F.Sc Math Part-II

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Unit#1 Functions And Limits

M.C.Q's

(1) If At least one Vertical line meets the Curve at More than one points Then Curve is... (LHR-2018)
(a) A Function (b) Not a function (c) One-to-one function (d) Onto

(2) If $f: X \to Y$ is a function, Then X is called (LHR-2005)
(a) Domain (b) Range (b) function (d) Dependent Variable

(3) $f(x) = ax+b, a \neq 0$ is (a) Cubic function (b) Linear function (c) Trigonometric function (d) Quadratic function (LHR-2019, Bahawalpur, Sargodha 2013)

(4) $f(x) = x^{2/3}$ is an =
(a) Even function (b) Odd function (c) Neither even nor odd (d) Cubic

(5) Let $f(x) = x^2 + \cos x$, Then $f(x)$ is = (LHR-2018)
(a) Odd function (b) Even function (c) Constant (d) None

(6) If y is expressed in Terms of a Variable x as $y = f(x)$, Then y is called = (LHR-2005)
(a) Identity Function (b) Constant Function (c) Explicit Function (d) Implicit

(7) The function $y = x^2 + 27$ is a/an = (LHR-2019)
(a) Even function (b) Implicit Function (c) Explicit Function (d) Both a & c

(8) $x = a\cos t, y = a\sin t$ are the Parametric Equations of a/an =
(a) Circle (b) Parabola (c) Ellipse (d) line (LHR-2012)

(9) For Parametric equations $x = at^2, y = 2at$ Represent the equation =
(a) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (b) $x^2 + y^2 = 1$ (c) $y^2 = 4ax$ (d) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (LHR-2006, 2008)

(10) $\operatorname{Cosech} x =$ (a) $\frac{e^x + e^{-x}}{2}$ (b) $\frac{e^x - e^{-x}}{2}$ (c) $\frac{2}{e^x - e^{-x}}$ (d) $\frac{e^x + e^{-x}}{e^x - e^{-x}}$ (LHR-2005)

(11) $\frac{e^x - e^{-x}}{2} =$ (a) $\sinh x$ (b) $\cosh x$ (c) $\cos x$ (d) $\sin x$

(12) $\cosh^2 x - \sinh^2 x =$

(a) -1 (b) 1 (c) 2 (d) 0 (LHR-2018)

(13) The Perimeter P of a Square as a function of its Area A

(a) $P = 4\sqrt{A}$ (b) $P = \sqrt{A}$ (c) $P = 4A$ (d) $P = \frac{1}{4}\sqrt{A}$ (LHR-2008, 2011)

(14) The Area A of a Circle as a function of its Circumference 'C'

(a) $A = \frac{1}{2\pi}C$ (b) $A = \frac{C}{4\pi}$ (c) $A = \frac{C^2}{4\pi}$ (d) $\frac{C^2}{\pi}$ (LHR-2013)

(15) The expression $\ln(x+\sqrt{x^2+1}) =$ (LHR-2016)

(a) $\sinh^{-1}x$ (b) $\cosh^{-1}x$ (c) $\tanh^{-1}x$ (d) $\mathrm{cosech}^{-1}x$

(16) Domain of $f(x) = x^2+1$ is (LHR-2018)

(a) $\mathbb{R}$ (b) $\mathbb{R}-\{1\}$ (c) $\mathbb{R}-\{1\}$ (d) $[1,\infty)$

(17) Range of $f(x) = \sqrt{x+1} =$ (LHR-2007)

(a) $[0,\infty)$ (b) $(1,\infty)$ (c) $[-1,\infty)$ (d) $(-\infty,\infty)$

(18) For Real Valued function $f(x) = 2x+1$, what will be $fof(x) = ?$

(a) $3x-4x$ (b) $2x-1$ (c) $2x+1$ (d) $4x+3$ (Fedral 2013, RWP-2013)

(19) If $f(x) = 2x+1$, Then $f^{-1}(x) = ?$ (LHR-2019)

(a) $2x-1$ (b) $1-2x$ (c) $x-\frac{1}{2}$ (d) $\frac{x-1}{2}$

(20) If $f(x) = -2x+8$, Then $f^{-1}(x) =$ (LHR-2013)

(a) $\frac{8+x}{2}$ (b) $\frac{x-8}{2}$ (c) $\frac{8-x}{2}$ (d) $\frac{2}{8-x}$

(21) $\lim_{x\to a} \frac{x^n-a^n}{x-a} =$ (a) na^{n-1} (b) na^n (c) na^{n+1} (d) None (Fedral 2013)

(22) $\lim_{x\to 2} \frac{x^5-32}{x-2} =$ (a) 40 (b) 60 (c) 80 (d) 120 (LHR-2016)

(23) $\lim_{x\to 0} \frac{a^x-1}{x} =$ (a) $\ln x$ (b) $\ln a$ (c) 1 (d) 0 (LHR-2008)

(24) $\lim_{x\to 0} \frac{e^x-1}{x} =$ (a) $\ln a$ (b) 0 (c) 1 (d) ∞ (FSD-Multan 2009)

(25) If $\lim_{x\to 0} \frac{f(x)-1}{x} = \ln a, a>0$ (a) a^{-x} (b) a^x (c) e^{-x} (d) e^x

(LHR-2015)

(26) $\lim_{x\to 0}\frac{\sin 7x}{x}=$ (a) 7 (b) $\frac{1}{7}$ (c) 1 (d) 0 (LHR-2016)

(27) $\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta}=$ (a) 0 (b) $\frac{0}{0}$ (c) 1 (d) ∞ (LHR-2017)

(28) $\lim_{x\to 0}\frac{\sin ax}{\sin bx}=$ (a) $\frac{a}{b}$ (b) $-\frac{a}{b}$ (c) $-\frac{b}{a}$ (d) $\frac{b}{a}$ (LHR FSD 2010)

(29) $\lim_{x\to 0}\frac{\sin 3x}{\sin 2x}=$ (a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{1}{6}$ (d) $\frac{1}{4}$ (LHR-2015)

(30) $\lim_{\theta\to 0}\frac{1-\cos p\theta}{1+\cos p\theta}=$ (a) 0 (b) $\frac{p}{q}$ (c) $\frac{p^2}{q^2}$ (d) -1 (LHR-2014) (DG Khan 2019)

(31) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=$ (a) e^{-1} (b) e (c) e^2 (d) e^3 (LHR-2007) (GJR-2006, 2008)

(32) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}=$ (a) 0 (b) e^{2n} (c) e^2 (d) e^n (FSD-2019)

(33) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{2}}=$ (a) e^{-1} (b) e^2 (c) $e^{1/2}$ (d) e^3 (LHR-2019)

(34) $\lim_{n\to\infty}\left(\frac{5n+1}{5n}\right)^n=$ (a) $e^{1/5}$ (b) e^5 (c) e^{-5} (d) $e^{-1/5}$ (LHR-2017)

(35) $\lim_{x\to 0}(1+x)^{1/x}=$ (a) e^x (b) ∞ (c) $e^{1/x}$ (d) e (LHR-2005)

(36) $\lim_{x\to 0}(1-x)^{1/x}=$ (a) e^x (b) ∞ (c) $e^{1/x}$ (d) e^{-1} (LHR-2009)

(37) $\lim_{x\to 0}\frac{e^{-x/2}}{1+e^{-x/2}}=$ (a) 1 (b) $\frac{1}{2}$ (c) 0 (d) ∞ (LHR-2013)

(38) If $f(x)$ is Continuous at $x=c$, Then (LHR-2013)
(a) $\lim_{x\to c} f(x)\neq 0$ (b) $\lim_{x\to c} f(x)=1$ (c) $\lim_{x\to c} f(x)=f(c)$ (d) None

(39) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n/2}=$ (a) e^{-1} (b) e^2 (c) $e^{1/2}$ (d) e^3 (LHR-2019)

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Page-18 **Example 1:-** If $P(x)=a_n x^n + a_{n-1}\cdot x^{n-1} + \dots + a_1 x + a_0$ is a polynomial function of degree n, Show that

$$\lim_{x\to c} P(x) = P(c)$$

Proof:- Using the Theorems on Limits, we have

$$\lim_{x\to c} P(x) = \lim_{x\to c}(a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0)$$

$$= \lim_{x\to c} a_n x^n + \lim_{x\to c} a_{n-1}x^{n-1} + \dots \lim_{x\to c} a_1 x + \lim_{x\to c} a_0$$

$$= a_n \lim_{x\to c} x^n + a_{n-1}\lim_{x\to c} x^{n-1} + \dots a_1 \lim_{x\to c} x + \lim_{x\to c} a_0$$

$$= a_n c^n + a_{n-1} c^{n-1} + \dots a_1 c + a_0$$

$$\lim_{x\to c} P(x) = P(c) \text{ Proved.}$$

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(Page 19) Prove that $\lim_{x\to a}\dfrac{x^n - a^n}{x-a} = na^{n-1}$, where n is an Integer and $a>0$.

Case-I Suppose n is a Positive Integer.

By Substituting $x=a$, we get $\left(\frac{0}{0}\right)$ form. So we make factors.

$$\because x^n - a^n = (x-a)(x^{n-1}\cdot a^0 + x^{n-2}\cdot a^1 + x^{n-3}\cdot a^2 + \dots x^{n-n}\cdot a^{n-1})$$

Key Point:-

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^4 - b^4 = (a-b)(a^3 + a^2 b + ab^2 + b^3)$$

$$\vdots \qquad \vdots$$

$$a^n - b^n = (a-b)(a^{n-1}\cdot b^0 + a^{n-2}\cdot b^1 + a^{n-3}\cdot b^2 + \dots a^{n-n}\cdot b^{n-1})$$

$$L.H.S = \lim_{x\to a}\frac{x^n - a^n}{x-a}$$

$$= \lim_{x\to a}\frac{(x-a)(x^{n-1}\cdot a^0 + x^{n-2}\cdot a^1 + x^{n-3}\cdot a^2 + \dots x^{n-n}\cdot a^{n-1})}{(x-a)}$$

$$= \lim_{x\to a}(x^{n-1} + a\cdot x^{n-2} + a^2\cdot x^{n-3} + \dots a^{n-1})$$

$$= \lim_{x\to a} x^{n-1} + a\lim_{x\to a} x^{n-2} + a^2\lim_{x\to a} x^{n-3} + \dots \lim_{x\to a} a^{n-1}$$

$$= a^{n-1} + a\cdot a^{n-2} + a^2\cdot a^{n-3} + \dots a^{n-1}$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + \dots a^{n-1} \text{ (n Terms)}$$

$$\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$$

Case-II: Suppose n is a negative Integer (Say $n=-m$), where m is a Positive Integer.

Now $\dfrac{x^n-a^n}{x-a}=\dfrac{x^{-m}-a^{-m}}{x-a}$

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$$

$$L.H.S=\lim_{x\to a}\frac{x^n-a^n}{x-a}$$

$$=\lim_{x\to a}\frac{x^{-m}-a^{-m}}{x-a},\quad \text{Put } n=-m$$

$$=\lim_{x\to a}\left[\frac{1}{x-a}\cdot(x^{-m}-a^{-m})\right]$$

$$=\lim_{x\to a}\left[\frac{1}{x-a}\cdot\left(\frac{1}{x^m}-\frac{1}{a^m}\right)\right]$$

$$=\lim_{x\to a}\left[\frac{1}{x-a}\cdot\left(\frac{a^m-x^m}{x^m\cdot a^m}\right)\right]$$

$$=\lim_{x\to a}\left[\frac{-1}{x-a}\cdot\left(\frac{x^m-a^m}{x^m\cdot a^m}\right)\right]$$

$$=-\lim_{x\to a}\left[\frac{1}{x-a}\times\frac{x^m-a^m}{x^m\cdot a^m}\right]$$

$$=-\lim_{x\to a}\left[\frac{1}{\cancel{(x-a)}}\times\frac{\cancel{(x-a)}(x^{m-1}+x^{m-2}\cdot a+x^{m-3}\cdot a^2+\cdots a^{m-1})}{x^m\cdot a^m}\right]$$

$$=-\lim_{x\to a}\left[\frac{x^{m-1}+x^{m-2}\cdot a+x^{m-3}a^2+\cdots a^{m-1}}{x^m\cdot a^m}\right]$$

By Applying Limit

$$=-\frac{[a^{m-1}+a^{m-2}\cdot a+a^{m-3}\cdot a^2+\cdots a^{m-1}]}{a^m\cdot a^m}$$

$$=-\frac{[a^{m-1}+a^{m-1}+a^{m-1}+\cdots a^{m-1}]}{a^{2m}}$$

$$=-ma^{m-1}\cdot a^{-2m}$$

$$=-ma^{m-1-2m}$$

$$=-ma^{-m-1}$$

$$=na^{n-1}\qquad \because \boxed{n=-m}$$

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Prove that $\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}=\frac{1}{2\sqrt{a}}$ (LHR-2019)

By Subsitituiting $x=0$, we get $\left(\frac{0}{0}\right)$ form,
So Rationalizing the Numerator.

$$\therefore \lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}=\lim_{x\to 0}\left(\frac{\sqrt{x+a}-\sqrt{a}}{x}\times\frac{\sqrt{x+a}+\sqrt{a}}{\sqrt{x+a}+\sqrt{a}}\right)$$

$$=\lim_{x\to 0}\left(\frac{x+\cancel{a}-\cancel{a}}{x(\sqrt{x+a}+\sqrt{a})}\right)$$

$$=\lim_{x\to 0}\left(\frac{\cancel{x}}{\cancel{x}\cdot(\sqrt{x+a}+\sqrt{a})}\right)$$

$$=\lim_{x\to 0}\left(\frac{1}{\sqrt{x+a}+\sqrt{a}}\right)$$

$$=\frac{1}{\sqrt{0+a}+\sqrt{a}}$$

$$\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}=\frac{1}{2\sqrt{a}}$$ Proved.

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Prove that $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n=e$ (LHR-2014-17)

By the Binomial Theorem

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}\cdot x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$

$$\left(1+\frac{1}{n}\right)^n=1+n\left(\frac{1}{n}\right)+\frac{\cancel{n}(n-1)}{2!}\cdot\left(\frac{1}{n}\right)^{\cancel{2}}+\frac{\cancel{n}(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^{\cancel{3}}+\cdots$$

$$=1+1+\frac{1}{2!}\cdot\left(\frac{n-1}{n}\right)+\frac{1}{3!}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)+\cdots$$

$$=1+1+\frac{1}{2!}\cdot\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots$$

When $n\to\infty$, $\frac{1}{n},\frac{2}{n},\frac{3}{n},\ldots$ all Tends to zero

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots$$

$$=1+1+0.5+0.166667+0.0416667$$

$$=2.718281$$ As Approximate Value of $e=2.718281$

$$\therefore \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)=e$$

Prove that $\lim\limits_{x\to 0}(1+x)^{1/x}=e$

We know that $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e \rightarrow (i)$

Put $n=\frac{1}{x}$, Then $\frac{1}{n}=x$

when $x\to 0$, $n\to\infty$

From(i) $\lim\limits_{x\to 0}(1+x)^{n}=e$ (Proved)

Prove that $\lim\limits_{x\to 0}\frac{a^x-1}{x}=\log_e a$ (LHR-2018) (Long Question)

$L.H.S=\lim\limits_{x\to 0}\frac{a^x-1}{x} \rightarrow *$

Put $a^x-1=y$

$a^x=1+y$

Taking ln on both sides

$\ln(a^x)=\ln(1+y)$

$x\ln a=\ln(1+y) \quad \because \ln x^n=n\ln x$

$x=\frac{\ln(1+y)}{\ln a}$

$a^x-1=y$

If $x\to 0$, $a^0-1=y$

$1-1=y$

$y\to 0$

$=\lim\limits_{x\to 0}\frac{a^x-1}{x}$

$=\lim\limits_{y\to 0}\frac{y}{\frac{\ln(1+y)}{\ln a}}$

$=\lim\limits_{y\to 0}\frac{y\cdot\ln a}{\ln(1+y)}$

$=\lim\limits_{y\to 0}\frac{\frac{y\cdot\ln a}{y}}{\frac{1}{y}\cdot\ln(1+y)}$

$=\lim\limits_{y\to 0}\frac{\ln a}{\ln(1+y)^{1/y}} \quad \because \ln x^n=n\ln x$

$=\lim\limits_{y\to 0}\frac{\ln a}{\ln(1+y)^{1/y}} \quad \because \lim\limits_{x\to 0}(1+x)^{1/x}=e$

$=\frac{\ln a}{\ln.\lim\limits_{y\to 0}(1+y)^{1/y}}$

$=\frac{\ln a}{\ln . e}$

$=\frac{\ln a}{1}$

$=\log_e a$ =R.H.S

Prove that $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right) = \log_e e = 1$

$$L.H.S = \lim_{x\to 0}\left(\frac{e^x-1}{x}\right) \to *$$

Put $e^x - 1 = y$

$$e^x = 1+y$$

Taking ln on both sides

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$$\ln e^x = \ln(1+y)$$

$$x \ln e = \ln(1+y) \quad \because \ln x^n = n\ln x$$

$$x = \ln(1+y) \quad \because \ln e = 1$$

$$e^x - 1 = y$$

If $x\to 0$, $e^0 - 1 = y$

$$1-1 = y$$

$$y \to 0$$

$$= \lim_{x\to 0}\left(\frac{e^x-1}{x}\right)$$

$$= \lim_{y\to 0}\left(\frac{y}{\ln(1+y)}\right)$$

$$= \lim_{y\to 0}\left(\frac{y/y}{\frac{1}{y}\ln(1+y)}\right)$$

$$= \lim_{y\to 0}\left(\frac{1}{\frac{1}{y}\ln(1+y)}\right)$$

$$= \lim_{y\to 0}\left(\frac{1}{\ln(1+y)^{1/y}}\right) \quad \because \ln x^n = n\ln x$$

$$= \frac{1}{\ln.\lim_{y\to 0}(1+y)^{1/y}}$$

$$= \frac{1}{\ln e}$$

$$= \frac{1}{1} = 1$$

Topic:- Continuity And Discontinuity of Function:

(1) Graphical Way to Check Continuity:-

Sketch The graph $y = x+3$

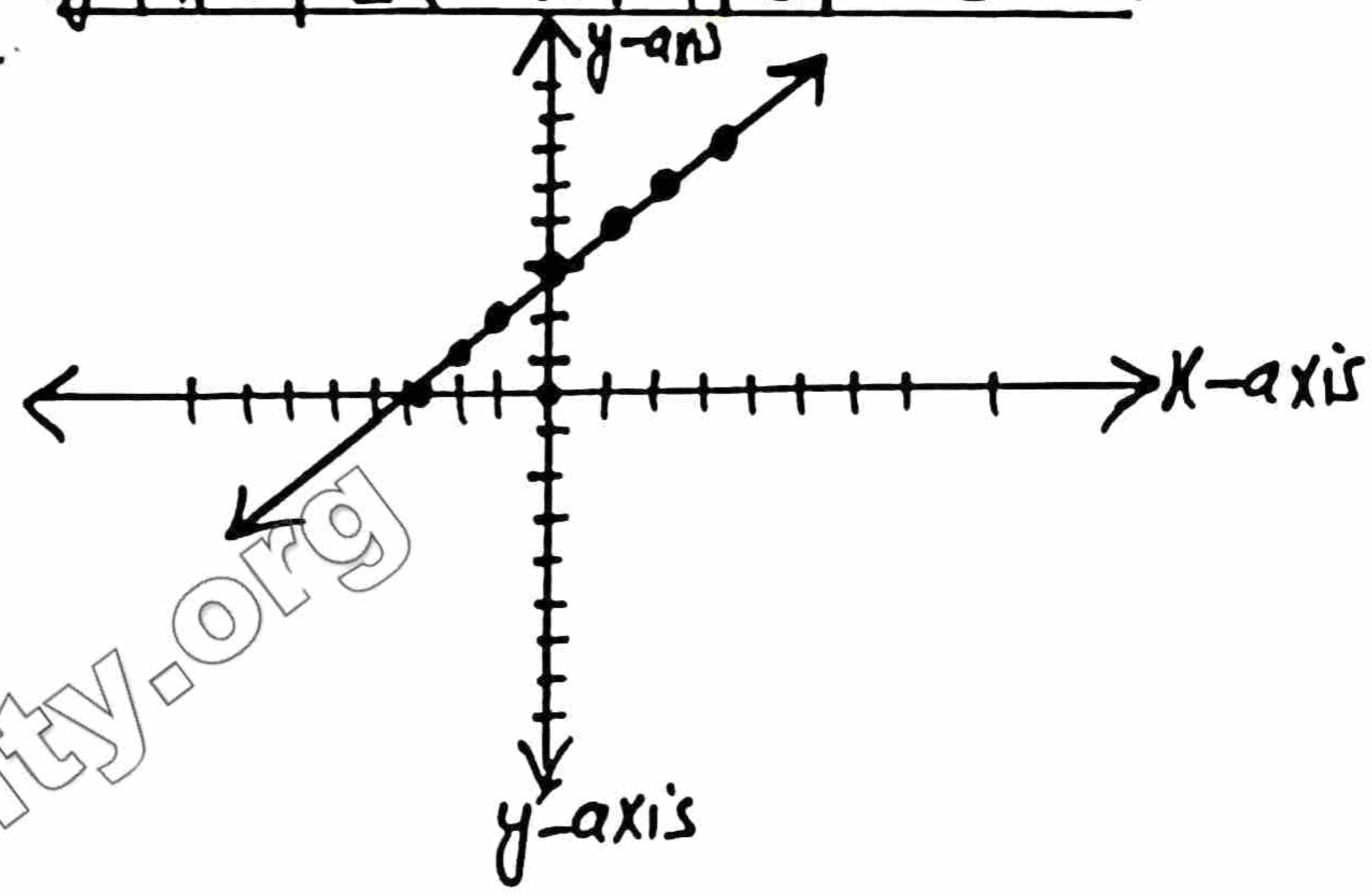
(i) If there is No break in the Graph of function Then function is Continuous.

(ii) A function Graph will be Continuous if has No Sharp edge.

(iii) All Trigonometric function are Continous, because They give Smooth Curves.

$y = x+3$

x	-3	-2	-1	0	1	2	3
$y = f(x)$	0	1	2	3	4	5	6

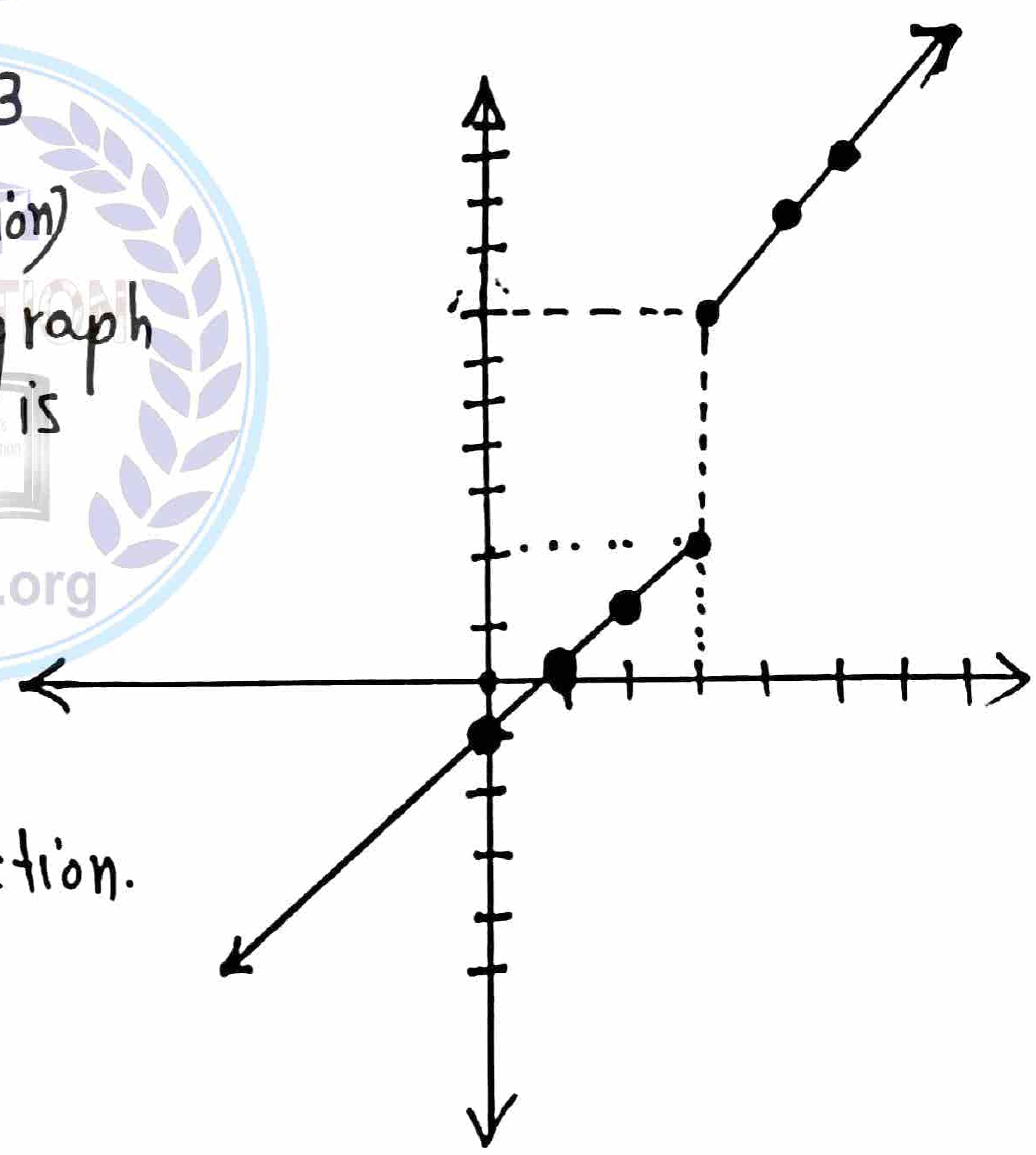


(ii) $f(x) = \begin{cases} x-1 & , \text{if } x<3 \\ 2x+1 & , \text{if } x \geq 3 \end{cases}$

(Piecewise function)

(i) If There is break in the graph of function, Then function is Discontinuous.

(ii) A function which is define in Two Rules is Called a Piecewise function.



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Algebric Way To Check Continuity:-

Continuous Function:-

A function is said to be Continuous at a Number "c" if and only if following Three Conditions are Satisfied.

(i) $f(c)$ is defined (ii) $\lim_{x \to c} f(x)$ Exists (iii) $\lim_{x \to c} f(x) = f(c)$

Discontinuous Function:-

If one or more of these three conditions fail to hold at "c", Then the function is said, to be Discontinuous at c.

The Left Hand Limit:-

$\lim_{x \to c^-} f(x) = L$ is read as the Limit of $f(x)$ equal to L as x approaches c from the Left. i-e for all x sufficiently close to c, but less than c.

The Right Hand Limit:-

$\lim_{x \to c^+} f(x) = M$ is read as the limit of $f(x)$ is equal to M as x approaches from the Right i-e for all x sufficiently close to c, but greater then c.

Criterion For Existence of Limit:-

$\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$ are equal

i-e

$\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$

The Sandwith Theorem:-

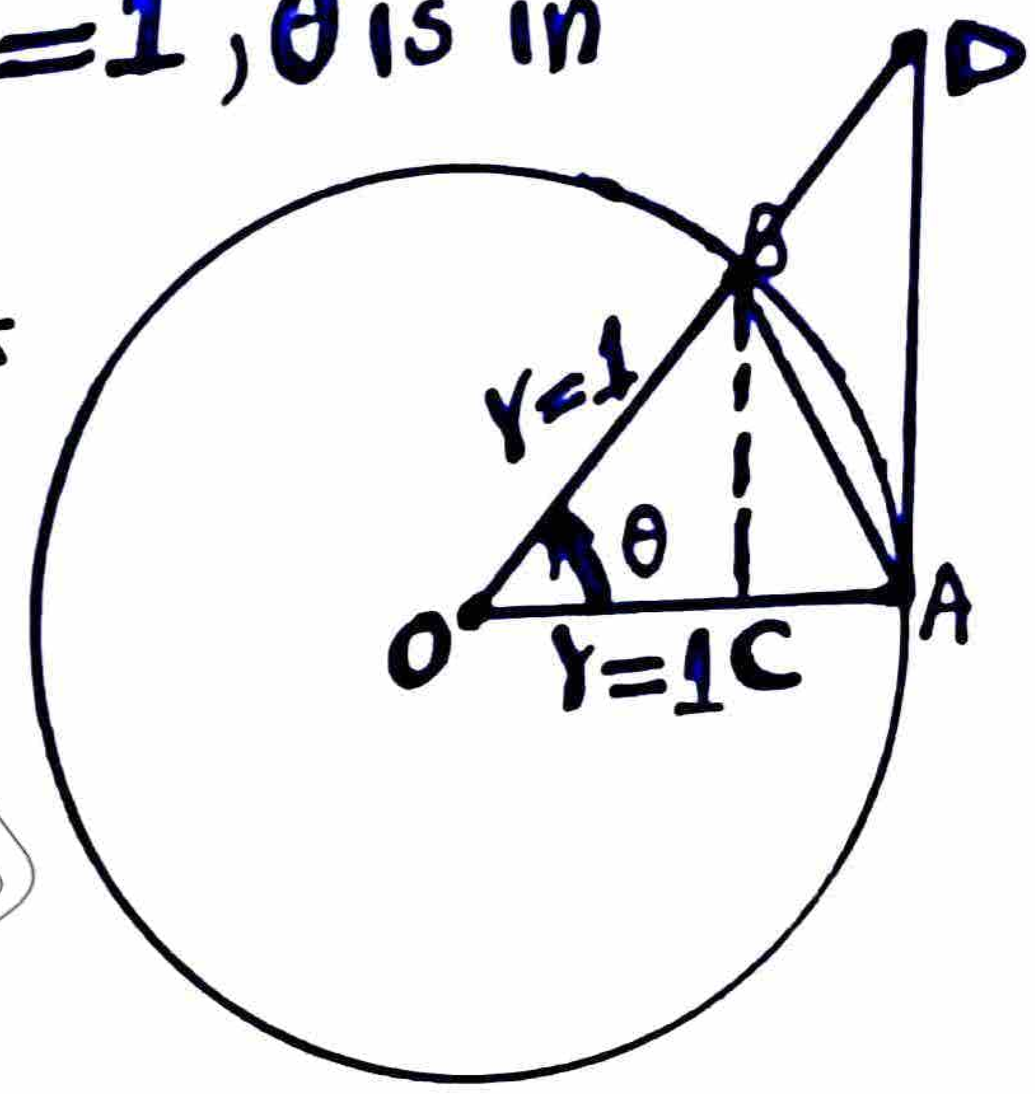
Let f, g and h be functions such that $f(x) \le g(x) \le h(x)$ for all numbers x in some open Interval Containing "c" except possibly c at itself.

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$

Then $\boxed{\lim_{x\to c} g(x) = L}$

Prove that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$, θ is in Radian.

Proof:-



Consider a Unit Circle. Take Two points A and B on the Circle Such that

$m\angle AOB = \theta$

Also $|OA| = |OB| = 1$

Join B with A. Produce OB upto D Such that $DA \perp OA$.

Draw $\perp$ from B on OA as BC.

Then Area of Triangle OAB < Area of Sector OAB < Area of Δ OAD $\longrightarrow (1)$

Area of Triangle $OAB = \frac{1}{2}(\text{base})(\text{height})$

$= \frac{1}{2}(OA)(BC)$

$= \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta$

Area of Sector $OAB = \frac{1}{2}r^2\theta = \frac{1}{2}\theta$

Area of Triangle $OAD = \frac{1}{2}(\text{base})(\text{height})$

$= \frac{1}{2}(OA)(AD)$

$= \frac{1}{2}(1)(AD) = \frac{1}{2}\tan\theta$

In Right ΔOCB

$\sin\theta = \frac{BC}{OB}$

$\sin\theta = \frac{BC}{1}$

$\boxed{\sin\theta = BC}$

In Right ΔOAD

$\tan\theta = \frac{AD}{OA}$

$\tan\theta = \frac{AD}{1}$

$\boxed{\tan\theta = AD}$

From (1) $\Rightarrow \frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$

Multiplying by 2

$\sin\theta < \theta < \frac{\sin\theta}{\cos\theta}$

Dividing both Sides by $\sin\theta$

$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$

Taking Reciprocal

$1 > \frac{\sin\theta}{\theta} > \frac{\cos\theta}{1}$

By Taking $\lim_{\theta\to 0}$

$\lim_{\theta\to 1} 1 > \lim_{\theta\to 0} \frac{\sin\theta}{\theta} > \lim_{\theta\to 0} \frac{\cos\theta}{1}$

$1 > \lim_{\theta\to 0} \frac{\sin\theta}{\theta} > 1$

Using Sandwitch Theorem

$\boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1}$

Chapter: 1

Functions and Limits

Short Questions

(i) Express $\lim_{x \to 0}\left(\frac{x}{1+x}\right)^{x}$ in terms of e. (LHR-2011)

Solution:

$$= \text{Limit}_{x \to 0}\left(\frac{1+x}{x}\right)^{-x}$$

$$= \lim_{x \to 0}\left(\frac{1}{x}+\frac{x}{x}\right)^{-x}$$

$$= \lim_{x \to 0}\left(\frac{1}{x}+1\right)^{-x}$$

$$= \lim_{x \to 0}\left(1+\frac{1}{x}\right)^{-x}$$

$$= \left[\lim_{x \to 0}\left(1+\frac{1}{x}\right)^{x}\right]^{-1}$$

$$= e^{-1}$$

Prove that identity:

$$\cosh^2 x + \sinh^2 x = \cosh 2x$$ (LHR-2011,16,18)

Solution:

$$\cosh^2 x + \sinh^2 x = \cosh 2x$$

We know that.

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{— I}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{— II}$$

Squaring equation I and II:

$$\sinh^2 x = \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$\sinh^2 x = \frac{e^{2x} + e^{-2x} - 2(e^x)(e^{-x})}{4}$$

$$\sinh^2 x = \frac{e^{2x} + e^{-2x} - 2}{4}$$

$$\cosh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2$$

$$\cosh^2 x = \frac{e^{2x} + e^{-2x} + 2(e^{2x})(e^{-2x})}{4}$$

$$\cosh^2 x = \frac{e^{2x} + e^{-2x} + 2}{4}$$

L.H.S = $\cosh^2 x + \sinh^2 x$

$$= \frac{e^{2x} + e^{-2x} + 2}{4} + \frac{e^{2x} + e^{-2x} - 2}{4}$$

$$= \frac{e^{2x} + e^{-2x} + 2 + e^{2x} + e^{-2x} - 2}{4}$$

$$= \frac{2e^{2x}+2e^{-2x}}{4}$$

$$= \frac{2(e^{2x}+e^{-2x})}{4}$$

$$= \frac{e^{2x}+e^{-2x}}{2}$$

$$= \cosh 2x$$

$$= R.H.S$$

$$\therefore \cosh^2 x + \sinh^2 x = \cosh 2x.$$

Q3

Evaluate $\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}}$. (LHR-2011) & (LHR-2021)

Solution:

$$\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}}$$

$$= \lim_{x\to 3} \frac{(\sqrt{x}+\sqrt{3})(\sqrt{x}-\sqrt{3})}{\sqrt{x}-\sqrt{3}}$$

$$= \lim_{x\to 3} (\sqrt{x}+\sqrt{3})$$

By applying limit:

$$= \sqrt{3}+\sqrt{3}$$

$$= 2\sqrt{3}$$

- **Alternate Solution**

$$\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}}$$

$$= \lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} \times \frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}}$$

$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x}+\sqrt{3})}{(x-3)}$$

$$= \lim_{x\to 3} (\sqrt{x}+\sqrt{3})$$

By applying limit:

$$= \sqrt{3}+\sqrt{3}$$

$$= 2\sqrt{3}$$

Define the terms: (LHR-2012)

• Even function:

"A function f is said to be an **even** if $f(-x) = f(x)$, for every number x in the domain of f."

Examples:

- $f(x) = x^2$
- $f(x) = \cos x$ are even functions of x.

Here, $f(-x) = (-x)^2 \Rightarrow x^2 = f(x)$

$f(-x) = \cos(-x) = \cos x = f(x)$ $\because \cos(-\theta) = \cos\theta$

• Odd function:

"A function f is said to be an **odd** if $f(-x) = -f(x)$, for every number x in the domain of f."

Examples:

- $f(x) = x^3$
- $f(x) = \sin x$ are odd functions of x.

Here, $f(-x) = (-x)^3 = -x^3 = -f(x)$

$f(-x) = \sin(-x) = -\sin x = -f(x)$.

15

Evaluate $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$. (LHR-2012)

Solution:

$$\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})}$$

By applying limit:

$$= \frac{1}{\sqrt{x+0}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

16

Prove that $\lim_{\theta\to 0} \frac{\tan\theta-\sin\theta}{\sin^3\theta} = \frac{1}{2}$ (LHR-2012)

Solution:

$$\lim_{\theta\to 0} \frac{\tan\theta-\sin\theta}{\sin^3\theta} = \frac{1}{2}$$

$$L.H.S = \lim_{\theta\to 0} \frac{\tan\theta-\sin\theta}{\sin^3\theta}$$

$$= \lim_{\theta\to 0} \frac{\frac{\sin\theta}{\cos\theta}-\sin\theta}{\sin^3\theta} \qquad \because \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \lim_{\theta\to 0} \frac{\frac{\sin\theta-\sin\theta\cos\theta}{\cos\theta}}{\sin^3\theta}$$

$$= \lim_{\theta\to 0} \frac{\sin\theta(1-\cos\theta)}{\cos\theta\cdot\sin^3\theta}$$

$$=\lim_{\theta\to 0}\frac{(1-\cos\theta)}{\cos\theta\cdot\sin^2\theta}$$

$$=\lim_{\theta\to 0}\frac{(1-\cos\theta)}{\cos\theta\cdot(1-\cos^2\theta)} \qquad \because \sin^2\theta+\cos^2\theta=1$$

$$=\lim_{\theta\to 0}\frac{(1-\cos\theta)}{\cos\theta\cdot[(1-\cos\theta)(1+\cos\theta)]}$$

$$=\lim_{\theta\to 0}\frac{1}{\cos\theta\,(1+\cos\theta)}$$

By applying limit:

$$=\frac{1}{\cos 0\,(1+\cos 0)}$$

$$=\frac{1}{1\cdot(1+1)}$$

$$=\frac{1}{2}$$

$$=R.H.S$$

$$\therefore \lim_{\theta\to 0}\frac{\tan\theta-\sin\theta}{\sin^3\theta}=\frac{1}{2}$$

17

If $f(x)=\sqrt{x+4}$ then find $f(x^2+4)$. (LHR 2013)

Solution:

$$f(x)=\sqrt{x+4} \quad \text{—— I}$$

$$f(x^2+4)=?$$

Replace x by (x^2+4) in equation I:

$$f(x^2+4)=\sqrt{x^2+4+4}$$

$$f(x^2+4=\sqrt{x^2+8}$$

8

Evaluate $\lim_{x\to\pi}\frac{\sin x}{\pi-x}$. (LHR-2013,2016)

Solution:

$$\lim_{x\to\pi}\frac{\sin x}{\pi-x}$$

$$=\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi-x} \qquad \because \sin(\pi-x)=\sin x$$

Put $\pi - x = \theta$

If $x \to \pi$ then $\pi - \pi = \theta$

and $\theta \to 0$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta}$$

$$= 1$$

9

Find $\lim_{x \to 0} \frac{\sin x^\circ}{x}$ (LHR-2013,2015)

Solution:

$$\lim_{x \to 0} \frac{\sin x^\circ}{x}$$

$$= \lim_{x \to 0} \frac{\sin x \times 1^\circ}{x}$$

$$= \lim_{x \to 0} \frac{\sin \frac{x\pi}{180}}{x}$$

• Multiplying and dividing by $\left(\frac{\pi}{180}\right)$:

$$= \frac{\pi}{180}\left(\lim_{x \to 0} \frac{\sin \frac{x\pi}{180}}{x\frac{\pi}{180}}\right)$$

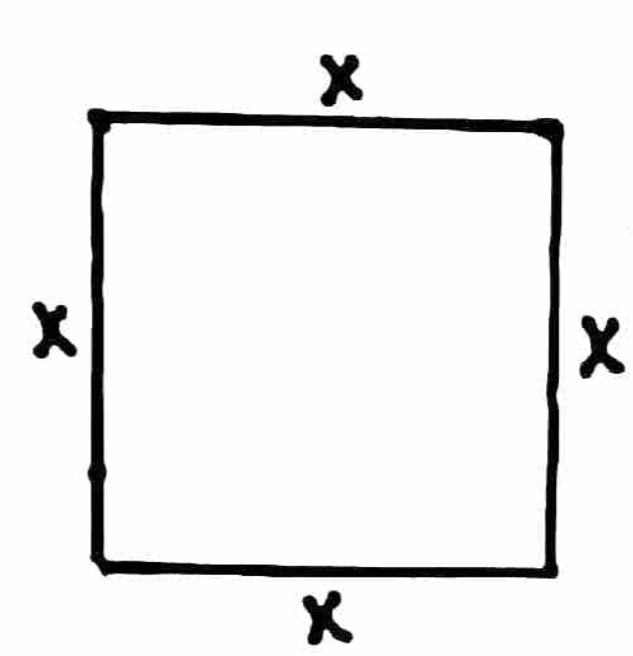
$$= \frac{\pi}{180}(1)$$

$$= \frac{\pi}{180}$$

10

The perimeter of square as a function of its area A, express. (LHR-2014,2017)

Solution: Let x be the length and width of square.



$$\text{Perimeter} = P = x + x + x + x$$

$$P = 4x \quad \text{—— I}$$

$$\text{Area} = A = (x)(x)$$

$$A = x^2$$

$$x = \sqrt{A} \quad \text{—— II}$$

Put equation II in I:

$$\boxed{P = 4\sqrt{A}}$$

11

Evaluate $\lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^n$. (LHR-2014)

Solution:

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$$

$$=\left[\lim_{n\to\infty}\left(1+\frac{1}{(-n)}\right)^{-n}\right]^{-1}$$

$$= e^{-1}$$

12

Show that $f(x) = x\sqrt{x^2+5}$ is odd function (LHR-2014) & (2019)

Solution: $f(x) = x\sqrt{x^2+5}$

Replace x by $(-x)$:

$$f(-x) = -x\sqrt{(-x)^2+5}$$

$$f(-x) = -x\sqrt{x^2+5}$$

$$f(-x) = -f(x)$$

Hence $f(x)$ is an odd function.

13

Show that $f(x) = x^{\frac{2}{3}} + 6$ is function (LHR-2015)

Solution: $f(x) = x^{\frac{2}{3}} + 6$

Replace x by $(-x)$:

$$f(-x) = (-x)^{\frac{2}{3}} + 6$$

$$f(-x) = [(-x)^2]^{\frac{1}{3}} + 6$$

$$f(-x) = (x^2)^{1/3} + 6$$

$$f(-x) = x^{\frac{2}{3}} + 6$$

$$f(-x) = f(x)$$

So, $f(x)$ is an even function.

14

Evaluate $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$. (LHR-2015, 2017)

Solution:

$$\lim_{x\to 0} \frac{\sin ax}{\sin bx}$$

Multiplying and Dividing numerator by(ax) and Denominator by (bx):

$$= \frac{\lim_{x\to 0} \sin \frac{ax}{ax} \cdot ax}{\lim_{x\to 0} \sin \frac{bx}{bx} \cdot bx}$$

$$= \frac{ax}{bx} \cdot \left(\frac{\lim_{x\to 0} \frac{\sin ax}{ax}}{\lim_{x\to 0} \frac{\sin bx}{bx}} \right)$$

$$= \frac{a}{b} \cdot \left(\frac{1}{1}\right)$$

$$= \frac{a}{b}$$

15

If $f(x) = \sin x$ then find $\frac{f(a+h) - f(a)}{h}$. (LHR-2015)

Solution:

$$f(a+h) = \sin(a+h)$$
$$f(a) = \sin a$$

$$\frac{f(a+h) - f(a)}{h} = \frac{\sin(a+h) - \sin a}{h}$$

$$= \frac{2\cos\left(\frac{a+h+a}{2}\right)\sin\left(\frac{a+h-a}{2}\right)}{h} \quad \because \sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}.$$

$$= \frac{2\cos\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \frac{2}{h}\cos\left(a+\frac{h}{2}\right)\sin\frac{h}{2}$$

16

Define continuity. (LHR-2015)

A function f is said to be **continuous** at a number "c" if and only if the following conditions are satisfied:

- $f(c)$ is defined.
- $\lim_{x \to c} f(x)$ exists
- $\lim_{x \to c} f(x) = f(c)$.

17

Find $f \circ f(x)$ if $f(x) = \sqrt{x+1}$. (LHR-2016)

Solution:

$$f(x) = \sqrt{x+1}$$

$$f \circ f(x) = ?$$

$$f \circ f(x) = f(f(x))$$

$$= f(\sqrt{x+1})$$

Replace x by $(\sqrt{x+1})$ in $f(x)$:

$$= \sqrt{\sqrt{x+1} + 1}$$

18

Evaluate $\lim_{\theta \to 0} \frac{\sin^2\theta}{\theta}$. (LHR-2016)

Solution:

$$\lim_{\theta \to 0} \frac{\sin^2\theta}{\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta^2} . \theta$$

$$= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta^2} . \lim_{\theta \to 0} \theta$$

$$= \left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\right)^2 . \lim_{\theta \to 0} \theta$$

$= (1)^2 \cdot (0)$

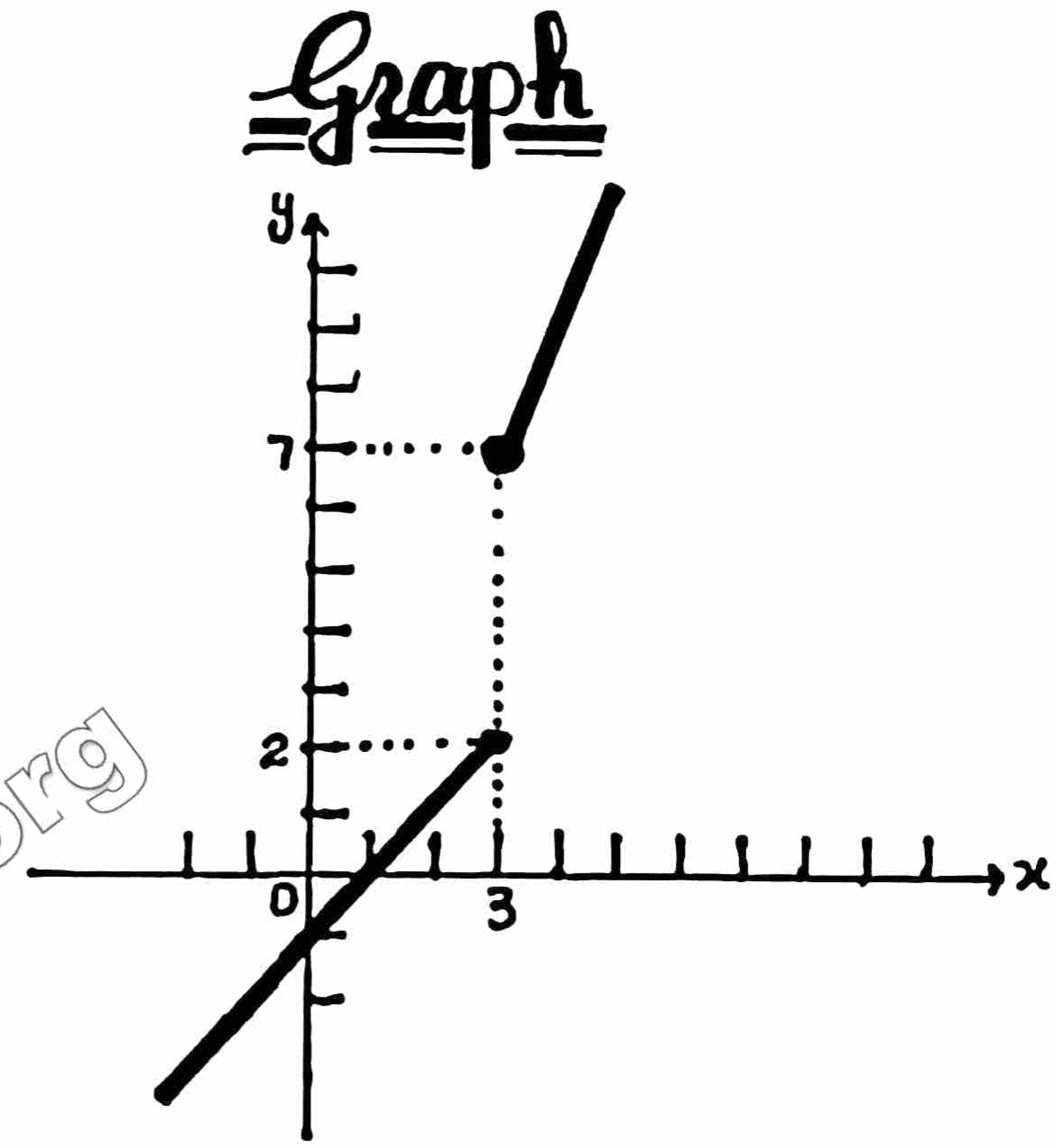
$= 0$

18

Discuss continuity of $f(x)$ at 3 when

$$f(x) = \begin{cases} x-1 & \text{if } x<3 \\ 2x+1 & \text{if } x \geq 3 \end{cases}$$ (LHR-2016)

Solution:



$$f(3) = 2(3)+1 = 7$$

$\Rightarrow$ Condition (i) satisfied.

$$L.H.L = \lim_{x\to3^-} f(x) = \lim_{x\to3^-} (x-1)$$
$$= 3-1$$
$$= 2$$

$$R.H.L = \lim_{x\to3^+} f(x)$$
$$= \lim_{x\to3^+} (2x+1)$$
$$= 2(3)+1$$
$$= 7$$

$\because \lim_{x\to3^-} f(x) \neq \lim_{x\to3^+} f(x)$

Condition (ii) is not satisfied.

$\therefore \lim_{x\to3} f(x)$ does not exist.

Hence $f(x)$ is not continuous at $x=3$.

19

Discuss continuity of $f(x)$ at 3 when

$$f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$ (LHR-2016)

Solution:

$$f(3) = 6$$

$\therefore$ the function (i) is defined at $x=3$.

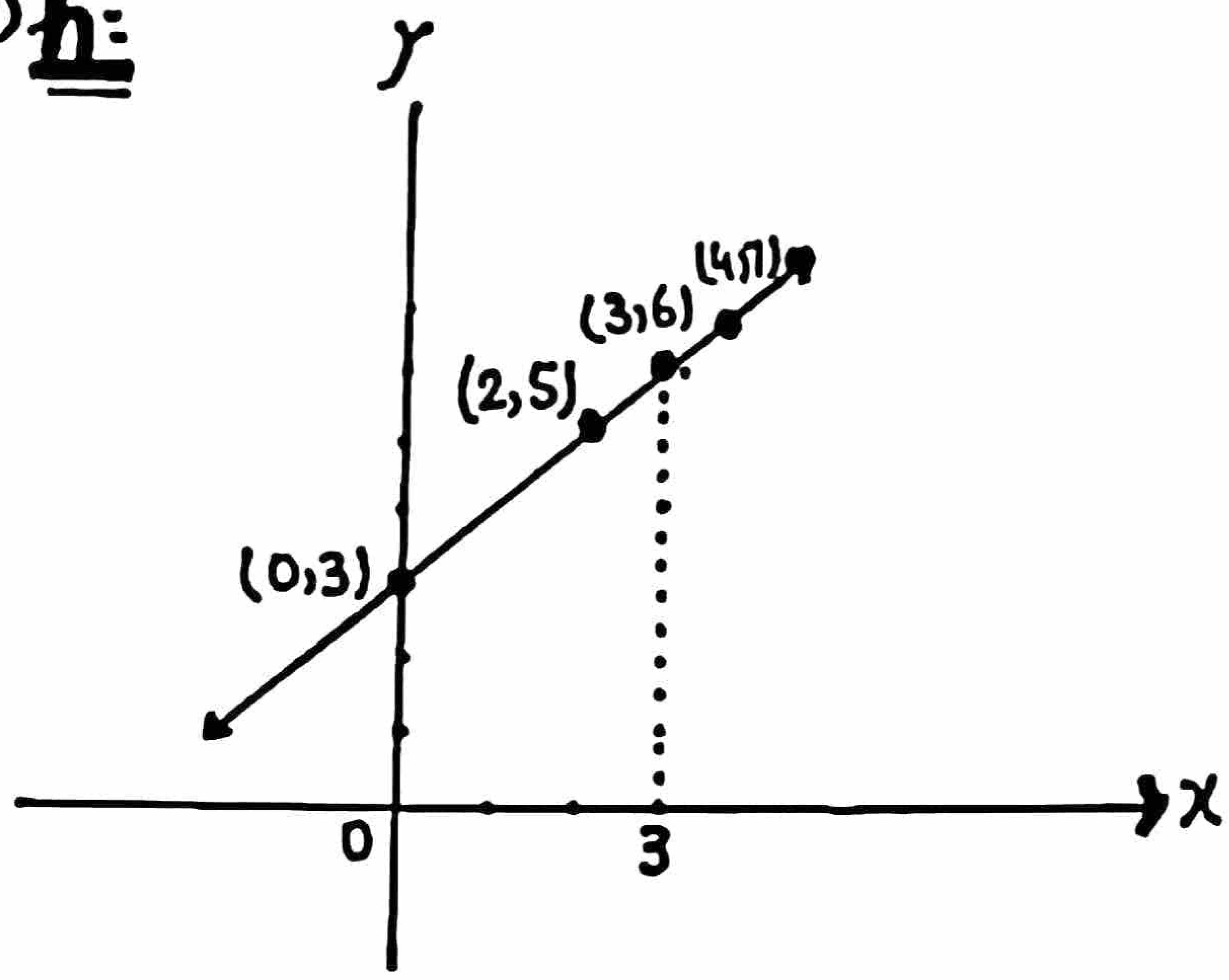
$$\lim_{x\to3} f(x) = \lim_{x\to3} \frac{x^2-9}{x-3}$$
$$= \lim_{x\to3} \frac{(x-3)(x+3)}{(x-3)}$$
$$= \lim_{x\to3} (x+3)$$

$=6$

$$\lim_{x\to 3} f(x) = f(3) = 6$$

$\therefore f(x)$ is continuous at $x=3$.

Graph:



2018

Show that $x=at^2$, $y=2at$ are parametric equations of parabola $y^2=4ax$ (LHR-2017)

Solution:

$x=at^2$ —— I

$y=2at$ —— II

From II $\Rightarrow$ $t=\dfrac{y}{2a}$

Putting value of "t" in I:

$$x=a\left(\frac{y}{2a}\right)^2$$

$$x=a\cdot\frac{y^2}{4a^2}$$

$$x=\frac{y^2}{4a}$$

$$\Rightarrow \quad y^2=4ax$$

Hence, $x=at^2$, $y=2at$ represent the equation of parabola.

121

If $f(x)=\sqrt{x+1}$ and $g(x)=\frac{1}{x^2}$. Find $g\circ f(x)$. (LHR-2017).

Solution:

$$f(x)=\sqrt{x+1}$$

$$g\circ f(x)=?$$

$$g\circ f(x)=g(f(x))$$

$$=g(\sqrt{x+1})$$

Replace x by $(\sqrt{x+1})$ in $g(x)$:

$$=\frac{1}{(\sqrt{x+1})^2}$$

$$=\frac{1}{x+1}$$

122

If $f(x)=(-x+9)^3$, find $f^{-1}(x)$. (LHR-2017)

Solution:

$$f(x)=(-x+9)^3$$

$$y=(-x+9)^3 \qquad \because f(x)=y$$

$$\Rightarrow \quad y^{\frac{1}{3}}=-x+9$$

$$x=9-y^{\frac{1}{3}}$$

$$f^{-1}(y)=9-y^{\frac{1}{3}} \qquad \because f^{-1}(y)=x$$

To find $f^{-1}(x)$, replace y by (x):

$$f^{-1}(x)=9-x^{\frac{1}{3}}$$

Evaluate $\lim_{x\to 0} \frac{\sec x-\cos x}{x}$. (LHR-2018)

Solution:

$$\lim_{x\to 0} \frac{\sec x-\cos x}{x}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\cos x} - \cos x}{x}$$

$$= \lim_{x\to 0} \frac{\frac{1-\cos^2 x}{\cos x}}{x}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x \cdot \cos x} \qquad \because \sin^2 x+\cos^2 x=1$$

Dividing Multiplying by (x):

$$= \lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \frac{x}{\cos x}$$

$$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 \cdot \lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{1}{\cos x}$$

$$= (1)^2 \cdot (0) \cdot \frac{1}{\cos(0)}$$

$$= (1)(0) \cdot \left(\frac{1}{1}\right)$$

$$= 0$$

Evaluate $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}$. (LHR-2017) & (LHR-2021)

Solution:

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}$$

$$= \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta}$$

$$= \lim_{\theta\to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)}$$

$$= \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)} \qquad \because \sin^2\theta+\cos^2\theta=1$$

$$= \lim_{\theta\to 0} \sin\theta \cdot \frac{\sin\theta}{\theta} \cdot \frac{1}{1+\cos\theta}$$

$$= \lim_{\theta\to 0} \sin\theta \cdot \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta\to 0} \frac{1}{1+\cos\theta}$$

$$=(0)(1)\cdot\frac{1}{1+1}$$
$$=0$$

25

Determine whether function $f(x)=\frac{x^3-x}{x^2+1}$ is even or odd. (LHR-2018)

Solution:

$$f(x)=\frac{x^3-x}{x^2+1}$$

Replace x by $(-x)$:

$$f(-x)=\frac{(-x)^3-(-x)}{(-x)^2+1}$$

$$f(-x)=\frac{-x^3+x}{x^2+1}$$

$$f(-x)=\frac{-(x^3-x)}{x^2+1}$$

$$f(-x)=-f(x)$$

Hence, $f(x)$ is an odd function.

26

State sandwitch theorem. (LHR-2018)

Let f, g and h be functions such that $f(x)\le g(x)\le h(x)$ for all numbers x in some open interval containing "c", except possible at c itself.

If:

$$\lim_{x\to c} f(x)=L \text{ and } \lim_{x\to c} h(x)=L, \text{ then}$$

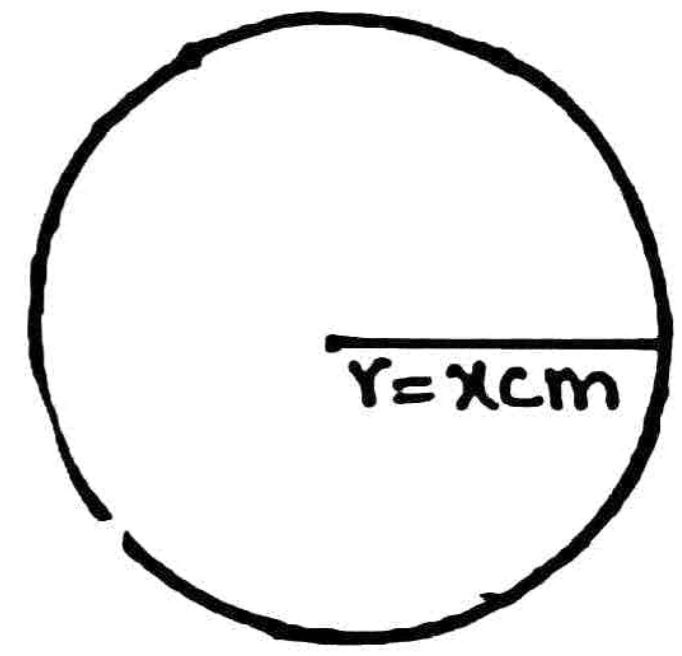
$$\lim_{x\to c} g(x)=L.$$

27

Express the area "A" of a circle as a function of its circumfnference "C". (LHR-2018,2021)

Solution:

Let x be the radius of the circle.



Area of circle $= A = \pi r^2$

$$A = \pi x^2 \text{ --- I} \qquad \because r = x\,cm$$

Circumference of circle $= C = 2\pi r$

$$C = 2\pi x \qquad \because r = x\,cm$$

$$x = \frac{C}{2\pi} \text{ --- II}$$

Put equation II in I:

$$A = \pi\left(\frac{C}{2\pi}\right)^2$$

$$A = \pi \cdot \frac{C^2}{4\pi^2}$$

$$\boxed{A = \frac{C^2}{4\pi}}$$

28

If $f(x) = \begin{cases} x+2, & x \leq -1 \\ c+2, & x > -1 \end{cases}$, find "c" so that $\lim\limits_{x \to -1} f(x)$ exists. (LHR-2018)

Solution:

$$L.H.L = \lim_{x \to -1^-} f(x)$$

$$= \lim_{x \to -1^-} (x+2)$$

$$= -1+2$$

$$= 1$$

$$R.H.L = \lim_{x \to -1^+} f(x)$$

$$= \lim_{x \to -1^+} (c+2)$$

$$= c+2$$

Given $\lim_{x \to -1} f(x)$ exists.

$$L.H.L = R.H.L$$

$$1 = c+2$$

$$\boxed{\Rightarrow \quad c = -1}$$

130

Find the domain and range of the function $g(x) = \sqrt{x+1}$. (LHR-2011)

Solution:

$$g(x) = \sqrt{x+1}$$

For real values of x:

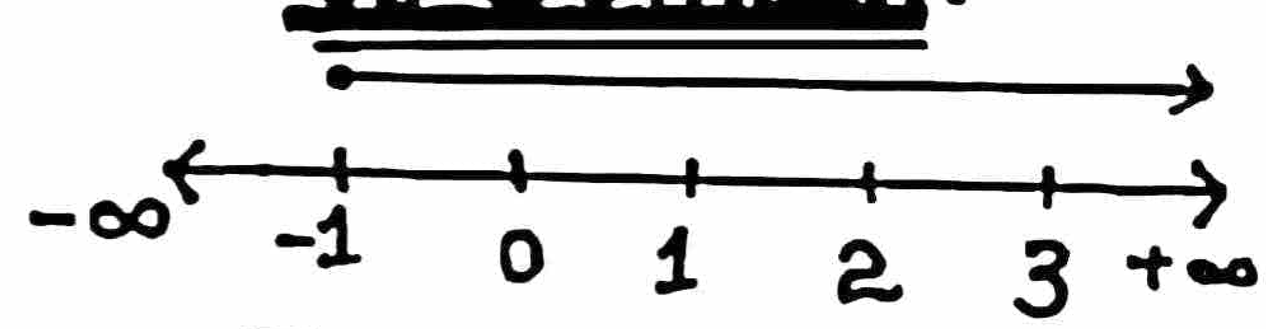
$$x+1 \geq 0$$

$$x \geq -1$$

Domain of $g = [-1, +\infty)$

Range of $g = [0, +\infty)$

For Domain:



For Range:

$$g(x) = \sqrt{x+1}$$

Put $x = -1$

$$g(x) = \sqrt{-1+1} = 0$$

131

Without finding the inverse, state domain and range of $f^{-1}(x)$ if $f(x) = (x-5)^2$, $x \geq 5$ (LHR-2013)

Solution:

$$f(x) = (x-5)^2 \quad , x \geq 5$$

Domain of $f = [5, +\infty)$

For Range of f:

Put $x=5$ in $f(x)$

$f(5) = (5-5)^2 = 0$

Range of f: $[0, +\infty)$

By definition of inverse function:

Domain of $f^{-1} = [0, +\infty)$

Range of $f^{-1} = [5, +\infty)$

321

Find the domain and range of $f^{-1}(x)$ where $f(x) = 2+\sqrt{x-1}$. (LHR-2015)

Solution:

$f(x) = 2+\sqrt{x-1}$

For domain of f:

$x-1 \geq 0$

$x \geq 1$

Domain of $f = [1, +\infty)$

For Range of f:

$f(x) = 2+\sqrt{x-1}$

At $x=1$

$f(x) = 2+\sqrt{1-1}$

$f(x) = 2$

Range of f: $[2, +\infty)$

Now,

By definition of inverse function:

Domain of $f^{-1} = [2, +\infty)$

Range of $f^{-1} = [1, \infty)$

133

Define an <u>even function</u> and give one example. (LHR-2012)

"A function is said to be an **even** if $f(-x) = f(x)$ for every number x in the domain of f"

<u>Examples:</u>

$f(x) = x^2$ and $f(x) = \cos x$ are even functions. Here,

- $f(-x) = (-x)^2 = x^2$
- $f(-x) = \cos(-x) = \cos x \quad \because \cos(-\theta) = \cos\theta$

134

Compute $\lim\limits_{n \to +\infty} \left(1 + \frac{4}{n}\right)^n$. (LHR-2014)

<u>Solution:</u>

$$\lim_{n \to +\infty} \left(1 + \frac{4}{n}\right)^n$$

$$= \left[\lim_{n \to +\infty} \left(1 + \frac{4}{n}\right)^{\frac{n}{4}}\right]^4$$

$$= e^4$$

135

For the real-valued function $f(x) = \frac{2x+1}{2x-1}$, $x>1$. Find $f^{-1}(x)$. (LHR-2021)

Solution:

$$f(x) = \frac{2x+1}{2x-1}$$

$$y = \frac{2x+1}{2x-1} \qquad \because f(x)=y$$

By cross multiplication:

$$2xy - y = 2x+1$$

$$2xy - 2x = 1+y$$

$$2x(y-1) = y+1$$

$$\Rightarrow \quad x = \frac{y+1}{2(y-1)}$$

$$f^{-1}(y) = \frac{y+1}{2(y-1)} \quad \because x = f^{-1}(y)$$

To find $f^{-1}(x)$, Replace y by "x", we get:

$$f^{-1}(x) = \frac{x+1}{2(x-1)}$$

136

Find the domain and range of $g(x) = |x-3|$. (LHR-2021)

Solution:

$$g(x) = |x-3|$$

$$g(x) = \begin{cases} (x-3) & \text{if } x-3 \geq 0 \Rightarrow x \geq 3 \\ -(x-3) & \text{if } x-3 < 0 \Rightarrow x < 3 \end{cases}$$

Domain of $g = (-\infty, +3) \cup [3, +\infty)$

$= (-\infty, +\infty)$

For range of g:

Put $x = 3$ in $g(x)$:

$\Rightarrow \quad g(x) = |3-3| = 0$

And $g(x)$ is positive for all other values of x in Domain of g, so:

Range of $g = [0, \infty)$

(37)

Find the domain and range of the function g defined by : $g(x) = \sqrt{x^2-4}$ (LHR-2021)

Solution:

$g(x) = \sqrt{x^2-4}$

For Domain of g:

$x^2 - 4 \geq 0$

$x^2 \geq 4$

$\pm x \geq 2$

OR

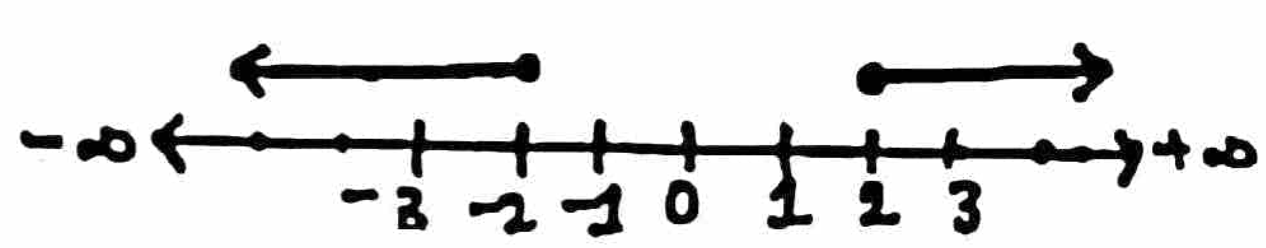
$x \geq 2$ or $x \leq -2$

Domain of $g = [2, \infty) \cup [-2, -\infty)$

Range of $g = [0, \infty)$

$= R^+$

- **For Domain:**



- **For Range:**

$g(x) = \sqrt{x^2-4}$

Put $x = 2$

$g(x) = \sqrt{4-4} = 0$

Put $x = 3$

$g(x) = \sqrt{9-4} = \sqrt{5}$

Put $x = -2$

$g(x) = \sqrt{4-4} = 0$

Put $x = -3$

$g(x) = \sqrt{9-4} = \sqrt{5}$

⋮

The real valued functions f and g are given. Find $f\circ g(x)$, if

$$f(x) = 3x^4 - 2x^2 \; ; \; g(x) = \frac{2}{\sqrt{x}}, \; x \neq 0 \quad \text{(LHR-2021)}$$

Solution:

$$f(x) = 3x^4 - 2x^2 \; ; \; g(x) = \frac{2}{\sqrt{x}}$$

$$f\circ g(x) = f(g(x))$$

$$= f\left(\frac{2}{\sqrt{x}}\right)$$

Replace x by $\left(\frac{2}{\sqrt{x}}\right)$ in $f(x)$:

$$= 3\left(\frac{2}{\sqrt{x}}\right)^4 - 2\left(\frac{2}{\sqrt{x}}\right)^2$$

$$= \frac{48}{x^2} - \frac{8}{x}$$

$$= \frac{48 - 8x}{x^2}$$

$$= \frac{8(6-x)}{x^2}$$

Evaluate $\lim\limits_{x\to 1} \dfrac{x^3 - 3x^2 + 3x - 1}{x^3 - x}$. (LHR-2021)

Solution:

$$\lim_{x\to 1} \frac{x^3 - 3x^2 + 3x - 1}{x^3 - x}$$

$$= \lim_{x\to 1} \frac{(x-1)^3}{x(x-1)} \qquad \because (x-y)^3 = x^3 - y^3 - 3xy(x-y)$$

$$= x^3 - y^3 - 3x^2y + 3xy^2$$

$$= \lim_{x\to 1} \frac{(x-1)^2}{x}$$

$$= \frac{(1-1)^2}{1}$$

$$= 0$$

Q 40

Define Explicit function. (LHR-2019)

If y is easily expressed in terms of the independent variable x, then y is called an **explicit function**.

Example: $y = 2x^2 + x - 1$ is an explicit function.

Q 41

Prove that $\lim\limits_{x \to 0} \dfrac{\sqrt{x+a} - \sqrt{a}}{x} = \dfrac{1}{2\sqrt{a}}$ (LHR-2019)

Solution:

$$\lim_{x \to 0} \frac{\sqrt{x+a} - \sqrt{a}}{x}$$

$$= \lim_{x \to 0} \frac{\sqrt{x+a} - \sqrt{a}}{x} \times \frac{\sqrt{x+a} + \sqrt{a}}{\sqrt{x+a} + \sqrt{a}} \quad \textbf{(Rationalizing)}$$

$$= \lim_{x \to 0} \frac{x+a-a}{x(\sqrt{x+a} + \sqrt{a})}$$

$$= \lim_{x \to 0} \frac{x}{x(\sqrt{x+a} + \sqrt{a})}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{x+a} + \sqrt{a}}$$

$$= \frac{1}{\sqrt{0+a} + \sqrt{a}}$$

$$= \frac{1}{\sqrt{a} + \sqrt{a}}$$

$$= \frac{1}{2\sqrt{a}}$$

$$\therefore \lim_{x \to 0} \frac{\sqrt{x+a} - \sqrt{a}}{x} = \frac{1}{2\sqrt{a}}$$

Proved.

Long Questions

Q 1

If $f(x)=\begin{cases} 3x & \text{when } x\le -2 \\ x^2-1 & \text{when } -2<x<2 \\ 3 & \text{when } x\ge 2 \end{cases}$

Discuss continuity at $x=2$ and $x=-2$. (LHR 2011+2013)

Solution:

$$f(x)=\begin{cases} 3x & \text{when } x\le -2 \\ x^2-1 & \text{when } -2<x<2 \\ 3 & \text{when } x\ge 2 \end{cases}$$

at $x=2$

$f(2)=3$

$\therefore f(2)$ is defined.

$$L.H.L = \lim_{x\to 2^-} f(x)$$

$$= \lim_{x\to 2^-} (x^2-1)$$

$$= (2)^2-1$$

$$= 3$$

$$R.H.L = \lim_{x\to 2^+} f(x)$$

$$= \lim_{x\to 2^+} 3$$

$$= 3$$

$L.H.L = R.H.L$

So, $\lim_{x\to 2} f(x)$ exists.

$$\lim_{x\to 2} f(x) = f(2)$$

As condition (i), (ii) and (iii) are satisfy, so it is continuous at $x=2$.

at $x=-2$

$$f(-2) = 3x$$
$$= 3(-2)$$
$$= -6$$

$\because f(-2)$ is defined.

$$\text{L.H.L} = \lim_{x\to -2^-} f(x)$$
$$= \lim_{x\to -2^-} 3x$$
$$= 3(-2)$$
$$= -6$$

$$\text{R.H.L} = \lim_{x\to -2^+} f(x)$$
$$= \lim_{x\to -2^+} (x^2-1)$$
$$= (-2)^2 - 1$$
$$= 4-1$$
$$= 3$$

$$\text{L.H.L} \neq \text{R.H.L}$$

So, $\lim_{x\to -2} f(x)$ does not exist.

$$\therefore \lim_{x\to -2} f(x) \neq f(-2)$$

As condition (iii) is not satisfy so it is discontinuous at $x=-2$.

Discuss the continuity of $f(x)=\begin{cases}2x+5 & \text{if } x\leq 2\\ 4x+1 & \text{if } x>2\end{cases}$, at $x=2$. (LHR-2011)

Solution:

$$f(x)=\begin{cases}2x+5 & \text{if } x\leq 2\\ 4x+1 & \text{if } x>2\end{cases}$$

$$f(x) = 2x+5$$

$$f(2) = 2(2)+5$$

$$f(2) = 9$$

$\therefore f(x)$ is defined.

$$L.H.L = \lim_{x\to 2^-} f(x)$$

$$= \lim_{x\to 2^-}(2x+5)$$

$$= 2(2)+5$$

$$= 9$$

$$R.H.L = \lim_{x\to 2^+} f(x)$$

$$= \lim_{x\to 2^+}(4x+1)$$

$$= 4(2)+1$$

$$= 8+1$$

$$= 9$$

$\Rightarrow \lim_{x\to 2} f(x)$ exist.

$$\lim_{x\to 2} f(x) = f(2).$$

As condition (i), (ii) and (iii) are satisfy so it is continuous at $x=2$.

If $f(x)=\begin{cases}\dfrac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} & , x\neq 2\\ k & , x=2\end{cases}$

Find the value of k, so that f is continuous at x = 2. [(LHR-2014,2015,2018) & 2021]

Solution:

$$f(x)=\begin{cases}\dfrac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} & , x\neq 2\\ k & , x=2\end{cases}$$

Given f is continuous at x = 2.

$$\lim_{x\to 2} f(x) = f(2)$$

$$\lim_{x\to 2}\left(\frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}\right) = k$$

$$\lim_{x\to 2}\left(\frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}\times\frac{\sqrt{2x+5}+\sqrt{x+7}}{\sqrt{2x+5}+\sqrt{x+7}}\right) = k \quad \text{(Rationalize)}$$

$$\lim_{x\to 2}\frac{(\sqrt{2x+5})^2-(\sqrt{x+7})^2}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})} = k$$

$$\lim_{x\to 2}\frac{2x+5-x-7}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})} = k$$

$$\lim_{x\to 2}\frac{(x-2)}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})} = k$$

$$\lim_{x\to 2}\frac{1}{\sqrt{2x+5}+\sqrt{x+7}} = k$$

By applying limit:

$$\frac{1}{\sqrt{2(2)+5}+\sqrt{2+7}} = k$$

$$\frac{1}{\sqrt{9}+\sqrt{9}} = k$$

$$\frac{1}{3+3} = k$$

$$\Rightarrow \boxed{k = \frac{1}{6}}$$

B14B

Evaluate $\lim_{x\to 0} \frac{1-\cos px}{1-\cos qx}$. (LHR-2012,2016)

Solution:

$$\lim_{x\to 0} \frac{1-\cos px}{1-\cos qx}$$

$$= \lim_{x\to 0} \frac{2\sin^2 \frac{px}{2}}{2\sin^2 \frac{qx}{2}}$$

$\because \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$

$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$

$2\sin^2\frac{\theta}{2} = 1-\cos\theta$

$2\sin^2\frac{px}{2} = 1-\cos px$

$2\sin^2\frac{qx}{2} = 1-\cos qx$

$$= \left(\frac{\lim_{x\to 0} 2\sin\frac{px}{2}}{\lim_{x\to 0} 2\sin\frac{qx}{2}}\right)^2$$

Dividing and Multiplying Numerator by $\left(\frac{px}{2}\right)$ and Denominator by $\left(\frac{qx}{2}\right)$.

$$= \left(\frac{\lim_{x\to 0} \frac{2\sin\frac{px}{2}}{\frac{px}{2}} \times \frac{px}{2}}{\lim_{x\to 0} \frac{2\sin\frac{qx}{2}}{\frac{qx}{2}} \times \frac{qx}{2}}\right)^2$$

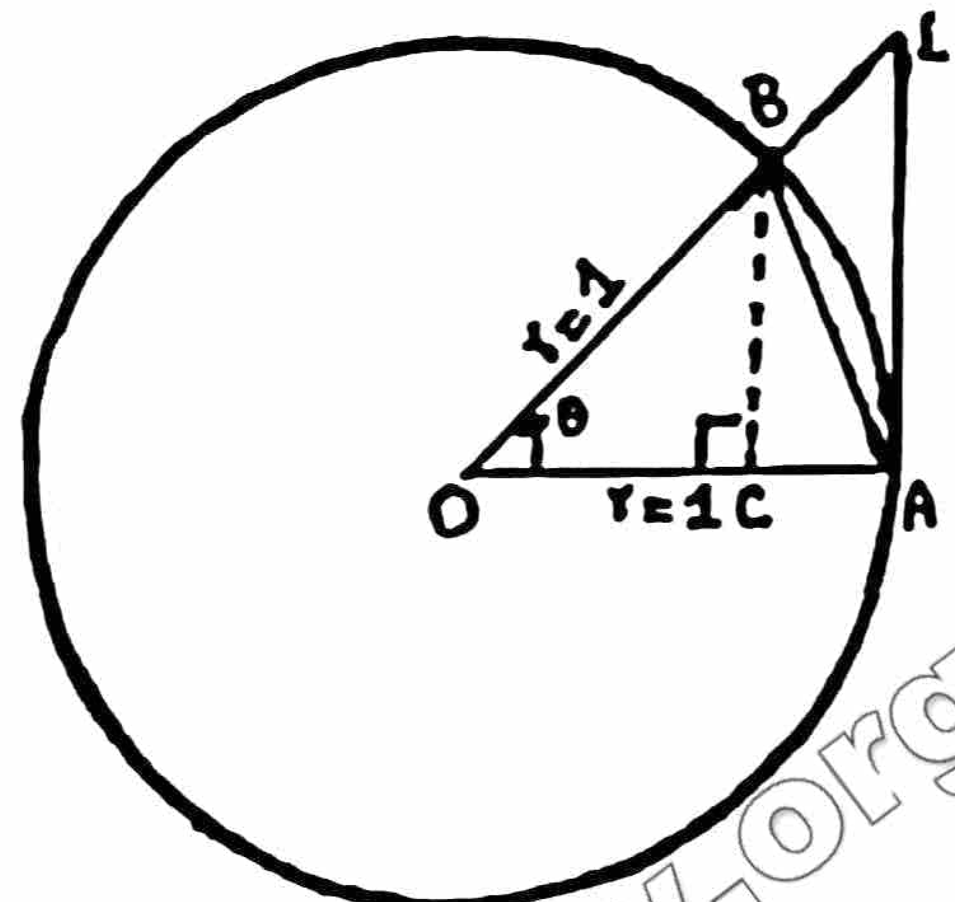
$$= \left(\frac{1}{1}\cdot\frac{p}{q}\right)^2$$

$$= \frac{p^2}{q^2}$$

Prove that $\lim_{x \to 0} \frac{\sin\theta}{\theta} = 1$. (LHR-2012)

Solution:

$$\lim_{x \to 0} \frac{\sin\theta}{\theta} = 1$$



Let:

OAB be the Area of sector with center "O" and radius r=1.

In $\Delta OAB = \frac{1}{2}(OA \times BC)$

In $\Delta OAB = \frac{1}{2}(1 \times \sin\theta)$

In $\Delta OAB = \frac{1}{2}\sin\theta$ —— I

$\because$ In right angle triangle OCB:

$\sin\theta = \frac{BC}{OB}$

$\sin\theta = \frac{BC}{1}$

$\sin\theta = BC$

Area of sector OAB $= \frac{1}{2}r^2\theta$

Area of sector OAB $= \frac{1}{2}(1)^2\theta$

Area of sector OAB $= \frac{1}{2}\theta$ —— II

In $\Delta OAD = \frac{1}{2}(OA \times AD)$

In $\Delta OAD = \frac{1}{2}(1 \times \tan\theta)$

In $\Delta OAD = \frac{1}{2}\tan\theta$ —— III

$\because$ In right angle triangle OAD:

$\tan\theta = \frac{AD}{OA}$

$\tan\theta = \frac{AD}{1}$

$\tan\theta = AD$

From figure it is clear:

Area of ΔOAB < Area of sector < Area of ΔOAD

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

Multiplying by "2":

$$\sin\theta < \theta < \tan\theta$$

Dividing by sinθ:

$$\frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta} \qquad \because \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Taking $\lim_{\theta\to 0}$:

$$\lim_{\theta\to 0} 1 > \lim_{\theta\to 0}\frac{\sin\theta}{\theta} > \lim_{\theta\to 0}\cos\theta$$

$$1 > \lim_{\theta\to 0}\frac{\sin\theta}{\theta} > \cos 0^\circ$$

$$1 > \lim_{\theta\to 0}\frac{\sin\theta}{\theta} > 1 \qquad \because \cos 0^\circ = 1$$

By sandwitch theorem,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Proved.

Evaluate $\lim_{\theta \to 0} \frac{\tan\theta - \sin\theta}{\sin^3\theta}$. (LHR-2013)

Solution:

$$\lim_{\theta \to 0} \frac{\tan\theta - \sin\theta}{\sin^3\theta}$$

$$= \lim_{\theta \to 0} \frac{\frac{\sin\theta}{\cos\theta} - \sin\theta}{\sin^3\theta} \qquad \because \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \lim_{\theta \to 0} \frac{\frac{\sin\theta - \sin\theta\cos\theta}{\cos\theta}}{\sin^3\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta(1-\cos\theta)}{\cos\theta\sin^3\theta}$$

$$= \lim_{\theta \to 0} \frac{(1-\cos\theta)}{\cos\theta \cdot \sin^2\theta}$$

$$= \lim_{\theta \to 0} \frac{(1-\cos\theta)}{\cos\theta \cdot (1-\cos^2\theta)} \qquad \because \sin^2\theta + \cos^2\theta = 1$$

$$= \lim_{\theta \to 0} \frac{(1-\cos\theta)}{\cos\theta(1-\cos\theta)(1+\cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{1}{\cos\theta(1+\cos\theta)}$$

$$= \frac{1}{\cos 0^\circ(1+\cos 0^\circ)}$$

$$= \frac{1}{1(1+1)}$$

$$= \frac{1}{1 \cdot (2)}$$

$$= \frac{1}{2}$$

BIIB

Discuss the continuity of $f(x)$ at $x=c$, $f(x)=\begin{cases} 3x-1 & \text{if } x<1 \\ 4 & \text{if } x=1 \\ 2x & \text{if } x>1 \end{cases}$, $c=1$ (LHR-2021)

Solution:

$$f(x)=\begin{cases} 3x-1 & \text{if } x<1 \\ 4 & \text{if } x=1 \\ 2x & \text{if } x>1 \end{cases}, \quad c=1$$

$$f(1)=4$$

(i) $\therefore f(1)$ is defined.

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-}(3x-1)$$
$$= 3(1)-1$$
$$= 2$$

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(2x)$$
$$= 2(1)$$
$$= 2$$

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x)$$

(ii) $\therefore \lim_{x\to1} f(x)$ exists.

(iii) $\lim_{x\to1} f(x) \neq f(1)$

As condition (iii) is not satisfy, so it is discontinuous at $x=1$.

Find the value of "m" and "n" so that

$$f(x)=\begin{cases} mx & \text{if } x<3 \\ n & \text{if } x=3 \\ -2x+9 & \text{if } x>3 \end{cases}$$ is continuous at $x=3$

(LHR-2019)

Solution:

$$f(x)=\begin{cases} mx & \text{if } x<3 \\ n & \text{if } x=3 \\ -2x+9 & \text{if } x>3 \end{cases}$$

(i) $f(3)=n$

$\because$ function is defined at $x=3$

(ii) $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} mx$

$= m(3)$

$= 3m$

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+}(-2x+9)$

$= -2(3)+9$

$= 3$

Given $f(x)$ is continuous at $x=3$.

$\Rightarrow \lim\limits_{x\to 3} f(x) = f(3)$

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) = f(3)$

$3m = 3 = n$

$\Rightarrow 3m=3$, $\boxed{n=3}$

$\boxed{m=1}$

Written by
Muhammad Asad Khalid

Prove that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

Solution:

(LHR-2014+2017)

By binomial series,

$$\left(1+\frac{1}{n}\right)^n = 1+n\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3+\cdots$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots$$

when $n\to\infty$, $\frac{1}{n},\frac{2}{n},\frac{3}{n},\ldots$ all tend to zero.

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5}+\cdots$$

$$= 1+1+0.5+0.166667+0.0416667+\cdots$$

$$= 2.718281$$

As approximate value of e is $= 2.718281$

$$\therefore \lim_{x\to+\infty}\left(1+\frac{1}{n}\right)^n = e$$

Find the values of m and n, so that given function f(x) is continuous.

(LHR-2015).

$$f(x) = \begin{cases} mx & \text{if } x<3 \\ n & \text{if } x=3 \\ -2x+9 & \text{if } x>3 \end{cases}$$

Solution:

(i)

$$f(3) = n$$

$\because$ function is defined at $x=3$.

(ii) $$L.H.L = \lim_{x\to 3^-} (f(x))$$

$$= \lim_{x\to 3} (mx)$$

$$= 3m$$

$$R.H.L = \lim_{x\to 3^+} (-2x+9)$$

$$= -2(3)+9$$

$$= 3$$

Given $f(x)$ is continuous at $x=3$.

$$\Rightarrow \lim_{x\to 3} f(x) = f(3)$$

$$\Rightarrow L.H.L = R.H.L = f(3)$$

$$3m = 3 = n$$

$$3m = 3 \quad , \quad \boxed{n = 3}$$

$$\boxed{m=1}$$

11

Let $f(x) = \dfrac{2x+1}{x-1}$; $x \neq 1$, find $f^{-1}(x)$ and verify that $f \circ f^{-1}(x) = x$ (LHR-2016)

Solution:

$$f^{-1}(x)$$

$$f(x) = \frac{2x+1}{x-1}$$

$$y = \frac{2x+1}{x-1} \quad \because f(x) = y$$

By cross multiplication:

$$yx - y = 2x+1$$

$$yx - 2x = y+1$$

$$x(y-2) = y+1$$

$$x = \frac{y+1}{y-2}$$

$$f^{-1}(y) = \frac{y+1}{y-2} \quad \because x = f^{-1}(y)$$

Replace (y) by (x):

$$f^{-1}(x) = \frac{x+1}{x-2}$$

Verification:

$$f(f^{-1}(x)) = f\left(\frac{x+1}{x-2}\right)$$

Replace x by $\left(\frac{x+1}{x-2}\right)$ in $f(x)$:

$$\frac{2\left(\frac{x+1}{x-2}\right)+1}{\left(\frac{x+1}{x-2}\right)-1}$$

$$= \frac{\frac{2x+\cancel{2}+x-\cancel{2}}{\cancel{(x-2)}}}{\frac{\cancel{x}+1-\cancel{x}+2}{\cancel{(x-2)}}}$$

$$= \frac{3x}{3} = x \implies f^{\circ}(f^{-1}(x)) = x$$

Hence proved.

(12)

Find $\lim_{\theta \to 0} \frac{1-\cos\theta}{\sin^2\theta}$. (LHR-2017)

Solution:

$$= \lim_{\theta \to 0} \frac{1-\cos\theta}{1-\cos^2\theta} \qquad \because \sin^2\theta + \cos^2\theta = 1$$

$$= \lim_{\theta \to 0} \frac{\cancel{(1-\cos\theta)}}{\cancel{(1-\cos\theta)}(1+\cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{1}{(1+\cos\theta)}$$

By applying limit:

$$= \frac{1}{(1+\cos(0))}$$

$$= \frac{1}{2}$$

(13)

Prove that $\lim_{x \to 0} \frac{a^x-1}{x} = \log_e a$. (LHR-2018)

Solution:

$$\lim_{x \to 0} \frac{a^x-1}{x} = \log_e a.$$

$$L.H.S = \lim_{x \to 0} \frac{a^x-1}{x} \quad \text{— I}$$

Let $a^x - 1 = y$ — II

$$a^x = y+1$$

Taking ln on both sides,

$$\ln(a^x) = \ln(y+1)$$

$$x \ln a = \ln(y+1) \quad \because \ln x^n = n \ln x$$

$$x = \frac{\ln(1+y)}{\ln(a)}$$

If $x \to 0$

$$a^x - 1 = y$$

$$a^0 - 1 = y$$

$$1 - 1 = y$$

$$\Rightarrow y = 0$$

$$y \to 0$$

$$= \lim_{y \to 0} \frac{y}{\frac{\ln(1+y)}{\ln(a)}} \quad \text{(from 1)}$$

$$= \lim_{y \to 0} \frac{y \ln(a)}{\ln(1+y)}$$

Dividing up and down by "y":

$$= \lim_{y \to 0} \frac{\frac{y}{y} \ln(a)}{\frac{\ln(1+y)}{y}}$$

$$= \lim_{y \to 0} \frac{\ln(a)}{\frac{1}{y} \ln(1+y)}$$

$$= \lim_{y \to 0} \frac{\ln a}{\ln(1+y)^{\frac{1}{y}}} \quad \because \ln x^n = n \ln x$$

$$= \frac{\ln a}{\ln \cdot \lim_{y \to 0} (1+y)^{\frac{1}{y}}}$$

$$= \frac{\ln a}{\ln e} \quad \because \lim_{y \to 0} (1+y)^{\frac{1}{y}} = e$$

$$= \frac{\ln a}{1} = \ln a \Rightarrow \log_e a.$$ Hence proved.