sqrt{2}}-\sqrt{2}x=y$

$$\frac{x+y-2x}{\sqrt{2}}=y$$

$$\Rightarrow y=\frac{-x+y}{\sqrt{2}}$$

Elements of parabola

From (2) $4a=2 \Rightarrow a=\frac{1}{2}$

Vertex:- $X=0, \quad Y-\frac{1}{2}=0$

$$\Rightarrow \frac{x+y}{\sqrt{2}}=0, \quad \frac{-x+y}{\sqrt{2}}=\frac{1}{2}$$

$$x+y=0, \quad -x+y=\frac{1}{\sqrt{2}}$$

Adding $\Rightarrow 2y=\frac{1}{\sqrt{2}} \Rightarrow y=\frac{1}{2\sqrt{2}}$

so $x+\frac{1}{2\sqrt{2}}=0 \Rightarrow x=-\frac{1}{2\sqrt{2}}$

so vertex of (1) $\left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

Foci:- $X=0, \quad Y-\frac{1}{2}=\frac{1}{2}$

$$\Rightarrow \frac{x+y}{\sqrt{2}}=0, \quad \frac{-x+y}{\sqrt{2}}=1$$

$$x+y=0, \quad -x+y=\sqrt{2}$$

Now adding it

$$2y=\sqrt{2} \Rightarrow y=\frac{1}{\sqrt{2}}$$

and $x+\frac{1}{\sqrt{2}}=0 \Rightarrow x=-\frac{1}{\sqrt{2}}$

so foci of (1) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Axis:- $X=0$

$$\Rightarrow \frac{x+y}{\sqrt{2}}=0 \Rightarrow x+y=0$$

Directrix:- $Y-\frac{1}{2}=\frac{-1}{2}$

$$\Rightarrow Y=0$$
$$\frac{-x+y}{\sqrt{2}}=0$$
or $-x+y=0$
$$\Rightarrow x-y=0$$

(iv) $x^2+xy+y^2-4=0$

Solution:-

$$x^2+xy+y^2-4=0 \longrightarrow (1)$$

$a=1, \; 2h=1, \; b=1$

If θ is angle of rotation; to remove the term xy then

$$\tan 2\theta=\frac{2h}{a-b}=\frac{1}{1-1}=\frac{1}{0}$$

$$2\theta=\tan^{-1}\left(\frac{1}{0}\right)=90^\circ \Rightarrow \theta=45^\circ$$

Eqs. of transformation are

$$x=X\cos 45^\circ-Y\sin 45^\circ=\frac{X-Y}{\sqrt{2}} \longrightarrow (i)$$
$$y=X\sin 45^\circ+Y\cos 45^\circ=\frac{X+Y}{\sqrt{2}} \longrightarrow (ii)$$

using (i) & (ii) in (1)

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2+\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right)+\left(\frac{X+Y}{\sqrt{2}}\right)^2-4=0$$

$$\frac{X^2+Y^2-2XY}{2}+\frac{X^2-Y^2}{2}+\frac{X^2+Y^2+2XY}{2}-4=0$$

$$X^2+Y^2-2XY+X^2-Y^2+X^2+Y^2+2XY-8=0$$

$$3X^2+Y^2-8=0 \Rightarrow 3X^2+Y^2=8$$

or $\frac{3X^2}{8}+\frac{Y^2}{8}=1$

$$\Rightarrow \frac{X^2}{8/3}+\frac{Y^2}{8}=1 \quad (\text{Ellipse})$$

Here $a^2=8, \quad b^2=8/3$

$$\Rightarrow a=2\sqrt{2}, \quad b=\frac{2\sqrt{2}}{3}$$

from (i) & (ii), we have

$$x - y = \sqrt{2}\,x$$
$$x + y = \sqrt{2}\,y$$
$$2x = \sqrt{2}(x+y)$$
$$\text{or } x = \frac{x+y}{\sqrt{2}}$$

Also $\frac{x+y}{\sqrt{2}} + y = \sqrt{2}\,y$

$$\text{or } y = \sqrt{2}\,y - \frac{x+y}{\sqrt{2}}$$
$$y = \frac{2y - x - y}{\sqrt{2}}$$
$$\Rightarrow y = \frac{-x+y}{\sqrt{2}}$$

Elements of Ellipse

Centre:– $x = 0, \quad y = 0$

$$\Rightarrow \frac{x+y}{\sqrt{2}} = 0, \quad \frac{-x+y}{\sqrt{2}} = 0$$

$$\Rightarrow x + y = 0, \quad -x + y = 0$$

For this, $x = 0, y = 0$

so centre of (1) $(0, 0)$

Vertices:– $x = 0, \quad y = \pm 2\sqrt{2}$

$$\Rightarrow \frac{x+y}{\sqrt{2}} = 0, \quad \frac{-x+y}{\sqrt{2}} = \pm 2\sqrt{2}$$

$$x + y = 0, \quad -x + y = \pm 4$$

$x + y = 0$	$x + y = 0$
$-x + y = 4$	$-x + y = -4$
$2y = 4$	$2y = -4$
$y = 2$	$y = -2$
and $x = -2$	so $x = 2$

so vertices of 1 are

$$(-2, 2) \;\&\; (2, -2)$$

Equations:–

Eq. of major axis;

$$x = 0 \Rightarrow \frac{x+y}{\sqrt{2}} = 0 \Rightarrow x + y = 0$$

Eq. of minor axis;

$$y = 0 \Rightarrow \frac{-x+y}{\sqrt{2}} = 0 \Rightarrow -x + y = 0$$

Now $c^2 = a^2 - b^2 = 8 - \frac{8}{3} = \frac{16}{3}$

$$\Rightarrow c = \frac{4}{\sqrt{3}}$$

Eccentricity:– $e = c/a$

$$\Rightarrow e = \frac{4/\sqrt{3}}{2\sqrt{2}} = \frac{4}{\sqrt{3}} \times \frac{1}{2\sqrt{2}} = \frac{2}{\sqrt{6}}$$

Foci:– $x = 0, \quad y = \pm \frac{4}{\sqrt{3}}$

$$\Rightarrow \frac{x+y}{\sqrt{2}} = 0, \quad \frac{-x+y}{\sqrt{2}} = \pm \frac{4}{\sqrt{3}}$$

$$\Rightarrow x + y = 0 \qquad -x + y = \pm \frac{4\sqrt{2}}{3}$$

$x + y = 0$	$x + y = 0$
$-x + y = \frac{4\sqrt{2}}{3}$	$-x + y = -\frac{4\sqrt{2}}{3}$
$2y = \frac{4\sqrt{2}}{3}$	$2y = -\frac{4\sqrt{2}}{3}$
$y = \frac{2\sqrt{2}}{3}$	$y = -\frac{2\sqrt{2}}{3}$
so $x = -\frac{2\sqrt{2}}{3}$	so $x = \frac{2\sqrt{2}}{3}$

Thus foci of (1) are

$$\left(-\frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}\right) \;\&\; \left(\frac{2\sqrt{2}}{3}, -\frac{2\sqrt{2}}{3}\right)$$

(v) $7x^2 - 6\sqrt{3}\,xy + 13y^2 - 16 = 0$

Solution:–

$$7x^2 - 6\sqrt{3}\,xy + 13y^2 - 16 = 0 \longrightarrow (1)$$

Let θ be the angle of rotation to remove the product term xy then

$$\tan 2\theta = \frac{-6\sqrt{3}}{7 - 13} = \sqrt{3}$$

$$\Rightarrow 2\theta = 60^\circ \Rightarrow \theta = 30^\circ$$

Now eqs. of transformation are

$$x = X\cos 30^\circ - Y\sin 30^\circ = \frac{\sqrt{3}X - Y}{2} \longrightarrow (i)$$

$$y = X\sin 30^\circ + Y\cos 30^\circ = \frac{X + \sqrt{3}Y}{2} \longrightarrow (ii)$$

using (i) & (ii) in (1)

$$7\left(\frac{\sqrt{3}X - Y}{2}\right)^2 - 6\sqrt{3}\left(\frac{\sqrt{3}X - Y}{2}\right)\left(\frac{X + \sqrt{3}Y}{2}\right) + 13\left(\frac{X + \sqrt{3}Y}{2}\right)^2 - 16 = 0$$

$$7\left(\frac{3X^2 + Y^2 - 2\sqrt{3}XY}{4}\right) - \frac{6\sqrt{3}}{4}\left(\sqrt{3}X^2 + 3XY - XY - \sqrt{3}Y^2\right) + \frac{13}{4}\left(X^2 + 3Y^2 + 2\sqrt{3}XY\right) - 16 = 0$$

$$21X^2 + 7Y^2 - 14\sqrt{3}XY - 18X^2 - 12\sqrt{3}XY + 8Y^2 + 13X^2 + 39Y^2 + 26\sqrt{3}XY - 64 = 0$$

$$16X^2 + 64Y^2 - 64 = 0$$

$$\Rightarrow X^2 + 4Y^2 = 4$$

$$\text{or } \frac{X^2}{4} + \frac{Y^2}{1} = 1 \qquad \text{(Ellipse)}$$

Here $a^2 = 4 \Rightarrow a = 2$, $b^2 = 1 \Rightarrow b = 1$

$c^2 = a^2 - b^2 = 4 - 1 = 3$

$\Rightarrow c = \sqrt{3}$

Now from (i) & (ii)

$\sqrt{3}X - Y = 2x$ —— (a)

$X + \sqrt{3}Y = 2y$ —— (b)

By $\sqrt{3}\times(a) \Rightarrow$ $3X - \sqrt{3}Y = 2\sqrt{3}x$

and add $X + \sqrt{3}Y = 2y$

$$4X = 2(\sqrt{3}x + y)$$

or $X = \dfrac{\sqrt{3}x + y}{2}$ put in (b)

$$\Rightarrow \sqrt{3}Y = 2y - \frac{\sqrt{3}x + y}{2}$$

$$= \frac{4y - \sqrt{3}x - y}{2}$$

$$\sqrt{3}Y = \frac{3y - \sqrt{3}x}{2}$$

$$\sqrt{3}Y = \frac{\sqrt{3}\sqrt{3}y - \sqrt{3}x}{2}$$

$$\text{or } Y = \frac{\sqrt{3}y - x}{2}$$

Elements of Ellipse

Centre:– $X = 0,\ Y = 0$

$$\frac{\sqrt{3}x + y}{2} = 0, \quad \frac{\sqrt{3}y - x}{2} = 0$$

$$\Rightarrow \sqrt{3}x + y = 0, \quad \sqrt{3}y - x = 0$$

For this $x = 0,\ y = 0$

so centre of 1 (0, 0)

Foci:– $X = \pm\sqrt{3},\ Y = 0$

$$\Rightarrow \frac{\sqrt{3}x + y}{2} = \pm\sqrt{3}, \quad \frac{\sqrt{3}y - x}{2} = 0$$

$$\sqrt{3}x + y = \pm 2\sqrt{3}, \quad \sqrt{3}y - x = 0$$

$\sqrt{3}x + y = 2\sqrt{3}$

$-x + \sqrt{3}y = 0$

Now we solve

$\sqrt{3}x + y = 2\sqrt{3}$

$-\sqrt{3}x + 3y = 0$

$4y = 2\sqrt{3}$

$y = \dfrac{\sqrt{3}}{2}$

so $x = \dfrac{3}{2}$

$\sqrt{3}x + y = -2\sqrt{3}$

$-x + \sqrt{3}y = 0$

Now we solve

$\sqrt{3}x + y = -2\sqrt{3}$

$-\sqrt{3}x + 3y = 0$

$4y = -2\sqrt{3}$

$y = -\dfrac{\sqrt{3}}{2}$

$x = -\dfrac{3}{2}$

so foci of (1) are

$$\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right) \ \& \ \left(-\frac{3}{2}, -\frac{\sqrt{3}}{2}\right)$$

Vertices:– $X = \pm 2,\ Y = 0$

$$\frac{\sqrt{3}x + y}{2} = \pm 2, \quad \frac{-x + \sqrt{3}y}{2} = 0$$

$\sqrt{3}x + y = 4$

$-x + \sqrt{3}y = 0$

Now we solve

$\sqrt{3}x + y = 4$

$-\sqrt{3}x + 3y = 0$

$4y = 4$

$y = 1$

so $x = \sqrt{3}$

$\sqrt{3}x + y = -4$

$-x + \sqrt{3}y = 0$

Now we solve

$\sqrt{3}x + y = -4$

$-\sqrt{3}x + 3y = 0$

$4y = -4$

$y = -1$

so $x = -\sqrt{3}$

so vertices of (1) are

$(\sqrt{3}, 1)$ & $(-\sqrt{3}, -1)$

Major axis:- $X = 0$

$$\rightarrow \frac{\sqrt{3}x + y}{2} = 0 \Rightarrow \sqrt{3}x + y = 0$$

Eccentricity:- $e = c/a$

$$\rightarrow e = \frac{\sqrt{3}}{2}$$

Directrices:- $X = \pm c/e^2$

$$\rightarrow \frac{\sqrt{3}x + y}{2} = \pm \frac{\sqrt{3}}{3/4}$$

$$\rightarrow \frac{\sqrt{3}x + y}{2} = \pm \frac{4}{\sqrt{3}}$$

or $\sqrt{3}x + y = \pm \frac{8}{\sqrt{3}}$

or $3x + \sqrt{3}y = \pm 8$ (Directrices of (1))

(vi) $4x^2 - 4xy + 7y^2 + 12x + 6y - 9 = 0$

Solution:—

$$4x^2 - 4xy + 7y^2 + 12x + 6y - 9 = 0 \longrightarrow (1)$$

If θ is the angle of rotation to remove xy term then

$$\tan 2\theta = \frac{2h}{a-b} = \frac{-4}{4-7} = \frac{4}{3}$$

$$\rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \frac{4}{3}$$

$$\rightarrow 3\tan\theta = 2 - 2\tan^2\theta$$

or $2\tan^2\theta + 3\tan\theta - 2 = 0$

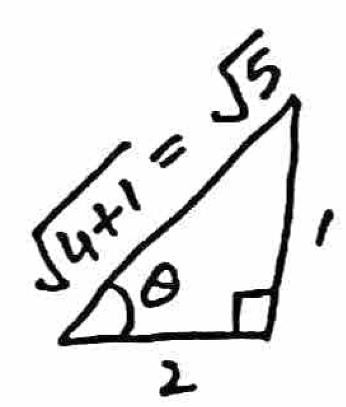
$$2\tan^2\theta + 4\tan\theta - \tan\theta - 2 = 0$$

$$2\tan\theta(\tan\theta + 2) - 1(\tan\theta + 2) = 0$$

$$(\tan\theta + 2)(2\tan\theta - 1) = 0$$

$\rightarrow \tan\theta = -2$, (Not admissible $\because 0 < \theta < 90°$) $\quad \tan\theta = \frac{1}{2}$

so $\tan\theta = \frac{1}{2}$



$$\rightarrow \sin\theta = \frac{1}{\sqrt{5}}, \quad \cos\theta = \frac{2}{\sqrt{5}}$$

Eqs. of transformation are

$$x = X\cos\theta - Y\sin\theta = \frac{2X - Y}{\sqrt{5}} \longrightarrow (i)$$

$$y = X\sin\theta + Y\cos\theta = \frac{X + 2Y}{\sqrt{5}} \longrightarrow (ii)$$

using (i) & (ii) in (1) we have

$$4\left(\frac{2X-Y}{\sqrt{5}}\right)^2 - 4\left(\frac{2X-Y}{\sqrt{5}}\right)\left(\frac{X+2Y}{\sqrt{5}}\right) + 7\left(\frac{X+2Y}{\sqrt{5}}\right)^2$$
$$+12\left(\frac{2X-Y}{\sqrt{5}}\right) + 6\left(\frac{X+2Y}{\sqrt{5}}\right) - 9 = 0$$

$$\frac{4}{5}(4X^2 + Y^2 - 4XY) - \frac{4}{5}(2X^2 + 4XY - XY - 2Y^2)$$
$$+\frac{7}{5}(X^2 + 4Y^2 + 4XY) + \frac{30X}{\sqrt{5}} - 9 = 0$$

$$16X^2 + 4Y^2 - \cancel{16XY} - 8X^2 - \cancel{12XY} + 8Y^2$$
$$+7X^2 + 28Y^2 + \cancel{28XY} + 30\sqrt{5}X - 45 = 0$$

$$15X^2 + 40Y^2 + 30\sqrt{5}X - 45 = 0$$

$$\rightarrow X^2 + 2\sqrt{5}X + \frac{8}{3}Y^2 - 3 = 0$$

$$\rightarrow X^2 + 2\sqrt{5}X + 5 + \frac{8}{3}Y^2 - 3 - 5 = 0$$

$$(X + \sqrt{5})^2 + \frac{8}{3}Y^2 = 8$$

$$\rightarrow \frac{(X+\sqrt{5})^2}{8} + \frac{Y^2}{3} = 1 \text{ (Ellipse)} \longrightarrow (2)$$

Here $a^2 = 8$, $\quad b^2 = 3$

$a = 2\sqrt{2}$, $\quad b = \sqrt{3}$

$c^2 = a^2 - b^2 = 8 - 3 = 5$

$c = \sqrt{5}$

From (i) & (ii)

$$2X - Y = \sqrt{5}x \longrightarrow (a)$$

$$X + 2Y = \sqrt{5}y \longrightarrow (b)$$

By $2\times(a) \rightarrow$ $\quad 4X - 2Y = 2\sqrt{5}x$

and adding $\quad X + 2Y = \sqrt{5}y$

$$5X = \sqrt{5}(2x + y)$$

$X = \frac{2x+y}{\sqrt{5}}$ put in (b)

$2Y = \sqrt{5}y - \frac{2x+y}{\sqrt{5}}$

$2Y = \frac{4y-2x}{\sqrt{5}}$

$Y = \frac{2y-x}{\sqrt{5}}$

Centre:- $X+\sqrt{5} = 0, \quad Y = 0$

$\frac{2x+y}{\sqrt{5}} + \sqrt{5} = 0, \quad \frac{2y-x}{\sqrt{5}} = 0$

$\frac{2x+y}{\sqrt{5}} = -\sqrt{5}, \quad 2y-x=0$

$2x+y = -5, \quad x = 2y$

$\rightarrow 2(2y)+y = -5$

$\rightarrow 5y = -5$

$y = -1$ and $x = -2$

so centre of (1) $(-2,-1)$

Foci:- $X+\sqrt{5} = \pm\sqrt{5}, \quad Y=0$

$\rightarrow \frac{2x+y}{\sqrt{5}} + \sqrt{5} = \pm\sqrt{5}, \quad \frac{-x+2y}{\sqrt{5}} = 0$

$\rightarrow 2x+y+5 = \pm 5, \quad -x+2y=0$

$2x+y+5=5$	$2x+y+5=-5$
$2x+y=0$	$2x+y=-10$
$-x+2y=0$	$-x+2y=0$
Now solving	Now solving
$2x+y=0$	$2x+y=-10$
$-2x+4y=0$	$-2x+4y=0$
$5y=0$	$5y=-10$
$y=0$	$y=-2$
so $x=0$	so $x=-4$

Thus foci of (1) $(0,0)$ & $(-4,-2)$

Major axis:- $Y=0$

$\frac{-x+2y}{\sqrt{5}} = 0$

$\rightarrow -x+2y=0$

Minor axis:- $X=0$

$\frac{2x+y}{\sqrt{5}} = 0 \rightarrow 2x+y=0$

Vertices:- $X+\sqrt{5} = \pm 2\sqrt{2}, \quad Y=0$

$\rightarrow \frac{2x+y}{\sqrt{5}} - \sqrt{5} = \pm 2\sqrt{2}, \quad \frac{-x+2y}{\sqrt{5}} = 0$

$\rightarrow -x+2y=0$

$\frac{2x+y}{\sqrt{5}} + \sqrt{5} = 2\sqrt{2}$	$\frac{2x+y}{\sqrt{5}} + \sqrt{5} = -2\sqrt{2}$
$2x+y+5 = 2\sqrt{10}$	$\rightarrow 2x+y = -5-2\sqrt{10}$
$2x+y = -5+2\sqrt{10}$	Now
Now	$2x+y = -5-2\sqrt{10}$
$2x+y = -5+2\sqrt{10}$	$-2x+4y=0$
$-2x+4y=0$	$5y = -5-2\sqrt{10}$
$5y = -5+2\sqrt{10}$	$y = -1-\frac{2\sqrt{10}}{5}$
$y = -1+\frac{2\sqrt{10}}{5}$	$= -1-\sqrt{\frac{40}{25}}$
$y = -1+\sqrt{\frac{4\times 10}{25}}$	$y = -1-\sqrt{\frac{8}{5}}$
$y = -1+\sqrt{\frac{8}{5}}$	Now
Now	$-x-2-2\sqrt{\frac{8}{5}}=0$
$-x+2\left(-1+\sqrt{\frac{8}{5}}\right)=0$	$-x = 2+\sqrt{\frac{32}{5}}$
$-x-2+2\sqrt{\frac{8}{5}}=0$	$x = -2-\sqrt{\frac{32}{5}}$
$-x = 2-\sqrt{\frac{32}{5}} = 0$	
or $x = -2+\sqrt{\frac{32}{5}}$	

So vertices of (1) are $\left(-2+\sqrt{\frac{32}{5}}, -1+\sqrt{\frac{8}{5}}\right)$ and $\left(-2-\sqrt{\frac{32}{5}}, -1-\sqrt{\frac{8}{5}}\right)$

Directrices:-

$X+\sqrt{5} = \pm a/e^2$

$\rightarrow X + \sqrt{5} = \pm \frac{\sqrt{5}}{5/8}$

$\frac{2x+y}{\sqrt{5}} + \sqrt{5} = \pm\frac{8}{\sqrt{5}}$

$\rightarrow 2x+y+5 = \pm 8$

(vii) $xy-4x-2y=0$

Solution:—

$$xy-4x-2y=0 \longrightarrow (1)$$

Here $a=0=b$, $h=\frac{1}{2}$ the angle through which axes be rotated is 45°

i.e., $\tan 2\theta = \frac{2(\frac{1}{2})}{0-0} = \frac{1}{0}$

$$2\theta = 90^\circ \rightarrow \theta = 45^\circ$$

Eqs. of transformation are

$$x = X\cos 45^\circ - Y\sin 45^\circ = \frac{X-Y}{\sqrt{2}} \rightarrow (i)$$

$$y = X\sin 45^\circ + Y\cos 45^\circ = \frac{X+Y}{\sqrt{2}} \rightarrow (ii)$$

using (i) and (ii) in (1)

$$\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right)-4\left(\frac{X-Y}{\sqrt{2}}\right)-2\left(\frac{X+Y}{\sqrt{2}}\right)=0$$

$$\frac{X^2-Y^2}{2}-4\left(\frac{X-Y}{\sqrt{2}}\right)-2\left(\frac{X+Y}{\sqrt{2}}\right)=0$$

$$X^2-Y^2-4\sqrt{2}X+4\sqrt{2}Y-2\sqrt{2}X-2\sqrt{2}Y=0$$

$$\rightarrow X^2-Y^2-6\sqrt{2}X+2\sqrt{2}Y=0$$

$$X^2-6\sqrt{2}X+18-(Y^2-2\sqrt{2}+2)=18-2$$

$$(X-3\sqrt{2})^2-(Y-\sqrt{2})^2=16$$

$$\frac{(X-3\sqrt{2})^2}{16}-\frac{(Y-\sqrt{2})^2}{16}=1 \text{ (Hyperbola)}$$

From (i) & (ii)

$$X-Y=\sqrt{2}x$$
$$X+Y=\sqrt{2}y$$

$$2X=\sqrt{2}(x+y) \rightarrow X=\frac{x+y}{\sqrt{2}}$$

and $Y=\sqrt{2}y-\frac{x+y}{\sqrt{2}}$

$$Y=\frac{-x+y}{\sqrt{2}}$$

Elements of Hyperbola

Centre:— $X-3\sqrt{2}=0$, $Y-\sqrt{2}=0$

$$\rightarrow \frac{x+y}{\sqrt{2}}=3\sqrt{2}, \quad \frac{-x+y}{\sqrt{2}}=\sqrt{2}$$

$$x+y=6, \quad -x+y=2$$

On solving, we get

$$2y=8 \rightarrow y=4$$

and $x=2$

so centre of (1) $(2,4)$

Eq. of focal axis:—

$$Y-\sqrt{2}=0$$

$$\rightarrow \frac{-x+y}{\sqrt{2}}=\sqrt{2} \rightarrow -x+y=2$$

Eq. of conjugate axis:—

$$X-3\sqrt{2}=0$$

$$\rightarrow \frac{1}{\sqrt{2}}(x+y)=3\sqrt{2}$$

$$\rightarrow x+y=6$$

Eccentricity:— $e=\frac{c}{a}$

$$e=\sqrt{\frac{a^2+b^2}{a^2}}=\sqrt{\frac{16+16}{16}}$$

$$e=\sqrt{2}$$

Foci:— $X-3\sqrt{2}=\pm 4\sqrt{2}$, $Y-\sqrt{2}=0$

$$\rightarrow X=3\sqrt{2}\pm 4\sqrt{2}, \quad Y=\sqrt{2}$$

$$\frac{x+y}{\sqrt{2}}=3\sqrt{2}\pm 4\sqrt{2}, \quad \frac{-x+y}{\sqrt{2}}=\sqrt{2}$$

$$-x+y=2$$

$$x+y=6\pm 8$$

$\rightarrow x+y=14$
$-x+y=2$
$2y=16$
$y=8$
so $x=6$

$x+y=-2$
$-x+y=2$
$2y=0 \rightarrow y=0$
so $x=-2$

so foci of (1) $(6, 8)$ & $(-2, 0)$

Vertices:— $X - 3\sqrt{2} = \pm 4\ ;\ Y - \sqrt{2} = 0$

$$\frac{x+y}{\sqrt{2}} = 3\sqrt{2} \pm 4\ , \qquad Y = \sqrt{2}$$

$$x+y = 6 \pm 4\sqrt{2}\ , \qquad \frac{-x+y}{\sqrt{2}} = \sqrt{2}$$

$$-x+y = 2$$

$$\begin{aligned} x+y &= 6+4\sqrt{2} \\ -x+y &= 2 \\ \hline 2y &= 8+4\sqrt{2} \\ y &= 4+2\sqrt{2} \\ \text{so } x &= 2+2\sqrt{2} \end{aligned}
\qquad
\begin{aligned} x+y &= 6-4\sqrt{2} \\ -x+y &= 2 \\ \hline 2y &= 8-4\sqrt{2} \\ y &= 4-2\sqrt{2} \\ \text{so } x &= 2-2\sqrt{2} \end{aligned}$$

vertices of (1)

$$(2+2\sqrt{2}, 4+2\sqrt{2}),\ (2-2\sqrt{2}, 4-2\sqrt{2})$$

(viii) $x^2 + 4xy - 2y^2 - 6 = 0$

Solution:—

$$x^2 + 4xy - 2y^2 - 6 = 0 \longrightarrow (1)$$

If θ be the angle of rotation to remove xy-term

then $\tan 2\theta = \dfrac{2h}{a-b} = \dfrac{2(2)}{1+2} = \dfrac{4}{3}$

$$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \frac{4}{3}$$

$$\Rightarrow 3\tan\theta = 2 - 2\tan^2\theta$$

$$2\tan^2\theta + 3\tan\theta - 2 = 0$$

$$2\tan^2\theta + 4\tan\theta - \tan\theta - 2 = 0$$

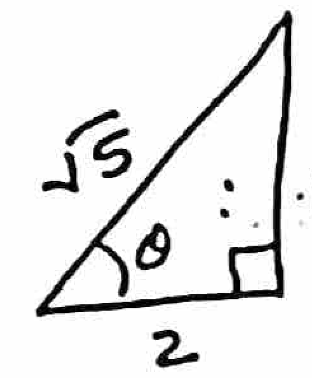
$$2\tan\theta(\tan\theta + 2) - 1(\tan\theta + 2) = 0$$

$$(\tan\theta + 2)(2\tan\theta - 1) = 0$$

$$\tan\theta = -2\ , \qquad \tan\theta = \frac{1}{2}$$

(Not admissible $\because 0 < \theta < 90^\circ$)

so $\tan\theta = \dfrac{1}{2}$



$$\Rightarrow \sin\theta = \frac{1}{\sqrt{5}}$$

$$\cos\theta = \frac{2}{\sqrt{5}}$$

Eqs. of transformation

$$x = X\cos\theta - Y\sin\theta = \frac{2X - Y}{\sqrt{5}} \longrightarrow (i)$$

$$y = X\sin\theta + Y\cos\theta = \frac{X + 2Y}{\sqrt{5}} \longrightarrow (ii)$$

using (i) & (ii) in (1)

$$\left(\frac{2X-Y}{\sqrt{5}}\right)^2 + 4\left(\frac{2X-Y}{\sqrt{5}}\right)\left(\frac{X+2Y}{\sqrt{5}}\right) - 2\left(\frac{X+2Y}{\sqrt{5}}\right)^2 - 6 = 0$$

$$\frac{1}{5}(4X^2 + Y^2 - 4XY) + \frac{4}{5}(2X^2 + 4XY - XY - 2Y^2) - \frac{2}{5}(X^2 + 4Y^2 + 4XY) - 6 = 0$$

$$4X^2 + Y^2 - 4XY + 8X^2 + 12XY - 8Y^2 - 2X^2 - 8Y^2 - 8XY - 30 = 0$$

$$10X^2 - 15Y^2 = 30$$

$$\frac{X^2}{3} - \frac{Y^2}{2} = 1$$

(Hyperbola)

From (i) & (ii),

$$2X - Y = \sqrt{5}\,x$$

$$X + 2Y = \sqrt{5}\,y$$

solving these eqs.

$$\begin{aligned} 4X - 2Y &= 2\sqrt{5}\,x \\ X + 2Y &= \sqrt{5}\,y \\ \hline 5X &= \sqrt{5}(2x+y) \end{aligned}$$

$$X = \frac{2x+y}{\sqrt{5}}$$

and $2Y = \sqrt{5}\,y - \dfrac{2x+y}{\sqrt{5}}$

$$2Y = \frac{-2x+4y}{\sqrt{5}}$$

or $$Y = -\frac{x+2y}{\sqrt{5}}$$

Elements of Hyperbola

Centre:- $X = 0,\ Y = 0$

$$\frac{2x+y}{\sqrt{5}} = 0, \quad \frac{-x+2y}{\sqrt{5}} = 0$$

$\Rightarrow 2x+y=0, \quad -x+2y=0$

For this $x=0, y=0$

so centre of (1) is $(0,0)$

Eq. of focal axis:-

$$Y=0 \Rightarrow \frac{-x+2y}{\sqrt{5}} = 0$$

$\Rightarrow -x+2y=0$

Eq. of conjugate axis:-

$$X=0 \Rightarrow \frac{2x+y}{\sqrt{5}} = 0$$

$\Rightarrow 2x+y=0$

Eccentricity:-

$$e = \frac{c}{a} = \sqrt{\frac{a^2+b^2}{a^2}} = \sqrt{\frac{3+2}{3}}$$

$$e = \sqrt{5/3}$$

Foci:- $X = \pm\sqrt{3}\cdot\sqrt{\frac{5}{3}},\ Y=0$

$$\Rightarrow \frac{2x+y}{\sqrt{5}} = \pm\sqrt{5}, \quad \frac{-x+2y}{\sqrt{5}} = 0$$

$2x+y = \pm 5, \quad -x+2y=0$

$2x+y = 5$
$-x+2y=0$
On solving

$2x+y = -5$
$-x+2y=0$
On solving

$2x+y=5$
$-2x+4y=0$
$5y=5$
$y=1$
so $x=2$

$2x+y=-5$
$-2x+4y=0$
$5y=-5$
$y=-1$
so $x=-2$

so foci of (1) are
$(2,1)$ & $(-2,-1)$

Vertices:- $X = \pm\sqrt{3},\ Y=0$

$$\frac{2x+y}{\sqrt{5}} = \pm\sqrt{3}, \quad \frac{-x+2y}{\sqrt{5}} = 0$$

$2x+y=\sqrt{15}$
$-x+2y=0$
By solving
$2x+y=\sqrt{15}$
$-2x+4y=0$
$5y=\sqrt{15}$
$y=\sqrt{\frac{3}{5}}$
so $-x=-2\sqrt{\frac{3}{5}}$
$x=2\sqrt{\frac{3}{5}}$

$2x+y=-\sqrt{15}$
$-x+2y=0$
By solving
$2x+y=-\sqrt{15}$
$-2x+4y=0$
$5y=-\sqrt{15}$
$y=-\sqrt{3/5}$
so $-x=2\sqrt{3/5}$
$x=-2\sqrt{3/5}$

vertices of (1) $\left(2\sqrt{\frac{3}{5}}, \sqrt{3/5}\right)$

and $\left(-2\sqrt{\frac{3}{5}}, -\sqrt{3/5}\right)$

(ix) $x^2-4xy-2y^2+10x+4y=0$

Solution:-

$x^2-4xy-2y^2+10x+4y=0 \longrightarrow (1)$

If θ be the angle of rotation to remove xy-term then

$$\tan 2\theta = \frac{2h}{a-b} = \frac{-4}{1+2} = \frac{-4}{3}$$

$$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = -\frac{4}{3}$$

$$\Rightarrow 3\tan\theta = -2 + 2\tan^2\theta$$

$$\text{or } 2\tan^2\theta - 3\tan\theta - 2 = 0$$

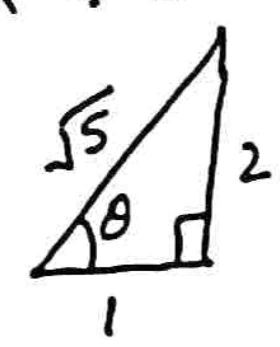
$$2\tan^2\theta - 4\tan\theta + \tan\theta - 2 = 0$$

$$2\tan\theta(\tan\theta - 2) + 1(\tan\theta - 2) = 0$$

$$(\tan\theta - 2)(2\tan\theta + 1) = 0$$

$\tan\theta - 2 = 0$, $\tan\theta = -\frac{1}{2}$ (Not admissible $\because 0 < \theta < 90°$)

so $\tan\theta = \frac{2}{1}$



$$\sin\theta = \frac{2}{\sqrt{5}}$$

$$\cos\theta = \frac{1}{\sqrt{5}}$$

Eqs. of transformation are

$$x = X\cos\theta - Y\sin\theta = \frac{X-2Y}{\sqrt{5}} \longrightarrow (i)$$

$$y = X\sin\theta + Y\cos\theta = \frac{2X+Y}{\sqrt{5}} \longrightarrow (ii)$$

using (i) & (ii) in (1)

$$\left(\frac{X-2Y}{\sqrt{5}}\right)^2 - 4\left(\frac{X-2Y}{\sqrt{5}}\right)\left(\frac{2X+Y}{\sqrt{5}}\right) - 2\left(\frac{2X+Y}{\sqrt{5}}\right)^2 + 10\left(\frac{X-2Y}{\sqrt{5}}\right) + 4\left(\frac{2X+Y}{\sqrt{5}}\right) = 0$$

$$\frac{1}{5}(X^2+4Y^2-4XY) - \frac{4}{5}(2X^2+XY-4XY-2Y^2) - \frac{2}{5}(4X^2+Y^2+4XY) + \frac{18X}{\sqrt{5}} - \frac{Y}{\sqrt{5}} = 0$$

$$X^2+4Y^2-\cancel{4XY}-8X^2+\cancel{12XY}+8Y^2-8X^2-2Y^2-\cancel{8XY}+18\sqrt{5}X-16\sqrt{5}Y = 0$$

$$\Rightarrow -15X^2 + 10Y^2 + 18\sqrt{5}X - 16\sqrt{5}Y = 0$$

$$(10Y^2 - 16\sqrt{5}Y) - (15X^2 - 18\sqrt{5}X) = 0$$

$$10\left(Y^2 - \frac{8}{\sqrt{5}}Y\right) - 15\left(X^2 - \frac{6}{\sqrt{5}}X\right) = 0$$

$$10\left(Y^2 - \frac{8}{\sqrt{5}}X + \left(\frac{4}{\sqrt{5}}\right)^2\right) - 15\left(X^2 - \frac{6}{\sqrt{5}}X + \left(\frac{3}{\sqrt{5}}\right)^2\right) = 10\left(\frac{4}{\sqrt{5}}\right)^2 - 15\left(\frac{3}{\sqrt{5}}\right)^2$$

$$10\left(Y - \frac{4}{\sqrt{5}}\right)^2 - 15\left(X - \frac{3}{\sqrt{5}}\right)^2 = \frac{180}{5} - \frac{135}{5}$$

$$10\left(Y - \frac{4}{\sqrt{5}}\right)^2 - 15\left(X - \frac{3}{\sqrt{5}}\right)^2 = 32 - 27$$

$$10\left(Y - \frac{4}{\sqrt{5}}\right)^2 - 15\left(X - \frac{3}{\sqrt{5}}\right)^2 = 5$$

$$\text{or } 2\left(Y - \frac{4}{\sqrt{5}}\right)^2 - 3\left(X - \frac{3}{\sqrt{5}}\right)^2 = 1$$

$$\text{or } \frac{\left(Y - \frac{4}{\sqrt{5}}\right)^2}{\frac{1}{2}} - \frac{\left(X - \frac{3}{\sqrt{5}}\right)^2}{\frac{1}{3}} = 1$$

($\therefore$ Hyperbola)

From (i) & (ii), we have

$$X - 2Y = \sqrt{5}x$$

$$2X + Y = \sqrt{5}y$$

Solving these eqs.

$$4X + 2Y = 2\sqrt{5}y$$

$$X - 2Y = \sqrt{5}x$$

$$5X = \sqrt{5}(x+2y)$$

$$\text{or } X = \frac{x+2y}{\sqrt{5}}$$

$$\text{and } Y = -\frac{2x-4y}{\sqrt{5}} + \sqrt{5}y$$

$$Y = \frac{-2x+y}{\sqrt{5}}$$

Elements of Hyperbola

Centre:– $X - \frac{3}{\sqrt{5}} = 0$, $Y - \frac{4}{\sqrt{5}} = 0$

$$\Rightarrow \frac{x+2y}{\sqrt{5}} = \frac{3}{\sqrt{5}}, \quad \frac{y-2x}{\sqrt{5}} = \frac{4}{\sqrt{5}}$$

$\Rightarrow x+2y=3, \quad -2x+y=4$

By solving

$$\begin{aligned} 2x+4y &= 6 \\ -2x+y &= 4 \\ \hline 5y &= 10 \Rightarrow y=2 \end{aligned}$$

so $x+2(2)=3 \Rightarrow x=-1$

so centre of (1) $(-1, 2)$

Eq. of focal axis:—

$$X-\frac{3}{\sqrt5}=0$$

$$\Rightarrow \frac{x+2y}{\sqrt5}=\frac{3}{\sqrt5} \Rightarrow x+2y=3$$

Eq. of conjugate axis:—

$$Y-\frac{4}{\sqrt5}=0 \Rightarrow Y=\frac{4}{\sqrt5}$$

$$\frac{y-2x}{\sqrt5}=\frac{4}{\sqrt5} \quad \text{or} \quad y-2x=4$$

Eccentricity:— $e=c/a$

$$\Rightarrow e=\sqrt{\frac{a^2+b^2}{a^2}}=\sqrt{\frac{1/2+1/3}{1/2}}$$

$$e=\sqrt{\frac{5}{6}\cdot\frac{2}{1}}=\sqrt{5/3}$$

Foci:— $Y-\frac{4}{\sqrt5}=\pm\sqrt{\frac{1}{2}}\sqrt{\frac{5}{3}}$

$$X-\frac{3}{\sqrt5}=0 \quad , \quad \frac{x+2y}{\sqrt5}=\frac{3}{\sqrt5}$$

Now $Y=\frac{4}{\sqrt5}\pm\sqrt{\frac{5}{6}}$ $\qquad x+2y=3$

$$\frac{y-2x}{\sqrt5}=\frac{4}{\sqrt5}\pm\sqrt{\frac{5}{6}}$$

$$\Rightarrow y-2x=4\pm\frac{5}{\sqrt6}$$

Now we solve

$y-2x=4+\frac{5}{\sqrt6}$

$x+2y=3$

$$\begin{aligned} -2x+y &= 4+\frac{5}{\sqrt6} \\ 2x+4y &= 6 \\ \hline 5y &= 10+\frac{5}{\sqrt6} \end{aligned}$$

$\Rightarrow y=2+\frac{1}{\sqrt6}$

and

$x+2\left(2+\frac{1}{\sqrt6}\right)=3$

$x+4+\frac{2}{\sqrt6}=3$

$x=-1-\frac{2}{\sqrt6}$

$y-2x=4-5/\sqrt6$

$x+2y=3$

$$\begin{aligned} -2x+y &= 4-\frac{5}{\sqrt6} \\ 2x+4y &= 6 \\ \hline 5y &= 10-\frac{5}{\sqrt6} \end{aligned}$$

$y=2-\frac{1}{\sqrt6}$

and

$x+2\left(2-\frac{1}{\sqrt6}\right)=3$

$x+4-\frac{2}{\sqrt6}=3$

$x=-1+\frac{2}{\sqrt6}$

so foci of (1) are

$$\left(-1-\frac{2}{\sqrt6}, 2+\frac{1}{\sqrt6}\right) \& \left(-1+\frac{2}{\sqrt6}, 2-\frac{1}{\sqrt6}\right)$$

Vertices:—

$$X-\frac{3}{\sqrt5}=0 \quad , \quad Y-\frac{4}{\sqrt5}=\pm\frac{1}{\sqrt2}$$

$$\frac{x+2y}{\sqrt5}=\frac{3}{\sqrt5} \quad , \quad \frac{-2x+y}{\sqrt5}=\frac{4}{\sqrt5}\pm\frac{1}{\sqrt2}$$

$$x+2y=3 \qquad -2x+y=4\pm\sqrt{\frac{5}{2}}$$

Now we solve

$x+2y=3$

$-2x+y=4+\sqrt{5/2}$

$$\begin{aligned} -2x+y &= 4+\sqrt{5/2} \\ 2x+4y &= 6 \\ \hline 5y &= 10+\sqrt{5/2} \end{aligned}$$

$y=2+\frac{1}{5}\sqrt{5/2}$

$y=2+\frac{1}{\sqrt{10}}$

and

$x+2\left(2+\frac{1}{\sqrt{10}}\right)=3$

$x=-1-\frac{2}{\sqrt{10}}$

$x+2y=3$

$-2x+y=4-\sqrt{5/2}$

$$\begin{aligned} -2x+y &= 4-\sqrt{5/2} \\ 2x+4y &= 6 \\ \hline 5y &= 10-\sqrt{5/2} \end{aligned}$$

$y=2-\frac{1}{5}\sqrt{5/2}$

$y=2-\frac{1}{\sqrt{10}}$

and

$x+2\left(2-\frac{1}{\sqrt{10}}\right)=3$

$x=-1+\frac{2}{\sqrt{10}}$

so vertices of (1) are

$$\left(-1-\frac{2}{\sqrt{10}}, 2+\frac{1}{\sqrt{10}}\right) \& \left(-1+\frac{2}{\sqrt{10}}, 2-\frac{1}{\sqrt{10}}\right)$$

Q2. Show that
i) $10xy+8x-15y-12=0$
and ii) $6x^2+xy-y^2-21x-8y+9=0$
each represents a pair of straight lines and find an equation of each line.

Solution:-

(i) $10xy+8x-15y-12=0$

Here $a=0,\ b=0,\ h=5$
$g=4,\ f=-\frac{15}{2},\ c=-12$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 5 & 4 \\ 5 & 0 & -15/2 \\ 4 & -\frac{15}{2} & -12 \end{vmatrix}$$

$$= 0-5(-60+30)+4\left(-\frac{75}{2}-0\right)$$

$$= 150-150 = 0$$

The given equation represents a degenerate conic which is a pair of lines.

The given eq. is

$$10xy+8x-15y-12=0$$

$$(10xy-15y)+(8x-12)=0$$

$$\Rightarrow 5y(2x-3)+4(2x-3)=0$$

$$(2x-3)(5y+4)=0$$

Eqs. of lines are

$$2x-3=0,\quad 5y+4=0$$

(ii) $6x^2+xy-y^2-21x-8y+9=0$

Here $a=6,\ b=1,\ h=\frac{1}{2}$
$g=-\frac{21}{2},\ f=-4,\ c=9$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 1/2 & -21/2 \\ 1/2 & -1 & -4 \\ -\frac{21}{2} & -4 & 9 \end{vmatrix}$$

$$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\begin{vmatrix} 12 & 1 & -21 \\ 1 & -2 & -8 \\ -21 & -8 & 18 \end{vmatrix} \begin{matrix} \text{By } 2R_1 \\ 2R_2 \\ 2R_3 \end{matrix}$$

$$= \frac{1}{8}\left[12(-36-64)-1(18-168)-21(-8-42)\right]$$

$$= \frac{1}{8}\left[12(-100)-1(-150)-21(-50)\right]$$

$$= \frac{1}{8}(-1200+150+1050)$$

$$= \frac{1}{8}(0) = 0$$

Hence given equation represents a pair of lines.
Now rearranging the given eq., we have

$$-y^2+xy-8y+6x^2-21x+9=0$$

$$\Rightarrow y^2-xy+8y-6x^2+21x-9=0$$

$$y^2-y(x-8)-3(2x^2-7x+3)=0$$

$$\therefore y = \frac{(x-8)\pm\sqrt{(x-8)^2+4(6x^2-21x+9)}}{2}$$

$$y = \frac{x-8\pm\sqrt{x^2+64-16x+24x^2-84x+36}}{2}$$

$$y = \frac{x-8 \pm \sqrt{25x^2-100x+100}}{2}$$

$$y = \frac{x-8 \pm 5\sqrt{x^2-4x+4}}{2}$$

$$y = \frac{x-8 \pm 5\sqrt{(x-2)^2}}{2}$$

$$y = \frac{x-8 \pm 5(x-2)}{2}$$

$$y = \frac{x-8+5x-10}{2}, \quad y = \frac{x-8-5x+10}{2}$$

$$y = \frac{6x-18}{2}, \quad y = \frac{-4x+2}{2}$$

$$y = 3x-9, \quad y = -2x+1$$

so required lines are

$y = 3x-9$ and $y = -2x+1$

Q3. Find an equation of tangent to each of the given conic at the indicated point.

(i) $3x^2-7y^2+2x-y-48=0$ at $(4,1)$

Solution:-

$$3x^2-7y^2+2x-y-48=0 \longrightarrow (1)$$

Diff (1) w.r.t 'x'

$$6x-14y\frac{dy}{dx}+2-\frac{dy}{dx}=0$$

$$(6x+2)-(14y+1)\frac{dy}{dx}=0$$

$$\frac{dy}{dx} = \frac{6x+2}{14y+1}$$

or $$\left.\frac{dy}{dx}\right|_{(4,1)} = \frac{6(4)+2}{14(1)+1} = \frac{26}{15}$$

Now eq. of tangent at $(4,1)$

$$y-1 = \frac{26}{15}(x-4)$$

$$\rightarrow 15y-15 = 26x-104$$

$$26x-15y-89=0$$

(ii) $x^2+5xy-4y^2+4=0$ at $y=-1$

Solution:-

$$x^2+5xy-4y^2+4=0 \longrightarrow (1)$$

Put $y=-1$

$$\rightarrow x^2-5x-4+4=0$$

$$x^2-5x=0$$

$$x(x-5)=0$$

$$x=0, \quad x=5$$

so points are $(0,-1)$ and $(5,-1)$

Now replacing x^2 by xx_1, y^2 by yy_1 and xy by $\frac{1}{2}(xy_1+yx_1)$ in (1)

$$xx_1+\frac{5}{2}(xy_1+yx_1)-4yy_1+4=0$$

At $(0,-1)$ eq. of tangent is

$$x(0)+\frac{5}{2}(x(-1)+y(0))-4y(-1)+4=0$$

$$\frac{5}{2}(-x)+4y+4=0$$

or $5x-8y-8=0$

At $(5,-1)$ eq. of tangent is

$$x(5)+\frac{5}{2}(x(-1)+5y)-4y(-1)+4=0$$

$$5x+\frac{5}{2}(-x+5y)+4y+4=0$$

$\Rightarrow 10x - 5x + 25y + 8y + 8 = 0$

$5x + 25y + 8y + 8 = 0$

or $5x + 33y + 8 = 0$

(iii) $x^2 + 4xy - 3y^2 - 5x - 9y + 6 = 0$

at $x = 3$

Solution:-

$x^2 + 4xy - 3y^2 - 5x - 9y + 6 = 0 \longrightarrow (1)$

Put $x = 3$

$9 + 12y - 3y^2 - 15 - 9y + 6 = 0$.

$-3y^2 + 3y = 0$

or $3y^2 - 3y = 0$

$3y(y-1) = 0$

$\Rightarrow y = 0, \; y = 1$

so points are $(3,0)$ & $(3,1)$

Now replacing x^2 by xx_1, y^2 by yy_1 and $2xy$ by $xy_1 + yx_1$

$$xx_1 + 2(xy_1 + yx_1) - 3yy_1 - \frac{5}{2}(x+x_1) - \frac{9}{2}(y+y_1) + 6 = 0$$

At $(3,0)$ Eq. of tangent is

$$x(3) + 2(x(0) + 3y) - 3y(0) - \frac{5}{2}(x+3) - \frac{9}{2}(y+0) + 6 = 0$$

$$3x + 6y - \frac{5}{2}x - \frac{15}{2} - \frac{9}{2}y + 6 = 0$$

$$\Rightarrow 6x + 12y - 5x - 15 - 9y + 12 = 0$$

$$x + 3y - 3 = 0$$

At $(3,1)$ Eq. of tangent is

$$x(3) + 2(x(1) + 3y) - 3y(1) - \frac{5}{2}(x+3) - \frac{9}{2}(y+1) + 6 = 0$$

$$3x + 2x + 6y - 3y - \frac{5}{2}x - \frac{15}{2} - \frac{9}{2}y - \frac{9}{2} + 6 = 0$$

$$6x + 4x + 12y - 6y - 5x - 15 - 9y - 9 + 12 = 0$$

$$5x - 3y - 12 = 0$$