F.Sc Math Part 2

Prof. Asad Khalid
M.Phil Math

Unit#7

Vectors

M.C.Q's

(1) Magnitude of vector $2\underline{i}+3\underline{j}+4\underline{k}=$

(a) 29 (b) $\sqrt{29}$ (c) 28 (d) $\sqrt{28}$ (LHR-2011, 2014)

(2) Length of the vector $2\underline{i}-\underline{j}+2\underline{k}=$ (LHR-2010)

(a) 6 (b) 4 (c) 3 (d) 5

(3) $|\cos\alpha\underline{i}+\sin\alpha\underline{j}+0\underline{k}|=$ (LHR-2012)

(a) 0 (b) 1 (c) -1 (d) 2

(4) A vector with magnitude 1 is called= (LHR-2015)

(a) Null Vector (b) Unit Vector (c) Zero Vector (d) Constant Vector

(5) If $\vec{v}$ is any vector then vector of magnitude 5 opposite to $\vec{v}$ is=

(a) $5\vec{v}$ (b) $-5\vec{v}$ (c) $5\frac{\underline{v}}{|\underline{v}|}$ (d) $-5\frac{\underline{v}}{|\underline{v}|}$ (LHR-2018)

(6) If α,β,γ be the direction angles of a vector then $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=$ (LHR-2016)

(a) -1 (b) 0 (c) 1 (d) 2

(7) The direction cosines of y-axis are= (LHR-2018)

(a) (0, 1, 0) (b) (1, 0, 0) (c) (0, 0, 1) (d) (1, 1, 1)

(8) Two vectors $\underline{a}$ and $\underline{b}$ are perpendicular if= (LHR-2019)

(a) $\underline{a}\times\underline{b}=0$ (b) $\underline{a}\cdot\underline{b}=0$ (c) $\underline{a}=\underline{b}$ (d) $\underline{a}=-\underline{b}$

(9) Zero vector is perpendicular to — (LHR-2011)

(a) Every Vector (b) Unit Vector only (c) Position vector (d) None

(10) If the vectors $2\underline{i}+4\underline{j}-7\underline{k}$ and $2\underline{i}+6\underline{j}+x\underline{k}$ are $\perp$ then $x=$

(a) 5 (b) 4 (c) 2 (d) 1 (D.G.Khan 2013, 2014)

(11) For any Two vectors $\underline{a}$ and $\underline{b}$, Projection of $\underline{a}$ along $\underline{b}$ is

(a) $\frac{\underline{a}\cdot\underline{b}}{a}$ (b) $\underline{a}\cdot\frac{\underline{b}}{|\underline{b}|}$ (c) $\frac{\underline{a}\cdot\underline{b}}{ab}$ (d) $\underline{a}\cdot\underline{b}$ (LHR-2018)

(12) The Angle between the Vectors $2\underline{i}+3\underline{j}+\underline{k}$ and $2\underline{i}-\underline{j}-\underline{k}$ is

(a) $90°$ (b) $45°$ (c) $60°$ (d) $30°$ (LHR-2016)

(13) If $\underline{a}$ and $\underline{b}$ have same direction Then $\underline{a}\cdot\underline{b}=$

(a) $ab\sin\theta$ (b) 0 (c) $-ab$ (d) ab (LHR-2010)

(14) The Non-zero vectors $\underline{a}$ and $\underline{b}$ are parallel if $\underline{a}\times\underline{b}=$

(a) 1 (b) -1 (c) 0 (d) ab (LHR-2019)

(15) $\vec{J}\times\vec{K}=$ (a) $\underline{i}$ (b) $-\underline{i}$ (c) 1 (d) 0 (LHR-2016, 2019)

(16) If $\underline{i},\underline{j},\underline{k}$ are Unit Vectors Then $\underline{k}\times\underline{j}=$ (GJR-2012) (AJK 2016, 2017)

(a) $\underline{i}$ (b) $-\underline{i}$ (c) 1 (d) 0

(17) The Area of Triangle having $\underline{a}$ and $\underline{b}$ as Two sides given

(a) $\bar{a}\times\bar{b}=0$ (b) $\frac{1}{2}|\bar{a}\times\bar{b}|$ (c) $\vec{a}\cdot\vec{b}$ (d) None (Fedral 2013)

(18) If Any Two Vectors of Scalar Triple Product are equal, Then its Value is ___ (LHR-2010)

(a) 1 (b) -1 (c) 2 (d) 0

(19) $\underline{i}\cdot(\underline{j}\times\underline{k})=$ (a) 0 (b) 1 (c) -1 (d) 229 (D.G Khan 2015)

(20) $\underline{j}\cdot(\underline{k}\times\underline{i})=$ (a) 0 (b) -1 (c) 1 (d) $\frac{1}{2}$ (LHR-2010 2011 2015)

(21) $[\underline{i}\ \underline{j}\ \underline{j}]=$ (a) 1 (b) -1 (c) 0 (d) 2 (LHR-2012)

(22) $2\underline{i}\cdot(2\underline{j}\times\underline{k})=$ (a) 0 (b) 2 (c) 4 (d) 6 (LHR-2014, 2015)

(23) $(\underline{i}\times\underline{k})\times\underline{j}=$ (a) 1 (b) $-\underline{j}$ (c) 0 (d) $\underline{i}$ (LHR-2016)

(24) The Triple Scalar Product of Vectors, Calculates the Volum

(a) Triangle (b) Parallelogram (c) Tetrahedron (d) Parallelpiped (LHR-2019)

(25) If $\underline{u}, \underline{v}$ and $\underline{w}$ are conterminous edges of a Tetrahedron Then Volume = (LHR-2011)

(a) $\frac{1}{6}[\underline{u}\ \underline{v}\ \underline{w}]$ (b) $\frac{1}{4}[\underline{u}\ \underline{v}\ \underline{w}]$ (c) $\frac{1}{3}[\underline{u}\ \underline{v}\ \underline{w}]$ (d) $\frac{1}{2}[\underline{u}\ \underline{v}\ \underline{w}]$

(26) The Work done by a force F during displacement $\underline{d}$ is equal to = (LHR-2014)

(a) $\underline{d} \times \underline{F}$ (b) $\underline{F} \times \underline{d}$ (c) $-\underline{F}\cdot\underline{d}$ (d) $\underline{F}\cdot\underline{d}$

(27) The Moment of force $\underline{F}$ about $\underline{r}$ = (LHR-2015)

(a) $\underline{r} \times \underline{F}$ (b) $\underline{F} \times \underline{r}$ (c) $\underline{r}\cdot\underline{F}$ (d) $\underline{F}\cdot\underline{r}$

(28) The dot Product of Vectors $\underline{u}$ and $\underline{v}$ is = (Federal 2016) (Multan 2011)

(a) $\cos\theta.\sin\theta$ (b) $|\underline{u}||\underline{v}|\cos\theta$ (c) $uv\cos\theta$ (d) $|\underline{u}||\underline{v}|\sin\theta$

(29) Direction Cosines of x-axis are = (DG Khan 2017)

(a) $(1,1,0)$ (b) $(1,0,1)$ (c) $(1,0,0)$ (d) $(0,0,1)$

(30) The direction Cosines of a Vector Parallel to z-axis are

(a) $(\pm1,0,0)$ (b) $(0,\pm1,0)$ (c) $(0,0,\pm1)$ (d) $(1,1,1)$

Prof. Asad Khalid
M.Phil Mathematics

Chapter no:07

Vectors

Short Questions

Find a real number α so that that the vectors $\underline{u} = 2\alpha\underline{i} + \underline{j} - \underline{k}$ and $\underline{v} = \underline{i} + \alpha\underline{j} + 4\underline{k}$ are perpendicular. (LHR Board 2011)

Solution:

As $\underline{u}$ and $\underline{v}$ are perpendicular

$\therefore \underline{u} \cdot \underline{v} = 0$

$(\alpha\underline{i} + 2\alpha\underline{j} - \underline{k}) \cdot (\underline{i} + \alpha\underline{j} + 3\underline{k}) = 0$

$\alpha + 2\alpha^2 - 3 = 0$

$2\alpha^2 + \alpha - 3 = 0$

$$2\alpha^2-2\alpha+3\alpha-3=0$$
$$2\alpha(\alpha-1)+3(\alpha-1)=0$$
$$(2\alpha+3)(\alpha-1)=0$$
$$2\alpha+3=0\ ,\ \alpha-1=0$$
$$\boxed{\alpha=\frac{-3}{2}}\qquad\boxed{\alpha=1}$$

Prove that $\underline{u}\cdot(\underline{v}\times\underline{w})+\underline{v}\cdot(\underline{w}\times\underline{u})+\underline{w}\cdot(\underline{u}\times\underline{v})=3\underline{u}\cdot(\underline{v}\times\underline{w})$

(LHR-2011)

Solution:

As we know that:

$$\underline{u}\cdot(\underline{v}\times\underline{w})=\underline{v}\cdot(\underline{w}\cdot\underline{u})=\underline{w}\cdot(\underline{u}\times\underline{v})$$

$$\begin{aligned}L.H.S &= \underline{u}\cdot(\underline{v}\times\underline{w})+\underline{v}\cdot(\underline{w}\times\underline{u})+\underline{w}\cdot(\underline{u}\times\underline{v})\\ &= \underline{u}\cdot(\underline{v}\times\underline{w})+\underline{u}\cdot(\underline{w}\times\underline{v})+\underline{u}\cdot(\underline{v}\times\underline{w})\\ &= 3\underline{u}\cdot(\underline{v}\times\underline{w})\\ &= R.H.S\end{aligned}$$

Hence proved.

Find $\underline{u}+\underline{v}+\underline{w}$; where $\underline{u}=2\underline{i}-7\underline{j}$; $\underline{v}=\underline{i}-6\underline{j}$; $\underline{w}=-\underline{i}+\underline{j}$

(LHR-2011)

Solution:

$$\underline{u}+\underline{v}+\underline{w}$$
$$=(2\underline{i}-7\underline{j})+(\underline{i}-6\underline{j})+(-\underline{i}+\underline{j})$$
$$=2\underline{i}-7\underline{j}+\underline{i}-6\underline{j}+\underline{i}+\underline{j}$$
$$=2\underline{i}-12\underline{j}$$

14

Find the cosine of angle between $\underline{a}$ & $\underline{b}$ if $\underline{a} = \underline{i} - 3\underline{j} + 4\underline{k}$; $\underline{b} = 4\underline{i} - \underline{j} + 3\underline{k}$

Solution:

We know that:

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| \cdot |\underline{b}|}$$

$$\cos\theta = \frac{(\underline{i} - 3\underline{j} + 4\underline{k}) \cdot (4\underline{i} - \underline{j} + 3\underline{k})}{\sqrt{(1)^2 + (-3)^2 + (4)^2} \cdot \sqrt{(4)^2 + (-1)^2 + (3)^2}}$$

$$\cos\theta = \frac{4 + 3 + 12}{\sqrt{1 + 9 + 16} \cdot \sqrt{16 + 1 + 9}}$$

$$\cos\theta = \frac{19}{\sqrt{26} \cdot \sqrt{26}}$$

$$\cos\theta = \frac{19}{(\sqrt{26})^2}$$

$$\cos\theta = \frac{19}{26}$$

15

Find a unit vector perpendicular to $\underline{u}$ and $\underline{v}$ if $\underline{u} = \underline{i} + \underline{j}$ and $\underline{v} = \underline{i} - \underline{j}$ (LHR-2011)

Solution:

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$$\underline{u} \times \underline{v} = \underline{i} \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} - \underline{j} \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + \underline{k} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$\underline{u} \times \underline{v} = \underline{i}(0+0) - \underline{j}(0-0) + \underline{k}(-1-1)$$

$$\underline{u} \times \underline{v} = 0 - 0 - 2\underline{k}$$

$$\underline{u} \times \underline{v} = -2\underline{k}$$

$$|\underline{a}\times\underline{b}| = \sqrt{(0)^2+(0)^2+(-2)^2}$$

$$|\underline{a}\times\underline{b}| = \sqrt{4}$$

$$|\underline{a}\times\underline{b}| = 2$$

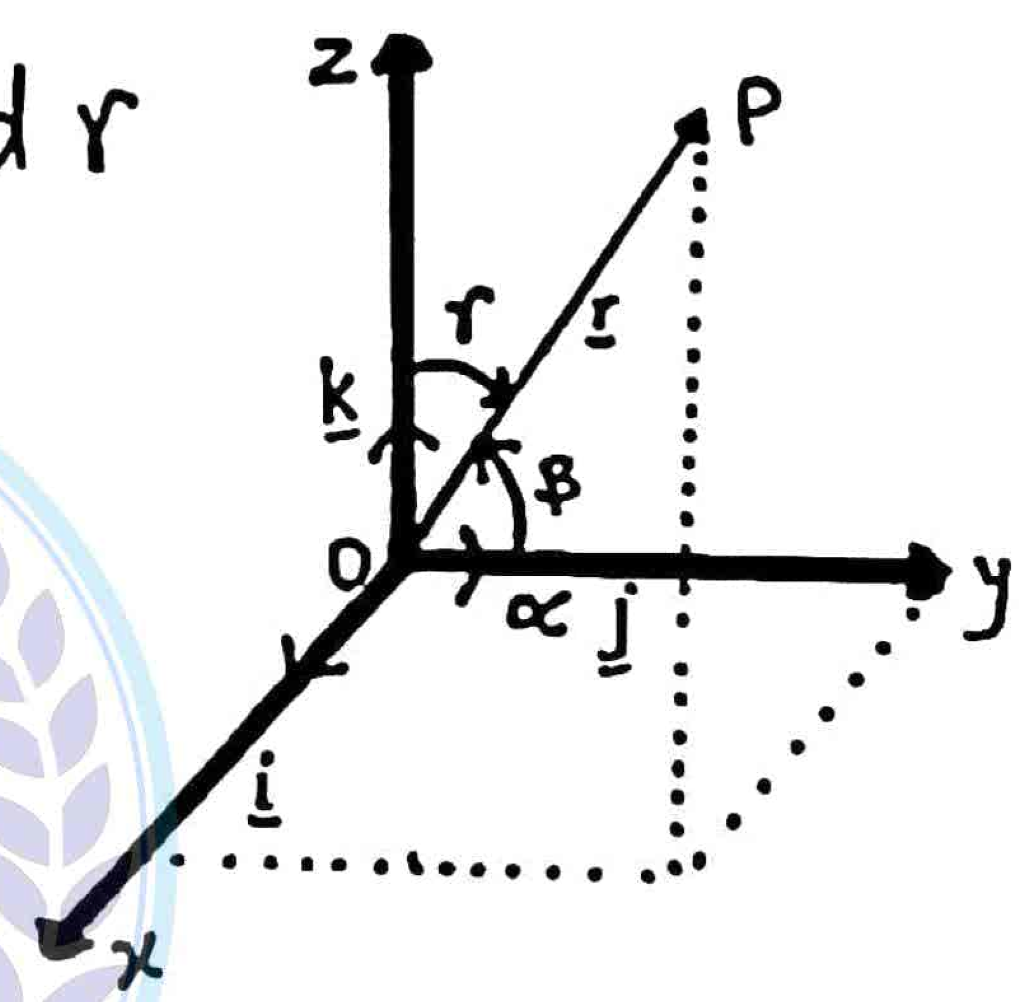
Required unit vector: $\dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|} = \dfrac{-2\underline{k}}{2} = -\underline{k}$

6

Direction angles of a vector. (LHR-2011)

Let $\underline{r} = \overrightarrow{OP} = x\underline{i}+y\underline{j}+z\underline{k}$ be a non-zero vector, let α, β, γ denote the angles formed between $\underline{r}$ and the unit coordinate vectors $\underline{i}, \underline{j}$ and $\underline{k}$ respectively, such that: $0 \le \alpha \le \pi$, $0 \le \beta \le \pi$ & $0 \le \gamma \le \pi$

The angles α, β and γ are called the direction angles.



7

Find a vector whose magnitude is 2 and is parallel to $-\underline{i}+\underline{j}+\underline{k}$. (LHR Board 2012+2014)

Solution: Let the required vector is $\underline{u}$.

$$|\underline{u}| = 2$$

Given vector $= \underline{v} = -\underline{i}+\underline{j}+\underline{k}$

As: $\underline{u} \parallel \underline{v}$

$$\underline{u} = |\underline{u}|\cdot\hat{\underline{v}}$$

(5)

$$\underline{u} = 2 \cdot \frac{\underline{v}}{|\underline{v}|}$$

$$\underline{u} = \frac{2(-\underline{i} + \underline{j} + \underline{k})}{\sqrt{(-1)^2 + (1)^2 + (1)^2}}$$

$$\underline{u} = \frac{-2\underline{i} + 2\underline{j} + 2\underline{k}}{\sqrt{3}}$$

$$\underline{u} = -\frac{2}{\sqrt{3}}\underline{i} + \frac{2}{\sqrt{3}}\underline{j} + \frac{2}{\sqrt{3}}\underline{k}$$

(8)

Find a vector perpendicular to each of the vector $\underline{a} = 2\underline{i} + \underline{j} + \underline{k}$; $\underline{b} = 4\underline{i} + 2\underline{j} - \underline{k}$. (LHR 2012) (2018 + 2014 + 2016)

Solution:

Let a vector perpendicular to both the vectors $\underline{a}$ and $\underline{b}$ is $\underline{a} \times \underline{b}$.

$$\therefore \quad \underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 4 & 2 & -1 \end{vmatrix}$$

$$\underline{a} \times \underline{b} = \underline{i} \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - \underline{j} \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} + \underline{k} \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix}$$

$$\underline{a} \times \underline{b} = \underline{i}(1-2) - \underline{j}(-2-4) + \underline{k}(4+4)$$

$$\underline{a} \times \underline{b} = -\underline{i} + 6\underline{j} + 8\underline{k}$$

Now,

$$\underline{a} \cdot (\underline{a} \times \underline{b}) = (2\underline{i} - \underline{j} + \underline{k}) \cdot (-\underline{i} + 6\underline{j} + 8\underline{k})$$

$$\underline{a} \cdot (\underline{a} \times \underline{b}) = -2 - 6 + 8$$

$$\underline{a} \cdot (\underline{a} \times \underline{b}) = 0$$

And

$$\underline{b} \cdot (\underline{a} \times \underline{b}) = (4\underline{i} + 2\underline{j} - \underline{k}) \cdot (-\underline{i} + 6\underline{j} + 8\underline{k})$$

$$\underline{b} \cdot (\underline{a} \times \underline{b}) = -4 + 12 - 8$$

$$\underline{b} \cdot (\underline{a} \times \underline{b}) = 0$$

Hence $\underline{a} \times \underline{b}$ is perpendicular to both vector $\underline{a}$ and $\underline{b}$.

(9)

A particle acted by constant force $4\underline{i}+\underline{j}-3\underline{k}$ and $3\underline{i}-\underline{j}-\underline{k}$ is displaced from A(1,2,3) to B(5,4,1). Find the work done.

Solution:

Let:

$\underline{F}_1 = 4\underline{i}+\underline{j}$; $\underline{F}_2 = 3\underline{i}-\underline{j}-\underline{k}$

$\underline{F} = \underline{F}_1 + \underline{F}_2$

$\underline{F} = 4\underline{i}+\underline{j}+0\underline{k}+3\underline{i}-\underline{j}-\underline{k}$

$\underline{F} = 7\underline{i}-0\underline{j}-4\underline{k}$

Given points are A(1,2,3) , B(5,4,1)

$\underline{d} = \overrightarrow{AB} = \text{P.v of } B - \text{P.v of } A$

$\underline{d} = (5,4,1)-(1,2,3)$

$\underline{d} = 4\underline{i}+2\underline{j}-2\underline{k}$

Work done $= W = \underline{F}\cdot\underline{d}$

$W = (7\underline{i}-0\underline{j}-4\underline{k})\cdot(4\underline{i}+2\underline{j}-2\underline{k})$

$W = 28-0-8$

$\boxed{W = 36\ \text{Nm}}$

(10)

Find a vector whose magnitude is 4 and is parallel to $2\underline{i}-3\underline{j}+6\underline{k}$. (LHR-2021 2012+2016 +2018)

Solution:

Let the required vector is $\underline{u}$.

$|\underline{u}| = 4$

Given vectors $= \underline{v} = 2\underline{i}-3\underline{j}+6\underline{k}$

As $\underline{u} \parallel \underline{v}$

$$\underline{u} = |\underline{u}| \cdot \hat{\underline{v}}$$

$$\underline{u} = 4 \cdot \frac{\underline{v}}{|\underline{v}|}$$

$$\underline{u} = \frac{4 \cdot (2\underline{i} - 3\underline{j} + 6\underline{k})}{\sqrt{(2)^2 + (-3)^2 + (6)^2}}$$

$$\underline{u} = \frac{8\underline{i} - 12\underline{j} + 24\underline{k}}{\sqrt{4+9+16}} = \frac{8\underline{i} - 12\underline{j} + 24\underline{k}}{\sqrt{49}}$$

$$\underline{u} = \frac{8}{7}\underline{i} - \frac{12}{7}\underline{j} + \frac{24}{7}\underline{k}$$

(11)

Find angle between the vectors

$\underline{u} = 2\underline{i} - \underline{j} + \underline{k}$; $\underline{v} = -\underline{i} + \underline{j}$ (LHR Board 2012+2015)

Solution:

$$\underline{u} \cdot \underline{v} = (2\underline{i} - \underline{j} + \underline{k}) \cdot (-\underline{i} + \underline{j})$$

$$\underline{u} \cdot \underline{v} = (-2 - 1 + 0)$$

$$\underline{u} \cdot \underline{v} = -3$$

$$\therefore |\underline{u}| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{6}$$

$$|\underline{v}| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

Now,

$$\cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| \cdot |\underline{v}|}$$

$$\cos\theta = \frac{-3}{\sqrt{6} \cdot \sqrt{2}} = \frac{-3}{\sqrt{12}} = \frac{-3}{\sqrt{4 \times 3}}$$

$$\cos\theta = \frac{-3}{2\sqrt{3}} = \frac{-\sqrt{3}}{2}$$

$$\therefore \theta = \frac{5\pi}{6}$$

(12)

If $\underline{a} + \underline{b} + \underline{c} = 0$ then prove that $\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$.

*(L.Q-LHR 2012) + (LHR-2012+2015+2017+2021)

Solution:

$$\underline{a} + \underline{b} + \underline{c} = 0$$

Taking cross product with $\underline{a}$:

$$\underline{a} \times (\underline{a}+\underline{b}+\underline{c})=0$$

$$\underline{a} \times \underline{a}+\underline{a} \times \underline{b}+\underline{a} \times \underline{c}=0$$

$$0+\underline{a} \times \underline{b}+\underline{a} \times \underline{c}=0$$

$$\underline{a} \times \underline{b}=-\underline{a} \times \underline{c}$$

$$\underline{a} \times \underline{b}=\underline{c} \times \underline{a} \text{ --- I} \quad \because -\underline{a} \times \underline{c}=\underline{c} \times \underline{a}$$

Taking cross product with b:

$$\underline{b} \times (\underline{a}+\underline{b}+\underline{c})=0$$

$$\underline{b} \times \underline{a}+\underline{b} \times \underline{b}+\underline{b} \times \underline{c}=0$$

$$\underline{b} \times \underline{c}=-\underline{b} \times \underline{a}$$

$$\underline{b} \times \underline{c}=\underline{a} \times \underline{b} \quad \because \underline{a} \times \underline{b}=-\underline{b} \times \underline{a}$$

$$\Rightarrow \underline{a} \times \underline{b}=\underline{b} \times \underline{c} \text{ --- II}$$

From I and II:

$$\underline{a} \times \underline{b}=\underline{b} \times \underline{c}=\underline{c} \times \underline{a}$$

(13)

Find the magnitude of $\underline{u}=2\underline{i}-7\underline{j}$.

(LHR-2013)

Solution:

$$|\underline{u}|=\sqrt{a^2+b^2}$$

$$|\underline{u}|=\sqrt{(2)^2+(-7)^2}$$

$$|\underline{u}|=\sqrt{53}$$

(14)

Find α so that $|\alpha\underline{i}+(\alpha-1)\underline{j}+2\underline{k}|=3$

(LHR-2013+2017)

Solution:

$$|\alpha\underline{i}+(\alpha-1)\underline{j}+2\underline{k}|=3$$

$$\sqrt{(\alpha)^2+(\alpha-1)^2+(2)^2}=3$$

$$\sqrt{\alpha^2+(\alpha-1)^2+(4)}=3$$

Taking square on both sides,

$$(\alpha)^2+(\alpha-1)^2+4=9$$
$$\alpha^2+\alpha^2-2\alpha+1+4=9$$
$$2\alpha^2-2\alpha+5-9=0$$
$$2\alpha^2-2\alpha-4=0$$
$$2(\alpha^2-\alpha-2)=0$$
$$\alpha^2-\alpha-2=0$$
$$\alpha^2+\alpha-2\alpha-2=0$$
$$\alpha(\alpha+1)-2(\alpha+1)=0$$
$$(\alpha-2)(\alpha+1)=0$$
$$(\alpha-2)=0\ ,\ (\alpha+1)=0$$
$$\boxed{\alpha=2}\ ,\ \boxed{\alpha=-1}$$

[15]

Find the projection $\underline{a}$ along $\underline{b}$ for $\underline{a}=\underline{i}-\underline{k}$, $\underline{b}=\underline{j}+\underline{k}$. (LHR-2013)+(LHR-2015)+(LHR-2018)

Solution:

Projection of $\underline{a}$ along $\underline{b}=\underline{a}\cdot\hat{\underline{b}}$

$$=(\underline{i}-\underline{k})\cdot\frac{(\underline{j}+\underline{k})}{\sqrt{(1)^2+(1)^2}} \qquad \because \hat{b}=\frac{b}{|b|}$$

$$=\frac{0+0-1}{\sqrt{2}}$$

$$=\frac{-1}{\sqrt{2}}$$

[16]

Vector Product of two vectors. (LHR-2013)

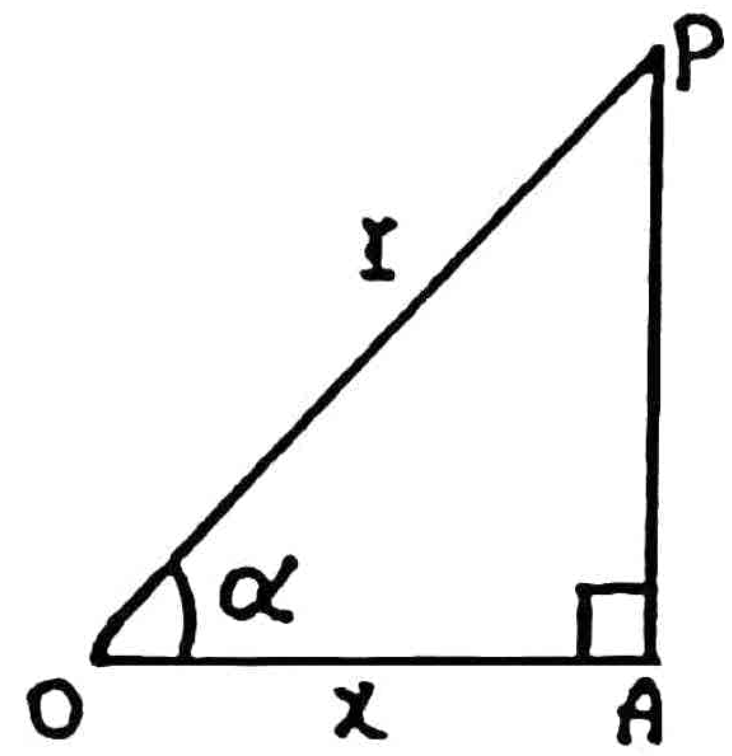
If $\underline{u}$ and $\underline{v}$ are two vectors then their vector product $(\underline{u}\times\underline{v})$ is defined as: $\underline{u}\times\underline{v}=(|\underline{u}||\underline{v}|\sin\theta)\hat{n}$

where θ is angle between $\underline{u}$ and $\underline{v}$ and $\hat{n}$ is unit vector.

117

Prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ where α,β,γ are direction angles of a vector.

Proof:



Let $\underline{r} = x\underline{i}+y\underline{j}+z\underline{k}$

$$|\underline{r}| = \sqrt{x^2+y^2+z^2} = r$$

$$\hat{\underline{r}} = \frac{\underline{r}}{|\underline{r}|} = \left[\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right]$$

As triangle ΔOAP is right triangle so,

$$\cos\alpha = \frac{\overline{OA}}{\overline{OP}} = \frac{x}{r}$$

Similarly, $\cos\beta = \frac{y}{r}$, $\cos\gamma = \frac{z}{r}$

Given:

$$L.H.S = \cos^2\alpha+\cos^2\beta+\cos^2\gamma$$

$$= \frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2}$$

$$= \frac{x^2+y^2+z^2}{r^2} = \frac{r^2}{r^2} = 1 = R.H.S$$

Hence proved.

118

If $\underline{u}=2\underline{i}-\underline{j}+\underline{k}$ and $\underline{v}=4\underline{i}+2\underline{j}-\underline{k}$ then $\underline{u}\times\underline{v}$. (LHR 2013)

Solution:

$$\underline{u}\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 4 & 2 & -1 \end{vmatrix}$$

$$\underline{u}\times\underline{v} = \underline{i}\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - \underline{j}\begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} + \underline{k}\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix}$$

$$\underline{u}\times\underline{v} = \underline{i}(1-2) - \underline{j}(-2-4) + \underline{k}(4+4)$$

$$\underline{u}\times\underline{v} = -\underline{i}+6\underline{j}+8\underline{k}$$

(19)

Find a vector from A to origin whose $\overline{AB} = 4\underline{i} - 2\underline{j}$ and $B(-2,5)$ (LHR-2014,2017)

Solution:

O (0,0) origin.

$$\vec{AO} = \vec{AB} + \vec{BO}$$

$$\vec{AB} = 4\underline{i} - 2\underline{j}$$

$$\vec{BO} = (0+2)\underline{i} + (0-5)\underline{j}$$

$$\vec{BO} = 2\underline{i} - 5\underline{j}$$

$$\vec{AO} = \vec{AB} + \vec{BO}$$

$$\vec{AO} = 4\underline{i} - 2\underline{j} + 2\underline{i} - 5\underline{j}$$

$$\vec{AO} = 6\underline{i} - 7\underline{j}$$

(20)

Find α so that vectors $2\underline{i} + \alpha\underline{j} + 5\underline{k}$ and $3\underline{i} + \underline{j} + \alpha\underline{k}$ are perpendicular. (LHR Board 2014+2017+2018) + (L.Q in LHR-2018)

Solution:

Let: $\underline{u} = 2\underline{i} + \alpha\underline{j} + 5\underline{k}$; $\underline{v} = 3\underline{i} + \underline{j} + \alpha\underline{k}$

It is given that $\underline{u}$ and $\underline{v}$ are perpendicular.

$$\therefore \underline{u} \cdot \underline{v} = 0$$

$$\Rightarrow (2\underline{i} + \alpha\underline{j} + 5\underline{k}) \cdot (3\underline{i} + \underline{j} + \alpha\underline{k}) = 0$$

$$\Rightarrow 6 + \alpha + 5\alpha = 0$$

$$6\alpha = -6$$

$$\therefore \boxed{\alpha = -1}$$

(21)

Find α so that $\alpha\underline{i} + \underline{j}$, $\underline{i} + \underline{j} + 3\underline{k}$, $2\underline{i} + \underline{j} - 2\underline{k}$ are coplaner. (LHR-2014).

Solution: As vectors are coplaner, so:

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = 0$$

$$\begin{vmatrix} \alpha & 1 & 0 \\ 1 & 1 & 3 \\ 2 & 1 & -2 \end{vmatrix} = 0$$

$$\alpha \begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} + 0 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 0$$

$$\alpha(-2-3) - 1(-2-6) + 0(1-2) = 0$$

$$-5\alpha + 8 + 0 = 0$$

$$-5\alpha = -8$$

$$\boxed{\alpha = \frac{8}{5}}$$

(22)

Find the sum of vectors $\overline{AB}$ and $\overline{CD}$, given the four points $A(1,-1)$, $B(2,0)$, $C(-1,3)$ and $D(-2,2)$.

(LHR-2014)

Solution:

$\overrightarrow{AB}$ = P.v of B − P.v of A

$\overrightarrow{AB} = (2,0) - (1,-1)$

$\overrightarrow{AB} = (2-1)\underline{i} + (0+1)\underline{j}$

$\overrightarrow{AB} = \underline{i} + \underline{j}$

$\overrightarrow{CD}$ = P.v of D − P.v of C

$\overrightarrow{CD} = (-2,2) - (-1,3)$

$\overrightarrow{CD} = (-2+1)\underline{i} + (2-3)\underline{j}$

$\overrightarrow{CD} = -\underline{i} - \underline{j}$

Now,

$\overrightarrow{AB} + \overrightarrow{CD}$

$= (\underline{i} + \underline{j}) + (-\underline{i} - \underline{j})$

$= \underline{i} + \underline{j} - \underline{i} - \underline{j}$

$= 0$ (Null vector)

123

Prove that $\underline{a}\times(\underline{b}+\underline{c})+\underline{b}\times(\underline{c}+\underline{a})+\underline{c}\times(\underline{a}+\underline{b})=0$

Solution: (LHR-2014)

$$L.H.S = \underline{a}\times(\underline{b}+\underline{c})+\underline{b}\times(\underline{c}+\underline{a})+\underline{c}\times(\underline{a}+\underline{b})$$

$$= \underline{a}\times\underline{b}+\underline{a}\times\underline{c}+\underline{b}\times\underline{c}+\underline{b}\times\underline{a}+\underline{c}\times\underline{a}+\underline{c}\times\underline{b}$$

$$= \underline{a}\times\underline{b}+\underline{a}\times\underline{c}+\underline{b}\times\underline{c}-\underline{a}\times\underline{b}-\underline{a}\times\underline{c}-\underline{b}\times\underline{c}$$

$$= 0$$

$$= R.H.S$$

Hence proved.

124

Prove that the vectors $\underline{i}-2\underline{j}+3\underline{k}, -2\underline{i}+3\underline{j}-4\underline{k}$ and $\underline{i}-3\underline{j}+5\underline{k}$ are coplanar. (LHR-2014+2017)

Solution:

Let: $\underline{a}=\underline{i}-2\underline{j}+3\underline{k}$; $\underline{b}=-2\underline{i}+3\underline{j}-4\underline{k}$; $\underline{c}=\underline{i}-3\underline{j}+5\underline{k}$

Now, $$\underline{a}\cdot(\underline{b}\times\underline{c})=\begin{vmatrix}1 & -2 & 3\\ -2 & 3 & -4\\ 1 & -3 & 5\end{vmatrix}$$

$$\underline{a}\cdot(\underline{b}\times\underline{c}) = 1\begin{vmatrix}3 & -4\\ -3 & 5\end{vmatrix}+2\begin{vmatrix}-2 & -4\\ 1 & 5\end{vmatrix}+3\begin{vmatrix}-2 & 3\\ 1 & -3\end{vmatrix}$$

$$\underline{a}\cdot(\underline{b}\times\underline{c}) = 1(15-12)+2(-10+4)+3(6-3)$$

$$\underline{a}\cdot(\underline{b}\times\underline{c}) = 3-12+9$$

$$\underline{a}\cdot(\underline{b}\times\underline{c}) = 0$$

As $\underline{a}\cdot(\underline{b}\times\underline{c})=0$ so the given three vectors are zero.

125

Find the area of triangle with vertices $A(1,-1,1), B(2,1,-1), C(-1,1,2)$.

Solution:

$\overrightarrow{AB}$ = P.v of B − P.v of A

$\overrightarrow{AB} = (2,1,-1)-(1,-1,1)$

$\overrightarrow{AB} = (2-1)\underline{i}+(1+1)\underline{j}+(-1-1)\underline{k}$

$\overrightarrow{AB} = \underline{i}+2\underline{j}-2\underline{k}$

$\overrightarrow{AC} = \text{P.v of } C - \text{P.v of } A$

$\overrightarrow{AC} = (-1,1,2)-(1,-1,1)$

$\overrightarrow{AC} = (-1-1)\underline{i}+(1+1)\underline{j}+(2-1)\underline{k}$

$\overrightarrow{AC} = -2\underline{i}+2\underline{j}+\underline{k}$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{vmatrix}$$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \underline{i}\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} - \underline{j}\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \underline{k}\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}$$

$\overrightarrow{AB}\times\overrightarrow{AC} = \underline{i}(2+4)-\underline{j}(1-4)+\underline{k}(2+4)$

$\overrightarrow{AB}\times\overrightarrow{AC} = 6\underline{i}+3\underline{j}+6\underline{k} \Rightarrow |\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{(6)^2+(3)^2+(6)^2} = 9$

Area of triangle: $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$= \frac{1}{2}(9)$

$= \frac{9}{2}$

(26)

Find the direction cosine for $\overrightarrow{PQ}$ where: $P(2,1,5)$ and $Q(1,3,1)$

Solution:

$\overrightarrow{PQ} = \text{P.v of } Q - \text{P.v of } P$

$\overrightarrow{PQ} = (1,3,1)-(2,1,5)$

$\overrightarrow{PQ} = (1-2)\underline{i}+(3-1)\underline{j}+(1-5)\underline{k}$

$\overrightarrow{PQ} = -\underline{i}+2\underline{j}-4\underline{k}$

$|\overrightarrow{PQ}| = \sqrt{(-1)^2+(2)^2+(-4)^2}$

$|\overrightarrow{PQ}| = \sqrt{21}$

Direction cosine are:

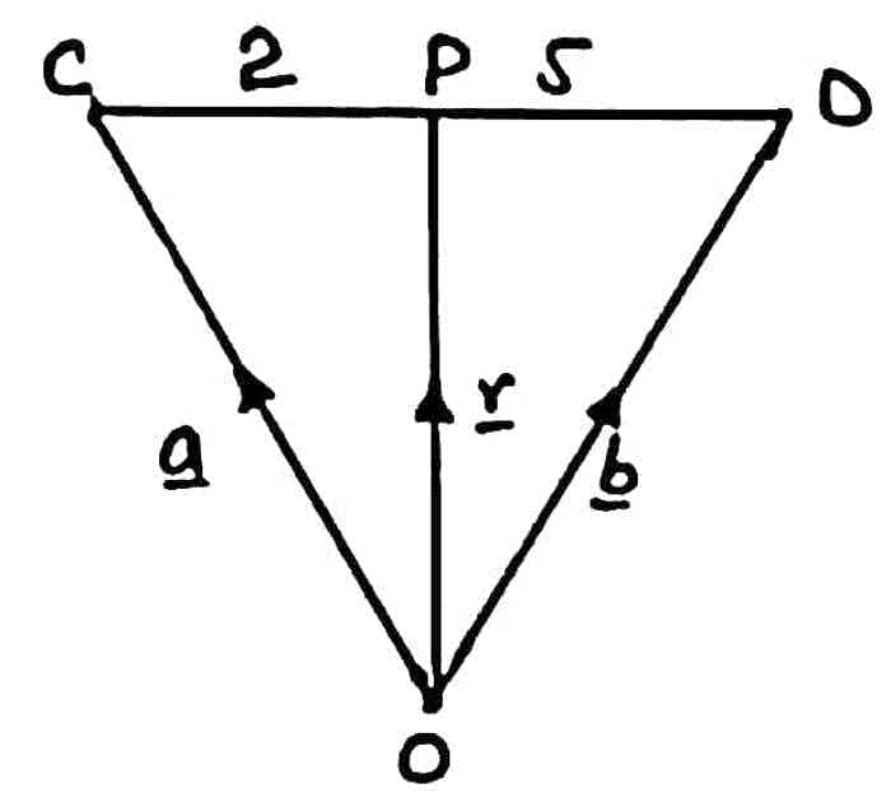
$$\cos\alpha = \frac{-1}{\sqrt{21}},\ \cos\beta = \frac{2}{\sqrt{21}},\ \cos\gamma = \frac{-4}{\sqrt{21}}$$

(27)

Find position vector of a point which divide the join of E with position vector $5\underline{j}$ and F with position vector $4\underline{i}+\underline{j}$ in the ratio 2:5.

(LHR-2015)

Solution:



$\underline{b}$ is p.v of D

so, $\underline{b} = 4\underline{i}+\underline{j}$

$p:q = 2:5$

$$\underline{r} = \frac{q\underline{a}+\underline{b}p}{p+q} = \frac{5(5\underline{j})+(4\underline{i}+\underline{j})2}{5+2}$$

$$\underline{r} = \frac{25\underline{j}+8\underline{i}+2\underline{j}}{7}$$

$$\underline{r} = \frac{27\underline{j}+8\underline{i}}{7} \Rightarrow \underline{r} = \frac{8}{7}\underline{i}+\frac{27}{7}\underline{j}$$

(28)

Find the direction cosines of $\underline{v} = 6\underline{i}-2\underline{j}+\underline{k}$

Solution: (LHR-2015)

$$|\underline{v}| = \sqrt{(6)^2+(-2)^2+(1)^2}$$

$$|\underline{v}| = \sqrt{36+4+1}$$

$$|\underline{v}| = \sqrt{41}$$

Direction cosines are:

$$\cos\alpha = \frac{6}{\sqrt{41}},\ \cos\beta = \frac{-2}{\sqrt{41}},\ \cos\gamma = \frac{1}{\sqrt{41}}$$

29

Find α so that vectors $\underline{u} = \alpha\underline{i} + 2\alpha\underline{j} - \underline{k}$ and $\underline{v} = \underline{i} + \alpha\underline{j} + 3\underline{k}$ are perpendicular.

(LHR-2015)

Solution:

As $\underline{u}$ and $\underline{v}$ are perpendicular.

$\therefore \quad \underline{u}\cdot\underline{v} = 0$

$$(\alpha\underline{i} + 2\alpha\underline{j} - \underline{k})\cdot(\underline{i} + \alpha\underline{j} + 3\underline{k}) = 0$$

$$\alpha + 2\alpha^2 - 3 = 0$$

$$2\alpha^2 + \alpha - 3 = 0$$

$$2\alpha^2 - 2\alpha + 3\alpha - 3 = 0$$

$$2\alpha(\alpha - 1) + 3(\alpha - 1) = 0$$

$$(2\alpha + 3)(\alpha - 1) = 0$$

$$2\alpha + 3 = 0, \quad \alpha - 1 = 0$$

$$\boxed{\alpha = -\frac{3}{2}}, \quad \boxed{\alpha = 1}$$

30

If O is the origin and $\overrightarrow{OP}$ and $\overrightarrow{AB}$ are equal, find the value of P when A(−3,7) and B(1,0). (LHR-2016)

Solution:

O(0,0) origin, Suppose P(x,y)

$$\overrightarrow{OP} = (x-0)\underline{i} + (y-0)\underline{j}$$

$$\overrightarrow{OP} = x\underline{i} + y\underline{j}$$

$$\overrightarrow{AB} = (1+3)\underline{i} + (0-7)\underline{j}$$

$$\overrightarrow{AB} = 4\underline{i} - 7\underline{j}$$

As $\overrightarrow{OP} = \overrightarrow{AB}$

So, $x\underline{i} + y\underline{j} = 4\underline{i} - 7\underline{j}$

$\Rightarrow \boxed{x = 4}, \quad \boxed{y = -7}$ So, P(4,−7)

131

Find the value of $3\underline{j}\cdot(\underline{k}\times\underline{i})$. (LHR-2016+2017)

Solution:

$$3\underline{j}\cdot(\underline{k}\times\underline{i}) = \begin{vmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$3\underline{j}\cdot(\underline{k}\times\underline{i}) = 0(0-0)-3(0-1)+0(0-0)$$

$$3\underline{j}\cdot(\underline{k}\times\underline{i}) = 0+3+0$$

$$3\underline{j}\cdot(\underline{k}\times\underline{i}) = 3$$

132

Find the direction cosines of $\underline{v} = 6\underline{i}-2\underline{j}+\underline{k}$ (LHR-2016)

Solution:

$$|\underline{v}| = \sqrt{(6)^2+(-2)^2+(1)^2}$$

$$|\underline{v}| = \sqrt{36+4+1}$$

$$|\underline{v}| = \sqrt{41}$$

Direction cosines are:

$$\cos\alpha = \frac{6}{\sqrt{41}}, \quad \cos\beta = \frac{-2}{\sqrt{41}}, \quad \cos\gamma = \frac{1}{\sqrt{41}}$$

133

If $\underline{a} = 2\underline{i}+\underline{j}-\underline{k}$ and $\underline{b} = \underline{i}-\underline{j}+\underline{k}$ find $\underline{a}\times\underline{b}$.

Solution: (LHR-2016)

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$\underline{a}\times\underline{b} = \underline{i}(1-1)-\underline{j}(2+1)+\underline{k}(-2-1)$$

$$\underline{a}\times\underline{b} = -3\underline{j}-3\underline{k}$$

34

Find the value of $2\underline{i}\times2\underline{j}\cdot\underline{k}$ (LHR-2016)

Solution:

$$2\underline{i}\times2\underline{j}\cdot\underline{k}=\begin{vmatrix}2&0&0\\0&2&0\\0&0&1\end{vmatrix}$$

$$2\underline{i}\times2\underline{j}\cdot\underline{k}=2(2-0)-0(0-0)+0(0-0)$$

$$2\underline{i}\times2\underline{j}\cdot\underline{k}=4$$

35

If $\underline{a}=3\underline{i}-2\underline{j}+\underline{k}$ and $\underline{b}=\underline{i}+\underline{j}$ find $\underline{b}\times\underline{a}$. (LHR-2017)

Solution:

$$\underline{b}\times\underline{a}=\begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\1&1&0\\3&-2&1\end{vmatrix}$$

$$\underline{b}\times\underline{a}=\underline{i}(1+0)-\underline{j}(1-0)+\underline{k}(-2-3)$$

$$\underline{b}\times\underline{a}=\underline{i}-\underline{j}-5\underline{k}$$

36

Find a vector of length 5, in the direction opposite that of $\underline{v}=\underline{i}-2\underline{j}+3\underline{k}$

(L.Q-LHR 2021) + (LHR-2017)

Solution:

Let required vector is $\underline{u}$.

$$|\underline{u}|=5$$

Given vector $=\underline{v}=\underline{i}-2\underline{j}+3\underline{k}$

As $\underline{u}\parallel\underline{v}$

$$\underline{u}=|\underline{u}|\cdot(-\hat{\underline{v}})$$

$$\underline{u} = \frac{-5(\underline{i}-2\underline{j}+3\underline{k})}{\sqrt{(1)^2+(-2)^2+(3)^2}}$$

$$\underline{u} = \frac{-5\underline{i}+10\underline{j}-15\underline{k}}{\sqrt{14}}$$

$$\underline{u} = \frac{-5}{\sqrt{14}}\underline{i} + \frac{10}{\sqrt{14}}\underline{j} - \frac{15}{\sqrt{14}}\underline{k}$$

(37)

Unit vector: (LHR-2018)

A vector whose magnitude is unity, is called <u>unit vector</u>.

It is written as $\hat{\underline{v}}$ and defined by $\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|}$

(38)

Find a unit vector in the direction of the vector $\underline{v} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$. (LHR-2018)

Solution:

$$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|} \quad ; \quad \underline{v} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$$

$$|\underline{v}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$|\underline{v}| = \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$$

$$\hat{\underline{v}} = \frac{\frac{1}{2}\underline{i} + \frac{\sqrt{3}}{\sqrt{2}}\underline{j}}{1} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$$

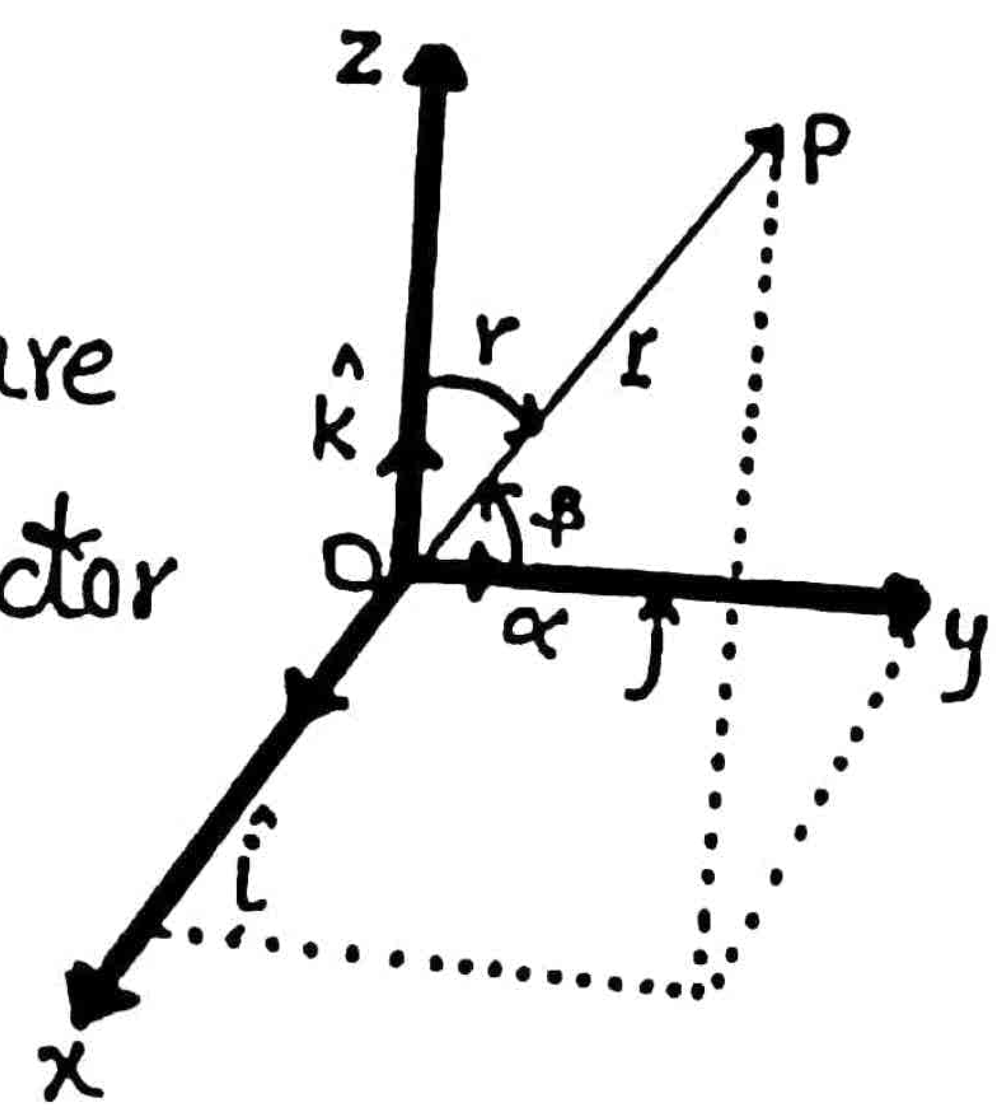
39

Direction angles and direction cosines. (LHR 2018)

Let $\underline{r} = \overrightarrow{OP} = x\underline{i} + y\underline{j} + z\underline{k}$ be a non-zero vector, Let α, β, γ denote the angles formed between $\underline{r}$ and the unit coordinate vectors $\underline{i}, \underline{j}$ and $\underline{k}$ respectively. such that: $0 \le \alpha \le \pi$, $0 \le \beta \le \pi$ and $0 \le \gamma \le \pi$

The angles α, β and γ called the direction angles.

The numbers $\cos\alpha, \cos\beta$ and $\cos\gamma$ are called direction cosines of the vector $\underline{r}$.



40

If $\overrightarrow{AB} = \overrightarrow{CD}$. Find the coordinates of point A when points B, C, D are (1,2), (-2,5), (4,11) respectivelly. (2019)

Solution:

Suppose $A(x, y)$

$\overrightarrow{AB} = (1-x)\underline{i} + (2-y)\underline{j}$

$\overrightarrow{CD} = (4+2)\underline{i} + (11-5)\underline{j}$

$\overrightarrow{CD} = 6\underline{i} + 6\underline{j}$

$\overrightarrow{AB} = \overrightarrow{CD}$

$(1-x)\underline{i} + (2-y)\underline{j} = 6\underline{i} + 6\underline{j}$

$\Rightarrow \boxed{x = -4}, \boxed{x = -5}$ So, $A(-4, -5)$

41

Show that the vectors $3\underline{i}-2\underline{j}+\underline{k}$, $\underline{i}-3\underline{j}+5\underline{k}$ and $2\underline{i}+\underline{j}-4\underline{k}$ form a triangle. (LHR- 2019)

Solution:

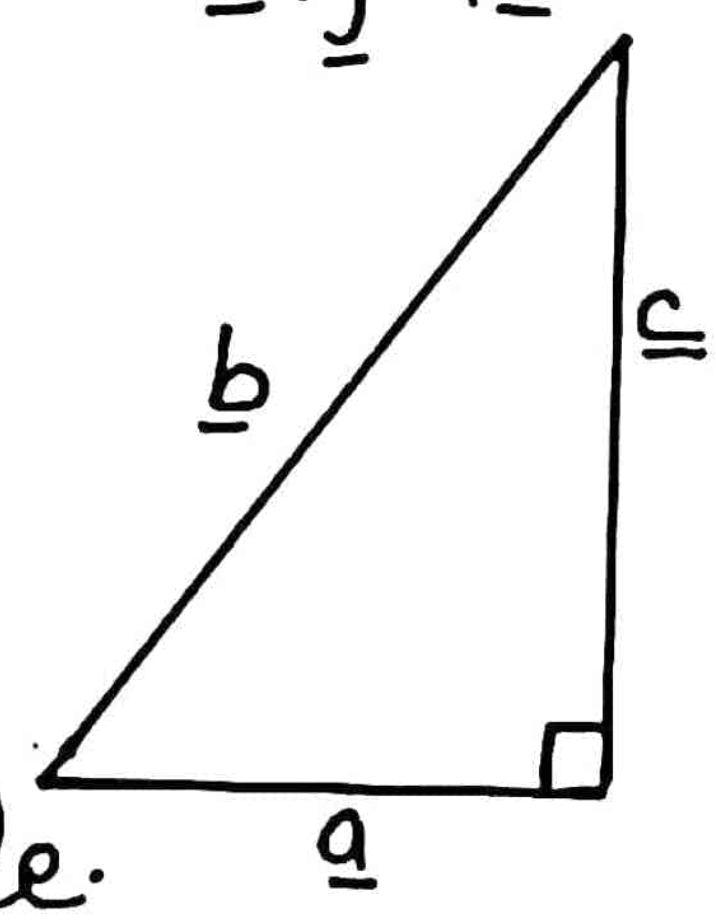
Let $\underline{a}=3\underline{i}-2\underline{j}+\underline{k}$, $\underline{b}=\underline{i}-3\underline{j}+5\underline{k}$, $\underline{c}=2\underline{i}+\underline{j}-4\underline{k}$

Now,

$$\underline{b}+\underline{c}= \underline{i}-3\underline{j}+5\underline{k}+2\underline{i}+\underline{j}-4k$$

$$\underline{b}+\underline{c}= 3\underline{i}-2\underline{j}+\underline{k}$$

$$\therefore \quad \mathbf{\underline{b}+\underline{c}=\underline{a}}$$



So, $\underline{a}$, $\underline{b}$ and $\underline{c}$ form triangle.

Now for right triangle;

$$\underline{a}\cdot\underline{c} = (3\underline{i}-2\underline{j}+\underline{k})\cdot(2\underline{i}+\underline{j}-4\underline{k})$$

$$\underline{a}\cdot\underline{c} = 6-2-4$$

$$\underline{a}\cdot\underline{c} = 0$$

So, $\underline{a}\perp\underline{c}$

Thus $\underline{a}$, $\underline{b}$ and $\underline{c}$ form right triangle.

42

Find a vector perpendicular to $\underline{a}=-\underline{i}-\underline{j}-\underline{k}$ and $\underline{b}=2\underline{i}-3\underline{j}+4\underline{k}$. (LHR 2019)

Solution:

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix}$$

$$\underline{a}\times\underline{b} = \underline{i}(-4-3)-\underline{j}(-4+2)+\underline{k}(3+2)$$

$$\underline{a} \times \underline{b} = -7\underline{i} + 2\underline{j} + 5\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(-7)^2 + (2)^2 + (5)^2}$$

$$|\underline{a} \times \underline{b}| = \sqrt{49 + 4 + 25}$$

$$|\underline{a} \times \underline{b}| = \sqrt{78}$$

Required unit vector: $\dfrac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \dfrac{-7\underline{i} + 2\underline{j} + 5\underline{k}}{\sqrt{78}}$

143

A force $F = 7\underline{i} + 4\underline{j} - 3\underline{k}$ is applied at $P(1, -2, 3)$. Find its moment about point $Q(2, 1, 1)$. (LHR-2019)

Solution:

$$\underline{r} = \overrightarrow{QP} = \text{P.v of } P - \text{P.v of } Q$$

$$\underline{r} = (1, -2, 3) - (2, 1, 1)$$

$$\underline{r} = -\underline{i} - 3\underline{j} + 2\underline{k}$$

Moment of arm $= M = \underline{r} \times \underline{F}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -3 & 2 \\ 7 & 4 & -3 \end{vmatrix}$$

$$= i(9-8) - \underline{j}(3-14) + \underline{k}(-4+21)$$

$$= \underline{i} + 11\underline{j} + 17\underline{k}$$

144

Find a unit vector of $\underline{v} = [3, -4]$ (LHR 2021)

Solution:

$$\underline{v} = 3\underline{i} - 4\underline{j}$$

$$|\underline{v}| = \sqrt{(3)^2+(-4)^2}$$

$$|\underline{v}| = \sqrt{25}$$

$$|\underline{v}| = 5$$

$$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|} = \frac{3\underline{i}-4\underline{j}}{5}$$

$$\hat{\underline{v}} = \frac{1}{5}[3,-4].$$

$$\hat{\underline{v}} = \left[\frac{3}{5}, -\frac{4}{5}\right]$$

145

Find a unit vector of $\underline{v} = \frac{-\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$

Solution:

(LHR-2021)

$$|\underline{v}| = \sqrt{\left(\frac{-\sqrt{3}}{2}\right)^2+\left(-\frac{1}{2}\right)^2}$$

$$|\underline{v}| = \sqrt{\frac{3}{4}+\frac{1}{4}}$$

$$|\underline{v}| = \sqrt{\frac{4}{4}} = 1$$

$$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|} = \frac{-\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}}{1}$$

$$\hat{\underline{v}} = \frac{-\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$$

146

Find a so that the vectors $\underline{v} = \underline{i} - 3\underline{j} + 4\underline{k}$ and $\underline{w} = a\underline{i} + 9\underline{j} - 12\underline{k}$ are parallel.

Solution

As $\underline{v}$ and $\underline{w}$ are parallel so, direction ratios are proportional.

$$\frac{1}{a}=\frac{-3}{9}=\frac{4}{-12}$$

$$\frac{1}{a}=\frac{-1}{3}=\frac{1}{-3}$$

$$\Rightarrow \boxed{a=-3}$$

B|47|B

Find two vector of length 2 parallel to vector $\underline{v}=2\underline{i}-4\underline{j}+4\underline{k}$.

Solution:

Let required vector is $\underline{u}$.

$|\underline{u}|=2$

Given vector $=\underline{v}=2\underline{i}-4\underline{j}+4\underline{k}$

As, $\underline{u} \parallel \underline{v}$

$$\underline{u}=|\underline{u}|(\pm\hat{\underline{v}})$$

$$\underline{u}=2\left(\pm\frac{\underline{v}}{|\underline{v}|}\right)$$

$$\underline{u}=\frac{\pm 2(2\underline{i}-4\underline{j}+4\underline{k})}{\sqrt{(2)^2+(-4)^2+(4)^2}}$$

$$\underline{u}=\frac{\pm(4\underline{i}-8\underline{j}+8\underline{k})}{6}$$

So two required vectors are:

Same direction	Opposite direction
$\frac{4\underline{i}-8\underline{j}+8\underline{k}}{6}$,	$-\frac{(4\underline{i}-8\underline{j}+8\underline{k})}{6}$
$\frac{2}{3}\underline{i}-\frac{4}{3}\underline{j}+\frac{4}{3}\underline{k}$,	$\frac{-2}{3}\underline{i}+\frac{4}{3}\underline{j}-\frac{4}{3}\underline{k}$

B|48|B

Find the cosine of angle between $\underline{u} = [2, -3, 1]$, $\underline{v} = [2, 4, 1]$ (LHR-2021)

Solution:

As we know that:

$$\cos\theta = \frac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|}$$

$$\cos\theta = \frac{(2\underline{i}-3\underline{j}+\underline{k})\cdot(2\underline{i}+4\underline{j}+\underline{k})}{\sqrt{(2)^2+(-3)^2+(1)^2}\cdot\sqrt{(2)^2+(4)^2+(1)^2}}$$

$$\cos\theta = \frac{4-12+1}{\sqrt{4+9+1}\cdot\sqrt{4+16+1}}$$

$$\cos\theta = \frac{-7}{\sqrt{14}\cdot\sqrt{21}} = \frac{-7}{\sqrt{294}} = \frac{-7}{\sqrt{6\times 49}}$$

$$\cos\theta = \frac{-7}{7\sqrt{6}} = \frac{-1}{\sqrt{6}}$$

B|49|B

If $\underline{v}$ is vector for which $\underline{v}\cdot\underline{i} = 0$, $\underline{v}\cdot\underline{j} = 0$, $\underline{v}\cdot\underline{k} = 0$ find $\underline{v}$. (LHR-2021)

Solution:

Let $\underline{v} = x\underline{i} + y\underline{j} + z\underline{k}$ —— I

Given : $\underline{v}\cdot\underline{i} = 0$

$$(x\underline{i} + y\underline{j} + z\underline{k})\cdot(\underline{i} + 0\underline{j} + 0\underline{k}) = 0$$

$$x + 0 + 0 = 0$$

$$\Rightarrow \boxed{x = 0}$$

Now,

$$\underline{v}\cdot\underline{j}=0$$

$$(x\underline{i}+y\underline{j}+z\underline{k})\cdot(0\underline{i}+\underline{j}+0\underline{k})=0$$

$$0+y+0=0$$

$$\boxed{y=0}$$

Now:

$$\underline{v}\cdot\underline{k}=0$$

$$(x\underline{i}+y\underline{j}+z\underline{k})\cdot(0\underline{i}+0\underline{j}+\underline{k})=0$$

$$0+0+z=0$$

$$\boxed{z=0}$$

Putting "x", "y" and "z" in I:

$$\underline{v}=0\underline{i}+0\underline{j}+0\underline{k}$$

$$\underline{v}=0 \text{ (Null vector)}$$

(50)

Compute $\underline{b}\times\underline{a}$, and show that $\underline{b}$ is perpendicular $\underline{b}\times\underline{a}$. (LHR-2021)

Solution: $\underline{a}=2\underline{i}+\underline{j}-\underline{k}$; $\underline{b}=\underline{i}-\underline{j}+\underline{k}$

$$\underline{b}\times\underline{a}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 1 & -1 & 1\\ 2 & 1 & -1\end{vmatrix}$$

$$\underline{b}\times\underline{a}=\underline{i}(1-1)-\underline{j}(-1-2)+\underline{k}(1+2)$$

$$\underline{b}\times\underline{a}=0\underline{i}+3\underline{j}+3\underline{k}$$

Now, $\underline{b}\cdot(\underline{b}\times\underline{a})=(\underline{i}-\underline{j}+\underline{k})\cdot(0\underline{i}+3\underline{j}+3\underline{k})$

$$\underline{b}\cdot(\underline{b}\times\underline{a})=0-3+3$$

$$\underline{b}\cdot(\underline{b}\times\underline{a})=0$$

So, $b\perp(\underline{b}\times\underline{a})$

(51)

Find the volume of parallelopiped.

Solution:

$\underline{u} = \underline{i} - 4\underline{j} - \underline{k}$; $\underline{v} = \underline{i} - \underline{j} - 2\underline{k}$; $\underline{w} = 2\underline{i} - 3\underline{j} + \underline{k}$

Volume of parallelopiped $= \underline{u} \cdot (\underline{v} \times \underline{w})$

$$= \begin{vmatrix} 1 & -4 & -1 \\ 1 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$

$= 1(-1-6) + 4(1+4) - 1(-3+2)$

$= -7 + 20 + 1$

$= 14$ cubic root.

(52)

Give a force $F = 2\underline{i} + \underline{j} - 3\underline{k}$ at a point A(1,−2,1). Find moment of arm about point B(2,0,−2). (LHR-2021)

Solution: $F = 2\underline{i} + \underline{j} - 3\underline{k}$

Given points are A(1,−2,1) and B(2,0,−2)

$\underline{r} = \overrightarrow{BA} = $ P.v of A − P.v of B

$\underline{r} = (1,-2,1) - (2,0,-2)$

$\underline{r} = -\underline{i} - 2\underline{j} + 3\underline{k}$

Moment of force: $M = \underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix}$

$= \underline{i}(6-3) - \underline{j}(3-6) + \underline{k}(-1+4)$

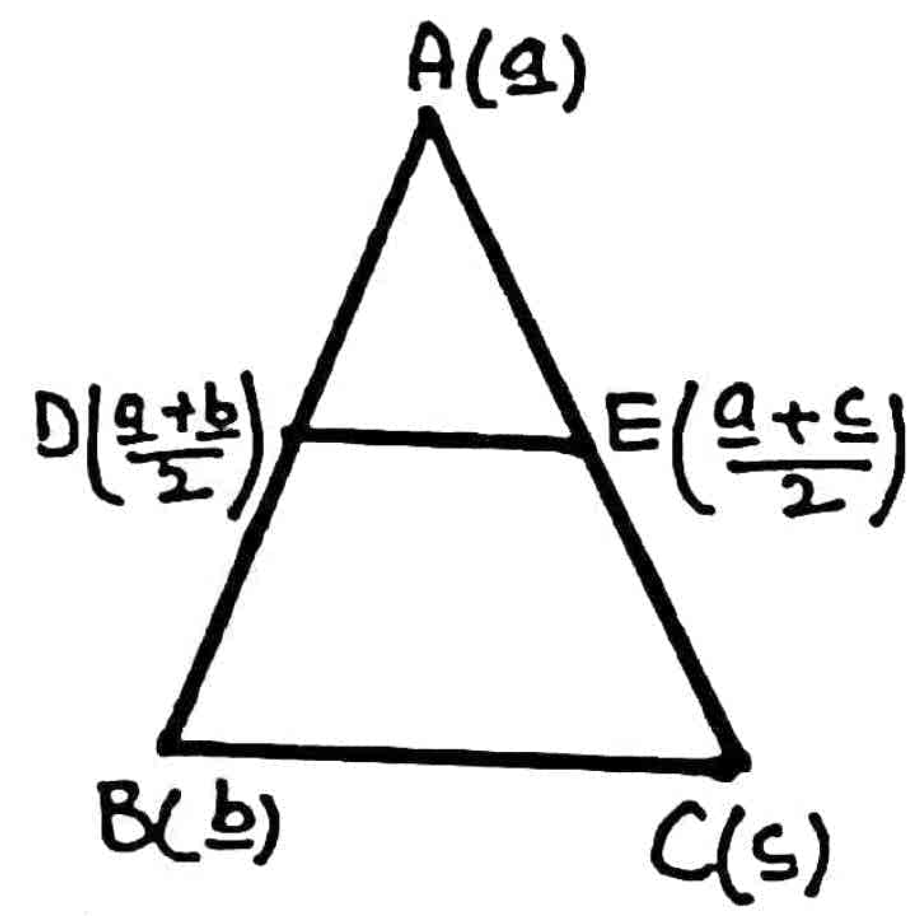
$= 3\underline{i} + 3\underline{j} + 3\underline{k}$

Long Questions

1

Prove that the line segment joining the mid points of two sides of triangle is parallel to the third side and half as long. (LHR-2011,2012,2013,2014,2018)

Solution:



Suppose that $\underline{a}, \underline{b}, \underline{c}$ are position vectors of A, B, C respectively.

Then, p.v of D $= \dfrac{\underline{a}+\underline{b}}{2}$

p.v of E $= \dfrac{\underline{a}+\underline{c}}{2}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$\overrightarrow{BC} = \underline{c} - \underline{b}$

$\overrightarrow{DE} = \overrightarrow{DE} - \overrightarrow{OD}$

$\overrightarrow{DE} = \left(\dfrac{\underline{a}+\underline{c}}{2}\right) - \left(\dfrac{\underline{a}+\underline{b}}{2}\right)$

$$\overrightarrow{DE} = \frac{1}{2}(\underline{a}+\underline{c}-\underline{a}-\underline{b})$$

$$\overrightarrow{DE} = \frac{1}{2}(\underline{c}-\underline{b})$$

So, $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$

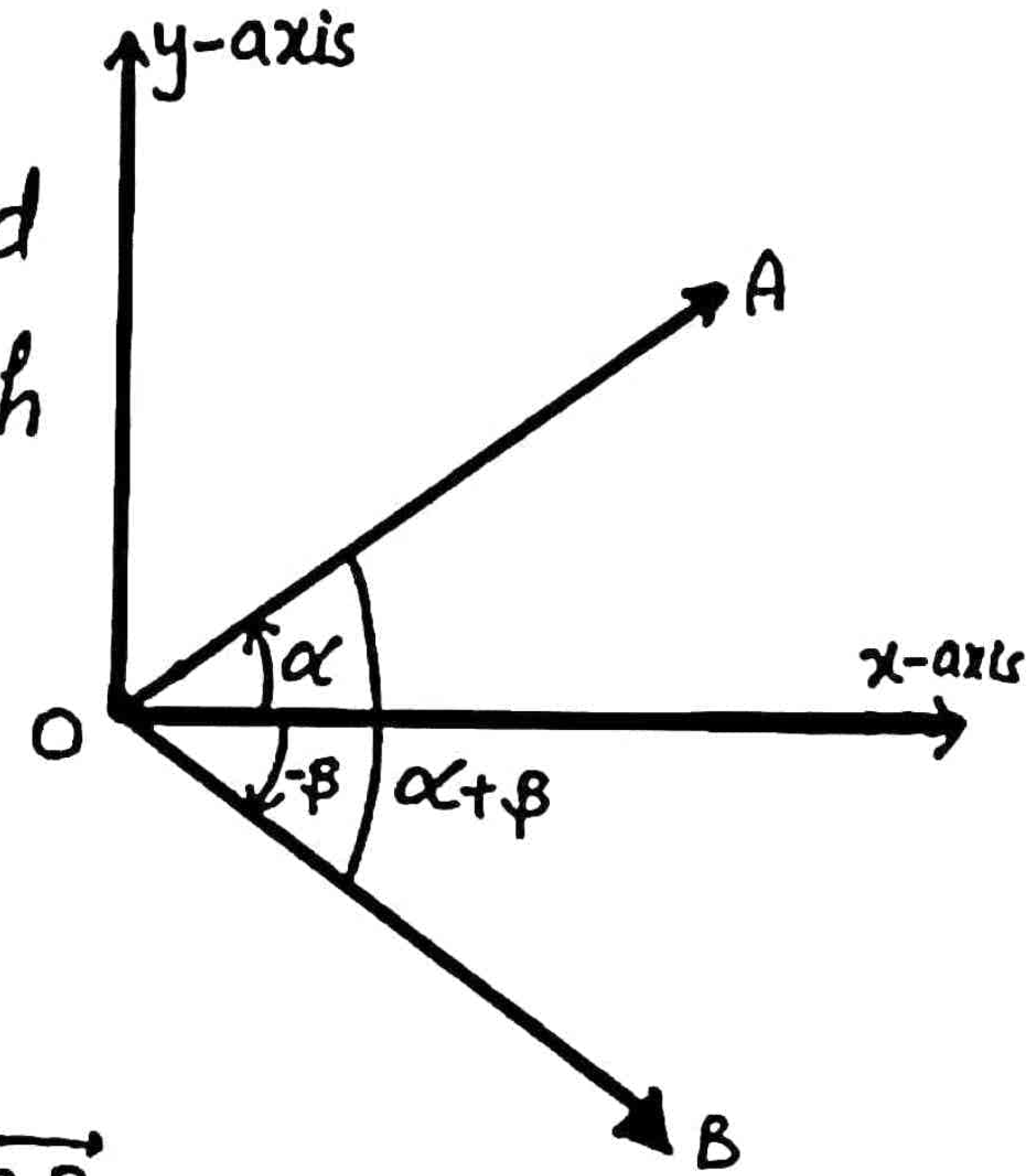
Hence proved.

Q121

Prove that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ by vector method. (LHR-2011).

Solution:

Consider two unit vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ making angles "α" and "$-\beta$" with positive x-axis.



Let:

$$\overrightarrow{OA} = \cos\alpha\underline{i} + \sin\alpha\underline{j}$$

$$\overrightarrow{OB} = \cos(-\beta)\underline{i} + \sin(-\beta)\underline{j}$$

$$\overrightarrow{OB} = \cos\beta\underline{i} - \sin\beta\underline{j} \quad \because \cos(-\theta) = \cos\theta$$

Taking dot product of $\overrightarrow{OA}$ with $\overrightarrow{OB}$.

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = (\cos\alpha\underline{i} + \sin\alpha\underline{j}) \cdot (\cos\beta\underline{i} - \sin\beta\underline{j})$$

$$|\overrightarrow{OA}| \cdot |\overrightarrow{OB}| \cdot \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$1 \cdot 1 \ \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Hence proved.

13

If $\underline{a} = 3\underline{i} - \underline{j} - 4\underline{k}$, $\underline{b} = -2\underline{i} - 4\underline{j} - 3\underline{k}$ and $\underline{c} = \underline{i} + 2\underline{j} - \underline{k}$, find a unit vector parallel to $3\underline{a} - 2\underline{b} + 4\underline{c}$. (LHR-2011, 2016)+(LHR-2012).

Solution:

Let: $\underline{u} = 3\underline{a} - 2\underline{b} + 4\underline{c}$

$$\underline{u} = 3(3\underline{i} - \underline{j} - 4\underline{k}) - 2(-2\underline{i} - 4\underline{j} - 3\underline{k}) + 4(\underline{i} + 2\underline{j} - \underline{k})$$

$$\underline{u} = 9\underline{i} - 3\underline{j} - 12\underline{k} + 4\underline{i} + 8\underline{j} + 6\underline{k} + 4\underline{i} + 8\underline{j} - 4\underline{k}$$

$$\underline{u} = 17\underline{i} + 13\underline{j} - 10\underline{k}$$

$$|\underline{u}| = \sqrt{(17)^2 + (13)^2 + (-10)^2}$$

$$|\underline{u}| = \sqrt{289 + 169 + 100}$$

$$|\underline{u}| = \sqrt{558}$$

$$\hat{\underline{u}} = \frac{\underline{u}}{|\underline{u}|} = \frac{17\underline{i} + 13\underline{j} - 10\underline{k}}{\sqrt{558}}$$

$$\hat{\underline{u}} = \frac{17}{\sqrt{558}}\underline{i} + \frac{13}{\sqrt{558}}\underline{j} - \frac{10}{\sqrt{558}}\underline{k}$$

14

Find a unit vector perpendicular to both vectors $\underline{a}$ and $\underline{b}$ where $\underline{a} = [-1, -1, -1]$ and $\underline{b} = [2, -3, 4]$. (LHR-2011)

Solution:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix}$$

$$= \underline{i}(-4-3) - \underline{j}(-4+2) + \underline{k}(3+2) = -7\underline{i} + 2\underline{j} + 5\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(-7)^2 + (2)^2 + (5)^2} = \sqrt{49 + 4 + 25} = \sqrt{78}$$

Required unit vector $= \dfrac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \dfrac{-7\underline{i} + 2\underline{j} + 5\underline{k}}{\sqrt{78}}$

$$= \frac{-7}{\sqrt{78}}\underline{i} + \frac{2}{\sqrt{78}}\underline{j} + \frac{5}{\sqrt{78}}\underline{k}$$

5

A force $\underline{F} = 4\underline{i} - 3\underline{k}$ passes through the point A(2, −2, 5). Find the moment of force about the point B(1, −3, 1).

Solution:

Given points are A(2, −2, 5), B(1, −3, 1)

$\underline{r} = \overrightarrow{BA} = $ P.v of A − P.v of B

$\underline{r} = (2, -2, 5) - (1, -3, 1)$

$\underline{r} = \underline{i} + \underline{j} + 4\underline{k}$

Moment of force $= \underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 4 \\ 4 & 0 & -3 \end{vmatrix}$

$= \underline{i}(-3-0) - \underline{j}(-3-16) + \underline{k}(0-4)$

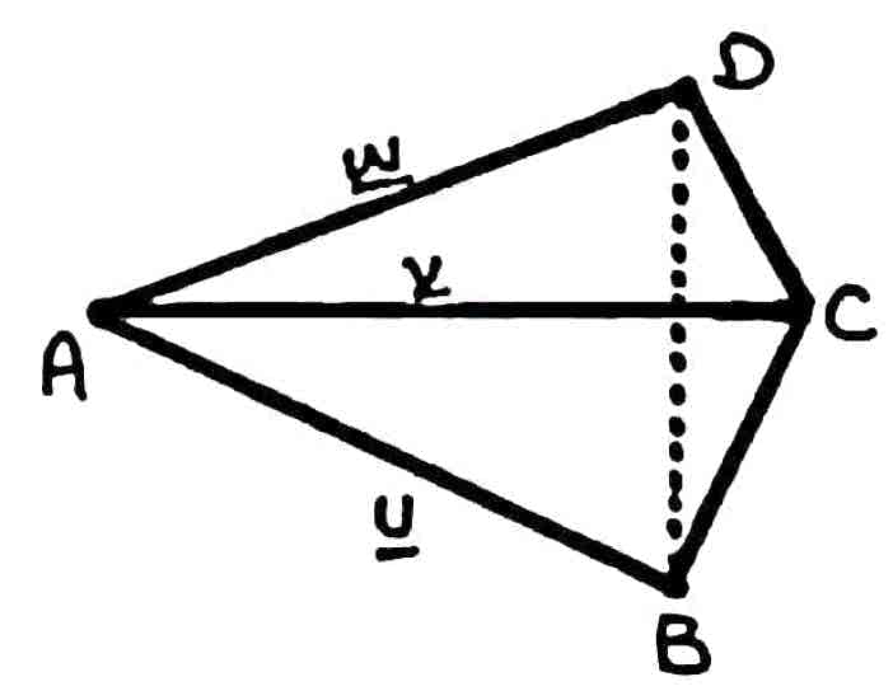
$= -3\underline{i} + 19\underline{j} - 4\underline{k}$

6

Find the volume of tetrahedron with vertices (0,1,2), (3,2,1), (1,2,1) and (5,5,6).

(LHR-2013, 2014, 2015).

Solution:



Let A(0,1,2), B(3,2,1), C(1,2,1) and D(5,5,6) be the vertices of tetrahedron.

$\overrightarrow{AB} = $ P.v of B − P.v of A

$\overrightarrow{AB} = (3,2,1) - (0,1,2)$

$\overrightarrow{AB} = 3\underline{i} + \underline{j} - \underline{k}$

$\overrightarrow{AC} = $ P.v of C − P.v of A

$\overrightarrow{AC} = (1,2,1) - (0,1,2)$

$\overrightarrow{AC} = \underline{i} + \underline{j} - \underline{k}$

$\overrightarrow{AD} = $ P.v of D − P.v of A

$\overrightarrow{AD} = (5\underline{i} + 5\underline{j} + 6\underline{k}) - (0\underline{i} + 1\underline{j} + 2\underline{k})$

$\overrightarrow{AD} = 5\underline{i} + 4\underline{j} + 4\underline{k}$

Volume of tetrahedron $= \frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]$

$$= \frac{1}{6}\begin{vmatrix} 3 & 1 & -1 \\ 1 & -1 & -1 \\ 5 & 4 & 4 \end{vmatrix}$$

$$= \frac{1}{6}\{3(4+4) - 1(4+5) - 1(4-5)\}$$

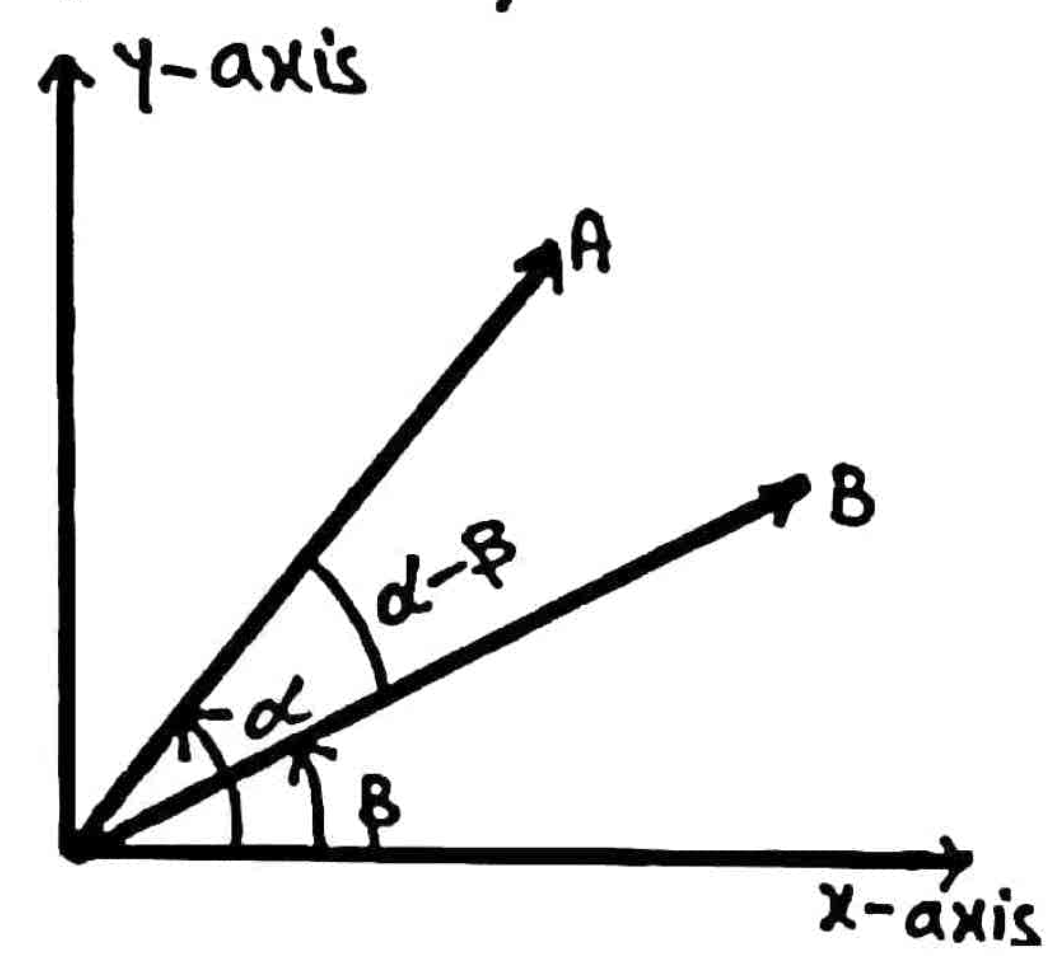
$$= \frac{1}{6}\{24 - 9 + 1\}$$

Volume of Tetrahedron $= \frac{16}{6} = \frac{8}{3}$ cubic units.

By using vectors Prove that

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$. (LHR-2013)+ (LHR-2019)

Solution:



Consider two unit vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ making angle "α" and "β" with positive x-axis.

$m\angle AOB = \alpha - \beta$

Let:

$\overrightarrow{OA} = \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}$

$\overrightarrow{OB} = \cos\beta\,\underline{i} + \sin\beta\,\underline{j}$

Taking cross product of $\overrightarrow{OB}$ with $\overrightarrow{OA}$.

$$\overrightarrow{OB} \times \overrightarrow{OA} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$$|\overrightarrow{OB}||\overrightarrow{OA}|\sin(\alpha-\beta)\underline{k} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(\sin\alpha\cos\beta - \cos\alpha\sin\beta).$$

$$1 \cdot 1\ \sin(\alpha-\beta)\underline{k} = \underline{k}(\sin\alpha\cos\beta - \cos\alpha\sin\beta).$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

(18)

A force of magnitude 6 units acting parallel to $2\underline{i}-2\underline{j}+\underline{k}$ displaces the point of application from (1,2,3) to (5,3,7). Find the work done.

(LHR-2014)

Solution:

Let, $\underline{u} = 2\underline{i}-2\underline{j}+\underline{k}$, A(1,2,3), B(5,3,7)

$$|\underline{u}| = \sqrt{(2)^2+(-2)^2+(1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\hat{\underline{u}} = \frac{\underline{u}}{|\underline{u}|} = \frac{2\underline{i}-2\underline{j}+\underline{k}}{3}$$

Now,

$$\underline{F} = 6\cdot\hat{\underline{u}} = \frac{6(2\underline{i}-2\underline{j}+\underline{k})}{3} = \frac{2(2\underline{i}-2\underline{j}+\underline{k})}{3}$$

$$\underline{F} = 4\underline{i}-4\underline{j}+2\underline{k}$$

$$\underline{d} = \overrightarrow{AB} = \text{P.v of } B - \text{P.v of } A$$

$$\underline{d} = (5,3,7)-(1,2,3)$$

$$\underline{d} = 4\underline{i}+\underline{j}+4\underline{k}$$

Work done $= \underline{F}\cdot\underline{d}$

$$W = (4\underline{i}-4\underline{j}+2\underline{k})\cdot(4\underline{i}+\underline{j}+4\underline{k})$$

$$W = 16-4+8$$

$$\boxed{W = 20 \text{ units.}}$$

(19)

The position vectors of points A, B, C and D are $2\underline{i}-\underline{j}+\underline{k}$, $3\underline{i}+\underline{j}$, $2\underline{i}+4\underline{j}-2\underline{k}$ and $-\underline{i}+2\underline{j}+\underline{k}$, respectively. Show that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

(LHR-2014).

Solution:

$$\overrightarrow{AB} = \text{P.v of } B - \text{P.v of } A$$

$$\overrightarrow{AB} = 3\underline{i}+\underline{j}+0\underline{k}-2\underline{i}+\underline{j}-\underline{k}$$

$\overrightarrow{AB} = \underline{i} + 2\underline{j} - \underline{k}$

$\overrightarrow{CD} = $ P.v of D − P.v of C

$\overrightarrow{CD} = -\underline{i} - 2\underline{j} + \underline{k} - 2\underline{i} - 4\underline{j} + 2\underline{k}$

$\overrightarrow{CD} = -3\underline{i} - 6\underline{j} + 3\underline{k}$

$\overrightarrow{CD} = -3(\underline{i} + 2\underline{j} - \underline{k})$

$\overrightarrow{CD} = -3\,\overrightarrow{AB}$

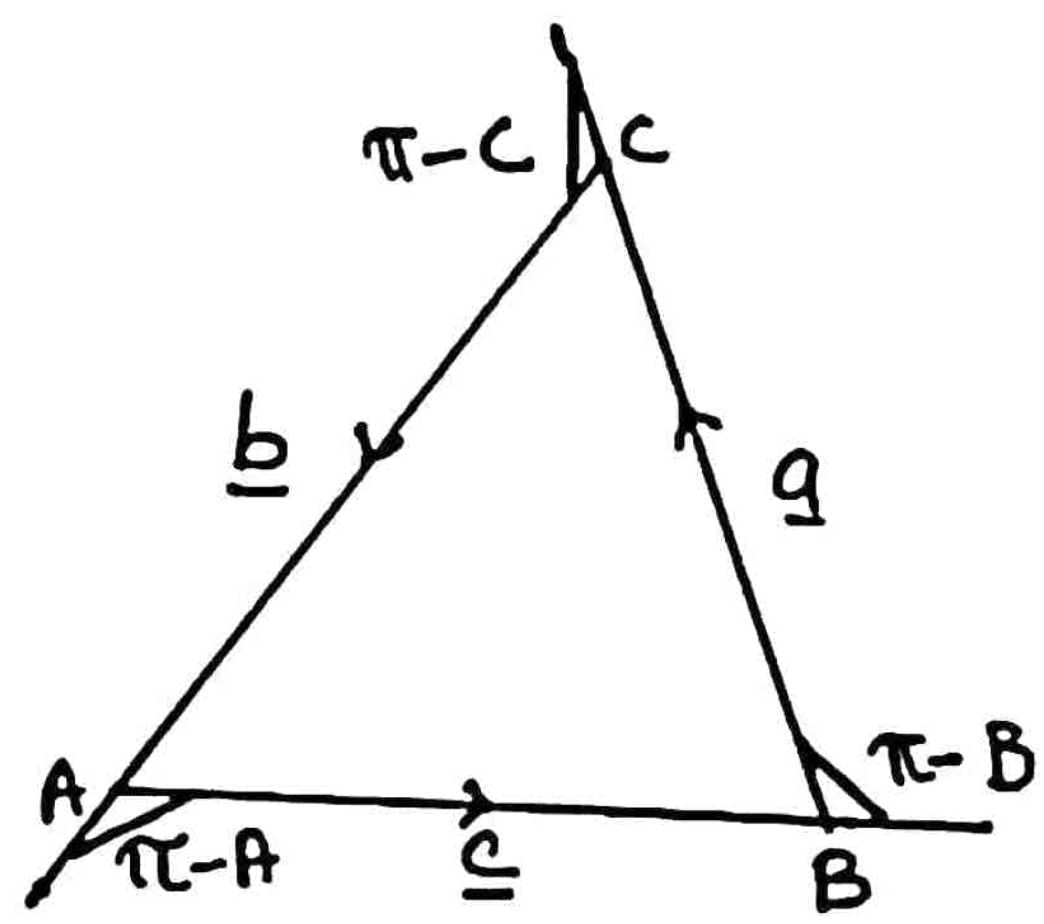
So, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel but in opposite direction

B(10)B

Using vector prove that $b = c\cos A + a\cos C$.

(LHR-2015)

Solution:



Consider:

$\overrightarrow{AB} = \underline{c}$

$\overrightarrow{BC} = \underline{a}$

$\overrightarrow{AC} = \underline{b}$

$\underline{a} + \underline{b} + \underline{c} = 0$

$\underline{b} = -\underline{a} - \underline{c}$

Taking dot product with $\underline{b}$:

$\underline{b}\cdot\underline{b} = -(\underline{a} + \underline{c})\cdot\underline{b}$

$b^2 = -(\underline{a}\cdot\underline{b} + \underline{b}\cdot\underline{c})$

$b^2 = -\left(|\underline{a}|\cdot|\underline{b}|\cos(\pi - C) + |\underline{b}|\cdot|\underline{c}|\cos(\pi - A)\right)$

$b^2 = -\left(a\cdot b(\cos C) - c\cdot b(-\cos A)\right)$

$b^2 = -(-ab\cos C + cb\cos A)$

$b^2 = ab\cos C - cb\cos A$

Dividing both sides by $\underline{b}$, we get:

$b = a\cos C - c\cos A$

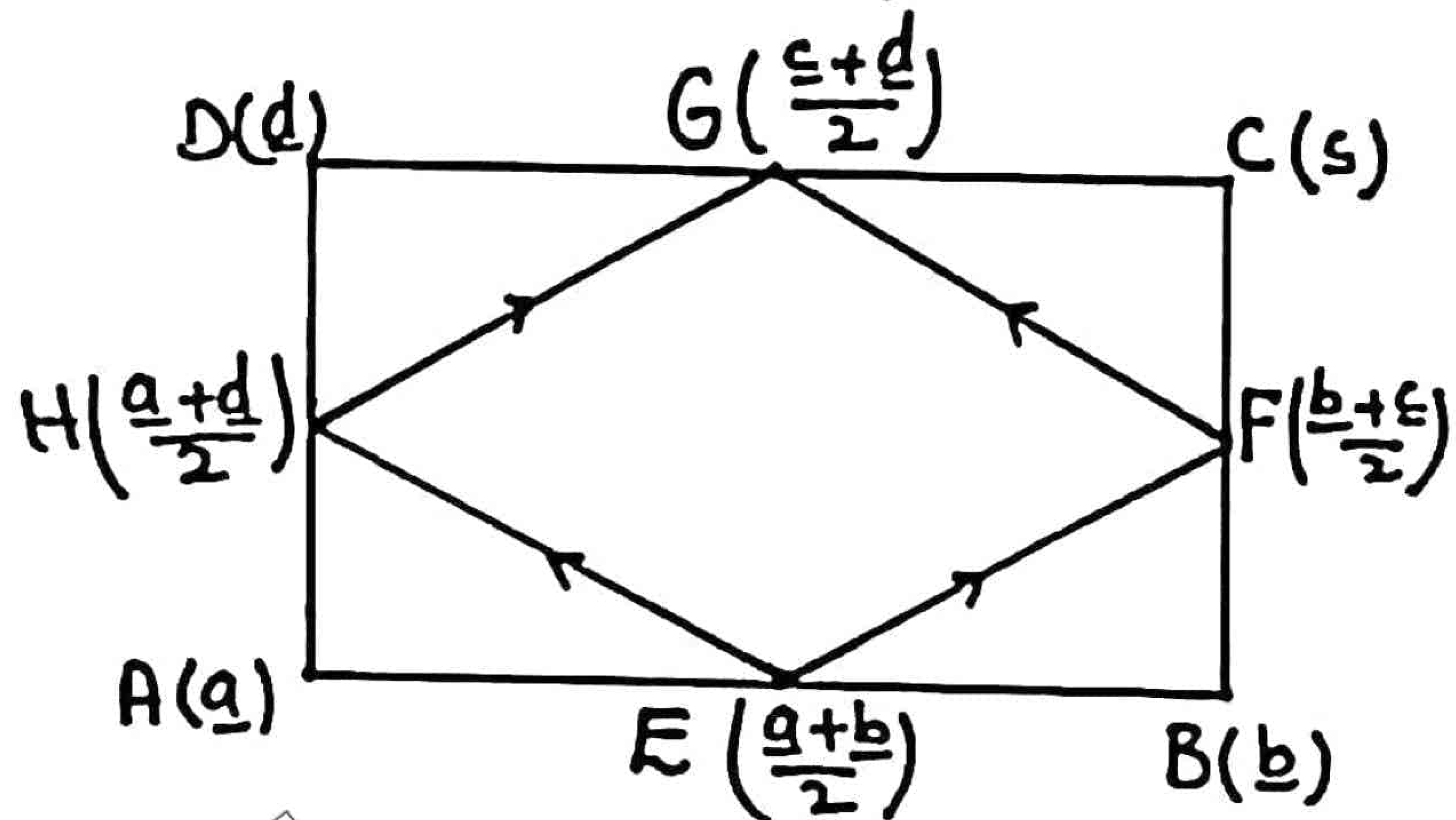
Hence proved.

(11)

Prove that the line segments joining the mid points of sides of quadrilateral taken in order form a parallelogram.

(LHR-2015)

Solution:



Suppose that $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ are position vectors of A, B, C, D respectively then,

P.v of E $= \left(\frac{\underline{a}+\underline{b}}{2}\right)$

P.v of F $= \left(\frac{\underline{b}+\underline{c}}{2}\right)$

P.v of G $= \left(\frac{\underline{c}+\underline{d}}{2}\right)$

P.v of H $= \left(\frac{\underline{a}+\underline{d}}{2}\right)$

$$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = \frac{\underline{b}+\underline{c}}{2} - \left(\frac{\underline{a}+\underline{b}}{2}\right) = \frac{\underline{b}+\underline{c}-\underline{a}-\underline{b}}{2} = \frac{\underline{c}-\underline{a}}{2} \quad \text{— I}$$

$$\overrightarrow{FG} = \overrightarrow{OG} - \overrightarrow{OF} = \left(\frac{\underline{c}+\underline{d}}{2}\right) - \left(\frac{\underline{b}+\underline{c}}{2}\right) = \frac{\underline{c}+\underline{d}-\underline{b}-\underline{c}}{2} = \frac{\underline{d}-\underline{b}}{2} \quad \text{— II}$$

$$\overrightarrow{HG} = \overrightarrow{OG} - \overrightarrow{OH} = \left(\frac{\underline{c}+\underline{d}}{2}\right) - \left(\frac{\underline{a}+\underline{d}}{2}\right) = \frac{\underline{c}+\underline{d}-\underline{a}-\underline{d}}{2} = \frac{\underline{c}-\underline{a}}{2} \quad \text{— III}$$

$$\overrightarrow{EH} = \overrightarrow{OH} - \overrightarrow{OE} = \left(\frac{\underline{a}+\underline{d}}{2}\right) - \left(\frac{\underline{a}+\underline{b}}{2}\right) = \frac{\underline{a}+\underline{d}-\underline{a}-\underline{b}}{2} = \frac{\underline{d}-\underline{b}}{2} \quad \text{— IV}$$

From equation I and III

$\overrightarrow{EF} = \overrightarrow{HG} \Rightarrow \overrightarrow{EF}$ is parallel to $\overrightarrow{HG}$

From equation II and IV:

$\overrightarrow{FG} = \overrightarrow{EH} \Rightarrow \overrightarrow{FG}$ is parallel to $\overrightarrow{EH}$

So,

EFGH is a parallelogram.

(12)

Find the constant "α" such that the vectors are coplanar if $\underline{i}-\underline{j}+\underline{k}$, $\underline{i}-2\underline{j}-3\underline{k}$ and $3\underline{i}-\alpha\underline{j}+5\underline{k}$. (LHR-2015)

Solution:

Given:

vectors are coplanar so,

$$\underline{a}\cdot(\underline{b}\times\underline{c})=0$$

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -\alpha & 5 \end{vmatrix}=0$$

$$1(-10-3\alpha)+1(5+9)+1(-\alpha+6)=0$$

$$-10-3\alpha+14-\alpha+6=0$$

$$-4\alpha-10=0$$

$$4\alpha=10$$

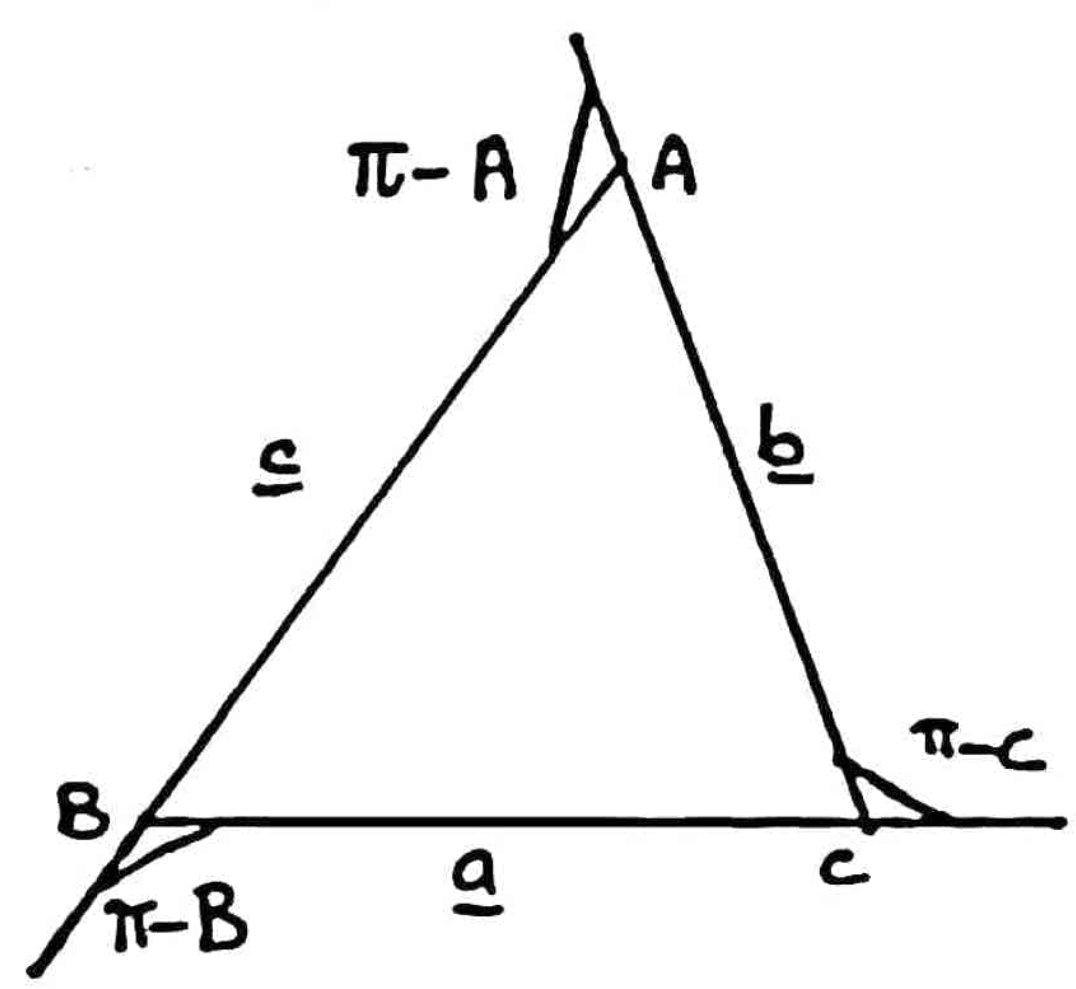
$$\alpha=\frac{10}{4} \Longrightarrow \boxed{\alpha=\frac{5}{2}}$$

(13)

Prove that in any triangle ABC by vector method $a^2=b^2+c^2-2bc\cos A$. (LHR-2016)

Solution:

Consider:



$\overline{AB}=\underline{c}$

$\overline{BC}=\underline{a}$

$\overline{CA}=\underline{b}$

$\underline{a}+\underline{b}+\underline{c}=0$

$$\underline{a} = -\underline{b} - \underline{c}$$

$$\underline{a} = -(\underline{b} + \underline{c})$$

Taking square on both sides,

$$\underline{a}\cdot\underline{a} = [-(\underline{b}+\underline{c})]^2$$

$$a^2 = (\underline{b}+\underline{c})\cdot(\underline{b}+\underline{c})$$

$$a^2 = \underline{b}\cdot\underline{b} + \underline{b}\cdot\underline{c} + \underline{c}\cdot\underline{b} + \underline{c}\cdot\underline{c}$$

$$a^2 = b^2 + 2\underline{b}\underline{c} + c^2 \qquad \because (\underline{b}\cdot\underline{c} = \underline{c}\cdot\underline{b})$$

$$a^2 = b^2 + c^2 + 2bc\cdot\cos(\pi - A)$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Hence proved.

(14)

Find the volume of tetrahedron whose vertices are A(2,1,8), B(3,2,9), C(2,1,4) and D(3,3,0). (LHR-2016+2018)

Solution:

$$\overrightarrow{AB} = \text{P.v of } B - \text{P.v of } A$$

$$\overrightarrow{AB} = (3,2,9) - (2,1,8)$$

$$\overrightarrow{AB} = \underline{i} + \underline{j} + \underline{k}$$

$$\overrightarrow{AC} = \text{P.v of } C - \text{P.v of } A$$

$$\overrightarrow{AC} = (2,1,4) - (2,1,8)$$

$$\overrightarrow{AC} = 0\underline{i} - 0\underline{j} - 4\underline{k}$$

$$\overrightarrow{AD} = \text{P.v of } D - \text{P.v of } A$$

$$\overrightarrow{AD} = (3,3,0) - (2,1,8)$$

$$\overrightarrow{AD} = \underline{i} + 2\underline{j} - 8\underline{k}$$

Now,

Volume of tetrahedron: $\frac{1}{6}[\overline{AB}\ \overline{AC}\ \overline{AD}]$

$$V = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & -8 \end{vmatrix}$$

$$V = \frac{1}{6}\left\{1(0+8)-1(0+4)+1(0-0)\right\}$$

$$V = \frac{1}{6}\left\{8-4\right\}$$

$$V = \frac{4}{6}$$

$$\boxed{V = \frac{2}{3}}$$

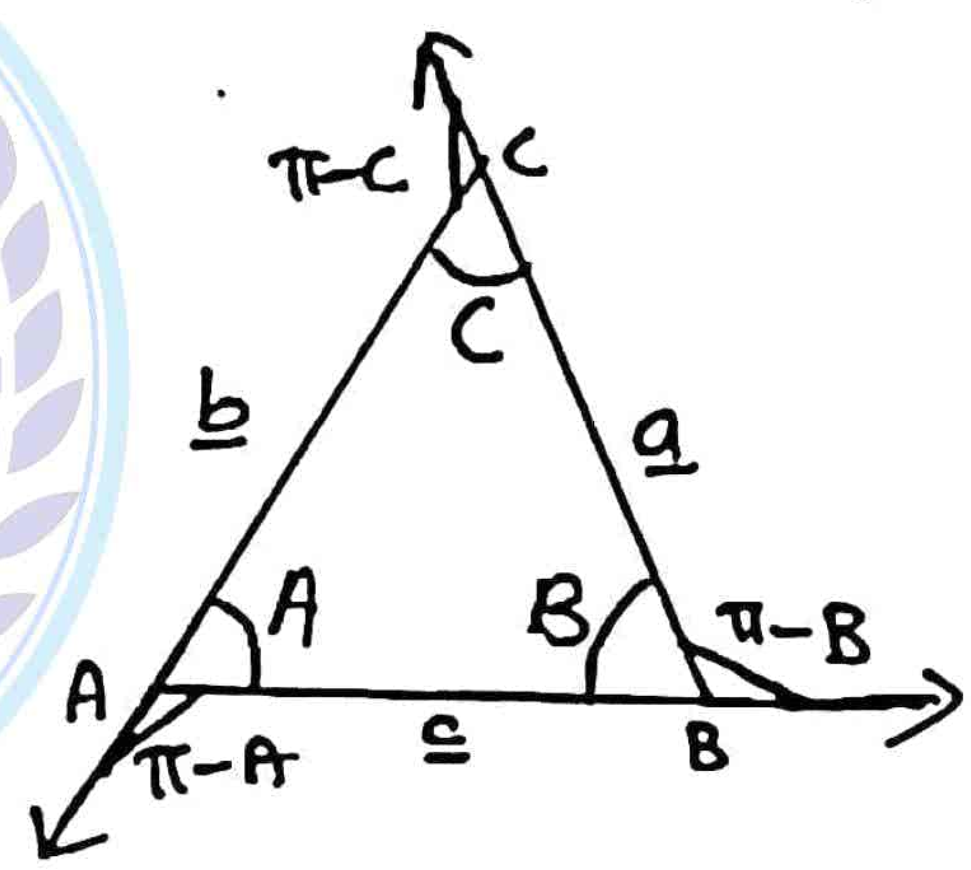
|15|

Prove that in any triangle ABC $b = c\cos A + a\cos C$ by using vectors.

Solution:

(LHR-2017)

Consider



$$\overrightarrow{AB} = \underline{c}$$
$$\overrightarrow{BC} = \underline{a}$$
$$\overrightarrow{AC} = \underline{b}$$
$$\underline{a}+\underline{b}+\underline{c} = 0$$
$$\underline{b} = -\underline{a}-\underline{c}$$

Taking dot product with $\underline{b}$:

$$\underline{b}\cdot\underline{b} = (-\underline{a}-\underline{c})\cdot\underline{b}$$

$$b^2 = -(\underline{a}\cdot\underline{b}+\underline{b}\cdot\underline{c})$$

$$b^2 = -\left[|\underline{a}|\cdot|\underline{b}|\cos(\pi-C)-|\underline{c}|\cdot|\underline{b}|\cos(\pi-A)\right]$$

$$b^2 = -a\cdot b(-\cos C) + cb\cos(-A)$$

$$b^2 = ab\cos C + cb\cos A$$

Dividing both sides by b, we get:

$$b = a\cos C + c\cos A$$

Hence proved.

16

Find the values of $2\underline{i}\times 2\underline{j}\cdot\underline{k}$ and $[\underline{k}\ \underline{i}\ \underline{j}]$ (LHR-2017)

Solution:

$$(2\underline{i}\times 2\underline{j})\cdot\underline{k} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2(2-0) - 0(0-0) + 0(0-0)$$

$$= 4$$

Now,

$$\underline{k}\cdot(\underline{i}\times\underline{j}) = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

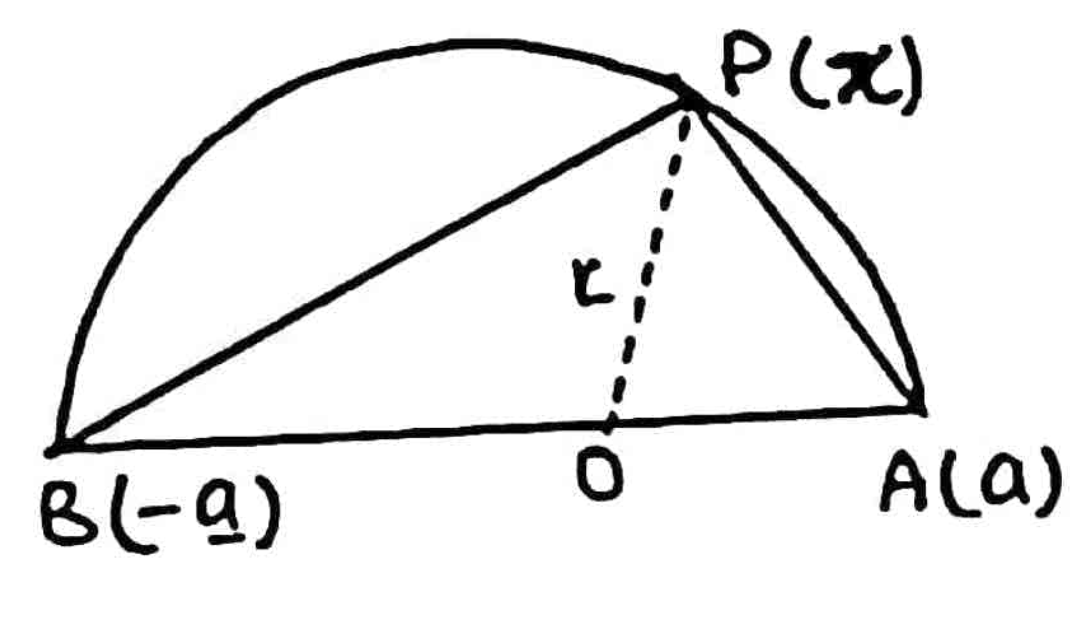
$$= 0(0-0) - 0(0-0) + 1(1-0)$$

$$= 1$$

17

Prove that the angle in cemi-circle is right triangle.

Proof:



Consider a semi-circle with end points $A(\underline{a})$ and $B(-\underline{b})$.

Let $P(\underline{x})$ be any point on semi circle with equal radii.

$$|\overrightarrow{OB}| = |\overrightarrow{OA}| = |\overrightarrow{OP}|$$

$\overrightarrow{AP} =$ P.v of P − P.v of A $= \underline{r} - \underline{a}$

$\overrightarrow{BP} =$ P.v of P − P.v of B $= \underline{r} - (-\underline{a}) = \underline{r} + \underline{a}$

Now, $\overrightarrow{AP} \cdot \overrightarrow{BP} = (\underline{r} - \underline{a}) \cdot (\underline{r} + \underline{a})$

$$\overrightarrow{AP} \cdot \overrightarrow{BP} = \underline{r} \cdot \underline{r} + \underline{r} \cdot \underline{a} - \underline{a} \cdot \underline{r} - \underline{a} \cdot \underline{a}$$

$$\overrightarrow{AP} \cdot \overrightarrow{BP} = r^2 + \underline{a} \cdot \underline{r} - \underline{a} \cdot \underline{r} - a^2 \qquad \therefore \underline{r} \cdot \underline{a} = \underline{a} \cdot \underline{r}$$

$$\overrightarrow{AP} \cdot \overrightarrow{BP} = r^2 - a^2$$

$$\overrightarrow{AP} \cdot \overrightarrow{BP} = a^2 - a^2 \qquad \therefore r = a$$

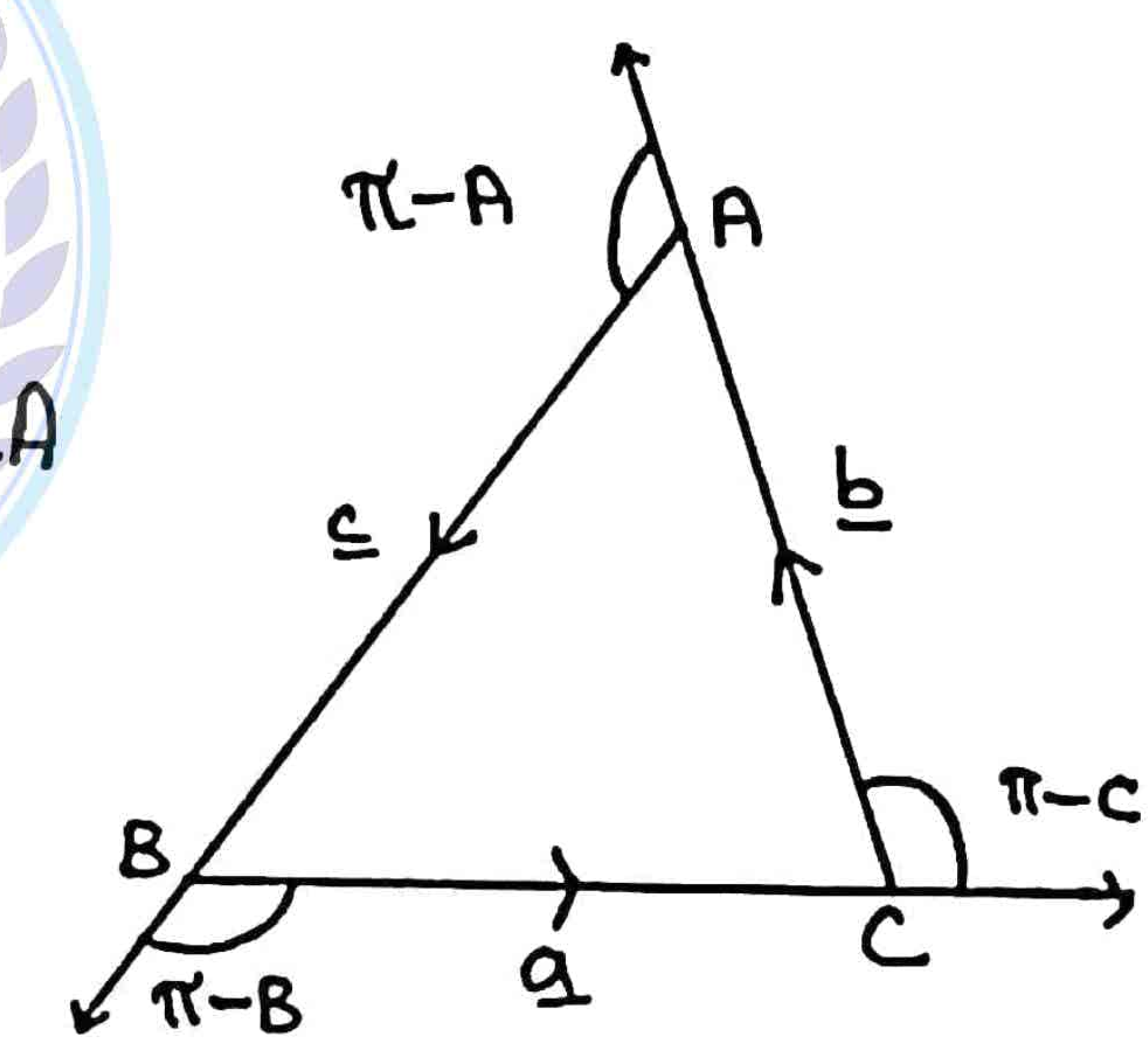
$$\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$$

So, $\overrightarrow{AP}$ is perpendicular $\overrightarrow{BP}$. Hence proved that angle between $\overrightarrow{AP}$ and $\overrightarrow{BP}$ is 90°.

(18)

By vector method, prove in any triangle ABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$. (LHR-2017)

Proof:



Suppose vectors $\underline{a}, \underline{b}$ and $\underline{c}$ are along the sides BC, CA and AB respectively of the triangle ABC.

$$\therefore \underline{a} + \underline{b} + \underline{c} = 0$$

$$\underline{b} + \underline{c} = -\underline{a} \quad \text{—— I}$$

Taking cross product with $\underline{c}$:

$$\underline{b} \times \underline{c} + \underline{c} \times \underline{c} = -\underline{a} \times \underline{c}$$

$$\underline{b} \times \underline{c} = \underline{c} \times \underline{a} \qquad \{\because \underline{c} \times \underline{a} = -\underline{a} \times \underline{c}$$

$$\Rightarrow |\underline{b} \times \underline{c}| = |\underline{c} \times \underline{a}|$$

$$|\underline{b}||\underline{c}| \sin(\pi - A) = |\underline{c}||\underline{a}| \sin(\pi - B)$$

$$\Rightarrow bc \sin A = ca \sin B$$

$$\Rightarrow b \sin A = a \sin B$$

$$\therefore \frac{b}{\sin B} = \frac{a}{\sin A} \quad \text{——— II}$$

Taking cross product of (i) with $\underline{b}$, we get.

$$\underline{b} \times \underline{b} + \underline{b} \times \underline{c} = -\underline{a} \times \underline{b}$$

$$\underline{b} \times \underline{c} = \underline{b} \times \underline{a} \qquad \because \underline{b} \times \underline{a} = -\underline{a} \times \underline{b}$$

$$|\underline{b}||\underline{c}| \sin(\pi - A) = |\underline{b}||\underline{a}| \sin(\pi - C)$$

$$\Rightarrow bc \sin A = ab \sin C$$

$$\Rightarrow c \sin A = a \sin C$$

$$\therefore \frac{a}{\sin A} = \frac{c}{\sin C}$$

From (i) and (ii),

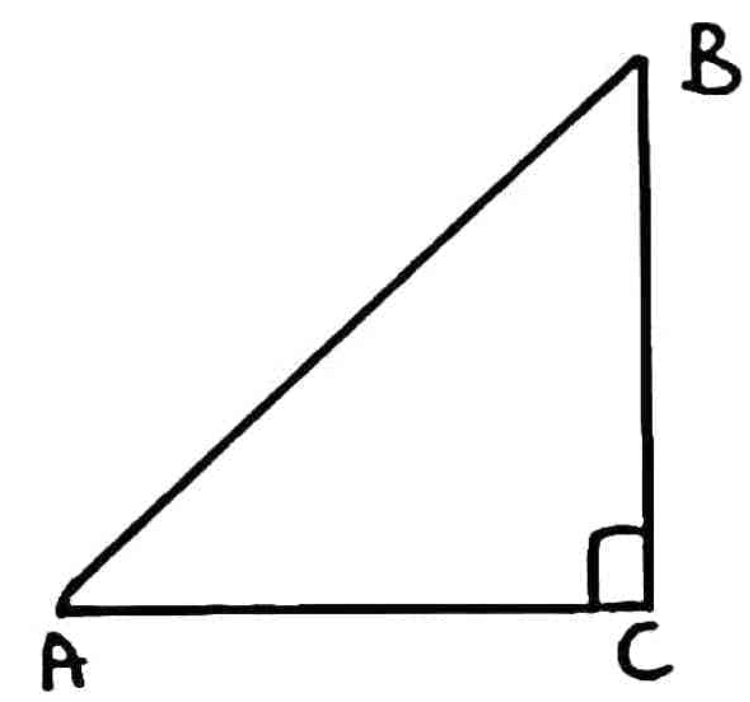
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Hence proved.

(19)

Find x so that points $A(1,-1,0)$, $B(-2,2,1)$ and $C(0,2,x)$ form triangle with angle at C. (LHR-2018)

Solution:



$\overrightarrow{CA}$ = P.v of A − P.v of C

$\overrightarrow{CA} = (1,-1,0) - (0,2,x)$

$\overrightarrow{CA} = (1,-3,-x)$

$\overrightarrow{CB}$ = P.v of B − P.v of C

$\overrightarrow{CB} = (-2,2,1) - (0,2,x)$

$$\overrightarrow{CB} = (-2, 0, 1-x)$$

Given ABC is a right angle triangle. So $\overrightarrow{CA} \perp \overrightarrow{CB}$

$$\Rightarrow \overrightarrow{CA} \cdot \overrightarrow{CB} = 0$$

$$(1, -3, -x) \cdot (-2, 0, 1-x) = 0$$

$$-2 - 0 + x^2 - x = 0$$

$$x^2 - x - 2 = 0$$

$$x^2 - 2x + x - 2 = 0$$

$$x(x-2) + 1(x-2) = 0$$

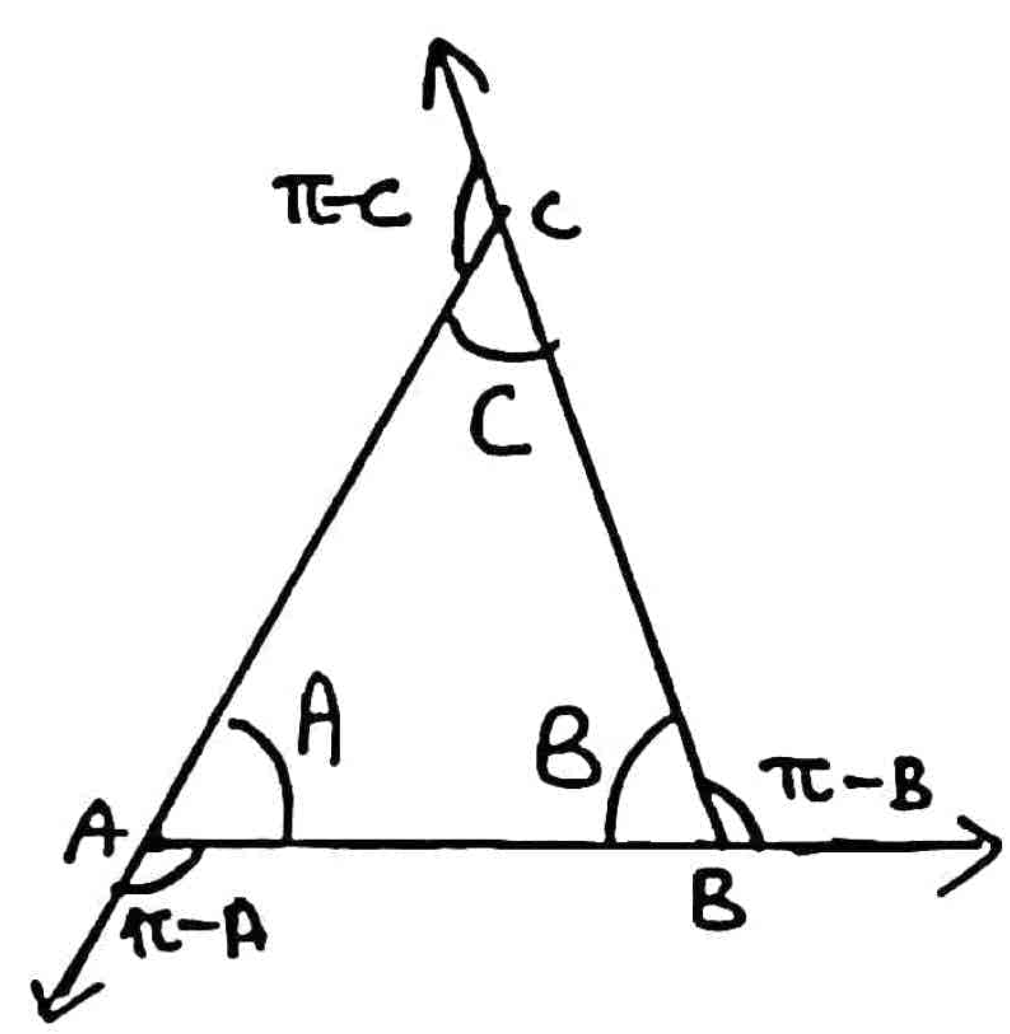
$$x + 1 = 0, \quad x - 2 = 0$$

$$\boxed{x = -1}, \quad \boxed{x = 2}$$

Q|20|B

Prove that $b^2 = c^2 + a^2 - 2ca\cos B$ by using vectors.

Consider:



$\overline{AB} = \underline{c}$

$\overline{BC} = \underline{a}$

$\overline{AC} = \underline{b}$

$$\underline{a} + \underline{b} + \underline{c} = 0$$

$$\underline{b} = -\underline{a} - \underline{c}$$

$$\underline{b} = -(\underline{a} + \underline{c})$$

Taking square on both sides,

$$b^2 = \left[-(\underline{a} + \underline{c})\right]^2$$

$$b^2 = (\underline{a} + \underline{c})^2$$

$$b^2 = (\underline{a} + \underline{c}) \cdot (\underline{a} + \underline{c})$$

$b^2 = \underline{a}\cdot\underline{a} + \underline{a}\cdot\underline{c} + \underline{c}\cdot\underline{a} + \underline{c}\cdot\underline{c}$

$b^2 = a^2 + ac + ac + c^2$ $\quad\because \underline{c}\cdot\underline{a} = \underline{a}\cdot\underline{c}$

$b^2 = a^2 + 2\,\underline{a}\cdot\underline{c} + c^2$

$b^2 = a^2 + c^2 + 2|\underline{a}||\underline{c}|\cos(\pi - B)$

$b^2 = a^2 + c^2 + 2ac(-\cos B)$

$b^2 = a^2 + c^2 - 2ac\cos B$

Hence proved