Unit#2 Differention
F.Sc Part-II M.C.Q's

Prof. Asad Khalid
M.Phil Mathematics

(1) Notation for Derivative was used by Newton is (LHR-2013)
(a) $\frac{dy}{dx}$ (b) Df (c) $\dot{f}(x)$ (d) $f'(x)$

(2) $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=$ (a) $f'(a)$ (b) $f''(x)$ (c) $f(x)$ (d) $f(x)$ (LHR-2009)

(3) $\frac{d}{dx}(x)^n=n(x)^{n-1}$, where (a) $n\in C$ (b) $n\in R$ (c) $n\in Q$ (d) $n\in N$ (LHR-2010)

(4) $\frac{dy}{dx}=x^3$ if (a) $y=x^3$ (b) $4x^4$ (c) $y=\frac{x^4}{4}$ (d) $y=3x^3$ (LHR-2016)

(5) $\frac{d}{dx}(ax^n+bx^m)=$ (a) $nax^{n-1}+mbx^{m-1}$ (b) $nax^{n-1}+mbx^{m+1}$ (c) $ax^{n-1}+bx^{m-1}$ (d) $nax^{n+1}+mbx^{m-1}$

(6) $\frac{d}{dx}(x^2+1)^2=$ (a) $2(x^2+1)$ (b) $\frac{(x^2+1)^2}{3}$ (c) $2x(x^2+1)$ (d) $4x(x^2+1)$ (LHR-2017)

(7) If $y=\sqrt{x}$, Then $\frac{dy}{dx}=$ (a) $\frac{1}{2\sqrt{x}}$ (b) $\frac{2}{\sqrt{x}}$ (c) $\frac{1}{\sqrt{x}}$ (d) $2\sqrt{x}$ (LHR-2019)

(8) If $f(x)=x^{2/3}$, Then $f'(8)=$ (a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) 3 (LHR-2014)

(9) $\frac{d}{dx}\left(\frac{a}{x}\right)=$ (a) $\frac{a}{x}$ (b) $-\frac{a}{x}$ (c) $\frac{a}{x^2}$ (d) $-\frac{a}{x^2}$ (LHR-2012)

(10) $\frac{d}{dx}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2=$ (a) $1-\frac{1}{2x}$ (b) $1+\frac{1}{x^2}$ (c) 0 (d) $1-\frac{1}{x^2}$ (LHR-2017)

(11) If $y=x-\frac{1}{x}$, Then $\frac{dy}{dx}=$ (a) $1+\frac{1}{x^2}$ (b) $1-\frac{1}{x^2}$ (c) $1+\frac{1}{x}$ (d) $1-\frac{1}{x}$ (LHR-2011)

(12) The Value of $\frac{dy}{dx}=\frac{-2}{x^3}$ at $x=-1$ (a) 4 (b) 5 (c) -2 (d) 2 (LHR-2019)

(13) If $x=f(\theta)$, $y=g(\theta)$, Then $\frac{dy}{dx}=$ (a) $\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$ (b) $\frac{dx}{d\theta}\cdot\frac{d\theta}{dy}$ (c) $\frac{d\theta}{dy}\cdot\frac{dx}{d\theta}$ (d) None (LHR-2018)

(14) If $\frac{d}{dx}(f(x))=\frac{x}{\sqrt{1-x^2}}$, Then $f'(\sin x)=$ (LHR-2012)
(a) $\sec^{-1}x$ (b) $\frac{1}{\cot x}$ (c) $\frac{1}{\tan x}$ (d) $\tan^{-1}x$

(15) If $f(x)=\cos x$, Then $f'(\pi)=$ (LHR-2019)
(a) -1 (b) 0 (c) $\frac{1}{2}$ (d) 1

(16) $\frac{d}{dx}(\cos x^2)=$ (a) $-2x\sin x^2$ (b) $2x\sin x^2$ (c) $-\frac{\sin x^2}{2x}$ (d) $-2\sin x\cos x$ (LHR-2019)

(17) $\frac{d}{dx}(\tan x) =$ (a) $\ln\cos x$ (b) $-\ln\cos x$ (c) $-\sec^2 x$ (d) $\sec^2 x$ (LHR-2014)

(18) $\frac{d}{dx}(\sqrt{\tan x}) =$ (a) $\frac{\sec^2 x}{2\sqrt{\tan x}}$ (b) $\frac{\sec^2 x}{\sqrt{\tan x}}$ (c) $\frac{\sec x}{\sqrt{\tan x}}$ (d) $\frac{\sqrt{\sec x}}{\tan x}$ (LHR-2011)

(19) $\frac{d}{dx}(-\cot x) =$ (a) $\operatorname{cosec}^2 x$ (b) $\sec^2 x$ (c) $\operatorname{cosec} x \cdot \sec x$ (d) $\sec x \cdot \tan x$

(20) $\frac{d}{dx}(\cot^2 2x) =$ (a) $4\cot 2x \operatorname{cosec} 2x$ (b) $-4\cot 2x \operatorname{cosec}^2 2x$
(c) $4\cot^2 2x \operatorname{cosec} 2x$ (d) $-4\cot 2x$ (LHR-2017)

(21) $\frac{d}{dx}(\sec x) =$ (a) $\sec x \cdot \tan x$ (b) $\sec^2 x$ (c) $-\sec x$ (d) $\tan^2 x$ (LHR-2010, 2016)

(22) If $y = \sec\left(\frac{3\pi}{2} - x\right)$, Then $y_1 =$ (a) $\operatorname{cosec} x \cdot \cot x$ (b) $-\operatorname{cosec} x \cdot \cot x$
(c) $\sec x \cdot \tan x$ (d) $-\sec x \cdot \tan x$ (LHR-2015)

(23) $\frac{d}{dx}(\operatorname{cosec}^2 x - \cot^2 x) =$ (a) $\cot^2 x + \operatorname{cosec}^2 x$ (b) $-\sec^2 x + \tan^2 x$
(c) 0 (d) $-2\operatorname{cosec} x \cot x$ (LHR-2013)

(24) $\frac{d}{dx}(\sin^{-1} x) =$ (a) $\cos^{-1} x$ (b) $\frac{1}{\sqrt{1-x^2}}$ (c) $\frac{-1}{\sqrt{1-x^2}}$ (d) $\frac{1}{\sqrt{1+x^2}}$ (LHR-2010, 2015, 2018)

(25) $\frac{d}{dx}(\tan^{-1} x) =$ (a) $\frac{1}{1+x^2}$ (b) $\frac{1}{1-x^2}$ (c) $\frac{-1}{1+x^2}$ (d) $\frac{1}{\sqrt{1-x^2}}$ (LHR-2012)

(26) $\frac{d}{dx}(a^{f(x)}) =$ (a) $f'(x)a^{f(x)} \cdot \ln a$ (b) $f'(x) \cdot a^{f(x)}$ (c) $\frac{f'(x) \cdot a^{f(x)}}{\ln a}$ (d) None (LHR-2009)

(27) If $y = a^x$, Then $\frac{dy}{dx} =$ (a) $a^x \cdot \ln a$ (b) $\frac{a^x}{\ln a}$ (c) $\frac{a^x}{\ln x}$ (d) xa^{x-1} (LHR-2011)

(28) $\frac{d}{dx}(e^{f(x)}) =$ (a) $e^{f(x)}$ (b) $e^{f(x)} \cdot f'(x)$ (c) $\frac{f(x)}{e^x}$ (d) $\frac{e^x}{f'(x)}$ (LHR-2016)

(29) If $f(x) = e^{mx}$, Then $f'(x) =$ (a) $\frac{e^{mx}}{a}$ (b) me^{mx} (c) $-\frac{e^{mx}}{a}$ (d) $-ae^{mx}$ (LHR-2015)

(30) If $y = 5e^{3x-4}$, Then $\frac{dy}{dx} =$ (a) $15e^{3x-4}$ (b) $5e^{3x-4}$ (c) $-5e^{3x-4}$ (d) e^{3x-4} (LHR-2010)

(31) $\frac{d}{dx}(e^{x+h}) =$ (a) $\frac{e^{x+h}}{\ln h}$ (b) $\frac{e^{x+h}}{\ln x}$ (c) e^{x+h} (d) he^{x+h} (LHR-2017)

(32) $\frac{d}{dx}(e^{\cos x}) =$ (a) $-\sin x \cdot e^{\cos x}$ (b) $\sin x \cdot e^{\cos x}$ (c) $\cos x e^{\sin x}$ (d) None (LHR-2015)

(33) $\frac{d}{dx}(3)^{3x} =$ (a) $3^{3x}.\ln 3$ (b) $3^{3x}.\ln 9$ (c) $3^{3x}.\ln 27$ (d) $3^{3x}.\ln 18$ (LHR-2012)

(34) $\frac{d}{dx}(\log_a x) =$ (a) $\frac{1}{x}$ (b) $\frac{1}{x}\ln a$ (c) $x\ln a$ (d) $\frac{1}{x\ln a}$ (LHR-2018)

(35) If $y = \ln(\sin x)$, Then $\frac{dy}{dx} =$ (a) $\tan x$ (b) $\cot x$ (c) $-\tan x$ (d) $-\cot x$ (LHR-2014, 2016)

(36) $\frac{d}{dx}(\sinh 2x) =$ (a) $2\cosh 2x$ (b) $2\sinh 2x$ (c) $-2\cosh 2x$ (d) $-2\sinh 2x$ (LHR-2012)

(37) $\frac{d}{dx}(\cosh x) =$ (a) $\sinh x$ (b) $-\sinh x$ (c) $\operatorname{cosech} x$ (d) $\coth x$ (LHR-2018)

(38) $\frac{d}{dx}(\operatorname{sech} x) =$ (a) $-\operatorname{sech} x.\tanh x$ (b) $-\sec x.\tan x$
(c) $\sec x.\tan x$ (d) $\operatorname{sech} x.\tanh x$ (LHR-2018)

(39) $\frac{d}{dx}(\sinh^{-1} x) =$ (a) $\frac{1}{\sqrt{1+x^2}}$ (b) $-\frac{1}{\sqrt{1-x^2}}$ (c) $\frac{1}{\sqrt{1-x^2}}$ (d) $-\frac{1}{\sqrt{1+x^2}}$ (LHR-2015)

(40) $\frac{d}{dx}(\tanh^{-1} x) =$ (a) $\frac{1}{1+x^2}$ (b) $\frac{1}{x^2-1}$ (c) $\frac{1}{1-x^2}$ (d) $\frac{1}{\sqrt{x^2-1}}$ (LHR-2011, 2016)

(41) If $f(x) = \sin x$, Then Value of $f'''(0) =$
(a) 1 (b) -1 (c) 0 (d) ∞ (LHR-2010)

(42) If $\sin 3x$, Then $y_2 =$ (a) $3\cos 3x$ (b) $9\cos 3x$ (c) $-9\sin 3x$ (d) $9\sin x$ (LHR-2014)

(43) If $y = \sin x$, Then (a) $y_4 \neq y$ (b) $y_4 = y$ (c) $y_4 = y_1$ (d) $y_4 = y_2$ (LHR-2009)

(44) If $y = \cos x$, Then $y_4 =$ (a) y_1 (b) y_2 (c) y_3 (d) y (LHR-2011)

(45) If $y = \cos x$, Then (a) $y_4 - y = 0$ (b) $y_4 + y = 0$ (c) $y_2 - y = 0$ (d) $y_3 - y = 0$ (LHR-2012)

(46) If $y = e^{-ax}$, Then $\frac{d^2y}{dx^2} =$ (a) $-a^2.e^{-ax}$ (b) $a^2.e^{ax}$
(c) $-a.e^{-ax}$ (d) ae^{-x} (LHR-2012)

(47) If $y = e^{2x}$, Then $y_2 =$ (a) $4.e^{2x}$ (b) $x.e^{2x-1}$ (c) $2.e^{2x}$ (d) e^{2x-1} (LHR-2019)

(48) The expansion $f(x)=f(0)+x\cdot f'(0)+\frac{x^2}{2!}f''(0)+\cdots$ is called

(a) Maclaurin's Theorem (b) Taylor Theorem (c) Binomial Theorem (d) None (LHR-2016)

(49) $\sin x=$ (a) $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$ (b) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$

(c) $-x+\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$ (d) $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

(LHR-2009)

(50) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$ is Maclaurin's Expansion of

(a) $\cos x$ (b) $\sin x$ (c) $\ln(1-x)$ (d) $\ln(1+x)$ (LHR-2012)

(51) If $f(x+h)=2^{x+h}$, Then $f'(x)=$

(a) 2^{x+h} (b) $\frac{2}{\ln 2}$ (c) $2^x.\ln 2$ (d) 2^x (LHR-2014)

(52) The Slope of Tangent Line to $y=f(x)$ at (x_1,y_1) is

(a) m (b) $\frac{y_2-y_1}{x_2-x_1}$ (c) $f'(x)$ (d) $-\frac{dy}{dx}$ (LHR-2013)

(53) Geometrical Meaning of Derivative is

(a) Slope of Tangent line (b) Slope of Normal line

(c) Slope of x-axis (d) Slope of y-axis (LHR-2010)

(54) A function $f(x)$ is said to have Relative Maxima at $x=c$ if $\forall x\in(c-\delta x, c+\delta x)$

(a) $f(c)\geq f(x)$ (b) $f(c)\leq f(x)$ (c) $f(c)=0$ (d) None (LHR-2018)

(55) If $f(x)=-3x^2$ has Maximum Value at (LHR-2011)

(a) $x=-2$ (b) $x=-1$ (c) $x=0$ (d) $x=1$

" Prof. Asad Khalid
M.Phill Mathematics

F.Sc (Part-II)
Unit 2

THE DERIVATIVE

In every day Life we come across problems of the type:

★ How fast is the Train moving after 10 Seconds?
★ At what rate is the population increasing?
★ What is the rate of change of Profit when Sales reach Rs 10,000?
★ What is the rate of change of Profit when Sales reach Rs 5000?

To Answer such problems we need derivative.

Use OF Derivative in Daily Life:

(1) Rate of flow of Water?
(2) Rate of Increase of height of Plant?
(3) Rate of change of Price in share Market?
(4) Slope of Mountain?
(5) Cricket Match Run Rate?

In above all we use Derivative.

History:- The ancient Greeks Knew the concepts of area, volume and centroids etc. Which are related to Integral Calculus. Later on, in the Seventeeth Century "Sir Isaac Newton" an English Mathematician (1642–1727) and "Gottfried Whihelm Leibniz" a German Mathematician (1646–1716) Considered the Problem of Instantaneous rates of change.

Mathematical Definition Of Derivative:-

Let f be a function defined in an open Interval Containing a point x. The Derivative of f at x, denoted by $f'(x)$ is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

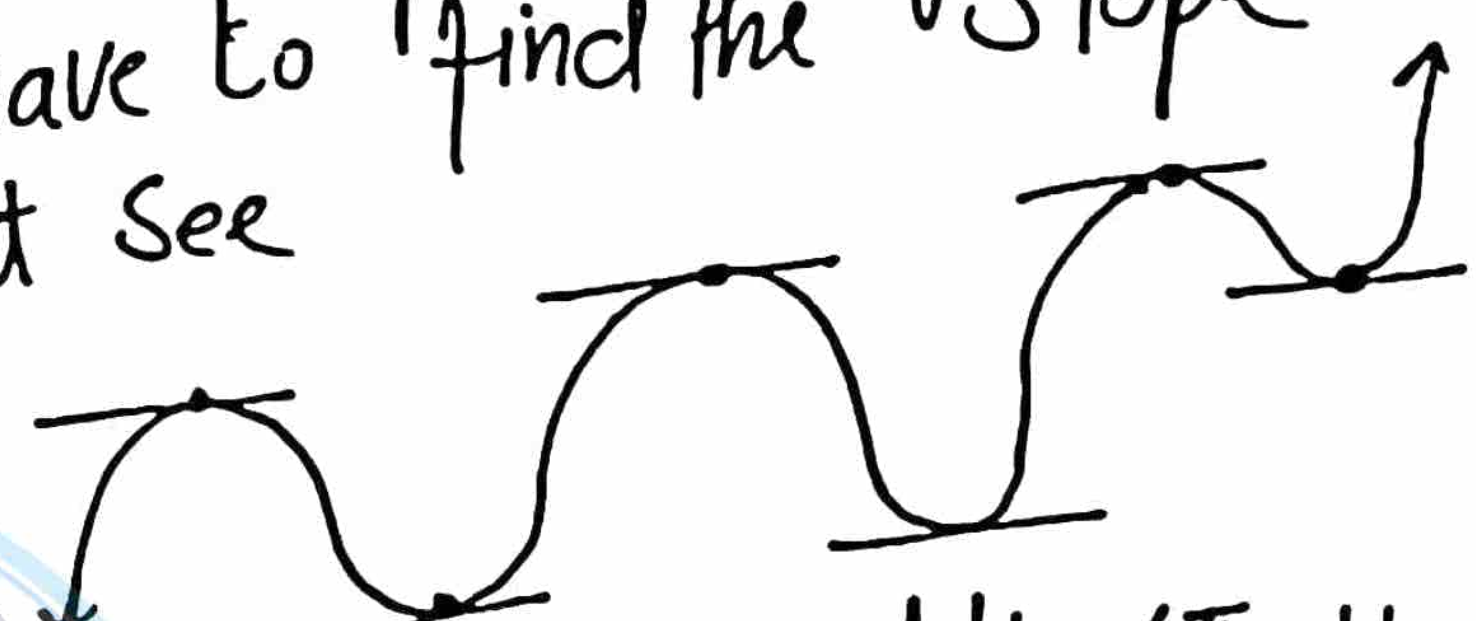
(Provided the limit exists)

Some Other Definition of Derivative: (For Unders

In Mathematics, Derivative is a way to show Rate of change, that is the amount by which a function is changing at one given point. OR (A function which tells us the Slope of the line tangent to Curve at any point.)

Geometrical Interpretation of Derivative:-

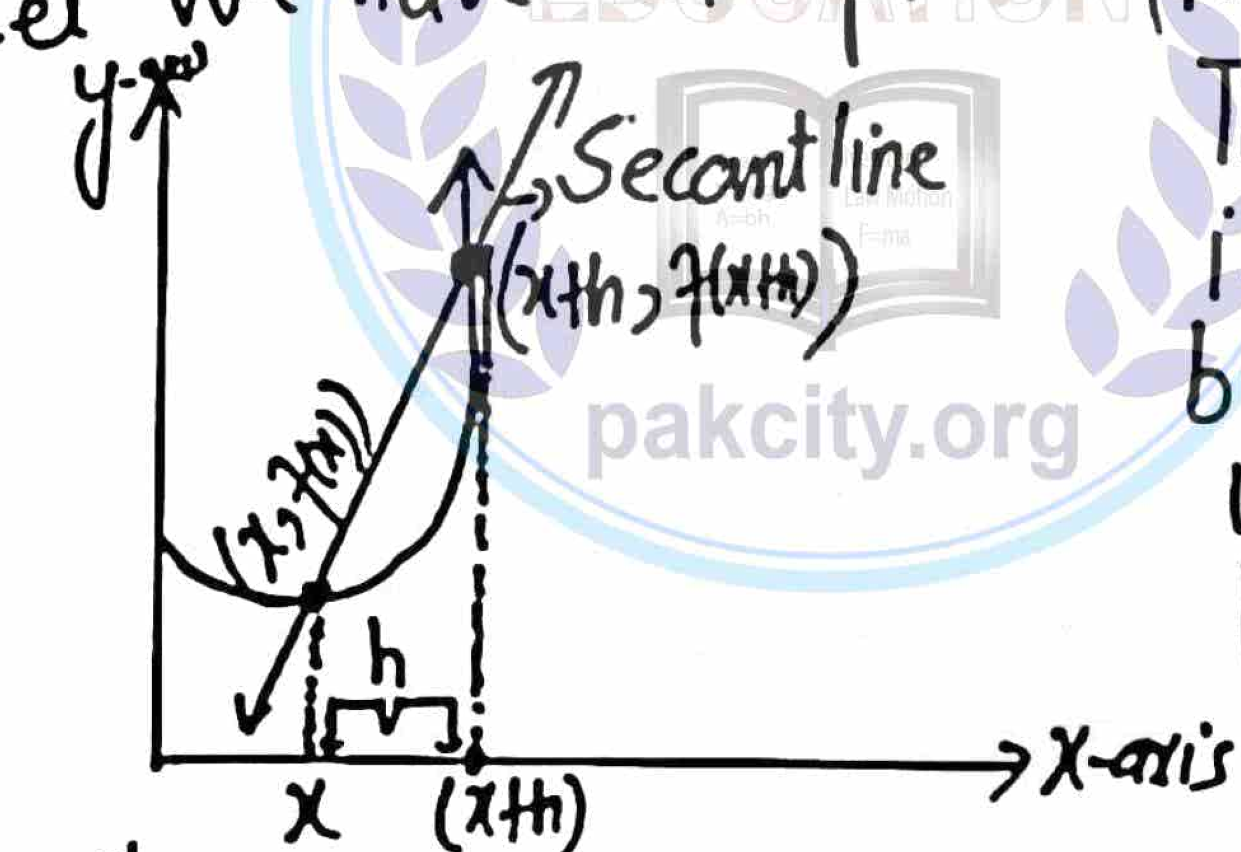
When we use the word Slope Means (Rise over Run) or (Vertical Change over the horizontal Change), if we have a Straight line we can find Slope because we have two points anywhere on the line. When we have to find the Slope of Curved Graph. Let See



We see that there is only one point but for finding Slope we require two points.

Tangent line (The line that touches the Curve at one point).

Let We have Graph.



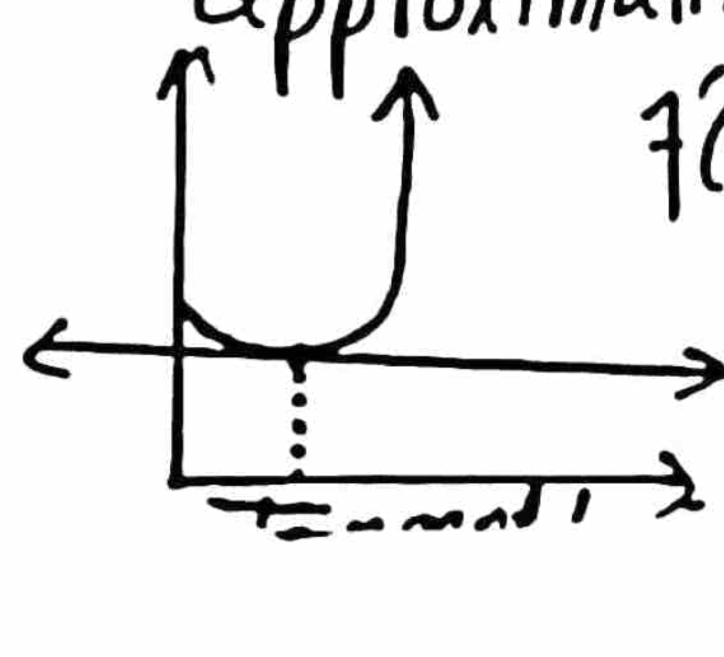
$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x+h) - f(x)}{x+h-x}$$

$$\text{Slope of Secant line} = \frac{f(x+h) - f(x)}{h}$$

(Difference Quotient)

(Note:-) The Slope that we have fined is a Rough estimation/approximation, but we don't Approximation, we want exact slope at that point. For this we close in x by Narrowing h to zero, picking a Right point that is close to h, so that horizontal Distance becomes Smaller and Smaller. The close Two points are, more better approximation get. So take $\lim_{h\to 0}$



$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

(Slope of Secant line becomes Slope of

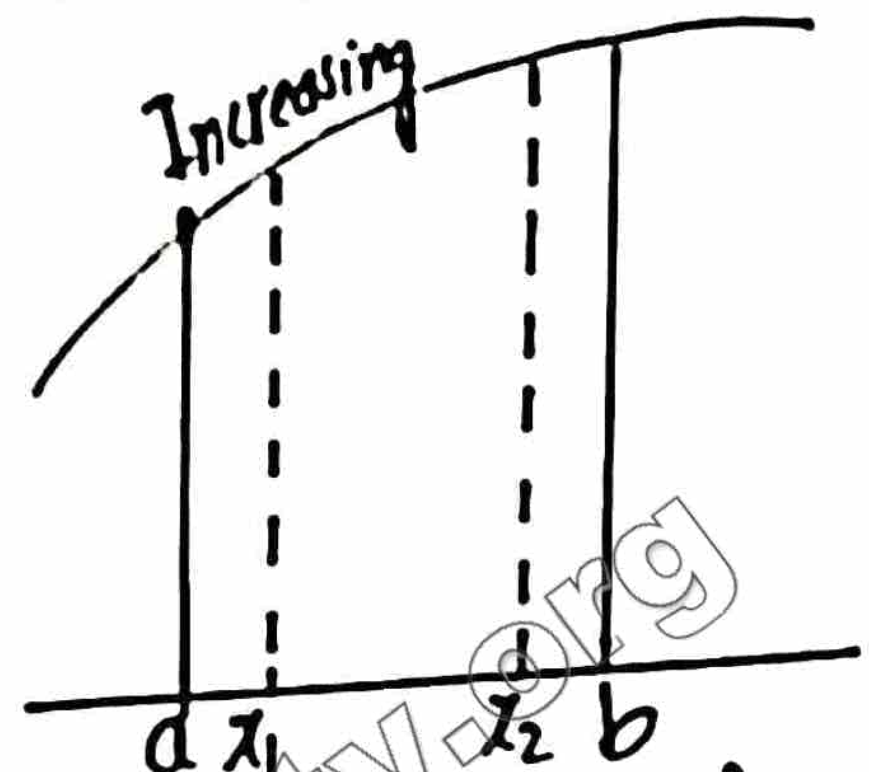
Application of Derivative

Increasing And Decreasing Functions

Increasing Function:-

Let f be defined on an Interval (a,b) and Let $x_1, x_2 \in (a,b)$. Then "f is Increasing on the Interval (a,b)" if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2$$



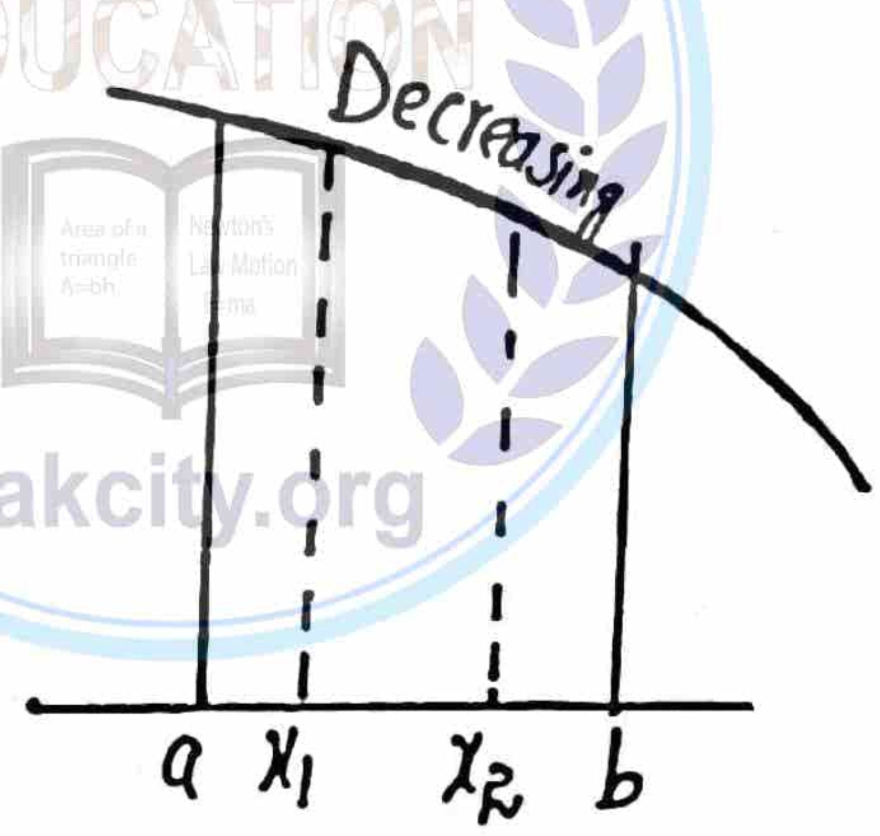
$$f(x_1) < f(x_2) \quad , \text{if } x_1 < x_2$$

Decreasing Function:-

Let f be defined on an Interval (a,b) and Let $x_1, x_2 \in (a,b)$, Then "f is decreasing on the Interval (a,b) if $f(x_1) > f(x_2)$, whenever $x_1 < x_2$

(Prepared By
Asad Ali Khalid
BS(H) Mathematics (GCUF)
M.Phill Mathematics (NTUF)
B.ed (AIOU).)

Khilji Academy
Mananwala).



$$f(x_1) > f(x_2) \quad , \text{if } \quad x_1 < x_2$$

Theorem:- Let f be a differentiable function on the open Interval (a,b). Then

(i) f is Increasing on (a,b) if $f'(x) > 0$ for each $x \in (a,b)$

(ii) f is Decreasing on (a,b) if $f'(x) < 0$ for each $x \in (a,b)$

pakcity.org

Derivative:- Slope of Tangent At a point of a Curve.

$$\left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \text{Slope of Tangent line}$$

$$f'(x) = \text{Tan}\boxed{\theta}$$

θ → Inclination → (Angle with +ve x-axis)

$\theta < 90^\circ \rightarrow \text{Tan}\theta > 0$

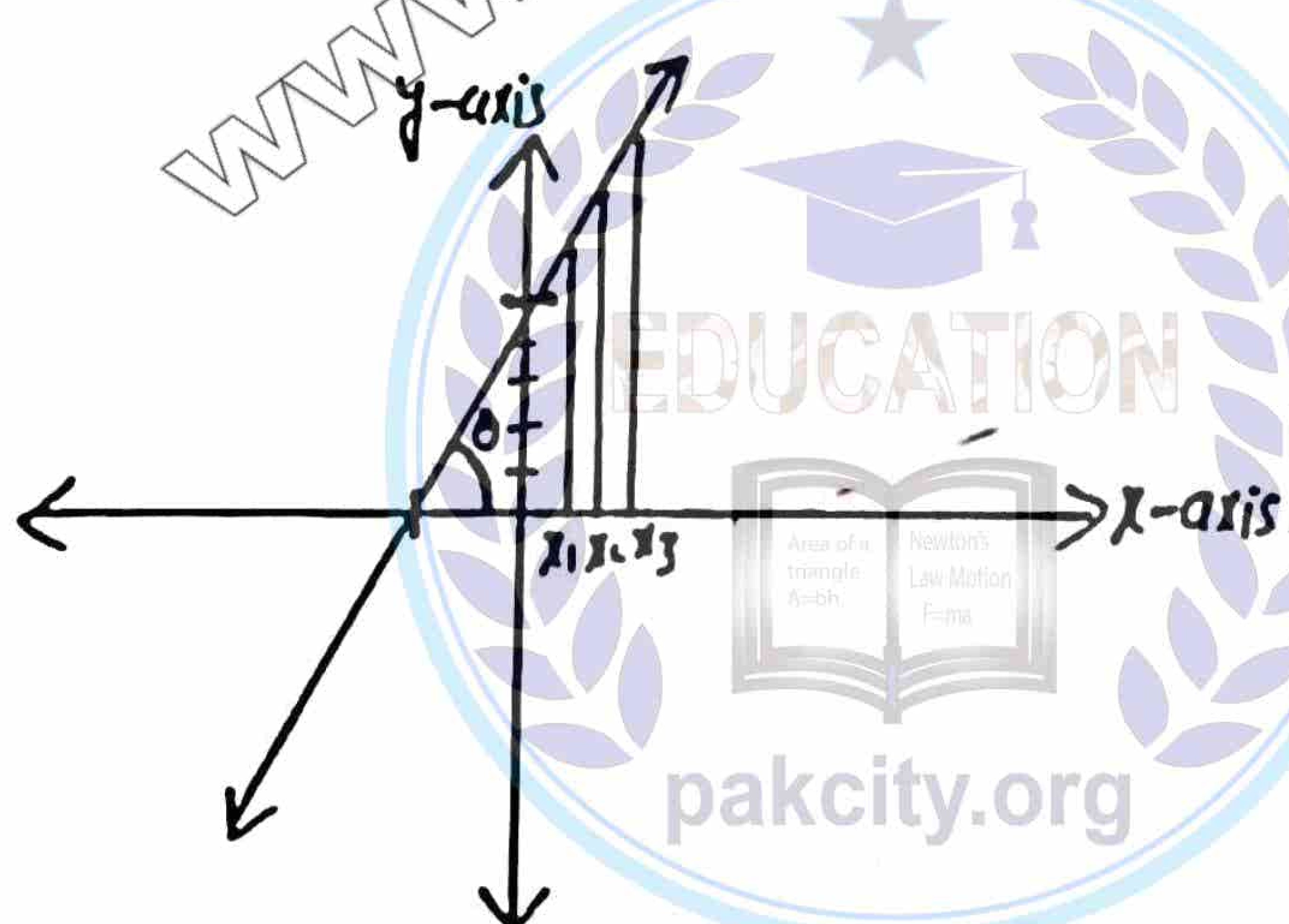
$\theta > 90^\circ \rightarrow \text{Tan}\theta < 0$

Examples:-

$y = x + 5$

$f(x) = \underset{m}{1}x + 5$ ← y-Intercept

$\boxed{y = mx + c}$ (Slope Intercept form of Straight Line)



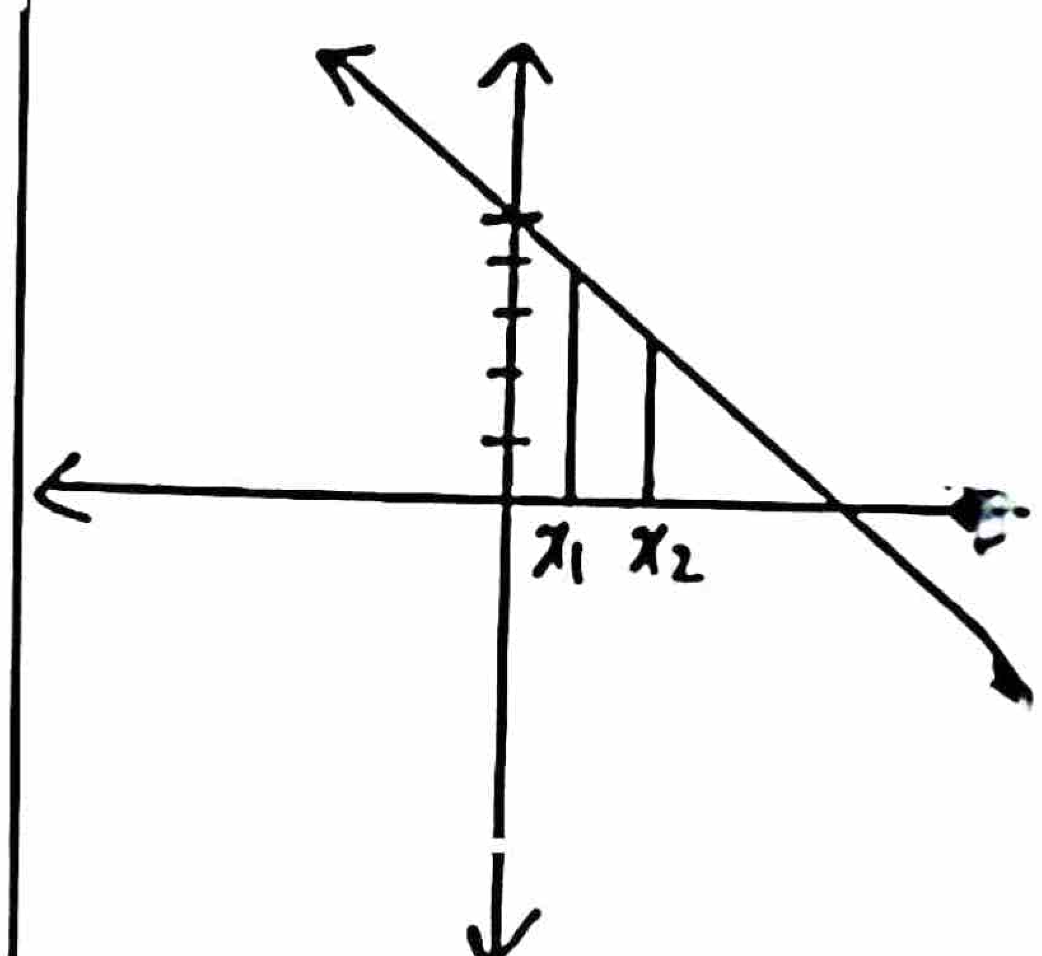
$f(x_1) < f(x_2)$ if $x_1 < x_2$

We See that As the values of x are Increasing The height of function Increasing So it is a Increasing Function.

$y = -x + 5$

$f(x) = -x + \boxed{5}$ ← y-Intercept

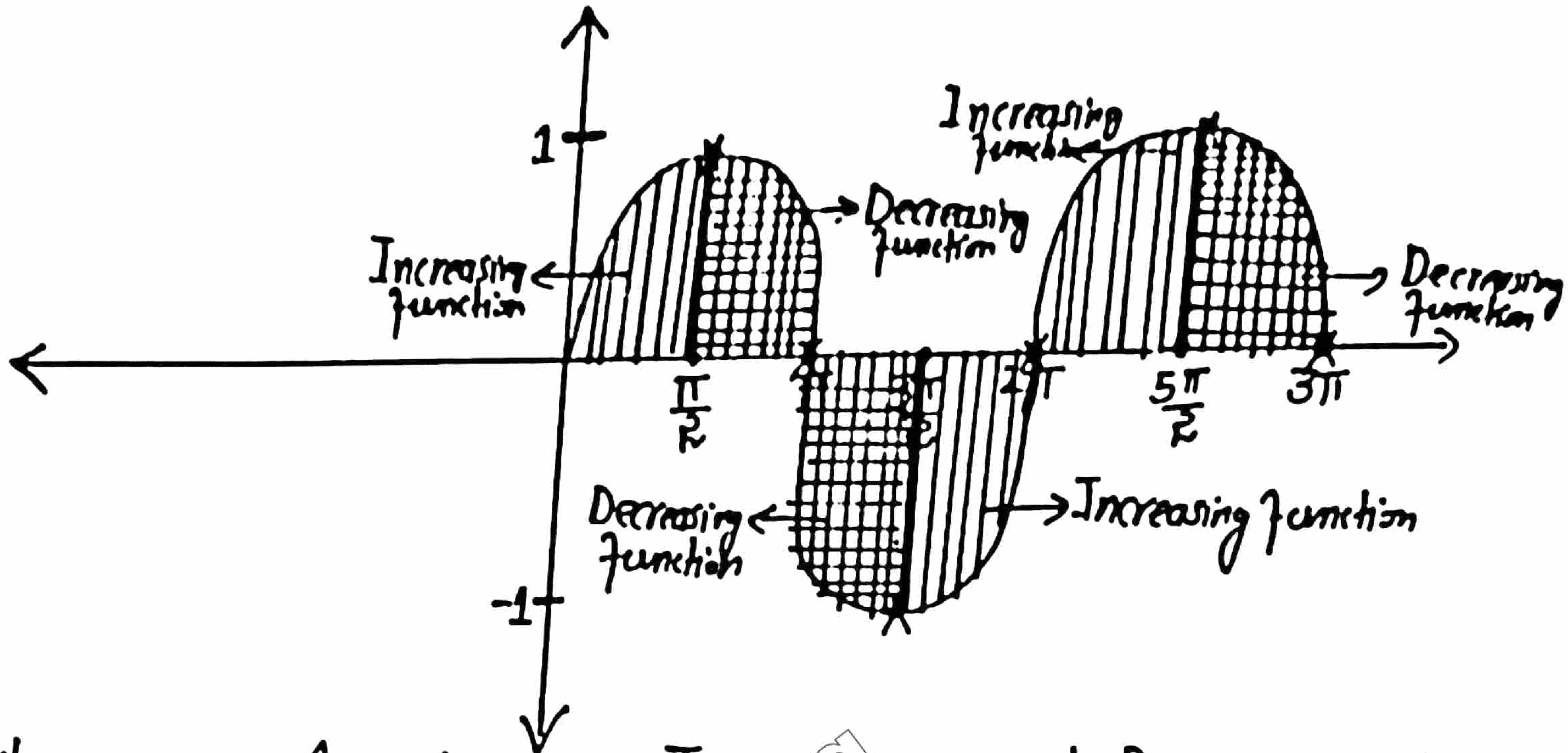
$y = mx + c$



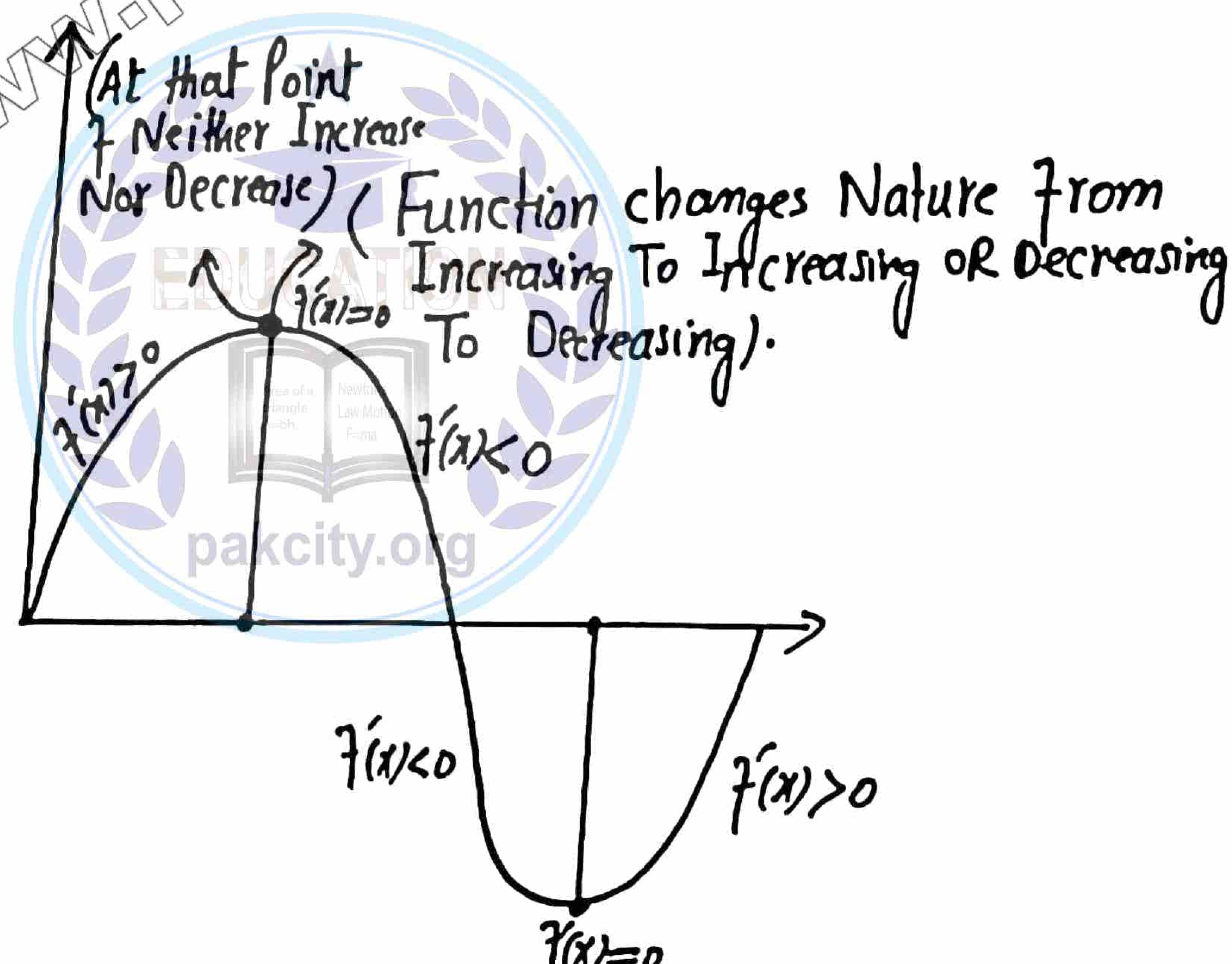
$f(x_1) > f(x_2), x_1 < x_2$

We see that As the Values of x are Decreasing, The height of the function decreases, So it is a Decreasing function.

Example:- $f(x) = \sin x$

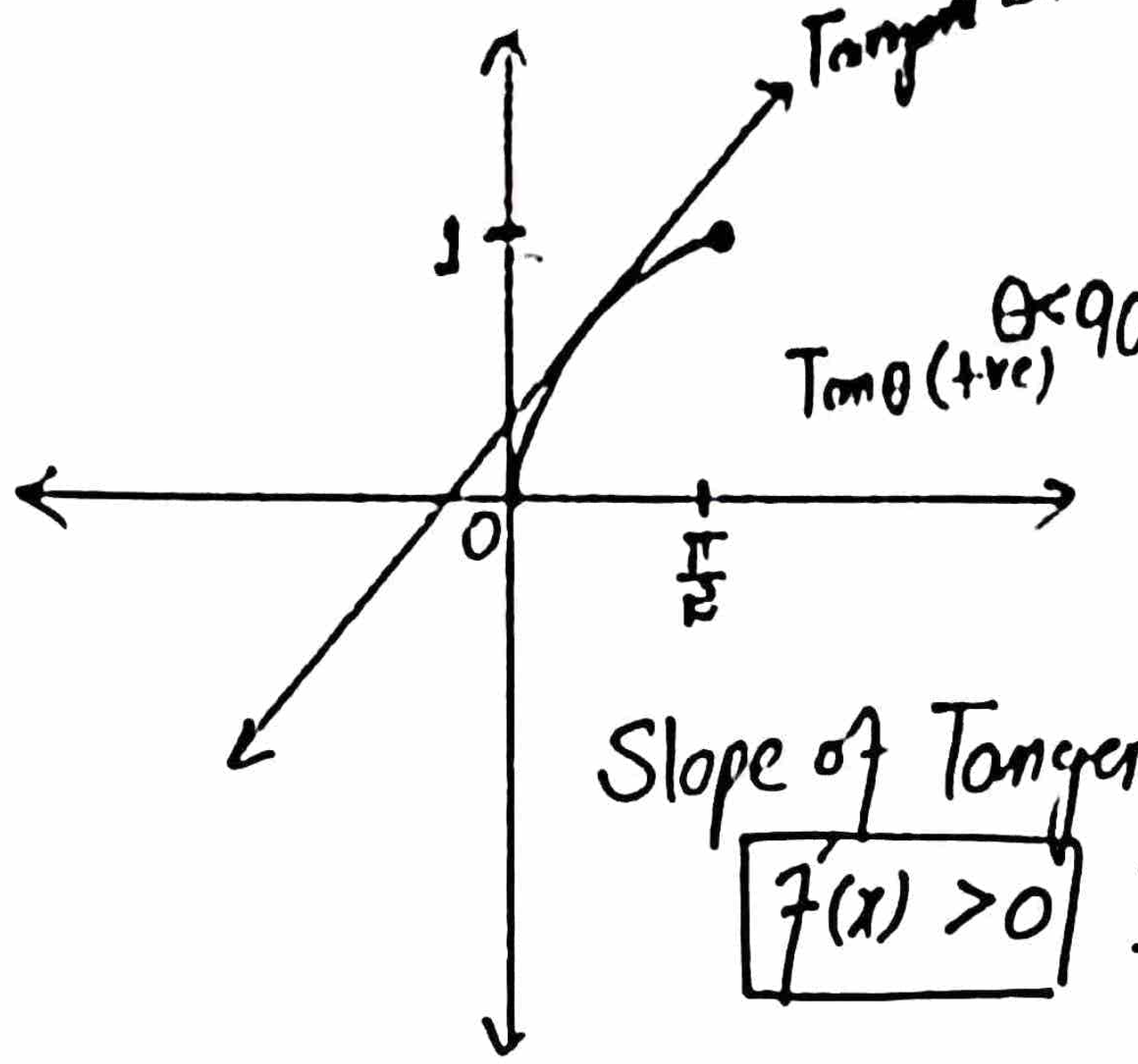


Whenever a function is Increasing and Decreasing in an specific Interval, Then we Say that function is (Neither Increasing Nor Decreasing).



Note:- Derivative is basically $\tan\theta$ Means put $\tan\theta$ at every point and see Angle, put Angle in $\tan\theta$ and see what is Answer.

We Restrict Domain $(0, \frac{\pi}{2})$

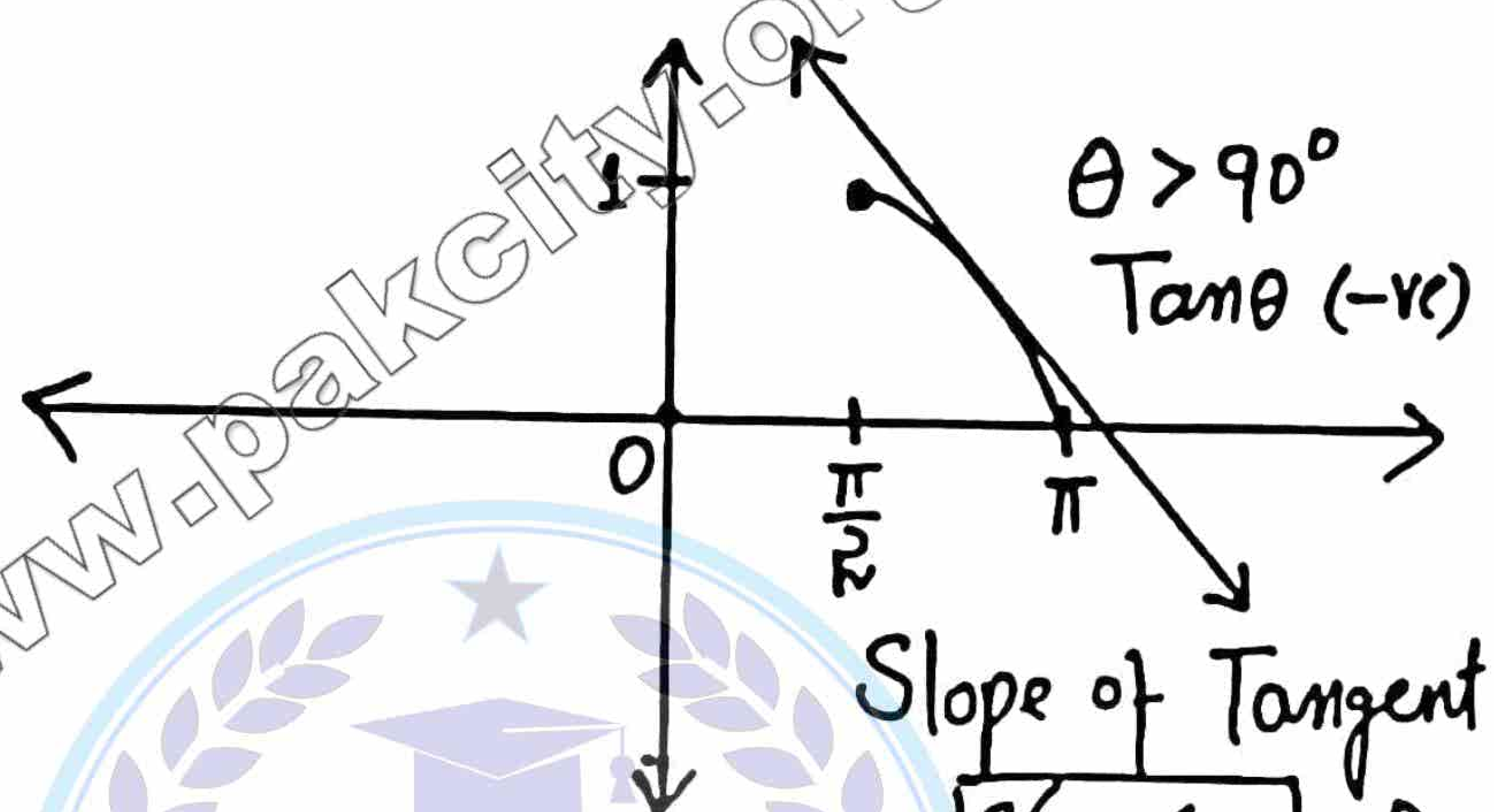


Note:- Angle should be along $+x$-axis in Anti-Clockwise direction.

Slope of Tangent line > 0 (+ve)

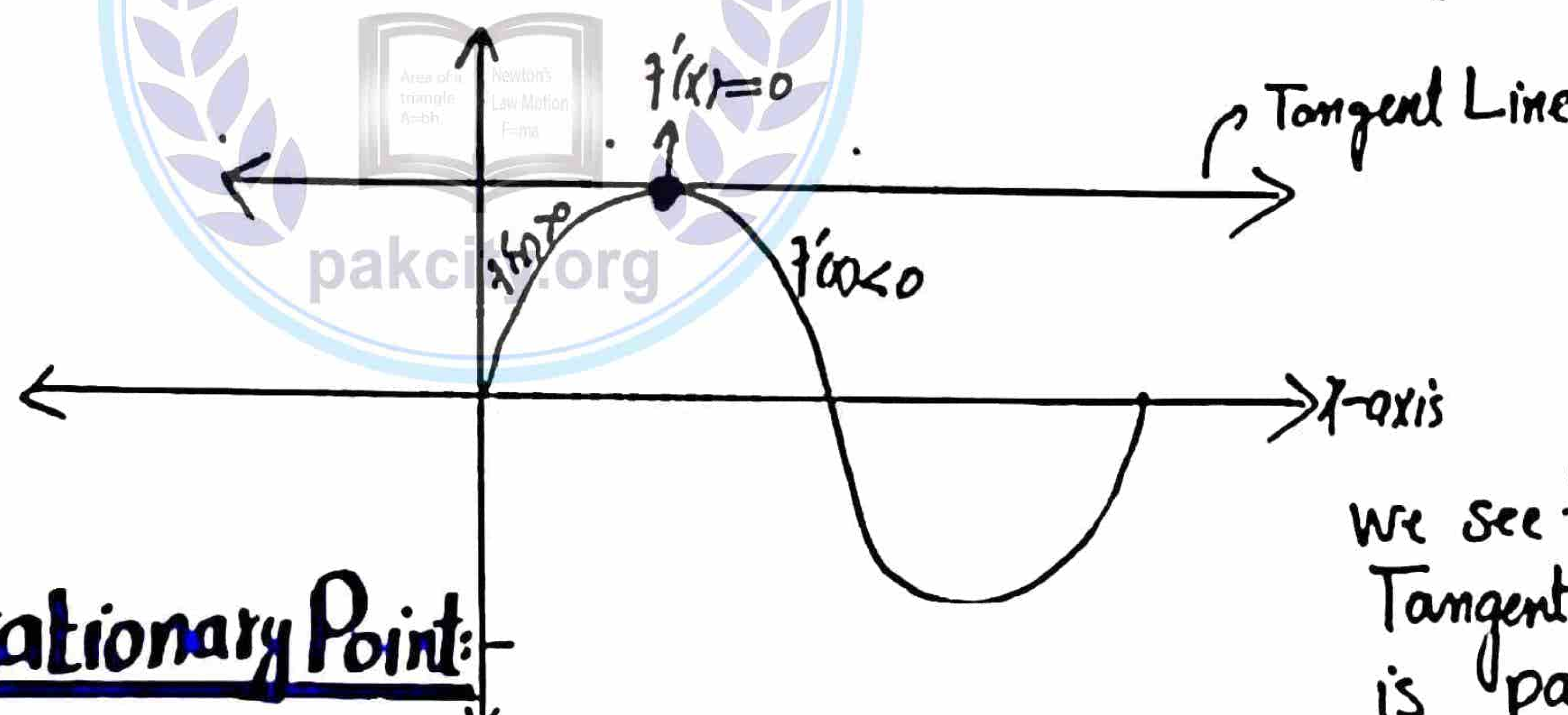
$\boxed{f'(x) > 0}$ Increasing function.

If Domain Restricted from $(\frac{\pi}{2}, \pi)$



Slope of Tangent line < 0 (−ve)

$\boxed{f'(x) < 0}$ Decreasing function.



We see that Tangent Line is parallel to x-axis.

$\boxed{\theta = 0}$

$\frac{dy}{dx} = \tan\theta = \tan 0$

$\boxed{\frac{dy}{dx} = 0}$

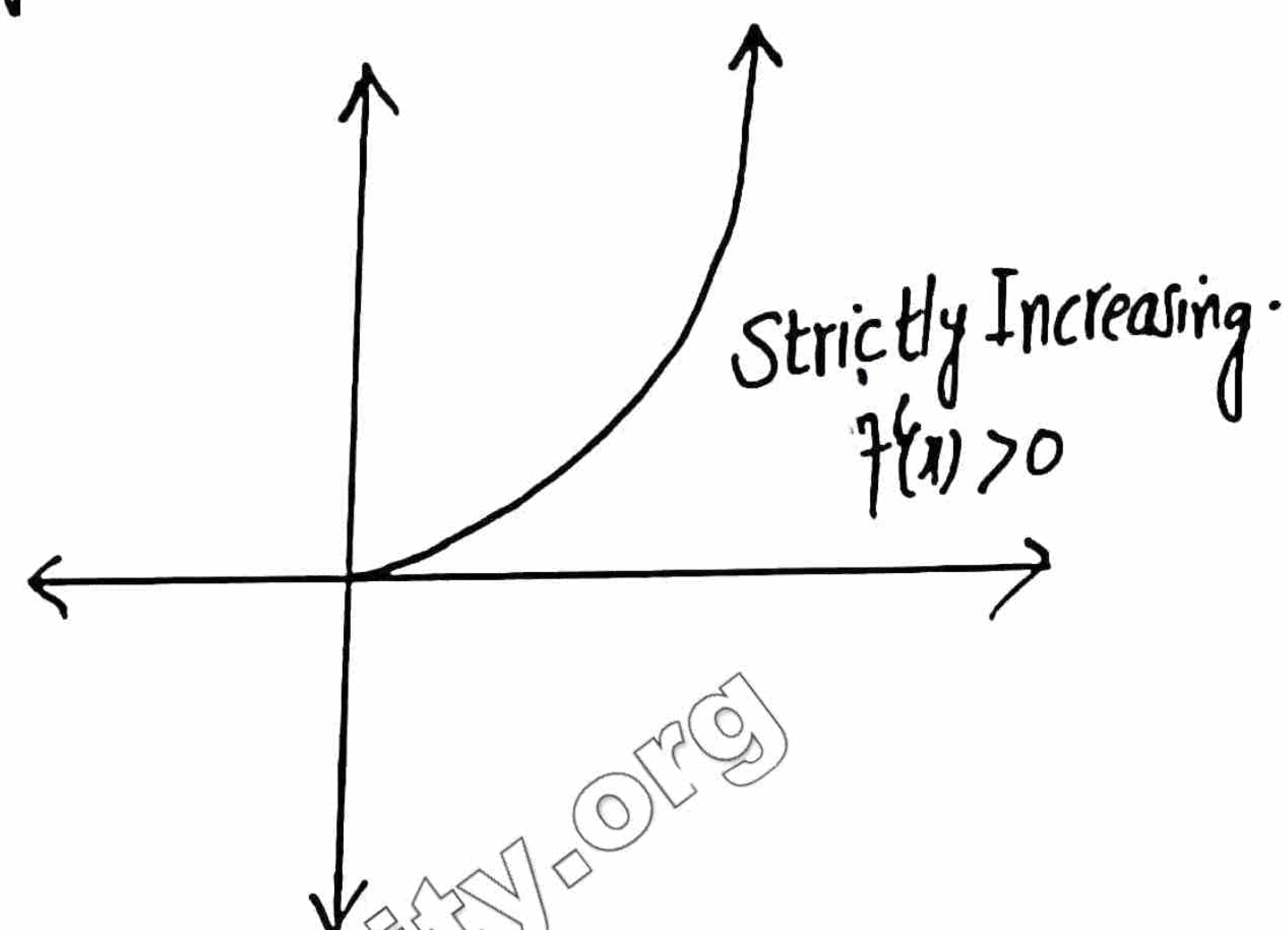
Stationary Point:-

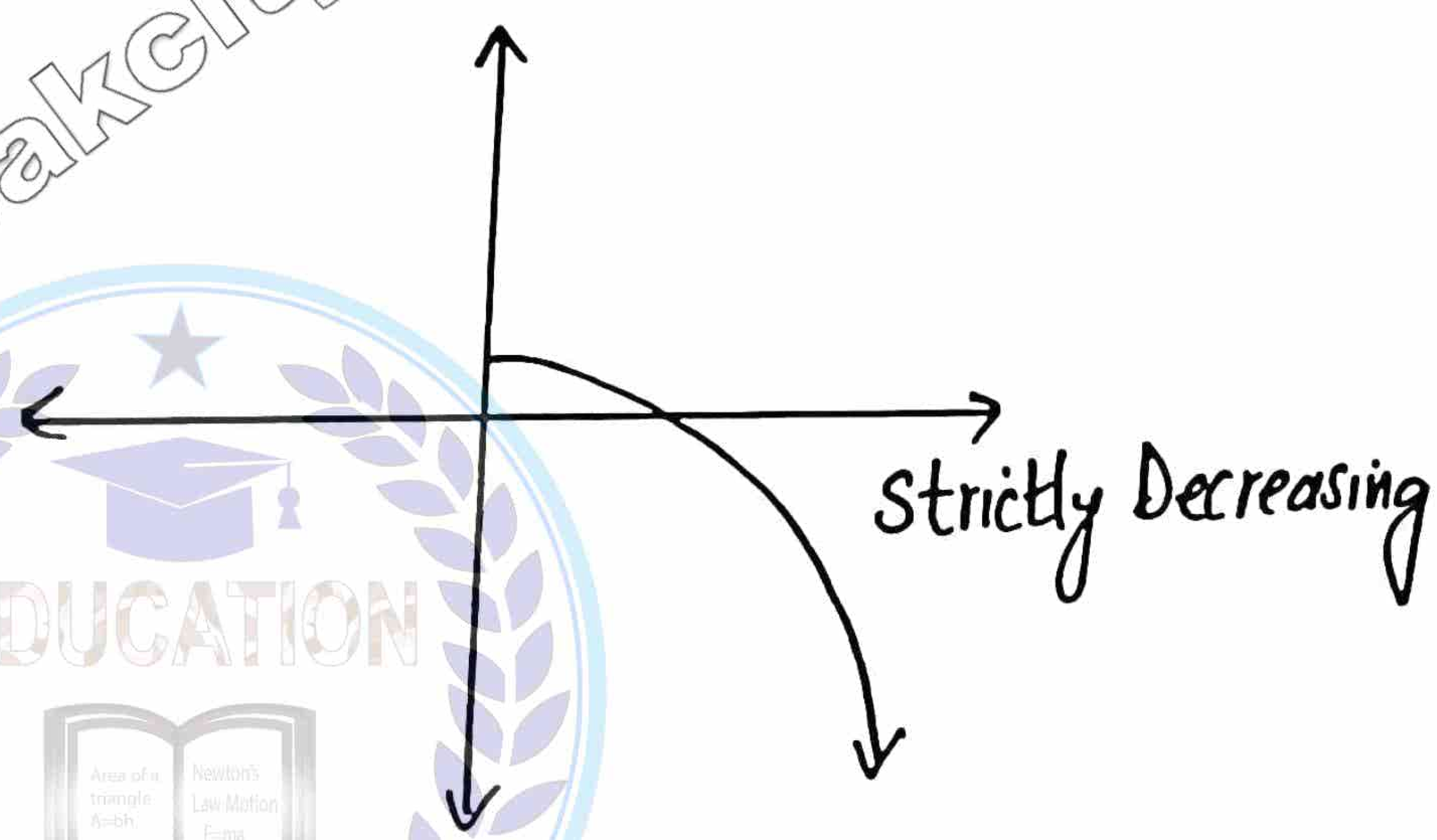
The Point At which The function is Neither Increasing Nor Decreasing i-e $f'(x) = 0$ is Known as Stationary Point.

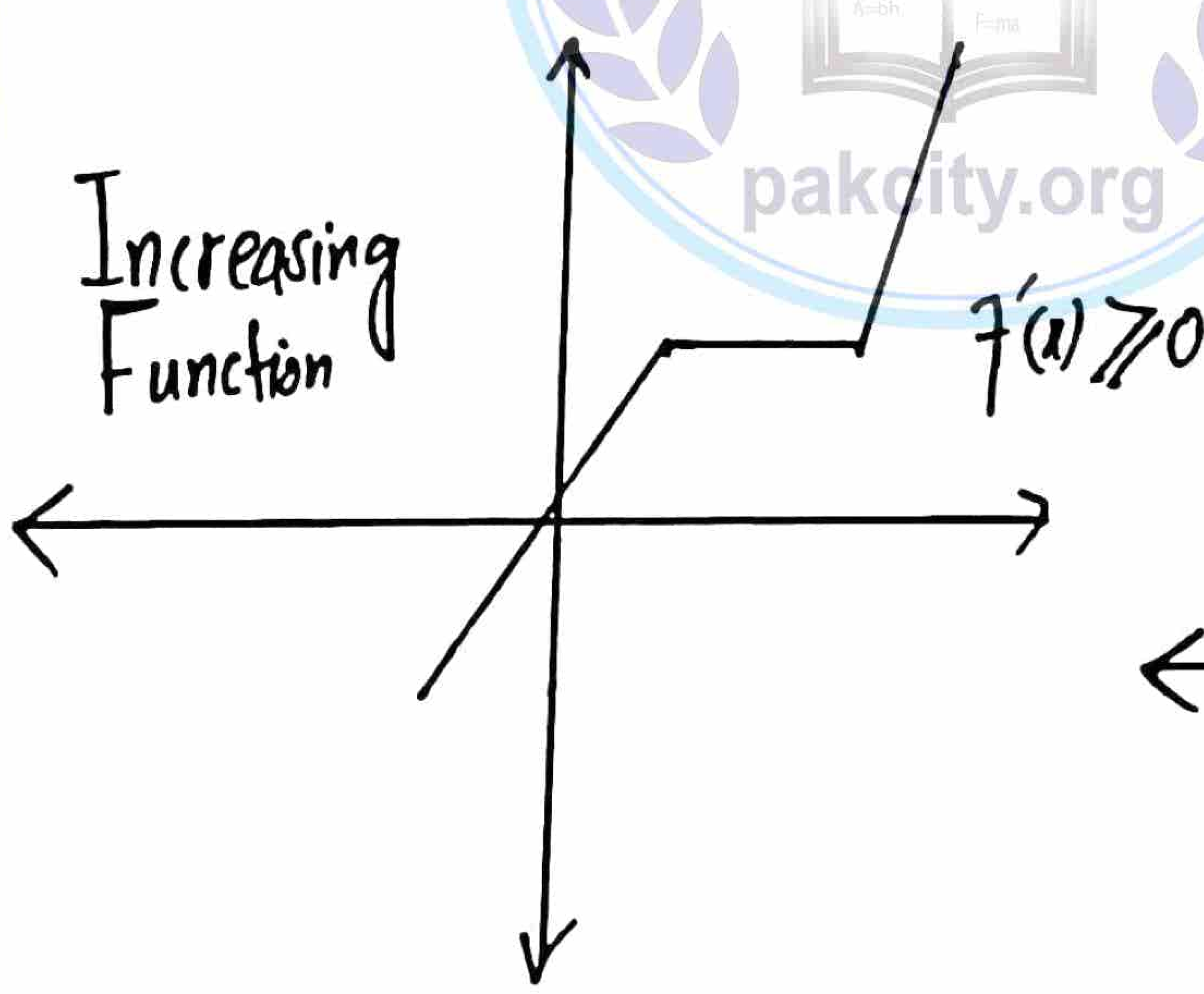
Key Points:-

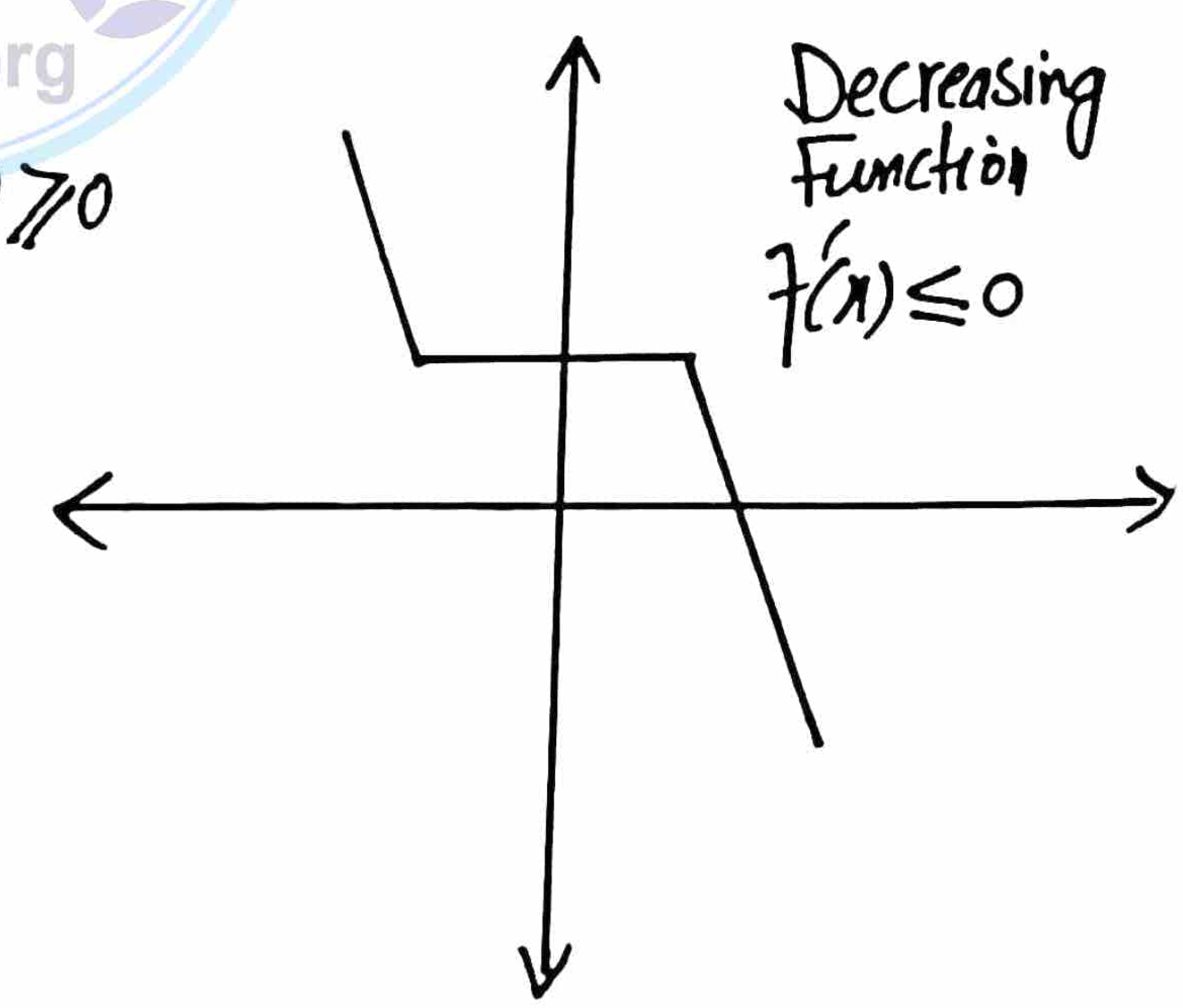
- ★ If $f'(x) > 0$, Then f is Increasing.
- ★ If $f'(x) < 0$, Then f is Decreasing.
- ★ If $f'(x) = 0$, Then f is Neither Increasing Nor Decreasing.

Different Types OF Graph:-









Increasing And Decreasing Function:-

Let $f(x)$ be differentiable in (a,b), Then

* If $f'(x) > 0$, $x \in (a,b)$, Then f is Increasing.
  ↳ Slope of Tangent line > 0
* If $f'(x) < 0$, $x \in (a,b)$, Then f is Decreasing.
  ↳ Slope of Tangent Line < 0
* If $f'(x) = 0$, $x \in (a,b)$
  ↳ Neither Increasing Nor Decreasing.

Question:

* Find the Values of x, where function is Increasing or Decreasing?
* Find the Interval where function is Increasing or Decreasing?
* Find the Values of x, where f is Neither Increase Nor Decrease?

Example:-

$$f(x) = x^2$$

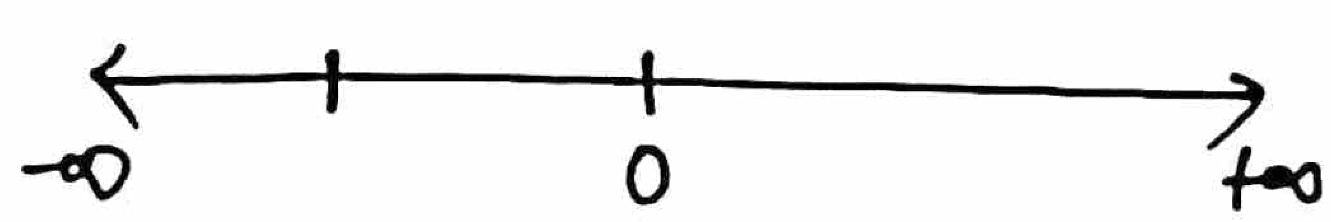
$$f'(x) = 2x$$

For Stationary Point

$$f'(x) = 0$$

$$2x = 0$$

$$\boxed{x = 0}$$



$(0, +\infty)$ Increasing function

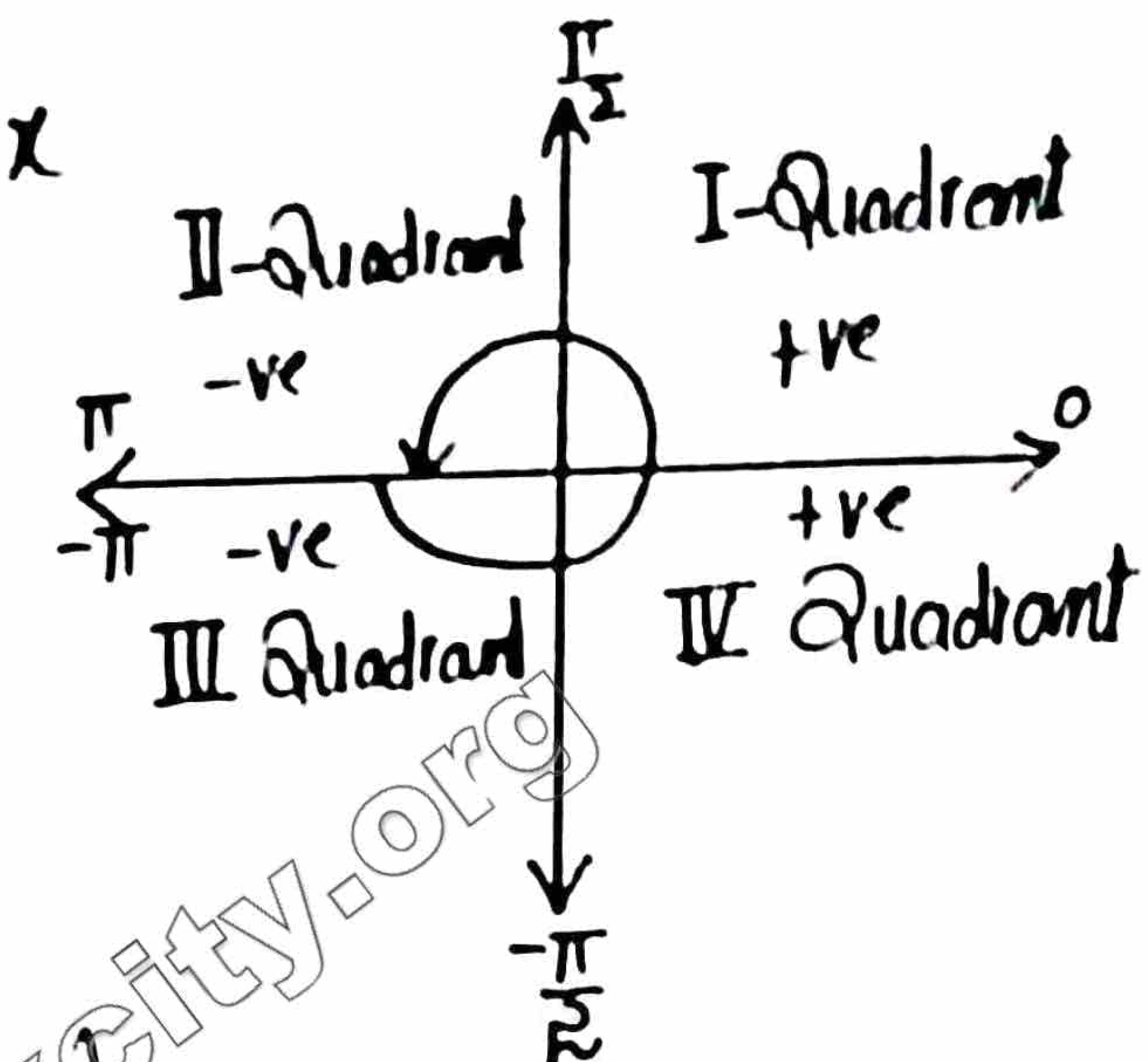
$(-\infty, 0)$ Decreasing function

Exercise 2.9

Determine the Interval in which f is Increasing or Decreasing for the Domain Mentional in each case.

(i) $f(x) = \sin x \quad , \quad x \in (-\pi, \pi)$

$f'(x) = \cos x$



For Increasing f

$\left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$

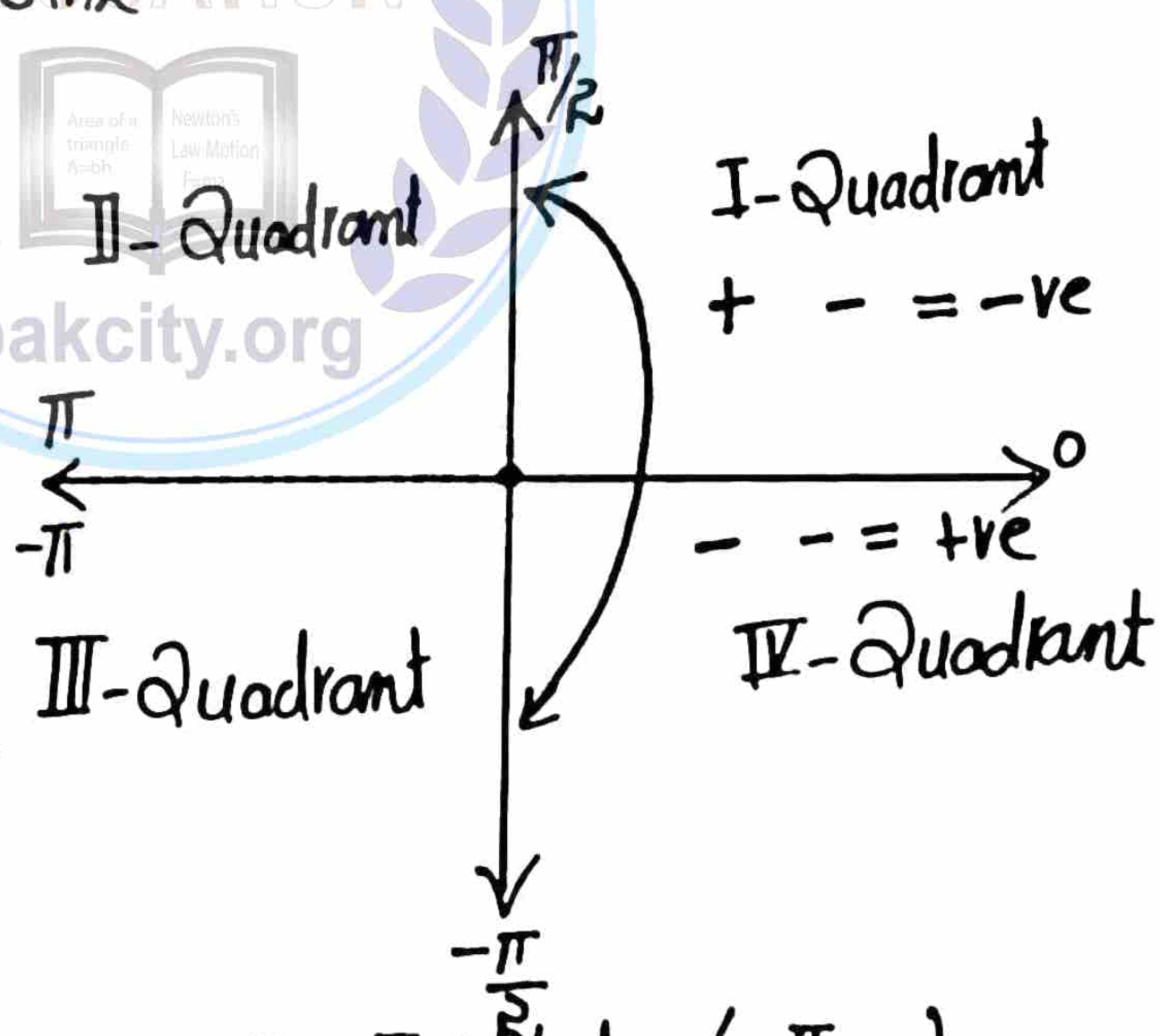
$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

For Decreasing f

$\left(-\pi, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

(ii) $f(x) = \cos x \quad , \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$f'(x) = -\sin x$



For Increasing f

$\left(-\frac{\pi}{2}, 0\right)$

For Decreasing f

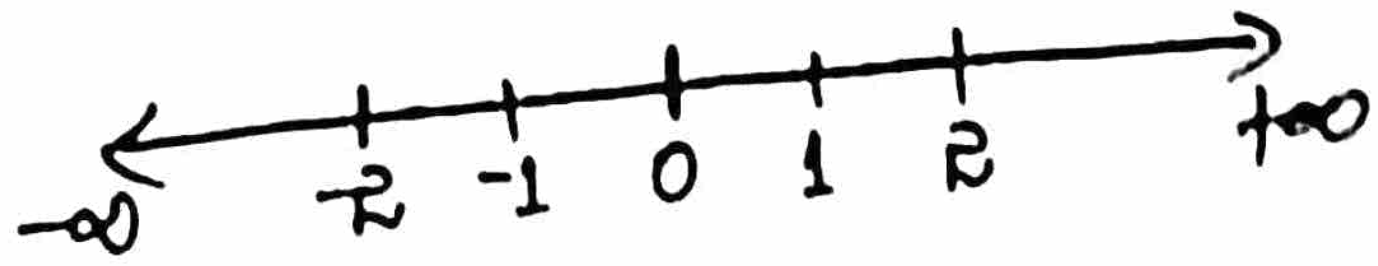
$\left(0, \frac{\pi}{2}\right)$

f is Increasing in the Interval $\left(-\frac{\pi}{2}, 0\right)$

f is Decreasing in the Interval $\left(0, \frac{\pi}{2}\right)$

(iii) $f(x) = 4 - x^2 \,,\; x \in (-2, 2)$

$f'(x) = -2x$



Put $f'(x) = 0$

$-2x = 0$

$\boxed{x = 0}$

For Increasing f :-

f is Increasing in the Interval $(-2, 0)$

For Decreasing f :-

f is Decreasing in the Interval $(0, 2)$

(iv) $f(x) = x^2 + 3x + 2 \;,\; x \in (-4, 1)$

$f'(x) = 2x + 3$

Put $f'(x) = 0$

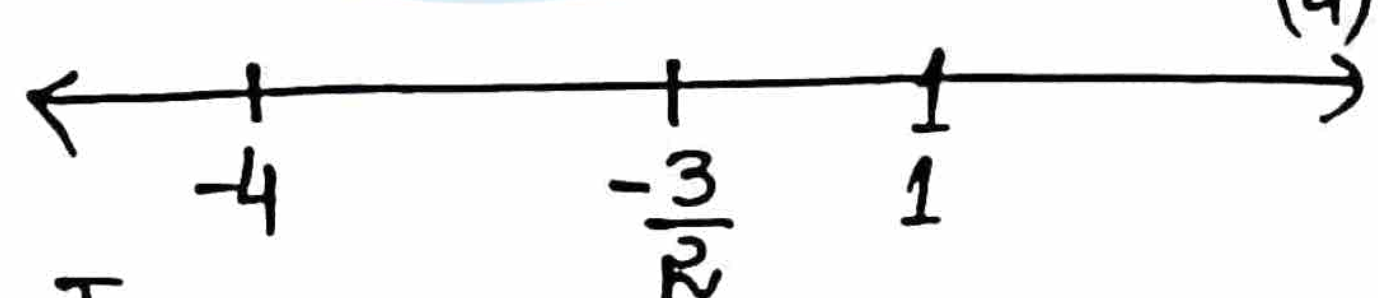
$2x + 3 = 0$

$2x = -3$

$\boxed{x = \frac{-3}{2}}$

Four Steps To find Increasing OR Decreasing Function:-

(1) Differentiate w.r.t "x".
(2) For Inc or Dec put $f'(x) = 0$
(3) Draw Number Line. $-\infty \longleftrightarrow +\infty$
(4) Interval Sign Result.



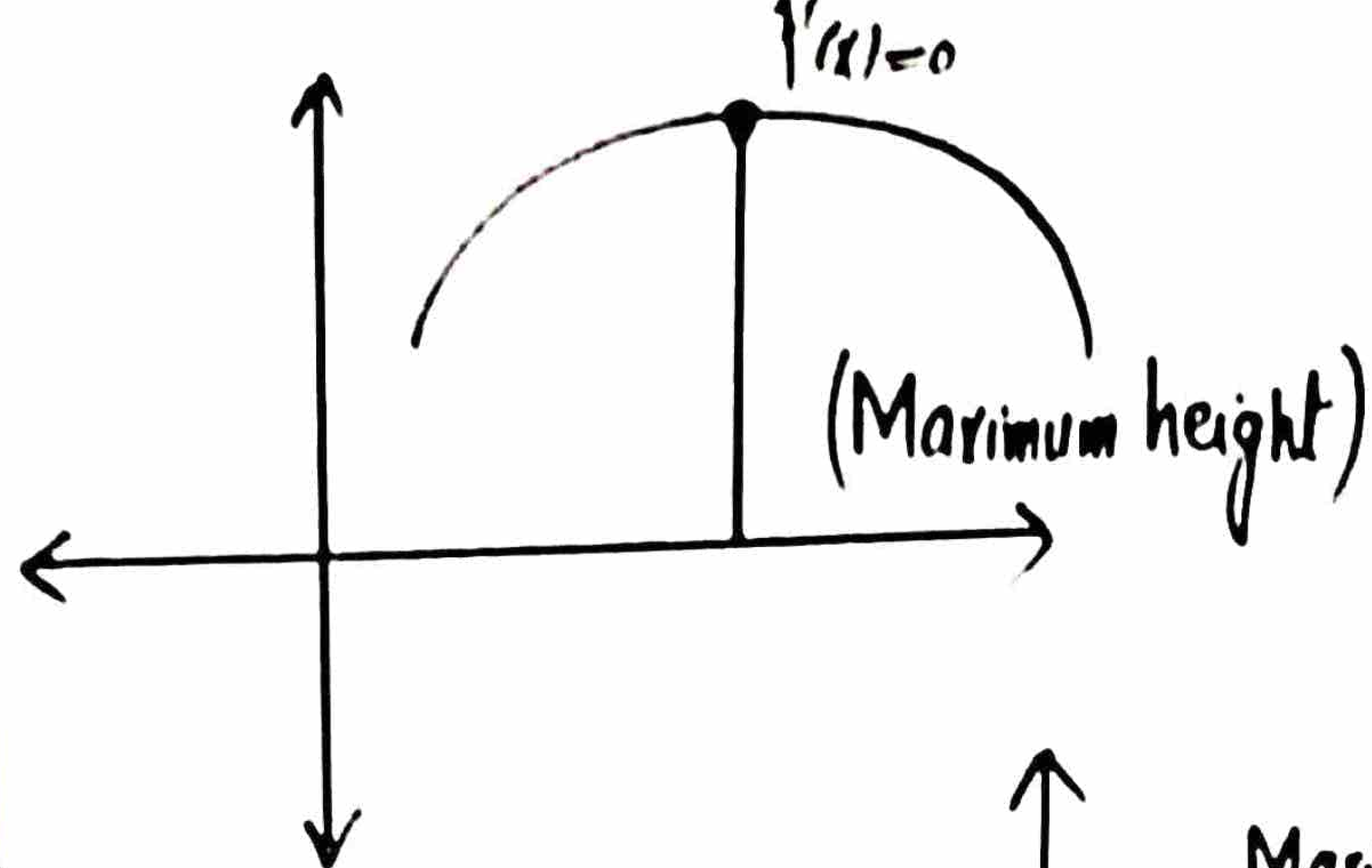
For Increasing f :-

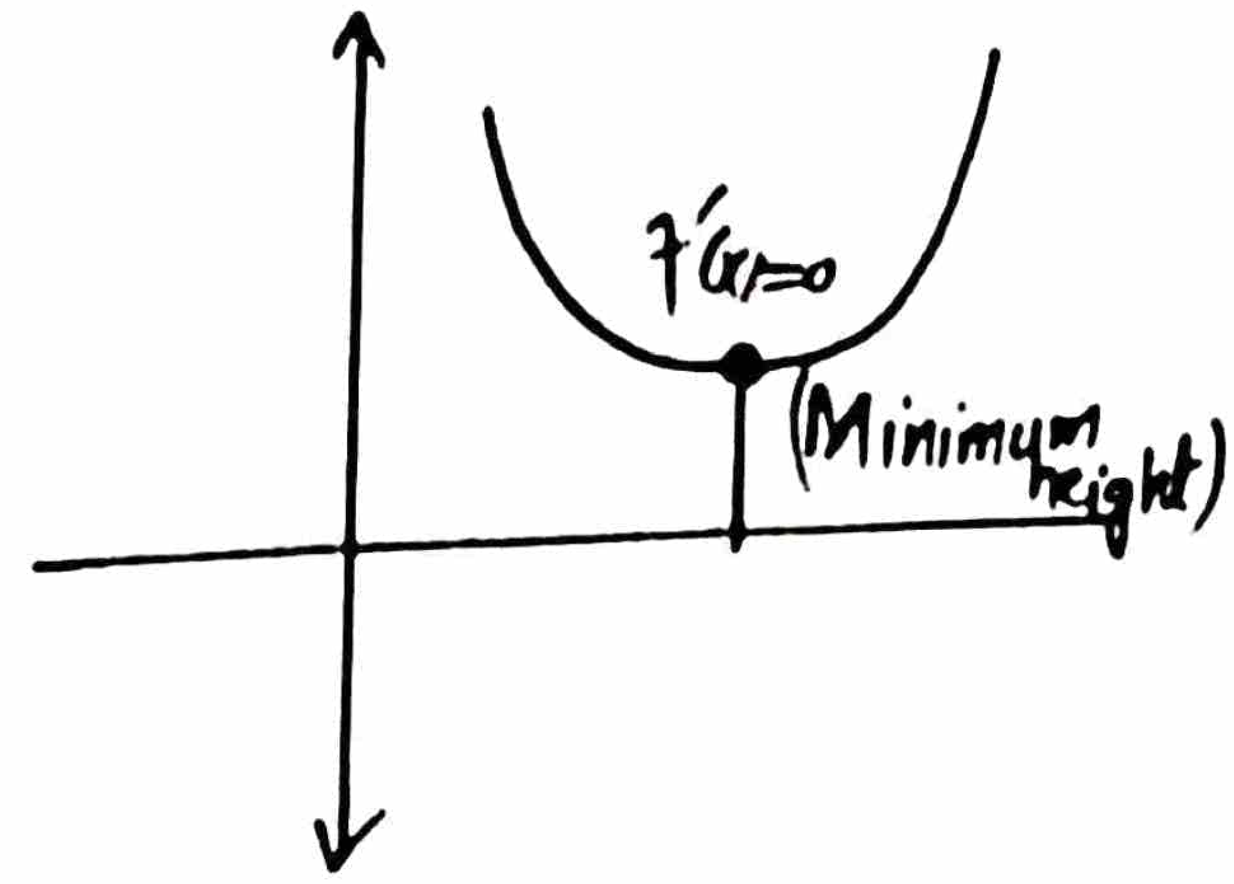
f is Increasing in the Interval $\left(-\frac{3}{2}, 1\right)$

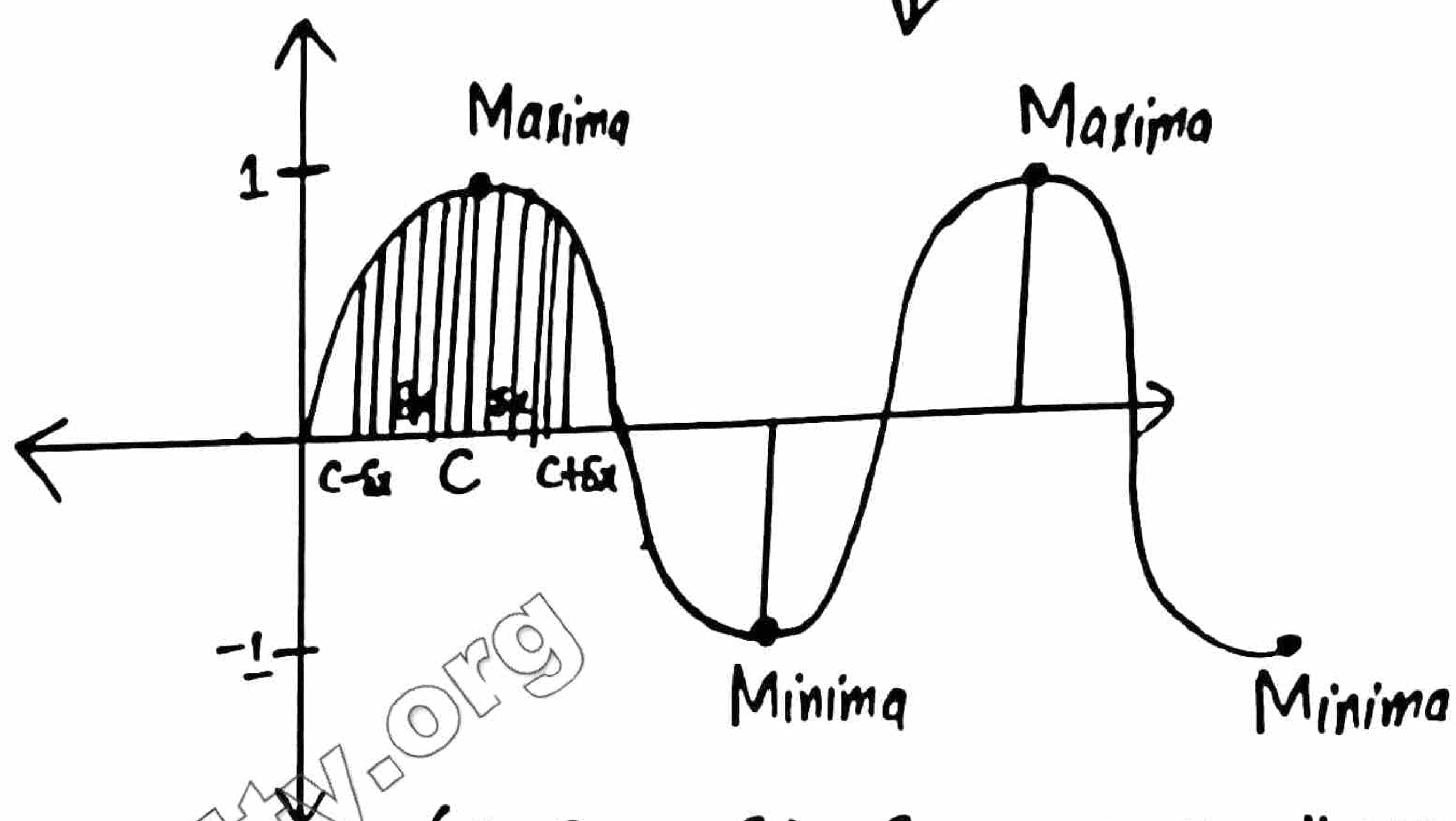
For Decreasing f :-

f is Decreasing in the Interval $\left(-4, -\frac{3}{2}\right)$

Relative Extrema:-





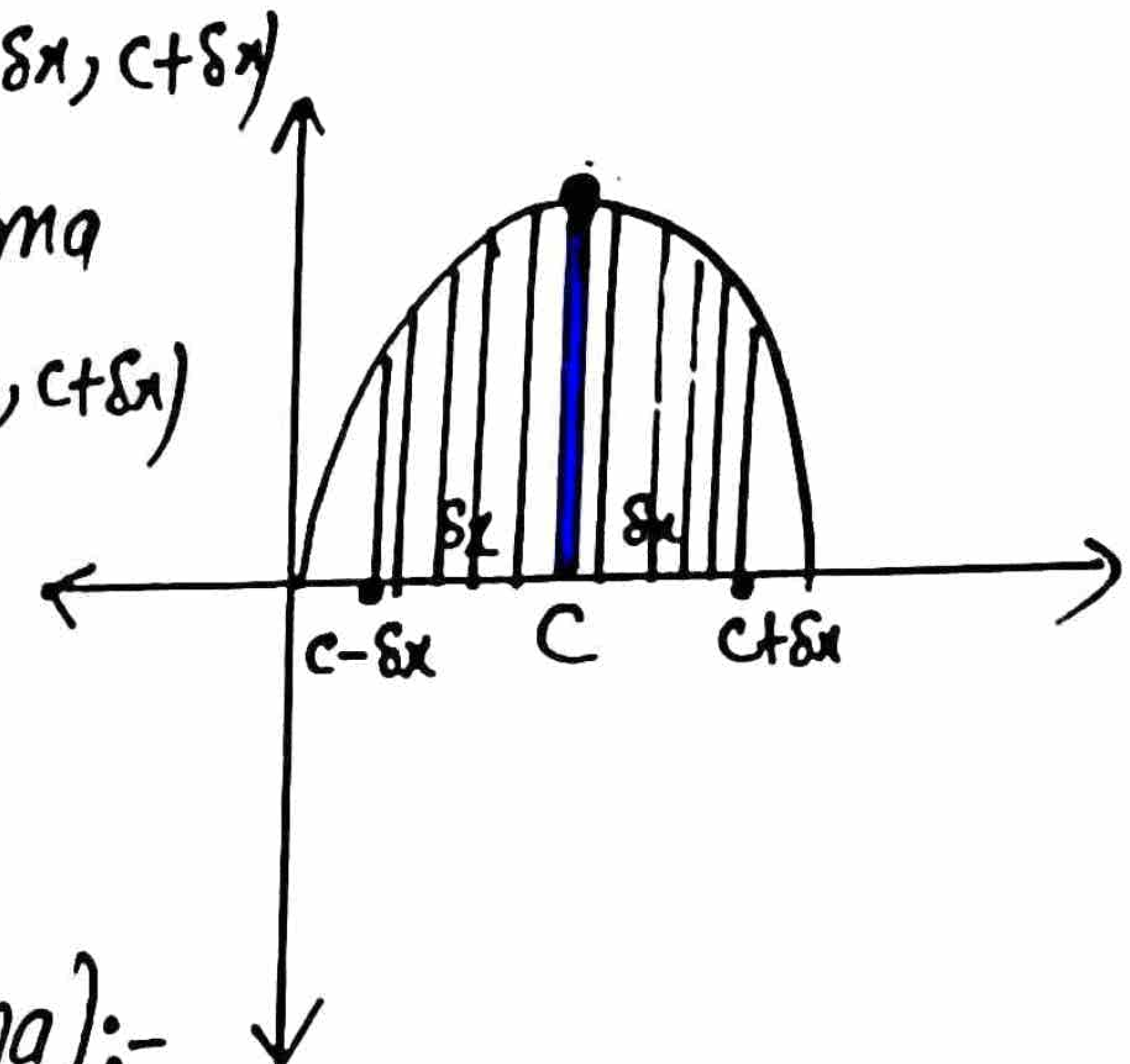


$(c-\delta x, c+\delta x)$, δx is very small +ve Number.

Relative Maxima (Local Maxima):-

A function $y=f(x)$, where $x \in (c-\delta x, c+\delta x)$ is said to be Relative Maxima or Local Maxima at $c \in (c-\delta x, c+\delta x)$ if

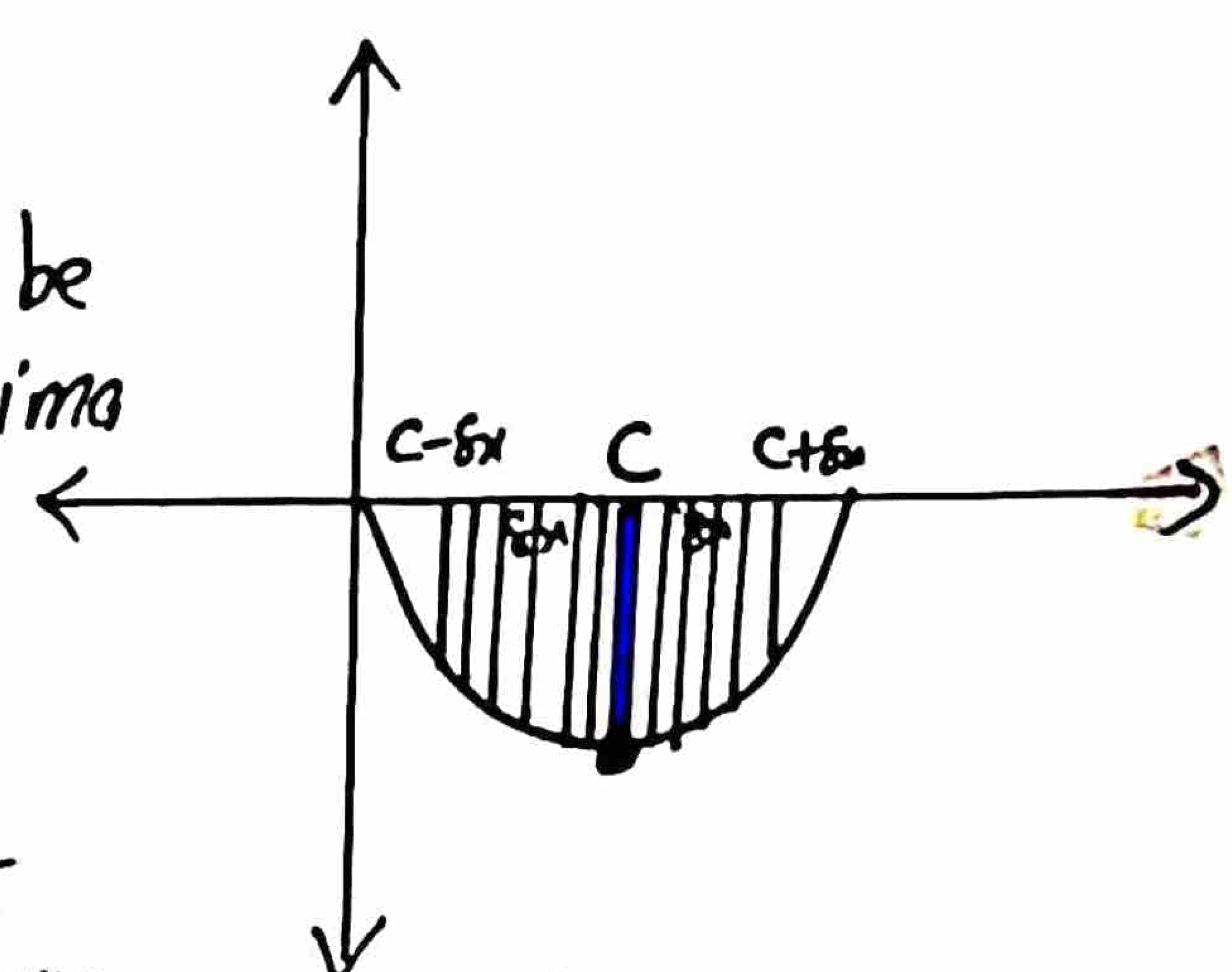
$$f(x) \leq f(c)$$



Relative Minima (Local Minima):-

A function $y=f(x)$, where $x \in (c-\delta x, c+\delta x)$ is said to be Relative Minima or Local Minima at $c \in (c-\delta x, c+\delta x)$ if

$$f(x) \geq f(c)$$



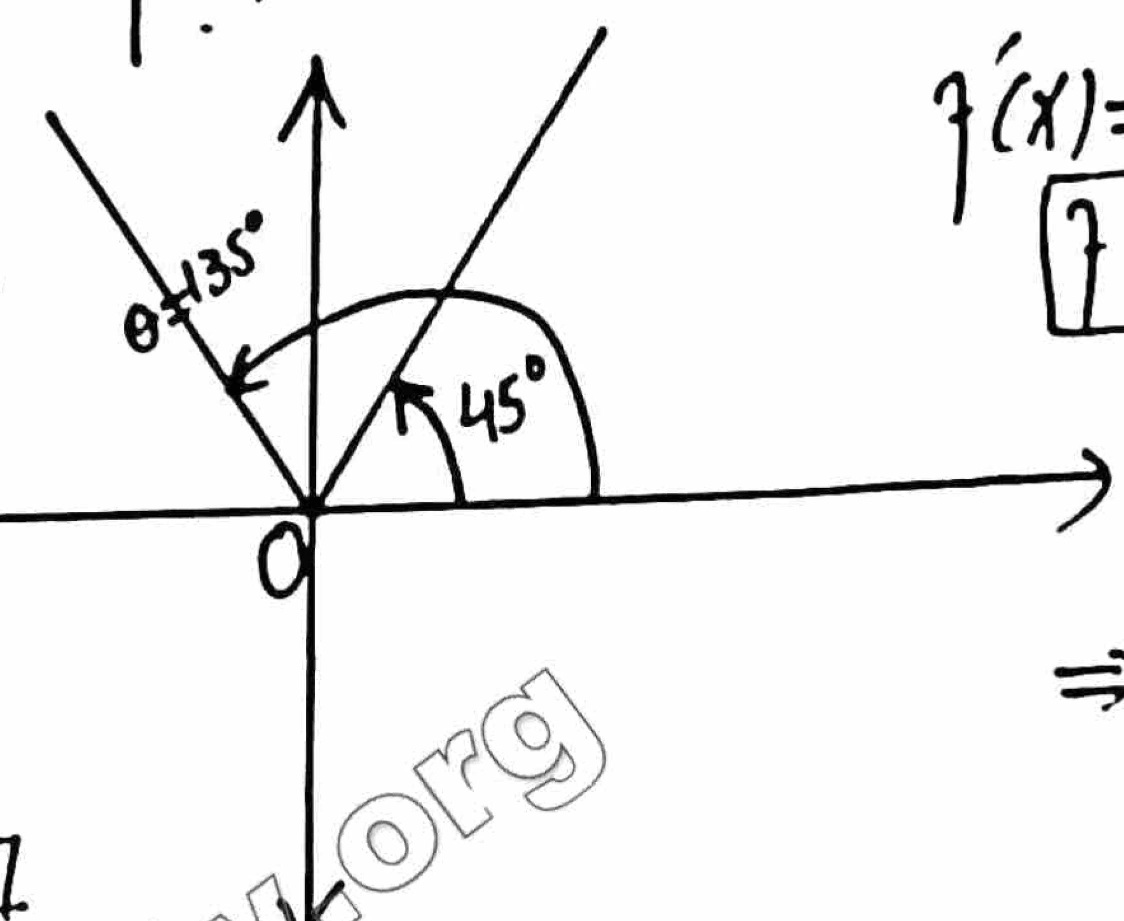
Turning Point:- Stationary point where function is either Maximum or Minimum is called Turning Point.

Critical Point:-

A function $y=f(x)$ $c \in D_f$ if $f'(c)=0$ or $f'(c)$ does not exist, The Number c is called Critical Value while the point $(c, f(c))$ on the graph of f is called Critical point.

Example:- $f(x) = |x|$

$f'(x) = \tan\theta$
$f'(x) = \tan 135°$
$\boxed{f'(x) = -1}$



$f'(x) = \tan\theta = \tan 45° = 1$
$\boxed{f'(x) = 1}$

$\boxed{x=0}$
$\boxed{y=0}$
$O(0,0)$ is Critical point.

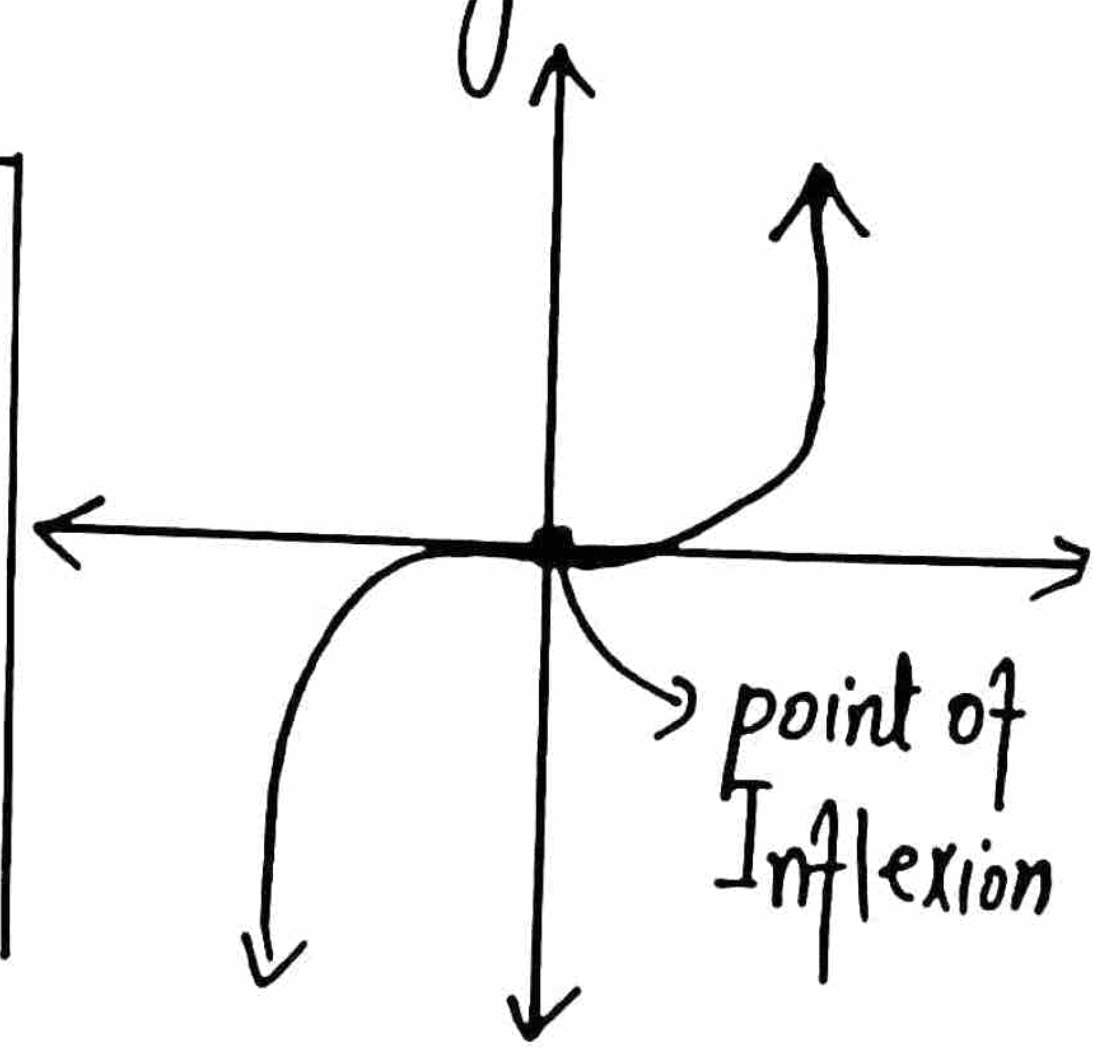
$\Rightarrow \left.\begin{array}{l} f'(x)=1 \\ f'(x)=-1 \end{array}\right\}$

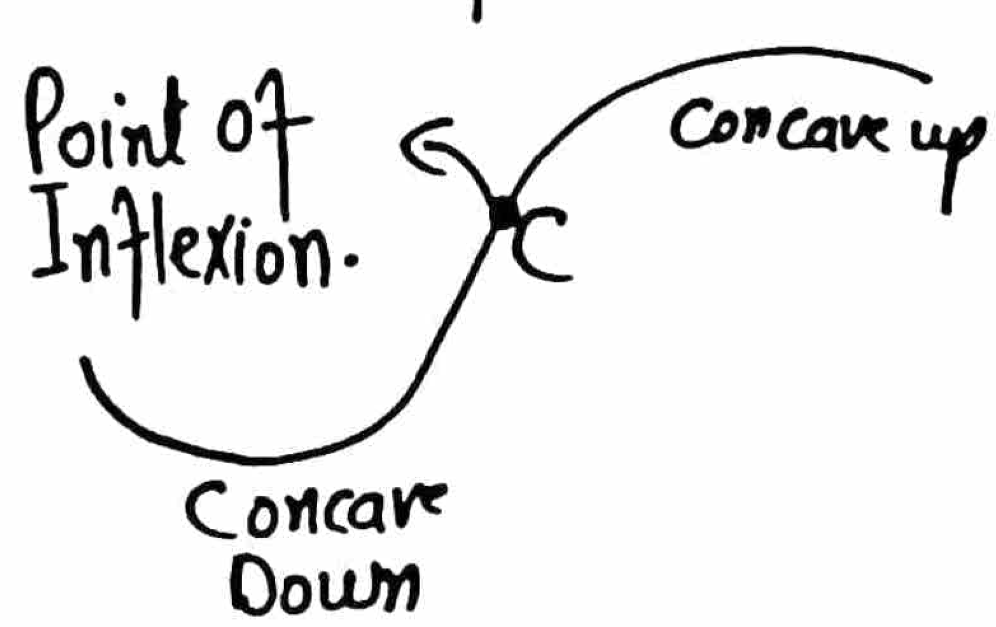
Derivative is not Unique.
First Derivative does not exist at $x=0$.

Point OF Inflexion:-

A Turning Point where the function does not change Sense, that is, the function changes from Increasing To Increasing or Decreasing To Decreasing is called Point of Inflexion.

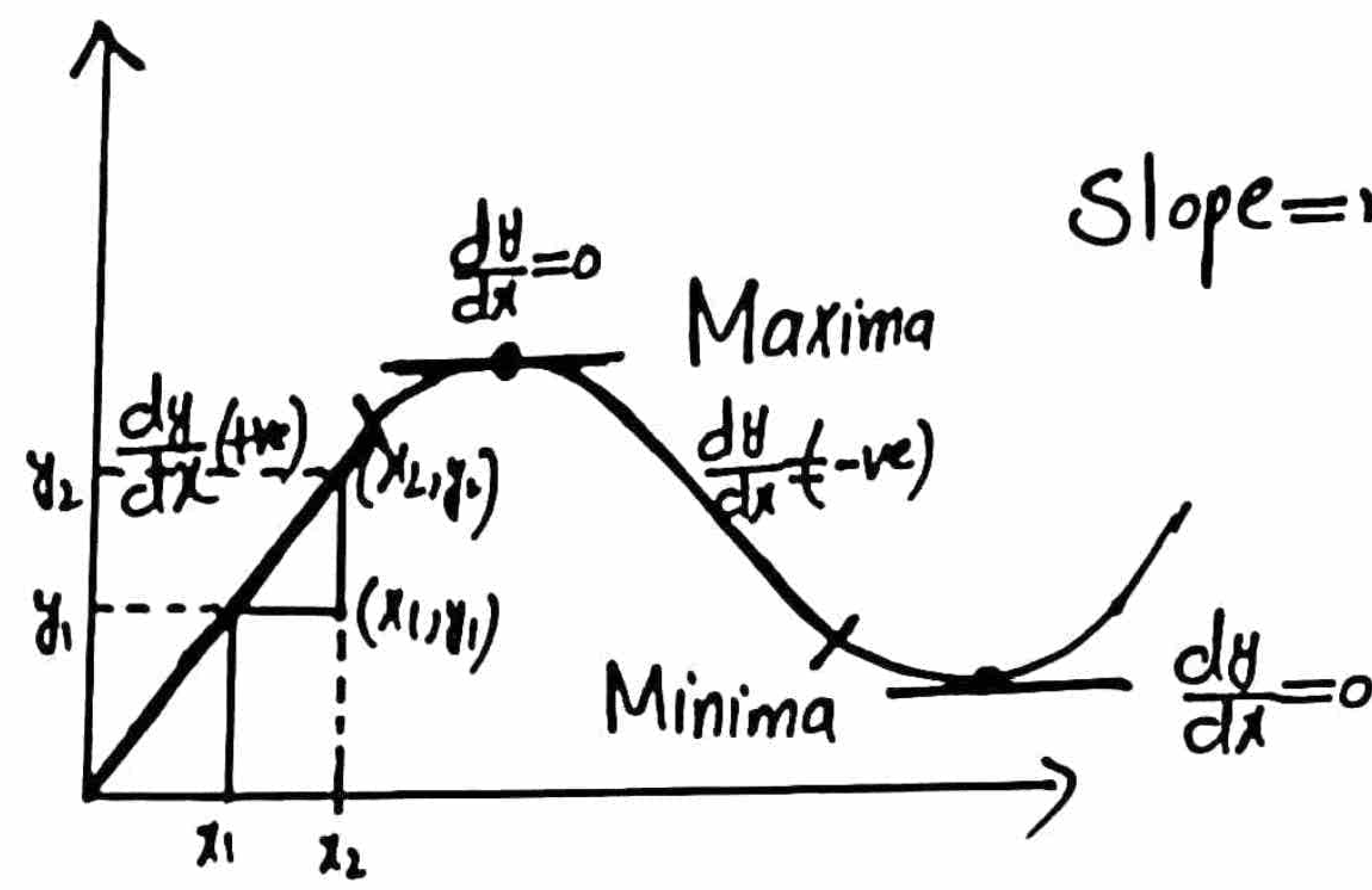
OR A point "c" on the Curve $y=f(x)$ is called point of Inflexion if f is Concave up on one side of c and Concave down on other side of c.





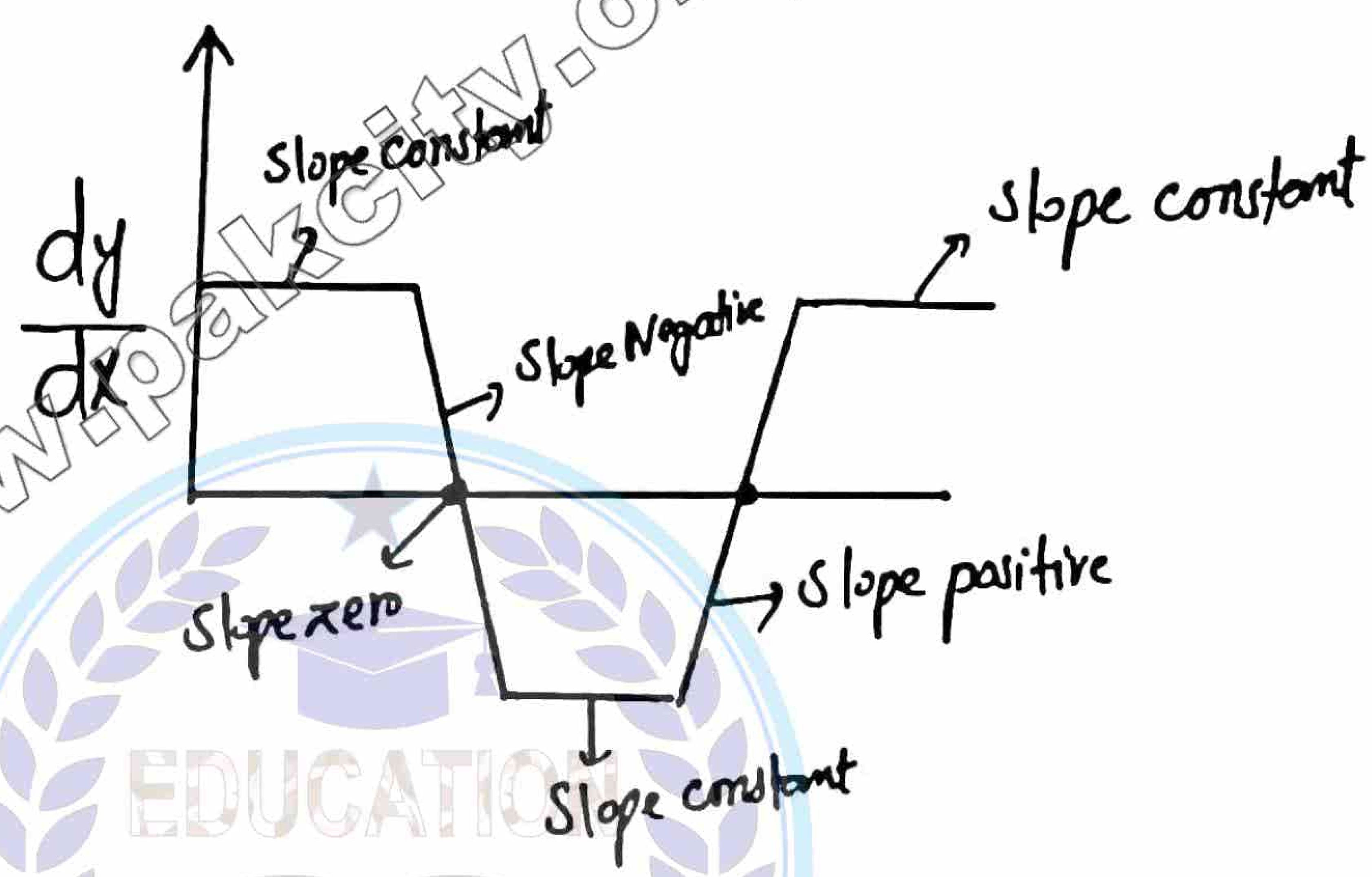
Differentiation
Graph

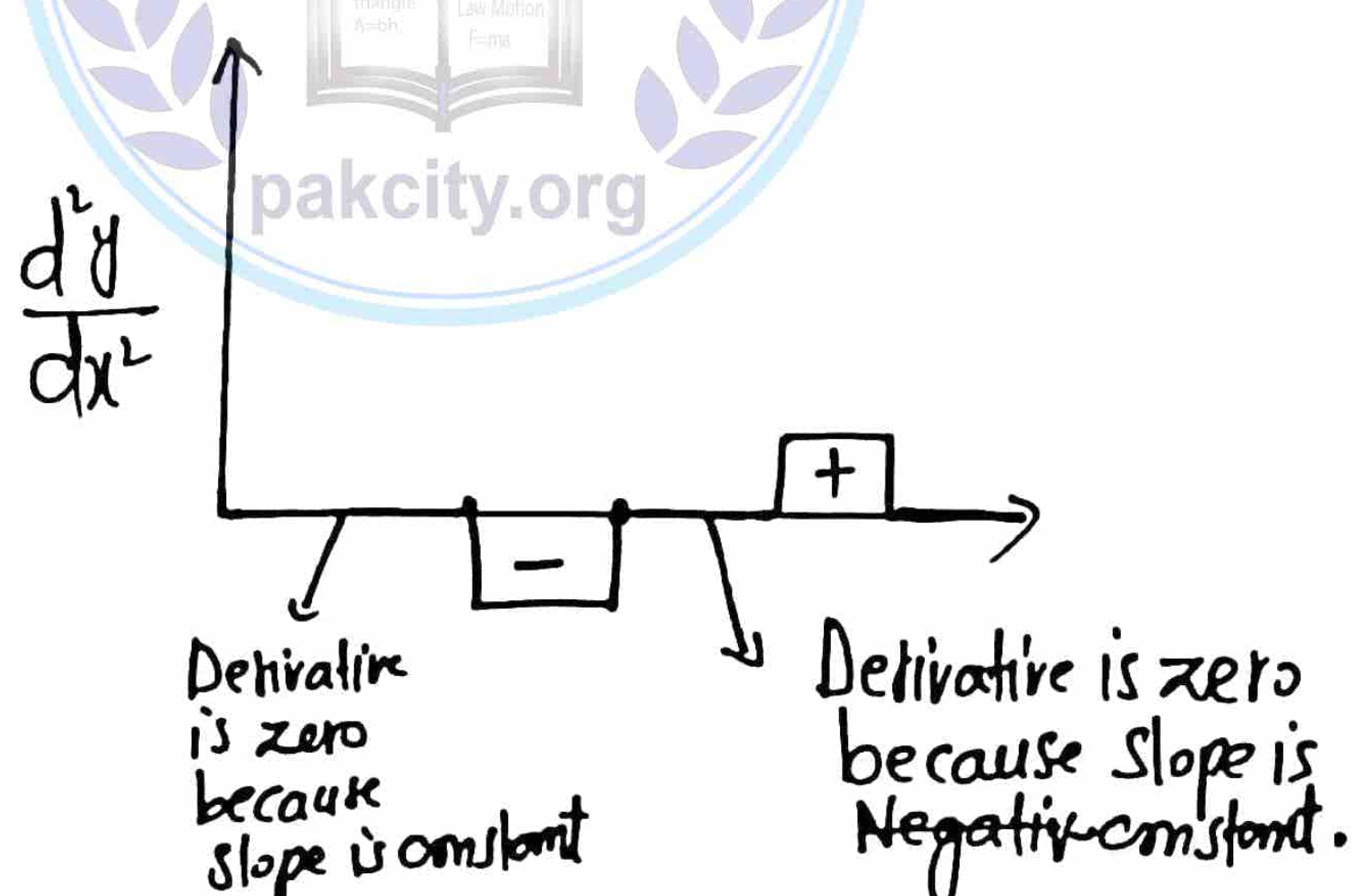
Relative Maxima And Relative Minima:-



$$Slope = m = \frac{y_2 - y_1}{x_2 - x_1}$$

Now If Draw Slope VS x Graph





We See that if $\frac{d^2y}{dx^2}$ is −ve, the function has Relative Maxima and if $\frac{d^2y}{dx^2}$ +ve function has Relative Minima.

Chapter: 02

Differentiation

Short Questions:

Q1

Find by definiton the derivate of $2x^2+1$. (LHR-2011)

Solution:

Let $y = 2x^2+1$

$$y+\delta y = 2(x+\delta x)^2+1$$

$$\delta y = 2(x^2+2x(\delta x)+(\delta x)^2)+1-(2x^2+1)$$

$$\delta y = 2x^2+4x\delta x+2(\delta x)^2+1-2x^2-1$$

$$\delta y = 4x\delta x+2(\delta x)^2$$

$$\delta y = \delta x(4x+2\delta x)$$

Dividing both sides by δx:

$$\frac{\delta y}{\delta x} = \frac{\delta x(4x+2\delta x)}{\delta x}$$

$$\frac{\delta y}{\delta x} = 4x+2\delta x$$

Taking limit $\delta x \to 0$ on both sides:

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} (4x+2\delta x)$$

$$\frac{dy}{dx} = 4x+2(0)$$

$$\boxed{\frac{dy}{dx} = 4x}$$

21

If $y = c\,f(x)$, prove that $\frac{dy}{dx} = c\,f'(x)$.

(LHR-2011)

Proof:

$$y = c\,f(x)$$

$$y+\delta y = c\,f(x+\delta x)$$

$$\delta y = c\,f(x+\delta x) - c\,f(x)$$

Dividing both sides by δx:

$$\frac{\delta y}{\delta x} = \frac{c\left[f(x+\delta x) - f(x)\right]}{\delta x}$$

Taking limit $\delta x \to 0$:

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left[c\left(\frac{f(x+\delta x) - f(x)}{\delta x}\right)\right]$$

$$\frac{dy}{dx} = c \cdot \lim_{\delta x \to 0} \left(\frac{f(x+\delta x) - f(x)}{\delta x} \right)$$

$$\boxed{\frac{dy}{dx} = c f'(x)}$$

Hence proved.

131

Differentiate $\frac{a+x}{a-x}$ w.r.t x.

(LHR-2011,2012,2014,2018)

Solution:

Let $y = \frac{a+x}{a-x}$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{a+x}{a-x}\right)$$

By using quotient rule,

$$\frac{dy}{dx} = \frac{(a-x)\frac{d}{dx}(a+x) - (a+x)\frac{d}{dx}(a-x)}{(a-x)^2}$$

$$\frac{dy}{dx} = \frac{(a-x)(1) - (a+x)(-1)}{(a-x)^2}$$

$$\frac{dy}{dx} = \frac{a-x+a+x}{(a-x)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{2a}{(a-x)^2}}$$

Find derivative of $(x^3+1)^9$. (LHR-2011)

Solution:

Let:

$y = (x^3+1)^9$; $u = x^3+1$

$y = u^9$

Differentiate w.r.t "u":

$$\frac{dy}{du} = \frac{d}{du}(u)^9$$

$$\frac{dy}{du} = 9(u)^{9-1}\frac{d}{du}(u)$$

$$\frac{dy}{du} = 9u^8 \Rightarrow \frac{dy}{du} = 9(x^3+1)^8$$

Differentiate w.r.t "x":

$$\frac{du}{dx} = \frac{d}{dx}(x^3+1)$$

$$\frac{du}{dx} = 3x^2 + 0$$

$$\frac{du}{dx} = 3x^2$$

By using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx} = 9(x^3+1)^8 \cdot 3x^2$$

$$\boxed{\frac{dy}{dx} = 27x^2(x^3+1)^8}$$

If $x = a\cos^3\theta$, $y = b\sin^3\theta$, show that $a\frac{dy}{dx} + b\tan\theta = 0$. (LHR-2011)

Solution:

$$x = a\cos^3\theta \quad ; \quad y = b\sin^3\theta$$

Differentiate w.r.t "θ":

$$\frac{dx}{d\theta} = a\,3(\cos\theta)^{3-1}\frac{d}{d\theta}(\cos\theta)$$

$$\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

$$\frac{dy}{d\theta} = b\,3(\sin\theta)^{3-1}\frac{d}{d\theta}(\sin\theta)$$

$$\frac{dy}{d\theta} = 3b\sin^2\theta(\cos\theta)$$

$$\frac{dy}{d\theta} = 3b\sin^2\theta\cos\theta$$

By using chain rule,

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$\frac{dy}{dx} = 3b\sin^2\theta\cos\theta \cdot \frac{-1}{3a\cos^2\theta\sin\theta}$$

$$\frac{dy}{dx} = \frac{-b\sin\theta}{a\cos\theta}$$

Multiplying both sides by "a":

$$a\frac{dy}{dx} = -b\frac{\sin\theta}{\cos\theta} \qquad \left\{\because \tan\theta = \frac{\sin\theta}{\cos\theta}\right.$$

$$\boxed{a\frac{dy}{dx} + b\tan\theta = 0}$$

Hence proved.

16

Differentiate w.r.t "x" $\ln(x^2+2x)$.

(LHR-2011, 2013, 2017)

Solution:

Let $y = \ln(x^2+2x)$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln(x^2+2x)\right)$$

$$\frac{dy}{dx} = \frac{1}{x^2+2x} \cdot \frac{d}{dx}(x^2+2x)$$

$$\frac{dy}{dx} = \frac{2x+2}{x^2+2x}$$

$$\boxed{\frac{dy}{dx} = \frac{2(x+1)}{x^2+2x}}$$

17

Find y_2 if $y = x^2e^{-x}$.

(LHR-2011+2021)

Solution:

$y = x^2e^{-x}$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}(x^2e^{-x})$$

$$y_1 = x^2\frac{d}{dx}e^{-x} + e^{-x}\frac{d}{dx}(x^2)$$

$$y_1 = x^2\, e^{-x}(-1) + e^{-x}(2x)$$

$$y_1 = -e^{-x}x^2 + 2e^{-x}x$$

$$y_1 = e^{-x}(2x - x^2)$$

Again differentiate w.r.t "x",

$$y_2 = \frac{d}{dx}\left(e^{-x}(-x^2+2x)\right)$$

$$y_2 = e^{-x}\frac{d}{dx}(-x^2+2x) + (2x-x^2)\frac{d}{dx}(e^{-x}) \quad \left(\begin{array}{l}\text{By using}\\ \text{Product rule}\end{array}\right)$$

$$y_2 = e^{-x}(-2x+2) + (2x-x^2)(e^{-x}(-1))$$

$$y_2 = e^{-x}(2-2x-2x+x^2)$$

$$\boxed{y_2 = e^{-x}(x^2-4x+2)}$$

18

Apply Maclaurin series expansion to prove $e^x = 1+x+\frac{x^2}{2}+\dots$ (LHR-2011)

Solution: Let $f(x) = e^x$

$$f(x) = e^x \;;\; f(0) = e^0 = 1$$

$$f'(x) = e^x \;;\; f'(0) = e^0 = 1$$

$$f''(x) = e^x \;;\; f''(0) = e^0 = 1$$

- By using Maclaurin series,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

$$f(x) = 1 + (1)x + \frac{1}{2!}x^2 + \dots$$

$$e^x = 1 + x + \frac{x^2}{2} + \dots$$

Hence proved.

Find derivative of $\sqrt{x}$ at $x=a$ by ab initio method. (LHR-2011)

Solution:

Let: $y=\sqrt{x}$

$$y+\delta y=\sqrt{x+\delta x}$$

$$\delta y=\sqrt{x+\delta x}-\sqrt{x}$$

$$\delta y=\sqrt{x+\delta x}-\sqrt{x}\times\frac{\sqrt{x+\delta x}+\sqrt{x}}{\sqrt{x+\delta x}+\sqrt{x}}$$

$$\delta y=\frac{x+\delta x-x}{\sqrt{x+\delta x}+\sqrt{x}}=\frac{\delta x}{\sqrt{x+\delta x}+\sqrt{x}}$$

Dividing both sides by δx:

$$\frac{\delta y}{\delta x}=\frac{\delta x}{\delta x\left(\sqrt{x+\delta x}+\sqrt{x}\right)}$$

$$\frac{\delta y}{\delta x}=\frac{1}{\sqrt{x+\delta x}+\sqrt{x}}$$

Taking limit $\delta x\to 0$,

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left(\frac{1}{\sqrt{x+\delta x}+\sqrt{x}}\right)$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{x+0}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

$$\boxed{\frac{dy}{dx}=\frac{1}{2\sqrt{a}}}\qquad\{\because x=a$$

Differentiate w.r.t "x" $\frac{2x-1}{\sqrt{x^2+1}}$. (LHR-2011)

Solution:

Let $y = \frac{2x-1}{\sqrt{x^2+1}}$

Differentiate w.r.t "x".

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x-1}{\sqrt{x^2+1}}\right)$$

By using quotient rule,

$$\frac{dy}{dx} = \frac{\sqrt{x^2+1}\,\frac{d}{dx}(2x-1)-(2x-1)\frac{d}{dx}(x^2+1)^{\frac{1}{2}}}{\left(\sqrt{x^2+1}\right)^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+1}\,(2)-(2x-1)\cdot\frac{1}{2\sqrt{x^2+1}}\frac{d}{dx}(x^2+1)}{(x^2+1)}$$

$$\frac{dy}{dx} = \frac{1}{(x^2+1)}\left[2\sqrt{x^2+1} - \frac{(2x-1)(\cancel{2}x)}{\cancel{2}\sqrt{x^2+1}}\right]$$

$$\frac{dy}{dx} = \frac{1}{(x^2+1)}\left[\frac{2(x^2+1)-x(2x-1)}{\sqrt{x^2+1}}\right]$$

$$\frac{dy}{dx} = \frac{2x^2+2-2x^2+\cancel{x}}{(x^2+1)^{\frac{3}{2}}}$$

$$\boxed{\frac{dy}{dx} = \frac{2+x}{(x^2+1)^{\frac{3}{2}}}}$$

11

Find $\frac{dy}{dx}$ if $y^2 - xy - x^2 + 4 = 0$. (LHR-2011, 2014)

Solution: $y^2 - xy - x^2 + 4 = 0$

Differentiating on both sides,

$$2y\frac{dy}{dx} - \left[x\frac{dy}{dx} + y\frac{d}{dx}(x)\right] - 2x + 0 = 0$$

$$2y\frac{dy}{dx} - x\frac{dy}{dx} - y - 2x = 0$$

$$\frac{dy}{dx}\left[2y - x\right] = 2x + y$$

$$\boxed{\frac{dy}{dx} = \frac{2x+y}{2y-x}}$$

12

Use Maclaurin series to expand $\cos x$. (LHR-2011, 2016)

Solution: Let: $f(x) = \cos x$

$f(x) = \cos x$; $f(0) = \cos 0^\circ = 1$

$f'(x) = -\sin x$; $f'(0) = -\sin 0^\circ = 0$

$f''(x) = -\cos x$; $f''(0) = -\cos 0^\circ = -1$

$f'''(x) = \sin x$; $f'''(0) = \sin 0^\circ = 0$

$f^{iv}(x) = \cos x$; $f^{iv}(0) = \cos 0^\circ = 1$

$f^{v}(x) = -\sin x$; $f^{v}(0) = -\sin 0^\circ = 0$

$f^{vi}(x) = -\cos x$; $f^{vi}(0) = -\cos 0 = -1$

Now,

By Maclaurin series;

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\cos x = 1 + (0)x + \frac{(-1)}{2!}x^2 + \frac{(0)}{3!}x^3 + \frac{(1)}{4!}x^4 + \frac{(0)}{5!}x^5 + \frac{(-1)}{6!}x^6 + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

13

Find y_4 if $y = \sin 3x$. (LHR-2011+2016)+(LHR-2012)

Solution:

$$y = \sin 3x$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{d}{dx}(\sin 3x)$$

$$y_1 = \cos 3x \frac{d}{dx}(3x) = 3\cos 3x$$

$$y_2 = \frac{d}{dx}(3\cos 3x)$$

$$y_2 = 3(-\sin 3x)\frac{d}{dx}(3x) = -3\sin 3x(3) = -9\sin 3x$$

$$y_3 = -9\frac{d}{dx}(\sin 3x) = -9(\cos 3x)\frac{d}{dx}(3x) = -27\cos 3x$$

$$y_4 = -27\frac{d}{dx}\cos 3x = -27(-\sin 3x)\frac{d}{dx}3x = +27\sin 3x(3)$$

$$\boxed{y_4 = 81\sin 3x}$$

Find $\frac{dy}{dx}$, if $y = \tanh(x^2)$. (LHR-2011+2013+2017)

Solution:

Let $u = x^2$

$y = \tanh u$; $u = x^2$

Differentiate w.r.t "u"

$$\frac{dy}{du} = \operatorname{sech}^2 u \frac{d}{du}(u)$$

$$\frac{dy}{du} = \operatorname{sech}^2 u$$

Differentiate w.r.t "x"

$$\frac{du}{dx} = \frac{d}{dx}(x^2)$$

$$\frac{du}{dx} = 2x$$

By using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \operatorname{sech}^2 u \cdot 2x$$

$$\boxed{\frac{dy}{dx} = 2x \operatorname{sech}^2 x^2} \quad \because u = x^2$$

Alternate Method (By formula):

$$y = \tanh(x^2) \qquad \left\{ \because \tanh x = \operatorname{sech}^2 x \frac{d}{dx}(x) \right.$$

$$\frac{dy}{dx} = \operatorname{sech}^2 x^2 \frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = \operatorname{sech}^2 x^2 (2x)$$

$$\boxed{\frac{dy}{dx} = 2x \operatorname{sech}^2 x^2}$$

15

Find Maclaurin series for a^x.

(LHR-2011)

Solution:

Let $f(x) = a^x$ then $f(0) = a^0 = 1$

$f'(x) = a^x \ln a$; $f'(0) = a^0 \ln a = \ln a$

$f''(x) = a^x (\ln a)^2$; $f''(0) = a^0 (\ln a)^2 = (\ln a)^2$

$f'''(x) = a^x (\ln a)^3$; $f'''(0) = a^0 (\ln a)^3 = (\ln a)^3$

$f^{iv}(x) = a^x (\ln a)^4$; $f^{iv}(0) = a^0 (\ln a)^4 = (\ln a)^0$

By Maclaurin series,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \dots$$

$$a^x = 1 + (\ln a)x + \frac{(\ln a)^2}{2!}x^2 + \frac{(\ln a)^3}{3!}x^3 + \frac{(\ln a)^4}{4!}x^4 + \dots$$

16

Let $y = (3t+2)^{-2}$, then find δy.

Solution:

(LHR-2012)

$$y = (3t+2)^{-2}$$

$$\delta y + y = \left(3(t+\delta t)+2\right)^{-2}$$

$$\delta y = (3t + 3\delta t + 2)^{-2} - y$$

$$\delta y = (3t+2+3\delta t)^{-2} - (3t+2)^{-2}$$

$$\delta y = (3t+2)^{-2}\left(1 + \frac{3\delta t}{3t+2}\right)^{-2} - (3t+2)^{-2}$$

$$\delta y = (3t+2)^{-2}\left[\left(1 + \frac{3\delta t}{3t+2}\right)^{-2} - 1\right]$$

By using binomial series,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$$

$$\delta y = (3t+2)^{-2}\left[1 + (-2)\left(\frac{3\delta t}{3t+2}\right) + \frac{(-2)(-2-1)}{2!}\left(\frac{3\delta t}{3t+2}\right)^2 + \cdots - 1\right]$$

$$\delta y = (3t+2)^{-2}\left[-2\left(\frac{3\delta t}{3t+2}\right) + \frac{(-2)(-2-1)}{2!}\left(\frac{3\delta t}{3t+2}\right)^2 + \cdots\right]$$

$$\delta y = (3t+2)^{-2}\left(\frac{3\delta t}{3t+2}\right)\left[-2 + \frac{6}{2!}\left(\frac{3\delta t}{3t+2}\right) + \cdots\right]$$

$$\delta y = (3t+2)^{-2-1}\, 3\delta t\left(-2 + \frac{6}{2!}\left(\frac{3\delta t}{3t+2}\right) + \cdots\right)$$

Dividing both sides by δt

$$\frac{\delta y}{\delta t} = \frac{(3t+2)^{-3}\, 3\delta t\left(-2 + \frac{6}{2!}\left(\frac{3\delta t}{3t+2}\right) + \cdots\right)}{\delta t}$$

$$\frac{\delta y}{\delta t} = (3t+2)^{-3}\, 3\left(-2 + \frac{6}{2!}\left(\frac{3\delta t}{3t+2}\right) + \cdots\right)$$

Applying limit $\delta t \to 0$,

$$\lim_{\delta t \to 0}\frac{\delta y}{\delta t} = \lim_{\delta t \to 0}\left((3t+2)^{-3}\, 3\left(-2 + \frac{6}{2!}\left(\frac{3\delta t}{3t+2}\right) + \cdots\right)\right)$$

$$\frac{dy}{dt} = 3(3t+2)^{-3}\lim_{\delta t \to 0}\left(-2 + \frac{6}{2!}\left(\frac{3\delta t}{3t+2}\right) + \cdots\right)$$

$$\frac{dy}{dt} = 3(3t+2)^{-3}\left(-2 + \frac{6}{2!}\left(\frac{3(0)}{3t+2}\right) + \cdots\right)$$

$$\frac{dy}{dt} = 3(3t+2)^{-3}(-2+0)$$

$$\frac{dy}{dt} = 3(3t+2)^{-3}(-2)$$

$$\frac{dy}{dt} = -6(3t+2)^{-3}$$

$$\boxed{\frac{dy}{dt} = \frac{-6}{(3t+2)^3}}$$

Find $\frac{dy}{dx}$ if $x=\theta+\frac{1}{\theta}$ and $y=\theta+1$.

(LHR-2012)

Solution:

$$x=\theta+\frac{1}{\theta} \quad ; \quad y=\theta+1$$

$$x=\frac{\theta^2+1}{\theta}$$

Differentiate w.r.t "θ":

$$\frac{dx}{d\theta}=\frac{\theta\frac{d}{d\theta}(\theta^2+1)-(\theta^2+1)\frac{d}{d\theta}(\theta)}{(\theta)^2}$$

$$\frac{dx}{d\theta}=\frac{\theta(2\theta)-(\theta^2+1)(1)}{\theta^2}$$

$$\frac{dx}{d\theta}=\frac{2\theta^2-\theta^2-1}{\theta^2}$$

$$\frac{dx}{d\theta}=\frac{\theta^2-1}{\theta^2}$$

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(\theta+1)$$

$$\frac{dy}{d\theta}=1+0$$

$$\frac{dy}{d\theta}=1$$

$$\frac{dy}{d\theta}=1$$

By using chain rule,

$$\frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$\frac{dy}{dx}=1\cdot\frac{\theta^2}{\theta^2-1}$$

$$\boxed{\frac{dy}{dx}=\frac{\theta^2}{\theta^2-1}}$$

18

If $y=\cot^{-1}\frac{x}{a}$, find $\frac{dy}{dx}$. (LHR-2012)

Solution:

$$y=\cot^{-1}\frac{x}{a}$$

Differentiate w.r.t "x":

$$\frac{dy}{dx}=\frac{-1}{1+\left(\frac{x}{a}\right)^2}\frac{d}{dx}\left(\frac{x}{a}\right)$$

$$\left\{\because \cot^{-1}x=\frac{-1}{1+x^2}\frac{d}{dx}(x)\right.$$

$$\frac{dy}{dx}=\frac{-1}{1+\frac{x^2}{a^2}}\frac{1}{a}\frac{d}{dx}(x)$$

$$\frac{dy}{dx}=\frac{-1}{\frac{a^2+x^2}{a^2}}\frac{1}{a}(1)$$

$$\frac{dy}{dx}=\frac{-a^{\cancel{2}}}{a^2+x^2}\cdot\frac{1}{\cancel{a}}$$

$$\boxed{\frac{dy}{dx}=\frac{-a}{a^2+x^2}}$$

19

Find $\frac{dy}{dx}$ if $y=\frac{x}{\ln x}$. (LHR-2012 +2019)

Solution:

$$y=\frac{x}{\ln x}$$

Differentiate w.r.t "x":

$$\frac{dy}{dx}=\frac{\ln x\frac{d}{dx}(x)-x\frac{d}{dx}\ln x}{(\ln x)^2}$$

$$\frac{dy}{dx}=\frac{\ln x(1)-x\cdot\frac{1}{x}}{(\ln x)^2}\Rightarrow\boxed{\frac{dy}{dx}=\frac{\ln x-1}{(\ln x)^2}}$$

pakcity.org

|20|

Find $\frac{dy}{dx}$ if $3x+4y+7=0.$

(LHR-2012,2016)

Solution:

$$3x+4y+7=0$$

Differentiate w.r.t "x",

$$\frac{d}{dx}(3x+4y+7)=0$$

$$3\frac{d}{dx}(x)+4\frac{dy}{dx}+\frac{d}{dx}(7)=0$$

$$3(1)+4\frac{dy}{dx}+0=0$$

$$4\frac{dy}{dx}=-3 \quad \Rightarrow \quad \boxed{\frac{dy}{dx}=\frac{-3}{4}}$$

|21|

If $y=\tan(p\tan^{-1}x)$, show that $(1+x^2)y_1-p(1+y^2)=0.$

(LHR-2012) & (LQ LHR-2017,2021)

Solution:

$$y=\tan(p\tan^{-1}x)$$

$$\tan^{-1}y=p\tan^{-1}x$$

Differentiate w.r.t "x",

$$\frac{1}{1+y^2}\cdot\frac{dy}{dx}=p\cdot\frac{1}{1+x^2}\frac{d}{dx}(x)$$

$$\frac{1}{1+y^2}\cdot y_1=p\cdot\frac{1}{1+x^2}$$

$$(1+x^2)y_1=p(1+y^2)$$

$$\boxed{(1+x^2)y_1-p(1+y^2)=0}$$

Hence proved.

(21)

Find $f'(x)$ for $f(x) = e^{\sqrt{x}-1}$.

(LHR-2012, 2013)

Solution:

$$f(x) = e^{\sqrt{x}-1}$$

Differentiate w.r.t "x",

$$f'(x) = e^{\sqrt{x}-1} \frac{d}{dx}(\sqrt{x}-1)$$

$$f'(x) = e^{\sqrt{x}-1}\left[\frac{1}{2}(x)^{\frac{1}{2}-1} - 0\right]$$

$$f'(x) = e^{\sqrt{x}-1} \cdot \frac{1}{2\sqrt{x}}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x}} e^{\sqrt{x}-1}}$$

(22)

Find y_2 if $y = 2x^5 - 3x^4 + 4x^3 + x - 2$.

(LHR-2012)

Solution:

$$y = 2x^5 - 3x^4 + 4x^3 + x - 2$$

Differentiate w.r.t "x",

$$y_1 = 10x^4 - 12x^3 + 12x^2 + 1 - 0$$

Again differentiate w.r.t "x":

$$y_2 = 10\frac{d}{dx}(x^4) - 12\frac{d}{dx}(x^3) + 12\frac{d}{dx}(x^2) + \frac{d}{dx}(1)$$

$$y_2 = 10 \cdot 4(x)^{4-1} - 12 \cdot 3(x)^{3-1} + 12 \cdot 2(x)^{2-1} + 0$$

$$\boxed{y_2 = 40x^3 - 36x^2 + 24x}$$

23

Apply Maclaurin series expansion to prove that $e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \dots$

(LHR-2012)

Solution: Let: $f(x) = e^{2x}$

$f(x) = e^{2x}$; $f(0) = e^{2(0)} = e^0 = 1$

$f'(x) = 2e^{2x}$; $f'(0) = 2e^{2(0)} = 2e^0 = 2$

$f''(x) = 4e^{2x}$; $f''(0) = 4e^{2(0)} = 4e^0 = 4$

$f'''(x) = 8e^{2x}$; $f'''(0) = 8e^{2(0)} = 8e^0 = 8$

By using Maclaurin series,

$$f(x) = f(0) + f'(0)x + f''\frac{(0)x^2}{2!} + f'''\frac{(0)}{3!}x^3 + \dots$$

$$e^{2x} = 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \dots$$

Hence proved.

24

Differentiate w.r.t x^4 to $x^2 - \frac{1}{x^2}$.

(LHR-2012+2018)

Solution: Let:

$y = x^2 - \frac{1}{x^2} = x^2 - x^{-2}$; $u = x^4$

Differentiate w.r.t "x".

$$\frac{dy}{dx} = 2x + \frac{2}{x^3} \quad ; \quad \frac{du}{dx} = 4x^3$$

$$\frac{dy}{dx} = \frac{2x^4+2}{x^3} = \frac{2(x^4+1)}{x^3}$$

By using chain rule, $\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dx}{du}$

$$\frac{dy}{dx} = \frac{\cancel{2}(x^4+1)}{x^3} \cdot \frac{1}{{}_2\cancel{4}x^3} \Rightarrow \boxed{\frac{dy}{dx} = \frac{x^4+1}{2x^6}}$$

Find by definition derivative of $f(x) = x^m$. (LHR-2012)

Solution:

$$f(x) = x^m$$

$$f(x+\delta x) = (x+\delta x)^m$$

$$f(x+\delta x) - f x = (x+\delta x)^m - f(x)$$

$$f(x+\delta x) - f(x) = (x+\delta x)^m - x^m$$

Using Binomial Theorem,

$$f(x+\delta x) - f(x) = \binom{m}{0}(x)^m(\delta x)^0 + \binom{m}{1}(x)^{m-1}(\delta x)^1 + \binom{m}{2}(x)^{m-2}(\delta x)^2 + \dots + \binom{m}{m}(x)^{m-m}(\delta x)^m - x^m$$

$$= (1)(x)^m 1 + \left(\frac{m!}{1!(m-1)!}\right)x^{m-1}(\delta x) + \left(\frac{m!}{2!(m-2)!}\right)x^{m-2}(\delta x)^2 + \dots + (1)(1)(\delta x)^m - x^m$$

$$= m x^{m-1}(\delta x) + \frac{m(m-1)}{2!}x^{m-2}(\delta x)^2 + \dots + (\delta x)^m$$

$$= \delta x\left(m x^{m-1} + \frac{m(m-1)}{2!}x^{m-2}(\delta x) + \dots + (\delta x)^{m-1}\right)$$

Dividing both sides by δx

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{\cancel{\delta x}\left(m x^{m-1} + \frac{m(m-1)}{2!}x^{m-2}(\delta x) + \dots + (\delta x)^{m-1}\right)}{\cancel{\delta x}}$$

Taking limit $\delta x \to 0$ on both sides,

$$\lim_{\delta x \to 0}\left(\frac{f(x+\delta x) - f(x)}{\delta x}\right) = \lim_{\delta x \to 0}\left(m x^{m-1} + \frac{m(m-1)}{2!}x^{m-2}(\delta x) + \dots + (\delta x)^{m-1}\right)$$

$$f'(x) = m x^{m-1} + 0 + \dots + 0$$

$$\boxed{f'(x) = m x^{m-1}}$$

(26)

Differentiate $\frac{2x-3}{2x+1}$ w.r.t "x". (LHR-2013)

Solution:

Let: $y = \frac{2x-3}{2x+1}$

Differentiate w.r.t "x",

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x-3}{2x+1}\right)$$

By using quotient rule,

$$\frac{dy}{dx} = \frac{(2x+1)\frac{d}{dx}(2x-3) - (2x-3)\frac{d}{dx}(2x+1)}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{(2x+1)(2) - (2x-3)(2)}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{4x+2-4x+6}{(2x+1)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{8}{(2x+1)^2}}$$

(27)

Differentiate $\sqrt{x+\sqrt{x}}$ w.r.t "x". (LHR-2013+2017)

Solution:

Let: $y = \sqrt{x+\sqrt{x}}$; $u = x+\sqrt{x}$

$y = \sqrt{u}$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = \frac{1}{2}(u)^{\frac{1}{2}-1} \quad ; \quad \frac{du}{dx} = \frac{d}{dx}(x+\sqrt{x})$$

$$\frac{dy}{dx} = \frac{1}{2}(u)^{-\frac{1}{2}} \quad ; \quad \frac{du}{dx} = 1 + \frac{1}{2}(x)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \quad ; \quad \frac{du}{dx} = 1 + \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+\sqrt{x}}} \quad ; \quad \frac{du}{dx} = \frac{2\sqrt{x}+1}{2\sqrt{x}}$$

By using Chain rule,

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \frac{2\sqrt{x}+1}{2\sqrt{x}} \Rightarrow \boxed{\frac{dy}{dx} = \frac{2\sqrt{x}+1}{4\sqrt{x}\sqrt{x+\sqrt{x}}}}$$

|28|

Find $\frac{dy}{dt}$ and $\frac{dx}{dt}$ when $x = at^2$, $y = 2at$.

(LHR-2013)

Solution: $x = at^2 \quad ; \quad y = 2at$

Differentiate w.r.t "t":

$$\frac{dx}{dt} = a\frac{d}{dt}(t)^2 \quad ; \quad \frac{dy}{dt} = 2a\frac{d}{dt}(t)$$

$$\frac{dx}{dt} = a(2t) \quad ; \quad \frac{dy}{dt} = 2a(1)$$

$$\boxed{\frac{dx}{dt} = 2at} \quad ; \quad \boxed{\frac{dy}{dt} = 2a}$$

129

If $y = xe^{\sin x}$ then find $\frac{dy}{dx}$.

Solution: (LHR-2013+2016+2018)

$$y = xe^{\sin x}$$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = \frac{d}{dx}(xe^{\sin x})$$

By using product rule,

$$\frac{dy}{dx} = x\frac{d}{dx}e^{\sin x} + e^{\sin x}\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x\, e^{\sin x}\frac{d}{dx}(\sin x) + e^{\sin x}(1)$$

$$\frac{dy}{dx} = xe^{\sin x}\cos x + e^{\sin x}$$

$$\boxed{\frac{dy}{dx} = e^{\sin x}(x\cos x + 1)}$$

130

Find the derivative of $\frac{1}{a}\sin^{-1}\left(\frac{a}{x}\right)$ w.r.t "x".

Solution: (LHR-2013) + (LHR-2017)

Let $y = \frac{1}{a}\sin^{-1}\left(\frac{a}{x}\right)$

Differentiate w.r.t "x":

$$\left\{\because \sin^{-1}x = \frac{1}{\sqrt{1-x^2}}\cdot\frac{d}{dx}(x)\right.$$

$$\frac{dy}{dx} = \frac{1}{a}\cdot\frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}}\cdot\frac{d}{dx}\left(\frac{a}{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{a}\cdot\frac{1}{\sqrt{\frac{x^2-a^2}{x^2}}}\cdot a\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = \frac{\cancel{x}}{\cancel{a}\sqrt{x^2-a^2}}\cdot\frac{-\cancel{a}}{x^{\cancel{2}}} \Rightarrow \boxed{\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-a^2}}}$$

Find y_1 if $x^3 - y^3 = a^3$. (LHR-2013 +2019)

Solution:

Differentiating on both sides,

$$\frac{d}{dx}(x^3-y^3) = \frac{d}{dx}(a^3)$$

$$3x^2 - 3y^2\frac{dy}{dx} = 0$$

$$-3y^2\frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{x^2}{y_2} \Rightarrow \boxed{y_1 = \frac{x^2}{y^2}}$$

Prove that $\frac{d}{d\theta}(\cos^2\theta + \sin^2\theta) = 0$. (LHR-2013)

Solution:

$$L.H.S = \frac{d}{d\theta}(\cos^2\theta + \sin^2\theta)$$

$$= \frac{d}{d\theta}\cos^2\theta + \frac{d}{d\theta}\sin^2\theta$$

$$= 2\cos\theta\frac{d}{d\theta}(\cos\theta) + 2\sin\theta\frac{d}{d\theta}(\sin\theta)$$

$$= -2\cos\theta\sin\theta + 2\sin\theta\cos\theta$$

$$= -2\sin\theta\cos\theta + 2\sin\theta\cos\theta$$

$$= 0$$

$$= R.H.S$$

$$\therefore \frac{d}{d\theta}(\cos^2\theta + \sin^2\theta) = 0$$

Hence proved.

133

If $y = \frac{1}{x^2}$ find $\frac{dy}{dx}$ by ab-initio method.

(LHR-2013)

Solution:

$$y = \frac{1}{x^2}$$

$$y + \delta y = \frac{1}{(x+\delta x)^2}$$

$$\delta y = \frac{1}{(x+\delta x)^2} - y \Rightarrow \delta y = \frac{1}{(x+\delta x)^2} - \frac{1}{x^2}$$

$$\delta y = \frac{1}{(x+\delta x)^2} - \frac{1}{x^2}$$

$$\delta y = \frac{x^2 - (x+\delta x)^2}{x^2(x+\delta x)^2}$$

$$\delta y = \frac{(x-(x+\delta x))(x+(x+\delta x))}{x^2(x+\delta x)^2}$$

$$\delta y = \frac{(x-x-\delta x)(x+x+\delta x)}{x^2(x+\delta x)^2}$$

$$\delta y = \frac{(-\delta x)(2x+\delta x)}{x^2(x+\delta x)^2}$$

Dividing by δx on both sides,

$$\frac{\delta y}{\delta x} = \frac{(-\delta x)(2x+\delta x)}{\delta x\,(x^2(x+\delta x)^2)} = \frac{-(2x+\delta x)}{x^2(x+\delta x)^2}$$

Taking limit $\delta x \to 0$ on both sides:

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\left(\frac{-(2x+\delta x)}{x^2(x+\delta x)^2}\right)$$

$$\frac{dy}{dx} = \frac{-(2x+0)}{x^2(x+0)^2} = \frac{-2x}{x^2(x^2)}$$

$$\frac{dy}{dx} = \frac{-2x}{x^4} \Rightarrow \boxed{\frac{dy}{dx} = \frac{-2}{x^3}}$$

34

State Maclaurin series expansion

If $f(x)$ is expanded in ascending power of x (LHR-2013)

then $f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''\frac{(0)}{3!}x^3 + \dots + f^{(n)}\frac{(0)}{n!}x^n + \dots$

This expansion of $f(x)$ is called the

Maclaurin series.

35

Differentiate $\frac{x^2+1}{x^2-3}$ w.r.t "x" (LHR-2013+2015)

Solution: Let $y = \left(\frac{x^2+1}{x^2-3}\right)$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-3}\right)$$

By using quotient rule,

$$\frac{dy}{dx} = \frac{(x^2-3)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-3)}{(x^2-3)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-3)(2x) - (x^2+1)(2x)}{(x^2-3)^2}$$

$$\frac{dy}{dx} = \frac{2x\,(x^2-3-x^2-1)}{(x^2-3)^2}$$

$$\frac{dy}{dx} = \frac{2x(-4)}{(x^2-3)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-8x}{(x^2-3)^2}}$$

36

Find $\frac{dy}{dx}$ if $x^2-4xy-5y=0$. (LHR-2013+2018)

Solution:

$$x^2-4xy-5y=0$$

$$2x-4\left[x\frac{dy}{dx}+y\frac{d}{dx}(x)\right]-5\frac{dy}{dx}=0$$

$$2x-4x\frac{dy}{dx}-4y-5\frac{dy}{dx}=0$$

$$-\frac{dy}{dx}(4x+5)=4y-2x$$

$$-\frac{dy}{dx}(4x+5)=-2(x-2y)$$

$$\boxed{\frac{dy}{dx}=\frac{2(x-2y)}{4x+5}}$$

37

Differentiate $\sin^3 x$ w.r.t $\cos^2 x$. (LHR-2013)

Solution:

Let:

$$y=\sin^3 x \quad ; \quad u=\cos^2 x$$

Differentiate w.r.t "x":

$$\frac{dy}{dx}=3\sin^2 x\cos x \quad ; \quad \frac{du}{dx}=-2\cos x\sin x$$

By using chain rule,

$$\frac{dy}{dx}=\frac{dy}{dx}\cdot\frac{dx}{du}$$

$$\frac{dy}{dx}=3\sin^2 x\cos x\cdot\frac{1}{-2\cos x\sin x}$$

$$\boxed{\frac{dy}{dx}=\frac{-3}{2}\sin x}$$

138

Differentiate w.r.t x $\sin^{-1}\sqrt{1-x^2}$.

(LHR-2013+2018)

Solution:

Let $y = \sin\sqrt{1-x^2}$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}}\,\frac{d}{dx}(1-x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-1+x^2}}\cdot\frac{1}{2\sqrt{1-x^2}}(-2x)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2}}\cdot\frac{-x}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-x}{x\sqrt{1-x^2}} \Rightarrow \boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}}$$

139

Find y_2 if $y = \ln(x^2-9)$.

(LHR-2013+2014)

Solution:

$$y = \ln(x^2-9)$$

$$y = \ln(x-3)(x+3)$$

$$y = \ln(x-3)+\ln(x+3)$$

Differentiate w.r.t "x",

$$y_1 = \frac{1}{(x-3)} + \frac{1}{x+3} \Rightarrow y_1 = (x-3)^{-1}+(x+3)^{-1}$$

$$y_2 = -(x-3)^{-2}-(x+3)^{-2}$$

$$\boxed{y_2 = \frac{-1}{(x-3)^2} - \frac{1}{(x+3)^2}}$$

140

Apply Maclaurin series expansion to prove that $\sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}+\cdots$

(LHR-2013+2015)

Solution:

Let $f(x) = \sqrt{1+x}$

$f(x) = (1+x)^{\frac{1}{2}}$; $f(0) = (1+0)^{\frac{1}{2}} = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$; $f'(0) = \frac{1}{2}(1+0)^{-\frac{1}{2}} = \frac{1}{2}$

$f''(x) = \frac{-1}{4}(1+x)^{-3/2}$; $f''(0) = \frac{-1}{4}(1+0)^{-3/2} = \frac{-1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$; $f'''(0) = \frac{3}{8}(1+0)^{-5/2} = \frac{3}{8}$

By using Maclaurin series,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{4\cdot 2!}x^2 + \frac{3}{8(3\cdot 2\cdot 1)}x^3 + \cdots$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots$$ Hence proved.

141

Find derivative of $f(x)=c$ by definition.

Solution:

(LHR-2014+2016)

$f(x) = c \Rightarrow f(x+\delta x) = c$

$f(x+\delta x) - f(x) = c - c$

$f(x+\delta x) - f(x) = 0$

Dividing by δx and Taking $\lim\limits_{\delta x \to 0}$ on both sides,

$$\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = \lim_{\delta x \to 0}\left(\frac{0}{\delta x}\right)$$

$\therefore$ $f'(x) = 0$, $\frac{d}{dx}(c) = 0$

142

If $y = x^4 + 2x^2 + 2$, prove that; $\frac{dy}{dx} = 4x\sqrt{y-1}$ (LHR-2014)

Solution:

$$y = x^4 + 2x^2 + 2$$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = 4x^3 + 4x$$

$$\frac{dy}{dx} = 4x(x^2+1)$$

$$\frac{dy}{dx} = 4x\sqrt{(x^2+1)^2}$$

$$\frac{dy}{dx} = 4x\sqrt{x^4+1+2x^2+1-1}$$

$$\frac{dy}{dx} = 4x\sqrt{x^4+2x^2+2-1}$$

$$\boxed{\frac{dy}{dx} = 4x\sqrt{y-1}}$$ Hence proved.

143

Differentiate $(1+x^2)^n$ w.r.t x^2. (LHR-2014)

Solution: Let:

$y = (1+x^2)^n$; $u = x^2$

Differentiate w.r.t "x":

$\frac{dy}{dx} = n\cdot(1+x^2)^{n-1}(2x)$; $\frac{du}{dx} = 2x$

By chain rule,

$$\frac{dy}{dx} = \frac{dy}{dx}\cdot\frac{dx}{du} = 2xn(1+x^2)^{n-1}\cdot\frac{1}{2x}$$

$$\boxed{\frac{dy}{dx} = n(1+x^2)^{n-1}}$$

144

Find $\frac{dy}{dx}$ if $x = y\sin y$. (LHR-2014)

Solution:

$$\frac{d}{dx}(x) = y\frac{d}{dx}\sin y + \sin y\frac{dy}{dx}$$

$$1 = y\cos y\frac{dy}{dx} + \sin\frac{dy}{dx}$$

$$1 = \frac{dy}{dx}(y\cos y + \sin y)$$

$$\boxed{\frac{dy}{dx} = \frac{1}{y\cos y + \sin y}}$$

145

Find $\frac{dy}{dx}$ if $y = x^2\ln\sqrt{x}$. (LHR-2014)

Solution:

$$y = x^2\ln\sqrt{x}$$

$$y = x^2\ln(x)^{\frac{1}{2}}$$

$$y = \frac{1}{2}x^2\ln x \qquad \left\{\because \ln x^x = x\ln x\right.$$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(x^2\ln x)$$

By product rule,

$$\frac{dy}{dx} = \frac{1}{2}\left(x^2\frac{d}{dx}\ln x + \ln x\frac{d}{dx}(x^2)\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\left(x^2\cdot\frac{1}{x} + \ln x\cdot 2x\right)$$

$$\frac{dy}{dx} = \frac{1}{2}(x + 2x\ln x)$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2}x + x\ln x}$$

146

If $y = x$ find $\frac{dy}{dx}$ by definition. (LHR-2014)

Solution:

$$y = x$$

$$y + \delta y = x + \delta x$$

$$\delta y = x + \delta x - y$$

$$\frac{\delta y}{d} = x + \delta x - x$$

$$\delta y = \delta x$$

Dividing by δx on both sides;

$$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x}$$

$$\frac{\delta y}{\delta x} = 1$$

Taking limit $\delta x \to 0$:

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} (1)$$

$$\boxed{\frac{dy}{dx} = 1} \Rightarrow \frac{d}{dx}(x) = 1$$

147

If $y = (2\sqrt{x}+2)(x-\sqrt{x})$ then find $\frac{dy}{dx}$.

Solution: (LHR-2014)

$$y = (2\sqrt{x}+2)(x-\sqrt{x})$$

$$y = 2(\sqrt{x}+1)\sqrt{x}(\sqrt{x}-1)$$

$$y = 2\sqrt{x}(x^2-1)$$

$$y = 2(x^{3/2} - x^{1/2})$$

Alternate:

$$y = (2\sqrt{x}+2)(x-\sqrt{x})$$

$$y = 2(\sqrt{x}+1)(x-\sqrt{x})$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = 2\frac{d}{dx}\left[(\sqrt{x}+1)(x-\sqrt{x})\right]$$

Differentiate w.r.t "x",

$$\frac{dy}{dx} = 2\left[\frac{3}{2}\sqrt{x} - \frac{1}{2\sqrt{2}}\right]$$

$$\frac{dy}{dx} = 2\cdot\frac{3}{2}\sqrt{x} - 2\cdot\frac{1}{2\sqrt{2}}$$

$$\boxed{\frac{dy}{dx} = \frac{3x-1}{\sqrt{x}}}$$

By product rule,

$$\frac{dy}{dx} = 2\left[(\sqrt{x}+1)\frac{d}{dx}(x-\sqrt{x}) + (x-\sqrt{x})\frac{d}{dx}(\sqrt{x}+1)\right]$$

$$\frac{dy}{dx} = 2\left[\sqrt{x}+1\left(1-\frac{1}{2\sqrt{x}}\right) + (x-\sqrt{x})\frac{1}{2\sqrt{x}}\right]$$

$$\frac{dy}{dx} = 2\left[(\sqrt{x}+1)\left(\frac{2\sqrt{x}-1}{2\sqrt{x}}\right) + \frac{x-\sqrt{x}}{2\sqrt{x}}\right]$$

$$\frac{dy}{dx} = 2\cdot\frac{1}{2\sqrt{x}}\left[(\sqrt{x}+1)(2\sqrt{x}-1)+(x-\sqrt{x})\right.$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}\left(2x-\sqrt{x}+2\sqrt{x}-1+x-\sqrt{x}\right.$$

$$\frac{dy}{dx} = \frac{3x-1}{\sqrt{x}}$$

B 148 B

Evaluate $\frac{dy}{dx}$ if $x^2+y^2-4x=5$

(LHR-2014+2017)

Solution:

$$x^2+y^2-4x=5$$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)-4\frac{d}{dx}(x)=\frac{d}{dx}(5)$$

$$2x+2y\frac{dy}{dx}-4=0$$

$$2y\frac{dy}{dx} = 4-2x$$

$$2y\frac{dy}{dx} = 2(2-x)$$

$$y\frac{dy}{dx} = 2-x$$

$$\boxed{\frac{dy}{dx} = \frac{2-x}{y}}$$

149

Find $\frac{dy}{dx}$ if $y = \sin^{-1}x$. (LHR-2014) + (LHR-2017) + (LHR-2019)

Solution:

$$y = \sin^{-1}x$$

$$x = \sin y$$

$$\frac{d}{dx}(x) = \frac{d}{dx}\sin y$$

$$1 = \cos y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} \qquad \left\{\because \sin^2\theta + \cos^2\theta = 1\right.$$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \left\{\because y = \sin^{-1}x\ ;\ x = \sin y\right.$$

150

Differentiate $(\ln x)^x$ w.r.t. x. (LHR-2014)

Solution:

Let $y = (\ln x)^x$

Taking ln on both sides:

$$\ln y = \ln(\ln x)^x$$

$$\ln y = x\ln(\ln x) \qquad \left\{\because \ln x^a = a\ln x\right.$$

Differentiate w.r.t. "x":

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\,\frac{1}{\ln x}\cdot\frac{1}{x} + \ln(\ln x)\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = y\left[\frac{1}{\ln x} + \ln(\ln x)\right]$$

$$\boxed{\frac{dy}{dx} = (\ln x)^x\left[\frac{1}{\ln x} + \ln(\ln x)\right]}$$

151

If $y = \cosh^{-1}(\sec x)$, find $\frac{dy}{dx}$. (LHR-2014)

Solution:

$$y = \cosh^{-1}\sec x$$

Let: $u = \sec x$

$$y = \cosh^{-1}u \quad ; u = \sec x$$

$$\frac{dy}{du} = \frac{1}{\sqrt{u^2-1}} \quad ; \frac{du}{dx} = \sec x \tan x$$

$$\frac{dy}{du} = \frac{1}{\sqrt{\sec^2 x-1}} \quad ; \frac{du}{dx} = \sec x \tan x$$

By using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\sec^2 x-1}}\cdot \sec x \tan x$$

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\tan x}$$

$$\boxed{\frac{dy}{dx} = \sec x}$$

Alternate:

$$y = \cosh^{-1}\sec x$$

Differentiate w.r.t. "x":

$$\frac{dy}{dx} = \frac{1}{\sqrt{\sec^2 x-1}}\cdot\frac{d}{dx}\sec x \quad \left\{\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}\right.$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\tan^2 x}}\cdot \sec x \tan x \quad \left\{\because 1+\tan^2 x = \sec^2 x\right.$$

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\tan x}$$

$$\boxed{\frac{dy}{dx} = \sec x}$$

152

Write first two terms of $f(x) = \frac{1}{x+1}$ by Maclaurin series. (LHR-2014)

Solution:

$$f(x) = (x+1)^{-1} \quad ; f(0) = 1$$

$$f'(x) = -1(x+1)^{-2}; \quad f'(0) = -1$$

$$f''(x) = 2(x+1)^{-3}; \quad f''(0) = 2$$

By using Maclaurin series;

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

$$\frac{1}{x+1} = 1 - x + \frac{2}{2\cdot 1}x^2 + \dots \implies \boxed{\frac{1}{x+1} = 1 - x + x^2 + \dots}$$

153

Find $\frac{dy}{dx}$ when $y = (x-5)(3-x)$. (LHR-2015)

Solution:

$$y = (x-5)(3-x)$$

$$y = 3x - x^2 - 15 + 5x$$

$$y = 8x - x^2 - 15$$

Differentiate w.r.t. "x":

$$\frac{dy}{dx} = 8 - 2x - 0$$

$$\boxed{\frac{dy}{dx} = 8 - 2x}$$

Alternate:

$$y = (x-5)(3-x)$$

Differentiate w.r.t. "x":

$$\frac{dy}{dx} = \frac{d}{dx}\left[(x-5)(3-x)\right]$$

$$\frac{dy}{dx} = (x-5)\frac{d}{dx}(3-x) + (3-x)\frac{d}{dx}(x-5)$$

$$\frac{dy}{dx} = (x-5)(-1) + (3-x)(1)$$

$$\frac{dy}{dx} = -x + 5 + 3 - x$$

$$\boxed{\frac{dy}{dx} = 8 - 2x}$$

154

Compute $\frac{dy}{dx}$ when $y = \frac{ax+b}{cx+d}$ (LHR-2015)

Solution:

$$y = \frac{ax+b}{cx+d}$$

$$\frac{dy}{dx} = \frac{(cx+d)\frac{d}{dx}(ax+b) - (ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$ (By using quotient rule)

$$\frac{dy}{dx} = \frac{(cx+d)(a) - (ax+b)(c)}{(cx+d)^2}$$

$$\frac{dy}{dx} = \frac{acx + ad - acx - bc}{(cx+d)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{ad-bc}{(cx+d)^2}}$$

155

Differentiate $\sec^{-1}x$ w.r.t x.

(LHR-2015)

Solution:

Let $y = \sec^{-1}x$

$\sec y = x$

$$\sec y \tan y \frac{dy}{dx} = \frac{d}{dx}(x)$$

$$\sec y \tan y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

$$\frac{dy}{dx} = \frac{1}{\sec y(\sqrt{\sec^2 y - 1})} \quad \because 1+\tan^2\theta = \sec^2\theta$$

$$\boxed{\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}} \quad \left\{\because \sec y = x\right.$$

156

If $y = \tan(p\tan^{-1}x)$, find $\frac{dy}{dx}$.

(LHR-2015)

Solution:

$y = \tan(p\tan^{-1}x)$

$\tan^{-1}y = p\tan^{-1}x$

$$\frac{1}{1+y^2} \cdot \frac{dy}{dx} = p \cdot \frac{1}{1+x^2}$$

$$\boxed{\frac{dy}{dx} = p\frac{(1+y^2)}{(1+x^2)}}$$

157

Find $f'(x)$ when $f(x) = x^3 e^{\frac{1}{x}}$. (LHR-2015)

Solution:

$$f(x) = x^3 e^{\frac{1}{x}}$$

Differentiate w.r.t. "x",

$$f'(x) = x^3 \frac{d}{dx}\left(e^{\frac{1}{x}}\right) + e^{\frac{1}{x}} \frac{d}{dx}(x^3)$$

$$f'(x) = x^3 . e^{\frac{1}{x}} . \frac{d}{dx}\left(\frac{1}{x}\right) + e^{\frac{1}{x}} 3x^2$$

$$f'(x) = x^3 . e^{\frac{1}{x}} . \left(\frac{-1}{x^2}\right) + 3x^2 e^{\frac{1}{x}}$$

$$f'(x) = -x e^{\frac{1}{x}} + 3x^2 e^{\frac{1}{x}}$$

$$\boxed{f'(x) = x e^{\frac{1}{x}} [3x - 1]}$$

158

Find $\frac{dy}{dx}$ when $y = a\cos(\ln x) + b\sin(\ln x)$ (LHR-2015)

Solution:

$$y = a\cos(\ln x) + b\sin(\ln x)$$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = a(-\sin \ln x) \frac{d}{dx} \ln x + b(\cos \ln x) \frac{d}{dx} \ln x$$

$$\frac{dy}{dx} = -a\sin \ln x \cdot \frac{1}{x} + b\cos \ln x \cdot \frac{1}{x}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{x}\left[b\cos \ln x - a\sin \ln x\right]}$$

59

Find $\frac{dy}{dx}$ when $y = \sinh^{-1}\left(\frac{x}{2}\right)$.

Solution: (LHR-2015+2017+2018)

$$y = \sinh^{-1}\left(\frac{x}{2}\right)$$

Differentiate w.r.t. "x":

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+\left(\frac{x}{2}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{2}\right) \quad \left\{\because \sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}\right.$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{4+x^2}{4}}} \cdot \frac{1}{2}\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{4+x^2}} \cdot \frac{1}{2}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{4+x^2}}}$$

60

Differentiate x^2 by definition.

Solution: (LHR-2015+2016)

Let: $y = x^2$

$$y + \delta y = (x+\delta x)^2$$

$$\delta y = (x+\delta x)^2 - x^2$$

$$\delta y = x^2 + \delta x^2 + 2x\delta x - x^2$$

$$\delta y = \delta x(\delta x + 2x)$$

Dividing both sides by δx:

$$\frac{\delta y}{\delta x} = \frac{\delta x(\delta x + 2x)}{\delta x}$$

$$\frac{\delta y}{\delta x} = \delta x + 2x$$

Taking limit $\delta x \to 0$:

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}(\delta x + 2x) \Rightarrow \frac{dy}{dx} = 0 + 2x$$

$$\boxed{\frac{dy}{dx} = 2x}$$

161

Differentiate $\sin x$ w.r.t. $\cot x$.

(LHR-2015+2016 +2021)

Solution:

Let:

$y = \sin x$; $u = \cot x$

Differentiate w.r.t "x":

$\frac{dy}{dx} = \cos x$; $\frac{du}{dx} = -\operatorname{cosec}^2 x$

By using chain rule,

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{dx} = \cos x \cdot \frac{-1}{\operatorname{cosec}^2 x} \Rightarrow \boxed{\frac{dy}{dx} = -\cos x \sin^2 x}$$

162

If $f(x) = \ln(e^x + e^{-x})$ then find $f'(x)$.

Solution:

(LHR-2015+2016)

$$f(x) = \ln(e^x + e^{-x})$$

Differentiate w.r.t "x":

$$f'(x) = \frac{1}{e^x + e^{-x}} \cdot \frac{d}{dx}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{e^x + e^{-x}} \cdot (e^x(1) + e^{-x}(-1))$$

$$f'(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$f'(x) = \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}}$$

$$f'(x) = \frac{\frac{e^{2x} - 1}{2}}{\frac{e^{2x} + 1}{2}} \Rightarrow \boxed{f'(x) = \frac{e^{2x} - 1}{e^{2x} + 1}}$$

163

Find y_2 if $x^2+y^2=a^2$. (LHR-2015)

Solution: $x^2+y^2=a^2$

$$2x+2y\frac{dy}{dx}=0$$

$$2yy_1=-2x$$

$$\Rightarrow \quad y_1=-\frac{x}{y}$$

Again, Differentiate w.r.t "x":

$$y_2=\frac{d}{dx}\left(-\frac{x}{y}\right)$$

By using quotient rule,

$$y_2=-\left[\frac{y\frac{d}{dx}(x)-x\frac{dy}{dx}}{y^2}\right]$$

$$y_2=-\left[\frac{y(1)-xy_1}{y^2}\right] \Rightarrow y_2=-\left[\frac{y-x\left(\frac{-x}{y}\right)}{y^2}\right]$$

$$y_2=-\left[\frac{y^2+x^2}{y^3}\right] \Rightarrow \boxed{y_2=\frac{-a^2}{y^3}} \qquad \{\because x^2+y^2=a^2$$

164

Find $\frac{dy}{dx}$ if $y=\ln\tanh x$. (LHR-2015 +2021)

Solution: $y=\ln\tanh x$

Differentiate w.r.t "x":

$$\frac{dy}{dx}=\frac{1}{\tanh x}\cdot\frac{d}{dx}(\tanh x)$$

$$\frac{dy}{dx}=\frac{1}{\tanh x}\cdot\operatorname{sech}^2 x$$

$$\frac{dy}{dx}=\frac{1}{\frac{\sinh x}{\cosh x}}\cdot\frac{1}{\cosh^2 x}=\frac{\cosh x}{\sinh x}\cdot\frac{1}{\cosh^2 x}=\frac{1}{\sinh x\cosh x}$$

$$\frac{dy}{dx}=\frac{2}{2\sinh x\cosh x}=\frac{2}{\sinh 2x} \Rightarrow \boxed{\frac{dy}{dx}=2\operatorname{cosech}2x}$$

65

Find $\frac{dy}{dx}$ if $xy + y^2 = 2$ (LHR-2015+2016+2018+2021)

Solution:

$$xy + y^2 = 2$$

Differentiate w.r.t "x":

$$x\frac{dy}{dx} + y\frac{d}{dx}(x) + 2y\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} + y(1) = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-y}{x+2y}}$$

66

Differentiate w.r.t "x" to $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$.

Solution: (LHR-2016+2018+2021)

Let $y = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$

$$y = x + \frac{1}{x} - 2 \Rightarrow y = x + x^{-1} - 2$$

Differentiate w.r.t. "x":

$$\frac{dy}{dx} = \frac{d}{dx}(x + x^{-1} - 2)$$

$$\frac{dy}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}(x)^{-1} - \frac{d}{dx}(2)$$

$$\frac{dy}{dx} = 1 - x^{-2} - 0$$

$$\frac{dy}{dx} = 1 - \frac{1}{x^2} \Rightarrow \boxed{\frac{dy}{dx} = \frac{x^2 - 1}{x^2}}$$

167

Differentiate $\cos\sqrt{x} + \sqrt{\sin x}$ w.r.t x.

(LHR-2016)

Solution:

Let $y = \cos\sqrt{x} + \sqrt{\sin x}$

Differentiate w.r.t "x",

$$\frac{dy}{dx} = -\sin\sqrt{x}\,\frac{d}{dx}(x)^{\frac{1}{2}} + \frac{1}{2\sqrt{\sin x}}\,\frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = -\sin\sqrt{x}\cdot\frac{1}{2\sqrt{x}} + \frac{\cos x}{2\sqrt{\sin x}}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2}\left[\frac{\cos x}{\sqrt{\sin x}} - \frac{\sin\sqrt{x}}{\sqrt{2}}\right]}$$

168

Find $\frac{dy}{dx}$ if $y = e^{-x}(x^3+2x^2+1)$.

(LHR-2016)

Solution:

$$y = e^{-x}(x^3+2x^2+1)$$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{-x}(x^3+2x^2+1)\right)$$

By Product rule,

$$\frac{dy}{dx} = e^{-x}\frac{d}{dx}(x^3+2x^2+1) + (x^3+2x^2+1)\frac{d}{dx}(e^{-x})$$

$$\frac{dy}{dx} = e^{-x}(3x^2+4x+0) + (x^3+2x^2+1)(e^{-x}(-1))$$

$$\frac{dy}{dx} = e^{-x}(3x^2+4x-x^3-2x^2-1)$$

$$\boxed{\frac{dy}{dx} = e^{-x}(-x^3+x^2+4x-1)}$$

169

Find $\frac{dy}{d\theta}$ if $y = \sin 2\theta - \cos 3\theta$.

(LHR-2016)

Solution:

$$y = \sin 2\theta - \cos 3\theta$$

Differentiate w.r.t "x":

$$\frac{dy}{d\theta} = \cos 2\theta \frac{d}{d\theta}(2\theta) - (-\sin 3\theta)\frac{d}{d\theta}(3\theta)$$

$$\frac{dy}{d\theta} = \cos 2\theta(2) + \sin 3\theta(3)$$

$$\boxed{\frac{dy}{d\theta} = 2\cos 2\theta + 3\sin 3\theta}$$

170

If $y = \cosh x$, prove that $\frac{dy}{dx} = \sinh x$.

Solution:

(LHR-2016)

$$y = \cosh x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{e^x + e^{-x}}{2}\right] \qquad \left\{\because \cosh x = \frac{e^x + e^{-x}}{2}\right.$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(e^x + e^{-x})$$

$$\frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x})$$

$$\frac{dy}{dx} = \frac{e^x - e^{-x}}{2}$$

$$\boxed{\frac{dy}{dx} = \sinh x} \qquad \left\{\because \sinh x = \frac{e^x - e^{-x}}{2}\right.$$

Hence proved.

171

Find derivative of $\sqrt{\frac{a-x}{a+x}}$ (LHR-2017)

Solution: Let $y=\sqrt{\frac{a-x}{a+x}}$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\frac{a-x}{a+x}}}\cdot\frac{d}{dx}\left(\frac{a-x}{a+x}\right)$$

$$\frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{a+x}{a-x}}\cdot\frac{(a+x)\frac{d}{dx}(a-x)-(a-x)\frac{d}{dx}(a+x)}{(a+x)^2}$$

$$\frac{dy}{dx}=\frac{1}{2}\cdot\frac{\sqrt{a+x}}{\sqrt{a-x}}\cdot\frac{(a+x)(-1)-(a-x)(1)}{(a+x)^2}$$

$$\frac{dy}{dx}=\frac{1}{2}\frac{\sqrt{a+x}}{\sqrt{a-x}}\cdot\frac{-a-x-a+x}{(a+x)^2}$$

$$\frac{dy}{dx}=\frac{1}{2}\cdot\frac{-2a}{\sqrt{a-x}\,(a+x)^{\frac{3}{2}}}\Rightarrow\boxed{\frac{dy}{dx}=\frac{-a}{\sqrt{a-x}\,(a+x)^{\frac{3}{2}}}}$$

172

Differentiate $\frac{ax^2+b}{ax^2+d}$. (LHR-2017)

Solution: Let $y=\frac{ax^2+b}{ax^2+d}$

$$\frac{dy}{dx}=\frac{(ax^2+d)\frac{d}{dx}(ax^2+b)-(ax^2+b)\frac{d}{dx}(ax^2+d)}{(ax^2+d)^2}$$

$$\frac{dy}{dx}=\frac{(ax^2+d)(2ax)-(ax^2+b)(2ax)}{(ax^2+d)^2}$$

$$\frac{dy}{dx}=\frac{2ax(ax^2+d-ax^2-b)}{(ax^2+d)^2}\Rightarrow\boxed{\frac{dy}{dx}=\frac{2ax(d-b)}{(ax^2+d)^2}}$$

173

Find $f'(x)$ if $f(x) = e^x(1+\ln x)$

Solution: (LHR-2017)

$$f(x) = e^x(1+\ln x)$$

$$f'(x) = e^x \frac{d}{dx}(1+\ln x) + (1+\ln x)\frac{d}{dx} e^x \quad \text{(By using Product rule)}$$

$$f'(x) = e^x \cdot \frac{1}{x} + (1+\ln x)e^x$$

$$f'(x) = e^x \left[\frac{1}{x} + (1+\ln x)\right]$$

$$\boxed{f'(x) = e^x \left[\frac{1+x(1+\ln x)}{x}\right]}$$

174

Differentiate $x^2 \sec 4x$. (LHR-2017)

Solution: Let:

$$y = x^2 \sec 4x$$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \sec 4x)$$

By using product rule,

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sec 4x) + \sec 4x \frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = x^2 (\sec 4x \tan x)\frac{d}{dx}(4x) + \sec 4x\,(2x)\frac{d}{dx}(x).$$

$$\frac{dy}{dx} = x^2\, 4\sec 4x \tan 4x + 2x \sec 4x$$

$$\boxed{\frac{dy}{dx} = 2x \sec 4x\,(2x \tan 4x + 1)}$$

175

If $y=(x^2+5)(x^3+7)$, find $\frac{dy}{dx}$. (LHR-2017 +2021)

Solution:

$$y=(x^2+5)(x^3+7)$$

$$y=(x^5+7x^2+5x^3+35)$$

Differentiate w.r.t "x".

$$\frac{dy}{dx}=5x^4+14x+15x^2+0$$

$$\boxed{\frac{dy}{dx}=5x^4+15x^2+14x}$$

Alternate: $y=(x^2+5)(x^3+7)$

Differentiate w.r.t "x".

$$\frac{dy}{dx}=\frac{d}{dx}\left[(x^2+5)(x^3+7)\right]$$

By product rule,

$$\frac{dy}{dx}=(x^2+5)\frac{d}{dx}(x^3+7)+(x^3+7)\frac{d}{dx}(x^2+5)$$

$$\frac{dy}{dx}=(x^2+5)(3x^2)+(x^3+7)(2x)$$

$$\frac{dy}{dx}=3x^4+15x^2+2x^4+14x$$

$$\boxed{\frac{dy}{dx}=5x^4+15x^2+14x}$$

176

If $y=\sinh^{-1}(ax+b)$ then find $\frac{dy}{dx}$.

(LHR-2017)

Solution:

$$y=\sinh^{-1}(ax+b)$$

Let: $u=ax+b$

$$y=\sinh^{-1}u \quad ; \quad u=ax+b$$

$$\frac{dy}{du}=\frac{1}{\sqrt{1+u^2}} \quad ; \quad \frac{du}{dx}=a$$

$$\frac{dy}{du}=\frac{1}{\sqrt{1+(ax+b)^2}} \quad ; \quad \frac{du}{dx}=a$$

By using chain rule,

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1+(ax+b)^2}}\cdot a$$

$$\boxed{\frac{dy}{dx}=\frac{a}{\sqrt{1+(ax+b)^2}}}$$

Alternate:

$$y=\sinh^{-1}(ax+b)$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1+(ax+b)^2}}\cdot\frac{d}{dx}(ax+b)$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1+(ax+b)^2}}\cdot(a)$$

$$\boxed{\frac{dy}{dx}=\frac{a}{\sqrt{(1+(ax+b)^2)}}}$$

177

If $y = x\cos y$, find $\frac{dy}{dx}$. (LHR-2017+2018)

Solution: $y = x\cos y$

Differentiate w.r.t. "x":

$$\frac{dy}{dx} = \frac{d}{dx}(x\cos y)$$

By using product rule,

$$\frac{dy}{dx} = x\frac{d}{dx}\cos y + \cos y\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x(-\sin y)\frac{dy}{dx} + \cos y(1)$$

$$\frac{dy}{dx} = -x\sin y\frac{dy}{dx} + \cos y$$

$$\frac{dy}{dx} + x\sin y\frac{dy}{dx} = \cos y$$

$$\frac{dy}{dx}(1 + x\sin y) = \cos y$$

$$\boxed{\frac{dy}{dx} = \frac{\cos y}{1 + x\sin y}}$$

178

Find $\frac{dy}{dx}$ if $y = \ln(x + \sqrt{x^2+1})$ (LHR-2018)

Solution: $y = \ln(x + \sqrt{x^2+1})$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}(x + \sqrt{x^2+1})$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\cdot\left(1 + \frac{1}{2\sqrt{x^2+1}}(2x)\right)$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) \Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}}$$

179

Find $\frac{dy}{dx}$ if $y = e^{-2x}\sin 2x$. (LHR-2018)

Solution: $y = e^{-2x}\sin 2x$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = \frac{d}{dx}(e^{-2x}\sin 2x)$$

By using product rule,

$$\frac{dy}{dx} = e^{-2x}\frac{d}{dx}(\sin 2x) + \sin 2x\frac{d}{dx}(e^{-2x})$$

$$\frac{dy}{dx} = e^{-2x}(\cos 2x)\frac{d}{dx}(2x) + \sin 2x(e^{-2x})\frac{d}{dx}(-2x)$$

$$\frac{dy}{dx} = e^{-2x}\cos 2x\,(2) + \sin 2x\, e^{-2x}(-2)$$

$$\boxed{\frac{dy}{dx} = 2e^{-2x}(\cos 2x - \sin 2x)}$$

180

Find $\frac{d^2y}{dx^2}$ if $y^3 + 3ax^2 + x^3 = 0$ (LHR-2018)

Solution: $y^3 + 3ax^2 + x^3 = 0$

$$3y^2 y_1 + 6ax + 3x^2 = 0$$

$$3y^2 y_1 = -6ax - 3x^2$$

$$3y^2 y_1 = -3(x^2 + 2ax)$$

$$y_1 = \frac{-(2ax + x^2)}{y^2}$$

Again differentiate w.r.t "x",

$$\frac{d}{dx}y_1 = -\frac{d}{dx}\left(\frac{2ax + x^2}{y^2}\right)$$

By using quotient rule,

$$\frac{d^2y}{dx^2} = -\left[\frac{y^2\frac{d}{dx}(2ax+x^2)-(2ax+x^2)\frac{d}{dx}(y^2)}{(y^2)^2}\right]$$

$$\frac{d^2y}{dx^2} = -\left[\frac{y^2(2a+2x)-(2ax+x^2)(2y)\frac{dy}{dx}}{y^4}\right]$$

$$\frac{d^2y}{dx^2} = -\left[\frac{2y^2(a+x)-2y(2ax+x^2)\left(-\left(\frac{2ax+x^2}{y^2}\right)\right)}{y^4}\right]$$

$$\frac{d^2y}{dx^2} = -\frac{2}{y^4}\left[\frac{y^3(a+x)+(2ax+x^2)^2}{y}\right]$$

$$\frac{d^2y}{dx^2} = -2\left[\frac{y^3(a+x)+x^2(2a+x)^2}{y^5}\right]$$

$$\frac{d^2y}{dx^2} = -2\left[\frac{(-x^3-3ax^2)(a+x)+x^2(4a^2+x^2+4ax)}{y^5}\right]$$

$$\frac{d^2y}{dx^2} = -2\left[\frac{-x^2(3a+x)(a+x)+x^2(4a^2+x^2+4ax)}{y^5}\right]$$

$$\frac{d^2y}{dx^2} = -2x^2\left[\frac{-(3a^2+3ax+ax+x^2)+(4a^2+x^2+4ax)}{y^5}\right]$$

$$\frac{d^2y}{dx^2} = -2x^2\left[\frac{-3a^2-4ax-x^2+4a^2+x^2+4ax}{y^5}\right]$$

$$\frac{d^2y}{dx^2} = \frac{-2x^2(a^2)}{y^5}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-2a^2x^2}{y^5}}$$

181

Find y_2 if $y = \cos^3 x$. (LHR-2018)

Solution:

$$y = \cos^3 x$$

Differentiate w.r.t "x",

$$y_1 = 3\cos^2 x \frac{d}{dx}(\cos x)$$

$$y_1 = 3\cos^2 x(-\sin x)$$

$$y_1 = -3(1-\sin^2 x)\sin x$$

$$y_1 = -3\sin x + 3\sin^3 x$$

Again, differentiate w.r.t "x":

$$y_2 = -3\cos x + 3 \cdot 3\sin^2 x(\cos x)$$

$$y_2 = -3\cos x + 9(1-\cos^2 x)\cos x$$

$$y_2 = -3\cos x + 9\cos x - 9\cos^3 x$$

$$\boxed{y_2 = 6\cos x - 9\cos^3 x}$$

182

Find $\frac{dy}{dx}$ if $y = \ln\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}}$. (LHR-2018)

Solution:

$$y = \ln\left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{2}} \quad ; \quad y = \frac{1}{2}\ln\left(\frac{x^2-1}{x^2+1}\right)$$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{\left(\frac{x^2-1}{x^2+1}\right)}\cdot\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{(x^2+1)}{(x^2-1)}\cdot\frac{(x^2+1)\frac{d}{dx}(x^2-1)-(x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{(x^2+1)}{(x^2-1)}\cdot\frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}}\cdot\frac{(x^2+1)}{(x^2-1)}\cdot\frac{\cancel{2}x(\cancel{x^2}+1-\cancel{x^2}+1)}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2+1)}{(x^2-1)} \cdot \frac{2x}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{2x}{(x^2-1)(x^2+1)}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2x}{x^4-1}}$$

183

Prove that $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$. (LHR-2018)

Solution: Let.

$$y = \cos^{-1}x$$

$$\cos y = x$$

Differentiate w.r.t "x":

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\sin y}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2 y}} \qquad \left\{\because \sin^2 x + \cos^2 x = 1\right.$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} \qquad \left\{\because x = \cos y\right.$$

$$\boxed{\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}}$$

Hence proved.

Find $\frac{dy}{dx}$ $x^2 \ln\frac{1}{x}$. (LHR-2021)

Solution:

Let $y = x^2 \ln\frac{1}{x}$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 \ln\frac{1}{x}\right)$$

By product rule,

$$\frac{dy}{dx} = x^2\frac{d}{dx}\ln\frac{1}{x} + \ln\frac{1}{x}\frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = x^2 \cdot \frac{1}{\frac{1}{x}}\frac{d}{dx}\cdot\frac{1}{x} + \ln\frac{1}{x}\cdot 2x$$

$$\frac{dy}{dx} = x^3\left(-\frac{1}{x^2}\right) + 2x\ln\frac{1}{x}$$

$$\frac{dy}{x} = -x + 2x\ln\frac{1}{x}$$

$$\boxed{\frac{dy}{dx} = x\left[2\ln\frac{1}{x} - 1\right]}$$

185

Find $f'(x)$ if $f(x) = \sqrt{\ln(e^{2x}+e^{-2x})}$. (LHR-2021)

Solution:

$$f(x) = \sqrt{\ln(e^{2x}+e^{-2x})}$$

$$f'(x) = \frac{1}{2\sqrt{\ln(e^{2x}+e^{-2x})}}\cdot\frac{d}{dx}\ln(e^{2x}+e^{-2x})$$

$$f'(x) = \frac{1}{2\sqrt{\ln(e^{2x}+e^{-2x})}}\cdot\frac{1}{e^{2x}+e^{-2x}}\frac{d}{dx}(e^{2x}+e^{-2x})$$

$$f'(x) = \frac{1}{2(e^{2x}+e^{-2x})\sqrt{\ln(e^{2x}+e^{-2x})}}\left(e^{2x}\cdot 2 + e^{2x}(-2)\right)$$

$$f'(x) = \frac{2(e^{2x}-e^{-2x})}{2(e^{2x}+e^{-2x})\sqrt{\ln(e^{2x}+e^{-2x})}} \Rightarrow \boxed{f'(x) = \frac{e^{2x}-e^{-2x}}{(e^{2x}+e^{-2x})\sqrt{\ln(e^{2x}+e^{-2x})}}}$$

Find $\frac{dy}{dx}$ if $y = \frac{1}{\sqrt{x+a}}$ (2019)

Solution: Let $y = (x+a)^{-\frac{1}{2}}$

$$y+\delta y = (x+\delta x+a)^{-\frac{1}{2}}$$

$$\delta y = (x+\delta x+a)^{-\frac{1}{2}} - (x+a)^{-\frac{1}{2}}$$

$$\delta y = (x+a)^{-\frac{1}{2}}\left(1+\frac{\delta x}{x+a}\right)^{-\frac{1}{2}} - (x+a)^{-\frac{1}{2}}$$

$$\delta y = (x+a)^{-\frac{1}{2}}\left\{\left(1+\frac{\delta x}{x+a}\right)^{-\frac{1}{2}} - 1\right\}$$

By using binomial series;

$$\delta y = (x+a)^{-\frac{1}{2}}\left\{1+\left(-\frac{1}{2}\right)\left(\frac{\delta x}{x+a}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x+a}\right)^2+\cdots-1\right\}$$

$$\delta y = (x+a)^{-\frac{1}{2}}\left\{\left(\frac{\delta x}{x+a}\right)\left(-\frac{1}{2}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x+a}\right)+\cdots\right)\right\}$$

$$\delta y = (x+a)^{-\frac{1}{2}-1}\cdot\delta x\left(-\frac{1}{2}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x+a}\right)+\cdots\right)$$

Dividing both sides by δx:

$$\frac{\delta y}{\delta x} = \frac{(x+a)^{-\frac{3}{2}}\cdot\delta x\left(-\frac{1}{2}+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x+a}\right)+\cdots\right)}{\delta x}$$

$$\frac{\delta y}{\delta x} = (x+a)^{-\frac{3}{2}}\left(-\frac{1}{2}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x+a}\right)+\cdots\right)$$

Taking limit on both sides;

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = (x+a)^{-\frac{3}{2}}\lim_{\delta x\to 0}\left(-\frac{1}{2}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x+a}\right)+\cdots\right)$$

$$\frac{dy}{dx} = (x+a)^{-\frac{3}{2}}\left(-\frac{1}{2}+0\right)$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{2(x+a)^{\frac{3}{2}}}}$$

(87)

Find $\frac{dy}{dx}$ if $y = (3x^2 - 2x + 7)^6$ (LHR-2019)

Solution: Let $3x^2 - 2x + 7 = u$

$y = u^6$; $u = 3x^2 - 2x + 7$

$\frac{dy}{du} = 6u^5$; $\frac{du}{dx} = 6x - 2$

$\frac{dy}{du} = 6(3x^2 - 2x + 7)^5$

By chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = 6(3x^2 - 2x + 7)^5 \cdot (6x - 2)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 6(6x - 2)(3x^2 - 2x + 7)^5}$$

(88)

Find $\frac{dy}{dx}$ if $y = \tan^3\theta \sec^2\theta$. (LHR-2019)

Solution: $y = \tan^3\theta \sec^2\theta$

Differentiate w.r.t "θ":

$$\frac{dy}{dx} = \tan^3\theta \frac{d}{d\theta}\sec^2\theta + \sec^2\theta \frac{d}{d\theta}\tan^3\theta \quad \text{(By using Product rule)}$$

$$\frac{dy}{dx} = \tan^3\theta \, 2\sec\theta \cdot \sec\theta\tan\theta + \sec^2\theta \, 3\tan^2\theta \cdot \sec^2\theta.$$

$$\frac{dy}{dx} = 2\sec^2\theta\tan^4\theta + 3\tan^2\theta\sec^4\theta$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \sec^2\theta\tan^2\theta(2\tan^2\theta + 3\sec^2\theta)}$$

(89)

Find $\frac{dy}{dx}$ if $y = \sinh^{-1}(x^3)$

(LHR-2019)

Solution:

$$y = \sinh^{-1}(x^3)$$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+(x^3)^2}}\frac{d}{dx}(x^3)$$

$$\boxed{\frac{dy}{dx} = \frac{3x^2}{\sqrt{1+x^6}}}$$

(90)

Define stationary point. (LHR-2011+2013+2014)

The point at which the function is neither increasing nor decreasing i.e. $f'(x)=0$ is known as stationary point.

(91)

What is Taylor theorem expansion of a function. (LHR-2012)

If x and h are two independent quantities and $f(x+h)$ can be expanded in ascending power of h as an infinite series, then:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \dots + \frac{f^n(x)}{n!}h^n + \dots$$

is called "Taylor Theorem."

192

Determine the interval in which $f(x) = x^2+3x+2\ ;\ x \in (-4,1)$ is increasing.

(LHR-2014)

Solution:

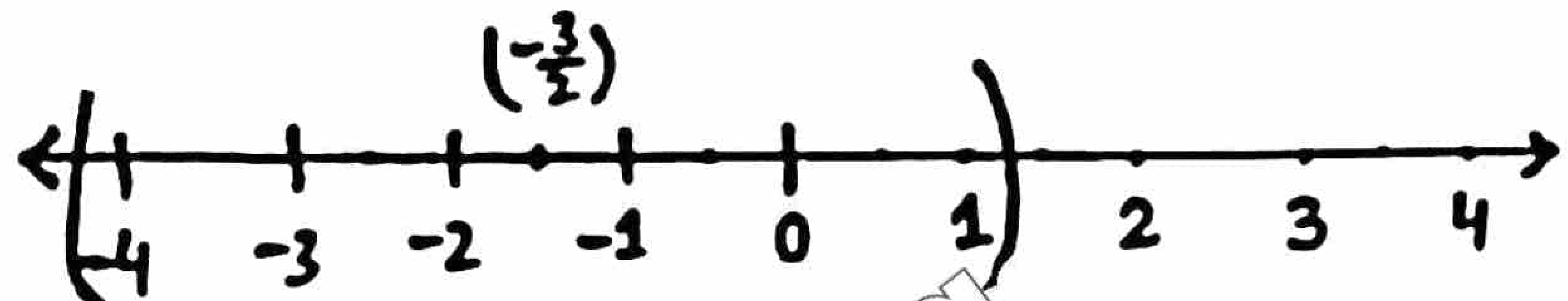
$$f(x) = x^2+3x+2 \quad ; x \in (-4,1)$$

$$f'(x) = 2x+3$$

For stationary point; Put $f'(x)=0$

$$2x+3=0$$

$$2x=-3 \quad \Rightarrow \boxed{x=-\frac{3}{2}}$$



For increasing function:

Given function is increasing in the interval $(-\frac{3}{2}, 1)$

At $x=-1$
$f'(x) = 2x+3$
$f'(x) = 2(-1)+3$
$f'(x) = 1$ +ve
increasing

193

Determine the interval in which $f(x) = 4-x^2\ ;\ x \in (-2,2)$ is increasing. (LHR-2017)

Solution:

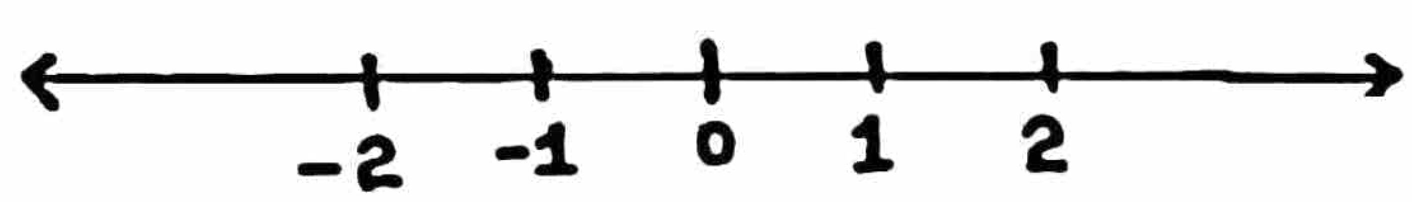
$$f(x) = 4-x^2 \quad ; x \in (-2,2)$$

$$f'(x) = -2x$$

For stationary point $f'(x)=0$:

$$-2x=0$$

$$\Rightarrow \boxed{x=0}$$



For increasing function:

Given function is increasing in the interval $(-2,0)$

$f'(x) = -2x$
at $x=-1$
$= -2(-1)$
$= 2 \rightarrow$ +ve $\Rightarrow$ increasing

94

Define derivative of a function.

(LHR-2012)

Let f be a function defined in an open interval containing a point x. The Derivative of f at x, denoted by $f'(x)$ is defined as:

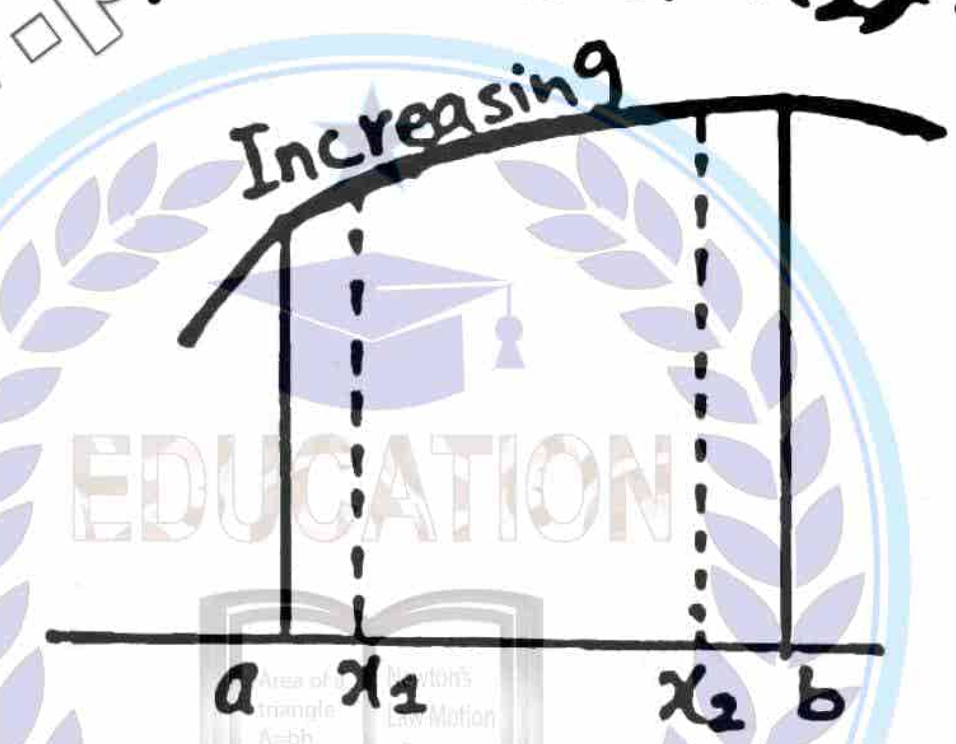
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

95

Find increasing and decreasing function. (LHR-2015+17+19)

Increasing function:

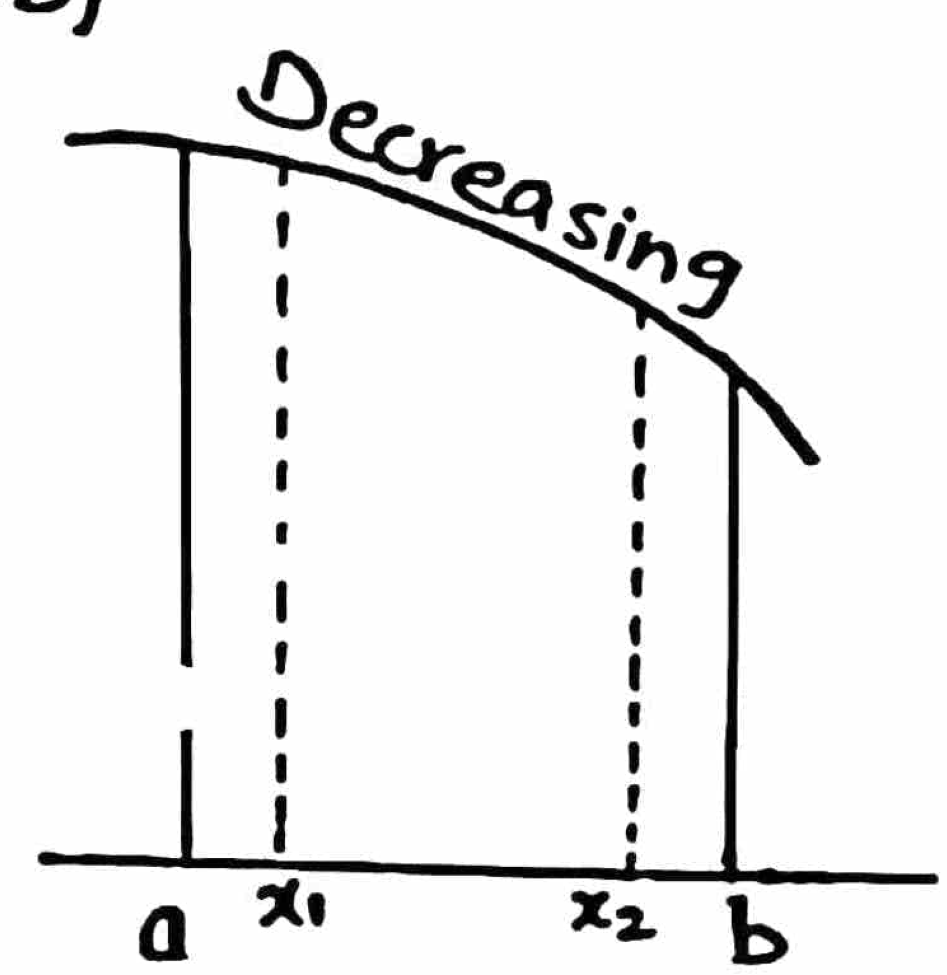
Let f be defined on an interval (a,b) and let $x_1, x_2 \in (a,b)$ then f is increasing on the interval (a,b) if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.



$f(x_2) > f(x_1)$ if $x_2 > x_1$

Decreasing function:

Let f be defined on an interval (a,b) and Let $x_1, x_2 \in (a,b)$ then f is decreasing on the interval (a,b) if $f(x_2) < f(x_1)$ whenever $x_2 < x_1$.



96

State Taylor series of a function f(x) at x = a. (LHR-2016)

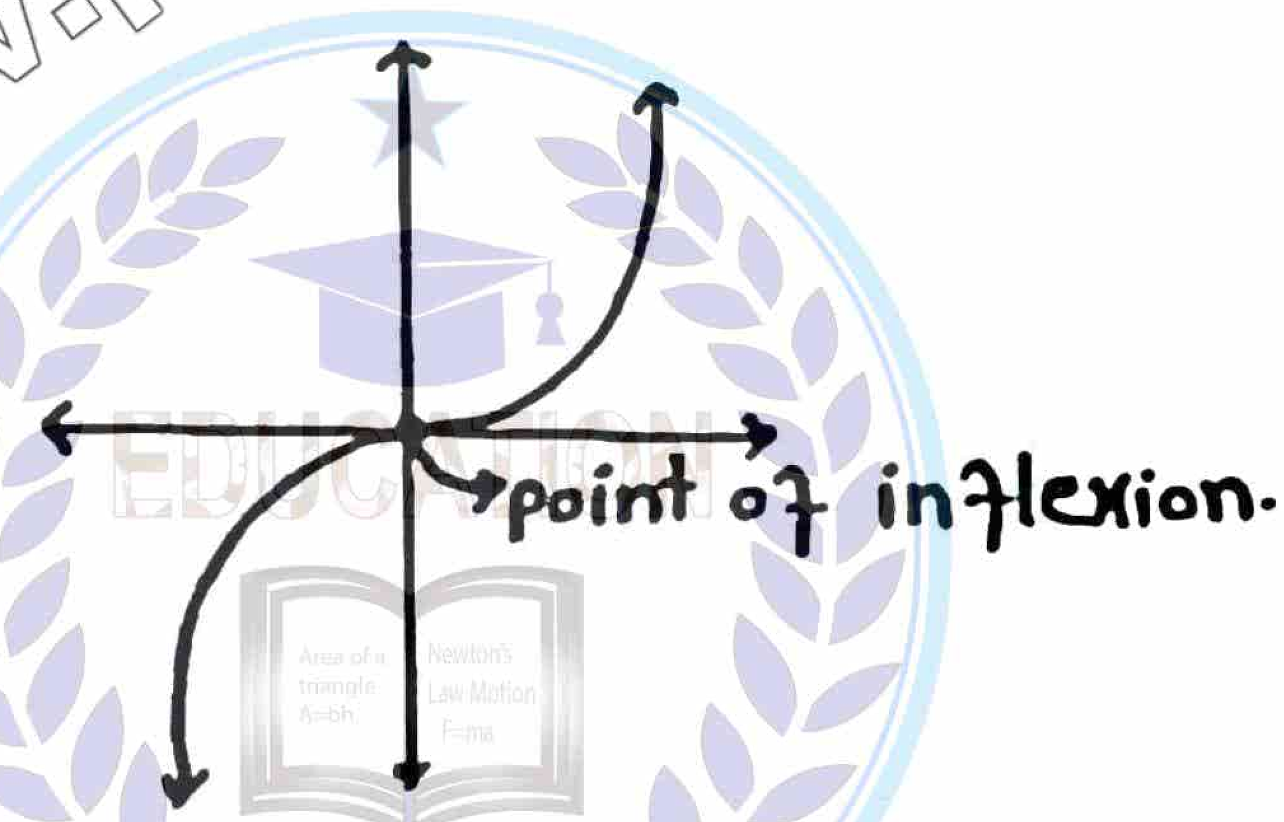
If f is defined in the interval containing 'a' and its derivatives of all orders exist at x = a, then we can expand f(x) as:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{n}(a)}{n!}(x-a)^n + \dots$$

97

Define point of inflexion. (LHR-2017)

A turning point where the function does not change sense, that is, the function changes from increasing to increasing or decreasing to decreasing is called "point of inflexion".



98

Define differentiation. (LHR-2018)

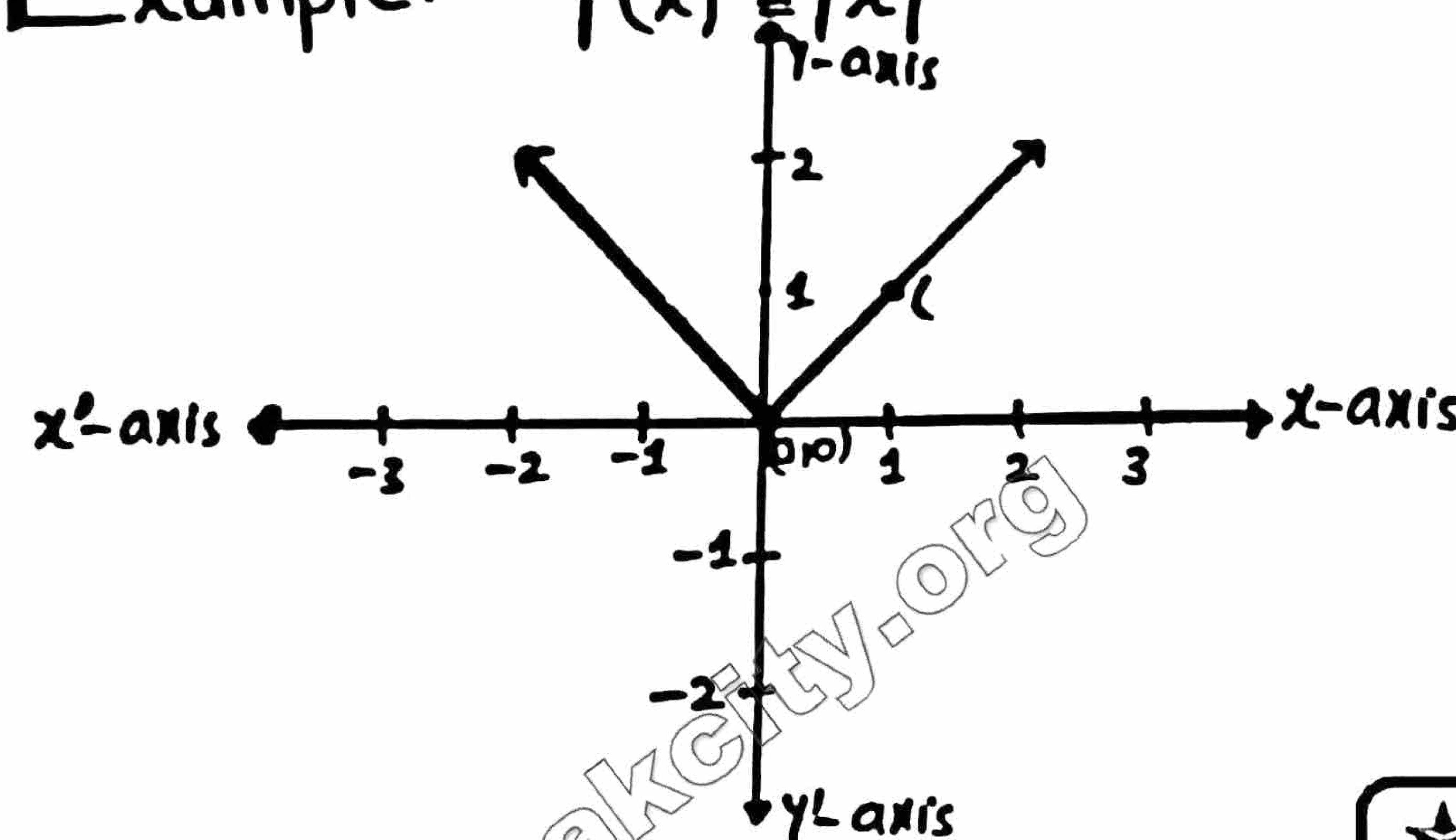
The process of finding f' is called "differentiation".

99

Define critical point. (LHR-2017)

A function $y = f(x)$, $c \in D_f$, if $f'(c) = 0$ or $f'(c)$ does not exist, the number c is called critical value while the point $(c, f(c))$ on the graph of f is called critical point.

Example: $f(x) = |x|$



100

Define power series. (LHR-2018)

A series of the form $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots + a_nx^n + \dots$ is called a power series expansion of a function $f(x)$, where $a_0, a_1, a_2, \dots, a_n \dots$ are constants and x is a variable.

Find critical values for $f(x)=5+3x-x^3$.

(LHR-2011)

Solution:

$$f(x)=5+3x-x^3$$

Differentiate w.r.t "x":

$$f'(x)=3-3x^2$$

Again, differentiate w.r.t "x":

$$f''(x)=-6x$$

For stationary point:

Put $f'(x)=0$

$$3-3x^2=0$$

$$-3x^2=-3$$

$$x^2=1 \Rightarrow x=\pm 1$$

at $x=1$:

$$f''(1)=-6(1)=-6<0$$

$\Rightarrow f(x)$ has relative maxima at $x=1$.

Now maximum value,

$$f(x)=5+3x-x^3$$

$$f(1)=5+3(1)-(1)^3$$

$$\boxed{f(1)=7}$$

at $x=-1$:

$$f''(-1)=-6(-1)=6>0$$

$\Rightarrow f(x)$ has relative minima at $x=-1$.

Now minimum value:

$$f(x)=5+3x-x^3$$

$$f(-1)=5+3(-1)-(-1)^3$$

$$\boxed{f(-1)=3}$$

pakcity.org

Divide 20 into two parts, so that the sum of their squares will be minimum. (LHR-2012)

Solution: Let the two parts of 20 are x and $20-x$ and the sum of their squares is y.

$$y = (x)^2 + (20-x)^2$$

$$y = x^2 + 400 + x^2 - 40x$$

$$y = 2x^2 - 40x + 400$$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = 4x - 40$$

Again, $\frac{d^2y}{dx^2} = 4$

For stationary point:

Put $f'(x) = 0$

$$4x - 40 = 0$$

$$x = 10 > 0$$

$\Rightarrow$ Given function has relative minima at $x=10$.

First part of $20 = x = 10$

Second part of $20 = 20 - x = 20 - 10 = 10$

Find the extreme value of $f(x) = x^2 - x - 2$ (LHR-2014+2018).

Solution:

Differentiate w.r.t "x":

$f'(x) = 2x - 1$

Again, differentiate w.r.t "x":

$f''(x) = 2$

For stationary point:

Put $f'(x) = 0$

$2x - 1 = 0$

$\boxed{x = \frac{1}{2}}$

At $x = \frac{1}{2}$

$f''\left(\frac{1}{2}\right) = 2 > 0$

So $f(x)$ has relative minima at $x = \frac{1}{2}$

Now minimum value.

$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 2$

$f\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} - 2$

$\boxed{f\left(\frac{1}{2}\right) = \frac{-9}{4}}$

Find critical values of $f(x) = \sin x + \cos x$. (LHR-2015)

Solution:

$$f(x) = \sin x + \cos x \text{ —— I}$$
$$f'(x) = \cos x - \sin x$$
$$f''(x) = -\sin x - \cos x$$

For stationary point:

$$f'(x) = 0$$
$$\cos x - \sin x = 0$$
$$\cos x = \sin x$$

Dividing both sides by "$\cos x$":

$$\frac{\cos x}{\cos x} = \frac{\sin x}{\cos x} \Rightarrow 1 = \tan x$$

$$x = \tan^{-1}(1) \Rightarrow x = \frac{\pi}{4}, 5\frac{\pi}{4}$$

at $x = \frac{\pi}{4}$:

$$f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$$
$$f''\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2} < 0$$

$\Rightarrow$ Given function is maximum.

Now for maximum value:

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$$
$$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

at $x = 5\frac{\pi}{4}$:

$$f''\left(5\frac{\pi}{4}\right) = -\sin\left(5\frac{\pi}{4}\right) - \cos\left(5\frac{\pi}{4}\right)$$
$$= -\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} > 0 \text{ (relative minima)}$$

Now for minimum value:

$$f\left(5\frac{\pi}{4}\right) = \sin\left(5\frac{\pi}{4}\right) + \cos\left(5\frac{\pi}{4}\right)$$
$$f\left(5\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$$

Long Questions

1

Prove that $y\frac{dy}{dx}+x=0$ if $x=\frac{1-t^2}{1+t^2}$, $y=\frac{2t}{1+t^2}$ (LHR-2012+2018)

Solution:

$$x=\frac{1-t^2}{1+t^2}\ ; \quad y=\frac{2t}{1+t^2}$$

Differentiate w.r.t "t".

$$\frac{dx}{dt}=\frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)\ ; \quad \frac{dy}{dt}=\frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$

By using quotient rule,

$$\frac{dx}{dt}=\frac{(1+t^2)\frac{d}{dt}(1-t^2)-(1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt}=\frac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2}$$

$$\frac{dx}{dt}=\frac{2t(-1-t^2-1+t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt}=\frac{-4t}{(1+t^2)^2}$$

$$\frac{dy}{dt}=\frac{(1+t^2)\frac{d}{dt}(2t)-2t\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt}=\frac{(1+t^2)(2)-2t(2t)}{(1+t^2)^2}$$

$$\frac{dy}{dt}=\frac{2+2t^2-4t^2}{(1+t^2)^2}$$

$$\frac{dy}{dt}=\frac{2-2t^2}{(1+t^2)^2} \Rightarrow \frac{dy}{dt}=\frac{2(1-t^2)}{(1+t^2)^2}$$

By chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{2(1-t^2)}{(1+t^2)^2} \cdot \frac{(1+t^2)^2}{-4t}$$

$$\frac{dy}{dx} = \frac{(1-t^2)}{-2t}$$

Given:
$$y\frac{dy}{dx} + x = 0$$

$$L.H.S = y\frac{dy}{dx} + x$$

$$= \frac{2t}{(1+t^2)} \cdot \frac{(1-t^2)}{-2t} + \frac{(1-t^2)}{(1+t^2)}$$

$$= \frac{-(1-t^2)}{(1+t^2)} + \frac{(1-t^2)}{(1+t^2)}$$

$$= 0$$
$$= R.H.S$$

$$\therefore \boxed{y\frac{dy}{dx} + x = 0}$$

Hence proved

If $y = (\cos^{-1}x)^2$, prove that $(1-x^2)y_2 - xy_1 - 2 = 0$

(LHR-2012 +2019)

Solution:

$$y = (\cos^{-1}x)^2$$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1}x)^2$$

$$\frac{dy}{dx} = 2\cos^{-1}x \frac{d}{dx}\left(\cos^{-1}x\right)$$

$$\frac{dy}{dx} = 2\cos^{-1}x \cdot \frac{-1}{\sqrt{1-x^2}} \cdot \frac{d}{dx}(x)$$

$$y_1 = 2\cos^{-1}x \cdot \frac{-1}{\sqrt{1-x^2}}(1)$$

$$y_1\sqrt{1-x^2} = -2\cos^{-1}x$$

Squaring on both sides:

$$y_1^2(1-x^2) = 4(\cos^{-1}x)^2$$

$$y_1^2(1-x^2) = 4y \qquad \{\because (\cos^{-1}x)^2 = y$$

Again differentiate w.r.t "x":

$$y_1^2(-2x) + (1-x^2)2y_1 \cdot y_2 - 4y_1 = 0$$

Dividing both sides by $2y_1$

$$-xy_1 + (1-x^2)y_2 - 2 = 0$$

$$\Rightarrow \boxed{(1-x^2)y_2 - xy_1 - 2 = 0}$$

Hence proved.

3

If $x=\sin\theta$ and $y=\sin m\theta$, show that

$(1-x^2)y_2 - xy_1 + m^2y = 0$ (LHR-2013)

Solution:

$$x=\sin\theta \Rightarrow \theta=\sin^{-1}x \quad \text{— I}$$

$$y=\sin m\theta \Rightarrow \theta=\frac{1}{m}\sin^{-1}y \quad \text{— II}$$

Comparing I and II:

$$\frac{1}{m}\sin^{-1}y=\sin^{-1}x$$

Differentiate w.r.t "x":

$$\frac{1}{m}\cdot\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

$$y_1\sqrt{1-x^2}=m\sqrt{1-y^2}$$

Squaring on both sides:

$$y_1^2(1-x^2)=m^2(1-y^2)$$

Again, differentiate w.r.t "x":

$$2y_1\cdot y_2(1-x^2)+y_1^2(-2x)=m^2(-2y)y_1$$

Dividing both sides by $2y_1$:

$$(1-x^2)y_2-xy_1=-m^2y$$

$$\Rightarrow \boxed{(1-x^2)y_2-xy_1+m^2y=0}$$

Hence proved.

(4)

Find $\frac{dy}{dx}$ if $x = a(\cos t + \sin t)$; $y = a(\sin t - t\cos t)$

(LHR-2013+2017)

Solution:

$x = a(\cos t + \sin t)$; $y = a(\sin t - t\cos t)$

Differentiate w.r.t "t":

$\frac{dx}{dt} = a(-\sin t + \cos t)$; $\frac{dy}{dt} = a\left(\cos t - (t(-\sin t) + \cos t(1))\right)$

$\frac{dx}{dt} = a(\cos t - \sin t)$; $\frac{dy}{dt} = a(\cos t + t\sin t - \cos t)$

; $\frac{dy}{dt} = at\sin t$

By chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = at\sin t \cdot \frac{1}{a(\cos t - \sin t)} \Rightarrow \boxed{\frac{dy}{dx} = \frac{t\sin t}{(\cos t - \sin t)}}$$

(5)

Differentiate $\frac{x^2+1}{x^2-1}$ w.r.t $\frac{x-1}{x+1}$. (LHR-2014)

Solution:

Let:

$y = \frac{x^2+1}{x^2-1}$ $u = \frac{x-1}{x+1}$

Differentiate w.r.t "x":

$\frac{dy}{dx} = \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$; $\frac{du}{dx} = \frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+1)}{(x+1)^2}$

$\frac{dy}{dx} = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$; $\frac{du}{dx} = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2}$

$$\frac{dy}{dx} = \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2} \quad ; \quad \frac{du}{dx} = \frac{x+1-x+1}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{-4x}{(x^2-1)^2} \quad ; \quad \frac{du}{dx} = \frac{2}{(x+1)^2}$$

By chain rule:

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\frac{dy}{dx} = \frac{-4x}{(x^2-1)^2} \cdot \frac{(x+1)^2}{2}$$

$$\frac{dy}{dx} = \frac{-2x(x+1)}{(x-1)(x+1)} \Rightarrow \boxed{\frac{dy}{dx} = \frac{-2x}{x-1}}$$

16

Differentiate $\cos\sqrt{x}$ from first principle

(LHR-2015)

Solution: Let:

$$y = \cos\sqrt{x}$$

$$y + \delta y = \cos\sqrt{x+\delta x}$$

$$\delta y = \cos\sqrt{x+\delta x} - \cos\sqrt{x} \Rightarrow \delta y = -2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)$$

Divide both sides by δx:

$$\frac{\delta y}{\delta x} = -2\left[\frac{\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x}\right]$$

$$\frac{\delta y}{\delta x} = -2\left[\frac{\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\sqrt{x+\delta x}+\sqrt{x}\right)\left(\sqrt{x+\delta x}-\sqrt{x}\right)}\right]$$

$$\frac{\delta y}{\delta x} = \frac{-2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\sqrt{x+\delta x}+\sqrt{x}\right)\, 2\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}$$

$$\frac{\delta y}{\delta x} = -\frac{\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)}{\sqrt{x+\delta x}+\sqrt{x}} \cdot \frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}$$

Taking limit $\delta x \to 0$ on both sides:

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = -\lim_{\delta x \to 0}\left(\frac{\sin\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}}{\sqrt{x+\delta x}+\sqrt{x}}\right) \cdot \lim_{\delta x \to 0}\left(\frac{\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}{\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}\right)$$

$$\frac{dy}{dx} = -\frac{\sin\frac{\sqrt{x+0}+\sqrt{x}}{2}}{\sqrt{x+0}+\sqrt{x}} \cdot 1 \qquad \left\{\because \lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1\right.$$

$$\frac{dy}{dx} = -\frac{\sin\frac{2\sqrt{x}}{2}}{2\sqrt{x}} \Rightarrow \boxed{\frac{dy}{dx} = -\frac{\sin\sqrt{x}}{2\sqrt{x}}}$$

B|7|B

Differentiate $x^2+\frac{1}{x^2}$ w.r.t. $x-\frac{1}{x}$. (LHR-2015)

Solution:

Let:

$$y = x^2+\frac{1}{x^2} \quad ; \quad u = x-\frac{1}{x}$$

Differentiate w.r.t "x":

$$\frac{dy}{dx^2} = 2x - 2x^{-3} \quad ; \quad \frac{du}{dx} = 1 + x^{-2}$$

$$\frac{dy}{dx} = 2\left(x - \frac{1}{x^3}\right) \quad ; \quad \frac{du}{dx} = 1+\frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{2(x^4-1)}{x^3} \quad ; \quad \frac{du}{dx} = \frac{x^2+1}{x^2}$$

By chain rule,

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dx}{du} = \frac{2(x^4-1)}{x^3} \cdot \frac{x^2}{x^2+1} = \frac{2x^2(x^2-1)(x^2+1)}{x^3(x^2+1)}$$

$$\frac{dy}{dx} = \frac{2(x^2-1)}{x} \Rightarrow \boxed{\frac{dy}{dx} = 2\left(x-\frac{1}{x}\right)}$$

Differentiate $\sin\sqrt{\frac{1+2x}{1+x}}$ w.r.t x. (LHR-2016)

Solution: Let $y = \sin\sqrt{\frac{1+2x}{1+x}}$

Differentiate w.r.t "x".

$$\frac{dy}{dx} = \cos\sqrt{\frac{1+2x}{1+x}}\ \frac{d}{dx}\left(\frac{1+2x}{1+x}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \cos\sqrt{\frac{1+2x}{1+x}} \cdot \frac{1}{2}\left(\frac{1+2x}{1+x}\right)^{-\frac{1}{2}}\ \frac{d}{dx}\left(\frac{1+2x}{1+x}\right)$$

$$\frac{dy}{dx} = \cos\sqrt{\frac{1+2x}{1+x}} \cdot \frac{1}{2\sqrt{\frac{1+2x}{1+x}}} \cdot \frac{(1+x)\frac{d}{dx}(1+2x)-(1+2x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$\frac{dy}{dx} = \cos\sqrt{\frac{1+2x}{1+x}} \cdot \frac{1}{2}\sqrt{\frac{1+x}{1+2x}} \cdot \frac{(1+x)(2)-(1+2x)(1)}{(1+x)^2}$$

$$\frac{dy}{dx} = \cos\sqrt{\frac{1+2x}{1+x}} \cdot \frac{1}{2}\sqrt{\frac{1+x}{1+2x}} \cdot \frac{2+2x-1-2x}{(1+x)^2}$$

$$\frac{dy}{dx} = \cos\sqrt{\frac{1+2x}{1+x}} \cdot \frac{1}{2\sqrt{1+2x}\,(1+x)^{\frac{3}{2}}}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{\cos\sqrt{\frac{1+2x}{1+x}}}{2\sqrt{1+2x}\,(1+x)^{\frac{3}{2}}}}$$

191

If $\frac{y}{x} = \tan^{-1}\left(\frac{x}{y}\right)$, then prove that $\frac{dy}{dx} = \frac{y}{x}$ (LHR-2016 +2019)

Solution: $\frac{y}{x} = \tan^{-1}\frac{x}{y}$

Differentiate w.r.t "x":

$$\frac{x\frac{dy}{dx} - y\frac{d}{dx}(x)}{x^2} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\frac{x\frac{dy}{dx} - y(1)}{x^2} = \frac{1}{\frac{y^2+x^2}{y^2}} \cdot \frac{y\frac{d}{dx}(x) - x\frac{dy}{dx}}{y^2}$$

$$\frac{x\frac{dy}{dx} - y}{x^2} = \frac{y^2}{x^2+y^2} \cdot \frac{y - x\frac{dy}{dx}}{y^2}$$

$$\frac{x\frac{dy}{dx} - y}{x^2} = \frac{y - x\frac{dy}{dx}}{x^2+y^2}$$

$$(x^2+y^2)\left(x\frac{dy}{dx} - y\right) = x^2\left(y - x\frac{dy}{dx}\right)$$

$$x^3\frac{dy}{dx} - x^2y + xy^2\frac{dy}{dx} - y^3 = x^2y - x^3\frac{dy}{dx}$$

$$x^3\frac{dy}{dx} + x^3\frac{dy}{dx} + xy^2\frac{dy}{dx} = x^2y + y^3 + x^2y$$

$$x\frac{dy}{dx}\left(x^2+x^2+y^2\right) = y\left(x^2+x^2+y^2\right)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{y}{x}}$$

Hence proved.

Find two positive integers whose sum is nine (9) and the product of one with other's square is maximum.

(LHR-2011)

Solution:

Let x and $9-x$ are the two positive integers and their product is y.

According to given condition;

$$y = x^2(9-x)$$

$$y = 9x^2 - x^3$$

Differentiate w.r.t "x":

$$\frac{dy}{dx} = 18x - 3x^2$$

$$\frac{d^2y}{dx^2} = 18 - 6x$$

For stationary point:

$$18x - 3x^2 = 0$$

$$3x(6-x) = 0$$

$x = 0$ (ignore) ; $\boxed{x = 6}$

at $x=6$: $f''(6) = 18 - 6x = 0$

$f''(6) = 18 - 6(6) = -18 < 0$ (relative maxima)

First number $= x = 6$

Second number $= 9 - x = 9 - 6 = 3$

Show that $y = \frac{\ln x}{x}$ has maximum value at $x = e$

(S.Q-LHR-2012) + L.Q*(LHR-2014+2011 +2021)

Solution:

Differentiate w.r.t "x":

$$\frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - \ln x(1)}{(x)^2}$$

$$\frac{dy}{dx} = \frac{1-\ln x}{x^2}$$

Again Differentiate w.r.t "x":

$$\frac{d^2y}{dx^2} = \frac{x^2\left(-\frac{1}{x}\right) - (1-\ln x)(2x)}{(x^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-x-2x+2x\ln x}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{-3x+2x\ln x}{x^4} = \frac{x(-3+2\ln x)}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{-3+2\ln x}{x^3}$$

At $x = e$:

$$\frac{d^2y}{dx^2} = \frac{-3+2\ln(e)}{(e)^3} = \frac{-3+2(1)}{e^3} = \frac{-1}{e^3} < 0$$

Hence proved that function has relative maxima.

Show that $y = x^x$ has maximum value at $x = \frac{1}{e}$.

Solution: $y = x^x$

Taking $\ln$ on both sides:

$\ln y = \ln (x)^x$

$\ln y = x \ln x$

Differentiate w.r.t "x":

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x (1)$$

$$\frac{dy}{dx} = y \left[1 + \ln x \right]$$

$$\frac{dy}{dx} = x^x (1 + \ln x)$$

Again: $$\frac{d^2y}{dx^2} = x^x \cdot \frac{1}{x} + (1 + \ln x) \frac{d}{dx}(x^x)$$

$$\frac{d^2y}{dx^2} = x^x \cdot \frac{1}{x} + (1 + \ln x) \frac{dy}{dx} \qquad \left\{ \because x^x = y \right.$$

$$\frac{d^2y}{dx^2} = x^x \cdot \frac{1}{x} + (1 + \ln x)\left(x^x (1 + \ln x)\right)$$

$$\frac{d^2y}{dx^2} = x^x \left[\frac{1}{x} + (1 + \ln x)^2 \right]$$

at $x = \frac{1}{e}$

$$\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{\frac{1}{e}} \left[\frac{1}{\frac{1}{e}} + \left(1 + \ln \frac{1}{e}\right)^2 \right]$$

$$\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{\frac{1}{e}} \left[e + (1 + \ln e^{-1})^2 \right]$$

$$\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{\frac{1}{e}}\left[e + (1 - \ln e)^2\right] \qquad \{\because \ln a^x = x \ln a$$

$$\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{\frac{1}{e}}\left[e + (1-1)^2\right] \qquad \{\because \ln e = 1$$

$$\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{\frac{1}{e}} \cdot e > 0$$

$\Rightarrow$ y has minimum value at $x = \frac{1}{e}$.

Hence proved.