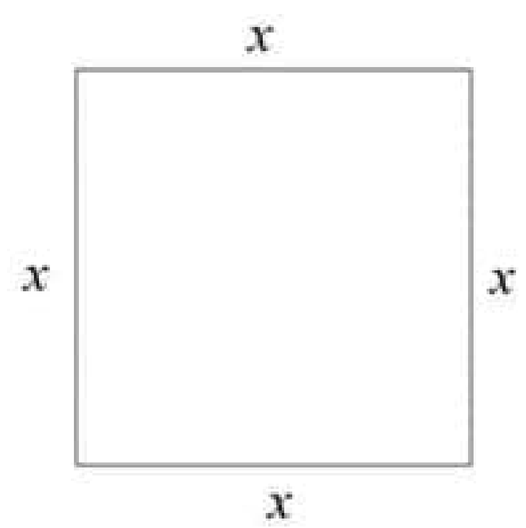
Function and Limits

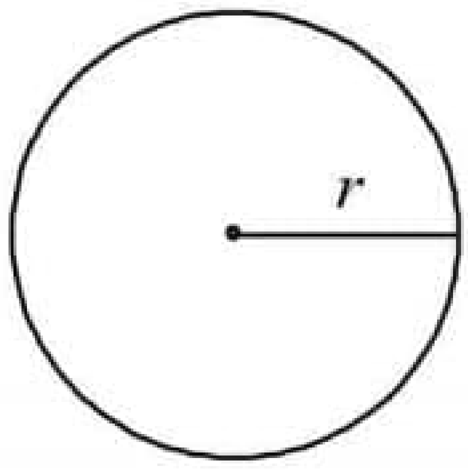
Concept of Functions:

Historically, the term function was first used by German mathematician Leibnitz (1646-1716) in 1673 to denote the dependence of one quantity on another e.g.

1) The area "A" of a square of side "x" is given by the formula $A=x^2$. As area depends on its side x, so we say that A is a function of x.



2) The area "A" of a circular disc of radius "r" is given by the formula $A=\pi r^2$ As area depends on its radius r, so we say that A is a function of r.



3) The volume "V" of a sphere of radius "r" is given by the formula $V=\frac{4}{3}\pi r^3$. As volume V of a sphere depends on its radius r, so we say that V is a function of r.

The Swiss mathematician, Leonard Euler conceived the idea of denoting function written as $y=f(x)$ and read as y is equal to f of x. $f(x)$ is called the value of f at x or image of x under f.

The variable x is called independent variable and the variable y is called dependent variable of f.

If x and y are real numbers then f is called real valued function of real numbers.

Domain of the function:

If the independent variable of a function is restricted to lie in some set, then this set is called the domain of the function e.g.
Dom of $f = \{0 \leq x \leq 5\}$

Range of the function:

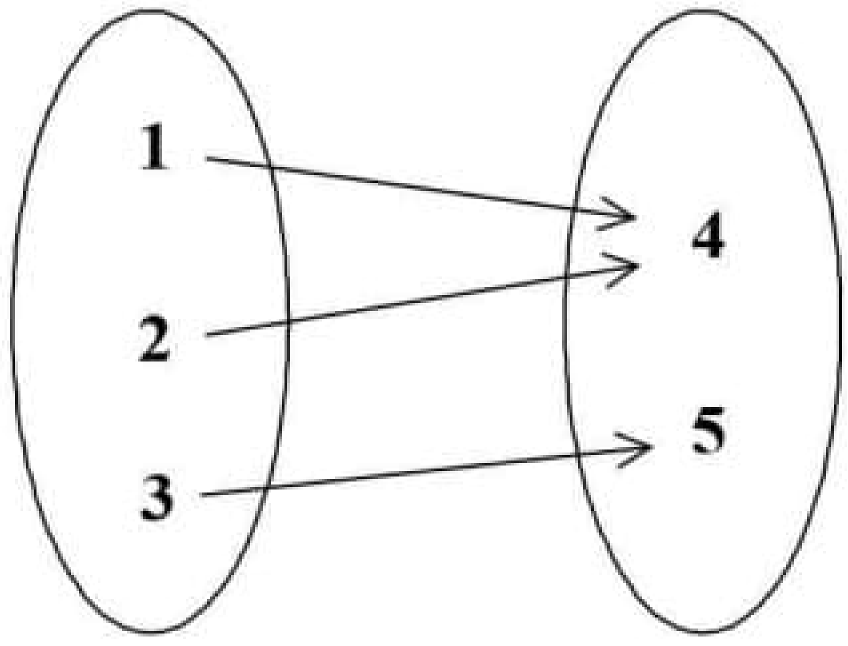
The set of all possible values of $f(x)$ as x varies over the domain of f is called the range of f e.g. $y = 100 - 4x^2$.
As x varies over the domain [0,5] the values of $y = 100 - 4x^2$ vary between y=0 (when x=5) and y = 100 (when x=0)
Range of $f = \{0 \leq y \leq 100\}$

Definition:

A function is a rule by which we relate two sets A and B (say) in such a way that each element of A is assigned with one and only one element of B. For example

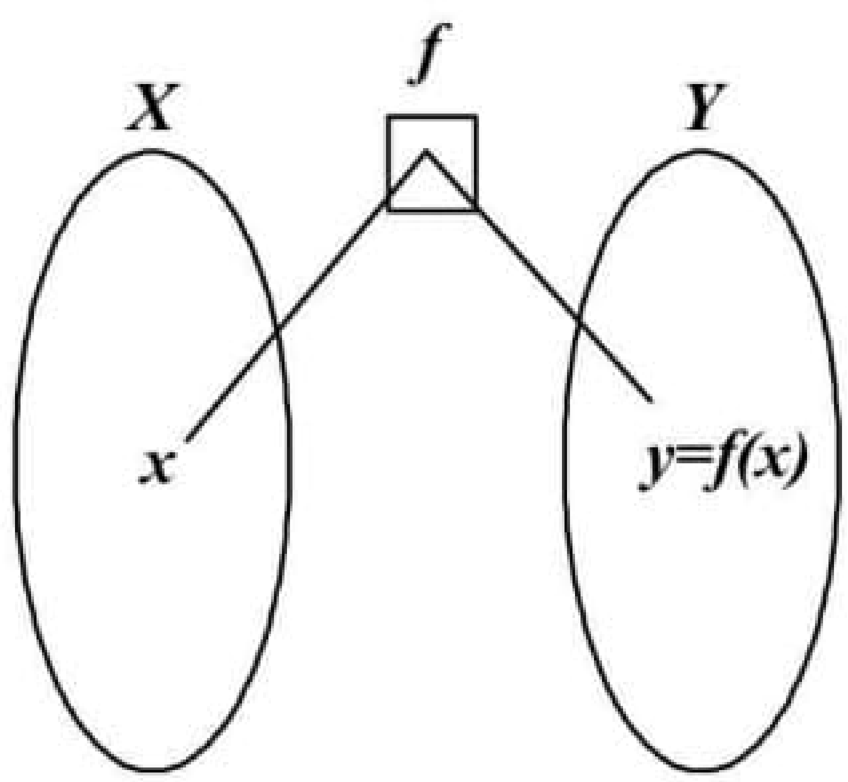
is a function from A to B.

its Domain = {1,2,3} and Range = {4,5}



In general:

A function f from a set 'X' to a set 'Y' is a rule that assigns to each element x in X one and only one element y in Y.(a unique element y in Y)



(f is function from X to Y)

If an element "y, of Y is associated with an element "x, of X, then we write y=f (x) &read as y" is equal to f of x. Here f(x) is called image of f at x or value of f at x .

Or if a quantity y depends on a quantity x in such a way that each value of x determines exactly one value of y. Then we say that y is a function of x.

The set x is called Domain of f. The set of corresponding elements y in y is called Range of f. we say that y is a function of x.

Exercise 1.1

Q1. (a) Given that $f(x) = x^2 - x$

i. $f(-2) = (-2)^2 - (-2) = 4 + 2 = 6$

ii. $f(0) = (0)^2 - (0) = 0$

iii. $f(x-1) = (x-1)^2 - (x-1) = x^2 - 2x + 1 - x + 1 = x^2 - 3x + 2$

iv. $f(x^2+4) = (x^2+4)^2 - (x^2+4) = x^4 + 8x^2 + 16 - x^2 - 4 = x^4 + 7x^2 + 12$

(b) Given that $f(x) = \sqrt{x+4}$

$i)\, f(-2)=\sqrt{-2+4}=\sqrt{2}$

$ii)\, f(0)=\sqrt{0+4}=\sqrt{4}=2$

$iii)\, f(x-1)=\sqrt{x-1+4}=\sqrt{x+3}$

$iv)\, f(x^2+4)=\sqrt{x^2+4+4}=\sqrt{x^2+8}$

Q2. Given that

i) $f(x)=6x-9$

$f(a+h)=6(a+h)-9=6a+6h-9$

$f(a)=6a-9$

Now $\dfrac{f(a+h)-f(a)}{h}=\dfrac{(6a+6h-9)-(6a-9)}{h}$

$=\dfrac{6a+6h-9-6a+9}{h}=\dfrac{6h}{h}=6$

ii) $f(x)=\sin x$ *given*

$\because \quad \sin\theta-\sin\varphi=2\cos\left(\dfrac{\theta+\varphi}{2}\right)\sin\left(\dfrac{\theta+\varphi}{2}\right)$

$f(a+h)=\sin(a+h)$ *and* $f(a)=\sin a$

Now $\dfrac{f(a+h)-f(a)}{h}=\dfrac{\sin(a+h)-\sin a}{h}$

$=\dfrac{1}{h}\left[\sin(a+h)-\sin a\right]$

$=\dfrac{1}{h}\left[2\cos\left(\dfrac{a+h+a}{2}\right)\sin\left(\dfrac{a+h-a}{2}\right)\right]=\dfrac{1}{h}\left[2\cos\left(\dfrac{2a+h}{2}\right)\sin\left(\dfrac{h}{2}\right)\right]$

$=\dfrac{1}{h}\left[2\cos\left(\dfrac{2a}{2}+\dfrac{h}{2}\right)\sin\left(\dfrac{h}{2}\right)\right]=\dfrac{2}{h}\cos\left(a+\dfrac{h}{2}\right)\sin\left(\dfrac{h}{2}\right)$

iii) Given that $f(x)=x^3+2x^2-1$

$$f(a+h)=(a+h)^3+2(a+h)^2-1=a^3+h^3+3ah(a+h)+2(a^2+2ah+h^2)-1$$

$$=a^3+h^3+3a^2h+3ah^2+2a^2+4ah+2h^2-1$$

$$f(a)=a^3+2a^2-1$$

Now $f(a+h)-f(a)$

$$=\frac{a^3+h^3+3a^2h+3ah^2+2a^2+4ah+2h^2-1-(a^3+2a^2-1)}{h}$$

$$=\frac{1}{h}\left[a^3+h^3+3a^2h+3ah^2+2a^2+4ah+2h^2-1-a^3-2a^2+1\right]$$

$$=\frac{1}{h}\left[h^3+3a^2h+3ah^2+4ah+2h^2\right]=\frac{h}{h}\left[h^2+3a^2+3ah+4a+2h\right]$$

$$=h^2+3a^2+3ah+4a+2h=h^2+3ah+2h+3a^2+4a=h^2+(3a+2)h+3a^2+4a$$

iv) Given that $f(x)=\cos x$

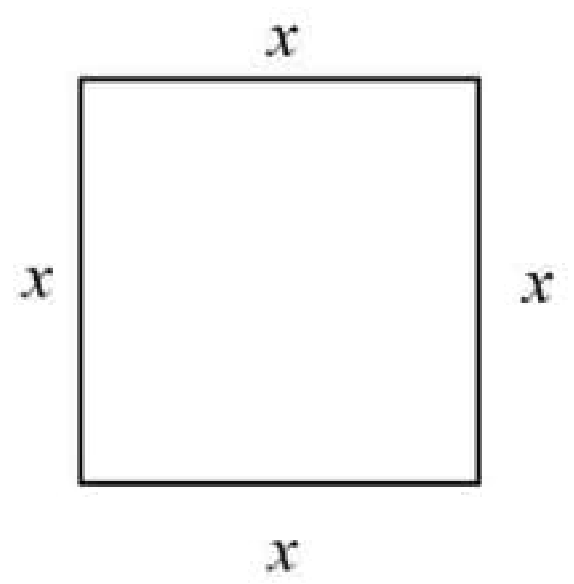
so $f(a+h)=\cos(a+h)$

and $f(a)=\cos a$

Now $\dfrac{f(a+h)-f(a)}{h}$

$$=\frac{\cos(a+h)-\cos a}{h}=\frac{1}{h}\left[-2\sin\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]=\frac{-2}{h}\sin\left(a+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)$$

Q3. (a) If 'x' unit be the side of square.



Then its perimeter $P=x+x+x+x=4x$ .... (1)

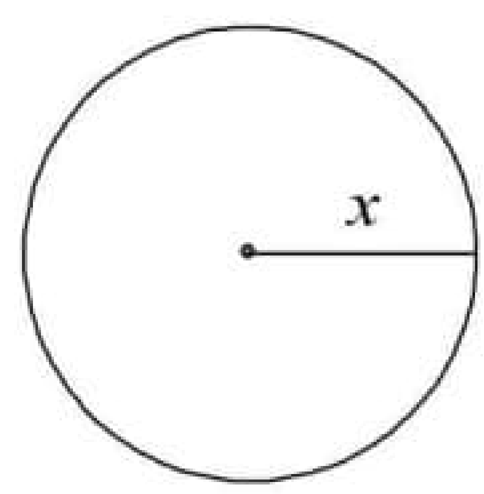
$A=\text{Area}=x\,.\,x=x^2$ .... (2)

From (2) $x=\sqrt{A}$ putting in (1)

$P=4\sqrt{A}$

$\therefore$ P is expressed as Area

(b) Let x units be the radius of circle



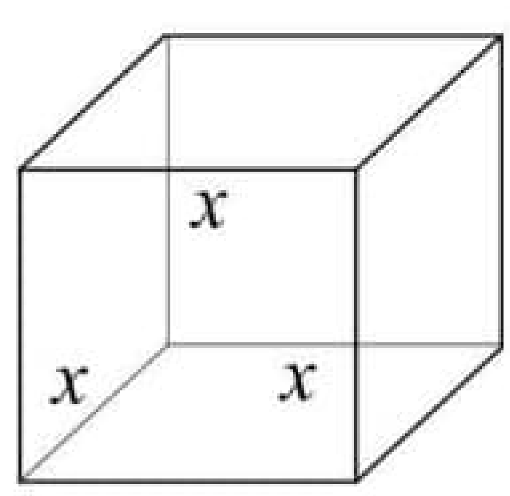
Then Area = $A=\pi x^2$ .... (1)

Circumference = $C=2\pi x$ .... (2)

From (2) $x=\dfrac{C}{2\pi}$ Putting in (1)

$$A=\pi\left(\frac{c}{2\pi}\right)^2=\pi\left(\frac{c^2}{4\pi^2}\right)=\frac{c^2}{4\pi}$$

$A=\dfrac{c^2}{4\pi}$ $\because$ *Area is a function of Circumference*



(c) Let x unit be each side of cube.

The Volume of Cube $=x\,.\,x\,.\,x=x^3$ .... (1)

Area of base $=A=x^2$ .... (2)

From (2) $x=\sqrt{A}$ Putting in (1)

$V=(\sqrt{A})^3=(A)^{3/2}$

$Q5.\quad f(x)=x^3-ax^2+bx+1$

If $f(2)=-3$ *and* $f(-1)=0$

$(2)^3-a(2)^2+b(2)+1=-3$

$8-4a+2b+1=-3$

$9-4a+2b=-3$

$12-4a+2b=0$

Dividing by -2

$-6+2a-b=0$ .... (1)

$(-1)^3-a(-1)^2+(-1)+1=0$

$-1-a-b+1=0$

$-a-b=0$

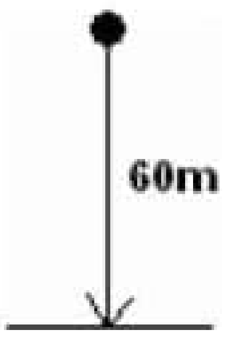
$a+b=0$ .... (2)

Solving (1) *and* (2)

$$\begin{array}{r}2a-b-6=0\\ a+b\quad=0\\ \hline 3a-6\quad=0\end{array}$$

$a=2$ *and* $(2)\Rightarrow b=-a \quad\Rightarrow b=-2$

$Q6.\quad h(x)=40-10x^2$



(a) $x=1\sec$

$h(1)=40-10(1)^2$

$=30m$

(b) $x=1.5\sec$

$h(1.5)=40-10(1.5)^2$

$=40-10(2.25)=40-22.5=17.5m$

(c) $x=1.7\sec$

$h(1.7)=40-10(1.7)^2$

$=40-10(2.89)=40-28.9=11.1m$

ii) Does the stone strike the ground = ?

$h(x)=0$

$40-10x^2=0$

$-10x^2=-40 \quad\Rightarrow x^2=4$

$x=\pm 2$

Stone strike the ground after 2 sec.

Graphs of Function

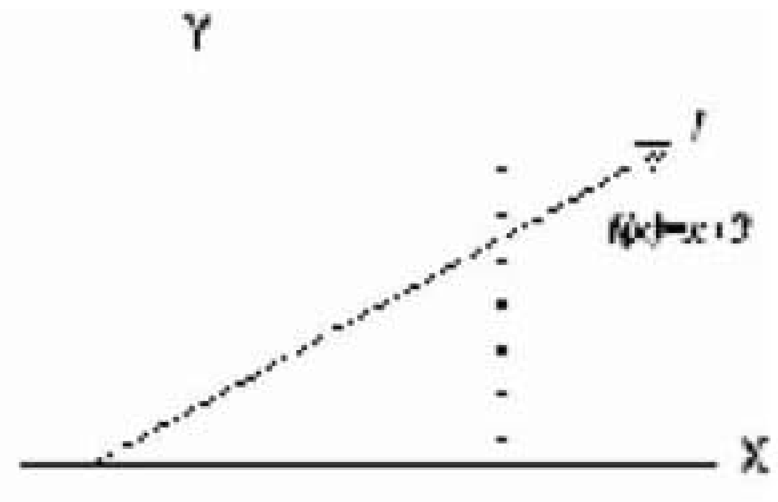
Definition:

The graph of a function *f* is the graph of the equation $y = f(x)$. It consists of the points in the Cartesian plane chose co-ordinates (x , y) are input - output pairs for *f* .

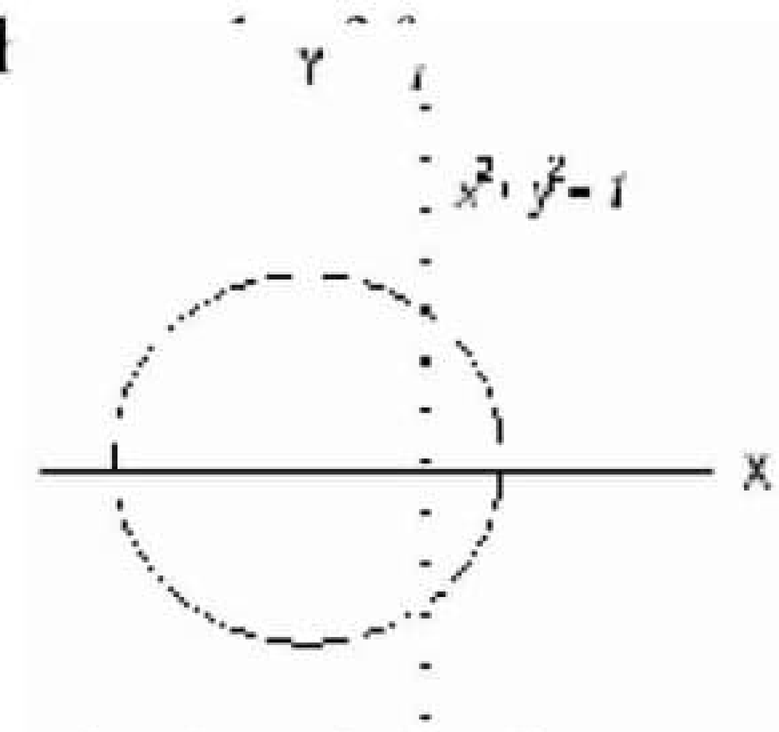
Note that not every curve we draw in the graph of a function. A function *f* can have only one value *f(x)* for each *x* in its domain.

Vertical Line Test

No vertical line can intersect the graph of a function more than once. Thus, a circle cannot be the graph of a function. Since some vertical lines intersect the circle Twice. If 'a' is the domain of the function *f* , then the vertical line x = a will intersect tl in the single point (a , *f(a)*).



It is graph of function



Is not a graph of function

Types of Function

ALGEBRAIC FUNCTIONS

Those functions which are defined by algebraic expressions.

1) Polynomial Functions:

$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots\ldots\ldots + a_1 x + a_0$ **Is a Polynomial Function for all x where** $a_0, a_1, a_2 \ldots\ldots a_n$ **are real numbers, and exponents are non-negative integer .** a_n **is called leading coefft of p(x) of degree n, Where** $a_n \neq 0$

$\Rightarrow$ *Degree of polynomial function is the* max *imum power of x in equation*

$P(x) = 2x^4 - 3x^3 + 2x - 1$ deg *ree* $= 4$

2) Linear Function: if the degree of polynomial fn is '1, is called linear function .i.e. p(x)=ax+b

or $\Rightarrow$ Degree of polynomial function is one.

$f(x) = ax + b \qquad a \neq 0$

$\because \qquad y = 5x + b$

3) Identity Function: For any set X, a function I: $X \to x$ of the form $y = x$ or *f(x)* = *x*. Domain and range of I is *x*. Note. I (x)= ax +b be a linear fn if a=1,b=0 then I(x)=x or y=x is called identity fn

4) Constant Function:

$C : X \to y$ *defined by* called constant fn

$C(x) = a \qquad \forall x \in X \text{ and } a \in y$

e.g. $C : R \to R$

$C(x) = 2 \text{ or } y = 2 \qquad \forall x \in R$

or

$f : X \to y$ If f (x)=c, (const) then f is

eg y=5

5) Rational Function:

$$R(x)=\frac{P(x)}{Q(x)}$$

Both $P(x)$ *and* $Q(x)$ *are polynomial and* $Q(x)\neq 0$

e.g. $R(x)=\dfrac{3x^2+4x+1}{5x^3+2x^2+1}$

Domain of rational function is the set of all real numbers for which $Q(x)\neq 0$

6) Exponential Function:

A function in which the variable appears as exponent (power) is called an exponential function.

i) $y=a^x \;\therefore x\in R \qquad a>0$

ii) $y=e^x \;\therefore x\in R \text{ and } e=2.178$

iii) $y=2^x$ or $y=e^{xh}$

are some exponential functions.

7) Logarithmic Function:

If $x=a^y$ *then* $y=\log_a^x$ x >0

$\because \quad a>0 \quad a\neq 1$

'a' is called the base of Logarithemic function

Then $y=\log_a^x$ *is Logarithmic function of base 'a'*

i) *If base* $=10$ *then* $y=\log_{10}^x$

is called common Logarithm of x

ii) *If base* $=e=2.718$

$y=\log_e^x=\ln x$ *is called natural* log

8) Hyperbolic Function:

We define as

i) $y=\sinh(x)=\dfrac{e^x-e^{-x}}{2}$ Sine hyperbolic function or hyperbolic sine function

$Dom=\{x/x\in R\}$ *and* $Range=\{y/y\in R\}$

ii) $y=\cosh(x)=\dfrac{e^x+e^{-x}}{2}$ *is called hyperbolic cosine function* $\Rightarrow$ $x\in R, y\in[1,\infty)$

iii) $y=Tanhx=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}=\dfrac{\sinh x}{\cosh x}$

iv) $y=\coth x=\dfrac{\cosh x}{\sinh x}$

v) $y=\sec hx=\dfrac{1}{\cosh x}=\dfrac{2}{e^x+e^{-x}}$ $x\in R$

vi) $y=\cos echx=\dfrac{1}{\sinh x}=\dfrac{2}{e^x-e^{-x}}$ $Dom=\{x\neq 0: x\in R$

9) Inverse Hyperbolic Function: (Study in B.Sc level)

i) $y=\sinh^{-1}x=\ln\left(x+\sqrt{x^2+1}\right)$ *for* $\forall x\in R$

ii) $y=\cosh^{-1}x=\ln\left(x+\sqrt{x^2-1}\right)$ *for* $\forall x\in R$ *and* $x>1$

iii) $y=Tanh^{-1}x=\dfrac{1}{2}\ln\left|\dfrac{1+x}{1-x}\right|$ $x\neq 1$ *and* $|x|<1$

iv) $y=\sec h^{-1}x=\ln\left(\dfrac{1}{x}+\dfrac{\sqrt{1-x^2}}{x}\right)$ $0<x\leq 1$

v) $y=\coth^{-1}x=\dfrac{1}{2}\left|\dfrac{x+1}{x-1}\right|$ $\because$ $|x|>1$

vi) $y=\cos ech^{-1}x=\ln\left(\dfrac{1}{x}+\dfrac{\sqrt{1+x^2}}{|x|}\right)$ $x\neq 0$

10) <u>Trigonometric Function:</u>

Functions	Domain(x)	Range(y)
i) $y=\sin x$	*All real numbers* $\because -\infty<x+\infty$	$-1\leq y\leq 1$
ii) $y=\cos x$	*All real numbers* $\because -\infty<x<\infty$	$-1\leq y\leq 1$
iii) $y=\tan x$	$x\in R-(2k+1)\dfrac{\pi}{2}$ $k\in Z$	$\because$ 'R' *all real numbers*
iv) $y=\cot x$	$x\in R-k\pi$ $k\in Z$	R
v) $y=\sec x$	$x\in R-(2k+1)\dfrac{\pi}{2}$ $k\in Z$	$R-(-1,1)$ *or* $R-(-1<y<1)$
vi) $y=\cos ecx$	$x\in R-(k\pi)$ $k\in Z$	$R-(-1<y<1)$

11) <u>Inverse Trigonometric Functions:</u>

Function	Dom(x)	Range(y)
$y=\sin^{-1}x\Leftrightarrow x=\sin y$	$-1\leq x\leq 1$	$-\dfrac{\pi}{2}\leq y\leq\dfrac{\pi}{2}$
$y=\cos^{-1}x\Leftrightarrow x=\cos y$	$-1\leq x\leq 1$	$0\leq y\leq\pi$

$y = Tan^{-1}x \Leftrightarrow x = Tany$	$x \in R$ or $-\infty < x < \infty$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = Sec^{-1}x \Leftrightarrow x = \sec y$	$x \in R - (-1,1)$	$y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$
$y = Co\sec^{-1} x \Leftrightarrow x = \cos ecy$	$x \in R - (-1,1)$	$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$y = Cot^{-1}x \Leftrightarrow x = \cot y$	$x \in R$	$0 < y < \pi$

12) <u>Explicit Function:</u>

If y is easily expressed in terms of x, then y is called an explicit function of x.

$\Rightarrow y = f(x)$ *e.g.* $y = x^3 + x + 1$ *etc.*

13) <u>Implicit Function:</u>

If x and y are so mixed up and y cannot be expressed in term of the independent variable x, Then y is called an implicit function of x. It can be written as. $f(x, y) = 0$

e.g. $x^2 + xy + y^2 = 2$ etc.

14) <u>Parametric Function:</u>

For a function $y = f(x)$ if both x & y are expressed in another variable say 't' or θ which is called a parameter of the given curve.

Such as:

i) $x = at^2$ Parametric parabola

$y = 2at$

ii) $x = a\cos t$ *Parametric equation of circle* $y^2 = 4a$

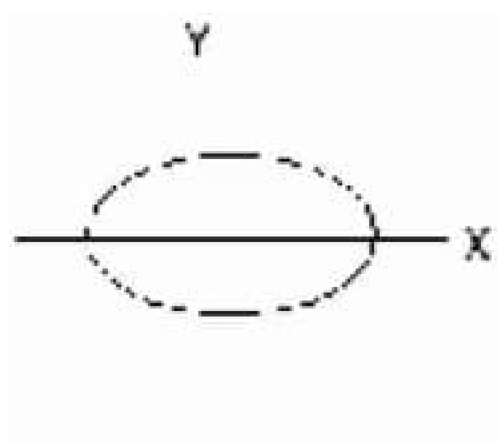
$y = a\sin t$

$x^2 + y^2 = a^2$

iii) $x = a\cos\theta$ *Parametric equation of Ellipse*

$y = b\sin\theta$

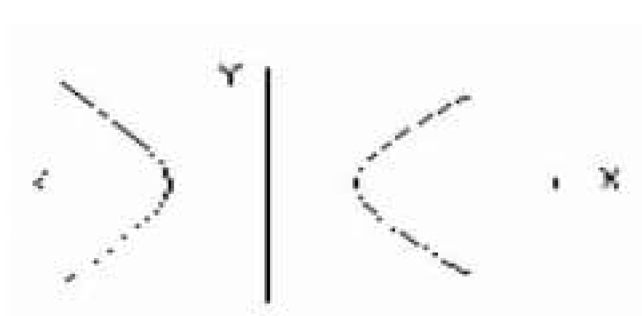
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



vi) $x = a\sec\theta$ *Parametric equation of hyperbola*

$y = b\tan\theta$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



Exercise 1.1

Q7. *Parabola* $\Rightarrow$ $y^2 = 4ax$ ....(1)

$x = at^2$ ....(i)

$y = 2at$ ....(ii)

To eliminating 't' from (ii) $t = \dfrac{y}{2a}$ *putting (i)*

$$x = a\left(\frac{y}{2a}\right)^2 \Rightarrow x = a\left(\frac{y^2}{4a^2}\right) \Rightarrow x = \frac{y^2}{4a}$$

$\Rightarrow$ $y^2 = 4ax$ *which is same as* (1)

which is equation of parabola.

ii) $x = a\cos\theta,$ $y = b\sin\theta$

$\Rightarrow$ $\dfrac{x}{a} = \cos\theta$..........(i) *and* $\dfrac{y}{b} = \sin\theta$..............(ii) *To eliminating* θ *from (i) and (ii)*

Squaring and adding (i) and (ii)

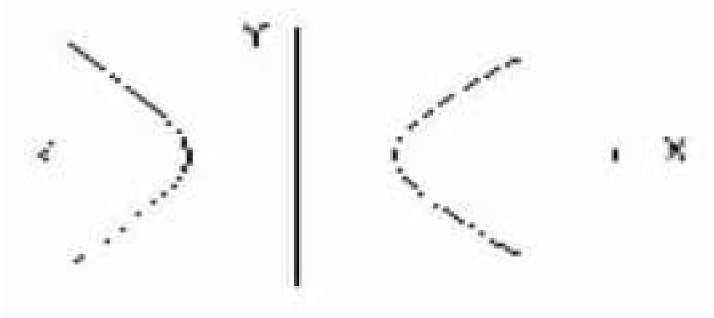
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y^2}{b}\right) = 1$$ *represent a Ellipse*

iii) $x = a\sec\theta,$ $y = b\tan\theta$

$\dfrac{x}{a} = \sec\theta$...............(i) $\dfrac{y}{b} = \tan\theta$..................(ii)

Squaring and Subtracting (i) *and* (ii)

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \sec^2\theta - \tan^2\theta \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 + \tan^2\theta - \tan^2\theta \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Which is equation of hyperbola

Q8. (i) $\sinh 2x = 2\sinh x\cosh x$

$$R.H.S = 2\sinh x\cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = 2\left(\frac{e^{2x} - e^{-2x}}{4}\right) = \frac{e^{2x} - e^{-2x}}{2}$$

$= \sinh 2x = L.H.S$

ii) $\sec^2 hx = 1 - \tan^2 hx$

$$R.H.S. = 1 - \tan^2 hx = 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = 1 - \left(\frac{e^{2x} + e^{-2x} - 2}{e^{2x} + e^{-2x} + 2}\right)$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{e^{2x} + e^{-2x} + 2} = \frac{4}{\left(e^x + e^{-x}\right)^2} = \frac{1}{\left(e^x + e^{-x}/2\right)^2}$$

$$= \frac{1}{\cosh^2 x} = \sec h^2 x = L.H.S$$

iii) $\cos eh^2 x = \coth^2 x - 1$

$$R.H.S = \coth^2 x - 1 = \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 - 1 = \frac{\left(e^x + e^{-x}\right)^2 - \left(e^x - e^{-x}\right)^2}{\left(e^x - e^{-x}\right)^2} = \frac{\left(e^{2x} + e^{-2x} + 2\right) - \left(e^{2x} + e^{-2x} - 2\right)}{\left(e^x - e^{-x}\right)^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{\left(e^x - e^{-x}\right)^2} = \frac{4}{\left(e^x - e^{-x}\right)^2} = \frac{1}{\left(e^x - e^{-x}/2\right)^2} = \frac{1}{\sinh^2 x} = \cos ech 2x = L.H.S$$

Q9. $f(x) = x^3 + x$

replace x by $-x$

$$f(-x) = (-x)^3 + (-x) = -x^3 - x = -\left[x^3 + x\right] = -f(x)$$

$\Rightarrow$ $f(x) = x^3 + x$ *is odd function*

ii) $f(x) = (x+2)^2$

replace x by $-x$

$$f(-x) = (-x+2)^2 \neq \pm f(x)$$

$f(x) = (x+2)^2$ *is neither even nor odd*

iii) $f(x) = x\sqrt{x^2+5}$

replace x by $-x$

$$f(-x) = (-x)\sqrt{(-x)^2+5} = -\left[x\sqrt{x^2+5}\right] = -f(x)$$

$f(x)$ *is odd function.*

iv) $f(x) = \dfrac{x-1}{x+1}$

replace x by $-x$

$$f(-x) = \frac{-x-1}{-x+1} = \frac{-(x+1)}{-(x-1)} = \frac{x+1}{x-1} \neq \pm f(x)$$

$f(x)$ *is neither even nor odd function.*

v) $f(x) = x^{\frac{2}{3}} + 6$

replace x by $-x$

$$f(-x) = (-x)^{\frac{2}{3}} + 6 = \left[(-x)^2\right]^{\frac{1}{3}} + 6 = x^{\frac{2}{3}} + 6 = f(x)$$

$f(x)$ *is an even function.*

$$= \frac{x+1}{x-1} \neq \pm f(x)$$

$\therefore$ f(x) is neither even nor odd function.

(v) $f(x) = x^{2/3} + 6$

$$f(-x) = (-x)^{2/3} + 6$$
$$= [(-x)^2]^{1/3} + 6$$
$$= (x^2)^{1/3} + 6$$
$$= x^{2/3} + 6$$
$$= f(x)$$

$\therefore$ f(x) is an even function.

(vi) $f(x) = \frac{x^3 - x}{x^2 + 1}$

$$f(-x) = \frac{(-x)^3 - (-x)}{(-x)^2 + 1}$$
$$= \frac{-x^3 + x}{x^2 + 1}$$
$$= \frac{-(x^3 - x)}{x^2 + 1}$$
$$= -f(x)$$

$\therefore$ f(x) is an odd function.

Composition of Functions:

Let f be a function from set X to set Y and g be a function from set Y to set Z. The composition of f and g is a function, denoted by gof, from X to Z and is defined by.

$$(gof)(x) = g(f(x)) = gf(x) \text{ for all } x \in X$$

Inverse of a Function:

Let f be one-one function from X onto Y. The inverse function of f, denoted by f^{-1}, is a function from Y onto X and is defined by.

$$x = f^{-1}(y), \ \forall\, y \in Y \text{ if and only if } y = f(x), \ \forall\, x \in X$$

EXERCISE 1.2

Q.1 **The real valued functions f and g are defined below. Find**

(a) fog (x) **(b)** gof (x) **(c)** fof (x) **(d)** gog (x)

(i) $f(x) = 2x + 1 \ ; \ g(x) = \frac{3}{x-1}, \quad x \neq 1$

(ii) $f(x) = \sqrt{x+1}$; $g(x) = \frac{1}{x^2}$, $x \neq 0$

(iii) $f(x) = \frac{1}{\sqrt{x-1}}$; $x \neq 1$; $g(x) = (x^2+1)^2$

(iv) $f(x) = 3x^4 - 2x^2$; $g(x) = \frac{2}{\sqrt{x}}$, $x \neq 0$

Solution:

(i) $f(x) = 2x + 1$; $g(x) = \frac{3}{x-1}$, $x \neq 1$

(a) $fog(x) = f(g(x))$

$$= f\left(\frac{3}{x-1}\right)$$

$$= 2\left(\frac{3}{x-1}\right) + 1$$

$$= \frac{6}{x-1} + 1$$

$$= \frac{6 + x - 1}{x-1}$$

$$= \frac{x+5}{x-1} \quad \text{Ans.}$$

(b) $gof(x) = g(f(x))$

$$= g(2x+1)$$

$$= \frac{3}{2x+1-1} = \frac{3}{2x} \quad \text{Ans.}$$

(c) $fof(x) = f(f(x))$

$$= f(2x+1)$$

$$= 2(2x+1) + 1$$

$$= 4x + 2 + 1$$

$$= 4x + 3 \quad \text{Ans.}$$

(d) $gog(x) = g(g(x))$

$$= g\left(\frac{3}{x-1}\right)$$

$$= \frac{3}{\frac{3}{x-1} - 1}$$

$$= \frac{3}{\frac{3-(x-1)}{x-1}}$$

$$= \frac{3(x-1)}{3-x+1}$$

$$= \frac{3(x-1)}{4-x} \quad \text{Ans.}$$

(ii) $\mathbf{f(x) = \sqrt{x+1} \;;\; g(x) = \frac{1}{x^2} \quad , \quad x \neq 0}$

(a) $fog(x) = f(g(x))$

$$= f\left(\frac{1}{x^2}\right)$$

$$= \sqrt{\frac{1}{x^2}+1}$$

$$= \sqrt{\frac{1+x^2}{x^2}} = \frac{\sqrt{1+x^2}}{x} \quad \text{Ans.}$$

(b) $gof(x) = g(f(x))$

$$= g\left(\sqrt{x+1}\right)$$

$$= \frac{1}{(\sqrt{x+1})^2} = \frac{1}{x+1} \quad \text{Ans.}$$

(c) $fof(x) = f(f(x))$

$$= f\left(\sqrt{x+1}\right)$$

$$= \sqrt{\sqrt{x+1}+1} \quad \text{Ans.}$$

(d) $gog(x) = g(g(x))$

$$= g\left(\frac{1}{x^2}\right)$$

$$= \frac{1}{\left(\frac{1}{x^2}\right)^2} = \frac{1}{\frac{1}{x^4}} = x^4 \quad \text{Ans.}$$

(iii) $\mathbf{f(x) = \frac{1}{\sqrt{x-1}} \;;\; x \neq 1 \;;\; g(x) = (x^2+1)^2}$

(a) $fog(x) = f(g(x))$

$$= f((x^2+1)^2)$$

$$= \frac{1}{\sqrt{(x^2+1)^2-1}}$$

$$= \frac{1}{\sqrt{x^4 + 1 + 2x^2 - 1}}$$

$$= \frac{1}{\sqrt{x^2(x^2 + 2)}} = \frac{1}{x\sqrt{x^2 + 2}} \quad \text{Ans.}$$

(b) $gof(x) = g(f(x))$

$$= g\left(\frac{1}{\sqrt{x - 1}}\right)$$

$$= \left[\left(\frac{1}{\sqrt{x - 1}}\right)^2 + 1\right]^2$$

$$= \left(\frac{1}{x - 1} + 1\right)^2 = \left(\frac{1 + x - 1}{x - 1}\right)^2$$

$$= \left(\frac{x}{x - 1}\right)^2 \quad \text{Ans.}$$

(c) $fof(x) = f(f(x))$

$$= f\left(\frac{1}{\sqrt{x - 1}}\right)$$

$$= \frac{1}{\sqrt{\frac{1}{\sqrt{x - 1}} - 1}}$$

$$= \frac{1}{\sqrt{\frac{1 - \sqrt{x - 1}}{\sqrt{x - 1}}}} = \sqrt{\frac{\sqrt{x - 1}}{1 - \sqrt{x - 1}}} \quad \text{Ans.}$$

(d) $gog(x) = g(g(x))$

$= g((x^2 + 1)^2)$

$= [\{(x^2 + 1)^2\}^2 + 1]^2$

$= [(x^2 + 1)^4 + 1]^2$ Ans.

(iv) $\mathbf{f(x) = 3x^4 - 2x^2 \;;\; g(x) = \frac{2}{\sqrt{x}} \;,\; x \neq 0}$

(a) $fog(x) = f(g(x))$

$$= f\left(\frac{2}{\sqrt{x}}\right)$$

$$= 3\left(\frac{2}{\sqrt{x}}\right)^4 - 2\left(\frac{2}{\sqrt{x}}\right)^2$$

$$= 3\left(\frac{16}{x^2}\right) - 2\left(\frac{4}{x}\right)$$

$$= \frac{48}{x^2} - \frac{8}{x}$$

$$= \frac{48 - 8x}{x^2}$$

$$= \frac{8(6 - x)}{x^2} \quad \text{Ans.}$$

(b) $gof(x) = g(f(x))$

$$= g(3x^4 - 2x^2)$$

$$= \frac{2}{\sqrt{3x^4 - 2x^2}}$$

$$= \frac{2}{\sqrt{x^2(3x^2 - 2)}} \quad = \frac{2}{x\sqrt{3x^2 - 2}} \quad \text{Ans.}$$

(c) $fof(x) = f(f(x))$

$$= f(3x^4 - 2x^2)$$

$$= 3(3x^4 - 2x^2)^4 - 2(3x^4 - 2x^2)^2 \quad \text{Ans.}$$

(d) $gog(x) = g(g(x))$

$$= g\left(\frac{2}{\sqrt{x}}\right)$$

$$= \frac{2}{\sqrt{2/\sqrt{x}}}$$

$$= 2\sqrt{\frac{\sqrt{x}}{2}}$$

$$= \sqrt{2} \times \sqrt{2}\ \frac{\sqrt{\sqrt{x}}}{\sqrt{2}}$$

$$= \sqrt{2\sqrt{x}} \quad \text{Ans.}$$

Q.2 For the real valued function, f defined below, find:

(a) $f^{-1}(x)$

(b) $f^{-1}(-1)$ **and verify** $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

(i) $f(x) = -2x + 8$ ***(Lahore Board 2007,2009)*** **(ii)** $f(x) = 3x^3 + 7$

(iii) $f(x) = (-x + 9)^3$ **(iv)** $f(x) = \frac{2x + 1}{x - 1}, \ x > 1$

Solution:

(i) $\mathbf{f(x) = -2x + 8}$

(a) Since $y = f(x)$

$\Rightarrow x = f^{-1}(y)$

Now,

$$f(x) = -2x + 8$$

$$y = -2x + 8$$

$$2x = 8 - y$$

$$x = \frac{8 - y}{2}$$

$$f^{-1}(y) = \frac{8 - y}{2}$$

Replacing y by x

$$f^{-1}(x) = \frac{8 - x}{2}$$

Replacing y by x.

$$\boxed{f^{-1}(x) = \frac{8 - x}{2}}$$

(b) Put, $x = -1$

$$f^{-1}(-1) = \frac{8 - (-1)}{2} = \frac{8 + 1}{2} = \frac{9}{2}$$

$$f\left(f^{-1}(x)\right) = f\left(\frac{8 - x}{2}\right)$$

$$= -2\left(\frac{8 - x}{2}\right) + 8$$

$$= -8 + x + 8$$

$$= x$$

$$f^{-1}\left(f(x)\right) = f^{-1}(-2x + 8)$$

$$= \frac{8 - (-2x + 8)}{2}$$

$$= \frac{8 + 2x - 8}{2}$$

$$= \frac{2x}{2} = x$$

$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$ Hence proved.

(ii) $\quad f(x) = 3x^3 + 7$

(a) Since $y = f(x)$

$\Rightarrow \quad x = f^{-1}(y)$

Now

$$f(x) = 3x^3 + 7$$

$$y = 3x^3 + 7$$

$$3x^3 = y - 7$$

$$x^3 = \frac{y-7}{3}$$

$$x = \left(\frac{y-7}{3}\right)^{\frac{1}{3}}$$

$$f^{-1}(y) = \left(\frac{y-7}{3}\right)^{\frac{1}{3}}$$

Replacing y by x

$$f^{-1}(x) = \left(\frac{x-7}{3}\right)^{\frac{1}{3}}$$

(b) Put $x = -1$

$$f^{-1}(-1) = \left(\frac{-1-7}{3}\right)^{\frac{1}{3}}$$

$$= \left(\frac{-8}{3}\right)^{\frac{1}{3}}$$

$$f\left(f^{-1}(x)\right) = f\left[\left(\frac{x-7}{3}\right)^{\frac{1}{3}}\right]$$

$$= 3\left[\left(\frac{x-7}{3}\right)^{\frac{1}{3}}\right]^3 + 7$$

$$= 3\left(\frac{x-7}{3}\right) + 7$$

$$= x - 7 + 7 = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}(3x^3 + 7)$$

$$= \left(\frac{3x^3 + 7 - 7}{3}\right)^{\frac{1}{3}}$$

$$= \left(\frac{3x^3}{3}\right)^{\frac{1}{3}}$$

$$= (x^3)^{\frac{1}{3}} = x$$

$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$ Hence proved.

(iii) $\mathbf{f(x) = (-x+9)^3}$

(a) Since $y = f(x)$

$x = f^{-1}(y)$

Now

$$f(x) = (-x+9)^3$$

$$y = (-x+9)^3$$

$$y^{\frac{1}{3}} = -x+9$$

$$x = 9 - y^{\frac{1}{3}}$$

Replacing y by x

$$f^{-1}(x) = 9 - x^{\frac{1}{3}}$$

(b) Put $x = -1$

$$f^{-1}(-1) = 9 - (-1)^{\frac{1}{3}}$$

$$f\left(f^{-1}(x)\right) = f\left(9 - x^{\frac{1}{3}}\right)$$

$$= [-(9 - x^{\frac{1}{3}}) + 9]^3$$

$$= (-9 + x^{\frac{1}{3}} + 9)^3$$

$$= \left(x^{\frac{1}{3}}\right)^3 = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}\left((-x+9)^3\right)$$

$$= 9 - [(-x+9)^3]^{\frac{1}{3}}$$

$$= 9 - (-x+9)$$

$$= 9 + x - 9$$

$$= x$$

$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$ Hence proved.

(iv) $f(x) = \dfrac{2x+1}{x-1}$, $x > 1$

(a) Since $y = f(x)$

$x = f^{-1}(y)$

Now

$$f(x) = \frac{2x+1}{x-1}$$

$$y = \frac{2x+1}{x-1}$$

$$y(x-1) = 2x+1$$

$$yx - y = 2x+1$$

$$yx - 2x = 1 + y$$

$$x(y-2) = y+1$$

$$x = \frac{y+1}{y-2}$$

$$f^{-1}(y) = \frac{y+1}{y-2}$$

Replacing y by x

$$f^{-1}(x) = \frac{x+1}{x-2}$$

(b) Put $x = -1$

$$f^{-1}(-1) = \frac{-1+1}{-1-2}$$

$$= \frac{0}{-3} = 0$$

$$f\left(f^{-1}(x)\right) = f\left(\frac{x+1}{x-2}\right)$$

$$= \frac{2\left(\frac{x+1}{x-2}\right)+1}{\frac{x+1}{x-2}-1}$$

$$= \frac{\frac{2(x+1)+(x-2)}{x-2}}{\frac{x+1-(x-2)}{x-2}}$$

$$= \frac{2x + 2 + x - 2}{x + 1 - x + 2}$$

$$= \frac{3x}{3} \quad = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}\left(\frac{2x + 1}{x - 1}\right)$$

$$= \frac{\frac{2x+1}{x-1} + 1}{\frac{2x+1}{x-1} - 2}$$

$$= \frac{\frac{2x + 1 + x - 1}{x - 1}}{\frac{2x + 1 - 2(x - 1)}{x - 1}}$$

$$= \frac{3x}{2x + 1 - 2x + 2}$$

$$= \frac{3x}{3} \quad = x$$

$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$ Hence proved.

Q.3 Without finding the inverse, state the domain and range of f^{-1}.

(i) $f(x) = \sqrt{x + 2}$ **(ii)** $f(x) = \frac{x - 1}{x - 4}, x \neq 4$

(iii) $f(x) = \frac{1}{x + 3}, x \neq -3$ **(iv)** $f(x) = (x - 5)^2, x \geq 5$

Solution:

(i) $f(x) = \sqrt{x + 2}$

Domain of f(x) = $[-2, +\infty)$

Range of f(x) = $[0, +\infty)$

Domain of $f^{-1}(x)$ = Range of f(x) = $[0, +\infty)$

Range of $f^{-1}(x)$ = Domain of f(x) = $[-2, +\infty)$

(ii) $f(x) = \frac{x - 1}{x - 4}, x \neq 4$

Domain of f(x) = $R - \{4\}$

Range of f(x) = $R - \{1\}$

Domain of $f^{-1}(x)$ = Range of f(x) = $R - \{1\}$

Range of $f^{-1}(x)$ = Domain of f(x) = $R - \{4\}$

(iii) $f(x) = \frac{1}{x+3}$, $x \neq -3$

Domain of f(x) $= R - \{-3\}$

Range of f(x) $= R - \{0\}$

Domain of $f^{-1}(x)$ = Range of f(x) $= R - \{0\}$

Range of $f^{-1}(x)$ = Domain of f(x) $= R - \{-3\}$

(iv) $f(x) = (x-5)^2$, $x \geq 5$ ***(Gujranwala Board 2007)***

Domain of f(x) $= [5, +\infty)$

Range of f(x) $= [0, +\infty)$

Domain of $f^{-1}(x)$ = Range of f(x) $= [0, +\infty)$

Range of $f^{-1}(x)$ = Domain of f(x) $= [5, +\infty)$

Limit of a Function:

Let a function f(x) be defined in an open interval near the number 'a' (need not at a) if, as x approaches 'a' from both left and right side of 'a', f(x) approaches a specific number 'L' then 'L', is called the limit of f(x) as x approaches a symbolically it is written as.

$\lim_{x \to a} f(x) = L$ read as "Limit of f(x) as $x \to a$, is L"

Theorems on Limits of Functions:

Let f and g be two functions, for which $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then

Theorem 1: The limit of the sum of two functions is equal to the sum of their limits.

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$$

Theorem 2: The limit of the difference of two functions is equal to the difference of their limits.

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L - M$$

Theorem 3: If K is any real numbers, then.

$$\lim_{x \to a} [kf(x)] = K \lim_{x \to a} f(x) = kL$$

Theorem 4: The limit of the product of the functions is equal to the product of their limits.

$$\lim_{x \to a} [f(x) \,.\, g(x)] = [\lim_{x \to a} f(x)] [\lim_{x \to a} g(x)] = LM$$

Theorem 5: The limit of the quotient of the functions is equal to the quotient of their limits provided the limit of the denominator is non-zero.

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M} \quad , \quad g(x) \neq 0,\ M \neq 0$$

Theorem 6: Limit of $[f(x)]^n$, where n is an integer.

$$\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n = L^n$$

The Sandwich Theorem:

Let f, g and h be functions such that $f(x) \leq g(x) \leq h(x)$ for all number x in some open interval containing "C", except possibly at C itself.

If, $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$

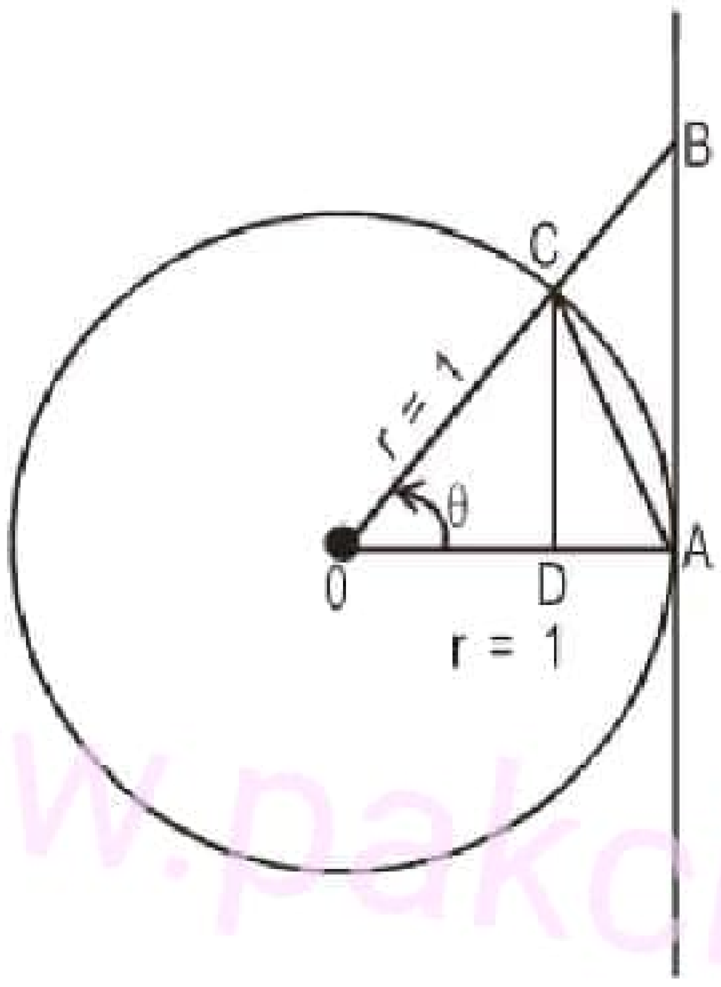
Prove that

If θ is measured in radian, then

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Proof:

Take θ a positive acute central angle of a circle with radius r = 1. OAB represents the sector of the circle.



$|OA| = |OC| = 1$ (radii of unit circle)

From right angle ΔODC

$$\text{Sin}\theta = \frac{|DC|}{|OC|} = |DC| \qquad (\because\ |OC| = 1)$$

From right angle ΔOAB

$$\text{Tan}\theta = \frac{|AB|}{|OA|} = AB \qquad (\because\ |OA| = 1)$$

In terms of θ, the areas are expressed as

$$\text{Area of } \Delta OAC = \frac{1}{2}|OA|\,|CD| = \frac{1}{2}(1)\sin\theta = \frac{1}{2}\sin\theta$$

Area of sector OAC $= \frac{1}{2}\ r^2\theta = \frac{1}{2}\ (1)(\theta) = \frac{1}{2}\theta$

Area of $\Delta OAB = \frac{1}{2}\ |OA|\ |AB| = \frac{1}{2}\ (1)\tan\theta = \frac{1}{2}\tan\theta$

From figure

Area of ΔOAB > Area of sector OAC > Area of ΔOAC

$$\frac{1}{2}\tan\theta > \frac{1}{2}\theta > \frac{1}{2}\sin\theta$$

$$\frac{1}{2}\ \frac{\sin\theta}{\cos\theta} > \frac{\theta}{2} > \frac{\sin\theta}{2}$$

As $\sin\theta$ is positive, so on division by $\frac{1}{2}\sin\theta$, we get.

$$\frac{1}{\cos\theta} > \frac{\theta}{\sin\theta} > 1 \qquad (0 < \theta < \pi/2)$$

i.e.

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

When, $\theta \to 0$, $\cos\theta \to 1$

Since $\frac{\sin\theta}{\theta}$ is sandwitched between 1 and a quantity approaching 1 itself.

So by the sandwitch theorem it must also approach 1.

i.e.

$$\boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1}$$

Theorem: Prove that

$$\lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = e$$

Proof:

Taking

$$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \ldots\ldots\ldots$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots\ldots\ldots$$

Taking $\lim_{n\to+\infty}$ on both sides.

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots\ldots\ldots$$

$$= 1 + 1 + 0.5 + 0.166667 + 0.0416667 + \ldots\ldots\ldots$$

$$= 2.718281 \ldots\ldots\ldots$$

As approximate value of e is $= 2.718281$

$$\therefore \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^2 = e$$

Deduction:

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

We know that.

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Put $x = \frac{1}{n}$ then $\frac{1}{x} = n$

As $n \to +\infty$, $x \to 0$

$$\therefore \lim_{n \to +\infty} (1 + x)^{1/x} = e$$

Theorem:

Prove that:

$$\lim_{x \to a} \frac{a^x - 1}{x} = \log_e a$$

Proof:

Taking,

$$\lim_{x \to a} \frac{a^x - 1}{x}$$

Let $a^x - 1 = y$

$$a^x = 1 + y$$

$$x = \log_a (1 + y)$$

As, $x \to a$, $y \to 0$

$$\lim_{x \to a} \frac{a^x - 1}{x} = \lim_{y \to 0} \frac{y}{\log_a(1 + y)}$$

$$= \lim_{y \to 0} \frac{1}{\frac{1}{y} \log_a(1 + y)} \qquad = \lim_{y \to 0} \frac{1}{\log_a(1 + y)^{\frac{1}{y}}}$$

$$= \frac{1}{\log_a e} \qquad \because \lim_{y \to 0} (1 + y)^{1/y} = e$$

$$= \log_e a$$

Deduction

$$\lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) = \log_e e = 1$$

We know that

$$\lim_{x \to 0} \left(\frac{a^x - 1}{x}\right) = \log_e a$$

Put $a = e$

$$\lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) = \log_e e = 1$$

Important results to remember

(i) $\lim_{x \to +\infty} (e^x) = \infty$ (ii) $\lim_{x \to -\infty} (e^x) = \lim_{x \to -\infty} \left(\frac{1}{e^{-x}}\right) = 0$

(iii) $\lim_{x \to \pm\infty} \left(\frac{a}{x}\right) = 0$, where a is any real number.

EXERCISE 1.3

Q.1 Evaluate each limit by using theorems of limits.

(i) $\lim_{x \to 3} (2x + 4)$ (ii) $\lim_{x \to 1} (3x^2 - 2x + 4)$

(iii) $\lim_{x \to 3} \sqrt{x^2 + x + 4}$ (iv) $\lim_{x \to 2} x\sqrt{x^2 - 4}$

(v) $\lim_{x \to 2} \left(\sqrt{x^3 + 1} - \sqrt{x^2 + 5}\right)$ (iv) $\lim_{x \to 2} \frac{2x^3 + 5x}{3x - 2}$

Solution:

(i)
$$\lim_{x \to 3} (2x + 4) = \lim_{x \to 3} (2x) + \lim_{x \to 3} (4)$$
$$= 2\lim_{x \to 3} x + 4$$
$$= 2(3) + 4 = 6 + 4 = 10 \quad \text{Ans.}$$

(ii)
$$\lim_{x \to 1} (3x^2 - 2x + 4) = \lim_{x \to 1} (3x^2) - \lim_{x \to 1} (2x) + \lim_{x \to 1} (4)$$
$$= 3\lim_{x \to 1} x^2 - 2\lim_{x \to 1} x + 4$$
$$= 3(1)^2 - 2(1) + 4$$
$$= 3 - 2 + 4$$
$$= 5 \quad \text{Ans.}$$

(iii)
$$\lim_{x \to 3} \sqrt{x^2 + x + 4} = [\lim_{x \to 3} (x^2 + x + 4)]^{1/2}$$

Important Limits

I. $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$, where n is integer and $a>0$.

II. $\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}=\frac{1}{2\sqrt{a}}$.

III. $\lim_{n\to 0}\left(1+\frac{1}{n}\right)^n=e$.

IV. $\lim_{x\to\infty}(1+x)^{\frac{1}{x}}=e$.

V. $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$, where $a>0$.

VI. $\lim_{x\to 0}\frac{e^x-1}{x}=\ln e=1$.

VII. If θ is measured in radian, then $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$.

Question # 1

(i) $\lim_{x\to 3}(2x+4)\ =\lim_{x\to 3}(2x)+\lim_{x\to 3}(4)\ =2\lim_{x\to 3}(x)+4\ =2(3)+4\ =10.$

(ii) $\lim_{x\to 1}\left(3x^2-2x+4\right)\ =3(1)^2-2(1)+4\ =3-2+4=5.$

(iii) $\lim_{x\to 3}\sqrt{x^2+x+4}\ =\sqrt{(3)^2+(3)+4}\ =\sqrt{9+3+4}\ =\sqrt{16}=4.$

(iv) $\lim_{x\to 2}x\sqrt{x^2-4}\ =2\sqrt{2^2-4}=0.$

(v) $\lim_{x\to 2}\left(\sqrt{x^3+1}-\sqrt{x^2+5}\right)\ =\lim_{x\to 2}\left(\sqrt{x^3+1}\right)-\lim_{x\to 2}\left(\sqrt{x^2+5}\right)$

$=\left(\sqrt{(2)^3+1}\right)-\left(\sqrt{(2)^2+5}\right)$

$=\sqrt{8+1}-\sqrt{4+5}\ =\sqrt{9}-\sqrt{9}\ =0.$

(vi) $\lim_{x\to -2}\frac{2x^3+5x}{3x-2}\ =\frac{2(-2)^3+5(-2)}{3(-2)-2}\ =\frac{-16-10}{-6-2}\ =\frac{-26}{-8}\ =\frac{13}{4}.$

Question # 2

(i) $\lim_{x\to -1}\frac{x^3-x}{x+1}\ =\lim_{x\to -1}\frac{x(x^2-1)}{x+1}\ =\lim_{x\to -1}\frac{x(x+1)(x-1)}{x+1}$

$=\lim_{x\to -1}x(x-1)\ =(-1)(-1-1)\ =2$

(ii) $\lim_{x\to 0}\left(\frac{3x^3+4x}{x^2+x}\right)\ =\lim_{x\to 0}\frac{x(3x^2+4)}{x(x+1)}\ =\lim_{x\to 0}\frac{3x^2+4}{x+1}\ =\frac{3(0)+4}{0+1}\ =4.$

(iii) $\lim\limits_{x\to 2}\dfrac{x^3-8}{x^2+x-6}$

$=\lim\limits_{x\to 2}\dfrac{x^3-(2)^3}{x^2+3x-2x-6} \quad =\lim\limits_{x\to 2}\dfrac{(x-2)(x^2+2x+4)}{x(x+3)-2(x+3)}$

$=\lim\limits_{x\to 2}\dfrac{(x-2)(x^2+2x+4)}{(x+3)(x-2)} \quad =\lim\limits_{x\to 2}\dfrac{(x^2+2x+4)}{(x+3)}$

$=\dfrac{(2)^2+2(2)+4)}{(2+3)} \quad =\dfrac{12}{5}$

(iv) $\lim\limits_{x\to 1}\dfrac{x^3-3x^2+3x-1}{x^3-x}$

$=\lim\limits_{x\to 1}\dfrac{\left(x-1\right)^3}{x(x^2-1)} \quad =\lim\limits_{x\to 1}\dfrac{\left(x-1\right)^3}{x(x-1)(x+1)}$

$=\lim\limits_{x\to 1}\dfrac{\left(x-1\right)^2}{x(x+1)} \quad =\lim\limits_{x\to 1}\dfrac{\left(1-1\right)^2}{(1)(1+1)} \quad =0$

(v) $\lim\limits_{x\to -1}\left(\dfrac{x^3+x^2}{x^2-1}\right) \quad =\lim\limits_{x\to -1}\dfrac{x^2\left(x+1\right)}{(x+1)(x-1)} \quad =\lim\limits_{x\to -1}\dfrac{x^2}{(x-1)}$

$=\dfrac{\left(-1\right)^2}{(-1-1)}=-\dfrac{1}{2}$

(vi) $\lim\limits_{x\to 4}\dfrac{2x^2-32}{x^3-4x^2} \quad =\lim\limits_{x\to 4}\dfrac{2(x^2-16)}{x^2(x-4)}$

$=\lim\limits_{x\to 4}\dfrac{2(x+4)\left(x-4\right)}{x^2(x-4)} \quad =\lim\limits_{x\to 4}\dfrac{2(x+4)}{x^2}$

$=\dfrac{2(4+4)}{4^2} \quad =\dfrac{16}{16} \quad =1.$

(vii) $\lim\limits_{x\to 2}\dfrac{\sqrt{x}-\sqrt{2}}{x-2} \quad =\lim\limits_{x\to 2}\dfrac{\sqrt{x}-\sqrt{2}}{x-2}\left(\dfrac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}\right)$

$=\lim\limits_{x\to 2}\dfrac{\left(\sqrt{x}\right)^2-\left(\sqrt{2}\right)^2}{\left(x-2\right)\left(\sqrt{x}+\sqrt{2}\right)} \quad =\lim\limits_{x\to 2}\dfrac{x-2}{\left(x-2\right)\left(\sqrt{x}+\sqrt{2}\right)}$

$=\lim\limits_{x\to 2}\dfrac{1}{\sqrt{x}+\sqrt{2}} \quad =\dfrac{1}{\sqrt{2}+\sqrt{2}} \quad =\dfrac{1}{2\sqrt{2}}$

(viii) $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h} \quad =\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$=\lim\limits_{h\to 0}\dfrac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \quad =\lim\limits_{h\to 0}\dfrac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$

$$=\lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \quad =\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{1}{\sqrt{x+0}+\sqrt{x}} \quad =\frac{1}{2\sqrt{x}}$$

(ix) $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x^m-a^m}$

$$=\lim_{x\to a}\frac{(x-a)\left(x^{n-1}+x^{n-2}a+x^{n-3}a^2+....+a^{n-1}\right)}{(x-a)\left(x^{m-1}+x^{m-2}a+x^{m-3}a^2+....+a^{m-1}\right)}$$

$$=\lim_{x\to a}\frac{\left(x^{n-1}+x^{n-2}a+x^{n-3}a^2+....+a^{n-1}\right)}{\left(x^{m-1}+x^{m-2}a+x^{m-3}a^2+....+a^{m-1}\right)}$$

$$=\frac{a^{n-1}+a^{n-2}a+a^{n-3}a^2+....+a^{n-1}}{a^{m-1}+a^{m-2}a+a^{m-3}a^2+....+a^{m-1}}$$

$$=\frac{a^{n-1}+a^{n-1}+a^{n-1}+....+a^{n-1}\ \ (n\text{ terms})}{a^{m-1}+a^{m-1}+a^{m-1}+....+a^{m-1}\ \ (m\text{ terms})}$$

$$=\frac{n\,a^{n-1}}{m\,a^{m-1}} \quad =\frac{n}{m}a^{n-1-m+1} \quad =\frac{n}{m}a^{n-m}$$

Law of Sine

If θ is measured in radian, then $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$

See proof on book at page 25

Question # 3

(i) $\lim\limits_{x\to 0}\dfrac{\sin 7x}{x}$

Put $t=7x \Rightarrow \dfrac{t}{7}=x$

When $x\to 0$ then $t\to 0$, so

$$\lim_{x\to 0}\frac{\sin 7x}{x} = \lim_{t\to 0}\frac{\sin t}{t/7}$$

$$= 7\lim_{t\to 0}\frac{\sin t}{t} = 7(1) = 7 \qquad \text{By law of sine.}$$

(ii) $\lim\limits_{x\to 0}\dfrac{\sin x^\circ}{x}$

Since $180^\circ=\pi$ rad $\Rightarrow 1^\circ=\dfrac{\pi}{180}$ rad $\Rightarrow x^\circ=\dfrac{x\pi}{180}$ rad

So $\lim\limits_{x\to 0}\dfrac{\sin x^\circ}{x} = \lim\limits_{x\to 0}\dfrac{\sin \pi x/180}{x}$

Now put $\frac{\pi x}{180}=t$ i.e. $x=\frac{180t}{\pi}$

When $x\to 0$ then $t\to 0$, so

$$\lim_{x\to 0}\frac{\sin \pi x/180}{x} = \lim_{x\to 0}\frac{\sin t}{180t/\pi}$$

$$= \frac{\pi}{180}\lim_{x\to 0}\frac{\sin t}{t} = \frac{\pi}{180}(1) = \frac{\pi}{180}$$ by law of sine

(iii) $$\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta} = \lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}$$

$$= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)} = \lim_{\theta\to 0}\frac{\sin^2\theta}{\sin\theta(1+\cos\theta)}$$

$$= \lim_{\theta\to 0}\frac{\sin\theta}{(1+\cos\theta)} = \frac{\sin(0)}{1+\cos(0)} = \frac{0}{1+1} = 0$$

(iv) $$\lim_{x\to\pi}\frac{\sin x}{\pi-x}$$

Put $t=\pi-x \Rightarrow x=\pi-t$

When $x\to\pi$ then $t\to 0$, so

$$\lim_{x\to\pi}\frac{\sin x}{\pi-x} = \lim_{t\to 0}\frac{\sin(\pi-t)}{t}$$

$$= \lim_{t\to 0}\frac{\sin t}{t}$$ $\because \sin(\pi-t)=\sin\left(2\cdot\frac{\pi}{2}-t\right)=\sin t$

$= 1$ By law of sine.

(v) $$\lim_{x\to 0}\frac{\sin ax}{\sin bx} = \lim_{x\to 0}\sin ax\cdot\frac{1}{\sin bx}$$

$$= \lim_{x\to 0}\sin ax\cdot\left(\frac{ax}{ax}\right)\frac{1}{\sin bx\cdot\left(\frac{bx}{bx}\right)} = \lim_{x\to 0}\frac{\sin ax}{ax}\cdot ax\frac{1}{\frac{\sin bx}{bx}\cdot bx}$$

$$= \frac{a}{b}\lim_{x\to 0}\frac{\sin ax}{ax}\cdot\frac{1}{\lim_{x\to 0}\frac{\sin bx}{bx}} = \frac{a}{b}\cdot(1)\cdot\frac{1}{(1)} = \frac{a}{b}$$ by law of sine

(vi) $$\lim_{x\to 0}\frac{x}{\tan x} = \lim_{x\to 0}\frac{x}{\frac{\sin x}{\cos x}} = \lim_{x\to 0}\frac{x}{\sin x}\cdot\cos x$$

$$= \lim_{x\to 0}\frac{1}{\frac{\sin x}{x}}\cdot\cos x = \frac{1}{\lim_{x\to 0}\frac{\sin x}{x}}\cdot\lim_{x\to 0}\cos x = \frac{1}{1}\cdot 1 = 1$$

(vii) $$\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$$

$$= \lim_{x\to 0}\frac{2\sin^2 x}{x^2}$$

$\because \sin^2 x = \frac{1-\cos 2x}{2}$

$\therefore 2\sin^2 x = 1-\cos 2x$

$$= 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 = 2\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2 = 2(1)^2 = 2$$

(vii) *Do yourself by rationalizing*

(viii) $\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\sin\theta$

$$= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\sin\theta = (1)\cdot(0) = 0$$

(x) $\lim_{x\to 0}\frac{\sec x - \cos x}{x}$

$$= \lim_{x\to 0}\frac{\frac{1}{\cos x} - \cos x}{x} = \lim_{x\to 0}\frac{\frac{1-\cos^2 x}{\cos x}}{x}$$

$$= \lim_{x\to 0}\frac{1-\cos^2 x}{x\cos x} = \lim_{x\to 0}\frac{\sin^2 x}{x\cos x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{\cos x}$$

$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{\cos x} = (1)\frac{\sin(0)}{\cos(0)} = (1)\cdot\frac{0}{1} = 0$$

(xi) $\lim_{x\to 0}\frac{1-\cos p\theta}{1-\cos q\theta}$

$$= \lim_{x\to 0}\frac{2\sin^2\frac{p\theta}{2}}{2\sin^2\frac{q\theta}{2}} \qquad \because \sin^2\frac{x}{2} = \frac{1-\cos x}{2}$$

$$= \lim_{x\to 0}\sin^2\frac{p\theta}{2}\cdot\frac{1}{\sin^2\frac{q\theta}{2}} = \lim_{x\to 0}\sin^2\frac{p\theta}{2}\cdot\frac{\left(\frac{p\theta}{2}\right)^2}{\left(\frac{p\theta}{2}\right)^2}\cdot\frac{1}{\sin^2\frac{q\theta}{2}\cdot\frac{\left(\frac{q\theta}{2}\right)^2}{\left(\frac{q\theta}{2}\right)^2}}$$

$$= \lim_{x\to 0}\frac{\sin^2\frac{p\theta}{2}}{\left(\frac{p\theta}{2}\right)^2}\cdot\left(\frac{p\theta}{2}\right)^2\cdot\frac{1}{\frac{\sin^2\frac{q\theta}{2}}{\left(\frac{q\theta}{2}\right)^2}\cdot\left(\frac{q\theta}{2}\right)^2} = \lim_{x\to 0}\left(\frac{\sin\frac{p\theta}{2}}{\frac{p\theta}{2}}\right)^2\cdot\frac{1}{\left(\frac{\sin\frac{q\theta}{2}}{\frac{q\theta}{2}}\right)^2}\cdot\frac{p^2\theta^2/4}{q^2\theta^2/4}$$

$$= \frac{p^2}{q^2}\left(\lim_{x\to 0}\frac{\sin\frac{p\theta}{2}}{\frac{p\theta}{2}}\right)^2\cdot\frac{1}{\left(\lim_{x\to 0}\frac{\sin\frac{q\theta}{2}}{\frac{q\theta}{2}}\right)^2} = \frac{p^2}{q^2}(1)^2\cdot\frac{1}{(1)^2} = \frac{p^2}{q^2}$$

(xii) $\lim\limits_{\theta\to 0}\dfrac{\tan\theta-\sin\theta}{\sin^3\theta}$

$$= \lim_{\theta\to 0}\frac{\frac{\sin\theta}{\cos\theta}-\sin\theta}{\sin^3\theta} \quad = \lim_{\theta\to 0}\frac{\frac{\sin\theta-\sin\theta\cos\theta}{\cos\theta}}{\sin^3\theta}$$

$$= \lim_{\theta\to 0}\frac{\sin\theta-\sin\theta\cos\theta}{\sin^3\theta\cos\theta} \quad = \lim_{\theta\to 0}\frac{\sin\theta(1-\cos\theta)}{\sin^3\theta\cos\theta}$$

$$= \lim_{\theta\to 0}\frac{1-\cos\theta}{\sin^2\theta\cos\theta} \quad = \lim_{\theta\to 0}\frac{1-\cos\theta}{\sin^2\theta\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}$$

$$= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\sin^2\theta\cos\theta(1+\cos\theta)} \quad = \lim_{\theta\to 0}\frac{\sin^2\theta}{\sin^2\theta\cos\theta(1+\cos\theta)}$$

$$= \lim_{\theta\to 0}\frac{1}{\cos\theta(1+\cos\theta)} \quad = \lim_{x\to 0}\frac{1}{\cos\theta(1+\cos\theta)}$$

$$= \frac{1}{\cos(1)(1+\cos(1))} \quad = \frac{1}{1\cdot(1+1)} \quad = \frac{1}{2}$$

Note:

a) $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$

b) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$ where $e = 2.718281...$

See proof of (a) and (b) on book at page 23

c) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x} = \log_e a$ or $\ln a$

Proof:

Put $y = a^x - 1$ .... (i)

When $x\to 0$ then $y\to 0$

Also from (i) $\quad 1+y = a^x$

Taking log on both sides

$$\ln(1+y) = \ln a^x \quad \Rightarrow \ln(1+y) = x\ln a \qquad \because \ln x^m = m\ln x$$

$$\Rightarrow x = \frac{\ln(1+y)}{\ln a}$$

Now $\lim\limits_{x\to 0}\dfrac{a^x-1}{x} = \lim\limits_{y\to 0}\dfrac{y}{\frac{\ln(1+y)}{\ln a}}$

$$= \lim_{y\to 0}\frac{y\ln a}{\ln(1+y)} \quad = \lim_{y\to 0}\frac{\ln a}{\frac{1}{y}\ln(1+y)}$$

$$= \lim_{y\to 0}\frac{\ln a}{\ln(1+y)^{\frac{1}{y}}} = \frac{\ln a}{\lim_{y\to 0}\ln(1+y)^{\frac{1}{y}}} \qquad \because \ln x^m = m\ln x$$

$$= \frac{\ln a}{\ln\left(\lim_{y\to 0}(1+y)^{\frac{1}{y}}\right)} = \frac{\ln a}{\ln(e)} \qquad \because \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

$$= \frac{\ln a}{1} = \ln a \qquad \because \ln e = 1$$

Question # 4

(i) $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{2n} = \left[\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n}\right]^2 = e^2$

(ii) $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{2}} = \left[\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{\frac{1}{2}} = e^{\frac{1}{2}} = \sqrt{e}$

(iii) $\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{n} = \left[\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{-n}\right]^{-1} = e^{-1} = \frac{1}{e}$

(iv) $\lim_{n\to+\infty}\left(1+\frac{1}{3n}\right)^{n}$

$$= \lim_{n\to+\infty}\left(1+\frac{1}{3n}\right)^{\frac{3n}{3}} = \left[\lim_{n\to+\infty}\left(1+\frac{1}{3n}\right)^{3n}\right]^{\frac{1}{3}} = e^{\frac{1}{3}}$$

(v) $\lim_{n\to+\infty}\left(1+\frac{4}{n}\right)^{n} = \lim_{n\to+\infty}\left(1+\frac{4}{n}\right)^{\frac{4n}{4}} = \left[\lim_{n\to+\infty}\left(1+\frac{4}{n}\right)^{\frac{n}{4}}\right]^{4} = e^4.$

(vi) $\lim_{x\to 0}(1+3x)^{\frac{2}{x}}$

$$= \lim_{x\to 0}(1+3x)^{\frac{6}{3x}} = \left[\lim_{x\to 0}(1+3x)^{\frac{1}{3x}}\right]^6 = e^6$$

(vii) $\lim_{x\to 0}(1+2x^2)^{\frac{1}{x^2}} = \lim_{x\to 0}(1+2x^2)^{\frac{2}{2x^2}} = \left[\lim_{x\to 0}(1+2x^2)^{\frac{1}{2x^2}}\right]^2 = e^2$

(viii) $\lim_{h\to 0}(1-2h)^{\frac{1}{h}} = \lim_{h\to 0}(1-2h)^{\frac{-2}{-2h}} = \left[\lim_{h\to 0}(1-2h)^{\frac{1}{-2h}}\right]^{-2} = e^{-2} = \frac{1}{e^2}$

(ix) $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^{x}$

$$= \lim_{x\to\infty}\left(\frac{1+x}{x}\right)^{-x} = \lim_{x\to\infty}\left(\frac{1}{x}+\frac{x}{x}\right)^{-x} = \lim_{x\to\infty}\left(\frac{1}{x}+1\right)^{-x}$$

$$=\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{-1} = e^{-1} = \frac{1}{e}$$

(x) $\lim_{x\to 0}\dfrac{e^{1/x}-1}{e^{1/x}+1}$; $x<0$

Put $x=-t$ where $t>0$

When $x\to 0$ then $t\to 0$, so

$$\lim_{x\to 0}\frac{e^{1/x}-1}{e^{1/x}+1} = \lim_{t\to 0}\frac{e^{-1/t}-1}{e^{-1/t}+1} = \frac{e^{-1/0}-1}{e^{-1/0}+1}$$

$$=\frac{e^{-\infty}-1}{e^{-\infty}+1} = \frac{0-1}{0+1} \qquad \because\ e^{-\infty}=\frac{1}{e^{\infty}}=\frac{1}{\infty}=0$$

$$=-1$$

(xi) $\lim_{x\to 0}\dfrac{e^{1/x}-1}{e^{1/x}+1}$; $x>0$

$$=\lim_{x\to 0}\frac{e^{1/x}\left(1-\frac{1}{e^{1/x}}\right)}{e^{1/x}\left(1+\frac{1}{e^{1/x}}\right)} = \lim_{x\to 0}\frac{\left(1-\frac{1}{e^{1/x}}\right)}{\left(1+\frac{1}{e^{1/x}}\right)}$$

$$=\frac{1-\frac{1}{e^{1/0}}}{1+\frac{1}{e^{1/0}}} = \frac{1-\frac{1}{e^{\infty}}}{1+\frac{1}{e^{\infty}}} = \frac{1-\frac{1}{\infty}}{1+\frac{1}{\infty}} = \frac{1-0}{1+0} = 1$$

Question # 1:

$(i)\qquad f(x)=2x^2+x-5 \qquad\qquad c=1$

$$\lim_{x\to1^-} f(x)=\lim_{x\to1^-}\left(2x^2+x-5\right)=2(1)^2+1-5=2+1-5=-2$$

$$\lim_{x\to1^+} f(x)=\lim_{x\to1^+}\left(2x^2+x-5\right)=2(1)^2+1-5=2+1-5=-2$$

$$\Rightarrow\qquad \lim_{x\to1^{-1}} f(x)=\lim_{x\to1^+} f(x)=-2 \qquad\qquad \therefore \qquad \lim_{x\to1} f(x)=-2$$

$(ii)\qquad f(x)=\dfrac{x^2-9}{x-3} \qquad\qquad C=-3$

$$\lim_{x\to-3^-} f(x)=\lim_{x\to-3^-}\frac{x^2-9}{x-3}=\frac{\lim\limits_{x\to-3^-}\left(x^2-9\right)}{\lim\limits_{x\to-3^-}(x-3)}=\frac{(-3)^2-9}{-3-3}=\frac{9-9}{-6}=\frac{0}{-6}=0$$

$$Now\qquad \lim_{x\to-3^+} f(x)=\lim_{x\to-3^+}\frac{x^2-9}{x-3}=\frac{\lim\limits_{x\to-3^+}\left(x^2-9\right)}{\lim\limits_{x\to-3^+}(x-3)}=\frac{(-3)^2-9}{-3-3}=\frac{9-9}{-6}=\frac{0}{-6}=0$$

$$\Rightarrow\qquad \lim_{x\to-3^-} f(x)=\lim_{x\to-3^+} f(x)=0 \qquad\qquad \therefore \qquad \lim_{x\to-3} f(x)=0$$

$(iii)\qquad f(x)=|x-5| \qquad\qquad C=5$

$$\lim_{x\to5^-} f(x)=\lim_{x\to5^-}|x-5|$$

$|x-5|=\pm(x-5)$

$-(x-5)$ $+(x-5)$

$-\infty$ 5 $+\infty$

$$=\lim_{x\to5^-}\left[-(x-5)\right]=-\lim_{x\to5}(x-5)=-(5-5)=0$$

$$\lim_{x\to5^+} f(x)=\lim_{x\to5^+}|x-5|=\lim_{x\to5^+}(x-5)=5-5=0$$

$$\Rightarrow\qquad \lim_{x\to5^-} f(x)=\lim_{x\to5^+} f(x)=0$$

$$\lim_{x\to5} f(x)=0$$

Question # 2:

Discuss the continuity of $\quad f(x) \quad at \quad x=c$

$$(i)\qquad f(x)=\begin{cases} 2x+5 & if \quad x\le 2\\ 4x+1 & if \quad x>2 \\ & c=2\end{cases}$$

We have to discuss the continuity of $f(x) \quad at \quad x=2$

$(a)\qquad f(2)=2(2)+5=4+5=9 \qquad\qquad ....(1)$

$(b)\qquad \lim\limits_{x\to2} f(x)=?$

$f(x)=2x+5$ $f(x)=4x+1$

$-\infty$ 2 $+\infty$

$$\lim_{x\to2^-} f(x)=\lim_{x\to2}(2x+5)=2(2)+5=4+5=9$$

$$and\qquad \lim_{x\to2^+} f(x)=\lim_{x\to2}(4x+1)=4(2)+1=8+1=9$$

$$\therefore\qquad \lim_{x\to2} f(x)=9 \qquad ....(2)$$

(c) *from* (1) *and* (2) *we get*

$$\lim_{x\to 2} f(x) = f(2)$$

$\therefore$ $f(x)$ *is continous at* $x = 2$

$$(ii)\qquad f(x) = \begin{cases} 3x-1 & \text{if } x<1 \\ 4 & x=1 \quad c=2 \\ 2x & x>1 \end{cases}$$

if $c = 2$ $\quad f(c) = f(2)$

is not defined so given function is discontinous

(ii) *Correction*

$$f(x) = \begin{cases} 3x-1 & \text{if } x<1 \\ 4 & \text{if } x=1 \\ 2x & \text{if } x>1 \end{cases}$$

$c = 1$ $(correction)$

$f(1) = 4$

$-\infty \quad 1 \quad +\infty$

$f(x) = 3x-1 \qquad f(x) = 2x$

(a) $f(1) = 4$ $(given)$

(b) $\lim_{x\to 1} f(x) = ?$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(3x-1) = 3(1)-1 = 2$$

and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(2x) = 2(1) = 2$

$\Rightarrow$ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2$

$\therefore$ $\lim_{x\to 1} f(x) = 2$ ....(2)

(c) *From* (1) *and* (2) *we get*

$$\lim_{x\to 1} f(x) \neq f(1)$$

$\therefore$ $f(x)$ *is discontinous at* $x = 1$

$$(iii)\qquad f(x) = \begin{cases} 3x-1 & \text{if } x<1 \\ \\ 2x & \text{if } x>1 \end{cases} \qquad c=1$$

(a) $f(1)$ *is not defined*

$\therefore$ $f(x)$ *is discontinous at* $x = 1$

Question # 3:

Given that

$$f(x) = \begin{cases} 3x & \text{if} & x \le -2 \\ x^2-1 & \text{if} & -2 < x < 2 \\ 3 & \text{if} & x \ge 2 \end{cases}$$

$-\infty \qquad +\infty$

$f(x) = 3x \quad -2 \quad f(x) = x^2-1 \quad 2 \quad f(x) = 3$

(i) *We check continuity at* $x = 2$

(a) $f(2) = 3$ ....(1) $(given)$

$(b)\quad \lim_{x\to 2} f(x) = ?$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2}(x^2-1) = (2)^2 - 1 = 4 - 1 = 3$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2}(3) = 3$

$\Rightarrow\quad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 3$

$\therefore\quad \lim_{x\to 2} f(x) = 3 \quad ....(2)$

$(c)\quad$ *From* (1) *and* (2), *we get*

$\lim_{x\to 2} f(x) = f(2)$

$\therefore\quad f(x)$ *is continuous at* $x = 2$

$(ii)\quad$ *At* $x = -2$

$(a)\quad f(-2) = 3(-2) = -6 \quad ....(1)$

$(b)\quad \lim_{x\to -2} f(x) = ?$

$\lim_{x\to -2^-} f(x) = \lim_{x\to -2}(3x) = 3(-2) = -6$

and $\lim_{x\to -2^+} f(x) = \lim_{x\to -2}(x^2-1) = (-2)^2 - 1 = 4 - 1 = 3$

$\Rightarrow\quad \lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x) \qquad \Rightarrow \lim_{x\to -2} f(x)$ *does not exist*

$\therefore\quad f(x)$ *is discontinuous at* $x = -2$

Question # 4:

Given that

$$f(x) = \begin{cases} x+2 & x \le -1 \\ c+2 & x > -1 \end{cases}$$

$c = ?$

$-\infty \qquad\qquad\qquad\qquad\qquad\qquad +\infty$

$f(x) = x+2 \qquad\qquad -1 \qquad\qquad f(x) = c+2$

$\because\quad \lim_{x\to -1} f(x)$ *exists*

$\therefore\quad \lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$

$\Rightarrow\quad \lim_{x\to -1}(x+2) = \lim_{x\to -1}(c+2)$

$\Rightarrow\quad -1+2 = c+2$

$\Rightarrow\quad 1 = c+2$

$\Rightarrow\quad c = 1-2 \qquad \Rightarrow \qquad c = -1$

Question # 5:

(i)

$$f(x) = \begin{cases} mx & if \quad x < 3 \\ n & if \quad x = 3 \\ -2x+9 & if \quad x > 3 \end{cases}$$

here $\quad f(3) = n \quad$ *(given)*

$\because\quad f(x)$ *is continuous at* $x = 3$

$\therefore\quad \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$

$\Rightarrow\quad \lim_{x\to 3}(mx) = \lim_{x\to 3}(-2x+9) = n$

$\Rightarrow \quad (m)(3) = -2(3) + 9 = n$

$\Rightarrow \quad 3m = -6 + 9 = n$

$\Rightarrow \quad 3m = 3 = n$

$\Rightarrow \quad 3m = 3 \quad , \quad n = 3$

$\Rightarrow \quad m = 1 \quad , \quad n = 3$

$$(ii) \quad f(x) = \begin{cases} mx & if \quad x < 4 \\ x^2 & if \quad x \geq 4 \end{cases}$$

$here \quad f(4) = (4)^2 = 16$

$\because \quad f(x) \quad \text{is continuous at} \quad x = 4$

$\because \quad \lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) = f(4)$

$\Rightarrow \quad \lim_{x \to 4} (mx) = \lim_{x \to 4} (x^2) = 16$

$\Rightarrow \quad 4m = (4)^2 = 16$

$\Rightarrow \quad 4m = 16 = 16 \quad \Rightarrow \quad 4m = 16$

$\Rightarrow \quad m = 4$

Question # 6:

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Given that

$$f(x) = \begin{cases} \dfrac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} & x \neq 2 \\ K & x = 2 \end{cases}$$

$K = ?$

$here \quad f(2) = K \quad given$

$\because \quad f(x) \quad \text{is continuous at} \quad x = 2$

$\because \quad \lim_{x \to 2} f(x) = f(2)$

$$\lim_{x \to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} = K \quad \Rightarrow \quad \lim_{x \to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} \frac{\sqrt{2x+5} + \sqrt{x+7}}{\sqrt{2x+5} + \sqrt{x+7}} = K$$

$$\Rightarrow \quad \lim_{x \to 2} \frac{\left(\sqrt{2x+5}\right)^2 - \left(\sqrt{x+7}\right)^2}{(x-2)\left(\sqrt{2x+5} + \sqrt{x+7}\right)} = K \quad \Rightarrow \quad \frac{(2x+5) - (x+7)}{(x-2)\left(\sqrt{2x+5} + \sqrt{x+7}\right)} = K$$

$$\Rightarrow \quad \frac{2x+5-x-7}{(x-2)\left(\sqrt{2x+5} + \sqrt{x+7}\right)} = K \quad \Rightarrow \quad \frac{(x-2)}{(x-2)\left(\sqrt{2x+5} + \sqrt{x+7}\right)} = K$$

$$\Rightarrow \quad \lim_{x \to 2} \frac{1}{\sqrt{2x+5} + \sqrt{x+7}} = K \quad \Rightarrow \quad \frac{1}{\lim_{x \to 2} \left[\sqrt{2x+5} + \sqrt{x+7}\right]} = K$$

$$\Rightarrow \quad \frac{1}{\sqrt{2(2)+5} + \sqrt{2+7}} = K \quad \Rightarrow \quad \frac{1}{\sqrt{9} + \sqrt{9}} = K$$

$$\Rightarrow \quad \frac{1}{3+3} = K$$

$$\Rightarrow \quad K = \frac{1}{6}$$

EXERCISE 1.5

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Q.1 Draw the graphs of the following equations.

(i) $x^2 + y^2 = 9$ **(ii)** $\frac{x^2}{16} + \frac{y^2}{4} = 1$

(iii) $y = e^{2x}$ **(iv)** $y = 3^x$

Solution:

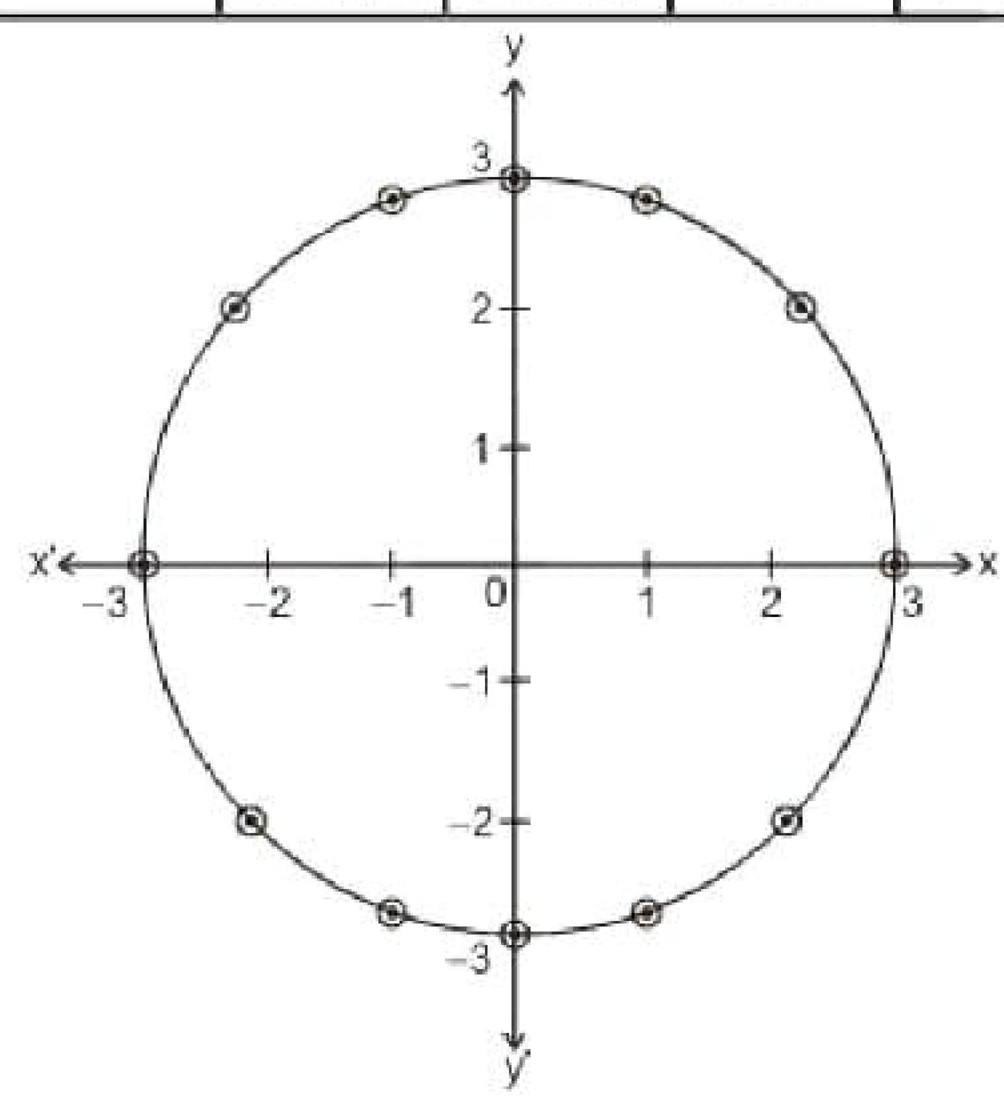
(i) $x^2 + y^2 = 9$

$$y^2 = 9 - x^2$$

$$y = \pm\sqrt{9 - x^2}$$

Its domain is $-3 \le x \le 3+$

x	−3	−2	−1	0	1	2	3
$y = \pm\sqrt{9 - x^2}$	0	± 2.2	± 2.8	± 3	± 2.8	± 2.2	0



(ii) $\frac{x^2}{16} + \frac{y^2}{4} = 1$

$$\frac{y^2}{4} = 1 - \frac{x^2}{16}$$

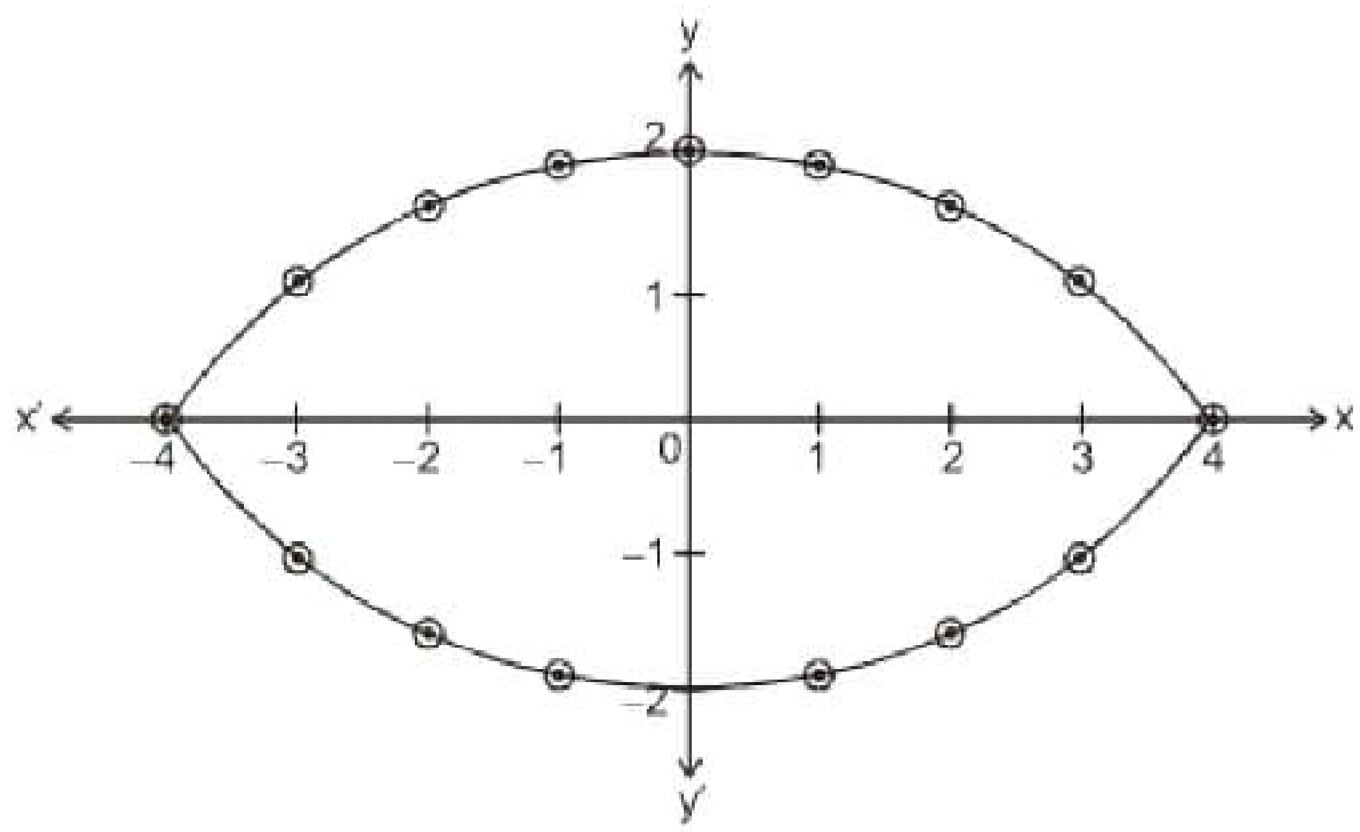
$$y^2 = 4\left(\frac{16 - x^2}{16}\right)$$

$$y^2 = \frac{16 - x^2}{4}$$

$$y = \pm\frac{\sqrt{16 - x^2}}{2}$$

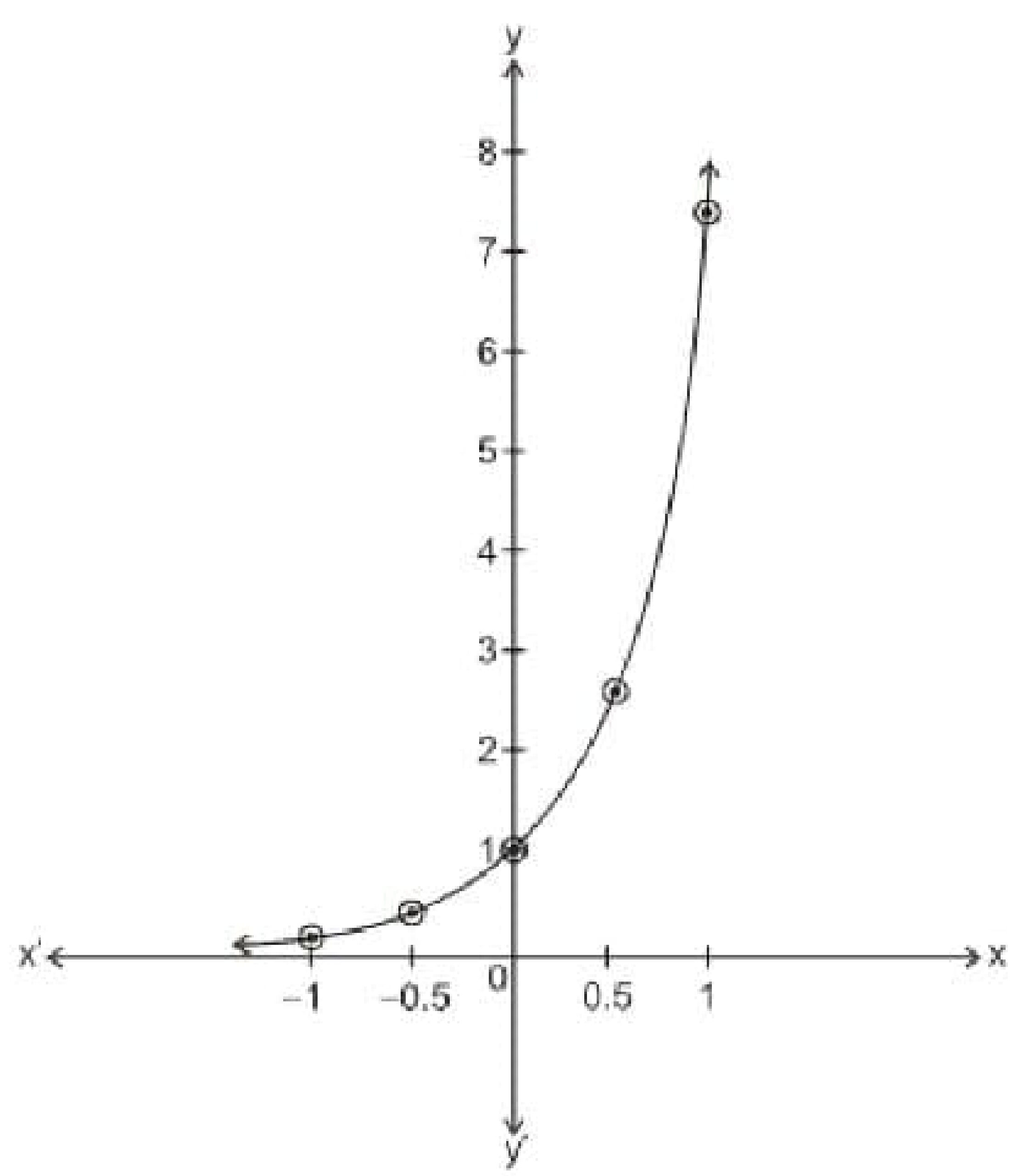
Its domain is $-4 \le x \le 4$.

x	−4	−3	−2	−1	0	1	2	3	4
$y = \pm \frac{\sqrt{9 - x^2}}{2}$	0	± 1.3	± 1.7	± 1.9	± 2	± 1.9	± 1.7	± 1.3	0



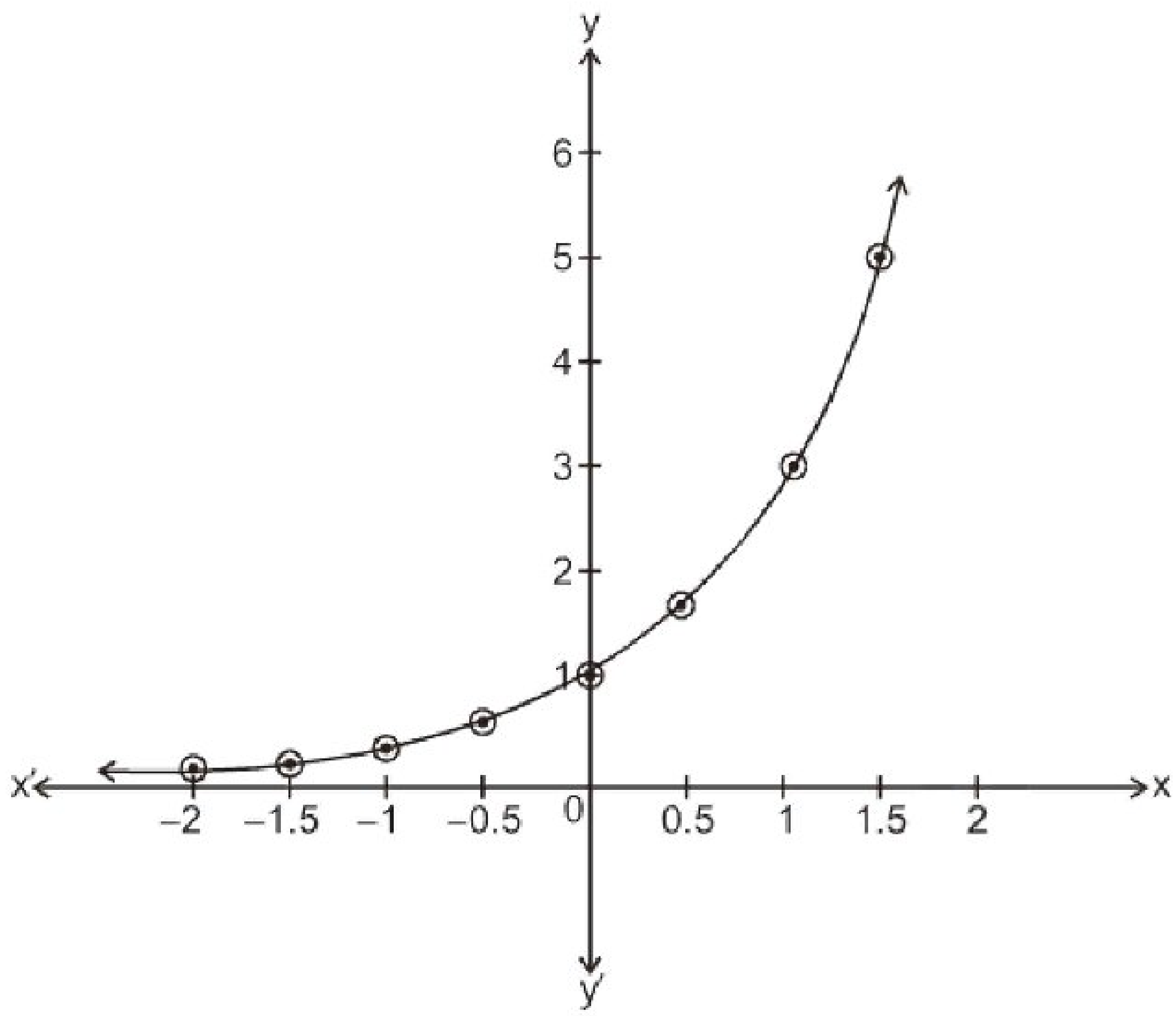
(iii) $y = e^{2x}$

x	−1	−0.5	0	0.5	1
$y = e^{2x}$	0.1	0.4	1	2.7	7.4



(iv) $y = 3^x$

x	−2	−1.5	−1	−0.5	0	0.5	1	1.5
$y = 3^x$	0.1	0.2	0.3	0.6	1	1.7	3	5.2



Q.2 Graph the curves that has the parametric equations given below.

(i) $x = t \quad , \quad y = t^2 \ , \quad -3 \le t \le 3$ **where 't' is a parameter**

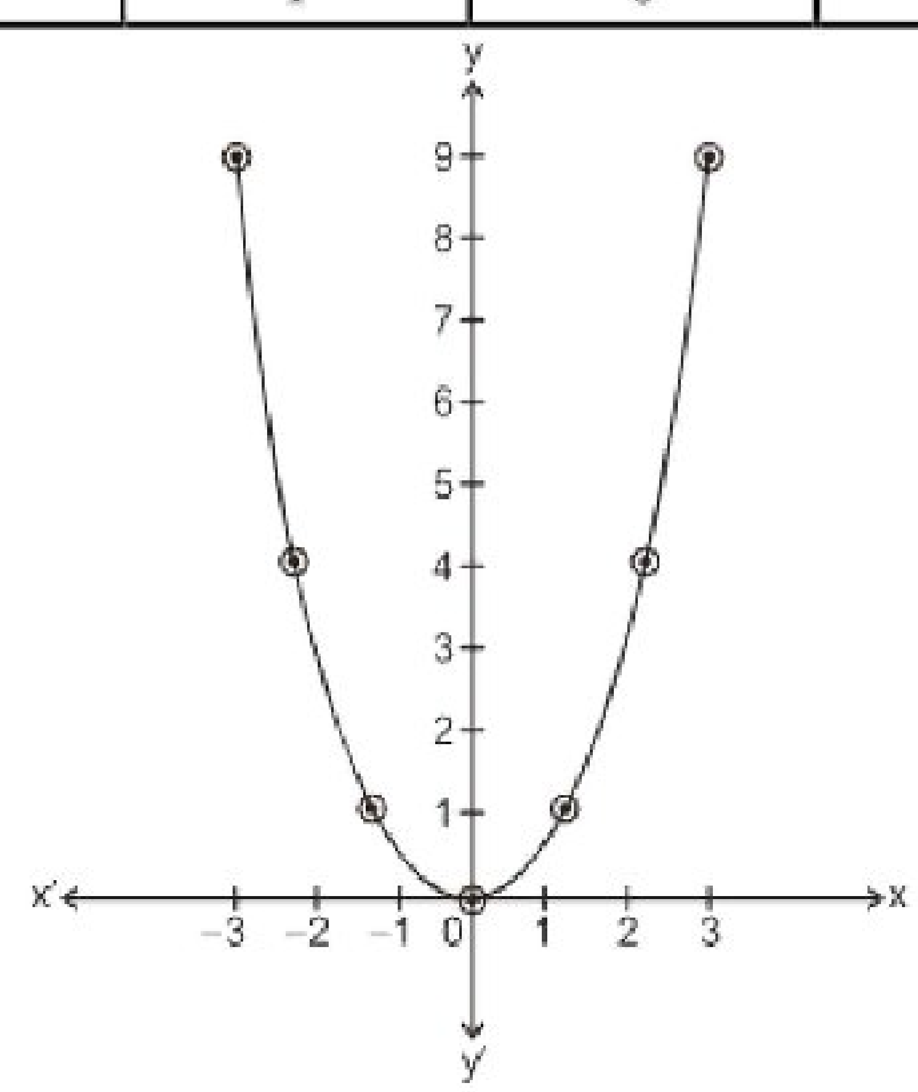
(ii) $x = t - 1 \ , \quad y = 2t - 1, \quad -1 < t < 5$ **where 't' is a parameter**

(iii) $x = \sec\theta \ , \quad y = \tan\theta$ **where 'θ' is a parameter**

Solution:

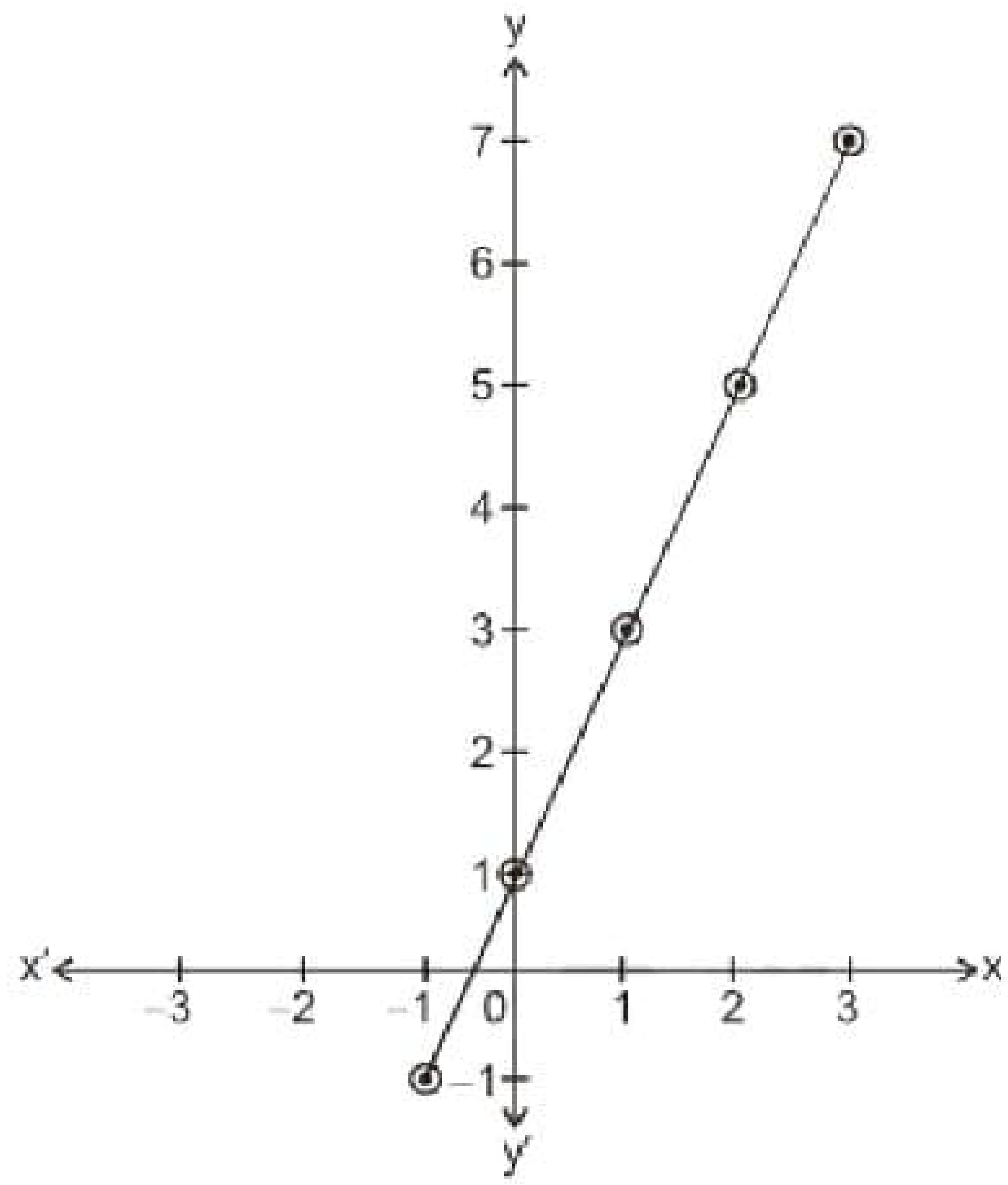
(i) $x = t \ , \ y = t^2 \ , \ -3 \le t \le 3$ where 't' is a parameter

t	−3	−2	−1	0	1	2	3
x = t	−3	−2	−1	0	1	2	3
$y = t^2$	9	4	1	0	1	4	9



(ii) $x = t - 1$, $y = 2t - 1$, $-1 < t < 5$ where 't' is a parameter

t	0	1	2	3	4
$x = t - 1$	-1	0	1	2	3
$y = 2t - 1$	-1	1	3	5	7



(iii) $x = \sec\theta$, $y = \tan\theta$ where 'θ' is a parameter

$x^2 = \sec^2\theta$, $y^2 = \tan^2\theta$

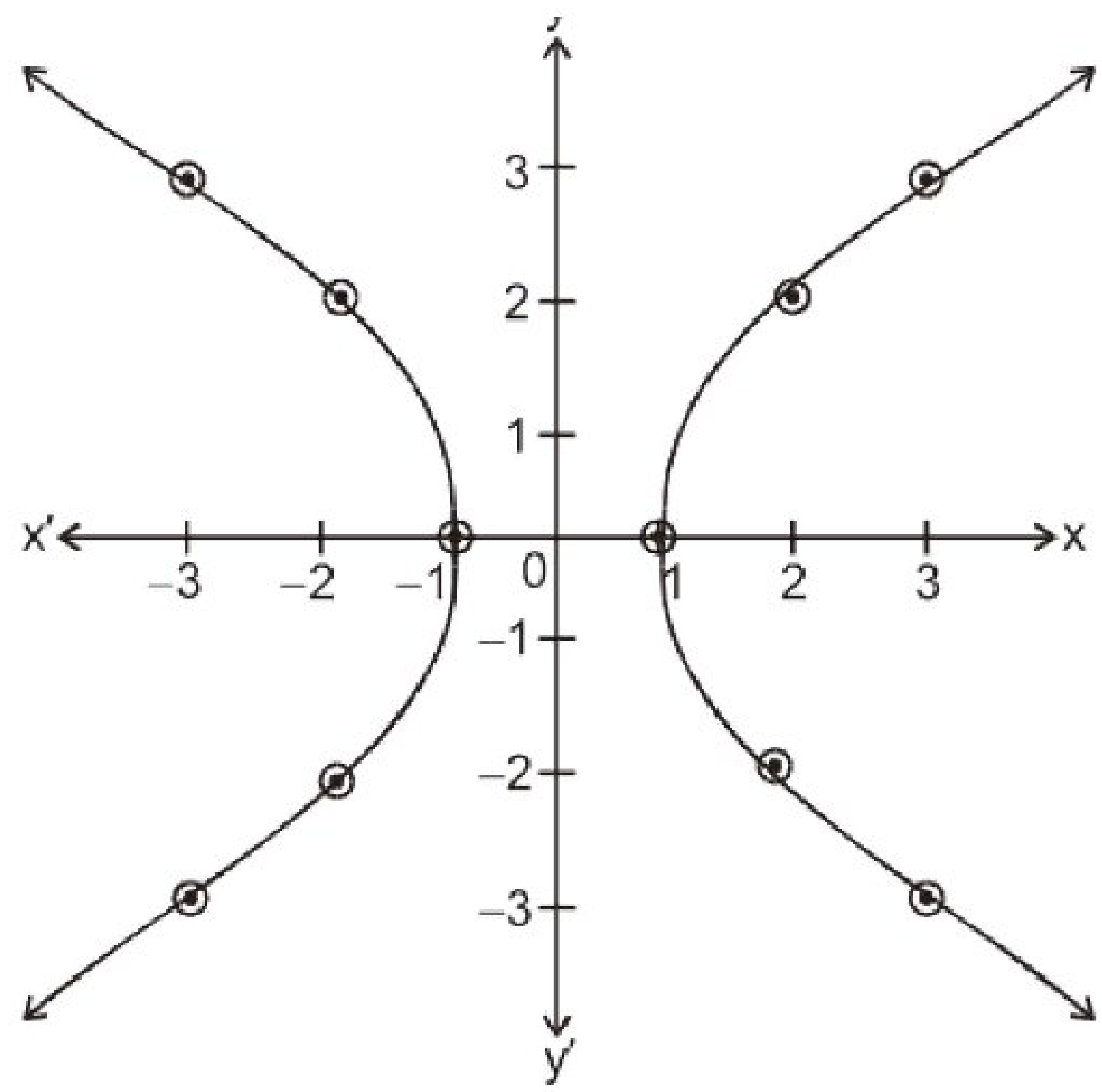
$x^2 - y^2 = \sec^2\theta - \tan^2\theta$

$x^2 - y^2 = 1$ $(\because\ 1 + \tan^2\theta = \sec^2\theta \Rightarrow 1 = \sec^2\theta - \tan^2\theta)$

$y^2 = x^2 - 1$

$y = \pm\sqrt{x^2 - 1}$

x	-3	-2	-1	1	2	3
$y = \sqrt{x^2 - 1}$	± 2.8	± 1.7	0	0	± 1.7	± 2.8



Q.3 **Draw the graphs of the functions defined below and find whether they are continuous.**

(i) $y = \begin{cases} x-1 & \text{if } x < 3 \\ 2x+1 & \text{if } x \geq 3 \end{cases}$ **(ii)** $y = \dfrac{x^2-4}{x-2}$, $x \neq 2$

(iii) $y = \begin{cases} x+3 & , x \neq 3 \\ 2 & , x = 3 \end{cases}$ **(iv)** $y = \dfrac{x^2-16}{x-4}$, $x \neq 4$

Solution:

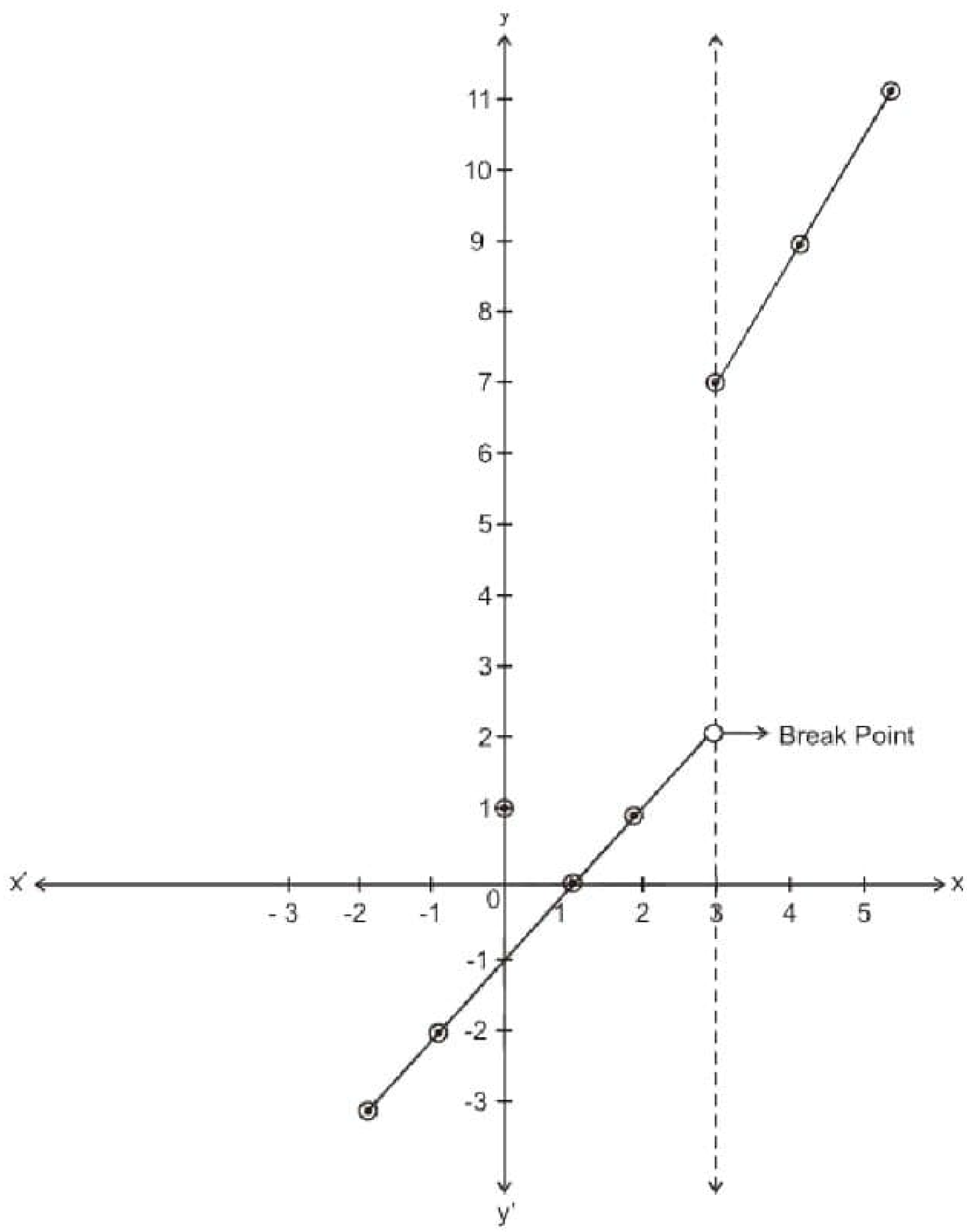
(i) $y = \begin{cases} x-1 & \text{if } x < 3 \\ 2x+1 & \text{if } x \geq 3 \end{cases}$

$y = x - 1 \quad , \quad x < 3$

x	–2	–1	0	1	2
y = x – 1	–3	–2	–1	0	1

$y = 2x + 1 \quad , \quad x \geq 3$

x	3	4	5
y = 2x + 1	7	9	11



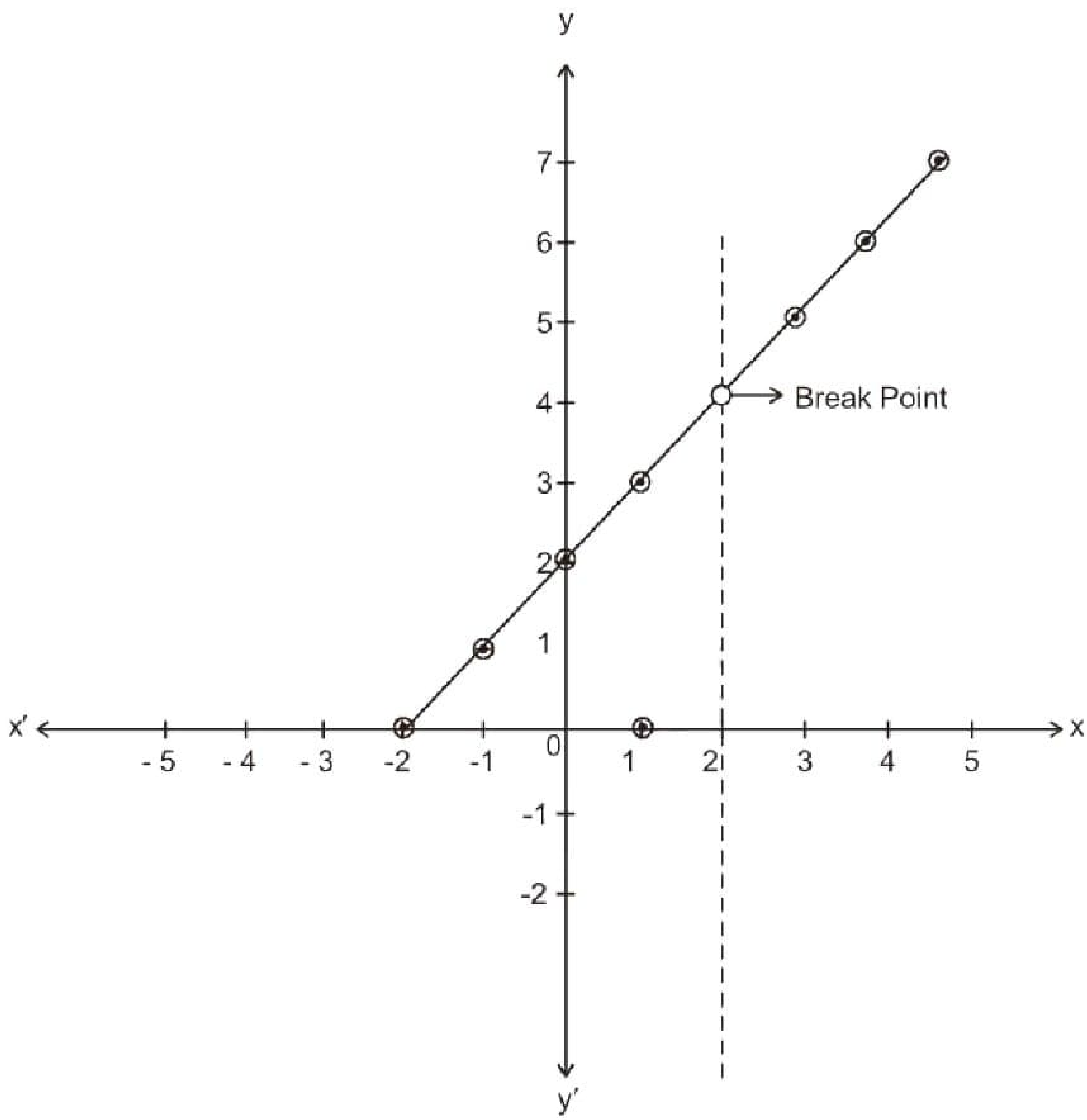
Since there is a break in a graph. So this function is not continuous.

(ii) $\quad \mathbf{y} = \dfrac{\mathbf{x^2 - 4}}{\mathbf{x - 2}} \quad , \quad \mathbf{x \neq 2}$

$$= \frac{(x+2)(x-2)}{x-2} \quad , \quad x \neq 2$$

$$y = x + 2 \,, \quad x \neq 2$$

x	– 3	– 2	– 1	0	1	3	4	5
y	– 1	0	1	2	3	5	6	7



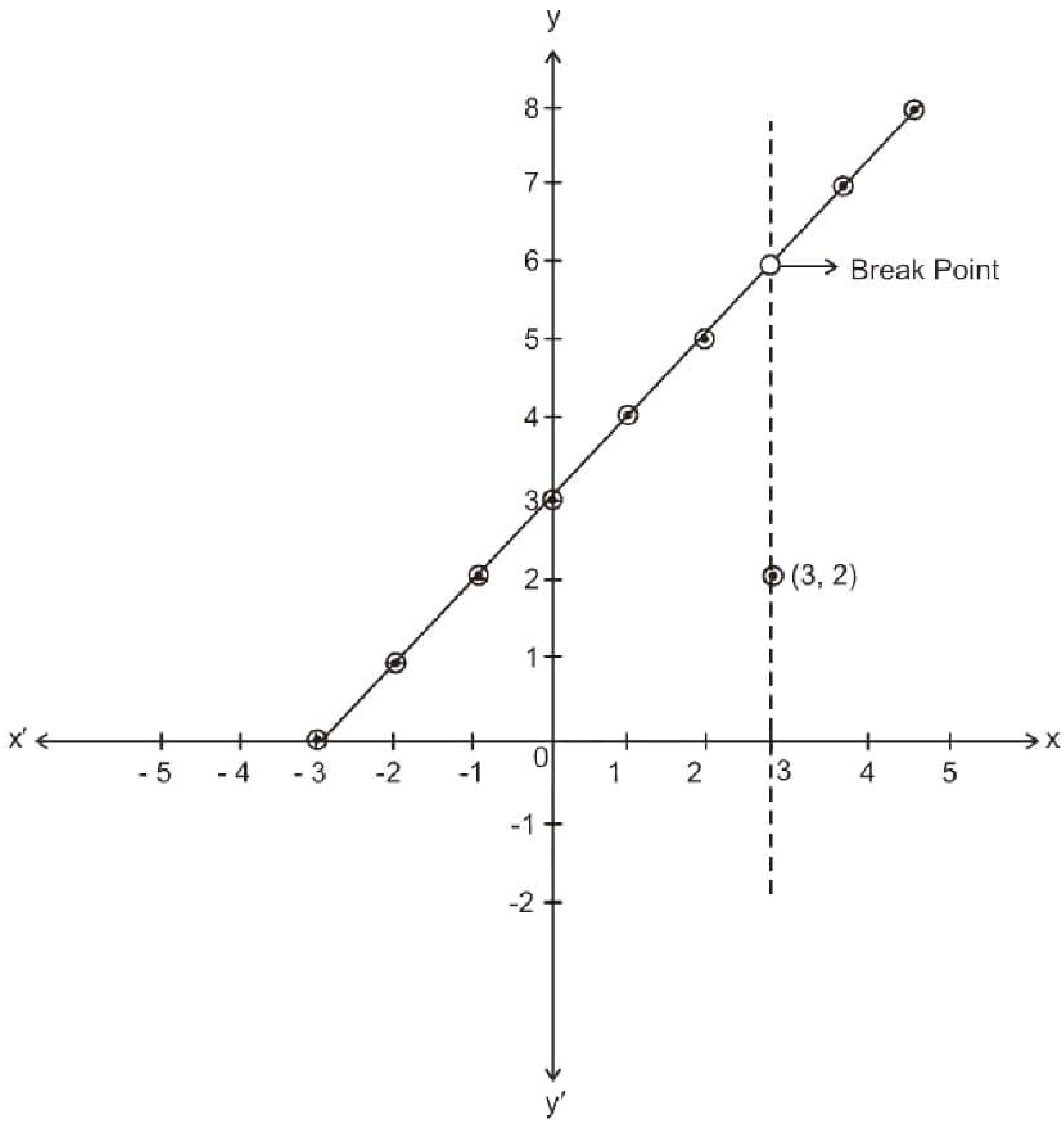
Since there is a break in a graph so this function is not continuous.

(iii) $y = \begin{cases} x + 3 & \text{if} \quad x \neq 3 \\ 2 & \text{if} \quad x = 3 \end{cases}$

$y = x + 3 \qquad \text{if} \quad x \neq 3$

x	– 3	– 2	– 1	0	1	3	4	5
y	0	1	2	3	4	5	7	8

$y = 2 \qquad \text{if} \quad x = 3$

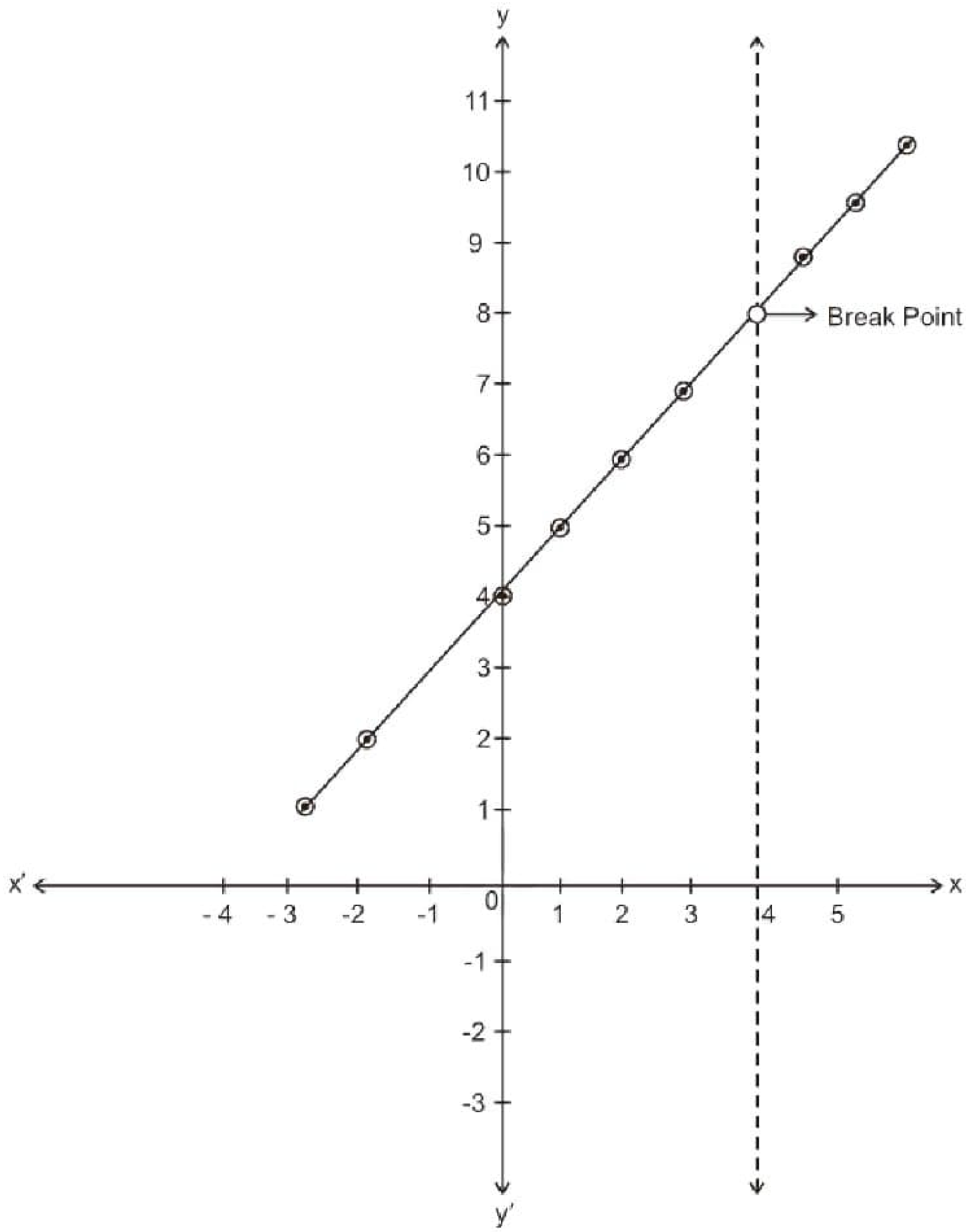


Since there is a break in a graph. So this function is not continuous at x = 3.

(iv) $y = \frac{x^2 - 16}{x - 4}$, $x \neq 4$

$= \frac{(x+4)(x-4)}{x-4}$, $x \neq 4$

x	− 3	− 2	− 1	0	1	2	3	5	6
y	1	2	3	4	5	6	7	9	10



Since there is a break in a graph. So this function is not continuous at x = 4.

Q.4 Find the graphical solution of the following equations.

(i) $x = \sin 2x$ **(ii)** $\frac{x}{2} = \cos x$ **(iii)** $2x = \tan x$

Solution:

(i) **Let** $y = x = \sin 2x$

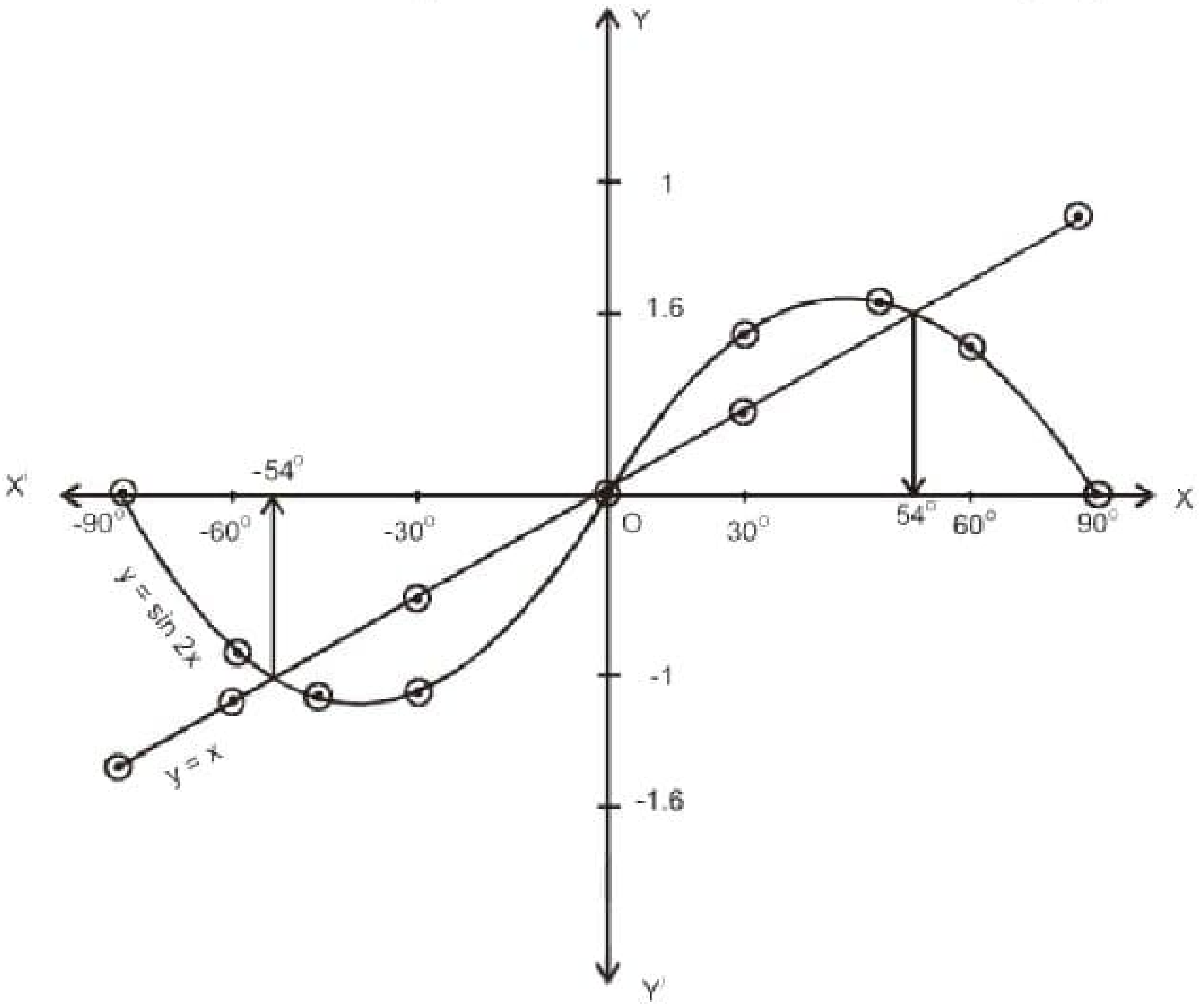
Therefore $y = x$ and $y = \sin 2x$

x	– 90°	– 60°	– 30°	0°	30°	60°	90°
y = x	– π/2 = – 1.6	– π/3 = – 1.05	– π/6 = – 0.52	0	π/6 = 0.52	π/3 = 1.05	π/2 = 1.6

$y = \sin 2x$

x	-90°	-60°	-30°	0°	30°	60°	90°
$y = \sin 2x$	0	-0.87	-0.87	0	0.87	0.87	0

The graphical solution is the points of intersection of two graphs, i.e. $x = 0^\circ$, 54°



(ii) **Let** $y = \frac{x}{2} = \cos x$

Therefore $y = \frac{x}{2}$ and $y = \cos x$

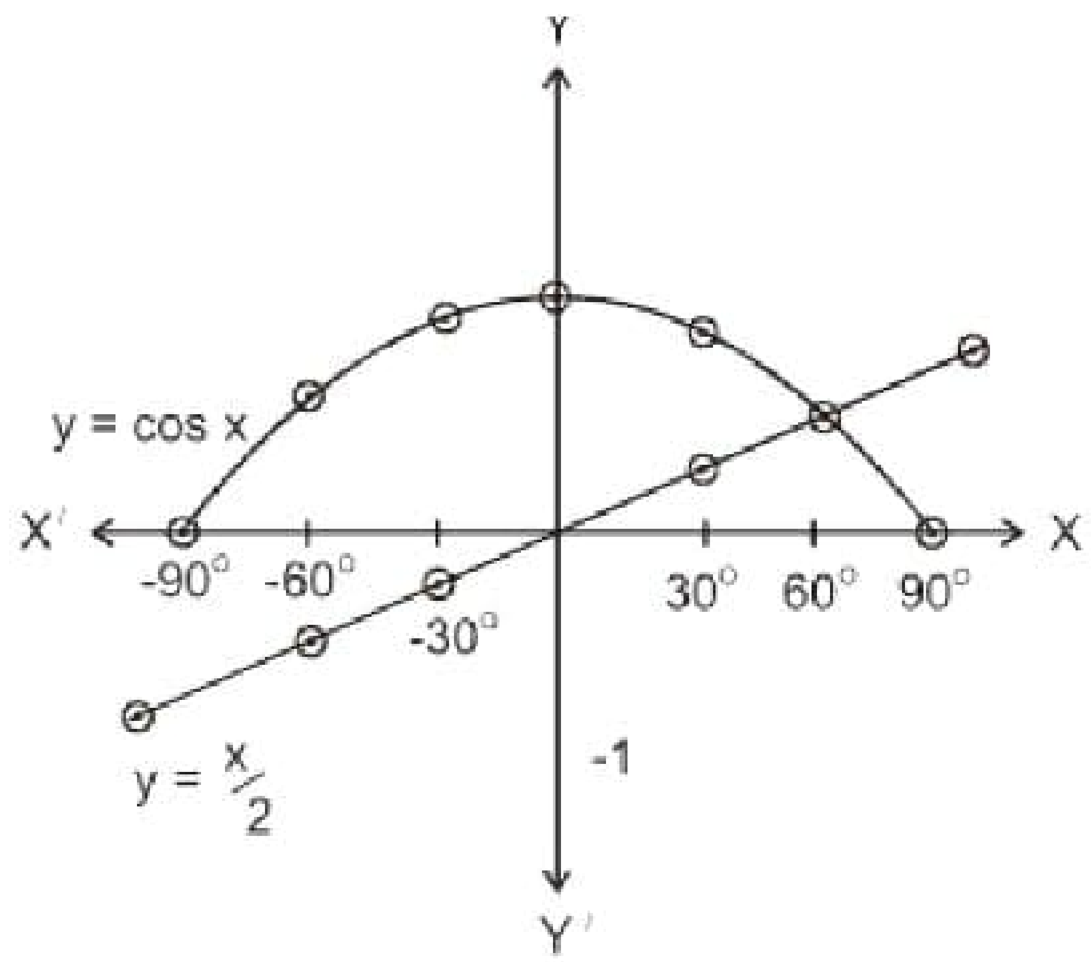
$y = \frac{x}{2}$

x	-90°	-60°	-30°	0°	30°	60°	90°
$y = \frac{x}{2}$	$-\pi/4$ $= -.79$	$-\pi/6$ $= -0.52$	$-\pi/12$ $= -0.26$	0	$\pi/6$ $= 0.26$	$\pi/6$ $= 0.52$	$\pi/4$ $= 0.79$

$y = \cos x$

x	-90°	-60°	-30°	0°	30°	60°	90°
$y = \cos x$	0	0.5	0.87	1	0.87	0.5	0

The graphical solution is the point on x-axis, which is just below the point of intersection of two graphs. Hence $x = 60^\circ$.



(iii) **Let** $\mathbf{y = 2x = \tan x}$

Therefore $y = 2x$ and $y = \tan x$

$y = 2x$

x	– 90°	– 60°	– 30°	0°	30°	60°	90°
y = 2x	– π = – 3.14	– 2π/3 = – 2.09	– π/3 = – 1.05	0	π/3 = 1.05	2π/3 = 2.09	π = 3.14

$y = \tan x$

x	– 90°	– 60°	– 30°	0°	30°	60°	90°
y = tan x	∞	– 1.73	– 0.58	0	0.58	1.73	∞

The graphical solution is the point of intersection of two graphs, i.e. x = 0°.