

## Chapter 3.

2<sup>nd</sup> year

## INTEGRATION

**Integration:**

The technique or method to find such a function whose derivative is given involves the inverse process of differentiation, called anti derivative or integration.

**Differential of variable:**

Let  $f$  be a differentiable function defined as

$$y = f(x) \Rightarrow y + \delta y = f(x + \delta x)$$

$$\Rightarrow \delta y = f(x + \delta x) - y \Rightarrow \delta y = f(x + \delta x) - f(x)$$

$$\text{Now } \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = f'(x)$$

$\therefore$  before the limit is reached,  $\frac{\delta y}{\delta x}$  differs from

$$f'(x) \text{ by small real number } \epsilon, \text{ i.e. } \frac{\delta y}{\delta x} = f'(x) + \epsilon$$

$\Rightarrow f'(x)$  is called differential of dependent variable  $y$  we denoted differential of  $y$  as  $dy$ .

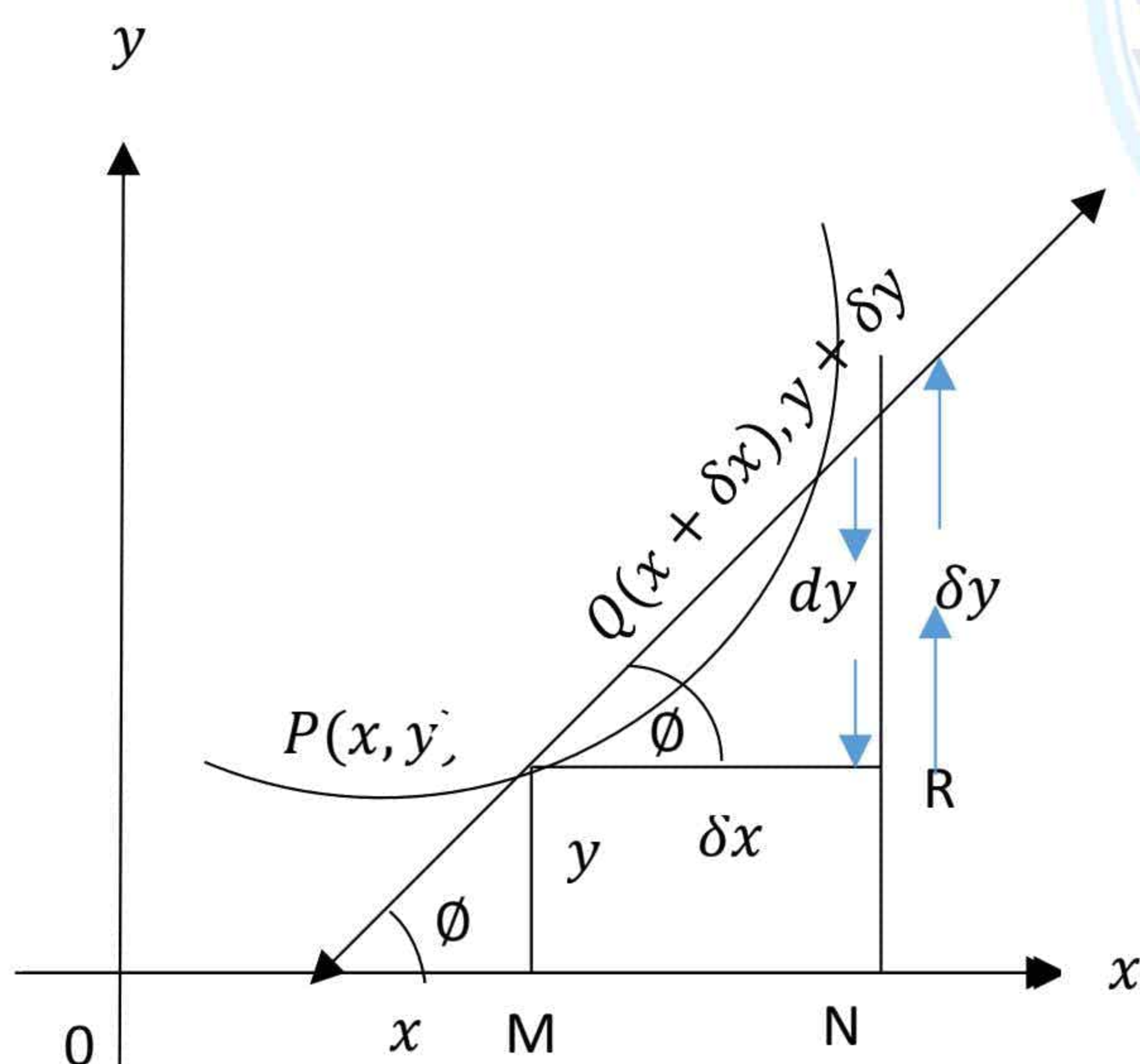
$$\text{so } dy = f'(x)\delta x \Rightarrow dx = \delta x = \frac{dy}{f'(x)}$$

**Note: 1.** The differential of  $x$  is denoted by  $dx$  and defined as  $dx = \delta x$

$$\text{i.e for } y = x \Rightarrow dy = \frac{d}{dx}(x)\delta x$$

$$\Rightarrow dy = 1 \cdot \delta x \Rightarrow dx = \delta x \quad \because y = x$$

**2.**  $f'(x)$  is used differential coefficient.

**Distinguishing between  $dy$  and  $\delta x$** 

Let us draw the graph of curve of a function  $y = f(x)$  Let  $P(x, y)$  and  $Q(x + \delta x, y + \delta y)$  be two neighbouring points on the curve at point  $P(x, y)$

s. that it makes an angle  $\phi$  with  $x$  - axis. Also Draw  $\perp PM$  and  $QN$  on  $x$  - axis also draw  $\perp PR$  on  $QN$  on  $x$  - axis. in fig.  $|PR| = dx$

$$|QR| = |QT| + |TR|$$

$$\Rightarrow \delta y = |QT| + |TR| \rightarrow (i)$$

$$\text{In } \triangle TPR, \quad \tan \phi dx = \frac{|TR|}{|PR|} = \frac{|TR|}{dx}$$

$$\Rightarrow |TR| = \tan \phi dx$$

$$\text{So (i)} \Rightarrow \delta y = \tan \phi dx + |QT|$$

$$\Rightarrow \delta y = \left(\frac{dy}{dx}\right) dx + |QT| \quad \because \frac{dy}{dx} = \tan \phi$$

$$\delta y = dy + |QT| \quad \because |QT| \text{ is very small}$$

So by neglecting  $|QT|$

$$\Rightarrow \delta y \approx dy$$

**Example:**

Find  $\delta y$  and  $dy$  of the function defined as

$$f(x) = x^2 \text{ when } x = 2 \text{ and } dx = 0.01$$

**Solution:**

$$\text{Let } y = f(x) \quad dy = ?$$

$$\Rightarrow y = x^2$$

$$\Rightarrow \frac{dy}{dx} = 2x \Rightarrow dy = 2dx$$

$$\text{Take } x = 2 \text{ and } dx = 0.01$$

$$dy = 2(2)(0.01) = 0.04$$

Now we find  $\delta y, y + \delta y = (x + \delta x)^2$

$$\Rightarrow \delta y = (x + \delta x)^2 - y, \quad y = (x)^2 = (2)^2 = 4$$

$$= (2 + 0.01)^2 - 4 \quad \because dx = \delta x = 0.01$$

$$\delta y = 4.041 - 4 = 0.0401$$

**Example:**

Use differentials find  $\frac{dy}{dx}$  when  $\frac{y}{x} - \ln x = \ln c$

**Solution:**

$$\frac{y}{x} - \ln x = \ln c$$

$$\Rightarrow d\left(\frac{y}{x} - \ln x\right) = d(\ln c)$$

$$\Rightarrow d\left(\frac{y}{x}\right) - d(\ln x) = 0$$

$$\Rightarrow \frac{xdy - ydx}{x^2} - \frac{1}{x} dx = 0$$

$$\Rightarrow \frac{xdy - ydx}{x^2} = \frac{1}{x} dx$$

$$\Rightarrow xdy - ydx = xdx$$

$$\Rightarrow xdy = xdx + ydx$$

$$\Rightarrow dy = \frac{x+y}{x} dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x}$$

## Exercise 3.1

**Q. 1: Find  $\delta y$  and  $dy$  in the following cases:**

i)  $y = x^2 - 1$

when  $x$  changes from 3 to 3.02

**SOLUTION:**

$y = x^2 - 1$  As  $x$  changes from 3 to 3.02, so

$y = x^2 - 1$

$d(y) = d(x^2 - 1)$

$dy = 2x dx - 0 = 2x dx$

Put value of  $x$  and  $dx$

$dy = 2(3)(0.02)$

$dy = 0.12$

Now

$y + \delta y = (x + \delta x)^2 - 1$

$\delta y = (x + \delta x)^2 - 1 - y$

Put value of  $y$

$\delta y = (x + \delta x)^2 - 1 - (x^2 - 1)$

$\delta y = (x + \delta x)^2 - 1 - x^2 + 1$

$\delta y = (x + \delta x)^2 - x^2$

$x = 3$

$\delta x = dx = 3.02 - 3 = 0.02$

Put value of  $x$  and  $\delta x$

$\delta y = (3 + 0.02)^2 - (3)^2$

$\delta y = 0.1204$

ii)  $y = x^2 + 2x$

when  $x$  changes from 2 to 1.8

**SOLUTION:**

$y = x^2 + 2x$

Now

$y = x^2 + 2x$

$d(y) = d(x^2 + 2x)$

$dy = 2x dx + 2dx$

Put value of  $x$  and  $dx$

$dy = 2(2)(-0.2) + 2(-0.2)$

$dy = -1.2$

Now

$y + \delta y = (x + \delta x)^2 + 2(x + \delta x)$

$\delta y = (x + \delta x)^2 + 2x + 2\delta x - y$

Put value of  $y$

$\delta y = (x + \delta x)^2 + 2x + 2\delta x - (x^2 + 2x)$

$\delta y = (x + \delta x)^2 + 2x + 2\delta x - x^2 - 2x$

$\delta y = (x + \delta x)^2 + 2\delta x - x^2$

$x = 2,$

$\delta x = dx = 1.8 - 2 = -0.2$

Put value of  $x$  and  $\delta x$

$\delta y = (2 - 0.2)^2 + 2(-0.2) - (2)^2$

$\delta y = -1.16$

iii)  $y = \sqrt{x}$

when  $x$  changes from 4 to 4.01

**SOLUTION:**

$y = \sqrt{x}$

Now

$y = \sqrt{x}$

$d(y) = d(\sqrt{x})$

$dy = \frac{1}{2\sqrt{x}} dx$

Put value of  $x$  and  $dx$

$dy = \frac{1}{2\sqrt{4}} (0.41)$

$dy = 0.1025$

Now.

$y + \delta y = \sqrt{x + \delta x}$

$\delta y = \sqrt{x + \delta x} - y$

Put value of  $y$

$\delta y = \sqrt{x + \delta x} - \sqrt{x}$

$x = 4,$

$\delta x = dx = 4.41 - 4 = 0.41$

Put value of  $x$  and  $\delta x$

$\delta y = \sqrt{4 + 0.41} - \sqrt{4}$

$\delta y = 0.1$

**Q. 2: Using differentials find  $\frac{dy}{dx}$  and  $\frac{dx}{dy}$  in the following equations.**

i)  $xy + x = 4$

Taking differentials on both sides

$d(xy + x) = d(4)$

$d(xy) + d(x) = 0$

$x dy + y dx + dx = 0$

$x dy + (y + 1)dx = 0$

$x dy = -(y + 1)dx$

$\frac{dy}{dx} = -\frac{y+1}{x}$  and

$\frac{dx}{dy} = -\frac{x}{y+1}$

ii)  $x^2 + 2y^2 = 16$

Taking differentials on both sides

$d(x^2 + 2y^2) = d(16)$

$d(x^2) + 2d(y^2) = 0$

$2x dx + 2 \cdot 2y^{2-1} \cdot dy = 0$

$2x dx + 4y dy = 0$

$4y dy = -2x dx$

$\frac{dy}{dx} = -\frac{2x}{4y} = -\frac{x}{2y}$  and

$\frac{dx}{dy} = -\frac{2y}{x}$

iii)  $x^4 + y^2 = xy^2$

Taking differentials on both sides

$d(x^4 + y^2) = d(xy^2)$

$d(x^4) + d(y^2) = (x)'(y^2) + (y^2)'x$

$4x^3 dx + 2y dy = dx \cdot y^2 + (2y dy)x$

$4x^3 dx + 2y dy = y^2 dx + 2xy dy$

$2y dy - 2xy dy = y^2 dx - 4x^3 dx$

$(2y - 2xy) dy = (y^2 - 4x^3) dx$

$\frac{Dy}{dx} = \frac{y^2 - 4x^3}{2y - 2xy}$  taking reciprocal

$\frac{dx}{dy} = \frac{2y - 2xy}{y^2 - 4x^3}$

iv)  $xy - \ln x = c$

Taking differentials on both sides

$d(xy - \ln x) = d(c)$

$d(xy) - d(\ln x) = 0$

$x dy + y dx - \frac{1}{x} dx = 0$

$$x dy = -y dx + \frac{1}{x} dx$$

$$x dy = -\left(y - \frac{1}{x}\right) dx$$

$$x dy = -\left(\frac{xy-1}{x}\right) dx$$

$$\frac{dy}{dx} = \frac{1-xy}{x^2} \quad \text{and}$$

$$\frac{dx}{dy} = \frac{x^2}{1-xy}$$

**Q. 3: Use differentials to approximate the values of:**

i)  $\sqrt[4]{17}$

**SOLUTION:**

Let  $y = \sqrt[4]{x} = x^{\frac{1}{4}}$

We take  $x = 16$ ,

$$\delta x = dx = 17 - 16 = 1$$

$$y = (16)^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = 2$$

Now  $y = x^{\frac{1}{4}}$

$$d(y) = d\left(x^{\frac{1}{4}}\right)$$

$$dy = \frac{1}{4} x^{\frac{1}{4}-1} dx$$

$$dy = \frac{1}{4} x^{-\frac{3}{4}} dx$$

Put  $x = 16$ ,  $dx = 1$

$$dy = \frac{1}{4} (16)^{-\frac{3}{4}} (1) = \frac{1}{4} (2^4)^{-\frac{3}{4}}$$

$$dy = \frac{1}{4} (2)^{-3} = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$$

$$dy = 0.03125$$

$$\begin{aligned} \text{Thus } \sqrt[4]{17} &\approx y + dy \\ &= 2 + 0.03125 \\ &= 2.03125 \end{aligned}$$

ii)  $(31)^{\frac{1}{5}}$

**SOLUTION:**

Let  $y = x^{\frac{1}{5}}$

We take  $x = 32$ ,

$$\delta x = dx = 31 - 32 = -1$$

$$y = (32)^{\frac{1}{5}} = (2^5)^{\frac{1}{5}} = 2$$

Now  $y = x^{\frac{1}{5}}$

$$d(y) = d\left(x^{\frac{1}{5}}\right)$$

$$dy = \frac{1}{5} x^{\frac{1}{5}-1} dx$$

$$dy = \frac{1}{5} x^{-\frac{4}{5}} dx$$

Put  $x = 32$ ,  $dx = -1$

$$dy = \frac{1}{5} (32)^{-\frac{4}{5}} (-1) = -\frac{1}{5} (2^5)^{-\frac{4}{5}}$$

$$dy = \frac{1}{5} (2)^{-4} = \frac{1}{5} \cdot \frac{1}{16} = \frac{1}{80}$$

$$dy = -0.0125$$

$$\begin{aligned} \text{Thus } (31)^{\frac{1}{5}} &\approx y + dy \\ &= 2 - 0.0125 \\ &= 1.9875 \end{aligned}$$

iii)  $\cos 29^\circ$

**SOLUTION:**

Let  $y = \cos x$

We take  $x = 30^\circ$ ,

$$\delta x = dx = 29^\circ - 30^\circ = -1^\circ = -0.01745$$

$$y = \cos 30^\circ = 0.866$$

Now  $y = \cos x$

$$d(y) = d(\cos x)$$

$$dy = -\sin x dx$$

$$dy = -\sin 30^\circ (-0.01745)$$

$$dy = -(0.5) (-0.01745)$$

$$dy = 0.0087$$

$$\begin{aligned} \text{Thus } \cos 29^\circ &\approx y + dy \\ &= 0.866 + 0.0087 \\ &= 0.8747 \end{aligned}$$

iv)  $\sin 61^\circ$

**SOLUTION:**

Let  $y = \sin x$

We take  $x = 60^\circ$ ,

$$\delta x = dx = 61^\circ - 60^\circ = 1^\circ = 0.01745$$

$$y = \sin 60^\circ = 0.866$$

Now  $y = \sin x$

$$d(y) = d(\sin x)$$

$$dy = \cos x dx$$

$$dy = \cos 60^\circ (0.01745)$$

$$dy = (0.5) (0.01745)$$

$$dy = 0.0087$$

$$\begin{aligned} \text{Thus } \sin 61^\circ &\approx y + dy \\ &= 0.866 + 0.0087 \\ &= 0.8747 \end{aligned}$$

**Q. 4: Find the approximate increase in the volume of a cube if the length of each edge changes from 5 to 5.02.**

**SOLUTION:**

Length of each edge of cube =  $x$  unit

Volume of a cube =  $L \cdot W \cdot H$

$$V = x \cdot x \cdot x$$

$$V = x^3$$

$$d(V) = (x^3)$$

$$dV = 3x^2 dx$$

when  $x$  changes from 5 to 5.02, so

$$x = 5, \quad dx = 5.02 - 5 = 0.02$$

$$dV = 3(5)^2 (0.02) = 1.5 \text{ cubic units}$$

**Q. 5: Find the approximate increase in the area of a circular disc if its diameter is increased from 44 cm to 44.4 cm.**

**SOLUTION:**

Let radius of circular disc =  $x$  cm

Area of a disc =  $\pi r^2$

$$A = \pi x^2$$

$$d(A) = d(\pi x^2)$$

$$dA = \pi \cdot 2x dx$$

As diameter changes from 44 to 44.4,

so radius changes from 22 to 22.4, so

$$x = 22, \quad dx = 22.2 - 22 = 0.2$$

$$dA = \pi(2)(22)(0.2)$$

$$dA = 27.646 \text{ cm}^2$$

## Integration as anti-derivative (inverse of derivative)

*Integration: v.v.v. important definition(\*\*\*)  
The inverse process of differentiation is called anti – differentiation or integration.*

Consider  $F(x)$  is antiderivative of a function if

$$\begin{aligned} F'(x) = f(x) \text{ then } \int f(x)dx &= \int F'(x)dx \\ &= \int \frac{d}{dx}F(x)dx \\ \int f(x)dx &= F(x) + c \end{aligned}$$

$\therefore \frac{d}{dx}$  and  $\int dx$  are inverse operations of each other.

\*The symbol  $\int \dots dx$  indicates that integrand is to be integrated w.r.t "x"

\*The anti-derivative of a function is also called integrated is called integrand of the integral.

\*The function which is to be integrated is called integrand of the integral.

### Some standard formulae for Anti-derivatives

$$\begin{aligned} \int 1dx &= x + c, \int x^n dx = \frac{x^{n+1}}{n+1} + c (n \neq -1) \\ \int \sin x dx &= -\cos x + c, \int \cos x dx = \sin x + c \\ \int \sec^2 x dx &= \tan x + c, \int \operatorname{cosec}^2 x dx = -\cot x + c \\ \int \sec x \tan x dx &= \sec x + c, \int \operatorname{cosec} x \cot x dx \\ &= -\operatorname{cosec} x \\ \int e^x dx &= e^x + c, \int a^x dx = \frac{1}{\ln a} \cdot a^x + c \\ \int \frac{1}{x} dx &= \ln|x| + c, x \neq 0, \int \tan x dx = \ln|\sec x| + c \\ &= -\ln|\cos x| + c \\ \int \cot x dx &= \ln|\sin x| + c \\ \int \sec x dx &= \ln|\sec x + \tan x| + c \\ \int \operatorname{cosec} x dx &= \ln|\operatorname{cosec} x - \cot x| + c \end{aligned}$$

Here  $c$  is constant of integration. These formulae can be verified by showing that the derivatives of right hand side of each w.r.t "x" is equal to the corresponding integral

**Examples:**

$$\begin{aligned} 1. \int x^5 dx & \quad \therefore \int x^n dx = \frac{x^{n+1}}{n+1} + c \\ &= \frac{x^{5+1}}{5+1} + c = \frac{x^6}{6} + c \end{aligned}$$

$$\begin{aligned} 2. \int \frac{1}{\sqrt{x^3}} dx & \\ &= \int \frac{1}{x^{\frac{3}{2}}} dx = \int x^{-\frac{3}{2}} dx = \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + c \\ &= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c = -\frac{2}{\sqrt{x}} + c \end{aligned}$$

$$\begin{aligned} 3. \int \frac{1}{(2x+3)^4} dx &= \\ \int (2x+3)^{-4} dx &= \frac{1}{2} \cdot \frac{(2x+3)^{-4+1}}{-4+1} + c \\ &= -\frac{1}{6(2x+3)^3} + c \end{aligned}$$

$$\begin{aligned} 4. \int \cos 2x dx & \quad \therefore \int \cos ax dx = \frac{\sin ax}{a} + c \\ &= \frac{\sin 2x}{2} + c \end{aligned}$$

$$\begin{aligned} 5. \int \sin 3x dx & \quad \therefore \int \sin ax dx = -\frac{\cos ax}{a} + c \\ &= -\frac{\cos 3x}{3} + c \end{aligned}$$

$$\begin{aligned} 6. \int \operatorname{cosec}^2 x dx &= -\cot x + c \\ 7. \int \sec 5x \tan 5x dx &= \frac{\sec 5x}{5} + c \quad \therefore \int \sec ax \tan ax dx \\ &= \frac{\sec ax}{a} + c \end{aligned}$$

$$\begin{aligned} 8. \int e^{ax+b} dx & \quad \therefore \int e^{ax} dx = \frac{e^{ax}}{a} + c \\ &= \frac{e^{ax+b}}{a} + c \end{aligned}$$

$$\begin{aligned} 9. \int 3^{\lambda x} dx &= \frac{3^{\lambda x}}{\lambda \ln 3} \quad \therefore \int e^{ax} dx = \frac{e^{ax}}{a} + c \end{aligned}$$

$$\begin{aligned} 10. \int \frac{1}{ax+b} dx &= \frac{1}{a} \ln(ax+b) + c \end{aligned}$$

$$1. \int af(x) dx = a \int f(x) dx$$

$$2. \int [f_1(x) \pm f_2(x)] dx = \int f_1(x) dx \pm \int f_2(x) dx$$

**Prove that**

$$\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c, (n \neq -1)$$

**Proof:**

$$\text{We know that } \frac{d}{dx} (f^{n+1}(x))$$

$$= (n+1) f^n(x) \cdot \frac{d}{dx} f(x)$$

$$\Rightarrow \frac{d}{dx} (f^{n+1}(x)) = (n+1) f^n(x) \cdot f'(x)$$

Taking integration

$$\int \frac{d}{dx} f^{(n+1)}(x) dx = (n+1) \int f^n(x) \cdot f'(x) dx$$

$$\Rightarrow f^{n+1}(x) = (n+1) \int f^n(x) f'(x) dx$$

$$\Rightarrow \int f^n(x) f'(x) dx = \frac{f^{n+1}(x)}{n+1} + c \text{ by def.}$$

Hence proved.

**Prove that**  $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$

**Proof:**

We know that

$$\frac{d}{dx} [\ln f(x)] = \frac{1}{f(x)} \cdot f'(x)$$

Taking integration both sides

$$\int \frac{d}{dx} [\ln f(x)] dx = \int \frac{1}{f(x)} \cdot f'(x) dx$$

$$\Rightarrow \ln f(x) = \int \frac{f'(x)}{f(x)} dx$$

$$\Rightarrow \int \frac{f'(x)}{f(x)} dx = \ln f(x) + c \text{ by definition}$$

$$(\int f(x) dx = F(x) + c)$$

Hence proved.

## Exercise 3.2

**Q. 1: Evaluate the following indefinite integrals:**

i)  $\int (3x^2 - 2x + 1) dx$

**SOLUTION:**

$$= \int 3x^2 dx - \int 2x dx + \int 1 dx$$

$$= 3 \int x^2 dx - 2 \int x dx + \int 1 dx$$

$$= 3 \cdot \frac{x^{2+1}}{2+1} - 2 \cdot \frac{x^{1+1}}{1+1} + x + c$$

$$= 3 \cdot \frac{x^3}{3} - 2 \cdot \frac{x^2}{2} + x + c$$

$$= x^3 - x^2 + x + c$$

ii)  $\int (\sqrt{x} + \frac{1}{\sqrt{x}}) dx$

**SOLUTION:**

$$= \int \sqrt{x} dx + \int \frac{1}{\sqrt{x}} dx$$

$$= \int x^{\frac{1}{2}} dx + \int x^{-\frac{1}{2}} dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + x + c$$

iii)  $\int x(\sqrt{x} + 1) dx$

**SOLUTION:**

$$= \int x(\sqrt{x} + 1) dx$$

$$= \int x\sqrt{x} dx + \int x dx$$

$$= \int x^{1+\frac{1}{2}} dx + \int x dx$$

$$= \int x^{\frac{3}{2}} dx + \int x dx$$

$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{x^{1+1}}{1+1} + c$$

$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^2}{2} + c$$

$$= \frac{2}{5} x^{\frac{5}{2}} + \frac{1}{2} x^2 + c$$

iv)  $\int (2x+3)^{\frac{1}{2}} dx$

**SOLUTION:**

$$= \int (2x+3)^{\frac{1}{2}} dx$$

× and ÷ by 2 to make derivative

$$= \frac{1}{2} \int (2x+3)^{\frac{1}{2}} \cdot 2 dx$$

$$= \frac{1}{2} \frac{(2x+3)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$= \frac{1}{2} \frac{(2x+3)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{1}{2} \cdot \frac{2}{3} (2x+3)^{\frac{3}{2}} + c$$

$$= \frac{1}{3} (2x+3)^{\frac{3}{2}} + c$$

v)  $\int (\sqrt{x} + 1)^2 dx$

**SOLUTION:**

$$= \int (\sqrt{x} + 1)^2 dx$$

$$= \int ((\sqrt{x})^2 + 2\sqrt{x} \cdot 1 + (1)^2) dx$$

$$= \int [x + 2\sqrt{x} + 1] dx$$

$$= \int x dx + 2 \int x^{\frac{1}{2}} dx + \int 1 dx$$

$$= \frac{x^{1+1}}{1+1} + 2 \cdot \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + x + c$$

$$= \frac{x^2}{2} + 2 \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + x + c$$

$$= \frac{1}{2} x^2 + 2 \cdot \frac{2}{3} x^{\frac{3}{2}} + x + c$$

$$= \frac{1}{2} x^2 + \frac{4}{3} x^{\frac{3}{2}} + x + c$$

vi)  $\int (\sqrt{x} - \frac{1}{\sqrt{x}})^2 dx$

**SOLUTION:**

$$= \int (\sqrt{x} - \frac{1}{\sqrt{x}})^2 dx$$

$$= \int \left[ (\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2\sqrt{x} \cdot \frac{1}{\sqrt{x}} \right] dx$$

$$= \int \left[ x + \frac{1}{x} - 2 \right] dx$$

$$= \int x dx + \int \frac{1}{x} dx - 2 \int 1 dx$$

$$= \frac{x^{1+1}}{1+1} + \ln x - 2x + c$$

$$= \frac{1}{2} x^2 + \ln x - 2x + c$$

**NOTE: FOR Q. (vi)**

AGAR FUNCTION OVER M HO AUR FUNCTION KI POWER 1 HO TU AP US PAR POWER RULE NI LAGA SAKTAY.

**FOR EXAMPLE:**

$$\int \frac{1}{x} dx = \int x^{-1} dx = \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} = \frac{1}{0} = \infty$$

IS M ANSWER  $\infty$  A GIA SO NOT SOLVED?

THEN AGAR FUNCTION KA DERIVATIVE UPER MAJOOD H T US K  $\ln$  K SATH LIKH DE.

vii)  $\int \frac{3x+2}{\sqrt{x}} dx$

**SOLUTION:**

$$\begin{aligned}
& \int \frac{3x+2}{\sqrt{x}} dx \\
&= \int \left[ \frac{3x}{\sqrt{x}} + \frac{2}{\sqrt{x}} \right] dx \\
&= \int \left[ \frac{3\sqrt{x}\sqrt{x}}{\sqrt{x}} + \frac{2}{\sqrt{x}} \right] dx \quad \because x = \sqrt{x} \cdot \sqrt{x} \\
&= \int \left[ 3\sqrt{x} + \frac{2}{\sqrt{x}} \right] dx \\
&= \int \left[ 3x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} \right] dx \\
&= 3 \int x^{\frac{1}{2}} dx + 2 \int x^{-\frac{1}{2}} dx \\
&= 3 \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 2 \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c \\
&= 3 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 2 \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\
&= 3 \frac{2}{3} x^{\frac{3}{2}} + 2 \cdot 2x^{\frac{1}{2}} + c \\
&= 2x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + c
\end{aligned}$$

$$\text{viii) } \int \frac{\sqrt{y}(y+1)}{y} dy$$

**SOLUTION:**

$$\begin{aligned}
& \int \frac{\sqrt{y}(y+1)}{y} dy \\
&= \int \frac{\sqrt{y}(y+1)}{\sqrt{y}\sqrt{y}} dy \\
&= \int \frac{y+1}{\sqrt{y}} dy \\
&= \int \left[ \frac{y}{\sqrt{y}} dx + \frac{1}{\sqrt{y}} dx \right] \\
&= \int \left[ \sqrt{y} dx + \frac{1}{\sqrt{y}} dx \right] \\
&= \int \left[ y^{\frac{1}{2}} dx + y^{-\frac{1}{2}} dx \right] \\
&= \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c \\
&= \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + \frac{y^{\frac{1}{2}}}{\frac{1}{2}} + c \\
&= \frac{2}{3} y^{\frac{3}{2}} + 2y^{\frac{1}{2}} + x + c
\end{aligned}$$

$$\text{ix) } \int \frac{(\sqrt{\theta}-1)^2}{\sqrt{\theta}} d\theta$$

**SOLUTION:**

$$\begin{aligned}
& \int \frac{(\sqrt{\theta}-1)^2}{\sqrt{\theta}} d\theta \\
&= \int \frac{(\sqrt{\theta})^2 + (1)^2 - 2\sqrt{\theta}}{\sqrt{\theta}} d\theta \\
&= \int \frac{\theta + 1 - 2\sqrt{\theta}}{\sqrt{\theta}} d\theta \\
&= \int \left[ \frac{\theta}{\sqrt{\theta}} + \frac{1}{\sqrt{\theta}} - \frac{2\sqrt{\theta}}{\sqrt{\theta}} \right] d\theta \\
&= \int \left[ \sqrt{\theta} + \frac{1}{\sqrt{\theta}} - 2 \right] d\theta \\
&= \int \theta^{\frac{1}{2}} d\theta + \int \theta^{-\frac{1}{2}} d\theta - 2 \int 1 d\theta \\
&= \frac{\theta^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{\theta^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - 2\theta + c \\
&= \frac{\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{\theta^{\frac{1}{2}}}{\frac{1}{2}} - 2\theta + c \\
&= \frac{2}{3} \theta^{\frac{3}{2}} + 2\theta^{\frac{1}{2}} - 2\theta + c
\end{aligned}$$

$$\text{x) } \int \frac{(1-\sqrt{x})^2}{\sqrt{x}} dx$$

**SOLUTION:**

$$\begin{aligned}
& \int \frac{(1-\sqrt{x})^2}{\sqrt{x}} dx \\
&= \int \frac{(\sqrt{x})^2 + (1)^2 - 2\sqrt{x}}{\sqrt{x}} dx \\
&= \int \frac{x+1-2\sqrt{x}}{\sqrt{x}} dx \\
&= \int \left[ \frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} - \frac{2\sqrt{x}}{\sqrt{x}} \right] dx \\
&= \int \left[ \sqrt{x} + \frac{1}{\sqrt{x}} - 2 \right] dx \\
&= \int x^{\frac{1}{2}} dx + \int x^{-\frac{1}{2}} dx - 2 \int 1 dx \\
&= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - 2x + c \\
&= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - 2x + c \\
&= \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 2x + c
\end{aligned}$$

$$\text{xi) } \int \frac{e^{2x}+e^x}{e^x} dx$$

**SOLUTION:**

$$\begin{aligned}
& \int \frac{e^{2x}+e^x}{e^x} dx \\
&= \int \left[ \frac{e^{2x}}{e^x} + \frac{e^x}{e^x} \right] dx \\
&= \int [e^x + 1] dx \\
&= \int e^x dx + \int 1 dx \\
&= \frac{e^x}{1} + x + c \\
&= e^x + x + c
\end{aligned}$$

**NOTE: DERIVATION M**

EXPONENTIAL FUNCTION KA JAB DERIVATIVE LATY H T FUNCTION AS IT AUR POWER KA DERIVATIVE MULTIPLY KARTY H. LAKIN INTEGRATION M DIVIDE KARE GAI.

**Q. 2: Evaluate:**

$$\text{i) } \int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$$

**SOLUTION:**

$$\begin{aligned}
& \int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}} \\
&= \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}} dx \\
&= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(\sqrt{x+a})^2 - (\sqrt{x+b})^2} dx = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{x+a-x-b} dx \\
&= \frac{1}{a-b} \int (\sqrt{x+a} - \sqrt{x+b}) dx \\
&= \frac{1}{a-b} \left\{ \int (x+a)^{\frac{1}{2}} dx + \int (x+b)^{\frac{1}{2}} dx \right\}
\end{aligned}$$

**using**  $\int [f(x)]^n \cdot f'(x) = \frac{[f(x)]^{n+1}}{n+1} + c$

$$\begin{aligned}
&= \frac{1}{a-b} \left\{ \frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{(x+b)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right\} + c \\
&= \frac{1}{a-b} \left\{ \frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}} \right\} + c \\
&= \frac{1}{a-b} \left\{ \frac{2}{3} (x+a)^{\frac{3}{2}} + \frac{2}{3} (x+b)^{\frac{3}{2}} \right\} + c \\
&= \frac{2}{3(a-b)} \left\{ (x+a)^{\frac{3}{2}} + (x+b)^{\frac{3}{2}} \right\} + c
\end{aligned}$$

$$\text{ii) } \int \frac{1-x^2}{1+x^2} dx$$

**SOLUTION:**

$$\begin{aligned} & \int \frac{1-x^2}{1+x^2} dx \\ &= \int \frac{2-1-x^2}{1+x^2} dx \\ &= \int \frac{2-(1+x^2)}{1+x^2} dx \\ &= \int \frac{2}{1+x^2} dx - \int \frac{1+x^2}{1+x^2} dx \\ &= 2 \int \frac{1}{1+x^2} dx - \int 1 dx \\ &= 2 \tan^{-1} x - x + c \end{aligned}$$

iii)  $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$

**SOLUTION:**

$$\begin{aligned} & \int \frac{dx}{\sqrt{x+a}+\sqrt{x}} \\ &= \int \frac{1}{\sqrt{x+a}+\sqrt{x}} \cdot \frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}} dx \\ &= \int \frac{\sqrt{x+a}-\sqrt{x}}{(\sqrt{x+a})^2-(\sqrt{x})^2} dx = \int \frac{\sqrt{x+a}-\sqrt{x}}{x+a-x} dx \\ &= \frac{1}{a} \int (\sqrt{x+a} - \sqrt{x}) dx \\ &= \frac{1}{a} \left\{ \int (x+a)^{\frac{1}{2}} dx + \int (x)^{\frac{1}{2}} dx \right\} \end{aligned}$$

using  $\int [f(x)]^n \cdot f'(x) = \frac{[f(x)]^{n+1}}{n+1} + c$

$$\begin{aligned} &= \frac{1}{a} \left\{ \frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{(x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right\} + c \\ &= \frac{1}{a} \left\{ \frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x)^{\frac{3}{2}}}{\frac{3}{2}} \right\} + c \\ &= \frac{1}{a} \left\{ \frac{2}{3} (x+a)^{\frac{3}{2}} + \frac{2}{3} (x)^{\frac{3}{2}} \right\} + c \\ &= \frac{2}{3a} \left\{ (x+a)^{\frac{3}{2}} + (x)^{\frac{3}{2}} \right\} + c \end{aligned}$$

iv)  $\int (a-2x)^{\frac{3}{2}} dx$

**SOLUTION:**

$$\begin{aligned} & \int (a-2x)^{\frac{3}{2}} dx \\ & \times \text{and } \div \text{ by } 2 \\ &= \frac{1}{-2} \int (a-2x)^{\frac{3}{2}} \cdot (-2) dx \\ &= -\frac{1}{2} \frac{(a-2x)^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c \\ &= -\frac{1}{2} \frac{(a-2x)^{\frac{5}{2}}}{\frac{5}{2}} + c \\ &= -\frac{1}{2} \cdot \frac{2}{5} (a-2x)^{\frac{5}{2}} + c \\ &= -\frac{1}{5} (a-2x)^{\frac{5}{2}} + c \end{aligned}$$

FUNCTION AS IT AUR POWER KE DERIVATIVE S DIVIDE KARNA H.

$$\int e^x dx = \frac{e^x}{1} + c = e^x + c$$

v)  $\int \frac{(1+e^x)^3}{e^x} dx$

**SOLUTION:**

$$\begin{aligned} & \int \frac{(1+e^x)^3}{e^x} dx \\ & \because (a+b)^3 = a^3 + b^3 + 3ab(a+b) \\ &= \int \frac{1^3+(e^x)^3+3(1)(e^x)(1+e^x)}{e^x} dx \\ &= \int \frac{1+e^{3x}+3e^x(1+e^x)}{e^x} dx \end{aligned}$$

$$\begin{aligned} &= \int \left[ \frac{1}{e^x} + \frac{e^{3x}}{e^x} + \frac{3e^x(1+e^x)}{e^x} \right] dx \\ &= \int [e^{-x} + e^{2x} + 3 + 3e^x] dx \\ &= \int e^{-x} dx + \int e^{2x} dx + 3 \int 1 dx + 3 \int e^x dx \\ &= \frac{e^{-x}}{-1} + \frac{e^{2x}}{2} + 3x + 3 \frac{e^x}{1} + c \\ &= -e^{-x} + \frac{1}{2} e^{2x} + 3x + 3e^x + c \end{aligned}$$

vi)  $\int \sin(a+b)x dx$

**SOLUTION:**

$$\begin{aligned} & \int \sin(a+b)x dx \\ &= \frac{-\cos(a+b)x}{a+b} + c \\ &= -\frac{1}{a+b} \cos(a+b)x + c \end{aligned}$$

DERIVATION M FUNCTION KA DERIVATIVE LENA HOTA H AUR SATH ANGLE KE DERIVATIVE KO MULTIPLY KARTY H. BUT INTEGRATION M ANGLE KE DERIVATIVE K DIVIDE KARE GAI.

vii)  $\int \sqrt{1-\cos 2x} dx$

**SOLUTION:**

$$\begin{aligned} & \int \sqrt{1-\cos 2x} dx \\ \text{As } \sin^2 x &= \frac{1-\cos 2x}{2} \\ \text{So } 1-\cos 2x &= 2\sin^2 x \\ &= \int \sqrt{2\sin^2 x} dx \\ &= \int \sqrt{2} \sqrt{\sin^2 x} dx \\ &= \sqrt{2} \int \sin x dx \\ &= \sqrt{2} (-\cos x) + c \\ &= -\sqrt{2} \cos x + c \end{aligned}$$

viii)  $\int \ln x \cdot \frac{1}{x} dx$

**SOLUTION:**

$$\begin{aligned} & \int \ln x \cdot \frac{1}{x} dx \\ \text{As } f(x) &= \ln x \\ \text{And } f'(x) &= \frac{1}{x}, \text{ so} \\ \text{using } \int [f(x)]^n &= \frac{[f(x)]^{n+1}}{n+1} \\ &= \frac{(\ln x)^{1+1}}{1+1} + c \\ &= \frac{(\ln x)^2}{2} + c \end{aligned}$$

ix)  $\int \sin^2 x dx$

**SOLUTION:**

$$\begin{aligned} & \int \sin^2 x dx \\ \text{As } \sin^2 x &= \frac{1-\cos 2x}{2} \\ &= \int \frac{1-\cos 2x}{2} dx \\ &= \frac{1}{2} \int (1-\cos 2x) dx \\ &= \frac{1}{2} \left[ x - \frac{\sin 2x}{2} \right] + c \\ &= \frac{1}{2} x - \frac{1}{4} \sin 2x + c \end{aligned}$$

x)  $\int \frac{1}{1+\cos x} dx$

**SOLUTION:**

$$\int \frac{1}{1+\cos x} dx$$

$$\text{As } \cos^2 \frac{x}{2} = \frac{1+\cos x}{2}$$

$$\text{So } 1 + \cos x = 2\cos^2 \frac{x}{2}$$

$$= \int \frac{1}{2\cos^2 \frac{x}{2}} dx$$

$$= \frac{1}{2} \int \sec^2 \frac{x}{2} dx$$

$$= \frac{1}{2} \frac{\tan \frac{x}{2}}{\frac{1}{2}} + c = \tan \frac{x}{2} + c$$

$\sin^2 x$ ,  $\cos^2 x$ ,  $\tan^2 x$ ,  $\cot^2 x$  in functions k derivative exist ni karty jab b ye function a jay t ap ye formula use kare.

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\tan^2 x = \sec^2 x - 1$$

$$\cot^2 x = \csc^2 x - 1$$

FUNCTIONS K DERIVATIVES K JO ANSWER H UN KI INTEGRATION HOTI H IS K ILAWA FUNCTIONS KI INTEGRATION NI H HOTI. E.G.

$\sin^2 x$ ,  $\cos^2 x$ ,  $\tan^2 x$ ,  $\cot^2 x$  IN KI INTEGRATION NI HOTI.  
( $\sin x$ )' =  $\cos x$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x$$

$$(\cot x)' = -\csc^2 x$$

$$(\sec x)' = \sec x \tan x$$

$$(\operatorname{cosec} x)' = -\csc x \cot x$$

FUNCTIONS K DERIVATIVES K JO ANSWER H UN KI INTEGRATION HOTI H IS K ILAWA FUNCTIONS KI INTEGRATION NI H HOTI. E.G.

$\sin^2 x$ ,  $\cos^2 x$ ,  $\tan^2 x$ ,  $\cot^2 x$  IN KI INTEGRATION NI HOTI.

$$\text{xi) } \int \frac{ax+b}{ax^2+2bx+c} dx$$

**SOLUTION:**

$$\int \frac{ax+b}{ax^2+2bx+c} dx$$

× & ÷ by 2 to make derivative uper

$$= \frac{1}{2} \int \frac{2(ax+b)}{ax^2+2bx+c} dx$$

$$= \frac{1}{2} \int \frac{2ax+2b}{ax^2+2bx+c} dx$$

$$\text{Using } \int \frac{f'(x)}{f(x)} = \ln[f(x)]$$

$$= \frac{1}{2} \ln(ax^2 + 2bx + c) + c$$

$$\text{xii) } \int \cos 3x \sin 2x dx$$

**SOLUTION:**

$$\int \cos 3x \sin 2x dx$$

× & ÷ by 2 to make formula

$$= \frac{1}{2} \int 2 \cos 3x \sin 2x dx$$

As  $2\cos \alpha \sin \beta = \sin(\alpha + \beta) - \sin(\alpha - \beta)$

$$= \frac{1}{2} \int [\sin(3x + 2x) - \sin(3x - 2x)] dx$$

$$= \frac{1}{2} \int [\sin(5x) - \sin(x)] dx$$

$$= \frac{1}{2} \left\{ \int \sin 5x dx - \int \sin x dx \right\}$$

$$= \frac{1}{2} \left\{ \frac{-\cos 5x}{5} - \frac{-\cos x}{1} \right\} + c$$

$$= -\frac{1}{2} \left\{ \frac{\cos 5x}{5} - \cos x \right\} + c$$

$$\text{xiii) } \int \frac{\cos 2x-1}{1+\cos 2x} dx$$

**SOLUTION:**

$$= \int \frac{\cos 2x-1}{1+\cos 2x} dx$$

$$= -\int \frac{1-\cos 2x}{1+\cos 2x} dx$$

$$\because \sin^2 x = \frac{1-\cos 2x}{2} \Rightarrow 2 \sin^2 x = 1 - \cos 2x$$

$$\because \cos^2 x = \frac{1+\cos 2x}{2} \Rightarrow 2 \cos^2 x = 1 + \cos 2x$$

$$= -\int \frac{2 \sin^2 x}{2 \cos^2 x} dx = -\int \tan^2 x dx$$

$$= -\int (\sec^2 x - 1) dx \quad \because 1 + \tan^2 x = \sec^2 x$$

$$= -\int \sec^2 x dx + \int 1 dx$$

$$= -\tan x + x + c$$

$$\text{xiv) } \int \tan^2 x dx$$

**SOLUTION:**

$$\int \tan^2 x dx$$

$$= \int (\sec^2 x - 1) dx \quad \because 1 + \tan^2 \theta = \sec^2 \theta$$

$$= \int \sec^2 x dx - \int 1 dx = \tan x - x + c$$

### Integration by method of substitution

Sometimes it is possible to convert an integral into standard form by a suitable change of a variable. This is called substitution method.

i.e Evaluate  $\int f(x)dx$  by method of substitution

$$\text{Let } x = \phi(t) \Rightarrow dx = \phi'(t)dt$$

$$\text{So } \int f(x)dx = \int f(\phi(t))\phi'(t)dt$$

**Some useful substitutions:**

$$1. \sqrt{a^2 - x^2}$$

$$\text{put } x = a \sin \theta$$

$$(\because 1 - \sin^2 \theta = \cos^2 \theta)$$

$$2. \sqrt{x^2 - a^2}$$

$$\text{put } x = a \sec \theta$$

$$(\because \sec^2 \theta - 1 = \tan^2 \theta)$$

$$3. \sqrt{a^2 + x^2}$$

$$\text{put } x = a \tan \theta$$

$$(\because \sec^2 \theta = 1 + \tan^2 \theta)$$

$$4. \sqrt{x+a} \text{ (or) } \sqrt{x-a}$$

$$\text{put } \sqrt{x+a} = t$$

$$\text{or } (\sqrt{x-a}) = t$$

$$5. \sqrt{2ax - x^2}$$

$$\text{put } x - a = a \sin \theta$$

$$6. \sqrt{2ax + x^2}$$

$$\text{put } x + a = a \sec \theta$$



## Exercise 3.3

Evaluate the following integrals:

**Q. 1:**  $\int \frac{-2x}{\sqrt{4-x^2}} dx$

**SOLUTION:**

$$\begin{aligned} & \int \frac{-2x}{\sqrt{4-x^2}} dx \\ &= \int (4-x^2)^{-\frac{1}{2}} (-2x) dx \\ & \text{Here } f(x) = 4-x^2 \\ & \quad f'(x) = -2x \\ &= \frac{(4-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \\ &= \frac{(4-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= 2\sqrt{4-x^2} + c \quad \because t = 4-x^2 \end{aligned}$$

**Q. 2:**  $\int \frac{dx}{x^2+4x+13}$

**SOLUTION:**

By completing square

$$\begin{aligned} &= \int \frac{dx}{x^2+4x+4-4+13} \\ &= \int \frac{dx}{(x+2)^2+9} \\ &= \int \frac{1}{(x+2)^2+(3)^2} dx \\ \because \int \frac{1}{a^2+x^2} dx &= \frac{1}{a} \tan^{-1} \frac{x}{a} \\ &= \frac{1}{3} \tan^{-1} \left( \frac{x+2}{3} \right) + c \end{aligned}$$

**Q. 3:**  $\int \frac{x^2}{4+x^2} dx$

**SOLUTION:**

$$\begin{aligned} & (+) \text{ and } (-) 4 \\ &= \int \frac{4+x^2-4}{4+x^2} dx \\ &= \int \left( \frac{4+x^2}{4+x^2} - \frac{4}{4+x^2} \right) dx \\ &= \int 1 dx - \int \frac{4}{4+x^2} dx \\ &= \int 1 dx - 4 \int \frac{1}{2^2+x^2} dx \\ &= x - 4 \cdot \frac{1}{2} \tan^{-1} \left( \frac{x}{2} \right) + c \\ &= x - 2 \tan^{-1} \left( \frac{x}{2} \right) + c \end{aligned}$$

**Q. 4:**  $\int \frac{1}{x \ln x} dx$

**SOLUTION:**

$$\begin{aligned} & \int \frac{1}{\ln x} \cdot \frac{1}{x} dx \\ & \text{As } f(x) = \ln x \\ & \text{And } f'(x) = \frac{1}{x}, \text{ so} \\ & \text{using } \int \frac{f'(x)}{[f(x)]} = \ln[f(x)] \\ &= \ln[\ln x] + c \end{aligned}$$

**Q. 5:**  $\int \frac{e^x}{e^x+3} dx$

**SOLUTION:**

$$\begin{aligned} & \int \frac{e^x}{e^x+3} dx \\ & \text{Here } f(x) = e^x \\ & \text{And } f'(x) = e^x, \text{ so} \end{aligned}$$

$$\text{using } \int \frac{f'(x)}{[f(x)]} = \ln[f(x)] + c$$

$$= \ln(e^x + 3) + c$$

**Q. 6:**  $\int \frac{x+b}{(x^2+2bx+c)^{\frac{1}{2}}} dx$

**SOLUTION:**

$$\begin{aligned} & \int \frac{x+b}{(x^2+2bx+c)^{\frac{1}{2}}} dx \\ & \int (x^2+2bx+c)^{-\frac{1}{2}} \cdot (x+b) dx \\ & \text{Here } f(x) = x^2+2bx+c \\ & \text{Here } f'(x) = 2x+2b = 2(x+b) \\ & \quad \times \text{ and } \div \text{ by } 2 \\ &= \frac{1}{2} \int (x^2+2bx+c)^{-\frac{1}{2}} \cdot 2(x+b) dx \\ &= \frac{1}{2} \frac{(x^2+2bx+c)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c \\ &= \frac{1}{2} \frac{(x^2+2bx+c)^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= \sqrt{x^2+2bx+c} + c \end{aligned}$$

**Q. 7:**  $\int \frac{\sec^2 x}{\sqrt{\tan x}} dx$

**SOLUTION:**

$$\begin{aligned} & \int \frac{\sec^2 x}{\sqrt{\tan x}} dx \\ &= \int (\tan x)^{-\frac{1}{2}} \sec^2 x \\ & \text{Here } f(x) = \tan x \\ & \text{Here } f'(x) = \sec^2 x \\ &= \frac{(\tan x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c \\ &= \frac{(\tan x)^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= 2\sqrt{\tan x} + c \end{aligned}$$

**Q. 8: (a) Show that**

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln(x + \sqrt{x^2-a^2}) + c$$

**SOLUTION:**

$$\begin{aligned} \text{L.H.S} &= \int \frac{dx}{\sqrt{x^2-a^2}} \\ \text{Put } x &= a \sec \theta \Rightarrow dx = a \sec \theta \tan \theta d\theta \\ &= \int \frac{a \sec \theta \tan \theta d\theta}{\sqrt{(a \sec \theta)^2 - a^2}} = \int \frac{a \sec \theta \tan \theta}{\sqrt{a^2 \sec^2 \theta - a^2}} d\theta \\ &= \int \frac{a \sec \theta \tan \theta}{\sqrt{a^2(\sec^2 \theta - 1)}} d\theta = \int \frac{\sec \theta \tan \theta}{\sqrt{\tan^2 \theta}} d\theta \\ &= \int \frac{\sec \theta \tan \theta}{\tan \theta} d\theta = \int \sec \theta d\theta \\ &= \ln|\sec \theta + \tan \theta| + c_1 \end{aligned}$$

Then back substitution:

$$x = a \sec \theta \Rightarrow \frac{x}{a} = \sec \theta$$

$$\text{And } 1 + \tan^2 = \sec^2 \theta$$

$$\tan^2 \theta = \sec^2 \theta - 1$$

$$\tan \theta = \sqrt{\sec^2 \theta - 1}$$

$$\tan \theta = \sqrt{\left(\frac{x}{a}\right)^2 - 1}$$

$$\tan \theta = \sqrt{\frac{x^2-a^2}{a^2}}$$

$$\tan \theta = \frac{\sqrt{x^2-a^2}}{a}$$

Now put values

$$= \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + c_1$$

$$= \ln \left| \frac{x + \sqrt{x^2 - a^2}}{a} \right| + c_1$$

Using  $\ln \frac{A}{B} = \ln A - \ln B$

$$= \ln |x + \sqrt{x^2 - a^2}| - \ln a + c_1$$

Where  $c = -\ln a + c_1$

$$= \ln |x + \sqrt{x^2 - a^2}| + c$$

**Q. 9:**  $\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$

**SOLUTION:**

$$\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$$

Put  $x = \tan \theta$

$$\Rightarrow d(x) = \sec^2 \theta d\theta$$

$$= \int \frac{\sec^2 \theta}{(1+\tan^2 \theta)^{\frac{3}{2}}} d\theta$$

$$= \int \frac{\sec^2 \theta}{(\sec^2 \theta)^{\frac{3}{2}}} d\theta$$

$$= \int \frac{\sec^2 \theta}{\sec^3 \theta} d\theta$$

$$= \int \frac{1}{\sec \theta} d\theta$$

$$= \int \cos \theta$$

$$= \frac{\sin \theta}{1} + c$$

$$= \frac{\sin \theta}{\cos \theta} \cdot \cos \theta + c$$

$$= \tan \theta \cdot \cos \theta + c$$

$$= \frac{\tan \theta}{\sec \theta} + c$$

$$= \frac{\tan \theta}{\sqrt{\sec^2 \theta}} + c$$

$$= \frac{\tan \theta}{\sqrt{1+\tan^2 \theta}} + c$$

Put  $\tan \theta = x$

$$= \frac{x}{\sqrt{1+x^2}} + c$$

**Q. 10:**  $\int \frac{1}{(1+x^2)\tan^{-1}x} dx$

**SOLUTION:**

$$\int \frac{1}{(1+x^2)\tan^{-1}x} dx$$

$$\int \frac{1}{\tan^{-1}x} \cdot \frac{1}{(1+x^2)} dx$$

Here  $f(x) = \tan^{-1}x$

$$\text{Here } f'(x) = \frac{1}{(1+x^2)}$$

$$\text{using } \int \frac{f'(x)}{[f(x)]} = \ln[f(x)] + c$$

$$= \ln|\tan^{-1}x| + c$$

**Q. 11:**  $\int \sqrt{\frac{1+x}{1-x}} dx$

**SOLUTION:**

By rationalizing

$$= \int \sqrt{\frac{1+x}{1-x}} \times \sqrt{\frac{1+x}{1+x}} dx$$

$$= \int \sqrt{\frac{1+x}{1-x}} \times \frac{1+x}{1+x} dx$$

$$= \int \sqrt{\frac{(1+x)^2}{1-x^2}} dx$$

$$= \int \frac{1+x}{\sqrt{1-x^2}} dx$$

$$= \int \frac{1}{\sqrt{1-x^2}} dx + \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= \sin^{-1} x + \frac{1}{-2} \int (1-x^2)^{-\frac{1}{2}} (-2x) dx$$

$$= \sin^{-1} x - \frac{1}{2} \cdot \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \sin^{-1} x - \frac{1}{2} \cdot \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \sin^{-1} x - \sqrt{1-x^2} + c$$

**Q. 12:**  $\int \frac{\sin \theta}{1+\cos^2 \theta} d\theta$

**SOLUTION:**

$$\int \frac{\sin \theta}{1+\cos^2 \theta} d\theta = \int \frac{1}{1+\cos^2 \theta} \sin \theta d\theta$$

$$\text{Put } \cos \theta = t \Rightarrow -\sin \theta d\theta = dt$$

$$\int \frac{1}{1+t^2} \cdot -dt = -\tan^{-1} t + c$$

Put  $t = \cos \theta$

$$= -\tan^{-1}(\cos \theta) + c$$

**Q. 13:**  $\int \frac{ax}{\sqrt{a^2-x^4}} dx$

**SOLUTION:**

$$\int \frac{ax}{\sqrt{a^2-x^4}} dx = a \int \frac{x}{\sqrt{a^2-(x^2)^2}} dx$$

$$\text{Put } x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{1}{2} dt$$

$$= \frac{a}{2} \int \frac{1}{\sqrt{a^2-t^2}} dt = \frac{a}{2} \sin^{-1} \frac{t}{a} + c$$

$$\text{using } \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a}$$

$$= \frac{a}{2} \sin^{-1} \frac{x^2}{a} + c \quad \because x^2 = t$$

**Q. 14:**  $\int \frac{dx}{\sqrt{7-6x-x^2}}$

**SOLUTION:**

$$\int \frac{dx}{\sqrt{7-6x-x^2}}$$

By completing square

$$= \int \frac{dx}{\sqrt{7-x^2-6x-9+9}}$$

$$= \int \frac{dx}{\sqrt{7-(x^2+6x+9)+9}}$$

$$= \int \frac{dx}{\sqrt{16-(x+3)^2}}$$

$$\text{Using } \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a} + c$$

$$= \sin^{-1} \frac{x+3}{4} + c$$

**Q. 15:**  $\int \frac{\cos x}{\sin x \ln \sin x} dx$

**SOLUTION:**

$$\int \frac{1}{\ln \sin x} \frac{\cos x}{\sin x} dx$$

Here  $f(x) = \ln \sin x$

$$\text{And } f'(x) = \frac{\cos x}{\sin x}, \text{ so}$$

$$\text{using } \int \frac{f'(x)}{[f(x)]} = \ln[f(x)] + c$$

$$= \ln[\ln \sin x] + c$$

**Q. 16:**  $\int \cos x \frac{\ln \sin x}{\sin x} dx$

**SOLUTION:**

$$\int \ln \sin x \cdot \frac{\cos x}{\sin x} dx$$

Here  $f(x) = \ln \sin x$

$$\text{And } f'(x) = \frac{\cos x}{\sin x}, \text{ so}$$

$$= \frac{[\ln \sin x]^{1+1}}{1+1} + c$$

$$= \frac{1}{2} [\ln \sin x]^2 + c$$

**Q. 17:**  $\int \frac{x dx}{4+2x+x^2}$

**SOLUTION:**

$$\int \frac{x dx}{4+2x+x^2}$$

$$= \frac{1}{2} \int \frac{2x}{4+2x+x^2} dx$$

$$= \frac{1}{2} \int \frac{2x+2-2}{4+2x+x^2} dx$$

$$= \frac{1}{2} \left\{ \int \frac{2x+2}{4+2x+x^2} dx - \int \frac{2}{4+2x+x^2} dx \right\}$$

$$= \frac{1}{2} \left\{ \ln(4+2x+x^2) - \int \frac{2}{x^2+2x+1^2+4-1^2} dx \right\}$$

$$= \frac{1}{2} \ln(4+2x+x^2) - \frac{1}{2} \int \frac{2}{(x+1)^2+(\sqrt{3})^2} dx$$

using  $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

$$= \frac{1}{2} \ln(x^2+2x+4) - \frac{1}{\sqrt{3}} \tan^{-1} \frac{(x+1)}{\sqrt{3}} + c$$

**Q. 18:**  $\int \frac{x}{x^4+2x^2+5} dx$

**SOLUTION:**

$$= \int \frac{x}{(x^2)^2+2x^2+5} dx$$

Put  $x^2 = t$

$$2x dx = dt$$

$$x dx = \frac{1}{2} dt$$

$$= \int \frac{\frac{1}{2}}{t^2+2t+5} dt$$

$$= \frac{1}{2} \int \frac{1}{t^2+2t+1+5-1} dt$$

$$= \frac{1}{2} \int \frac{1}{(t+1)^2+2^2} dt$$

$$= \frac{1}{2} \cdot \frac{1}{2} \tan^{-1} \frac{t+1}{2} + c$$

Put  $x^2 = t$

$$= \frac{1}{4} \tan^{-1} \frac{x^2+1}{2} + c$$

**Q. 19:**  $\int \left[ \cos \left( \sqrt{x} - \frac{x}{2} \right) \times \left( \frac{1}{\sqrt{x}} - 1 \right) \right] dx$

**SOLUTION:**

$$\int \left[ \cos \left( \sqrt{x} - \frac{x}{2} \right) \times \left( \frac{1}{\sqrt{x}} - 1 \right) \right] dx$$

Put  $\sqrt{x} - \frac{x}{2} = t$

$$\Rightarrow d \left( \sqrt{x} - \frac{x}{2} \right) = d(t)$$

$$\frac{1}{2\sqrt{x}} - \frac{1}{2} = dt$$

$$\frac{1}{2} \left( \frac{1}{\sqrt{x}} - 1 \right) = dt$$

$$\left( \frac{1}{\sqrt{x}} - 1 \right) = 2dt$$

$$= \int [\cos t \times 2 dt]$$

$$= 2 \int [\cos t dt]$$

$$= 2 \frac{\sin t}{1} + c$$

Put value of  $t$

$$= 2 \sin \left( \sqrt{x} - \frac{x}{2} \right) + c$$

**Q. 20:**  $\int \frac{x+2}{\sqrt{x+3}} dx$

[Q. 19: solve on page 9]

**SOLUTION:**

$$\int \frac{x+2}{\sqrt{x+3}} dx = \int \frac{x+2+1-1}{\sqrt{x+3}} dx = \int \frac{x+3}{\sqrt{x+3}} dx - \int \frac{1}{\sqrt{x+3}} dx =$$

$$\int \sqrt{x+3} dx - \int \frac{1}{\sqrt{x+3}} dx = \int (x+3)^{\frac{1}{2}} \cdot 1 dx - \int (x+3)^{-\frac{1}{2}} \cdot 1 dx$$

Now integrate

$$\frac{(x+3)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{(x+3)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x+3)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \frac{2}{3} (x+3)^{\frac{3}{2}} + 2 \sqrt{x+3} + c$$

**Q. 21:**  $\int \frac{\sqrt{2}}{\sin x + \cos x} dx$

**SOLUTION:**

$$\int \frac{1}{\frac{1}{\sqrt{2}}(\cos x + \sin x)} dx$$

$$= \int \frac{1}{\cos x \cdot \frac{1}{\sqrt{2}} + \sin x \cdot \frac{1}{\sqrt{2}}} dx$$

$$= \int \frac{1}{\cos x \cos 45^\circ + \sin x \sin 45^\circ} dx$$

using  $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

$$= \int \frac{1}{\cos(x-45^\circ)} dx$$

$$= \int \sec(x-45^\circ) dx$$

using  $\int \sec x dx = \ln|\sec x + \tan x| + c$

$$\ln|\sec(x-45^\circ) + \tan(x-45^\circ)| + c$$

**Q. 22:**  $\int \frac{dx}{\frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x}$

**SOLUTION:**

$$= \int \frac{1}{\sin x \cdot \frac{1}{2} + \cos x \cdot \frac{\sqrt{3}}{2}} dx$$

$$= \int \frac{1}{\sin x \cdot \cos 60^\circ + \cos x \cdot \sin 60^\circ} dx$$

using  $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$$= \int \frac{1}{\sin(x+60^\circ)} dx$$

$$= \int \operatorname{cosec}(x+60^\circ) dx$$

using  $\int \operatorname{cosec} x dx = \ln|\operatorname{cosec} x - \cot x| + c$

$$= \ln|\operatorname{cosec}(x-60^\circ) + \cot(x-60^\circ)| + c$$

### Integration by parts.

We know that for two functions  $f$  and  $g$

$$\frac{d}{dx} (f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

$$\Rightarrow f(x)g'(x) = \frac{d}{dx} (f(x)g(x)) - f'(x)g(x)$$

Taking integrations w.r.t  $x$  we get

$$\int f(x)g'(x) dx = \int \left[ \frac{d}{dx} f(x)g(x) - f'(x)g(x) \right] dx$$

$$= \int \left( \frac{d}{dx} (f(x)g(x)) \right) - \int f'(x)g(x) dx$$

$$\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx$$

Or  $\int f(x)g'(x) dx = f(x) \int g'(x) dx - \int (g'(x)dx) f'(x) dx$

In other words.

$$\int (1st \text{ function})(2nd \text{ function}) dx$$

$$= (1st \text{ funct.}) \cdot \int (2nd \text{ funct.}) dx$$

$$- \int (\text{integrated funct.}) \frac{d}{dx} (\text{1st function}) dx$$

This is called "integrations by parts"

### Some basic rules for Integration by parts.

\*some the function as 2<sup>nd</sup> function whose integration is known or possible.

\*if integration of both given functions are known but one of the given function is polynomial functions then whose polynomial function as first function.

\*if integration of both given function are known but no one is polynomial function. Then we may choose any function as 1<sup>st</sup>.

\*if we are given only one function whose integration is unknown or cannot be easily find.

$$i. e, \sin^{-1} x, \cos^{-1} x, \sqrt{a^2 - x^2}, \frac{1}{\sqrt{x^2 - a^2}} e. t. c$$

Then we take 1 as 2<sup>nd</sup> function.

### "Review above Rules"

$\int x^n \cos x dx$	1 <sup>st</sup> function $x^n$	2 <sup>nd</sup> function $\cos x$
$\int x^n \sin x dx$	$x^n$	$\sin x$
$\int x^n \sin^{-1} x dx$	$\sin^{-1} x$	$x^n$
$\int x^n \tan^{-1} x dx$	$\tan^{-1} x$	$x^n$
$\int e^x \sin x dx$	$e^x$ or $\sin x$	$\sin x$ or $e^x$
$\int \ln x x^n dx$	$x^n$	$\ln x$
$\int \tan^{-1} x dx$	$\tan^{-1} x$	1
$\int \sqrt{a^2 + x^2} dx$	$\sqrt{a^2 + x^2}$	1

You may remember the word "ILATE"

I=inverse function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential functions.

### \*Remember useful formulas\*

$$1. \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left( \frac{x}{a} \right) + c$$

$$2. \int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln |x + \sqrt{x^2 - a^2}| + c$$

$$3. \int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln |x + \sqrt{x^2 + a^2}| + c$$

Prove that  $\int e^x (f(x) + f'(x)) dx = e^x f(x) + c$

Prove:  $\int e^x f(x) dx = f(x) e^x - \int e^x f'(x) dx$

$$\Rightarrow \int e^x f(x) dx + \int e^x f'(x) dx = e^x f(x)$$

$$\Rightarrow \int e^x (f(x) + f'(x)) dx = e^x f(x) + c$$

Hence proved.

## Exercise 3.4

i)  $\int x \sin x dx$

**SOLUTION:**

$$\int x \sin x dx$$

Here  $U = x$ ,  $V = \sin x$

$$\text{Using } \int U.V = U \cdot \int V dx - \int [U' \cdot \int V dx] dx$$

$$= x \cdot \int \sin x dx - \int [(x)' \cdot \int \sin x dx] dx$$

$$= x \cdot (-\cos x) - \int [1 \cdot (-\cos x)] dx$$

$$= -x \cos x - \int [-\cos x] dx$$

$$= -x \cos x + \int [\cos x] dx$$

$$= -x \cos x + \sin x + c$$

$$= \sin x - x \cos x + c$$

ii)  $\int \ln x dx$

**SOLUTION:**

$$\int \ln x \cdot 1 dx$$

Here  $U = \ln x$ ,  $V = 1$

$$\text{Using } \int U.V = U \cdot \int V dx - \int [U' \cdot \int V dx] dx$$

$$= \ln x \cdot \int 1 dx - \int [(\ln x)' \cdot \int 1 dx] dx$$

$$= \ln x \cdot x - \int \left[ \frac{1}{x} \cdot x \right] dx$$

$$= \ln x \cdot x - \int 1 dx$$

$$= x \ln x - x + c$$

iii)  $\int x \ln x dx$

**SOLUTION:**

$$\int x \ln x dx$$

Here  $U = \ln x$ ,  $V = x$

$$\text{Using } \int U.V = U \cdot \int V dx - \int [U' \cdot \int V dx] dx$$

$$= \ln x \cdot \int x dx - \int [(\ln x)' \cdot \int x dx] dx$$

$$= \ln x \cdot \frac{x^2}{2} - \int \left[ \frac{1}{x} \cdot \frac{x^2}{2} \right] dx$$

$$= \ln x \cdot \frac{x^2}{2} - \frac{1}{2} \int x dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + c$$

$$= \frac{x^2}{2} \left( \ln x - \frac{1}{2} \right) + c$$

iv)  $\int x^2 \ln x dx$

**SOLUTION:**

$$\int x^2 \ln x dx$$

Here  $U = \ln x$ ,  $V = x^2$

$$\text{Using } \int U.V = U \cdot \int V dx - \int [U' \cdot \int V dx] dx$$

$$= \ln x \cdot \int x^2 dx - \int [(\ln x)' \cdot \int x^2 dx] dx$$

$$= \ln x \cdot \frac{x^3}{3} - \int \left[ \frac{1}{x} \cdot \frac{x^3}{3} \right] dx$$

$$= \ln x \cdot \frac{x^3}{3} - \frac{1}{3} \int x^2 dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \cdot \frac{x^3}{3} + c$$

$$= \frac{x^3}{3} \left( \ln x - \frac{1}{3} \right) + c$$

v)  $\int x^3 \ln x dx$

**SOLUTION:**

$$\int x^3 \ln x dx$$

Here  $U = \ln x$ ,  $V = x^3$

$$\text{Using } \int U.V = U \cdot \int V dx - \int [U' \cdot \int V dx] dx$$

$$= \ln x \cdot \int x^3 dx - \int [(\ln x)' \cdot \int x^3 dx] dx$$

$$= \ln x \cdot \frac{x^4}{4} - \int \left[ \frac{1}{x} \cdot \frac{x^4}{4} \right] dx$$

$$= \ln x \cdot \frac{x^4}{4} - \frac{1}{4} \int x^3 dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \cdot \frac{x^4}{4} + c$$

$$= \frac{x^4}{4} \left( \ln x - \frac{1}{4} \right) + c$$

vi)  $\int x^4 \ln x \, dx$ **SOLUTION:**

$$\int x^4 \ln x \, dx$$

Here  $U = \ln x$ ,  $V = x^4$

Using  $\int U.V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$

$$= \ln x \cdot \int x^4 \, dx - \int [(\ln x)' \cdot \int x^4 \, dx] \, dx$$

$$= \ln x \cdot \frac{x^5}{5} - \int \left[ \frac{1}{x} \cdot \frac{x^5}{5} \right] \, dx$$

$$= \ln x \cdot \frac{x^5}{5} - \frac{1}{5} \int x^4 \, dx$$

$$= \frac{x^5}{5} \ln x - \frac{1}{5} \cdot \frac{x^5}{5} + c$$

$$= \frac{x^5}{5} \left( \ln x - \frac{1}{5} \right) + c$$

vii)  $\int \tan^{-1} x \, dx$ **SOLUTION:**

$$\int 1 \cdot \tan^{-1} x \, dx$$

Here  $U = \tan^{-1} x$ ,  $V = 1$

Using  $\int U.V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$

$$= \tan^{-1} x \cdot \int 1 \, dx - \int [(\tan^{-1} x)' \cdot \int 1 \, dx] \, dx$$

$$= \tan^{-1} x \cdot x - \int \left[ \frac{1}{1+x^2} \cdot x \right] \, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \ln|1+x^2| + c$$

viii)  $\int x^2 \sin x \, dx$ **SOLUTION:**

$$\int x^2 \sin x \, dx$$

Here  $U = x^2$ ,  $V = \sin x$

Using  $\int U.V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$

$$= x^2 \cdot \int \sin x \, dx - \int [(x^2)' \cdot \int \sin x \, dx] \, dx$$

$$= x^2 \cdot (-\cos x) - \int [2x \cdot (-\cos x)] \, dx$$

$$= -x^2 \cos x + 2 \int [x \cdot \cos x] \, dx$$

Using  $\int U.V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$

$$= -x^2 \cos x + 2 \{ x \cdot \int \cos x \, dx - \int [(x)' \cdot \int \cos x \, dx] \, dx \}$$

$$= -x^2 \cos x + 2 \{ x \cdot \sin x - \int [1 \cdot \sin x] \, dx \}$$

$$= -x^2 \cos x + 2x \cdot \sin x - 2 \int \sin x \, dx$$

$$= -x^2 \cos x + 2x \cdot \sin x - 2(-\cos x) + c$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

ix)  $\int x^2 \tan^{-1} x \, dx$ **SOLUTION:**

$$\int x^2 \cdot \tan^{-1} x \, dx$$

Here  $U = \tan^{-1} x$ ,  $V = x^2$

Using  $\int U.V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$

$$= \tan^{-1} x \cdot \int x^2 \, dx - \int [(\tan^{-1} x)' \cdot \int x^2 \, dx] \, dx$$

$$= \tan^{-1} x \cdot \frac{x^3}{3} - \int \left[ \frac{1}{1+x^2} \cdot \frac{x^3}{3} \right] \, dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} \, dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \left( x - \frac{x}{1+x^2} \right) \, dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int x \, dx + \frac{1}{3 \cdot 2} \int \frac{2x}{1+x^2} \, dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \cdot \frac{x^2}{2} + \frac{1}{6} \ln|1+x^2| + c$$

$$\frac{x}{1+x^2\sqrt{x^3}}$$

$$\frac{\pm x^3 \pm 1}{-1}$$

$$\therefore \int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + c$$

x)  $\int x \tan^{-1} x \, dx$ **SOLUTION:**

$$\int x \cdot \tan^{-1} x \, dx$$

Here  $U = \tan^{-1} x$ ,  $V = x$

Using  $\int U.V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$

$$= \tan^{-1} x \cdot \int x \, dx - \int [(\tan^{-1} x)' \cdot \int x \, dx] \, dx$$

$$= \tan^{-1} x \cdot \frac{x^2}{2} - \int \left[ \frac{1}{1+x^2} \cdot \frac{x^2}{2} \right] \, dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left( \frac{1+x^2-1}{1+x^2} \right) \, dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{1+x^2}{1+x^2} \, dx - \frac{1}{2} \int \frac{1}{1+x^2} \, dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int 1 \, dx - \frac{1}{2} \tan^{-1} x$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} x - \frac{1}{2} \tan^{-1} x + c$$

$$= \left( \frac{1}{2} \tan^{-1} x \right) (x^2 + 1) - \frac{1}{2} x + c$$

xi)  $\int x^3 \tan^{-1} x \, dx$ **SOLUTION:**

$$\int x^3 \cdot \tan^{-1} x \, dx$$

Here  $U = \tan^{-1} x$ ,  $V = x^3$

Using  $\int U.V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$

$$= \tan^{-1} x \cdot \int x^3 \, dx - \int [(\tan^{-1} x)' \cdot \int x^3 \, dx] \, dx$$

$$= \tan^{-1} x \cdot \frac{x^4}{4} - \int \left[ \frac{1}{1+x^2} \cdot \frac{x^4}{4} \right] \, dx$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \frac{x^4}{1+x^2} \, dx$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \left( x^2 - 1 + \frac{1}{1+x^2} \right) \, dx$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int x^2 \, dx + \frac{1}{4} \int 1 \, dx - \frac{1}{4} \int \frac{1}{1+x^2} \, dx$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \cdot \frac{x^3}{3} + \frac{1}{4} x - \frac{1}{4} \tan^{-1} x + c$$

$$= \frac{1}{4} \left[ x^4 \tan^{-1} x - \frac{x^3}{3} + x - \tan^{-1} x \right] + c$$

$$= \frac{1}{4} \left[ (x^4 - 1) \tan^{-1} x - \frac{x^3}{3} + x \right] + c$$

$$= \frac{1}{4} \left[ x^4 \tan^{-1} x - \frac{x^3}{3} + x - \tan^{-1} x \right] + c$$

xii)  $\int x^3 \cos x \, dx$ **SOLUTION:**

$$\int x^3 \cos x \, dx$$

Here  $U = x^3$ ,  $V = \cos x$

Using  $\int U.V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$

$$= x^3 \cdot \int \cos x \, dx - \int [(x^3)' \cdot \int \cos x \, dx] \, dx$$

$$= x^3 \cdot (\sin x) - \int [3x^2 \cdot (\sin x)] \, dx$$

$$= x^3 \sin x - 3 \int [x^2 \sin x] \, dx$$

Using  $\int U.V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$

$$= x^3 \sin x - 3 \{ x^2 \cdot \int \sin x \, dx - \int [(x^2)' \cdot \int \sin x \, dx] \, dx \}$$

$$= x^3 \sin x - 3 \{ x^2 \cdot (-\cos x) - \int [2x \cdot (-\cos x)] \, dx \}$$

$$= x^3 \sin x + 3x^2 \cos x - 6 \int x \cos x \, dx$$

$$= x^3 \sin x + 3x^2 \cos x - 6 \{ x \int \cos x \, dx - \int [(x)' \cdot \int \cos x \, dx] \, dx \}$$

$$= x^3 \sin x + 3x^2 \cos x - 6 \{ x \sin x - \int [1 \cdot \sin x] \, dx \}$$

$$= x^3 \sin x + 3x^2 \cos x - 6 \{ x \sin x - \int \sin x \, dx \}$$

$$= x^3 \sin x + 3x^2 \cos x - 6 \{ x \sin x - (-\cos x) \} + c$$

$$= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + c$$

$$= (x^3 - 6x) \sin x + (3x^2 - 6) \cos x + c$$

$$\text{xiii) } \int \sin^{-1} x \, dx$$

**SOLUTION:**

$$\int 1 \cdot \sin^{-1} x \, dx$$

$$\text{Here } U = \sin^{-1} x, V = 1$$

$$\text{Using } \int U \cdot V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$$

$$= \sin^{-1} x \cdot \int 1 \, dx - \int [(\sin^{-1} x)' \cdot \int 1 \, dx] \, dx$$

$$= \sin^{-1} x \cdot x - \int \left[ \frac{1}{\sqrt{1-x^2}} \cdot x \right] \, dx \quad \text{skip}$$

$$= x \sin^{-1} x - \frac{1}{-2} \int (1-x^2)^{-\frac{1}{2}} (-2x) \, dx$$

$$= x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + c$$

$$\text{xiv) } \int x \sin^{-1} x \, dx$$

**SOLUTION:**

$$\int x \cdot \sin^{-1} x \, dx$$

$$\text{Here } U = \sin^{-1} x, V = x$$

$$\text{Using } \int U \cdot V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$$

$$= \sin^{-1} x \cdot \int x \, dx - \int [(\sin^{-1} x)' \cdot \int x \, dx] \, dx$$

$$= \sin^{-1} x \cdot \frac{x^2}{2} - \int \left[ \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} \right] \, dx$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} \, dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} \, dx + c$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} \, dx + c$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1-x^2}{\sqrt{1-x^2}} \, dx - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} \, dx + c$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \sqrt{1-x^2} \, dx - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} \, dx + c$$

$$\text{Using } \int \sqrt{a^2-x^2} \, dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2-x^2} + c$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \left\{ \frac{1}{2} \sin^{-1} x + \frac{x}{2} \sqrt{1-x^2} \right\} - \frac{1}{2} \sin^{-1} x + c$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{4} \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} - \frac{1}{2} \sin^{-1} x + c$$

$$= \frac{x^2}{2} \sin^{-1} x + \left( \frac{1}{4} - \frac{1}{2} \right) \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} + c$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{4} \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} + c$$

$$\text{xv) } \int e^x \sin x \cos x \, dx$$

**SOLUTION:**

$$\text{Let } I = \int e^x \sin x \cos x \, dx$$

Multiply and divide by 2

$$I = \frac{1}{2} \int e^x 2 \sin x \cos x \, dx$$

$$I = \frac{1}{2} \int e^x \sin 2x \, dx$$

$$\text{Here } U = \sin 2x, V = e^x$$

$$\text{Using } \int U \cdot V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$$

$$I = \sin 2x \int e^x \, dx - \int [(\sin 2x)' \cdot \int e^x \, dx] \, dx$$

$$I = \sin 2x e^x - \int [\cos 2x \cdot 2 \cdot e^x] \, dx$$

$$I = \sin 2x e^x - 2 \int \cos 2x e^x \, dx$$

$$\text{Using } \int U \cdot V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$$

$$I = \sin 2x e^x - 2 \{ \cos 2x \int e^x \, dx - \int [(\cos 2x)' \cdot \int e^x \, dx] \, dx \}$$

$$I = \sin 2x e^x - 2 \{ \cos 2x e^x - \int (-\sin 2x) e^x \, dx \}$$

$$I = e^x \sin 2x - 2 \cos 2x e^x - 2 \int 2 \sin x \cos x e^x \, dx$$

$$I = e^x \sin 2x - 2 \cos 2x e^x - 4 \int \sin x \cos x e^x \, dx$$

$$\text{Put } I = \int e^x \sin x \cos x \, dx$$

$$I = e^x \sin 2x - 2 \cos 2x e^x - 4I$$

$$5I = e^x (\sin 2x - 2 \cos 2x)$$

$$I = \frac{e^x}{5} (\sin 2x - 2 \cos 2x)$$

$$\text{xvi) } \int x \sin x \cos x \, dx$$

**SOLUTION:**

$$\int x \sin x \cos x \, dx = \frac{1}{2} \int x \cdot 2 \sin x \cos x \, dx = \frac{1}{2} \int x \sin 2x \, dx$$

$$\text{Here } U = x, V = \sin 2x$$

$$\text{Using } \int U \cdot V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$$

$$= \frac{1}{2} [x \cdot \int \sin 2x \, dx - \int [(x)'] \cdot \int \sin 2x \, dx] \, dx$$

$$= \frac{1}{2} x \cdot \left( -\frac{\cos 2x}{2} \right) - \frac{1}{2} \int \left[ 1 \left( -\frac{\cos 2x}{2} \right) \right] \, dx =$$

$$-\frac{1}{4} x \cos 2x + \frac{1}{4} \int \cos 2x \, dx$$

$$= -\frac{1}{4} x \cos 2x + \frac{1}{4} \frac{\sin 2x}{2} + c = \frac{1}{4} \left[ -x \cos 2x + \frac{\sin 2x}{2} \right] + c$$

$$= \frac{1}{4} \left[ -x \cos 2x + \frac{2 \sin x \cos x}{2} \right] + c$$

$$= \frac{1}{4} \left[ -x \cos 2x + \sin x \cos x \right] + c = \frac{1}{4} [\sin x \cos x - x \cos 2x] + c$$

$$\text{xvii) } \int x \cos^2 x \, dx$$

**SOLUTION:**

$$\int x \cos^2 x \, dx = \int x \cdot \frac{1+\cos 2x}{2} \, dx \quad \text{As } \cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{2} \int x \cdot (1 + \cos 2x) \, dx = \frac{1}{2} \int (x + x \cos 2x) \, dx$$

$$= \frac{1}{2} \int x \, dx + \frac{1}{2} \int x \cos 2x \, dx = \frac{1}{2} \frac{x^2}{2} + \frac{1}{2} \int x \cos 2x \, dx$$

$$\text{Here } U = x, V = \cos 2x$$

$$\text{Using } \int U \cdot V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$$

$$= \frac{x^2}{4} + \frac{1}{2} [x \cdot \int \cos 2x \, dx - \int [x'] \cdot \int \cos 2x \, dx] \, dx$$

$$= \frac{x^2}{4} + \frac{1}{2} \left[ x \cdot \frac{\sin 2x}{2} - \int \left[ 1 \cdot \frac{\sin 2x}{2} \right] \, dx \right]$$

$$= \frac{x^2}{4} + \frac{1}{2} x \cdot \frac{\sin 2x}{2} - \frac{1}{4} \int \sin 2x \, dx$$

$$= \frac{x^2}{4} + \frac{x \sin 2x}{4} - \frac{1}{4} \frac{-\cos 2x}{2}$$

$$= \frac{1}{4} \left( x^2 + x \sin 2x + \frac{1}{2} \cos 2x \right) + c$$

$$\text{xviii) } \int x \sin^2 x \, dx$$

**SOLUTION:**

$$\int x \sin^2 x \, dx = \int x \cdot \frac{1-\cos 2x}{2} \, dx \quad \text{As } \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{1}{2} \int x \cdot (1 - \cos 2x) \, dx = \frac{1}{2} \int (x - x \cos 2x) \, dx$$

$$= \frac{1}{2} \int x \, dx - \frac{1}{2} \int x \cos 2x \, dx = \frac{1}{2} \frac{x^2}{2} - \frac{1}{2} \int x \cos 2x \, dx$$

$$\text{Here } U = x, V = \cos 2x$$

$$\text{Using } \int U \cdot V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$$

$$= \frac{x^2}{4} - \frac{1}{2} [x \cdot \int \cos 2x \, dx - \int [x'] \cdot \int \cos 2x \, dx] \, dx$$

$$\begin{aligned}
&= \frac{x^2}{4} - \frac{1}{2} \left[ x \cdot \frac{\sin 2x}{2} - \int \left[ 1 \cdot \frac{\sin 2x}{2} \right] dx \right] \\
&= \frac{x^2}{4} - \frac{1}{2} x \cdot \frac{\sin 2x}{2} + \frac{1}{4} \int \sin 2x dx \\
&= \frac{x^2}{4} - \frac{x \sin 2x}{4} + \frac{1}{4} \left( \frac{-\cos 2x}{2} \right) + c \\
&= \frac{1}{4} \left( x^2 - x \sin 2x - \frac{1}{2} \cos 2x \right) + c
\end{aligned}$$

**xix)  $\int (\ln x)^2 dx$**

**SOLUTION:**

$$\int (\ln x)^2 \cdot 1 dx$$

Here  $U = (\ln x)^2$ ,  $V = 1$

Using  $\int U \cdot V = U \cdot \int V dx - \int [U' \cdot \int V dx] dx$

$$= (\ln x)^2 \cdot \int 1 dx - \int [((\ln x)^2)' \cdot \int 1 dx] dx$$

$$= (\ln x)^2 \cdot x - \int \left[ 2(\ln x) \frac{1}{x} \cdot x \right] dx$$

$$= x(\ln x)^2 - 2 \int \ln x dx$$

$$= x(\ln x)^2 - 2 \left[ \int (\ln x) \cdot 1 dx \right]$$

Here  $U = \ln x$ ,  $V = 1$

Using  $\int U \cdot V = U \cdot \int V dx - \int [U' \cdot \int V dx] dx$

$$= x(\ln x)^2 - 2 \left[ \ln x \cdot \int 1 dx - \int [(\ln x)' \cdot \int 1 dx] dx \right]$$

$$= x(\ln x)^2 - 2 \left[ \ln x \cdot x - \int \left[ \frac{1}{x} \cdot x \right] dx \right]$$

$$= x(\ln x)^2 - 2 \left[ \ln x \cdot x - \int 1 dx \right]$$

$$= x(\ln x)^2 - 2 \left[ x \ln x - x \right] + c$$

$$= x(\ln x)^2 - 2x \ln x + 2x + c$$

$$= x \ln x (\ln x - 2) + 2x + c$$

**xx)  $\int \ln(\tan x) \sec^2 x dx$**

**SOLUTION:**

$$\int \ln(\tan x) \sec^2 x dx$$

Here  $U = \ln(\tan x)$ ,  $V = \sec^2 x$

Using  $\int U \cdot V = U \cdot \int V dx - \int [U' \cdot \int V dx] dx$

$$= \ln(\tan x) \cdot \int \sec^2 x dx -$$

$$\int [(\ln(\tan x))' \cdot \int \sec^2 x dx] dx$$

$$= \ln(\tan x) \cdot \tan x - \int \left[ \frac{\sec^2 x}{\tan x} \cdot \tan x \right] dx$$

$$= \tan x \cdot \ln(\tan x) - \int \sec^2 x dx$$

$$= \tan x \cdot \ln(\tan x) - \tan x + c$$

**xxi)  $\int \frac{x \cdot \sin^{-1} x}{\sqrt{1-x^2}} dx$**

**SOLUTION:**

$$\int \frac{x \cdot \sin^{-1} x}{\sqrt{1-x^2}} dx = \frac{1}{-2} \int \sin^{-1} x \left[ (1-x^2)^{-\frac{1}{2}} (-2x) \right]$$

Here  $U = \sin^{-1} x$ ,  $V = (1-x^2)^{-\frac{1}{2}} (-2x)$

Using  $\int U \cdot V = U \cdot \int V dx - \int [U' \cdot \int V dx] dx$

$$= -\frac{1}{2} \left\{ \sin^{-1} x \int (1-x^2)^{-\frac{1}{2}} (-2x) dx -$$

$$\int [(\sin^{-1} x)' \cdot \int (1-x^2)^{-\frac{1}{2}} (-2x) dx] dx \right\}$$

$$= -\frac{1}{2} \left\{ \sin^{-1} x \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \int \left[ \frac{1}{\sqrt{1-x^2}} \cdot \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right] dx \right\} =$$

$$-\frac{1}{2} \left\{ \sin^{-1} x \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} - \int \left[ \frac{1}{\sqrt{1-x^2}} \cdot \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} \right] dx \right\}$$

$$= -\frac{1}{2} \left\{ 2 \sin^{-1} x \sqrt{1-x^2} - \int [2] dx \right\} =$$

$$-\frac{1}{2} \left\{ 2 \sin^{-1} x \sqrt{1-x^2} - 2 \int 1 dx \right\} =$$

$$-\sin^{-1} x \sqrt{1-x^2} + x + c$$

$$= x - \sqrt{1-x^2} \sin^{-1} x + c$$

**Q.2: Evaluate the following integrals:**

i)  $\int \tan^4 x dx$

**SOLUTION:**

$$\int \tan^4 x dx$$

$$= \int \tan^2 x \cdot \tan^2 x dx$$

$$= \int \tan^2 x (\sec^2 x - 1) dx$$

$$= \int \tan^2 x \sec^2 x dx - \int (\sec^2 x - 1) dx$$

$$= \frac{\tan^3 x}{3} - \int \sec^2 x dx + \int 1 dx$$

$$= \frac{\tan^3 x}{3} - \tan x + x + c$$

ii)  $\int \sec^4 x dx$

**SOLUTION:**

$$\int \sec^4 x dx$$

$$= \int \sec^2 x \cdot \sec^2 x dx$$

$$= \int \sec^2 x (1 + \tan^2 x) dx$$

$$= \int \sec^2 x dx + \int \sec^2 x \tan^2 x dx$$

$$= \tan x + \frac{\tan^3 x}{3} + c$$

$$= \tan x + \frac{1}{3} \tan^3 x + c$$

iv)  $\int \tan^3 x \sec x dx$

**SOLUTION:**

$$\int \tan^3 x \sec x dx$$

$$= \int \tan^2 x \tan x \sec x dx$$

$$= \int (\sec^2 x - 1) \tan x \sec x dx$$

$$= \int \sec^2 x \sec x \tan x dx - \int \sec x \tan x dx$$

$$= \frac{1}{3} \sec^3 x - \sec x + c$$

v)  $\int x^3 e^{5x} dx$

**SOLUTION:**

$$\int x^3 e^{5x} dx$$

Here  $U = x^3$ ,  $V = e^{5x}$

Using  $\int U \cdot V = U \cdot \int V dx - \int [U' \cdot \int V dx] dx$

$$= x^3 \int e^{5x} dx - \int [(x^3)' \cdot \int e^{5x} dx] dx$$

$$= x^3 \frac{e^{5x}}{5} - \int 3x^2 \frac{e^{5x}}{5} dx$$

$$= x^3 \frac{e^{5x}}{5} - \frac{3}{5} \left\{ \int x^2 e^{5x} dx \right\}$$

Again integrating by parts

$$= x^3 \frac{e^{5x}}{5} - \frac{3}{5} \left\{ x^2 \int e^{5x} dx - \int [(x^2)' \cdot \int e^{5x} dx] dx \right\}$$

$$= x^3 \frac{e^{5x}}{5} - \frac{3}{5} \left\{ x^2 \frac{e^{5x}}{5} - \int 2x \frac{e^{5x}}{5} dx \right\}$$

$$= x^3 \frac{e^{5x}}{5} - \frac{3}{25} x^2 e^{5x} + \frac{6}{25} \int x e^{5x} dx$$

Again integrating by parts

$$= x^3 \frac{e^{5x}}{5} - \frac{3}{25} x^2 e^{5x} + \frac{6}{25} \left\{ x \int e^{5x} dx -$$

$$\int [(x)' \cdot \int e^{5x} dx] dx \right\}$$

$$= x^3 \frac{e^{5x}}{5} - \frac{3}{25} x^2 e^{5x} + \frac{6}{25} \left\{ x \frac{e^{5x}}{5} - \int 1 \cdot \frac{e^{5x}}{5} dx \right\}$$

$$= x^3 \frac{e^{5x}}{5} - \frac{3}{25} x^2 e^{5x} + \frac{6}{125} x e^{5x} - \frac{6}{125} \int e^{5x} dx$$

$$= x^3 \frac{e^{5x}}{5} - \frac{3}{25} x^2 e^{5x} + \frac{6}{125} x e^{5x} - \frac{6}{125} \frac{e^{5x}}{5} + c$$

$$= \frac{e^{5x}}{5} \left( x^3 - \frac{3}{5}x^2 + \frac{6}{25}x - \frac{6}{125} \right) + c$$

$$\text{vi) } \int e^{-x} \sin 2x \, dx$$

**SOLUTION:**

$$\text{Let } I = \int \sin 2x e^{-x} \, dx$$

$$\text{Here } U = \sin 2x, V = e^{-x}$$

$$\text{Using } \int U.V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$$

$$I = \sin 2x \int e^{-x} \, dx - \int [(\sin 2x)' \int e^{-x} \, dx] \, dx$$

$$I = \sin 2x \frac{e^{-x}}{-1} - \int [(\cos 2x \cdot 2) \frac{e^{-x}}{-1}] \, dx$$

$$I = -e^{-x} \sin 2x + 2 \int e^{-x} \cos 2x \, dx$$

Again integrating by parts

$$I = -e^{-x} \sin 2x + 2 \{ \cos 2x \int e^{-x} \, dx - \int [(\cos 2x)' \int e^{-x} \, dx] \, dx \}$$

$$I = -e^{-x} \sin 2x + 2 \left\{ \cos 2x \frac{e^{-x}}{-1} - \int [(-\sin 2x \cdot 2) \frac{e^{-x}}{-1}] \, dx \right\}$$

$$I = -e^{-x} \sin 2x - 2 \cos 2x e^{-x} - 4 \int e^{-x} \sin 2x \, dx$$

$$I = -e^{-x} \sin 2x - 2 \cos 2x e^{-x} - 4I + c_1$$

$$5I = -e^{-x} \sin 2x - 2 \cos 2x e^{-x} + c_1$$

$$I = -\frac{2}{5} \cos 2x e^{-x} - \frac{1}{5} e^{-x} \sin 2x + \frac{c_1}{5}$$

$$I = -\frac{2}{5} e^{-x} (\cos 2x + \frac{1}{2} e^{-x} \sin 2x) + c \quad \text{where } c = \frac{c_1}{5}$$

$$\text{vii) } \int e^{2x} \cos 3x \, dx$$

**SOLUTION:**

$$\text{Let } I = \int e^{2x} \cos 3x \, dx$$

$$\text{Here } U = \cos 3x, V = e^{2x}$$

$$\text{Using } \int U.V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$$

$$I = \cos 3x \int e^{2x} \, dx - \int [(\cos 3x)' \int e^{2x} \, dx] \, dx$$

$$I = \cos 3x \frac{e^{2x}}{2} - \int [(-\sin 3x \cdot 3) \frac{e^{2x}}{2}] \, dx$$

$$I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{2} \int [\sin 3x e^{2x}] \, dx$$

Again integrating by parts

$$I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{2} \{ \sin 3x \int e^{2x} \, dx - \int [(\sin 3x)' \int e^{2x} \, dx] \, dx \}$$

$$I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{2} \left\{ \sin 3x \frac{e^{2x}}{2} - \int \left[ \cos 3x \cdot 3 \cdot \frac{e^{2x}}{2} \right] \, dx \right\}$$

$$I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{4} \sin 3x e^{2x} - \frac{9}{4} \int \cos 3x e^{2x} \, dx$$

$$I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{4} \sin 3x e^{2x} - \frac{9}{4} I + c_1$$

$$I + \frac{9}{4} I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{4} \sin 3x e^{2x} + c_1$$

$$\frac{13}{4} I = \cos 3x \frac{e^{2x}}{2} + \frac{3}{4} \sin 3x e^{2x} + c_1$$

$$I = \frac{4}{13} \frac{e^{2x}}{2} (\cos 3x + \frac{3}{2} \sin 3x) + \frac{4}{13} c_1$$

$$I = \frac{2}{13} e^{2x} \left( \cos 3x + \frac{3}{2} \sin 3x \right) + c \quad \text{where } \frac{4}{13} c_1 = c$$

$c$

$$I = \frac{3}{13} e^{2x} (\sin 3x + \frac{2}{3} \cos 3x) + c$$

$$\text{viii) } \int \operatorname{cosec}^3 x \, dx$$

**SOLUTION:**

$$\text{Let } I = \int \operatorname{cosec}^2 x \operatorname{cosec} x \, dx$$

$$\text{Here } U = \operatorname{cosec} x, V = \operatorname{cosec}^2 x$$

$$\text{Using } \int U.V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$$

$$= \operatorname{cosec} x \int \operatorname{cosec}^2 x \, dx - \int [(\operatorname{cosec} x)' \cdot \int \operatorname{cosec}^2 x \, dx] \, dx$$

$$I = \operatorname{cosec} x (-\cot x) - \int [(-\operatorname{cosec} x \cot x)(-\cot x)] \, dx$$

$$I = -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x \cot^2 x \, dx$$

$$I = -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x (\operatorname{cosec}^2 x - 1) \, dx$$

$$I = -\operatorname{cosec} x \cot x - \int \operatorname{cosec}^3 x \, dx + \int \operatorname{cosec} x \, dx$$

$$I = -\operatorname{cosec} x \cot x - I + \int \operatorname{cosec} x \, dx$$

$$2I = -\operatorname{cosec} x \cot x + \ln |\operatorname{cosec} x - \cot x| + c_1$$

$$I = -\frac{1}{2} [\cot x \operatorname{cosec} x - \ln |\operatorname{cosec} x - \cot x|] + \frac{1}{2} c_1$$

$$I = -\frac{1}{2} [\cot x \operatorname{cosec} x - \ln |\operatorname{cosec} x - \cot x|] + c$$

**TIT BIT:**

Jab pure quadratic equation h aur us ka derivative b majood na h t substitution s solve karty h aur substitution m trigonometry functions hi let karty lakin j c s start hu w let nai karny nai t book answer ni aye ga ut jin pure quadratic equation walay questions ki power  $\frac{1}{2}$  h t un k ap by parts integration k method s b kar saktay h.

$$\text{Q.3: Show that } \int e^{ax} \sin bx \, dx = \frac{1}{\sqrt{a^2 + b^2}} e^{ax} \sin \left( bx - \tan^{-1} \left( \frac{b}{a} \right) \right) + c$$

**SOLUTION:**

$$\text{Let } I = \int e^{ax} \sin bx \, dx$$

$$\text{Here } U = \sin bx, V = e^{ax}$$

$$\text{Using } \int U.V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$$

$$I = \sin bx \int e^{ax} \, dx - \int [(\sin bx)' \cdot \int e^{ax} \, dx] \, dx$$

$$I = \sin bx \frac{e^{ax}}{a} - \int \left[ \cos bx \cdot b \cdot \frac{e^{ax}}{a} \right] \, dx$$

$$I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a} \int \cos bx e^{ax} \, dx$$

Again integrating by parts

$$I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a} \{ \cos bx \int e^{ax} \, dx - \int [(\cos bx)' \cdot \int e^{ax} \, dx] \, dx \}$$

$$I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a} \left\{ \cos bx \frac{e^{ax}}{a} - \int \left[ -\sin bx \cdot b \cdot \frac{e^{ax}}{a} \right] \, dx \right\}$$

$$I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a^2} \cos bx e^{ax} - \frac{b^2}{a^2} \int \sin bx e^{ax} \, dx$$

$$I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a^2} \cos bx e^{ax} - \frac{b^2}{a^2} I + c_1$$

$$I + \frac{b^2}{a^2} I = \sin bx \frac{e^{ax}}{a} - \frac{b}{a^2} \cos bx e^{ax} + c_1$$

$$\left( \frac{a^2 + b^2}{a^2} \right) I = e^{ax} \left( \sin bx \frac{1}{a} - \frac{b}{a^2} \cos bx \right) + c_1$$

$$I = \frac{a^2}{a^2 + b^2} e^{ax} \left( \sin bx \frac{1}{a} - \frac{b}{a^2} \cos bx \right) + \frac{a^2}{a^2 + b^2} c_1$$

$$I = \frac{1}{a^2 + b^2} e^{ax} (a \sin bx - b \cos bx) + c$$

$$\text{Let } a = r \cos \theta \quad (1), \quad b = r \sin \theta \quad (2)$$

$$\text{Squaring and adding (1) and (2)}$$

$$\text{dividing (1) and (2)}$$

$$a^2 + b^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta$$

$$\frac{r \sin \theta}{r \cos \theta} = \frac{b}{a}$$

$$a^2 + b^2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$$\tan \theta = \frac{b}{a}$$

$$a^2 + b^2 = r^2 \implies r = \sqrt{a^2 + b^2} \quad \theta = \tan^{-1} \left( \frac{b}{a} \right)$$

Now Put values in (A)

$$I = \frac{1}{a^2 + b^2} e^{ax} (r \cos \theta \sin bx - r \sin \theta \cos bx) + c$$

$$I = \frac{r}{a^2 + b^2} e^{ax} (\cos \theta \sin bx - \sin \theta \cos bx) + c$$

(Take r common)

$$I = \frac{\sqrt{a^2 + b^2}}{a^2 + b^2} e^{ax} (\sin bx \cos \theta - \cos bx \sin \theta) + c$$

(Put value r)

$$I = \frac{1}{\sqrt{a^2 + b^2}} e^{ax} (\sin(bx - \theta)) + c$$

$$\text{Using } \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$



$$I = \frac{1}{\sqrt{a^2+b^2}} e^{ax} \sin(bx - \tan^{-1}(\frac{b}{a})) + c$$

Put  $\theta = \tan^{-1}(\frac{b}{a})$  have proved.

**Q.4: Evaluate the following indefinite integrals:**

i)  $\int \sqrt{a^2 - x^2} dx$

**SOLUTION:**

Let  $I = \int \sqrt{a^2 - x^2} . 1 dx$

Here  $U = \sqrt{a^2 - x^2}$ ,  $V = 1$

Using  $\int U.V = U . \int V dx - \int [U'. \int V dx] dx$

$$I = \sqrt{a^2 - x^2} \int 1 dx - \int [(\sqrt{a^2 - x^2})' . \int 1 dx] dx$$

$$I = \sqrt{a^2 - x^2} . x - \int \left[ \frac{-2x}{2\sqrt{a^2 - x^2}} . x \right] dx$$

$$I = x \sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2 - x^2}} dx$$

$$I = x \sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}} dx$$

$$I = x \sqrt{a^2 - x^2} - \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx + \int \frac{a^2}{\sqrt{a^2 - x^2}} dx$$

$$I = x \sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2} dx + a^2 \int \frac{1}{\sqrt{a^2 - x^2}} dx$$

Using  $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + c$

$$I = x \sqrt{a^2 - x^2} - I + a^2 \sin^{-1} \frac{x}{a} + c_1$$

$$I + I = x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} + c_1$$

$$2I = x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} + c_1$$

$$I = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{c_1}{2}$$

$$I = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c \quad \text{where } \frac{c_1}{2} = c$$

ii)  $\int \sqrt{x^2 - a^2} dx$

**SOLUTION:**

Let  $I = \int \sqrt{x^2 - a^2} . 1 dx$

Here  $U = \sqrt{x^2 - a^2}$ ,  $V = 1$

Using  $\int U.V = U . \int V dx - \int [U'. \int V dx] dx$

$$I = \sqrt{x^2 - a^2} \int 1 dx - \int [(\sqrt{x^2 - a^2})' . \int 1 dx] dx$$

$$I = \sqrt{x^2 - a^2} . x - \int \left[ \frac{2x}{2\sqrt{x^2 - a^2}} . x \right] dx$$

$$I = x \sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}} dx$$

$$I = x \sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}} dx$$

$$I = x \sqrt{x^2 - a^2} - \int \frac{x^2 - a^2}{\sqrt{x^2 - a^2}} dx - \int \frac{a^2}{\sqrt{x^2 - a^2}} dx$$

$$I = x \sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2} dx - a^2 \int \frac{1}{\sqrt{x^2 - a^2}} dx$$

Using  $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln|x + \sqrt{x^2 - a^2}| + c$

$$I = x \sqrt{x^2 - a^2} - I - a^2 \ln|x + \sqrt{x^2 - a^2}| + c_1$$

$$I + I = x \sqrt{x^2 - a^2} - a^2 \ln|x + \sqrt{x^2 - a^2}| + c_1$$

$$2I = x \sqrt{x^2 - a^2} - a^2 \ln|x + \sqrt{x^2 - a^2}| + c_1$$

$$I = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + \frac{c_1}{2}$$

$$I = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + c$$

where  $\frac{c_1}{2} = c$

iii)  $\int \sqrt{4 - 5x^2} dx$

**SOLUTION:**

Let  $I = \int \sqrt{4 - 5x^2} . 1 dx$

Here  $U = \sqrt{4 - 5x^2}$ ,  $V = 1$

Using  $\int U.V = U . \int V dx - \int [U'. \int V dx] dx$

$$I = \sqrt{4 - 5x^2} \int 1 dx - \int [(\sqrt{4 - 5x^2})' . \int 1 dx] dx$$

$$I = \sqrt{4 - 5x^2} . x - \int \left[ \frac{-10x}{2\sqrt{4 - 5x^2}} . x \right] dx$$

$$I = x \sqrt{4 - 5x^2} - \int \frac{-5x^2}{\sqrt{4 - 5x^2}} dx$$

$$I = x \sqrt{4 - 5x^2} - \int \frac{4 - 5x^2 - 4}{\sqrt{4 - 5x^2}} dx$$

$$I = x \sqrt{4 - 5x^2} - \int \frac{4 - 5x^2}{\sqrt{4 - 5x^2}} dx + \int \frac{4}{\sqrt{4 - 5x^2}} dx$$

$$I = x \sqrt{4 - 5x^2} - \int \sqrt{4 - 5x^2} dx + 4 \int \frac{1}{\sqrt{5(\frac{4}{5} - x^2)}} dx$$

$$I = x \sqrt{4 - 5x^2} - \int \sqrt{4 - 5x^2} dx + \frac{4}{\sqrt{5}} \int \frac{1}{\sqrt{(\frac{4}{5})^2 - x^2}} dx$$

Using  $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + c$

$$I = x \sqrt{4 - 5x^2} - I + \frac{4}{\sqrt{5}} \sin^{-1} \left( \frac{x}{\frac{2}{\sqrt{5}}} \right) + c_1$$

$$2I = x \sqrt{4 - 5x^2} + \frac{4}{\sqrt{5}} \sin^{-1} \left( \frac{\sqrt{5}x}{2} \right) + c_1$$

$$I = \frac{x}{2} \sqrt{4 - 5x^2} + \frac{2}{\sqrt{5}} \sin^{-1} \left( \frac{\sqrt{5}x}{2} \right) + \frac{c_1}{2}$$

$$I = \frac{x}{2} \sqrt{4 - 5x^2} + \frac{2}{\sqrt{5}} \sin^{-1} \left( \frac{\sqrt{5}x}{2} \right) + c \quad \text{where } \frac{c_1}{2} = c$$

iv)  $\int \sqrt{3 - 4x^2} dx$

**SOLUTION:**

Let  $I = \int \sqrt{3 - 4x^2} . 1 dx$

Here  $U = \sqrt{3 - 4x^2}$ ,  $V = 1$

Using  $\int U.V = U . \int V dx - \int [U'. \int V dx] dx$

$$I = \sqrt{3 - 4x^2} \int 1 dx - \int [(\sqrt{3 - 4x^2})' . \int 1 dx] dx$$

$$I = \sqrt{3 - 4x^2} . x - \int \left[ \frac{-8x}{2\sqrt{3 - 4x^2}} . x \right] dx$$

$$I = x \sqrt{3 - 4x^2} - \int \frac{-4x^2}{\sqrt{3 - 4x^2}} dx$$

$$I = x \sqrt{3 - 4x^2} - \int \frac{3 - 4x^2 - 3}{\sqrt{3 - 4x^2}} dx$$

$$I = x \sqrt{3 - 4x^2} - \int \frac{3 - 4x^2}{\sqrt{3 - 4x^2}} dx + \int \frac{3}{\sqrt{3 - 4x^2}} dx$$

$$I = x \sqrt{3 - 4x^2} - \int \sqrt{3 - 4x^2} dx + 3 \int \frac{1}{\sqrt{4(\frac{3}{4} - x^2)}} dx$$

$$I = x \sqrt{3 - 4x^2} - \int \sqrt{3 - 4x^2} dx + \frac{3}{2} \int \frac{1}{\sqrt{(\frac{\sqrt{3}}{2})^2 - x^2}} dx$$

Using  $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + c$

$$I = x \sqrt{3 - 4x^2} - I + \frac{3}{2} \sin^{-1} \left( \frac{x}{\frac{\sqrt{3}}{2}} \right) + c_1$$

$$2I = x \sqrt{3 - 4x^2} + \frac{3}{2} \sin^{-1} \left( \frac{2x}{\sqrt{3}} \right) + c_1$$

$$I = \frac{x}{2} \sqrt{3 - 4x^2} + \frac{3}{4} \sin^{-1} \left( \frac{2x}{\sqrt{3}} \right) + \frac{c_1}{2}$$

$$I = \frac{x}{2} \sqrt{3 - 4x^2} + \frac{3}{4} \sin^{-1} \left( \frac{2x}{\sqrt{3}} \right) + c \quad \text{where } \frac{c_1}{2} = c$$

v)  $\int \sqrt{x^2 + 4} dx$

**SOLUTION:**

Let  $I = \int \sqrt{x^2 + 4} . 1 dx$

$$\text{Here } U = \sqrt{x^2 + 4}, V = 1$$

$$\text{Using } \int U.V = U \cdot \int V dx - \int [U' \cdot \int V dx] dx$$

$$I = \sqrt{x^2 + 4} \int 1 dx - \int [(\sqrt{x^2 + 4})' \cdot \int 1 dx] dx$$

$$I = \sqrt{x^2 + 4} \cdot x - \int \left[ \frac{2x}{2\sqrt{x^2 + 4}} \cdot x \right] dx$$

$$I = x\sqrt{x^2 + 4} - \int \frac{x^2}{\sqrt{x^2 + 4}} dx$$

$$I = x\sqrt{x^2 + 4} - \int \frac{x^2 + 4 - 4}{\sqrt{x^2 + 4}} dx$$

$$I = x\sqrt{x^2 + 4} - \int \frac{x^2 + 4}{\sqrt{x^2 + 4}} dx + \int \frac{4}{\sqrt{x^2 + 4}} dx$$

$$I = x\sqrt{x^2 + 4} - \int \sqrt{x^2 + 4} dx + 4 \int \frac{1}{\sqrt{x^2 + 4}} dx$$

$$\text{Using } \int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln|x + \sqrt{x^2 + a^2}| + c$$

$$I = x\sqrt{x^2 + 4} - I + 4\ln|x + \sqrt{x^2 + 2^2}| + c_1$$

$$I + I = x\sqrt{x^2 + 4} + 4\ln|x + \sqrt{x^2 + 2^2}| + c_1$$

$$2I = x\sqrt{x^2 + 4} + 4\ln|x + \sqrt{x^2 + 2^2}| + c_1$$

$$I = \frac{x}{2}\sqrt{x^2 + 4} + \frac{4}{2}\ln|x + \sqrt{x^2 + 2^2}| + \frac{c_1}{2}$$

$$I = \frac{x}{2}\sqrt{x^2 + 4} + 2\ln|x + \sqrt{x^2 + 2^2}| + c \quad \text{where } \frac{c_1}{2} = c$$

$$\text{vi) } \int x^2 e^{ax} dx$$

**SOLUTION:**

$$\int x^2 e^{ax} dx$$

$$\text{Here } U = x^2, V = e^{ax}$$

$$\text{Using } \int U.V = U \cdot \int V dx - \int [U' \cdot \int V dx] dx$$

$$= x^2 \int e^{ax} dx - \int [(x^2)' \cdot \int e^{ax} dx] dx$$

$$= x^2 \frac{e^{ax}}{a} - \int 2x \frac{e^{ax}}{a} dx$$

$$= x^2 \frac{e^{ax}}{a} - \frac{2}{a} \{ \int x e^{ax} dx \}$$

Again integrating by parts

$$= x^2 \frac{e^{ax}}{a} - \frac{2}{a} \left\{ x \frac{e^{ax}}{a} - \int 1 \cdot \frac{e^{ax}}{a} dx \right\}$$

$$= x^2 \frac{e^{ax}}{a} - \frac{2}{a^2} x e^{ax} + \frac{2}{a^2} \int e^{ax} dx$$

$$= x^2 \frac{e^{ax}}{a} - \frac{2}{a^2} x e^{ax} + \frac{2}{a^2} \frac{e^{ax}}{a} + c$$

Take common  $\frac{e^{ax}}{a}$

$$= \frac{e^{ax}}{a} \left( x^2 - \frac{2x}{a} + \frac{2}{a^2} \right) + c$$

**Q. 5: Evaluate the following integrals:**

$$\text{i) } \int e^x \left( \frac{1}{x} + \ln x \right) dx$$

**SOLUTION:**

$$= \int e^{1 \cdot x} \left( 1 \cdot \ln x + \frac{1}{x} \right) dx$$

$$\because \int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$$

$$= e^{1 \cdot x} \ln x + c$$

$$= e^x \ln x + c$$

$$\text{ii) } \int e^x (\cos x + \sin x) dx$$

**SOLUTION:**

$$= \int e^{1 \cdot x} (1 \cdot \sin x + \cos x) dx$$

$$\because \int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$$

$$= e^{1 \cdot x} \sin x + c$$

$$= e^x \sin x + c$$

$$\text{iii) } \int e^{ax} \left( a \sec^{-1} x + \frac{1}{x\sqrt{x^2-1}} \right) dx$$

**SOLUTION:**

$$\int e^{ax} \left( a \sec^{-1} x + \frac{1}{x\sqrt{x^2-1}} \right) dx$$

$$\because \int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$$

$$= e^{ax} \sec^{-1} x + c$$

$$\text{iv) } \int e^{3x} \left( \frac{3 \sin x - \cos x}{\sin^2 x} \right) dx$$

**SOLUTION:**

$$= \int e^{3x} \left( \frac{3 \sin x}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \right) dx$$

$$= \int e^{3x} \left( \frac{3}{\sin x} - \frac{\cos x}{\sin x \cdot \sin x} \right) dx$$

$$= \int e^{3x} (3 \operatorname{cosec} x - \cot x \operatorname{cosec} x) dx$$

$$\because \int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$$

$$= e^{3x} \operatorname{cosec} x + c$$

$$\text{v) } \int e^{2x} (-\sin x + 2 \cos x) dx$$

**SOLUTION:**

$$= \int e^{2x} (2 \cos x - \sin x) dx$$

$$= \int e^{2x} (2 \cos x + (-\sin x)) dx$$

$$\because \int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$$

$$= e^{2x} \cos x + c$$

$$\text{vi) } \int \frac{x e^x}{(1+x)^2} dx$$

**SOLUTION:**

$$= \int e^x \left[ \frac{1+x-1}{(1+x)^2} \right] dx$$

$$= \int e^x \left[ \frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2} \right] dx$$

$$= \int e^x \left[ \frac{1}{1+x} - \frac{1}{(1+x)^2} \right] dx$$

$$= e^x \frac{1}{1+x} + c$$

$$\text{vii) } \int e^{-x} (\cos x - \sin x) dx$$

**SOLUTION:**

$$= \int e^{-x} (-\sin x + \cos x) dx$$

$$= \int e^{-1 \cdot x} (-1 \cdot \sin x + \cos x) dx$$

$$\because \int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$$

$$= e^{-1 \cdot x} \sin x + c$$

$$= e^{-x} \sin x + c$$

$$\text{viii) } \int \frac{e^{m \tan^{-1} x}}{1+x^2} dx$$

**SOLUTION:**

$$= \int e^{m \tan^{-1} x} \frac{1}{1+x^2} dx$$

$$\text{Put } y = \tan^{-1} x$$

$$dy = \frac{1}{1+x^2} dx$$

$$= \int e^{my} dy = \frac{e^{my}}{m} + c = \frac{e^{m \tan^{-1} x}}{m} + c \quad \text{Put } y =$$

$$\tan^{-1} x$$

$$\text{ix) } \int \frac{2x}{1-\sin x} dx$$

**SOLUTION:**

$$\int \frac{2x}{1-\sin x} \times \frac{1+\sin x}{1+\sin x} dx = \int \frac{2x(1+\sin x)}{1-\sin^2 x} dx =$$

$$\int \frac{2x(1-\sin x)}{\cos^2 x} dx = \int 2x \left( \frac{1}{\cos^2 x} + \frac{\sin x}{\cos x \cos x} \right) dx$$

$$= \int 2x (\sec^2 x + \tan x \sec x) dx = \int 2x \sec^2 x dx - \int 2x \tan x \sec x dx$$

$$\text{Here } U = 2x, V = \sec^2 x \quad \text{and} \quad U = 2x, V = \tan x \sec x$$

$$\text{Using } \int U.V = U \cdot \int V dx - \int [U' \cdot \int V dx] dx$$

$$= [2x \tan x - \int 2(1) \tan x \, dx] + [2x \sec x - \int 2(1) \sec x \, dx]$$

$$= 2x \cdot \tan x - 2 \ln|\sec x| + 2x \cdot \sec x - 2 \ln|\sec x + \tan x| + c$$

$$x) \int \frac{e^{x(1+x)}}{(2+x)^2} dx$$

**SOLUTION:**

$$= \int e^x \left[ \frac{2-1+x}{(2+x)^2} \right] dx$$

$$= \int e^x \left[ \frac{(2+x)-1}{(2+x)^2} \right] dx$$

$$= \int e^x \left[ \frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2} \right] dx$$

$$= \int e^x \left[ \frac{1}{2+x} - \frac{1}{(2+x)^2} \right] dx$$

$$\int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$$

$$= e^x \cdot \frac{1}{2+x} + c$$

$$xi) \int \left( \frac{1-\sin x}{1-\cos x} \right) e^x dx$$

**SOLUTION:**

$$= \int e^x \left( \frac{1-2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}} \right) dx$$

$$= \int e^x \left( \frac{1}{2 \sin^2 \frac{x}{2}} - \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}} \right) dx$$

$$= \int e^x \left( \frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} - \cot \frac{x}{2} \right) dx$$

$$= \int e^x \left( -\cot \frac{x}{2} + \frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} \right) dx$$

$$\int e^{ax} [a f(x) + f'(x)] dx = e^{ax} f(x) + c$$

$$= e^x \left( -\cot \frac{x}{2} \right) + c = -e^x \cot \frac{x}{2} + c$$

## Integration involving Partial Fraction

If  $P(x), Q(x)$  are two polynomial function and  $Q(x) \neq 0$ . In rational fraction  $\frac{P(x)}{Q(x)}$  can be factorized into linear and Quadratic (irreducible) factors then the rational function is written as a sum of simpler rational functions, each of which can be integrated by methods already known.

Here we discuss examples of the three cases of partial fraction and then apply integrated.

### Case1.

when  $Q(x)$  contain non-repeated linear factors. e.g;

$$\frac{P(x)}{(x-a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

$$\text{Or } \frac{-x+6}{(x-2)(x-3)(x-4)} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{x-4} \text{ e.t.c}$$

### Case2.

when  $Q(x)$  contain non repeated and repeats linear factors.

$$\frac{P(x)}{(x-a)(x+b)^2} = \frac{A}{x-a} + \frac{B}{x+b} + \frac{C}{(x+b)^2}$$

$$\frac{2x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \text{ e.t.c}$$

### Case3.

When  $Q(x)$  contain non repeated irreducible quadratic factors.

$$\frac{P(x)}{(x+b)(x^2+c)} = \frac{A}{x+b} + \frac{Bx+C}{x^2+c}$$

$$\frac{1}{(x-1)(x^2+1+2x)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1+2x}$$

## Exercise 3.5

Evaluate the following integrals.

$$Q1. \int \frac{3x+1}{x^2-x-6} dx$$

$$\text{Solution: } \int \frac{3x+1}{x^2-x-6} dx$$

Now

$$\frac{3x+1}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$\Rightarrow 3x+1 = A(x+2) + B(x-3) \rightarrow (i)$$

$$\text{Put } x-3=0 \Rightarrow x=3 \text{ in (i)}$$

$$3(3)+1 = A(3+2) + B(0) \Rightarrow 5A = 10 \Rightarrow A = 2$$

$$\text{Put } x+2=0 \Rightarrow x=-2 \text{ in (i)}$$

$$3(-2)+1 = A(0) + B(-2-3) \Rightarrow -5B = -6+1$$

$$\Rightarrow -5B = -5 \Rightarrow B = 1$$

$$\text{so } \frac{3x+1}{x^2-x-6} = \frac{2}{x-3} + \frac{1}{x+2}$$

$$\Rightarrow \int \frac{3x+1}{x^2-x-6} dx = 2 \int \frac{1}{x-3} + \int \frac{1}{x+2} dx$$

$$= 2 \ln|x-3| + \ln|x+2| + c$$

$$Q2. \int \frac{5x+8}{(x+3)(2x-1)} dx$$

$$\text{Solution: } \int \frac{5x+8}{(x+3)(2x-1)} dx$$

Now.

$$\frac{5x+8}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{2x-1}$$

$$\Rightarrow 5x+8 = A(2x-1) + B(x+3) \rightarrow (i)$$

$$\text{Put } 2x-1=0 \Rightarrow x = \frac{1}{2} \text{ in (i)}$$

$$\Rightarrow 5\left(\frac{1}{2}\right) + 8 = A(0) + B\left(\frac{1}{2} + 3\right)$$

$$\Rightarrow \frac{5+16}{2} = B\left(\frac{1+6}{2}\right) \Rightarrow 7B = 21 \Rightarrow B = 3$$

$$\text{Put } x+3=0 \Rightarrow x=-3 \text{ in (i)}$$

$$\Rightarrow 5(-3) + 8 = A(2(-3)-1) + B(0)$$

$$\Rightarrow -15+8 = -7A \Rightarrow -7 = -7A \Rightarrow A = 1$$

$$\text{So } \frac{5x+8}{(x+3)(2x-1)} = \frac{1}{x+3} + \frac{3}{2x-1}$$

$$\int \frac{5x+8}{(x+3)(2x-1)} dx = \int \frac{1}{x+3} dx + 3 \int \frac{1}{2x-1} dx$$

$$= \ln|x+3| + \frac{3}{2} \int \frac{2}{2x-1} dx$$

$$\int \frac{5x+8}{(x+3)(2x-1)} dx = \ln|x+3| + \frac{3}{2} \ln|2x-1| + c$$

**Q3.**  $\int \frac{x^2+3x-34}{x^2+2x-15} dx$

**Solution:**  $\int \frac{x^2+3x-34}{x^2+2x-15} dx$

So  $\int \left(1 + \frac{x-19}{x^2+2x-15}\right) dx$

$= \int 1 dx + \int \frac{x-19}{x^2+2x-15} dx$

Now  $\frac{x-19}{x^2+2x-15} = \frac{A}{x-3} + \frac{B}{x+5} \rightarrow (i)$

$\Rightarrow x-19 = A(x+5) + B(x-3) \rightarrow (ii)$

put  $x-3=0 \Rightarrow x=3$  in (ii)  $= x^2+5x-3x-15$

$\Rightarrow 3-19 = A(3+5) + B(0)$   $x(x+5) - 3(x+5)$

$\Rightarrow -16 = 8A \Rightarrow A = -2$   $(x-3)(x+5)$

put  $x+5=0 \Rightarrow x=-5$  in (ii)

$\Rightarrow -5-19 = A(0) + B(-5-3) \Rightarrow -24 = -8B$

$\Rightarrow B = 3$

(i)  $\Rightarrow \frac{x-19}{x^2+2x-15} = -\frac{2}{x-3} + \frac{3}{x+5}$

Thus,

$\int \frac{x^2+3x-34}{x^2+2x-15} dx = \int 1 dx + \int \frac{-2}{x-3} dx + \int \frac{3}{x+5} dx$

$= x - 2 \ln|x-3| + 3 \ln|x+5| + c$

**Q4.**  $\int \frac{(a-b)x}{(x-a)(x-b)} dx, (a > b)$

**Solution:**  $\int \frac{(a-b)x}{(x-a)(x-b)} dx$

Now

$\frac{(a-b)x}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$

$\Rightarrow (a-b)x = A(x-b) + B(x-a) \rightarrow (i)$

Put  $x-a=0 \Rightarrow x=a$  in (i)

$\Rightarrow (a-b).a = A(a-b) + B(a-a)$

$(a-b).b = A(0) + B(a-b) \Rightarrow A = a$

Put  $x-b=0 \Rightarrow x=b$  in (i)

$\Rightarrow (a-b).b = A(0) + B(b-a)$

$(a-b).b = -B(a-b)$

$b = -B$

$B = -b$

Thus

$\frac{(a-b)x}{(x-a)(x-b)} = \frac{a}{x-a} + \frac{-b}{x-b}$

$\int \frac{(a-b)x}{(x-a)(x-b)} dx = \int \frac{a}{x-a} dx - \int \frac{b}{x-b} dx$

$= a \ln|x-a| - b \ln|x-b| + c$

**Q5.**  $\int \frac{3-x}{1-x-6x^2} dx$

**Solution:**  $\int \frac{3-x}{1-x-6x^2} dx$

Now

$\frac{3-x}{1-x-6x^2} = \frac{A}{2x+1} + \frac{B}{1-3x}$

$\Rightarrow 3-x = A(1-3x) + B(2x+1) \rightarrow (i)$

Put  $2x+1=0 \Rightarrow x=-\frac{1}{2}$  in (i)

$\therefore 1-x-6x^2$   
 $= -6x^2 - x + 1$   
 $-3x(2x+1) + 1(2x+1)$   
 $(2x+1)(1-3x)$

$\Rightarrow 3 - \left(-\frac{1}{2}\right) = A\left(1 - 3\left(-\frac{1}{2}\right) + B(0)\right)$

$\Rightarrow 3 + \frac{1}{2} = A\left(1 + \frac{3}{2}\right) \Rightarrow \frac{7}{2} = A\left(\frac{5}{2}\right)$

$\Rightarrow A = \frac{7}{5}$

Put  $1-3x=0 \Rightarrow 1=3x \Rightarrow x=\frac{1}{3}$  in (i)

$\Rightarrow 3 - \frac{1}{3} = A(0) + B\left(2\left(\frac{1}{3}\right) + 1\right)$

$\frac{9-1}{3} = B\left(\frac{2+3}{3}\right) \Rightarrow 8 = 5B \Rightarrow \frac{8}{5}$

So

$\frac{3-x}{1-x-6x^2} = \frac{7/5}{2x+1} + \frac{8/5}{1-3x}$

$\therefore \int \frac{3-x}{1-x-6x^2} dx = \frac{7}{5} \int \frac{1}{2x+1} dx + \frac{8}{5} \int \frac{1}{1-3x} dx$

$= \frac{7}{10} \int \frac{2}{2x+1} dx - \frac{8}{15} \int \frac{-3}{1-3x} dx$

$= \frac{7}{10} \ln|2x+1| - \frac{8}{5} \ln|1-3x| + C$

**Q.6**  $\int \frac{2x}{x^2-a^2} dx$

**Solution:**  $\int \frac{2x}{x^2-a^2} dx$

Now

$\frac{2x}{x^2-a^2} = \frac{A}{x-a} + \frac{B}{x+a} \quad \because x^2-a^2 = (x-a)(x+a)$

$\Rightarrow 2x = A(x+a) + B(x-a) \rightarrow (i)$

Put  $x-a=0 \Rightarrow x=a$  in (i)

$\Rightarrow 2a = A(a+a) + B(0) \Rightarrow 2a = 2A \Rightarrow A = 1$

Put  $x+a=0 \Rightarrow x=-a$  in (i)

$\Rightarrow 2(-a) = A(0) + B(-a-a) \Rightarrow -2a = -2aB$   
 $\Rightarrow B = 1$

So  $\frac{2x}{x^2-a^2} = \frac{1}{x-a} + \frac{1}{x+a}$

$\int \frac{2x}{x^2-a^2} dx = \int \frac{1}{x-a} dx + \int \frac{1}{x+a} dx$

$= \ln|x-a| + \ln|x+a| + c$

$= \ln|(x-a)(x+a)| + c$

$= \ln|x^2-a^2| + c$

**Q.7**  $\int \frac{1}{6x^2+5x-4} dx$

**Solution:**  $\int \frac{1}{6x^2+5x-4} dx$

Now

$\frac{1}{(2x-1)(3x+4)} = \frac{A}{2x-1} + \frac{B}{3x+4}$

$\Rightarrow 1 = A(3x+4) + B(2x-1) \rightarrow (i)$

Put  $2x-1=0 \Rightarrow x=\frac{1}{2}$  in (i)

$\Rightarrow 1 = A\left(3\left(\frac{1}{2}\right) + 4\right) + B(0) \Rightarrow 1 = A\left(\frac{3+8}{2}\right)$

$\Rightarrow 3 = -11B \Rightarrow B = -\frac{3}{11}$

Put  $3x+4=0 \Rightarrow x=-\frac{4}{3}$  in (i)

$\Rightarrow 1 = A(0) + B\left(2\left(-\frac{4}{3}\right) - 1\right) \Rightarrow 1 = B\left(\frac{-8-3}{3}\right)$

$$\Rightarrow 3 = -11B \Rightarrow B = -\frac{3}{11}$$

So

$$\frac{1}{(2x-1)(3x+4)} = \frac{A}{2x-1} + \frac{B}{3x+4}$$

$$1 = A(3x+4) + B(2x-1) \rightarrow (i)$$

$$\text{Put } 2x-1=0 \Rightarrow x = \frac{1}{2} \text{ put in (i)}$$

$$\Rightarrow 1 = A\left(3\left(\frac{1}{2}\right) + 4\right) \Rightarrow 1 = A\left(\frac{3}{2} + 4\right)$$

$$\Rightarrow 1 = A\left(\frac{3+8}{2}\right) \Rightarrow 1 = A\left(\frac{11}{2}\right) \Rightarrow A = \frac{2}{11}$$

$$\text{Put } 3x+4=0 \Rightarrow x = -\frac{4}{3} \text{ put in (i)}$$

$$\Rightarrow 1 = A(0) + B\left(2\left(-\frac{4}{3}\right) - 1\right) \Rightarrow 1 = B\left(\frac{-8-3}{3}\right)$$

$$\Rightarrow 3 = -11B \Rightarrow B = -\frac{3}{11}$$

$$\text{So, } \frac{1}{6x^2+5x-4} = \frac{\frac{2}{11}}{2x-1} + \frac{-3/11}{3x+4}$$

$$\Rightarrow \int \frac{1}{6x^2+5x-4} dx = \frac{1}{11} \int \frac{2}{2x-1} dx - \frac{1}{11} \int \frac{3}{3x+4} dx$$

$$= \frac{1}{11} \ln|2x-1| - \frac{1}{11} \ln|3x+4| + c$$

$$= \frac{1}{11} \ln \left| \frac{2x-1}{3x+4} \right| + c$$

$$\text{Q.8} \int \frac{2x^2-3x^2-x-7}{2x^2-3x-2} dx$$

$$\frac{x}{2x^2-3x-2} = \frac{x}{\sqrt{2x^3-3x^2-x-7} \pm 2x^3 \pm 3x^2 \mp 2x}$$

$$\frac{x}{x-7}$$

$$\int \frac{2x^2-3x^2-x-7}{2x^2-3x-2} dx = \int \left( x + \frac{x-7}{2x^2-3x-2} \right) dx$$

$$= \int x dx + \int \frac{x-7}{2x^2-3x-2} dx$$

Now

$$\frac{x-7}{(x-2)(2x+1)} = \frac{A}{x-2} + \frac{B}{2x+1}$$

$$x-7 = A(2x+1) + B(x-2) \rightarrow (i) \quad \because 2x^2-3x-2 = 2x^2-4x+x-2$$

$$\Rightarrow \text{Put } x-2=0 \Rightarrow x=2 \text{ in (i)} \quad (x-2)(3x+1)$$

$$\Rightarrow 2-7 = A(2(2)+1) + B(0) \Rightarrow -5 = 5A \Rightarrow A = -\frac{5}{5} = -1$$

$$A = -1$$

$$\text{Put } 2x+1=0 \Rightarrow x = -\frac{1}{2} \text{ in (i)}$$

$$\Rightarrow -\frac{1}{2} - 7 = A(0) + B\left(-\frac{1}{2} - 2\right) \Rightarrow \frac{-1-14}{2} = B\left(\frac{-1-4}{2}\right)$$

$$\Rightarrow -15 = -5B \Rightarrow B = 3$$

So

$$\frac{x-7}{2x^2-3x-2} = \frac{-1}{x-2} + \frac{3}{2x+1}$$

$$\text{Thus } \int \frac{2x^2-3x^2-x-7}{2x^2-3x-2} dx = \int x dx = \int \frac{1}{x-2} dx + 3 \int \frac{1}{2x+1} dx$$

$$= \frac{x^2}{2} - \ln|x-2| + \frac{3}{2} \int \frac{2}{2x+1} dx$$

$$= \frac{x^2}{2} - \ln|x-2| + \frac{3}{2} \ln|2x+1| + c$$

$$\text{Q.9} \int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} dx$$

$$\text{Solution: } \int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} dx$$

Now

$$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$3x^2-12x+11 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \rightarrow (i)$$

$$\text{Put } x-1=0 \Rightarrow x=1 \text{ in (i)}$$

$$\Rightarrow 3(1)^2 - 12(1) + 11 = A(1-2)(1-3) + B(0) + C(0)$$

$$= 3 - 12 + 11 = A(-1)(-2)$$

$$\Rightarrow 2 = 2A \Rightarrow A = 1$$

$$\text{Put } x-2=0 \Rightarrow x=2 \text{ in (i)}$$

$$\Rightarrow 3(2)^2 - 12(2) + 11 = A(0) + B(2-1)(2-3) + C(0)$$

$$\Rightarrow 12 - 24 + 11 = -B$$

$$\Rightarrow -1 = -B \Rightarrow B = 1$$

$$\text{Put } x-3=0 \Rightarrow x=3 \text{ in (i)}$$

$$\Rightarrow 3(3)^2 - 12(3) + 11 = A(0) + B(0) + C(3-1)(3-2)$$

$$\Rightarrow 3(9) - 36 + 11 = C(2)(1)$$

$$\Rightarrow 27 - 36 + 11 = 2C$$

$$\Rightarrow 2 = 2C \Rightarrow C = 1$$

So

$$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}$$

$$\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} dx$$

$$= \int \frac{1}{x-1} dx + \int \frac{1}{x-2} dx + \int \frac{1}{x-3} dx$$

$$= \ln|x-1| + \ln|x-2| + \ln|x-3| + c$$

$$\text{Q.10} \int \frac{2x-1}{x(x-1)(x-3)} dx$$

$$\text{Solution: } \int \frac{2x-1}{x(x-1)(x-3)} dx$$

Now

$$\frac{2x-1}{x(x-1)(x-3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-3}$$

$$2x-1 = A(x-1)(x-3) + B(x)(x-3) + C(x)(x-1) \rightarrow (i)$$

$$\text{Put } x=0 \text{ in (i)}$$

$$2(0) - 1 = A(0-1)(0-3) + B(0)(C(0))$$

$$\Rightarrow -1 = A(-1)(-3) \Rightarrow -1 = 3A \Rightarrow A = -\frac{1}{3}$$

$$\text{Put } x-1=0 \Rightarrow x=1 \text{ in (i)}$$

$$\Rightarrow 2(1) - 1 = A(0) + B(1)(1-3) + C(0)$$

$$\Rightarrow 1 = B(-2) \Rightarrow B = -\frac{1}{2}$$

$$\text{Put } x-3=0 \Rightarrow x=3 \text{ in (i)}$$

$$\Rightarrow 2(3) - 1 = A(0) + B(0) + C(3)(3-1)$$

$$\Rightarrow 5 = 6C \Rightarrow C = \frac{5}{6}$$

So

$$\frac{2x-1}{x(x-1)(x-3)} = \frac{-1}{3x} + \frac{-1}{2(x-1)} + \frac{5}{6(x-3)}$$

$$\frac{2x-1}{x(x-1)(x-3)} = -\frac{1}{3x} - \frac{1}{2(x-1)} + \frac{5}{6(x-3)}$$

$$\int \frac{x-1}{x(x-1)(x-3)} dx$$

$$= -\frac{1}{3} \int \frac{1}{x} dx - \frac{1}{2} \int \frac{1}{x-1} + \frac{5}{6} \int \frac{1}{x-3}$$

$$= -\frac{1}{3} \ln|x| + \frac{1}{2} \ln|x-1| + \frac{5}{6} \ln|x-3| + c$$

**Q.11**  $\int \frac{5x^2+9x+6}{(x^2-1)(2x+3)} dx$

**Solution:**  $\int \frac{5x^2+9x+6}{(x^2-1)(2x+3)} dx$

Now

$$\frac{5x^2+9x+6}{(x^2-1)(2x+3)} = \frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+3}$$

$$5x^2+9x+6 = A(x+1)(2x+3) + B(x-1)(2x+3) + C(x+1)(x-1)$$

Put  $x+1=0 \Rightarrow x=-1$  in (i)

$$5(-1)^2+9(-1)+6 = A(0) + B(-1-1)(2(-1)+3) + C(0)$$

$$\Rightarrow 5-9+6 = B(-2)(1)$$

$$\Rightarrow 2 = -2B \Rightarrow B = -1$$

Put  $x-1=0 \Rightarrow x=1$  in (i)

$$\Rightarrow 5(1)^2+9(1)+6 = A(1+1)(2(1)+3) + B(0) + C(0)$$

$$\Rightarrow 5+9+6 = A(2)(5) \Rightarrow 20 = A10 \Rightarrow A = 2$$

Put  $2x+3=0 \Rightarrow x=-\frac{3}{2}$  in (i)

$$\Rightarrow 5\left(-\frac{3}{2}\right)^2 + 9\left(-\frac{3}{2}\right) + 6 = A(0) + B(0) + C\left(-\frac{3}{2}+1\right)\left(-\frac{3}{2}-1\right)$$

$$5\left(\frac{9}{4}\right) + \left(-\frac{27}{2}\right) + 6 = C\left(\frac{-3+2}{2}\right)\left(\frac{-3-2}{2}\right)$$

$$\frac{45}{4} - \frac{27}{2} + 6 = C\left(-\frac{1}{2}\right)\left(-\frac{5}{2}\right)$$

$$\frac{45-54+24}{4} = C\frac{5}{4}$$

$$\Rightarrow 15 = 5C \Rightarrow C = 3$$

$$\frac{5x^2+9x+6}{(x^2-1)(2x+3)} = \frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)}$$

$$= \frac{2}{x-1} + \frac{-1}{x+1} + \frac{3}{2x+3}$$

$$\therefore \int \frac{5x^2+9x+6}{(x^2-1)(2x+3)} dx = 2 \int \frac{1}{x-1} dx - \int \frac{1}{x+1} dx + \frac{3}{2} \int \frac{2}{2x+3} dx$$

$$= 2 \ln|x-1| - \ln|x+1| + \frac{3}{2} \ln|2x+3| + c$$

**Q.12.**  $\int \frac{4+7x}{(1+x)^2(2+3x)} dx$

Solution:

$$\int \frac{4+7x}{(1+x)^2(2+3x)} dx$$

Now

$$\frac{4+7x}{(1+x)^2(2+3x)} = \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{2x+3}$$

$$\Rightarrow 4+7x = A(1+x)(2x+3) + B(2x+3) + C(1+x)^2 \rightarrow (i)$$

Put  $1+x=0 \Rightarrow x=-1$  in (i)

$$\Rightarrow 4+7(-1) = A(0) + B(-2+3) + C(0)$$

$$\Rightarrow -3 = B \Rightarrow B = -3$$

Put  $2+3x=0 \Rightarrow x=-\frac{2}{3}$  in (i)

$$4+7\left(-\frac{2}{3}\right) = A(0) + B(0) + C\left(1-\frac{2}{3}\right)^2$$

$$4 - \frac{14}{3} = C\left(\frac{3-2}{3}\right)^2$$

$$\frac{12-14}{3} = C\left(\frac{1}{9}\right)$$

$$\Rightarrow -\frac{2}{3} = \frac{1}{9}C \Rightarrow -\frac{2}{3} \times \frac{9}{1} = C \Rightarrow C = -6$$

From (i)

$$4+7x = A(2+3x+2x+3x^2) + 2B + 3Bx + C(1+2x+x^2)$$

$$\Rightarrow 4+7x = 2A + 5Ax + 3x^2A + 2B + 3Bx + C + 2Cx + cx^2$$

Equating coefficient of  $x^2$

$$0 = 3A + C \Rightarrow 3A = -C \Rightarrow 3A = -(-6)$$

$$\Rightarrow 3A = 6 \Rightarrow A = \frac{6}{3} = 2 \Rightarrow A = 2$$

So,

$$\frac{4+7x}{(1+x)^2(2+3x)} = \frac{2}{1+x} + \frac{3}{(1+x)^2} - \frac{6}{2x+3}$$

$$\int \frac{4+7x}{(1+x)^2(2+3x)} dx = 2 \int \frac{1}{1+x} dx + 3 \int (1+x^2)^{-2} dx + \frac{6}{3} \int \frac{3}{2+3x} dx$$

$$= 2 \ln|1+x| + \frac{3(1+x)^{-1}}{-1} - 2 \ln|2+3x| + c$$

$$\ln|1+x|^2 - \frac{3}{1+x} - \ln|2+3x|^2 + c$$

**Q.13**  $\int \frac{2x^2}{(x-1)^2(x+1)} dx$

Solution:

Now

$$\frac{2x^2}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$\Rightarrow 2x^2 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

Put  $x-1=0 \Rightarrow x=1$  in (i)

$$\Rightarrow 2(1)^2 = A(0) + B(1+1) + C(0)$$

$$\Rightarrow 2 = 2B \Rightarrow B = 1$$

Put  $x+1=0 \Rightarrow x=-1$  in (i)

$$\Rightarrow 2(-1)^2 = A(0) + B(0) + C(-1-1)^2$$

$$2 = 4C \Rightarrow C = \frac{1}{2}$$

From (i)

$$2x^2 = A(x^2 - 1) + Bx + B + C(x^2 + 1 - 2x)$$

$$\Rightarrow 2x^2 = Ax^2 - A + Bx + B + Cx^2 + C - 2Cx$$

Equating coefficients of  $x^2$ , we have

$$\Rightarrow 2 = A + C \Rightarrow 2 = A + \frac{1}{2} \Rightarrow A = 2 - \frac{1}{2}$$

$$\Rightarrow A = \frac{3}{2}$$

So,

$$\frac{2x^2}{(x-1)^2(x+1)} = \frac{3/2}{(x-1)} + \frac{1}{(x-1)^2} + \frac{1/2}{(x+1)}$$

$$\int \frac{2x^2}{(x-1)^2(x+1)} dx$$

$$= \frac{3}{2} \int \frac{1}{x-1} dx + \int (x-1)^{-2} dx + \frac{1}{2} \int \frac{1}{x+1} dx$$

$$= \frac{3}{2} \ln|x-1| + \frac{(x-1)^{-1}}{-1} + \frac{1}{2} \ln|x+1| + c$$

$$= \frac{3}{2} \ln|x-1| - \frac{1}{x-1} + \frac{1}{2} \ln|x+1| + c$$

**Q.14**  $\int \frac{1}{(x-1)(x+1)^2} dx$

**Solution:**  $\int \frac{1}{(x-1)(x+1)^2} dx$

Now

$$\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\Rightarrow 1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1) \rightarrow (i)$$

Put  $x - 1 = 0 \Rightarrow x = 1$  in (i)

$$1 = A(1+1)^2 \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$$

Put  $x + 1 = 0 \Rightarrow x = -1$  in (i)

$$1 = C(-1-1)$$

$$\Rightarrow 1 = -2C \Rightarrow C = -\frac{1}{2}$$

From (i)

$$\Rightarrow 1 = A(x^2 + 2x + 1) + B(x^2 - 1) + Cx - C$$

$$\Rightarrow 1 = Ax^2 + 2Ax + A + Bx^2 - B + Cx - C$$

Equating coefficient of  $x^2$ , we have

$$0 = A + B \Rightarrow 0 = \frac{1}{4} + B \Rightarrow B = -\frac{1}{4}$$

$$\frac{1}{(x-1)(x+1)^2} = \frac{1/4}{x-1} + \frac{-1/4}{x+1} + \frac{-1/2}{(x+1)^2}$$

$$\int \frac{1}{(x-1)(x+1)^2} dx = \frac{1}{4} \int \frac{1}{x-1} dx - \frac{1}{4} \int \frac{1}{x+1} dx - \frac{1}{2} \int (x+1)^{-2} dx$$

$$= \frac{1}{4} \ln|x-1| - \frac{1}{4} \ln|x+1| - \frac{1}{2} \frac{(x+1)^{-1}}{-1} + C$$

$$= \left\{ \frac{1}{4} \ln|x-1| - \frac{1}{4} \ln|x+1| \right\} + \frac{1}{2(x+1)} + C$$

**Q.15**  $\int \frac{x+4}{x^3-3x^2+4} dx$

**Solution:**  $\int \frac{x+4}{x^3-3x^2+4} dx$

Now

$$\because x^3 - 3x^2 + 4 = x^3 + x^2 - 4x^2 + 4$$

$$= x^2(x+1) - 1(x^2-1)$$

$$= x^2(x+1) - 4(x-1)(x+1)$$

$$= (x+1)(x^2-4x+4)$$

$$\Rightarrow x^3 - 3x^2 + 4 = (x+1)(x-2)^2$$

Now

$$\frac{x+4}{x^3-3x^2+4} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$\Rightarrow x+4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$

→ (i)

Put  $x + 1 = 0 \Rightarrow x = -1$  in (i)

$$\Rightarrow -1 + 4 = A(-1-2)^2 + B(0) + C(0)$$

$$\Rightarrow 3 = 9A \Rightarrow A = \frac{1}{3}$$

Put  $x - 2 = 0 \Rightarrow x = 2$  in (i)

$$\Rightarrow 2 + 4 = A(0) + B(0) + C(2+1)$$

$$\Rightarrow 6 = 3C \Rightarrow C = 2$$

From (i)

$$x+4 = A(x^2-4x+4) + B(x^2-2x+x-2) + Cx + C$$

$$\Rightarrow x+4 = Ax^2 - 4Ax + 4A + Bx^2 - Bx - 2B + Cx + C$$

Equating coefficients of  $x^2$

$$\Rightarrow 0 = A + B \Rightarrow 0 = \frac{1}{3} + B \Rightarrow B = -\frac{1}{3}$$

$$\frac{x+4}{x^3-3x^2+4} = \frac{1/3}{x+1} + \frac{-1/3}{x-2} + \frac{2}{(x-2)^2}$$

$$\int \frac{x+4}{x^3-3x^2+4} dx = \frac{1}{3} \int \frac{1}{x+1} dx - \frac{1}{3} \int \frac{1}{x-2} dx + 2 \int (x-2)^{-2} dx$$

$$= \frac{1}{3} \ln|x+1| - \frac{1}{3} \ln|x-2| + 2 \frac{(x-2)^{-1}}{-1} + c$$

$$= \frac{1}{3} \ln|x+1| - \frac{1}{3} \ln|x-2| - \frac{2}{x-2} + c$$

$$= \frac{1}{3} \{ \ln|x+1| - \ln|x-2| \} - \frac{2}{x-2} + c$$

**Q.16.**  $\int \frac{x^3-6x^2+25}{(x+1)^2(x-2)^2} dx$

**Solution:**

$$\frac{x^3-6x^2+25}{(x+1)^2(x-2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

$$\Rightarrow x^3 - 6x^2 + 25 = A(x+1)(x-2)^2 + B(x-2)^2 + C(x+1)^2(x-2) + D(x+1)^2 \rightarrow (i)$$

Put  $x + 1 = 0 \Rightarrow x = -1$  in (i)

$$\Rightarrow (-1)^3 - 6(-1)^2 + 25 = A(0) + B(-1-2)^2 + C(0) + D(0)$$

$$-1 - 6 + 25 = 9B$$

$$9B = 18 \Rightarrow B = 2$$

Put  $x - 2 = 0 \Rightarrow x = 2$  in (i)

$$\Rightarrow (2)^3 - 6(2)^2 + 25 = D(2+1)^2$$

$$\Rightarrow 8 - 24 + 25 = 9D$$

$$9 = 9D \Rightarrow D = 1$$

From (i)

$$\begin{aligned}
 x^3 - 6x^2 + 25 &= A(x+1)(x^2 - 4x + 4) + B(x^2 - 4x + 4) \\
 &+ C(x^2 + 1 + 2x)(x-2) + D(x^2 + 1 + 2x) \\
 &= A(x^3 - 4x^2 + 4x + x^2 - 4x + 4) + Bx^2 - 4Bx + 4B \\
 &+ C(x^3 - 2x^2 + x - 2 + 2x^2 - 4x) + Dx^2 + D + 2Dx \\
 &= Ax^3 - 3Ax^2 + 4A + Bx^2 - 4Bx + 4B + Cx^3 \\
 &\quad - 3Cx + Dx^2 + D + 2Dx
 \end{aligned}$$

Equating coefficients of  $x^3$  and  $x^2$

For  $x^3$

$$1 = A + C \rightarrow (ii)$$

$$\text{For } x^2 \quad -6 = -3A + B + D$$

$$-6 = -3A + 2 + 1$$

$$-6 - 3 = -3A \Rightarrow -9 = -3A \Rightarrow A = 3 \text{ put in (ii)}$$

$$1 = 3 + C \Rightarrow C = 1 - 3 = -2 \Rightarrow C = -2$$

$$\frac{x^3 - 6x^2 + 25}{(x+1)^2(x-2)^2} = \frac{3}{x+1} + \frac{2}{(x+1)^2} - \frac{2}{(x-2)} + \frac{1}{(x-2)^2}$$

$$\int \frac{x^3 - 6x^2 + 25}{(x+1)^2(x-2)^2} dx$$

$$= 3 \int \frac{1}{x+1} dx + 2 \int (x+1)^{-2} dx - 2 \int \frac{1}{x-2} dx + \int (x-2)^{-2} dx$$

$$= 3 \ln|x+1| + 2 \frac{(x+1)^{-1}}{-1} - 2 \ln|x-2| + \frac{(x-2)^{-1}}{-1} + C$$

$$= 3 \ln|x+1| - \frac{2}{x+1} - 2 \ln|x-2| - \frac{1}{x-2} + C$$

$$\text{Q.17} \int \frac{x^3 + 22x^2 + 14x - 17}{(x-3)(x+2)^3} dx$$

Solution:

$$\frac{x^3 + 22x^2 + 14x - 17}{(x-3)(x+2)^3}$$

$$= \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$$

$$\Rightarrow x^3 + 22x^2 + 14x - 17 = A(x+2)^3 + B(x-3)(x+2)^2 + C(x-3)(x+2) + D(x-3) \rightarrow (i)$$

$$\text{Put } x-3=0 \Rightarrow x=3 \text{ in (i)}$$

$$\Rightarrow (3)^3 + 22(3)^2 + 14(3) - 17 = A(3+2)^3$$

$$\Rightarrow 27 + 198 + 42 - 17 = 125A$$

$$\Rightarrow 250 = 125A \Rightarrow A = 2$$

$$\text{Put } x+2=0 \Rightarrow x=-2 \text{ in (i)}$$

$$\Rightarrow (-2)^3 + 22(-2)^2 + 14(-2) - 17 = D(-2-3)$$

$$-8 + 88 - 28 - 17 = -5D \Rightarrow 35 = -5D$$

$$D = -7$$

From (i)

$$x^3 + 22x^2 + 14x - 17$$

$$= A[x^3 + 6x^2 + 12x + 8] + B(2-3)(x^2 + 4x + 4)$$

$$+ C(x^2 + 2x - 3x - 6) + Dx - 3D$$

$$= Ax^3 + 6Ax^2 + 12Ax + 8A$$

$$+ B(x^3 + 4x^2 + 4x - 3x^2 - 12x - 12)$$

$$+ Cx^2 - Cx - 6C + Dx - 3D$$

Equating coefficients of  $x^2$  and  $x^3$

$$\text{For } x^3; 1 = A + B \Rightarrow 1 = 2 + B \Rightarrow B = -1$$

$$\text{For } x^2; 22 = 6A + B + C \Rightarrow 22 = 6(2) - 1 + C$$

$$\Rightarrow C = 22 - 12 + 1 = 11 \Rightarrow C = 11$$

So

$$\frac{x^3 + 22x^2 + 14x - 17}{(x-3)(x+2)^3}$$

$$= \frac{2}{x-3} + \frac{1}{x+2} + \frac{11}{(x+2)^2} - \frac{7}{(x+2)^3}$$

$$\begin{aligned}
 &\int \frac{x^3 + 22x^2 + 14x - 17}{(x-3)(x+2)^3} dx \\
 &= 2 \int \frac{1}{x-3} - \int \frac{1}{x+2} dx + \int (x+2)^{-2} dx - 7 \int (x+2)^{-3} dx \\
 &= 2 \ln|x-3| - \ln|x+2| + 11 \frac{(x+2)^{-1}}{-1} - 7 \frac{(x+2)^{-2}}{-2} + c \\
 &= 2 \ln|x-3| - \ln|x+2| - \frac{11}{x+2} + \frac{7}{2} \frac{1}{(x+2)^2} + c
 \end{aligned}$$

$$\text{Q.18} \int \frac{x-2}{(x+1)(x^2+1)} dx$$

Solution:

$$\frac{x-2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow x-2 = A(x^2+1) + (Bx+C)(x+1) \rightarrow (i)$$

$$\text{Put } x+1=0 \Rightarrow x=-1 \text{ in (i)}$$

$$\Rightarrow -1-2 = A((-1)^2+1)$$

$$-3 = 2A \Rightarrow A = -\frac{3}{2}$$

From (i)

$$\Rightarrow x-2 = Ax^2 + A + Bx^2 + Bx + Cx + C$$

Equating coefficients of  $x^2$  and  $x$

$$\text{For } x^2; 0 = A + B \Rightarrow 0 = -\frac{3}{2} + B \Rightarrow B = \frac{3}{2}$$

$$\text{For } x; 1 = B + C \Rightarrow 1 = \frac{3}{2} + C$$

$$\Rightarrow C = 1 - \frac{3}{2} = -\frac{1}{2} \Rightarrow C = -\frac{1}{2}$$

$$\frac{x-2}{(x+1)(x^2+1)} = \frac{-3/2}{x+1} + \frac{3/2x - 1/2}{x^2+1}$$

$$\int \frac{x-2}{(x+1)(x^2+1)} dx = -\frac{3}{2} \int \frac{1}{x+1} dx + \frac{1}{2} \int \frac{3x-1}{x^2+1} dx$$

$$= -\frac{3}{2} \ln|x+1| + \frac{1}{2} \int \frac{3x}{x^2+1} dx - \frac{1}{2} \int \frac{1}{x^2+1} dx$$

$$= -\frac{3}{2} \ln|x+1| + \frac{3}{2} \cdot \frac{1}{2} \ln|x^2+1| - \frac{1}{2} \tan^{-1} x$$

$$= -\frac{3}{2} \ln|x+1| + \frac{3}{4} \ln|x^2+1| - \frac{1}{2} \tan^{-1} x + c$$

$$\text{Q.19} \int \frac{x}{(x-1)(x^2+1)} dx$$

$$\text{Solution: } \int \frac{x}{(x-1)(x^2+1)} dx$$

Now

$$\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow x = A(x^2+1) + (Bx+C)(x-1) \rightarrow (i)$$

$$\text{Put } x-1=0 \Rightarrow x=1 \text{ in (i)}$$

$$\Rightarrow 1 = A((1)^2+1)$$

$$\Rightarrow A = \frac{1}{2}$$

From (i)

$$\Rightarrow x = Ax^2 + A + Bx^2 - Bx + Cx - C$$

Equating coefficients of  $x^2$  and  $x$  we have



$$\text{For } x^2; \Rightarrow 0 = A + B \Rightarrow 0 = \frac{1}{2} + B \Rightarrow B = -\frac{1}{2}$$

$$\text{For } x; 1 = -B + C \Rightarrow 1 = -\left(-\frac{1}{2}\right) + C$$

$$\Rightarrow 1 = \frac{1}{2} + C \Rightarrow 1 - \frac{1}{2} = C \Rightarrow C = \frac{1}{2}$$

So,

$$\begin{aligned} \frac{x}{(x-1)(x^2+1)} &= \frac{1/2}{x-1} + \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1} \\ \int \frac{x}{(x-1)(x^2+1)} dx &= \frac{1}{2} \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{x-1}{x^2+1} dx \\ &= \frac{1}{2} \ln|x-1| - \frac{1}{4} \int \frac{2x-2}{x^2+1} dx \\ &= \frac{1}{2} \ln|x-1| - \frac{1}{4} \int \frac{2x}{x^2+1} dx + \frac{1}{2} \int \frac{1}{x^2+1} dx \\ &= \frac{1}{2} \ln|x-1| - \frac{1}{4} \ln|x^2+1| + \frac{1}{2} \tan^{-1}x + c \end{aligned}$$

$$\text{Q.20 } \int \frac{9x-7}{(x+3)(x^2+1)} dx$$

$$\text{Solution: } \int \frac{9x-7}{(x+3)(x^2+1)} dx$$

Now

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow 9x-7 = A(x^2+1) + (Bx+C)(x+3) \rightarrow (i)$$

$$\text{Put } x+3=0 \Rightarrow x=-3 \text{ in (i)}$$

$$\Rightarrow 9(-3)-7 = A((-3)^2+1)$$

$$-27-7 = 10A \Rightarrow -34 = 10A \Rightarrow A = -\frac{34}{10}$$

$$\Rightarrow A = -\frac{17}{5}$$

From (i)

$$9x-7 = Ax^2 + A + Bx^2 + 3Bx + Cx + 3C$$

Equating coefficients of  $x^3$  and  $x$ 

$$\text{For } x^2; \Rightarrow 0 = A + B \Rightarrow 0 = -\frac{17}{5} + B \Rightarrow B = \frac{17}{5}$$

$$\text{and for } x; 3B + C = 9 \Rightarrow 3\left(\frac{17}{5}\right) + C = 9 \Rightarrow \frac{51}{5} + C = 9$$

$$C = 9 - \frac{51}{5} = \frac{45-51}{5} \Rightarrow C = -\frac{6}{5}$$

So

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{-17/5}{x+3} + \frac{17/5 x - 6/5}{x^2+1}$$

$$\int \frac{9x-7}{(x+3)(x^2+1)} dx$$

$$= -\frac{17}{5} \int \frac{1}{x+3} dx + \frac{17}{5} \int \frac{x}{x^2+1} dx - \frac{6}{5} \int \frac{1}{x^2+1} dx$$

$$= -\frac{17}{5} \ln|x+3| + \frac{17}{10} \ln|x^2+1| - \frac{6}{5} \tan^{-1}x + c$$

$$\text{Q.21. } \int \frac{1+4x}{(x-3)(x^2+4)} dx$$

$$\text{Solution: } \int \frac{1+4x}{(x-3)(x^2+4)} dx$$

Now

$$\frac{1+4x}{(x-3)(x^2+4)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+4}$$

$$\Rightarrow 1+4x = A(x^2+4) + (Bx+C)(x-3) \rightarrow (i)$$

$$\text{Put } x-3=0 \Rightarrow x=3 \text{ in (i)}$$

$$1+4(3) = A((3)^2+4) + B(3) + C(0)$$

$$\Rightarrow 13 = A(9+4) \Rightarrow 13 = 13A \Rightarrow A = 1$$

From (i)

$$1+4x = Ax^2 + 4a + Bx^2 - 3Bx + Cx - 3C$$

Equating Coefficients of  $x^2$  and  $x$ 

$$\Rightarrow 0 = A + B \text{ for } x^2$$

$$0 = 1 + B \Rightarrow B = -1$$

$$\Rightarrow 4 = -3B + C \Rightarrow 4 - 3 = C \Rightarrow C = 1$$

So

$$\frac{1+4x}{(x-3)(x^2+4)} = \frac{1}{x-3} + \frac{(-1)x+1}{x^2+4}$$

$$\int \frac{1+4x}{(x-3)(x^2+4)} dx$$

$$= \int \frac{1}{x-3} dx - \int \frac{x}{x^2+4} dx + \int \frac{1}{x^2+4} dx$$

$$= \ln|x-3| - \frac{1}{2} \int \frac{2x}{x^2+4} dx + \int \frac{1}{x^2+4} dx$$

$$= \ln|x-3| - \frac{1}{2} \ln|x^2+4| + \frac{1}{2} \tan^{-1} \frac{x}{2} + c$$

Q.22

$$\int \frac{12}{x^3+8} dx$$

Solution:

$$\int \frac{12}{x^3+8} dx \quad \because a^3 - b^3 = (a+b)(a^2 - ab + b^2)$$

Now

$$\frac{12}{x^3+8} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4}$$

$$\Rightarrow 12 = A(x^2-2x+4) + (Bx+C)(x+2) \rightarrow (i)$$

$$\text{Put } x+2=0 \Rightarrow x=-2 \text{ in (i)}$$

$$\Rightarrow 12 = A(4+4+4) \Rightarrow 12 = 12A \Rightarrow A = 1$$

From (i)

$$12 = Ax^2 - 2Ax + 4A + Bx^2 + 2Bx + Cx + 2C$$

Equating coefficients of  $x^2$  and  $x$  we have

$$\text{for } x^2; 0 = A + B \Rightarrow 0 = 1 + B \Rightarrow B = -1$$

$$\text{for } x; 0 = -2(1) + 2(-1) + C \Rightarrow 0 = -2 - 2 + C$$

$$\Rightarrow C = 4$$

So

$$\frac{12}{x^3+8} = \frac{1}{x+2} + \frac{-x+4}{x^2-2x+4}$$

$$\int \frac{12}{x^3+8} dx = \int \frac{1}{x+2} dx - \int \frac{x-4}{x^2-2x+4} dx$$

$$= \int \frac{1}{x+2} dx - \frac{1}{2} \int \frac{2x-8}{x^2-2x+4} dx$$

$$= \ln|x+2| - \frac{1}{2} \int \frac{2x-2-6}{x^2-2x+4} dx$$

$$= \ln|x+2| - \frac{1}{2} \int \frac{2x-2}{x^2-2x+4} dx + \frac{6}{2} \int \frac{1}{x^2-2x+4} dx$$

$$= \ln|x+2| - \frac{1}{2} \ln|x^2-2x+4| + 3 \int \frac{1}{x^2-2x+4} dx$$

$$= \ln|x+2| - \frac{1}{2} \ln|x^2-2x+4| + 3 \int \frac{1}{(x-1)^2 + \sqrt{3}} dx$$

$$= \ln|x+2| - \frac{1}{2} \ln|x^2-2x+4| + \frac{3}{\sqrt{3}} \tan^{-1} \left( \frac{x-1}{\sqrt{3}} \right) + c$$

$$= \ln|x+2| - \frac{1}{2} \ln|x^2-2x+4| + \sqrt{3} \tan^{-1} \left( \frac{x-1}{\sqrt{3}} \right) + c$$

Q23.  $\int \frac{9x^2+6}{x^3-8} dx$

Solution:

$$\int \frac{9x^2+6}{x^3-8} dx$$

Now

$$\frac{9x^2+6}{x^3-8} = \frac{9x^2+6}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$$

$$\Rightarrow 9x+6 = A(x^3+2x+4) + (Bx+C)(x-2) \rightarrow (i)$$

Put  $x-2=0 \Rightarrow x=2$  in (i)

$$9(2)+6 = A[(2)^2+2(2)+4] + B(2)+C(0)$$

$$\Rightarrow 24 = 12A \Rightarrow A = 2$$

From (i)

$$9x+6 = Ax^2+2Ax+4A+Bx^2-2Bx+Cx-2C$$

Equating coefficient of  $x^2$  and  $x$

For  $x^2$ ;  $0 = A+B \Rightarrow 0 = 2+B \Rightarrow B = -2$

For  $x$ ;  $9 = 2A-2B+C \Rightarrow 9 = 2(2)-2(-2)+C$

$$\Rightarrow 9 = 4+4+C \Rightarrow 9-8 = C \Rightarrow C = 1$$

So

$$\frac{9x^2+6}{(x-2)(x^2+2x+4)} = \frac{2}{x-2} + \frac{-2x+1}{x^2+2x+4}$$

$$\int \frac{9x^2+6}{x^3-8} dx = 2 \int \frac{1}{x-2} dx - \int \frac{2x-1}{x^2+2x+4} dx$$

$$= -2 \int \frac{1}{x-2} dx - \int \frac{2x+2-2-1}{x^2+2x+4} dx$$

$$= -2 \int \frac{1}{x-2} dx - \int \frac{2x+2}{x^2+2x+4} dx + 3 \int \frac{1}{x^2+2x+4} dx$$

$$= 2 \ln|x-2| - \ln|x^2+2x+4| + 3 \int \frac{1}{(x+1)^2+(\sqrt{3})^2} dx$$

$$2 \ln|x-2| - \ln|x^2+2x+4| + \frac{3}{\sqrt{3}} \tan^{-1} \left( \frac{x+1}{\sqrt{3}} \right) + C$$

$$2 \ln|x-2| - \ln|x^2+2x+4| + \sqrt{3} \tan^{-1} \left( \frac{x+1}{\sqrt{3}} \right) + c$$

Q24  $\int \frac{2x^2+5x+3}{(x-1)^2(x^2+4)} dx$

Solution:  $\int \frac{2x^2+5x+3}{(x-1)^2(x^2+4)} dx$

Now

$$\frac{2x^2+5x+3}{(x-1)^2(x^2+4)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4}$$

$$\Rightarrow 2x^2+5x+3 = A(x-1)(x^2+4) + B(x^2+4) + (Cx+D)(x-1)^2 \rightarrow (i)$$

Put  $x-1=0 \Rightarrow x=1$  in (i)

$$\Rightarrow 2(1)^2+5(1)+3 = B(1+4)$$

$$2+5+3 = 5B \Rightarrow 10 = 5B \Rightarrow B = 2$$

From (i)

$$2x^2+5x+3 = A(x^3+4x-x^2-4) + Bx^2+4B + (Cx+D)(x^2+1+2x)$$

$$Ax^3+4Ax-Ax^2-4A+Bx^2+4B+Cx^3+Cx-2Cx^2+Dx^2+D-2Dx$$

Equating coefficient of  $x^3, x^2$  and  $x$  we get

For  $x^3 \Rightarrow 0 = A+C \Rightarrow C = -A \rightarrow (ii)$

For  $x^2$ ;  $2 = -A+B-2C+D$

Put  $B = 2$  and  $C = -A$

$$2 = -A+2-2(-A)+D$$

$$\Rightarrow 2-2 = -A+2A+D \Rightarrow 0 = A+D$$

$$\Rightarrow D = -A \rightarrow (iii)$$

For  $x$ ;  $5 = 4A+C-2D$  put  $C = -A$  and  $D = -A$

$$\Rightarrow 5 = 4A-A-2(-A)$$

$$5 = 3A+2A \Rightarrow 5 = 5A \Rightarrow A = 1$$

So (ii)  $\Rightarrow C = -1$  and (iii)  $\Rightarrow D = -1$

Thus

$$\frac{2x^2+5x+3}{(x-1)^2(x^2+4)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{(-1)x-1}{x^2+4}$$

$$\Rightarrow \int \frac{2x^2+5x+3}{(x-1)^2(x^2+4)} dx = \int \frac{1}{x-1} dx + 2 \int (x-1)^{-2} dx - \int \frac{x+1}{x^2+4} dx$$

$$= \int \frac{1}{x-1} dx + 2 \int (x-1)^{-2} dx - \int \frac{x}{x^2+4} dx - \int \frac{1}{x^2+4} dx$$

$$= \ln|x-1| + \frac{2(x-1)^{-1}}{-1} - \frac{1}{2} \int \frac{2x}{x^2+4} dx - \int \frac{1}{(x)^2+(2)^2} dx$$

$$= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2} \ln|x^2+4| - \frac{1}{2} \tan^{-1} \frac{x}{2} + c$$

Q25.

$$\int \frac{2x^2-x-7}{(x+2)^2(x^2+x+1)} dx$$

Solution:

$$\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2+x+1}$$

$$\Rightarrow 2x^2-x-7 = A(x+2)(x^2+x+1) + B(x^2+x+1) + (Cx+D)(x+2)^2$$

Put  $x+2=0 \Rightarrow x=-2$

$$\Rightarrow 2(-2)^2-2(-2)-7 = B((-2)^2+(-2)+1)$$

$$\Rightarrow 8+2-7 = B(4-2+1)$$

$$\Rightarrow 3 = 3B \Rightarrow B = 1$$

From (i)

$$2x^2-x-7 = A(x^3+x^2+x+2x^2+2) + Bx^2+Bx + B + C(Cx+D)(x^2+4x+4)$$

$$= Ax^3+3Ax^2+3A+Bx^2+Bx+B+Cx^3+4Cx^2+4Cx+Dx^2+4Dx+4D$$

Equating coefficients of  $x^3, x^2$  and  $x$

for  $x^3$ ;  $2+3A+B+4C+D$

Put  $B = 1, C = -A \rightarrow (ii)$

For  $x^2$ ;  $2 = 3A+B+4C+D \Rightarrow 2-1 = -A+D$

$$\Rightarrow D = A+1 \rightarrow (iii)$$

For  $x$ ;  $-1 = 3A+B+4C+4D$

Put  $B = 1, C = -A, D = A+1$

$$\Rightarrow -1 = 3A+1-4A+4A+4$$

$$-1-1-4 = 3A \Rightarrow -6 = 3A \Rightarrow A = -2$$

So (ii)  $\Rightarrow C = 2$  and (iii)  $\Rightarrow B = -1$

Thus,

$$\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)} = \frac{-2}{x+2} + \frac{1}{(x+2)^2} + \frac{2x-1}{x^2+x+1}$$

$$\int \frac{2x^2-x-7}{(x+2)^2(x^2+x+1)} dx$$

$$= -2 \int \frac{1}{x+2} dx + \int (x+2)^{-2} dx + \int \frac{2x+1-2}{x^2+x+1} dx$$

$$= -2 \ln|x+2| + \frac{(x+2)^{-1}}{-1} + \int \frac{2x+1}{x^2+x+1} dx - 2 \int \frac{1}{x^2+x+1} dx$$

$$\begin{aligned}
&= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - 2 \int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}} \\
&= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - 2 \int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}} dx \\
&= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - 2 \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\
&= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - \frac{2}{\frac{\sqrt{3}}{2}} \tan^{-1} \frac{\left(x+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + c \\
&= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - \frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}}\right) + c
\end{aligned}$$

Q.26  $\int \frac{3x+1}{(4x^2+1)(x^2-x+1)} dx$

Solution:  $\int \frac{3x+1}{(4x^2+1)(x^2-x+1)} dx$

$$\begin{aligned}
\therefore \frac{3x+1}{(4x^2+1)(x^2-x+1)} &= \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{x^2-x+1} \\
\Rightarrow 3x+1 &= (Ax+B)(x^2-x+1) + (Cx+D)(4x^2+1) \\
3x+1 &= Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + 4Cx^3 \\
&\quad + Cx + 4Dx^2 + D
\end{aligned}$$

Equating coefficients of  $x^3, x^2, x$  and constants terms.

For  $x^3$ ;  $0 = A + 4C \rightarrow (i)$

For  $x^2$ ;  $0 = -A + B + 4D \rightarrow (ii)$

for  $x$ ;  $3 = A + B + C \rightarrow (iii)$

For constant term;  $1 = B + D \rightarrow (iv)$

From (i)  $A = -4C$  and (iv)  $\Rightarrow B = 1 - D$

Put in (ii) and (iii)

$$\Rightarrow 0 = -(-4C) + (1 - D) + 4D \text{ and } 3 = -4C - (1 - D) + C$$

$$0 = 4C + 1 + 4D \quad 3 = -4C - 1 + D + C$$

$$0 = 4C + 3D + 1 \rightarrow (v) \quad 0 = -3C + D - 4$$

$$\Rightarrow D = 3C + 4 \text{ put in (iv)}$$

$$\Rightarrow 0 = 4C + 3(3C + 4) + 1$$

$$0 = 4C + 3(3C + 4) + 1$$

$$0 = 4C + 9C + 12 + 1 \Rightarrow 0 = 13C + 13$$

$$\Rightarrow -13C = 12 \Rightarrow C = -1$$

As  $A = -4C \Rightarrow A = -4(-1) \Rightarrow A = 4 \therefore C = -1$

As  $D = 3C + 4 \Rightarrow D = 3(-1) + 4 = -3 + 4$

$$\Rightarrow D = 1$$

As  $B = 1 - D = 1 - 1 = 0 \Rightarrow B = 0$

Thus

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{4x+0}{4x^2+1} + \frac{(-1)x+1}{x^2-x+1}$$

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{1}{2} \frac{8x}{4x^2+1} + \frac{(-1)(x-1)}{x^2-x+1}$$

$$\int \frac{3x+1}{(4x^2+1)(x^2-x+1)} dx = \frac{1}{2} \int \frac{8x}{4x^2+1} - \frac{1}{2} \int \frac{2x-2}{x^2-x+1} dx$$

$$= \frac{1}{2} \ln|4x^2+1| - \frac{1}{2} \int \frac{2x-1-1}{x^2-x+1} dx$$

$$= \frac{1}{2} \ln|x^2+1| - \frac{1}{2} \int \frac{2x-1}{x^2-x+1} dx + \frac{1}{2} \int \frac{1}{x^2-x+1} dx$$

$$= \frac{1}{2} \ln|4x^2+1| - \frac{1}{2} \ln|x^2-x+1| = \frac{1}{2} \int \frac{1}{x^2-x+\frac{1}{4}+\frac{3}{4}} dx$$

$$\begin{aligned}
&= \frac{1}{2} \ln|4x^2+1| - \frac{1}{2} \ln|x^2-x+1| + \frac{1}{2} \int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\
&= \frac{1}{2} \ln|4x^2+1| - \frac{1}{2} \ln|x^2-x+1| + \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \frac{\left(x-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + c \\
&= \frac{1}{2} \ln|4x^2+1| - \frac{1}{2} \ln|x^2-x+1| + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2x-1}{\sqrt{3}}\right) + c
\end{aligned}$$

Q27.  $\int \frac{4x+1}{(x^2+4)(x^2+4x+5)} dx$

Solution:

$$\begin{aligned}
&\int \frac{4x+1}{(x^2+4)(x^2+4x+5)} dx \\
\therefore \frac{4x+1}{(x^2+4)(x^2+4x+5)} &= \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+4x+5} \\
\Rightarrow 4x+1 &= (Ax+B)(x^2+4x+5) + (Cx+D)(x^2+4) \\
\Rightarrow 4x+1 &= Ax^3 + 4Ax^2 + 5Ax + Bx^2 + 4Bx \\
&\quad + 5B + Cx^3 + 4Cx + Dx^2 + 4D
\end{aligned}$$

Equating coefficients of  $x^3, x^2, x$  and constant term.

Put  $x^3$ ;  $0 = A + C \rightarrow (i)$

for  $x^2$ ;  $0 = 4A + B + D \rightarrow (ii)$

for  $x$ ;  $4 = 5A + 4B + 4C \rightarrow (iii)$

For constant term  $1 = 5B + 4D \rightarrow (iv)$

From (i)  $\Rightarrow A = -C$  and (iv)  $\Rightarrow 5B = 1 - 4D$

$$B = \frac{1-4D}{5} \text{ put in (ii) and (iii)}$$

so (ii)  $\Rightarrow 0 = 4(-C) + \frac{1-4D}{5} + D$  and (iii)  $\Rightarrow 4 = 5(-C) + 4\left(\frac{-4D}{5}\right) + 4C$

$$\Rightarrow 0 = -4C + \frac{1-4D}{5} + D \Rightarrow 20 = -25C + 4 - 16D + 20C$$

$$0 = -20C + 1 - 4D + 5D \Rightarrow 16D = -5C + 4 - 20$$

$$\Rightarrow 0 = -20C + D + 1 \Rightarrow D = \frac{-5C - 16}{16} \rightarrow (vi)$$

$$\Rightarrow D = 20C - 1 \rightarrow (v)$$

By (v) and (vi)  $\Rightarrow 20C - 1 = \frac{-5C - 16}{16}$

$$\Rightarrow 320C - 16 = -5C - 16 \Rightarrow 320C + 5C = 0$$

$$\Rightarrow 320C = 0 \Rightarrow C = 0$$

As  $a = -C \Rightarrow A = 0$

As  $D = 20C - 1 \Rightarrow D = 20(0) - 1 \Rightarrow D = -1$

$$\text{As } B = \frac{1-4D}{5} \Rightarrow B = \frac{1-4(-1)}{5} = \frac{5}{5} = 1$$

$$B = 1$$

So

$$\frac{4x+1}{(x^2+4)(x^2+4x+5)} = \frac{0x+1}{x^2+4} + \frac{0x+(-1)}{x^2+4x+5}$$

$$\int \frac{4x+1}{(x^2+4)(x^2+4x+5)} = \int \frac{1}{x^2+4} dx - \int \frac{1}{x^2+4x+4+1} dx$$

$$= \int \frac{1}{x^2+4} dx - \int \frac{1}{x^2+4x+4+1} dx$$

$$= \frac{1}{2} \tan^{-1} \frac{x}{2} - \int \frac{1}{(x-2)^2 + (1)^2} dx$$

$$= \frac{1}{2} \tan^{-1} \frac{x}{2} - \tan^{-1}(x-2) + c$$

$$\text{Q28. } \int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)} dx$$

$$\begin{aligned} \therefore \frac{6a^2}{(x^2+a^2)(x^2+4a^2)} &= \frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{x^2+4a^2} \\ \Rightarrow 6a^2 &= (Ax+B)(x^2+4a^2) + (Cx+D)(x^2+a^2) \\ \Rightarrow 6a^2 &= Ax^3 + 4a^2Ax + Bx^2 + 4Ba^2 + Cx^3 \\ &\quad + Ca^2x + Dx^2 + Da^2 \end{aligned}$$

Equating coefficients of  $x^3, x^2, x$  and constants term.

$$\text{Put } x^3; 0 = A + C \rightarrow (i)$$

$$\text{for } x^2; 0 = B + D \rightarrow (ii)$$

$$\begin{aligned} \text{for } x; 0 &= 4a^2A + a^2C \Rightarrow 0 = (4A + C)a^2 \\ &\Rightarrow 4A + C \rightarrow (iii) \end{aligned}$$

For constant term  $1 = 5B + 4D \rightarrow (iv)$

From (i)  $\Rightarrow A = -C$  and (iv)  $\Rightarrow B = -D$

Put in (iii) and (iv) so

$$(iii) 4(-C) + C = 0 \Rightarrow -4C + C = 0 \Rightarrow -3C = 0 \Rightarrow C = 0$$

$$(iv) 4(-D) + D = 6 \Rightarrow -4D + D = 6 \Rightarrow -3D = 6 \Rightarrow D = -2$$

As  $A = -C \Rightarrow A = 0 \therefore C = 0$

$$\text{As } B = -D \Rightarrow B = -(-2) \Rightarrow B = 2 \therefore D = -2$$

So

$$\begin{aligned} \frac{6a^2}{(x^2+a^2)(x^2+4a^2)} &= \frac{0x+2}{x^2+a^2} + \frac{0x+(-2)}{x^2+4a^2} \\ \int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)} dx &= 2 \int \frac{1}{x^2+a^2} dx - 2 \int \frac{1}{x^2+(2a)^2} dx \\ &= \frac{2}{a} \tan^{-1} \frac{x}{a} - \frac{1}{a} \tan^{-1} \frac{x}{2a} + c \\ &= \frac{2}{a} \tan^{-1} \frac{x}{a} - \frac{1}{a} \tan^{-1} \frac{x}{2a} + c \end{aligned}$$

$$\text{Q29. } \int \frac{2x^2-2}{(x^4+x^2+1)(x^2-x+1)} dx$$

Solution:

$$\begin{aligned} \int \frac{2x^2-2}{(x^2+x^2+1)(x^2-x+1)} &= \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1} \\ 2x^2-2 &= (Ax+B)(x^2-x+1) + (Cx+D)(x^2+x+1) \\ &= Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + Cx^3 + Cx^2 + Cx \\ &\quad + Dx^2 + Dx + D \end{aligned}$$

Equating coefficients of  $x^3, x^2, x$  and constant term.

$$\text{for } x^3; 0 = A + C \rightarrow (i)$$

$$\text{for } x^2; 0 = A - B + C + D \rightarrow (ii)$$

$$\text{for } x; 2 = -A + B + C + D \rightarrow (iii)$$

$$\begin{aligned} \text{for constant term; } -2 &= -A + C - 2 \\ \Rightarrow 2 + 2 &= -A + C \Rightarrow -A + C = 4 \rightarrow (v) \end{aligned}$$

Put  $A + C = 0$  in (ii)  $\Rightarrow 0 = -B + D \rightarrow (vi)$

$$\text{Now by (i) + (v) } \Rightarrow 2C = 4 \Rightarrow C = 2$$

$$\text{as } A + C = 0 \Rightarrow A + 2 = 0 \Rightarrow A = -2$$

Now by (iv) + (vi)  $\Rightarrow 2D = -D \Rightarrow D = -1$

$$\text{As } B + D = -2 \Rightarrow B - 1 = -2 \Rightarrow B = -1$$

So;

$$\frac{2x^2-2}{(x^2+x^2+1)(x^2-x+1)} = \frac{-2x-1}{x^2+x+1} + \frac{2x-1}{x^2-x+1}$$

$$\begin{aligned} \int \frac{2x^2-2}{(x^2+x+1)(x^2-x+1)} &= - \int \frac{2x+1}{x^2+x+1} dx + \int \frac{2x-1}{x^2-x+1} dx \\ &= -\ln|x^2+x+1| + \ln|x^2-x+1| + c \\ &= \ln \left| \frac{x^2-x+1}{x^2+x+1} \right| + c \end{aligned}$$

$$\text{Q 30. } \int \frac{3x-8}{(x^2-x+2)(x^2+x+2)} dx$$

Solution:  $\int \frac{3x-8}{(x^2-x+2)(x^2+x+2)} dx$

$$\begin{aligned} \therefore \frac{3x-8}{(x^2-x+2)(x^2+x+2)} &= \frac{Ax+B}{x^2-x+2} + \frac{Cx+D}{x^2+x+2} \\ 3x-8 &= (Ax+B)(x^2+x+2) + (Cx+D)(x^2-x+2) \\ &= Ax^3 + Ax^2 + 2Ax + Bx^2 + Bx + 2B + Cx^3 - Cx^2 + 2Cx \\ &\quad + Dx^2 - Dx + 2D \end{aligned}$$

Equating coefficients of  $x^3, x^2, x$  and constant term.

$$\text{for } x^3; 0 = A + C \rightarrow (i)$$

$$\text{for } x^2; 0 = A + B - C + D \rightarrow (ii)$$

$$\text{for } x; 3 = 2A + B + 2C - D \rightarrow (iii)$$

$$\text{for constant term; } -8 = 2B + 2D \Rightarrow B + D = -4 \rightarrow (iv)$$

From (i)  $\Rightarrow A = -C$  and from (iv)  $\Rightarrow B = -4 - D$

Put in (ii) and (iii) so

$$(ii) \Rightarrow 0 = -C + (-4 - D) - C + D$$

$$0 = -C - 4 - B - C + D$$

$$0 = -2C - 4$$

$$\Rightarrow 2C = -4 \Rightarrow C = -2 \text{ as } A = -C \Rightarrow A = 2$$

$$(iii) \Rightarrow 3 = 2(-C) - 4 - D + 2C - D$$

$$3 = -2C - 4 + 2C - 2D$$

$$\Rightarrow 3 + 4 = -2D \Rightarrow D = -\frac{7}{2}$$

$$\text{As } B = -4 - D = -4 - \left(-\frac{7}{2}\right) = -4 + \frac{7}{2} = \frac{-8+7}{2} = -\frac{1}{2}$$

$$1/2 \Rightarrow B = -\frac{1}{2}$$

So

$$\frac{3x-8}{(x^2-x+2)(x^2+x+2)} = \frac{2x-1/2}{x^2-x+2} + \frac{-2x+(-7/2)}{x^2+x+2}$$

$$\begin{aligned} \int \frac{3x-8}{(x^2-x+2)(x^2+x+2)} &= - \int \frac{2x+1-1-1/2}{x^2-x+2} dx + \int \frac{2x+1-1+7/2}{x^2+x+2} dx \\ &= \int \frac{2x-1+1/2}{x^2-x+2} dx + \int \frac{2x+1+5/2}{x^2+x+2} dx \\ &= \int \frac{2x-1}{x^2-x+2} dx + \frac{1}{2} \int \frac{1}{x^2-x+2} dx - \int \frac{2x+1}{x^2+x+2} dx \\ &\quad - \int \frac{5/2}{x^2+x+2} dx \\ &= \ln|x^2-x+2| + \frac{1}{2} \int \frac{dx}{x^2-x+\frac{1}{4}-\frac{1}{4}+2} - \ln|x^2+x+2| \\ &\quad - \frac{5}{2} \int \frac{1}{x^2+x+\frac{1}{4}-\frac{1}{4}+2} dx \\ &= \ln|x^2-x+2| + \frac{1}{2} \int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \frac{7}{4}} - \ln|x^2+x+2| \\ &\quad - \frac{5}{2} \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{7}{4}} dx \end{aligned}$$

$$\begin{aligned}
&= \ln|x^2 - x + 1| + \frac{1}{2} \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{7}{4}} dx - \ln|x^2 + x + 2| \\
&\quad - \frac{5}{2} \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{7}{4}} dx \\
&= \ln|x^2 - x + 1| + \frac{1}{2} \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} dx - \ln|x^2 + x + 1| \\
&\quad - \frac{5}{2} \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} dx \\
&= \ln|x^2 - x + 1| + \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{7}}{2}} \tan^{-1} \left( \frac{x - \frac{1}{2}}{\frac{\sqrt{7}}{2}} \right) - \ln|x^2 + x + 1| \\
&\quad - \frac{5}{2} \cdot \frac{1}{\frac{\sqrt{7}}{2}} \tan^{-1} \left( \frac{x + \frac{1}{2}}{\frac{\sqrt{7}}{2}} \right) \\
&= \ln|x^2 - x + 1| + \frac{1}{\sqrt{7}} \tan^{-1} \left( \frac{2x - 1}{\sqrt{7}} \right) - \ln|x^2 + x + 1| \\
&\quad - \frac{5}{\sqrt{7}} \tan^{-1} \left( \frac{2x + 1}{\sqrt{7}} \right) + c
\end{aligned}$$

Q31.  $\int \frac{3x^3 + 4x^2 + 9x + 5}{(x^2 + x + 1)(x^2 + 2x + 3)} dx$

Solution:  $\int \frac{3x^3 + 4x^2 + 9x + 5}{(x^2 + x + 1)(x^2 + 2x + 3)} dx$

$$\begin{aligned}
\because \frac{3x^3 + 4x^2 + 9x + 5}{(x^2 + x + 1)(x^2 + 2x + 3)} &= \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{x^2 + 2x + 3} \\
\Rightarrow 3x^3 + 4x^2 + 9x + 5 &= (Ax + B)(x^2 + 2x + 3) + (Cx + D)(x^2 + x + 1) \\
ax^3 + 2Ax^2 + 3Ax + Bx^2 + 2Bx + 3B + Cx^3 + Cx^2 + Cx + Dx^2 &+ Dx + D
\end{aligned}$$

Equation coefficients of  $x^3, x^2, x$  and constant term.

for  $x^3; 3 = A + C \rightarrow (i)$

For  $x^2; 4 = 2A + B + C + D \rightarrow (ii)$

For  $x; 9 = 3A + 2B + C + D \rightarrow (iii)$

For constant term;  $5 = 3B + 2B + C + D \rightarrow (iv)$

From (i)  $\Rightarrow A = 3 - C$  and from (iv)  $\Rightarrow D = 5 - 3B$

Put in (ii) and (iii)

(ii)  $\Rightarrow 4 = 2(3 - C) + B + C + 5 - 3B$

$4 = 6 - 2C + B + C + 5 - 3B$

$4 - 6 - 5 = -C - 2B \Rightarrow -7 = -(C + 2B)$

$\Rightarrow C + 2B = 7 \rightarrow (v)$

(iii)  $\Rightarrow 9 = 3(3 - C) + 2B + C + 5 - 3B$

$\Rightarrow 9 = 9 - 3C + 2B + C + 5 - 3B$

$\Rightarrow 9 - 9 - 5 = -2C - B \Rightarrow B = -2C + 5$  put in (v)

$\Rightarrow C + 2(-2C + 5) = 7 \Rightarrow C - 4C + 10 = 7$

$\Rightarrow -3C + 10 + 7 \Rightarrow -3C = 7 - 10$

$\Rightarrow -3C = -3 \Rightarrow C = 1$

As  $B = 5 - 2C = 5 - 2(1) = 3 \Rightarrow B = 3$

As  $D = 5 - 3B = 5 - 3(3) = 5 - 9 = -4 \Rightarrow D = -4$

As  $A = 3 - C = 3 - 1 = 2 \Rightarrow A = 2$

So

$$\frac{3x^3 + 4x^2 + 9x + 5}{(x^2 + x + 1)(x^2 + 2x + 3)} = \frac{2x + 3}{x^2 + x + 1} + \frac{x - 4}{x^2 + 2x + 3}$$

$$\begin{aligned}
\int \frac{3x^3 + 4x^2 + 9x + 5}{(x^2 + x + 1)(x^2 + 2x + 3)} dx \\
= \int \frac{2x + 1 + 2}{x^2 + x + 1} dx + \frac{1}{2} \int \frac{2x - 8}{x^2 + 2x + 3} dx
\end{aligned}$$

$$\begin{aligned}
&= \int \frac{2x + 1}{x^2 + x + 1} dx + 2 \int \frac{1}{x^2 + x + 1} dx + \frac{1}{2} \int \frac{2x + 2 - 10}{x^2 + 2x + 3} dx \\
&= \ln|x^2 + x + 1| + 2 \int \frac{1}{x^2 + x + \frac{1}{4} + \frac{3}{4}} dx + \frac{1}{2} \int \frac{2x + 2}{x^2 + 2x + 3} dx \\
&\quad - 5 \int \frac{1}{x^2 + 2x + 3} dx \\
&= \ln|x^2 + x + 1| + 2 \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx + \frac{1}{2} \ln|x^2 + 2x + 3| \\
&\quad - 5 \int \frac{1}{x^2 + 2x + 1 + 2} dx \\
&= \ln|x^2 + x + 1| + 2 \cdot \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \left( \frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + \frac{1}{2} \ln|x^2 + 2x + 3| \\
&\quad - 5 \int \frac{1}{(x + 1)^2 + (\sqrt{2})^2} dx \\
&= \ln|x^2 + x + 1| + \frac{4}{\sqrt{3}} \tan^{-1} \left( \frac{2x + 1}{\sqrt{3}} \right) + \frac{1}{2} \ln|x^2 + 2x + 3| \\
&\quad - \frac{5}{\sqrt{2}} \tan^{-1} \left( \frac{x + 1}{\sqrt{2}} \right) + c \\
&= \ln|x^2 + x + 1| + \ln|x^2 + 2x + 3|^{\frac{1}{2}} + \frac{4}{\sqrt{3}} \tan^{-1} \left( \frac{2x + 1}{\sqrt{3}} \right) \\
&\quad - \frac{5}{2} \tan^{-1} \left( \frac{x + 1}{\sqrt{2}} \right) + c \\
&= \ln|x^2 + x + 1| \sqrt{x^2 + 2x + 3} + \frac{4}{\sqrt{3}} \tan^{-1} \left( \frac{2x + 1}{\sqrt{3}} \right) \\
&\quad - \frac{5}{2} \tan^{-1} \left( \frac{x + 1}{\sqrt{2}} \right) + c
\end{aligned}$$

**The Definite integrals:**

If  $\int f(x)dx = \phi(x) + c$ ,

then the integral of  $f(x)$  from  $a$  to  $b$  is denoted by  $\int_a^b f(x)dx$

And read  $a$  as definite integral of  $f(x)$  here  $a$  is called lower limit and  $b$  is called upper limit.

\*the interval  $[a, b]$  is called range of integration.

We evaluate  $\int_a^b f(x)dx$  as;

Consider  $\int f(x)dx = \phi(x) + c$

$$\begin{aligned}\Rightarrow \int_a^b f(x)dx &= [\phi(x) + c]_a^b \\ &= [\phi(b) + c] - [\phi(a) + c] \\ &= \phi(b) + c - \phi(a) - c\end{aligned}$$

$$\Rightarrow \int_a^b f(x)dx = \phi(b) - \phi(a)$$

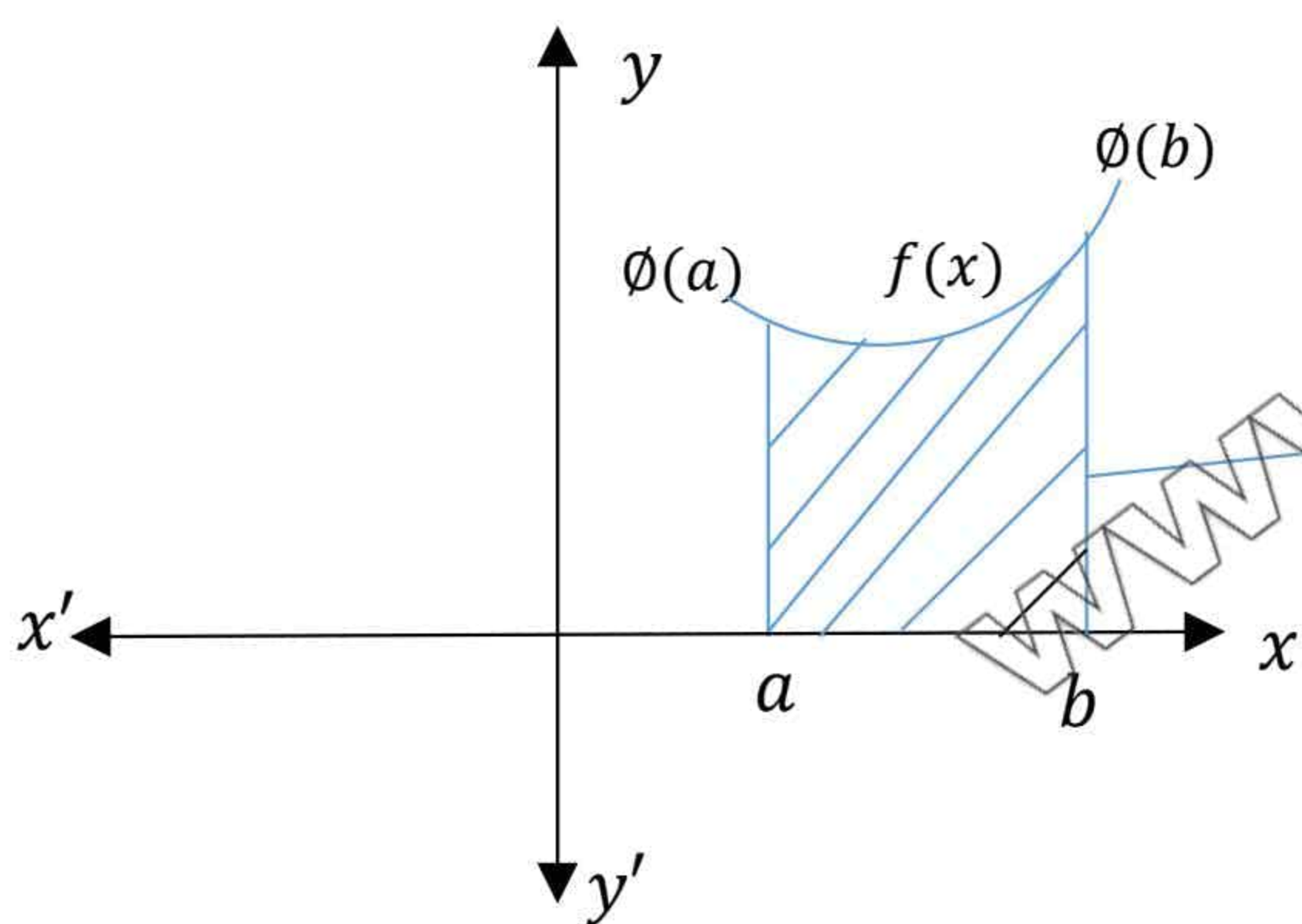
**Note:** if the lower limit is a constant and upper limit is a variable, then the integral is a function of the upper limit.

$$\int_a^x f(t)dt = [\phi(t)]_a^x = \phi(x) - \phi(a)$$

**The area under the curve**

$$\int_a^b f(x)dx = \phi(b) - \phi(a)$$

Represented the "area of region" bounded under the curve of function  $f(x)$  the  $x$ -axis and between two ordinates  $x = a, x = b$  as shown in figure.

**Fundamental theorem of calculus:**

If  $f(x)$  is continuous  $\forall x \in [a, b]$  and  $\phi'(x) = f(x)$

$$\int_a^b f(x)dx = \phi(b) - \phi(a)$$

Is called fundamental theorem of integral calculus.

**Properties of Definite integral**

$$\begin{aligned}\int_a^b f(x)dx &= -\int_b^a f(x)dx \\ &= \phi(b) - \phi(a) \\ &= -[\phi(a) - \phi(b)] \\ &= -\int_b^a f(x)dx\end{aligned}$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$(b) \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad a < c < b$$

**Proof:**

$$\int_a^c f(x)dx = \phi(c) - \phi(a)$$

$$\int_c^b f(x)dx = \phi(b) - \phi(c)$$

$$\begin{aligned}\int_a^c f(x)dx + \int_c^b f(x)dx &= \\ &= \phi(c) - \phi(a) + \phi(b) - \phi(c) \\ &= \phi(b) - \phi(a)\end{aligned}$$

$$\begin{aligned}\Rightarrow \int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx \\ &= \int_a^b f(x)dx\end{aligned}$$

$$(c) \int_a^a f(x)dx = 0$$

**Proof:**

$$\begin{aligned}\int_a^a f(x)dx &= \phi(a) - \phi(a) \\ &= 0\end{aligned}$$

$$\Rightarrow \int_a^a f(x)dx = 0$$

$$\text{Also member } \int_a^b cf(x)dx = c \int_a^b f(x)dx$$

$$\text{and } \int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

## Exercise 3.6

Evaluate the following indefinite integrals:

**Q. 1:**  $\int_1^2 (x^2 + 1) dx$

**SOLUTION:**

$$\begin{aligned} & \int_1^2 (x^2 + 1) dx \\ &= \left[ \frac{x^3}{3} + x \right]_1^2 \\ &= \left( \frac{2^3}{3} + 2 \right) - \left( \frac{1^3}{3} + 1 \right) \\ &= \left( \frac{8}{3} + 2 \right) - \left( \frac{1}{3} + 1 \right) \\ &= \left( \frac{8+6}{3} \right) - \left( \frac{1+3}{3} \right) \\ &= \frac{14}{3} - \frac{4}{3} \\ &= \frac{14-4}{3} = \frac{10}{3} \end{aligned}$$

**Q. 2:**  $\int_{-1}^1 (x^{\frac{1}{3}} + 1) dx$

**SOLUTION:**

$$\begin{aligned} &= \int_{-1}^1 (x^{\frac{1}{3}} + 1) dx \\ &= \left[ \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + x \right]_{-1}^1 \\ &= \left[ \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + x \right]_{-1}^1 \\ &= \left[ \frac{3}{4} x^{\frac{4}{3}} + x \right]_{-1}^1 \\ &= \left( \frac{3}{4} (1)^{\frac{4}{3}} + 1 \right) - \left( \frac{3}{4} (-1)^{\frac{4}{3}} + (-1) \right) \\ &= \left( \frac{3}{4} \cdot 1 + 1 \right) - \left( \frac{3}{4} \cdot 1 - 1 \right) \\ &= \left( \frac{3+4}{4} \right) - \left( \frac{3-4}{4} \right) \\ &= \frac{7}{4} - \frac{-1}{4} = \frac{7}{4} + \frac{1}{4} \\ &= \frac{7+1}{4} = \frac{8}{4} = 2 \end{aligned}$$

**Q. 3:**  $\int_{-2}^0 \frac{1}{(2x-1)^2} dx$

**SOLUTION:**

$$\begin{aligned} &= \int_{-2}^0 (2x-1)^{-2} dx \\ &= \frac{1}{2} \int_{-2}^0 (2x-1)^{-2} \cdot 2 dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \left| \frac{(2x-1)^{-2+1}}{-2+1} \right|_{-2}^0 \\ &= \frac{1}{2} \left| \frac{(2x-1)^{-1}}{-1} \right|_{-2}^0 \\ &= -\frac{1}{2} \left| \frac{1}{2x-1} \right|_{-2}^0 \\ &= -\frac{1}{2} \left[ \left( \frac{1}{2(0)-1} \right) - \left( \frac{1}{2(-2)-1} \right) \right] \\ &= -\frac{1}{2} \left[ \left( \frac{1}{-1} \right) - \left( \frac{1}{-5} \right) \right] \\ &= -\frac{1}{2} \left[ -1 + \frac{1}{5} \right] = -\frac{1}{2} \left[ \frac{-5+1}{5} \right] \\ &= -\frac{1}{2} \left[ \frac{-4}{5} \right] = \frac{2}{5} \end{aligned}$$

**Q. 4:**  $\int_{-6}^2 \sqrt{3-x} dx$

**SOLUTION:**

$$\begin{aligned} &= \int_{-6}^2 (3-x)^{\frac{1}{2}} dx \\ &= (-1) \int_{-6}^2 (3-x)^{\frac{1}{2}} (-1) dx \\ &= - \left| \frac{(3-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right|_{-6}^2 \\ &= - \left| \frac{(3-x)^{\frac{3}{2}}}{\frac{3}{2}} \right|_{-6}^2 \\ &= -\frac{2}{3} \left| (3-x)^{\frac{3}{2}} \right|_{-6}^2 \\ &= -\frac{2}{3} \left[ (3-2)^{\frac{3}{2}} - (3-(-6))^{\frac{3}{2}} \right] \\ &= -\frac{2}{3} \left[ (1)^{\frac{3}{2}} - (9)^{\frac{3}{2}} \right] \\ &= -\frac{2}{3} \left[ 1 - (3^2)^{\frac{3}{2}} \right] \\ &= -\frac{2}{3} [1 - 27] = \frac{52}{3} \end{aligned}$$

**Q. 5:**  $\int_1^{\sqrt{5}} \sqrt{(2t-1)^3} dt$

**SOLUTION:**

$$\begin{aligned} &= \int_1^{\sqrt{5}} (2t-1)^{\frac{3}{2}} dt \\ &= \frac{1}{2} \int_1^{\sqrt{5}} (2t-1)^{\frac{3}{2}} \cdot 2 dt \\ &= \frac{1}{2} \left| \frac{(2t-1)^{\frac{3}{2}+1}}{\frac{3}{2}+1} \right|_1^{\sqrt{5}} = \frac{1}{2} \left| \frac{(2t-1)^{\frac{5}{2}}}{\frac{5}{2}} \right|_1^{\sqrt{5}} \\ &= \frac{1}{2} \cdot \frac{2}{5} \left[ \left( (2\sqrt{5}-1)^{\frac{5}{2}} \right) - \left( (2(1)-1)^{\frac{5}{2}} \right) \right] \end{aligned}$$

$$= \frac{1}{5} \left[ (2\sqrt{5} - 1)^{\frac{5}{2}} - 1 \right]$$

**Q. 6:**  $\int_2^{\sqrt{5}} x\sqrt{x^2-1} dx$

**SOLUTION:**

$$= \int_2^{\sqrt{5}} (x^2 - 1)^{\frac{1}{2}} x dx$$

$$= \frac{1}{2} \int_2^{\sqrt{5}} (x^2 - 1)^{\frac{1}{2}} \cdot 2x dx$$

$$= \frac{1}{2} \left| \frac{(x^2-1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right|_2^{\sqrt{5}} = \frac{1}{2} \left| \frac{(x^2-1)^{\frac{3}{2}}}{\frac{3}{2}} \right|_2^{\sqrt{5}}$$

$$= \frac{1}{2} \cdot \frac{2}{3} \left[ \left( (\sqrt{5})^2 - 1 \right)^{\frac{3}{2}} - \left( (2)^2 - 1 \right)^{\frac{3}{2}} \right]$$

$$= \frac{1}{3} \left[ \left( (4)^{\frac{3}{2}} \right) - \left( 3^{\frac{3}{2}} \right) \right] = \frac{1}{3} \left[ \left( (2^2)^{\frac{3}{2}} \right) - \left( 3^{\frac{3}{2}} \right) \right]$$

$$= \frac{1}{3} [8 - 3\sqrt{3}] = \frac{8}{3} - \sqrt{3}$$

**Q. 7:**  $\int_1^2 \frac{x}{x^2+2} dx$

**SOLUTION:**

$$= \frac{1}{2} \int_1^2 \frac{2x}{x^2+2} dx$$

$$= \frac{1}{2} |\ln(x^2 + 2)|_1^2$$

$$= \frac{1}{2} [\ln(2^2 + 2) - \ln(1^2 + 2)]$$

$$= \frac{1}{2} [\ln(6) - \ln(3)]$$

$$= \frac{1}{2} \left[ \ln \left( \frac{6}{3} \right) \right] = \frac{1}{2} \ln 2$$

$$= \ln 2^{\frac{1}{2}} = \ln \sqrt{2}$$

**Properties of natural log**

$$\ln(AB) = \ln A + \ln B$$

$$\ln \left( \frac{A}{B} \right) = \ln A - \ln B$$

$$\ln A^B = B \ln A$$

**Q. 8:**  $\int_2^3 \left( x - \frac{1}{x} \right)^2 dx$

**SOLUTION:**

$$= \int_2^3 \left( x^2 + \frac{1}{x^2} - 2x \cdot \frac{1}{x} \right) dx$$

$$= \int_2^3 (x^2 + x^{-2} - 2) dx$$

$$= \left| \frac{x^{2+1}}{2+1} + \frac{x^{-2+1}}{-2+1} - 2x \right|_2^3$$

$$= \left| \frac{x^3}{3} + \frac{x^{-1}}{-1} - 2x \right|_2^3 = \left| \frac{x^3}{3} - \frac{1}{x} - 2x \right|_2^3$$

$$= \left( \frac{3^3}{3} - \frac{1}{3} - 2(3) \right) - \left( \frac{2^3}{3} - \frac{1}{2} - 2(2) \right)$$

$$= \left( \frac{27}{3} - \frac{1}{3} - 6 \right) - \left( \frac{8}{3} - \frac{1}{2} - 4 \right)$$

$$= \left( \frac{27-1-18}{3} \right) - \left( \frac{16-3-24}{6} \right)$$

$$= \left( \frac{8}{3} \right) - \left( \frac{-11}{6} \right) = \frac{16+11}{6} = \frac{27}{6} = \frac{9}{2}$$

**Q. 9:**  $\int_{-1}^1 \left( x + \frac{1}{2} \right) \sqrt{x^2 + x + 1} dx$

**SOLUTION:**

$$= \int_{-1}^1 (x^2 + x + 1)^{\frac{1}{2}} \left( \frac{2x+1}{2} \right) dx$$

$$= \frac{1}{2} \int_{-1}^1 (x^2 + x + 1)^{\frac{1}{2}} (2x + 1) dx$$

$$= \frac{1}{2} \left[ \frac{(x^2+x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right]_{-1}^1 = \frac{1}{2} \left[ \frac{(x^2+x+1)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{-1}^1$$

$$= \frac{1}{2} \cdot \frac{2}{3} \left[ \left( (1^2 + 1 + 1)^{\frac{3}{2}} \right) - \left( ((-1)^2 + (-1) + 1)^{\frac{3}{2}} \right) \right]$$

$$= \frac{1}{3} \left[ \left( (3)^{\frac{3}{2}} \right) - \left( (1 - 1 + 1)^{\frac{3}{2}} \right) \right]$$

$$= \frac{1}{3} \left[ \left( (3)^{\frac{3}{2}} \right) - \left( (1)^{\frac{3}{2}} \right) \right]$$

$$= \frac{1}{3} [3\sqrt{3} - 1] = \sqrt{3} - \frac{1}{3} \quad \text{ANS.}$$

$$\frac{3^{\frac{3}{2}}}{3^{\frac{3}{2}}} = \left( 3^{\frac{1}{2}} \right)^3 = (\sqrt{3})^3 = (\sqrt{3})^2 (\sqrt{3})^1 = 3\sqrt{3}$$

**Q. 10:**  $\int_0^3 \frac{dx}{x^2+9}$

**SOLUTION:**

$$= \int_0^3 \frac{1}{3^2+x^2} dx = \left| \frac{1}{3} \tan^{-1} \frac{x}{3} \right|_0^3$$

$$= \frac{1}{3} \left[ \left( \tan^{-1} \frac{3}{3} \right) - \left( \tan^{-1} \frac{0}{3} \right) \right]$$

$$= \frac{1}{3} [(\tan^{-1} 1) - (\tan^{-1} 0)]$$

$$= \frac{1}{3} \left[ \frac{\pi}{4} - 0 \right] = \frac{\pi}{12}$$

**Q. 11:**  $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos t dt$

**SOLUTION:**

$$= |\sin t|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \left( \sin \frac{\pi}{3} \right) - \left( \sin \frac{\pi}{6} \right)$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$$



$$Q.12: \int_1^2 \left(x + \frac{1}{x}\right)^{\frac{1}{2}} \left(1 - \frac{1}{x^2}\right) dx$$

**SOLUTION:**

$$\text{Here } f(x) = x + \frac{1}{x}$$

$$f'(x) = 1 - \frac{1}{x^2}$$

$$\left| \frac{\left(x + \frac{1}{x}\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right|_1^2 = \left| \frac{\left(x + \frac{1}{x}\right)^{\frac{3}{2}}}{\frac{3}{2}} \right|_1^2$$

$$= \frac{2}{3} \left[ \left( \left(2 + \frac{1}{2}\right)^{\frac{3}{2}} \right) - \left( \left(1 + \frac{1}{1}\right)^{\frac{3}{2}} \right) \right]$$

$$= \frac{2}{3} \left[ \left( \left(\frac{5}{2}\right)^{\frac{3}{2}} \right) - \left( (2)^{\frac{3}{2}} \right) \right]$$

$$= \frac{2}{3} \left[ \frac{5}{2} \sqrt{\frac{5}{2}} - 2\sqrt{2} \right]$$

$$= \frac{2}{3} \cdot \frac{5\sqrt{5}}{2\sqrt{2}} - \frac{2}{3} \cdot 2\sqrt{2}$$

$$= \frac{5\sqrt{5}}{3\sqrt{2}} - \frac{4\sqrt{2}}{3} = \frac{5\sqrt{5}-4(2)}{3\sqrt{2}}$$

$$= \frac{5\sqrt{5}-8}{3\sqrt{2}}$$

$$Q.13: \int_1^2 \ln x \, dx$$

**SOLUTION:**

Consider

$$\int \ln x \, dx = \int \ln x \cdot 1 \, dx$$

$$\text{Here } U = \ln x, \quad V = 1$$

$$\text{Using } \int U \cdot V = U \cdot \int V \, dx - \int [U' \cdot \int V \, dx] \, dx$$

$$= \ln x \cdot \int 1 \, dx - \int [(\ln x)' \cdot \int 1 \, dx] \, dx$$

$$= \ln x \cdot x - \int \left[ \frac{1}{x} \cdot x \right] \, dx$$

$$= \ln x \cdot x - \int 1 \, dx$$

$$= x \ln x - x + c$$

Taking limits

$$\int_1^2 \ln x \, dx = [x \ln x - x]_1^2$$

$$= (2 \ln 2 - 2) - (1 \ln 1 - 1)$$

$$= (2 \ln 2 - 2) - (1(0) - 1)$$

$$= (2 \ln 2 - 2)$$

$$Q.14: \int_0^2 \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right) dx$$

**SOLUTION:**

$$\int_0^2 \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right) dx = \left| \frac{e^{\frac{x}{2}}}{\frac{1}{2}} - \frac{e^{-\frac{x}{2}}}{-\frac{1}{2}} \right|_0^2 = \left| 2e^{\frac{x}{2}} + 2e^{-\frac{x}{2}} \right|_0^2$$

$$= 2 \left[ \left( e^{\frac{2}{2}} + e^{-\frac{2}{2}} \right) - \left( e^{\frac{0}{2}} + e^{-\frac{0}{2}} \right) \right]$$

$$= 2 \left[ (e^1 + e^{-1}) - (e^0 + e^0) \right]$$

$$= 2 \left[ e + \frac{1}{e} - 1 - 1 \right] = 2 \left[ e + \frac{1}{e} - 2 \right]$$

$$= 2 \left[ \frac{e^2 + 1 - 2e}{e} \right] = \frac{2}{e} (e^2 + 1^2 - 2e)$$

$$Q.15: \int_0^{\frac{\pi}{4}} \frac{\cos \theta + \sin \theta}{\cos 2\theta + 1} d\theta$$

**SOLUTION:**

$$= \int_0^{\frac{\pi}{4}} \frac{\cos \theta + \sin \theta}{1 + \cos 2\theta} dx = \int_0^{\frac{\pi}{4}} \frac{\cos \theta + \sin \theta}{2 \cos^2 \theta} d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \left[ \frac{\cos \theta}{\cos^2 \theta} + \frac{\sin \theta}{\cos^2 \theta} \right] dx = \frac{1}{2} \int_0^{\frac{\pi}{4}} \left[ \frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta \cdot \cos \theta} \right] d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} [\sec \theta + \sec \theta \tan \theta] d\theta$$

$$= \frac{1}{2} \left[ \ln |\sec \theta + \tan \theta| + \sec \theta \right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2} \left[ \left( \ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| + \sec \frac{\pi}{4} \right) - \left( \ln |\sec 0 + \tan 0| + \sec 0 \right) \right]$$

$$= \frac{1}{2} \left[ \left( \ln |\sqrt{2} + 1| + \sqrt{2} \right) - \left( \ln |1 + 0| + 1 \right) \right]$$

$$= \frac{1}{2} \left[ \left( \ln |\sqrt{2} + 1| + \sqrt{2} \right) - (0 + 1) \right]$$

$$= \frac{1}{2} \left[ \ln |\sqrt{2} + 1| + \sqrt{2} - 1 \right]$$

$$Q.16: \int_0^{\frac{\pi}{6}} \cos^3 \theta \, d\theta$$

**SOLUTION:**

$$= \int_0^{\frac{\pi}{6}} \cos \theta \cos^2 \theta \, d\theta = \int_0^{\frac{\pi}{6}} \cos \theta (1 - \sin^2 \theta) \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} (\cos \theta - \sin^2 \theta \cos \theta) \, d\theta = \left| \sin \theta - \frac{\sin^3 \theta}{3} \right|_0^{\frac{\pi}{6}}$$

$$= \left( \sin \frac{\pi}{6} - \frac{\sin^3 \frac{\pi}{6}}{3} \right) - \left( \sin 0 - \frac{\sin^3 0}{3} \right)$$

$$= \left( \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} \right) - \left( 0 - \frac{0}{3} \right) = \left( \frac{1}{2} - \frac{1}{24} \right) - (0) = \frac{12-1}{24} = \frac{11}{24}$$

$$Q.17: \int_0^{\frac{\pi}{6}} \cos^2 \theta \cot^2 \theta \, d\theta$$

**SOLUTION:**

$$\int_0^{\frac{\pi}{6}} \cos^2 \theta (\operatorname{cosec}^2 \theta - 1) \, d\theta = \int_0^{\frac{\pi}{6}} (\cos^2 \theta \operatorname{cosec}^2 \theta -$$

$$\cos^2 \theta) \, d\theta =$$

$$\int_0^{\frac{\pi}{6}} \left( \frac{\cos^2 \theta}{\sin^2 \theta} - \cos^2 \theta \right) d\theta = \int_0^{\frac{\pi}{6}} (\cot^2 \theta - \cos^2 \theta) \, d\theta$$

$$\begin{aligned}
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\operatorname{cosec}^2 \theta - 1 - \frac{1+\cos 2\theta}{2}) d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\frac{2\operatorname{cosec}^2 \theta - 2 - 1 - \cos 2\theta}{2}) d\theta \\
&= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (2\operatorname{cosec}^2 \theta - 3 - \cos 2\theta) d\theta \\
&= \frac{1}{2} \left[ -2 \cot \theta - 3\theta - \frac{\sin 2\theta}{2} \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} \\
&= \frac{1}{2} \left[ \left( -2 \cot \frac{\pi}{4} - 3 \frac{\pi}{4} - \frac{\sin 2(\frac{\pi}{4})}{2} \right) - \left( -2 \cot \frac{\pi}{6} - 3 \frac{\pi}{6} - \frac{\sin 2(\frac{\pi}{6})}{2} \right) \right] \\
&= \frac{1}{2} \left[ \left( -2(1) - \frac{3\pi}{4} - \frac{1}{2} \sin \frac{\pi}{2} \right) - \left( -2\sqrt{3} - \frac{\pi}{2} - \frac{1}{2} \sin \frac{\pi}{3} \right) \right] \\
&= \frac{1}{2} \left[ \left( -2 - \frac{3\pi}{4} - \frac{1}{2} \cdot 1 \right) - \left( -2\sqrt{3} - \frac{\pi}{2} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \right) \right] \\
&= \frac{1}{2} \left[ -2 - \frac{3\pi}{4} - \frac{1}{2} + 2\sqrt{3} + \frac{\pi}{2} + \frac{\sqrt{3}}{4} \right] \\
&= \frac{1}{2} \left[ -2 - \frac{1}{2} + 2\sqrt{3} + \frac{\sqrt{3}}{4} - \frac{3\pi}{4} + \frac{\pi}{2} \right] \\
&= \frac{1}{2} \left[ -2 - \frac{1}{2} + 2\sqrt{3} + \frac{\sqrt{3}}{4} - \frac{3\pi}{4} + \frac{\pi}{2} \right] \\
&= \frac{1}{2} \left[ \frac{-8-2+8\sqrt{3}+\sqrt{3}-3\pi+2\pi}{4} \right] \\
&= \frac{1}{2} \left[ \frac{-10+9\sqrt{3}-\pi}{4} \right] \\
&= \frac{-10+9\sqrt{3}-\pi}{8}
\end{aligned}$$

**Q. 18:**  $\int_0^{\frac{\pi}{4}} \cos^4 t \, dt$

**SOLUTION:**

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \cos^4 t \, dt &= \int_0^{\frac{\pi}{4}} (\cos^2 t)^2 \, dt = \int_0^{\frac{\pi}{4}} \left( \frac{1+\cos 2t}{2} \right)^2 \, dt \\
&= \int_0^{\frac{\pi}{4}} \frac{1+\cos^2 2t+2\cos 2t}{4} \, dt \\
&= \frac{1}{4} \int_0^{\frac{\pi}{4}} \left( 1 + \frac{1+\cos 4t}{2} + 2\cos 2t \right) \, dt \\
&= \frac{1}{4} \int_0^{\frac{\pi}{4}} \left( \frac{2+1+\cos 4t+4\cos 2t}{2} \right) \, dt = \frac{1}{8} \int_0^{\frac{\pi}{4}} (3 + \cos 4t + 4\cos 2t) \, dt \\
&= \frac{1}{8} \left[ 3t + \frac{\sin 4t}{4} + 4 \frac{\sin 2t}{2} \right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{8} \left[ \left( 3 \cdot \frac{\pi}{4} + \frac{\sin 4(\frac{\pi}{4})}{4} + 4 \frac{\sin 2(\frac{\pi}{4})}{2} \right) - \left( 3(0) + \frac{\sin 4(0)}{4} + 4 \frac{\sin 2(0)}{2} \right) \right] \\
&= \frac{1}{8} \left[ \left( \frac{3\pi}{4} + \frac{\sin \pi}{4} + 2 \sin \left( \frac{\pi}{2} \right) \right) - \left( 0 + \frac{\sin 0}{4} + 4 \frac{\sin 0}{2} \right) \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8} \left[ \left( \frac{3\pi}{4} + \frac{0}{4} + 2 \cdot 1 \right) - \left( 0 + \frac{0}{4} + 4 \cdot \frac{0}{2} \right) \right] = \frac{1}{8} \left[ \left( \frac{3\pi}{4} + 0 + 2 \right) - 0 \right] \\
&= \frac{1}{8} \left[ \left( \frac{3\pi}{4} + 2 \right) \right] = \frac{1}{8} \left[ \left( \frac{3\pi+8}{4} \right) \right] = \frac{3\pi+8}{32}
\end{aligned}$$

**Q. 19:**  $\int_0^{\frac{\pi}{3}} \cos^2 \theta \sin \theta \, d\theta$

**SOLUTION:**

$$\begin{aligned}
\int_0^{\frac{\pi}{3}} \cos^2 \theta \sin \theta \, d\theta &= - \int_0^{\frac{\pi}{3}} \cos^2 \theta (-\sin \theta) \, d\theta = \\
&= - \left[ \frac{\cos^3 \theta}{3} \right]_0^{\frac{\pi}{3}} = - \frac{1}{3} \left[ \left( \cos^3 \frac{\pi}{3} \right) - \left( \cos^3 0 \right) \right] = - \frac{1}{3} \left[ \cos^3 \frac{\pi}{3} - \cos^3 0 \right] \\
&= - \frac{1}{3} \left[ \left( \frac{1}{2} \right)^3 - (1)^3 \right] = - \frac{1}{3} \left[ \frac{1}{8} - 1 \right] = - \frac{1}{3} \left[ \frac{1-8}{8} \right] = \\
&= - \frac{1}{3} \left[ \frac{-7}{8} \right] = \frac{7}{24}
\end{aligned}$$

**Q. 20:**  $\int_0^{\frac{\pi}{4}} (1 + \cos^2 \theta) \tan^2 \theta \, d\theta$

**SOLUTION:**

$$\begin{aligned}
&= \int_0^{\frac{\pi}{4}} (\tan^2 \theta + \tan^2 \theta \cos^2 \theta) \, d\theta \\
&= \int_0^{\frac{\pi}{4}} (\tan^2 \theta + \tan^2 \theta \cos^2 \theta) \, d\theta = \\
&= \int_0^{\frac{\pi}{4}} \left( \tan^2 \theta + \frac{\sin^2 \theta}{\cos^2 \theta} \cos^2 \theta \right) \, d\theta \\
&= \int_0^{\frac{\pi}{4}} (\tan^2 \theta + \sin^2 \theta) \, dt = \int_0^{\frac{\pi}{4}} \left[ \sec^2 \theta - 1 + \frac{1-\cos 2\theta}{2} \right] \, d\theta \\
&= \int_0^{\frac{\pi}{4}} \left[ \sec^2 \theta - 1 + \frac{1}{2} - \frac{\cos 2\theta}{2} \right] \, d\theta = \int_0^{\frac{\pi}{4}} \left[ \sec^2 \theta - \frac{1}{2} - \frac{1}{2} \cos 2\theta \right] \, d\theta \\
&= \left[ \tan \theta - \frac{1}{2} \theta - \frac{1}{2} \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{4}} = \left[ \tan \theta - \frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right]_0^{\frac{\pi}{4}} \\
&= \left[ \left( \tan \frac{\pi}{4} - \frac{1}{2} \frac{\pi}{4} - \frac{1}{4} \sin 2 \left( \frac{\pi}{4} \right) \right) - \left( \tan 0 - \frac{1}{2} 0 - \frac{1}{4} \sin 2(0) \right) \right] \\
&= \left[ \left( 1 - \frac{\pi}{8} - \frac{1}{4} \sin \frac{\pi}{2} \right) - \left( 0 - 0 - \frac{1}{4} \sin(0) \right) \right] \\
&= \left[ \left( 1 - \frac{\pi}{8} - \frac{1}{4} \sin \frac{\pi}{2} \right) - (0) \right] = \left[ \left( 1 - \frac{\pi}{8} - \frac{1}{4} (1) \right) - (0) \right]
\end{aligned}$$

**Q. 21:**  $\int_0^{\frac{\pi}{4}} \frac{\sec \theta}{\sin \theta + \cos \theta} \, d\theta$

**SOLUTION:**

Divide up and down by  $\cos \theta$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec \theta}{\sin \theta + \cos \theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{\frac{\sec \theta}{\cos \theta}}{\frac{\sin \theta + \cos \theta}{\cos \theta}} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec \theta \sec \theta}{\tan \theta + 1} d\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta}{\tan \theta + 1} d\theta$$

Here  $f(x) = \tan \theta + 1 \Rightarrow f'(x) = \sec^2 \theta$

$$= [\ln|\tan \theta + 1|]_0^{\frac{\pi}{4}}$$

$$= \left[ (\ln|\tan \frac{\pi}{4} + 1|) - (\ln|\tan 0 + 1|) \right]$$

$$= [(\ln|1 + 1|) - (\ln|0 + 1|)]$$

$$= \ln|2| - \ln|1| = \ln 2 - 0 = \ln 2$$

**Q.22:**  $\int_{-1}^5 |x - 3| dx$

**SOLUTION:**

$$\int_{-1}^5 |x - 3| dx$$

$$= \int_{-1}^3 |x - 3| dx + \int_3^5 |x - 3| dx$$

$$= \int_{-1}^3 -(x - 3) dx + \int_3^5 (x - 3) dx$$

$$= -\int_{-1}^3 (x - 3) \cdot 1 dx + \int_3^5 (x - 3) \cdot 1 dx$$

$$= -\left| \frac{(x-3)^2}{2} \right|_{-1}^3 + \left| \frac{(x-3)^2}{2} \right|_3^5$$

$$= -\frac{1}{2} [((3-3)^2) - ((-1-3)^2)] + \frac{1}{2} [((5-3)^2) - ((3-3)^2)]$$

$$= -\frac{1}{2} [0 - 16] + \frac{1}{2} [4 - 0]$$

$$= -\frac{1}{2} [0 - 16] + \frac{1}{2} [4 - 0] = 8 + 2 = 10$$

**Q.23:**  $\int_{\frac{1}{8}}^1 \frac{(x^{\frac{1}{3}} + 2)^2}{x^{\frac{2}{3}}} dx$

$$\int_{\frac{1}{8}}^1 \frac{(x^{\frac{1}{3}} + 2)^2}{x^{\frac{2}{3}}} dx = 3 \int_{\frac{1}{8}}^1 (x^{\frac{1}{3}} + 2)^2 \cdot \frac{1}{3} x^{-\frac{2}{3}} dx =$$

$$3 \left| \frac{(x^{\frac{1}{3}} + 2)^{2+1}}{2+1} \right|_{\frac{1}{8}}^1 = \frac{3}{3} \left| (x^{\frac{1}{3}} + 2)^3 \right|_{\frac{1}{8}}^1 = \left( \left( (1)^{\frac{1}{3}} + 2 \right)^3 \right) -$$

$$\left( \left( \left( \frac{1}{8} \right)^{\frac{1}{3}} + 2 \right)^3 \right)$$

$$= ((1 + 2)^3) - \left( \left( (2^{-3})^{\frac{1}{3}} + 2 \right)^3 \right) = (3)^3 - \left( \frac{1}{2} + 2 \right)^3 =$$

$$27 - \left( \frac{1+4}{2} \right)^3 = 27 - \frac{125}{8} = \frac{216-125}{8} = \frac{91}{8}$$

$$\int_1^3 \frac{x^2-2}{x+1} dx \quad (\text{Improper fraction}) \quad \frac{x-1}{\sqrt{x^2-2}}$$

$$= \int_1^3 \left( Q + \frac{R}{D} \right) dx \quad \frac{\pm x^2 \pm x}{-x-2} \quad \frac{\pm x \pm 1}{-1}$$

$$= \int_1^3 \left( x - 1 - \frac{1}{1+x} \right) dx$$

$$= \left| \frac{x^2}{2} - x - \ln|x+1| \right|_1^3$$

$$= \left( \frac{3^2}{2} - 3 - \ln|3+1| \right) - \left( \frac{1^2}{2} - 1 - \ln|1+1| \right)$$

$$= \left( \frac{9}{2} - 3 - \ln 4 \right) - \left( \frac{1}{2} - 1 - \ln 2 \right)$$

$$= \frac{9}{2} - 3 - \ln 4 - \frac{1}{2} + 1 + \ln 2$$

$$= \frac{9}{2} - 2 - \frac{1}{2} - \ln 4 + \ln 2$$

$$= \frac{9-4-1}{2} - \ln 2^2 + \ln 2$$

$$= 2 - 2 \ln 2 + \ln 2 = 2 - \ln 2$$

**Q.25:**  $\int_2^3 \frac{3x^2-2x+1}{(x-1)(x^2+1)} dx$

**SOLUTION:**

$$\int_2^3 \frac{3x^2-2x+1}{(x-1)(x^2+1)} dx = \int_2^3 \frac{3x^2-2x+1}{x^3-x^2+x-1} dx = |\ln|x^3 - x^2 + x - 1||_2^3$$

$$= (\ln|3^3 - 3^2 + 3 - 1|) - (\ln|2^3 - 2^2 + 2 - 1|)$$

$$= (\ln|27 - 9 + 3 - 1|) - (\ln|8 - 4 + 2 - 1|)$$

$$= \ln 20 - \ln 5 = \ln \frac{20}{5} = \ln 4$$

**Q.26:**  $\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x} dx$

**SOLUTION:**

$$= \int_0^{\frac{\pi}{4}} \left( \frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x} \right) dx$$

$$= \int_0^{\frac{\pi}{4}} \left( \frac{\sin x}{\cos x \cos x} - \frac{1}{\cos^2 x} \right) dx$$

$$= \int_0^{\frac{\pi}{4}} (\sec x \tan x - \sec^2 x) dx$$

$$= |\sec x - \tan x|_0^{\frac{\pi}{4}}$$

$$= \left( \sec \frac{\pi}{4} - \tan \frac{\pi}{4} \right) - (\sec 0 - \tan 0)$$

$$= (\sqrt{2} - 1) - (1 + 0)$$

$$= \sqrt{2} - 1 - 1 = \sqrt{2} - 2$$

**Q.27:**  $\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x} dx$

**SOLUTION:**

$$= \int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx = \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{1-\sin^2 x} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{\cos^2 x} dx = -\int_0^{\frac{\pi}{4}} \frac{\sin x - 1}{\cos^2 x} dx$$

$$= -\int_0^{\frac{\pi}{4}} \left( \frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x} \right) dx$$

$$\begin{aligned}
&= -\int_0^{\frac{\pi}{4}} \left( \frac{\sin x}{\cos x \cos x} - \frac{1}{\cos^2 x} \right) dx \\
&= -\int_0^{\frac{\pi}{4}} (\sec x \tan x - \sec^2 x) dx \\
&= -|\sec x - \tan x|_0^{\frac{\pi}{4}} \\
&= -\left[ \left( \sec \frac{\pi}{4} - \tan \frac{\pi}{4} \right) - (\sec 0 - \tan 0) \right] \\
&= -[(\sqrt{2} - 1) - (1 + 0)] \\
&= -\sqrt{2} + 1 + 1 = 2 - \sqrt{2}
\end{aligned}$$

**Q. 28:**  $\int_0^1 \frac{3x}{\sqrt{4-3x}} dx$

**SOLUTION:**

$$\begin{aligned}
\int_0^1 \frac{3x}{\sqrt{4-3x}} dx &= -\int_0^1 \frac{-3x}{\sqrt{4-3x}} dx = -\int_0^1 \frac{4-3x-4}{\sqrt{4-3x}} dx = \\
&= -\int_0^1 \frac{4-3x}{\sqrt{4-3x}} dx - \int_0^1 \frac{-4}{\sqrt{4-3x}} dx = -\int_0^1 \sqrt{4-3x} dx + \\
&= 4 \int_0^1 \frac{1}{\sqrt{4-3x}} dx \\
&= -\frac{1}{-3} \int_0^1 (4-3x)^{\frac{1}{2}} (-3) dx + \frac{4}{-3} \int_0^1 (4- \\
&= \frac{1}{3} \left| \frac{(4-3x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right|_0^1 - \frac{4}{3} \left| \frac{(4-3x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right|_0^1 = \frac{1}{3} \left| \frac{(4-3x)^{\frac{3}{2}}}{\frac{3}{2}} \right|_0^1 - \\
&= \frac{4}{3} \left| \frac{(4-3x)^{\frac{1}{2}}}{\frac{1}{2}} \right|_0^1 = \frac{1}{3} \cdot \frac{2}{3} \left| (4-3x)^{\frac{3}{2}} \right|_0^1 - \frac{4}{3} \cdot \frac{2}{1} \left| (4-3x)^{\frac{1}{2}} \right|_0^1 \\
&= \frac{2}{9} \left| (4-3x)^{\frac{3}{2}} \right|_0^1 - \frac{8}{3} \left| (4-3x)^{\frac{1}{2}} \right|_0^1 = \frac{2}{9} \left[ \left( (4-3(1))^{\frac{3}{2}} \right) - \left( (4-3(0))^{\frac{3}{2}} \right) \right] - \frac{8}{3} \left[ \left( (4-3(1))^{\frac{1}{2}} \right) - \left( (4-3(0))^{\frac{1}{2}} \right) \right] \\
&= \frac{2}{9} \left[ (1)^{\frac{3}{2}} - (4)^{\frac{3}{2}} \right] - \frac{8}{3} \left[ (1)^{\frac{1}{2}} - (4)^{\frac{1}{2}} \right] = \frac{2}{9} [1 - (2^2)^{\frac{3}{2}}] - \frac{8}{3} \left[ (1)^{\frac{1}{2}} - (2^2)^{\frac{1}{2}} \right] = \frac{2}{9} [1 - 2^3] - \frac{8}{3} [1 - 2^1] \\
&= \frac{2}{9} [1 - 8] - \frac{8}{3} [1 - 2] = \frac{2}{9} [-7] - \frac{8}{3} [-1] = \frac{-14}{9} + \frac{8}{3} = \frac{-14+24}{9} = \frac{10}{9}
\end{aligned}$$

**Q. 29:**  $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x(2+\sin x)} dx$

**SOLUTION:**

$$\begin{aligned}
&= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{2}{\sin x(2+\sin x)} \cdot \cos x dx = \\
&= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{(2+\sin x) - \sin x}{\sin x(2+\sin x)} \cdot \cos x dx \\
&= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[ \frac{(2+\sin x)}{\sin x(2+\sin x)} - \frac{\sin x}{\sin x(2+\sin x)} \right] \cos x dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[ \frac{1}{\sin x} - \frac{1}{2+\sin x} \right] \cos x dx = \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[ \frac{\cos x}{\sin x} - \frac{\cos x}{2+\sin x} \right] dx \\
&= \frac{1}{2} \left[ \ln|\sin x| - \ln|2+\sin x| \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&= \frac{1}{2} \left[ \left( \ln \left| \sin \frac{\pi}{2} \right| - \ln \left| 2 + \sin \frac{\pi}{2} \right| \right) - \left( \ln \left| \sin \frac{\pi}{6} \right| - \ln \left| 2 + \sin \frac{\pi}{6} \right| \right) \right] \\
&= \frac{1}{2} \left[ (\ln(1) - \ln|2+1|) - \left( \ln \frac{1}{2} - \ln \left| 2 + \frac{1}{2} \right| \right) \right] \\
&= \frac{1}{2} \left[ 0 - \ln 3 - \ln \frac{1}{2} + \ln \frac{5}{2} \right] = \frac{1}{2} [-\ln 3 - (\ln 1 - \ln 2) + (\ln 5 - \ln 2)] \\
&= \frac{1}{2} [-\ln 3 - \ln 1 + \ln 2 + \ln 5 - \ln 2] = \frac{1}{2} [-\ln 3 - 0 + \ln 5] \\
&= \frac{1}{2} [\ln 5 - \ln 3] = \frac{1}{2} \left[ \ln \frac{5}{3} \right] = \frac{1}{2} \ln \frac{5}{3}
\end{aligned}$$

**Q. 30:**  $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)(2+\cos x)} dx$

**SOLUTION:**

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)(2+\cos x)} dx &= \int_0^{\frac{\pi}{2}} \frac{1}{(1+\cos x)(2+\cos x)} \sin x dx \\
&= \int_0^{\frac{\pi}{2}} \frac{(2+\cos x) - (1+\cos x)}{(1+\cos x)(2+\cos x)} \sin x dx \\
&= \int_0^{\frac{\pi}{2}} \left[ \frac{(2+\cos x)}{(1+\cos x)(2+\cos x)} - \frac{(1+\cos x)}{(1+\cos x)(2+\cos x)} \right] \sin x dx \\
&= \int_0^{\frac{\pi}{2}} \left[ \frac{1}{(1+\cos x)} - \frac{1}{(2+\cos x)} \right] \sin x dx \\
&= \int_0^{\frac{\pi}{2}} \left[ \frac{\sin x}{(1+\cos x)} - \frac{\sin x}{(2+\cos x)} \right] dx \\
&= \frac{-1}{-1} \int_0^{\frac{\pi}{2}} \left[ \frac{\sin x}{(1+\cos x)} - \frac{\sin x}{(2+\cos x)} \right] dx \\
&= \frac{1}{-1} \int_0^{\frac{\pi}{2}} \left[ \frac{-\sin x}{(1+\cos x)} - \frac{-\sin x}{(2+\cos x)} \right] dx \\
&= -[\ln|1+\cos x| - \ln|2+\cos x|]_0^{\frac{\pi}{2}} \\
&= -\left[ \left( \ln \left| 1 + \cos \frac{\pi}{2} \right| - \ln \left| 2 + \cos \frac{\pi}{2} \right| \right) - \left( \ln|1+\cos 0| - \ln|2+\cos 0| \right) \right] \\
&= -[(\ln|1+0| - \ln|2+0|) - (\ln|1+1| - \ln|2+1|)] \\
&= -[\ln(1) - \ln(2) - \ln(2) + \ln(3)] \\
&= -[0 - 2\ln(2) + \ln 3] = 2\ln(2) - \ln(3) \\
&= \ln(2)^2 - \ln(3) = \ln 4 - \ln(3) = \ln \frac{4}{3}
\end{aligned}$$

$$\therefore \begin{pmatrix} a \ln b = \ln b^a \\ \because \ln a - \ln b \\ = \ln \frac{a}{b} \end{pmatrix}$$

**Application of definite integral:  
Area under the curve:**

**Case I.** if  $f(x) \geq 0 \forall x \in [a, b]$  then curve lies above  $x - axis$ .

$$A = \int_a^b f(x) dx \text{ where } a < b$$

$A$  is area of region above  $x - axis$ , under the curve of function  $y = f(x)$  from  $a$  to  $b$

**Case II** if  $f(x) \leq 0 \forall x \in [a, b]$  then curve lies below

$$x - axis. \text{ so } A = - \int_a^b f(x) dx \text{ where } a < b$$

$A$  is area of region below  $x - axis$ , under the curve of function  $y = f(x)$  from  $a$  to  $b$

## Exercise 3.7

**Q. 1:** Find the area between the  $x - axis$  and the curve  $y = x^2 + 1$  from  $x = 1$  to  $x = 2$

**SOLUTION:**

Given  $y = x^2 + 1$  As  $y = x^2 + 1 > 0$  in  $[1, 2]$ , therefore curve is above  $x - axis$

$$\begin{aligned} \text{Required Area} &= \int_1^2 y \, dx = \int_1^2 (x^2 + 1) \, dx \\ &= \left[ \frac{x^3}{3} + x \right]_1^2 = \left( \frac{8}{3} + 2 \right) - \left( \frac{1}{3} + 1 \right) \\ &= \left( \frac{2^3}{3} + 2 \right) - \left( \frac{1^3}{3} + 1 \right) \\ &= \left( \frac{8+6}{3} \right) - \left( \frac{1+3}{3} \right) = \frac{14}{3} - \frac{4}{3} \\ &= \frac{14-4}{3} = \frac{10}{3} \text{ square unit} \end{aligned}$$

**Q. 2:** Find the area above the  $x - axis$  and under the curve  $y = 5 - x^2$  from  $x = -1$  to  $x = 2$

**SOLUTION:**

Given  $y = 5 - x^2$  As  $y = 5 - x^2 > 0$  in  $[-1, 2]$ , therefore curve is above  $x - axis$

$$\begin{aligned} \text{Required Area} &= \int_{-1}^2 y \, dx = \int_{-1}^2 (5 - x^2) \, dx \\ &= \left[ 5x - \frac{x^3}{3} \right]_{-1}^2 \\ &= \left( 5 \cdot 2 - \frac{2^3}{3} \right) - \left( 5(-1) - \frac{(-1)^3}{3} \right) \\ &= \left( 10 - \frac{8}{3} \right) - \left( -5 - \frac{-1}{3} \right) \\ &= \left( \frac{30-8}{3} \right) - \left( \frac{-15+1}{3} \right) = \frac{22}{3} - \frac{-14}{3} \\ &= \frac{22+14}{3} = \frac{36}{3} = 12 \text{ sq. unit} \end{aligned}$$

**Q. 3:** Find the area below the curve  $y = 3\sqrt{x}$  and above the  $x - axis$  between  $x = 1$  to  $x = 4$

**SOLUTION:**

Given  $y = 3\sqrt{x}$  As  $y = 3\sqrt{x} > 0$

in  $[1, 4]$ , therefore curve is above  $x - axis$

$$\begin{aligned} \text{Required Area} &= \int_1^4 y \, dx = 3 \int_1^4 \sqrt{x} \, dx \\ &= 3 \int_1^4 (x)^{\frac{1}{2}} \cdot 1 \, dx = 3 \left[ \frac{(x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right]_1^4 \\ &= 3 \left[ \frac{(x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^4 = 3 \cdot \frac{2}{3} \left[ (x)^{\frac{3}{2}} \right]_1^4 \\ &= 2 \left( (4)^{\frac{3}{2}} - (1)^{\frac{3}{2}} \right) = 2 \left( (2^2)^{\frac{3}{2}} - 1 \right) \\ &= 2(2^3 - 1) = 2(8 - 1) = 2(7) \\ &= 14 \text{ sq. unit} \end{aligned}$$

**Q. 4:** Find the area bounded by cos function from

$$x = -\frac{\pi}{2} \text{ to } x = \frac{\pi}{2}$$

**SOLUTION:**

Given  $y = \cos x$  As  $y = \cos x \geq 0$  in  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ , therefore curve is above  $x - axis$

$$\begin{aligned} \text{Required Area} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx = |\sin x|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) = 1 - (-1) \\ &= 1 + 1 = 2 \text{ sq. unit} \end{aligned}$$

**Q. 5:** Find the area between the  $x - axis$  and the curve  $y = 4x - x^2$ .

**SOLUTION:**

Put  $y = 0$   $4x - x^2 = 0$   
 $x(4 - x) = 0$   
 $x = 0$  or  $4 - x = 0$   
 $x = 4$

As  $y = 4x - x^2 \geq 0$  in  $[0, 4]$ , therefore curve is above  $x - axis$

$$\begin{aligned} \text{Required Area} &= \int_0^4 y \, dx = \int_0^4 (4x - x^2) \, dx \\ &= \left[ 4 \frac{x^2}{2} - \frac{x^3}{3} \right]_0^4 \\ &= \left( 2 \cdot 4^2 - \frac{4^3}{3} \right) - \left( 0^2 - \frac{0^3}{3} \right) \\ &= \left( 32 - \frac{64}{3} \right) - (0 - 0) = \frac{96 - 64}{3} \\ &= \frac{32}{3} \text{ sq. unit} \end{aligned}$$

**Q. 6:** Determine the area bounded by the parabola  $y = x^2 + 2x - 3$  and the  $x$ -axis.

**SOLUTION:**

Given  $y = x^2 + 2x - 3$

As  $y = x^2 + 2x - 3 \geq 0$

0 in  $[-3, 1]$ , therefore curve is below  $x$ -axis

$$\text{Required Area} = - \int_{-3}^1 y \, dx = - \int_{-3}^1 (x^2 + 2x - 3) \, dx$$

$$= - \left[ \frac{x^3}{3} + 2 \frac{x^2}{2} - 3x \right]_{-3}^1$$

$$= - \left[ \frac{x^3}{3} + x^2 - 3x \right]_{-3}^1$$

$$= - \left[ \left( \frac{1^3}{3} + 1^2 - 3 \cdot 1 \right) - \left( \frac{(-3)^3}{3} + (-3)^2 - 3(-3) \right) \right]$$

$$= - \left[ \left( \frac{1}{3} + 1 - 3 \right) - \left( \frac{-27}{3} + 9 + 9 \right) \right]$$

$$= - \left[ \frac{1}{3} + 1 - 3 + \frac{27}{3} - 9 - 9 \right]$$

$$= - \left[ \frac{1}{3} + \frac{27}{3} - 20 \right]$$

$$= - \left[ \frac{1+27-60}{3} \right]$$

$$= - \left[ \frac{-32}{3} \right] = \frac{32}{3} \text{ sq. unit}$$

Put $y = 0$ $x^2 + 2x - 3 = 0$ $x^2 + 3x - x - 3 = 0$ $x(x+3) - 1(x+3) = 0$ $(x+3) - (x-1) = 0$ $x+3 = 0$ or $x-1 = 0$ $x = -3$ , $x = 0$
--

**Q. 7:** Find the area bounded by the curve  $y = x^3 + 1$ , the  $x$ -axis and line  $x = 2$

**SOLUTION:**

Given  $y = x^3 + 1$  Put  $y = 0$   $x^3 + 1 = 0$

$$(x+1)(x^2 -$$

$$x+1) = 0$$

$$x+1 =$$

$$0 \text{ or } x^2 - x + 1 = 0 \text{ (Solve itself)}$$

$$x = -1$$

After solving this equation gives imaginary roots so neglect.

As  $y = x^3 + 1 \geq 0$

0 in  $[-1, 2]$ , therefore curve is above  $x$ -axis

$$\text{Required Area} = \int_{-1}^2 y \, dx = \int_{-1}^2 (x^3 + 1) \, dx$$

$$= \left[ \frac{x^4}{4} + x \right]_{-1}^2$$

$$= \left( \frac{2^4}{4} + 2 \right) - \left( \frac{(-1)^4}{4} + (-1) \right)$$

$$= (4 + 2) - \left( \frac{1}{4} - 1 \right) = (6) - \left( \frac{1-4}{4} \right)$$

$$= 6 - \frac{-3}{4} = 6 + \frac{3}{4} = \frac{24+3}{4}$$

$$= \frac{27}{4} \text{ square unit.}$$

**Q. 8:** Find the area bounded by the curve  $y = x^3 - 4x$ , and the  $x$ -axis.

**SOLUTION:**

Put $y = 0$ $x^3 - 4x = 0$ $x(x^2 - 4) = 0$ $x(x-2)(x+2) = 0$ $x = 0$ or $x-2 = 0$ , or $x+2 = 0$ $x = 0$ or $x = 2$ , or $x = -2$
--

Given  $y = x^3 - 4x$

As  $y = x^3 - 4x \geq 0$

0 in  $[-2, 0]$ , therefore the curve is above  $x$ -axis

As  $y = x^3 - 4x \leq 0$

0 in  $[0, 2]$ , therefore the curve is below  $x$ -axis

$$\text{Required Area} = \int_{-2}^0 y \, dx - \int_0^2 y \, dx$$

$$= \int_{-2}^0 (x^3 - 4x) \, dx - \int_0^2 (x^3 - 4x) \, dx$$

$$= \left[ \frac{x^4}{4} - 4 \frac{x^2}{2} \right]_{-2}^0 - \left[ \frac{x^4}{4} - 4 \frac{x^2}{2} \right]_0^2$$

$$= \left[ \left( \frac{0^4}{4} - 2(0)^2 \right) - \left( \frac{(-2)^4}{4} - 2(-2)^2 \right) \right]$$

$$- \left[ \left( \frac{(2)^4}{4} - 2(2)^2 \right) - \left( \frac{(0)^4}{4} - 2(0)^2 \right) \right]$$

$$= [(0 - 0) - (4 - 8)]$$

$$- [(4 - 8) - (0 - 0)] = 0 + 4 + 4 + 0$$

$$= 8 \text{ square unit.}$$

**Q. 9:** Find the area between the curve  $y = x(x-1)(x+1)$ , and the  $x$ -axis.

**SOLUTION:**

Given  $y = x(x-1)(x+1) = x^3 - x$

As  $y = x^3 - x \geq 0$

0 in  $[-1, 0]$ , therefore the curve is above  $x$ -axis

As  $y = x^3 - x \leq 0$

0 in  $[0, 1]$ , therefore the curve is below  $x$ -axis

$$\text{Required Area} = \int_{-1}^0 y \, dx - \int_0^1 y \, dx$$

$$= \int_{-1}^0 (x^3 - x) \, dx - \int_0^1 (x^3 - x) \, dx$$

$$= \left[ \frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^0 - \left[ \frac{x^4}{4} - \frac{x^2}{2} \right]_0^1$$

$$= \left[ \left( \frac{0^4}{4} - \frac{(0)^2}{2} \right) - \left( \frac{(-1)^4}{4} - \frac{(-1)^2}{2} \right) \right]$$

$$- \left[ \left( \frac{(1)^4}{4} - \frac{(1)^2}{2} \right) - \left( \frac{(0)^4}{4} - \frac{(0)^2}{2} \right) \right]$$

$$= \left[ (0 - 0) - \left( \frac{1}{4} - \frac{1}{2} \right) \right]$$

$$- \left[ \left( \frac{1}{4} - \frac{1}{2} \right) - (0 - 0) \right]$$

$$= 0 - \frac{1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2} + 0$$

$$= \frac{-1 + 2 - 1 + 2}{4} = \frac{1}{2}$$

**Q. 10:** Find the area above the  $x$ -axis, bounded by the curve  $y^2 = 3 - x$  from  $x = -1$  to  $x = 2$

**SOLUTION:**

Given  $y^2 = 3 - x \Rightarrow y = \sqrt{3 - x}$  As  $y =$

$\sqrt{3 - x} \geq 0$  in  $[-1, 2]$ , therefore curve is above  $x$ -axis

$$\begin{aligned}
 \text{Required Area} &= \int_{-1}^2 y \, dx = \int_{-1}^2 \sqrt{3-x} \, dx \\
 &= -\int_{-1}^2 (3-x)^{\frac{1}{2}} (-1) \, dx \\
 &= -\left[ \frac{(3-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right]_{-1}^2 = -\frac{2}{3} \left[ (3-x)^{\frac{3}{2}} \right]_{-1}^2 \\
 &= -\frac{2}{3} \left[ \left( (3-2)^{\frac{3}{2}} \right) - \left( (3-(-1))^{\frac{3}{2}} \right) \right] \\
 &= -\frac{2}{3} \left[ 1 - \left( (4)^{\frac{3}{2}} \right) \right] = -\frac{2}{3} [1 - 8] \\
 &= \frac{14}{3} \text{ sq. unit}
 \end{aligned}$$

**Q. 11: Find the area between the x-axis and the curve  $y = \cos \frac{1}{2}x$  from  $x = -\pi$  to  $\pi$ .**

**SOLUTION:**

Given  $y = \cos \frac{1}{2}x$  As  $y = \cos \frac{1}{2}x \geq 0$  in  $[-\pi, \pi]$ , therefore curve is above x-axis

$$\begin{aligned}
 \text{Required Area} &= \int_{-\pi}^{\pi} y \, dx = \int_{-\pi}^{\pi} \cos \frac{1}{2}x \, dx \\
 &= \left[ \frac{\sin \frac{1}{2}x}{\frac{1}{2}} \right]_{-\pi}^{\pi} = 2 \left[ \left( \sin \frac{1}{2}(\pi) \right) - \left( \sin \frac{1}{2}(-\pi) \right) \right] \\
 &= 2[1 - (-1)] = 4
 \end{aligned}$$

**Q. 12: Find the area between the x-axis and the curve  $y = \sin 2x$  from  $x = 0$  to  $\frac{\pi}{3}$ .**

**SOLUTION:**

Given  $y = \sin 2x$  As  $y = \sin 2x \geq 0$  in  $[0, \frac{\pi}{3}]$ , therefore curve is above x-axis

$$\begin{aligned}
 \text{Required Area} &= \int_0^{\frac{\pi}{3}} y \, dx = \int_0^{\frac{\pi}{3}} \sin 2x \, dx \\
 &= \left[ \frac{-\cos 2x}{2} \right]_0^{\frac{\pi}{3}} = -\frac{1}{2} \left[ \left( \cos \frac{2\pi}{3} \right) - \left( \cos 2(0) \right) \right] \\
 &= -\frac{1}{2} \left[ -\frac{1}{2} - 1 \right] = \frac{3}{4} \text{ sq. unit}
 \end{aligned}$$

**Q. 13: Find the area between the x-axis and the curve  $y = \sqrt{2ax - x^2}$  when  $a > 0$ .**

**SOLUTION:**

Given  $y = \sin 2x$  As  $y = \sin 2x \geq 0$  in  $[0, \frac{\pi}{3}]$ , therefore curve is above x-axis

$$\begin{aligned}
 \text{Required Area} &= \int_0^{2a} y \, dx = \int_0^{2a} \sqrt{2ax - x^2} \, dx = \\
 &= \int_0^{2a} \sqrt{a^2 - a^2 + 2ax - x^2} \, dx \\
 &= \int_0^{2a} \sqrt{a^2 - (a^2 - 2ax + x^2)} \, dx \\
 &= \int_0^{2a} \sqrt{a^2 - (x-a)^2} \, dx
 \end{aligned}$$

Using formula  $\int \sqrt{a^2 - x^2} \, dx$

$$= \frac{a^2}{2} \sin^{-1} \left( \frac{x}{a} \right) + \frac{x}{2} \sqrt{a^2 - x^2} + c$$

$$= \left[ \frac{a^2}{2} \sin^{-1} \left( \frac{x-a}{a} \right) + \left( \frac{x-a}{2} \right) \sqrt{a^2 - (x-a)^2} \right]_0^{2a}$$

$$\begin{aligned}
 &= \left( \frac{a^2}{2} \sin^{-1} \left( \frac{2a-a}{a} \right) + \left( \frac{2a-a}{2} \right) \sqrt{a^2 - (2a-a)^2} \right) \\
 &\quad - \left( \frac{a^2}{2} \sin^{-1} \left( \frac{0-a}{a} \right) \right. \\
 &\quad \left. + \left( \frac{0-a}{2} \right) \sqrt{a^2 - (0-a)^2} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \left( \frac{a^2}{2} \sin^{-1}(1) + \left( \frac{a}{2} \right) \sqrt{a^2 - a^2} \right) - \left( \frac{a^2}{2} \sin^{-1}(-1) + \right. \\
 &\quad \left. \left( \frac{-a}{2} \right) \sqrt{a^2 - a^2} \right) = \left( \frac{a^2}{2} \cdot \frac{\pi}{2} + 0 \right) - \left( \frac{a^2}{2} \left( -\frac{\pi}{2} \right) - 0 \right) = \frac{a^2\pi}{4} + \frac{a^2\pi}{4}
 \end{aligned}$$

$$= \frac{a^2\pi + a^2\pi}{4} = \frac{2a^2\pi}{4} = \frac{a^2\pi}{2} \text{ square unit.}$$

➤ **Differential equation:**

An equation containing at least one derivative of a dependent variable with respect to an independent variable is called differential equation. e.g.

$$y \frac{dy}{dx} + 2x = 0 \quad \text{and} \quad x \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2x = 0$$

➤ **Order of Differential equation:**

The order of the differential equation is the order of the highest derivative in the equation.

$$y \frac{dy}{dx} + 2x = 0 \quad (1\text{st order differential equation})$$

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2x = 0 \quad (2\text{nd order differential equation})$$

➤ **Degree of Differential equation:**

The degree of a differential equation is the greatest power of the highest order derivative in the equation.

$$x \frac{d^4y}{dx^4} + \frac{dy}{dx} - x \left( \frac{d^2y}{dx^2} \right)^4 + \frac{dy}{dx} + 2x = 0$$

(1st degree differential equation)

$$x \left( \frac{d^4y}{dx^4} \right)^3 + \frac{dy}{dx} - x \left( \frac{d^2y}{dx^2} \right)^4 + \frac{dy}{dx} + 2x = 0$$

(3rd degree differential equation)

➤ **General solution:**

The solution of differential equation which contains arbitrary constants is called general solution.

➤ **Particular solution:**

The solution obtained from general solution by applying the initial conditions is called particular solution.

➤ **Initial value conditions:**

The arbitrary constants involving in the solution of a differential equation can be determined by the some even conditions. such conditions are called Initial value conditions.

**TIP BIT:** General solution of the differential equation of order  $n$  contains  $n$  arbitrary constants which can be determined by  $n$  initial values conditions.

## Exercise 3.8

**Q. 1:** Check that each of the following equations written against the differential equation is its solution.

i)  $x \frac{dy}{dx} = 1 + y$

**Prove that  $y = cx - 1$**

**SOLUTION:**

$$x \frac{dy}{dx} = 1 + y$$

Separating the variables:

$$x dy = (1 + y) dx$$

$$\frac{1}{1+y} dy = \frac{1}{x} dx$$

Integrating both sides

$$\int \frac{1}{1+y} dy = \int \frac{1}{x} dx$$

$$\ln|1 + y| = \ln|x| + \ln|c|$$

$$\ln|1 + y| = \ln|cx|$$

$$1 + y = cx$$

$$y = cx - 1$$

ii)  $x^2(2y + 1) \frac{dy}{dx} - 1 = 0$

**prove that  $y^2 + y = c - \frac{1}{x}$**

**SOLUTION:**

$$x^2(2y + 1) \frac{dy}{dx} - 1 = 0$$

Separating the variables:

$$x^2(2y + 1) \frac{dy}{dx} = 1$$

$$(2y + 1) dy = \frac{1}{x^2} dx$$

Integrating both sides

$$\int (2y + 1) dy = \int x^{-2} dx$$

$$2 \frac{y^2}{2} + y = \frac{x^{-2+1}}{-2+1}$$

$$y^2 + y = \frac{x^{-1}}{-1} + c$$

$$y^2 + y = -\frac{1}{x} + c$$

iii)  $y \frac{dy}{dx} - 1 = 0$

**prove that  $y^2 + y = c - \frac{1}{x}$**

**SOLUTION:**

$$x^2(2y + 1) \frac{dy}{dx} - 1 = 0$$

Separating the variables:

$$x^2(2y + 1) \frac{dy}{dx} = 1$$

$$(2y + 1) dy = \frac{1}{x^2} dx$$

Integrating both sides

$$\int (2y + 1) dy = \int x^{-2} dx$$

$$2 \frac{y^2}{2} + y = \frac{x^{-2+1}}{-2+1}$$

$$y^2 + y = \frac{x^{-1}}{-1} + c$$

$$y^2 + y = -\frac{1}{x} + c$$

iv)  $\frac{1}{x} \frac{dy}{dx} - 2y = 0$

**Prove that  $y = ce^{x^2}$**

**SOLUTION:**

$$\frac{1}{x} \frac{dy}{dx} = 2y$$

Separating the variables:

$$\frac{1}{y} dy = 2x dx$$

Integrating both sides

$$\int \frac{1}{y} dy = \int 2x dx$$

$$\ln|y| = 2 \frac{x^2}{2} + c_1$$

$$\ln|y| = x^2 + c_1$$

$$e^{\ln|y|} = e^{x^2+c_1}$$

$$y = e^{x^2} + e^{c_1}$$

$$y = e^{x^2} + c$$

v)  $\frac{dy}{dx} = \frac{y^2+1}{e^{-x}}$

**Prove  $y = \tan(e^x + c)$**

**SOLUTION:**

$$\frac{dy}{dx} = \frac{y^2+1}{e^{-x}}$$

Separating the variables:

$$\frac{1}{y^2+1} dy = \frac{1}{e^{-x}} dx$$

$$\frac{1}{1+y^2} dy = e^x dx$$

Integrating both sides

$$\int \frac{1}{1+y^2} dy = \int e^x dx$$

$$\tan^{-1} y = e^x + c$$

$$y = \tan(e^x + c)$$

Q. 2:  $\frac{dy}{dx} = -y$

**SOLUTION:**

$$\frac{dy}{dx} = -y$$

Separating the variables:

$$\frac{1}{y} dy = -dx$$

Integrating both sides

$$\int \frac{1}{y} dy = -\int 1 dx$$

$$\ln y = -x + c_1$$

$$e^{\ln y} = e^{-x+c_1}$$

$$y = e^{-x} e^{c_1}$$

$$y = c e^{-x}$$

Q. 3:  $y dx + x dy = 0$

**SOLUTION:**

$$y dx + x dy = 0$$

Separating the variables:

$$x dy = -y dx$$

$$\frac{1}{y} dy = -\frac{1}{x} dx$$

Integrating both sides

$$\int \frac{1}{y} dy = -\int \frac{1}{x} dx$$

$$\ln y = -\ln x + \ln c$$

$$\ln y + \ln x = +\ln c$$

$$\ln(xy) = +\ln c$$

$$xy = c$$

Q. 4:  $\frac{dy}{dx} = \frac{1-x}{y}$

**SOLUTION:**

$$\frac{dy}{dx} = \frac{1-x}{y}$$



Separating the variables:

$$y dy = (1 - x) dx$$

Integrating both sides

$$\int y dy = \int (1 - x) dx$$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + c_1$$

$$y^2 = 2x - x^2 + 2c_1$$

$$y^2 = x(2 - x) + c$$

**Q. 5:**  $\frac{dy}{dx} = \frac{y}{x^2}$

**SOLUTION:**

$$\frac{dy}{y} = \frac{1}{x^2} dx$$

Separating the variables:

$$\frac{1}{y} dy = \frac{1}{x^2} dx$$

Integrating both sides

$$\int \frac{1}{y} dy = \int x^{-2} dx$$

$$\ln y = \frac{x^{-2+1}}{-2+1} + c_1$$

$$\ln y = \frac{x^{-1}}{-1} + c_1$$

$$\ln y = -\frac{1}{x} + c_1$$

$$e^{\ln y} = e^{-\frac{1}{x} + c_1}$$

$$y = e^{-\frac{1}{x}} e^{c_1}$$

$$y = c e^{-\frac{1}{x}}$$

**Q. 6:**  $\sin y \csc x \frac{dy}{dx} = 1$

**SOLUTION:**

$$\sin y \csc x \frac{dy}{dx} = 1$$

Separating the variables:

$$\sin y \frac{1}{\sin x} dy = dx$$

$$\sin y dy = \sin x dx$$

Integrating both sides

$$\int \sin y dy = \int \sin x dx$$

$$-\cos y = -\cos x + c_1$$

$$-\cos y = -(\cos x - c_1)$$

$$\cos y = \cos x - c_1$$

$$\cos y = \cos x + c$$

**Q. 7:**  $x dy + y(x - 1)dx = 0$

**SOLUTION:**

$$x dy + y(x - 1)dx = 0$$

Separating the variables:

$$x dy = -y(x - 1)dx$$

$$\frac{1}{y} dy = -\left(\frac{x-1}{x}\right) dx$$

$$\frac{1}{y} dy = -\left(1 - \frac{1}{x}\right) dx$$

Integrating both sides

$$\int \frac{1}{y} dy = \int -1 + \frac{1}{x} dx$$

$$\ln y = -x + \ln x + \ln c$$

$$\ln y = \ln(xc) - x$$

$$\ln y - \ln(xc) = -x$$

$$\ln\left(\frac{y}{xc}\right) = -x$$

$$e^{\ln\left(\frac{y}{xc}\right)} = e^{-x}$$

$$\frac{y}{xc} = e^{-x}$$

$$y = cx e^{-x}$$

**Q. 8:**  $\frac{x^2+1}{y+1} = \frac{x}{y} \cdot \frac{dy}{dx}$

**SOLUTION:**

$$\frac{x^2+1}{y+1} = \frac{x}{y} \cdot \frac{dy}{dx}$$

Separating the variables:

$$(x^2 + 1)y dx = x(y + 1)dy$$

$$x(y + 1) dy = (x^2 + 1)y dx$$

$$\frac{y+1}{y} dy = \frac{x^2+1}{x} dx$$

$$1 + \frac{1}{y} dy = x + \frac{1}{x} dx$$

Integrating both sides

$$\int \left(1 + \frac{1}{y}\right) dy = \int \left(x + \frac{1}{x}\right) dx$$

$$y + \ln y = \frac{x^2}{2} + \ln(x) + \ln c$$

$$y + \ln y = \frac{x^2}{2} + \ln(xc)$$

$$\ln y - \ln(xc) = \frac{x^2}{2} - y$$

$$\ln\left(\frac{y}{xc}\right) = \frac{x^2}{2} - y$$

$$e^{\ln\left(\frac{y}{xc}\right)} = e^{\frac{x^2}{2} - y}$$

$$\frac{y}{xc} = e^{\frac{x^2}{2}} e^{-y}$$

$$\frac{y}{e^{-y}} = xc e^{\frac{x^2}{2}}$$

$$y e^y = c x e^{\frac{x^2}{2}}$$

**Q. 9:**  $\frac{1}{x} \frac{dy}{dx} = \frac{1}{2} (1 + y^2)$

**SOLUTION:**

$$\frac{1}{x} \frac{dy}{dx} = \frac{1}{2} (1 + y^2)$$

Separating the variables:

$$\frac{1}{1+y^2} dy = \frac{1}{2} (x) dx$$

Integrating both sides

$$\int \frac{1}{1+y^2} dy = \frac{1}{2} \int x dx$$

$$\tan^{-1} y = \frac{1}{2} \frac{x^2}{2} + c$$

$$y = \tan\left(\frac{x^2}{4} + c\right)$$

**Q. 10:**  $2x^2 y \frac{dy}{dx} = x^2 - 1$

**SOLUTION:**

$$2x^2 y \frac{dy}{dx} = x^2 - 1$$

Separating the variables:

$$y dy = \frac{x^2-1}{2x^2} dx$$

$$y dy = \frac{1}{2} \left(\frac{x^2-1}{x^2}\right) dx$$

Integrating both sides

$$\int y dy = \frac{1}{2} \int \left(1 - \frac{1}{x^2}\right) dx$$

$$\int y dy = \frac{1}{2} \int (1 - x^{-2}) dx$$

$$\frac{y^2}{2} = \frac{1}{2} \left(x - \frac{x^{-1}}{-1}\right) + c$$

$$y^2 = x + \frac{1}{x} + c$$

**Q. 11:**  $\frac{dy}{dx} + \frac{2xy}{2y+1} = x$

**SOLUTION:**

$$\frac{dy}{dx} + \frac{2xy}{2y+1} = x$$

Separating the variables:

$$\frac{dy}{dx} = x - \frac{2xy}{2y+1}$$

$$\frac{dy}{dx} = \frac{x(2y+1) - 2xy}{2y+1}$$

$$\frac{dy}{dx} = \frac{2xy + x - 2xy}{2y+1}$$

$$\frac{dy}{dx} = \frac{x}{2y+1}$$

$$(2y+1)dy = xdx$$

Integrating both sides

$$\int (2y+1) dy = \int x dx$$

$$2 \frac{y^2}{2} + y = \frac{x^2}{2} + c$$

$$y(y+1) = \frac{x^2}{2} + c$$

**Q. 12:**  $(x^2 - yx^2) \frac{dy}{dx} + y^2 + xy^2$

**SOLUTION:**

$$(x^2 - yx^2) \frac{dy}{dx} + y^2 + xy^2$$

Separating the variables:

$$x^2(1-y) \frac{dy}{dx} = -y^2(1+x)$$

$$x^2(1-y)dy = -y^2(1+x)dx$$

$$\frac{1-y}{y^2} dy = -\frac{1+x}{x^2} dx$$

$$\frac{1}{y^2} - \frac{y}{y^2} dy = -\left(\frac{1}{x^2} + \frac{x}{x^2}\right) dx$$

$$\frac{1}{y^2} - \frac{1}{y} dy = -\left(\frac{1}{x^2} + \frac{1}{x}\right) dx$$

$$\left(y^{-2} - \frac{1}{y}\right) dy = -\left(x^{-2} + \frac{1}{x}\right) dx$$

Integrating both sides

$$\int \left(y^{-2} - \frac{1}{y}\right) dy = -\int \left(x^{-2} + \frac{1}{x}\right) dx$$

$$\frac{y^{-2+1}}{-2+1} - \ln y = -\left(\frac{x^{-2+1}}{-2+1} + \ln x\right) + c_1$$

$$\frac{y^{-1}}{-1} - \ln y = -\left(\frac{x^{-1}}{-1} + \ln x\right) + c_1$$

$$-\frac{1}{y} - \ln y = -\left(-\frac{1}{x} + \ln x\right) + c_1$$

$$\ln y + \frac{1}{y} = \left(-\frac{1}{x} + \ln x\right) - c_1$$

$$\ln y + \frac{1}{y} = \ln x - \frac{1}{x} + c$$

**Q. 13:**  $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

**SOLUTION:**

$$\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$$

Separating the variables:

$$\sec^2 y \tan x dy = \sec^2 x \tan y dx$$

$$\frac{\sec^2 y}{\tan y} dy = \frac{\sec^2 x}{\tan x} dx$$

Integrating both sides

$$\int \frac{\sec^2 y}{\tan y} dy = -\int \frac{\sec^2 x}{\tan x} dx$$

$$\ln(\tan y) = -\ln(\tan x) + \ln c$$

$$\ln(\tan y) + \ln(\tan x) = \ln c$$

$$\ln(\tan y \tan x) = \ln c$$

$$\tan y \tan x = c$$

**Q. 14:**  $\left(y - x \frac{dy}{dx}\right) = 2\left(y^2 + \frac{dy}{dx}\right)$

**SOLUTION:**

$$\left(y - x \frac{dy}{dx}\right) = 2\left(y^2 + \frac{dy}{dx}\right)$$

Separating the variables:

$$y - x \frac{dy}{dx} = 2y^2 + 2 \frac{dy}{dx}$$

$$-x \frac{dy}{dx} - 2 \frac{dy}{dx} = 2y^2 - y$$

Multiplying both sides by  $-1$

$$x \frac{dy}{dx} + 2 \frac{dy}{dx} = y - 2y^2$$

$$(x+2) \frac{dy}{dx} = y - 2y^2$$

$$(x+2)dy = y(1-2y) dx$$

$$\frac{1}{y(1-2y)} dy = \frac{1}{x+2} dx$$

Integrating both sides

$$\int \frac{(1-2y)+2y}{y(1-2y)} dy = \int \frac{1}{x+2} dx$$

$$\int \left[\frac{(1-2y)}{y(1-2y)} + \frac{2y}{y(1-2y)}\right] dy = \int \frac{1}{x+2} dx$$

$$\int \left[\frac{1}{y} + \frac{2}{(1-2y)}\right] dy = \int \frac{1}{x+2} dx$$

$$\int \frac{1}{y} dy - \int \frac{2}{2y-1} dy = \int \frac{1}{x+2} dx$$

$$\ln(y) + \ln(2y-1) = \ln(x+2) + \ln(c)$$

$$\ln\left(\frac{y}{2y-1}\right) = \ln c(x+2)$$

$$\frac{y}{2y-1} = c(x+2)$$

**Q. 15:**  $1 + \cos x \tan y \frac{dy}{dx} = 0$

**SOLUTION:**

$$1 + \cos x \tan y \frac{dy}{dx} = 0$$

Separating the variables:

$$\cos x \tan y \frac{dy}{dx} = -1$$

$$\tan y dy = -\frac{1}{\cos x} dx$$

Integrating both sides

$$\int \frac{-\sin y}{\cos y} dy = \int \sec x dx$$

$$\ln(\cos y) = \ln(\sec x + \tan x) + \ln c$$

$$\ln(\cos y) = \ln[c(\sec x + \tan x)]$$

$$\cos y = c(\sec x + \tan x)$$

**Q. 16:**  $y - x \frac{dy}{dx} = 3\left(1 + x \frac{dy}{dx}\right)$

**SOLUTION:**

$$y - x \frac{dy}{dx} = 3\left(1 + x \frac{dy}{dx}\right)$$

Separating the variables:

$$y - x \frac{dy}{dx} = 3 + 3x \frac{dy}{dx}$$

$$y - 3 = 3x \frac{dy}{dx} + x \frac{dy}{dx}$$

$$y - 3 = 4x \frac{dy}{dx}$$

$$4x \frac{dy}{dx} = y - 3$$

$$\frac{1}{y-3} dy = \frac{1}{4x} dx$$

Integrating both sides

$$\int \frac{1}{y-3} dy = \frac{1}{4} \int \frac{1}{x} dx$$

$$\ln(y-3) = \frac{1}{4} \ln x + \ln c$$

$$\ln(y-3) = \ln x^{\frac{1}{4}} + \ln c$$

$$\ln(y-3) = \ln\left(cx^{\frac{1}{4}}\right)$$

$$y-3 = cx^{\frac{1}{4}}$$

$$y = 3 + cx^{\frac{1}{4}}$$

**Q. 17:**  $\sec x + \tan y \frac{dy}{dx} = 0$

**SOLUTION:**

$$\sec x + \tan y \frac{dy}{dx} = 0$$

Separating the variables:

$$\tan y \frac{dy}{dx} = -\sec x$$

$$\tan y dy = -\sec x dx$$

Integrating both sides

$$\int \tan y dy = -\int \sec x dx$$

$$\int \frac{-\sin y}{\cos y} dy = \int \sec x dx$$

$$\ln(\cos y) = \ln(\sec x + \tan x) + \ln c$$

$$\ln(\cos y) = \ln[c(\sec x + \tan x)]$$

$$\cos y = c(\sec x + \tan x)$$

**Q. 18:**  $(e^x + e^{-x}) \frac{dy}{dx} = e^x - e^{-x}$

**SOLUTION:**

$$(e^x + e^{-x}) \frac{dy}{dx} = e^x - e^{-x} \text{ Separating the variables:}$$

$$dy = \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

Integrating both sides

$$\int 1 dy = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$$y = \ln(e^x + e^{-x}) + c$$

**Q. 19:** Find the general solution of the following equation

$$\frac{dy}{dx} - x = xy^2. \text{ Also find the}$$

Particular solution if  $y = 1$

when  $x = 0$

**SOLUTION:**

$$\frac{dy}{dx} - x = xy^2$$

Separating the variables:

$$\frac{dy}{dx} = xy^2 + x$$

$$\frac{dy}{dx} = x(y^2 + 1)$$

$$\frac{1}{y^2+1} dy = x dx$$

Integrating both sides

$$\int \frac{1}{1+y^2} dy = \int x dx$$

$$\tan^{-1} y = \frac{x^2}{2} + c$$

(General solution) (1)

$$\text{At } x = 0, y = 1$$

$$\tan^{-1}(1) = \frac{(0)^2}{2} + c$$

$$\frac{\pi}{4} = c \text{ (Put in 1)}$$

$$\tan^{-1} y = \frac{x^2}{2} + \frac{\pi}{4}$$

(Particular solution)

**Q. 20:** Solve the differential

$$\text{equation } \frac{dx}{dt} = 2x \text{ given that}$$

$x = 4$  when  $t = 0$

**SOLUTION:**

$$\frac{dx}{dt} = 2x$$

Separating the variables:

$$dx = 2x dt$$

$$\frac{1}{x} dx = 2 dt$$

Integrating both sides

$$\int \frac{1}{x} dy = 2 \int 1 dx$$

$$\ln x = 2t + c_1$$

$$e^{\ln x} = e^{2t+c_1}$$

$$x = e^{2t} e^{c_1}$$

$$x = ce^{2t} \text{ Where } e^{c_1} = c$$

(General solution) (1)

$$\text{At } x = 4, t = 0$$

$$4 = ce^{2(0)}$$

$$4 = ce^0$$

$$4 = c \text{ Put in (1) } \because e^0 = 1$$

$$x = 4e^{2t}$$

(Particular solution)

**Q. 21:** Solve the differential

$$\text{equation } \frac{ds}{dt} + 2st = 0. \text{ Also find the}$$

Particular solution if  $s = 4e$

when  $t = 0$

**SOLUTION:**

$$\frac{ds}{dt} + 2st = 0$$

Separating the variables:

$$\frac{ds}{dt} = -2st$$

$$\frac{1}{s} ds = -2t dt$$

Integrating both sides

$$\int \frac{1}{s} ds = -\int 2t dt$$

$$\ln s = -2 \frac{t^2}{2} + c_1$$

$$\ln s = -t^2 + c_1$$

$$s = e^{-t^2+c_1}$$

$$s = e^{-t^2} e^{c_1}$$

$$s = ce^{-t^2} \text{ Where } e^{c_1} = c$$

(General solution) (1)

$$\text{At } s = 4e, t = 0$$

$$4e = ce^{-(0)^2}$$

$$4e = ce^0$$

$$4e = c \text{ Put in (1) } \because e^0 = 1$$

$$s = 4e \cdot e^{-t^2}$$

$$s = 4e^{1-t^2}$$

(Particular solution)

**Q22.** In a culture, bacteria increase number of bacteria present. If bacteria are 200 initially and are doubled in 2 hours, find the number of bacteria present four hours later.

**Solution:**

Let  $P$  be numbers of bacteria then

$$\frac{dP}{dt} \propto P$$

$$\Rightarrow \frac{dP}{dt} = kP$$

$$\Rightarrow \frac{1}{P} dP = k dt$$

take integral

$$\Rightarrow \int \frac{1}{p} dP = k \int dt$$

$$\Rightarrow \ln P = kt + \ln c$$

$$\Rightarrow \ln p - \ln c = kt$$

$$\Rightarrow \ln \frac{p}{c} = kt$$

$$\Rightarrow \frac{p}{c} = e^{kt}$$

$$\Rightarrow p = ce^{kt} \rightarrow (i)$$

put  $p=200$ ,  $t=0$  (condition1)

$$200 = ce^{k(0)} = ce^0$$

$$\Rightarrow c = 200 \quad \because e^0 = 1$$

So (i)  $p = 200e^{kt} \rightarrow (ii)$

Put  $p = 400$  when  $t = 2$  (conditionII)

$$\text{so (i)} \Rightarrow 400 = 200e^{kt}$$

$$\Rightarrow 2 = e^{kt} \Rightarrow \ln 2 = \ln e^{2k}$$

$$\Rightarrow 2k = \ln 2$$

$$\Rightarrow k = \frac{1}{2} \ln 2$$

$$\text{So (ii)} \Rightarrow p = 200e^{\frac{1}{2} \ln 2 t}$$

$$\Rightarrow p = 200e^{\frac{\ln 2}{2}(4)} \quad \text{for } t = 4$$

$$\Rightarrow p = 200^{2 \ln 2} = 200e^{\ln 2^2} = 200e^{\ln 4}$$

$$\Rightarrow p = 200(4) \Rightarrow p = 800$$

Which is required number of bacteria present four latter.

**Q.23** a ball is thrown vertically upward with a velocity of 2450cm/sec neglecting air resistance, find

i. Velocity of ball at any time  $t$

ii. Distance traveled in any time  $t$

iii. Maximum height attained by the ball

**Solution:**

Let  $v$  is velocity and  $g$  is acceleration, so

$$i) \frac{dv}{dt} = -g \quad \text{for upward}$$

$$\Rightarrow dv = -g dt$$

$$\Rightarrow \int dv = -g \int dt$$

$$\Rightarrow v = -gt + c_1$$

Put  $v = 2450$ ,  $t = 0$  so

$$2450 = -g(0) + c_1 \Rightarrow c_1 = 2450$$

$$v = -gt + 2450 \quad \because g = 9.8 \text{m/sec}$$

$$\text{Thus } v = -980t + 2450 \Rightarrow g = 980 \text{cm/sec}$$

ii) let  $h$  be height so

$$v = \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = v$$

$$\Rightarrow \frac{dh}{dt} = -980t + 2450$$

$$\Rightarrow dh = -980t dt + 2450 dt$$

$$\Rightarrow \int dh = -980 \int t dt + 2450 \int dt$$

$$\Rightarrow h = -980 \frac{t^2}{2} + 2450t + c_2$$

put  $h = 0$ ,  $t = 0$

$$0 = -490(0)^2 + 2450(0) + c_2$$

$$\Rightarrow c_2 = 0$$

$$\text{so } h = -490t^2 + 2450t$$

(iii) For max. height,  $v = 0$

$$\text{So } 0 = -980t^2 + 2450 \text{ from (i)}$$

$$\Rightarrow 980t = \frac{2450}{980}$$

$$\Rightarrow t = \frac{5}{2}$$

$$\text{So } h = 2450 \left(\frac{5}{2}\right) - 490 \left(\frac{5}{2}\right)^2$$

$$= 6125 - 3062.5$$

$$\Rightarrow h = 3062.5$$

So max. height = 3062.5cm

$$\Rightarrow \text{max height} = 30.6 \text{m } (\div \text{ by } 100)$$