

Chapter - 2

Vectors and Equilibrium

2.1 Basic Concepts of Vectors:

Scalar:

"Physical quantity which is completely specified by magnitude only is a scalar."

For example:

mass, length, time, work, pressure energy etc.

Vector:

"Physical quantity which is completely specified by magnitude and direction is called a vector."

For example:

displacement, velocity, acceleration, Force, momentum, torque, impulse, weight etc.

Vector Representation:

Symbolically: A vector is represented by a bold face letter such as **A**, **d**, **r**, **v**. A vector may be represented by putting an arrow above the letter such as $\vec{A}$, $\vec{d}$, $\vec{r}$, $\vec{v}$.

Magnitude of a vector is represented by a light face letter A, d, r, v as $|\vec{A}|$, $|\vec{d}|$, $|\vec{r}|$, $|\vec{v}|$.

Graphically:

Graphically a vector is represented by a straight line of definite length with an arrow at one end. The length of the line on a suitable scale represents the magnitude and arrow-head gives the direction.

Rectangular Coordinate System:

Two lines drawn perpendicular to each other are called rectangular coordinate axis. The point of intersection of lines is called as origin O. This system of coordinate is called

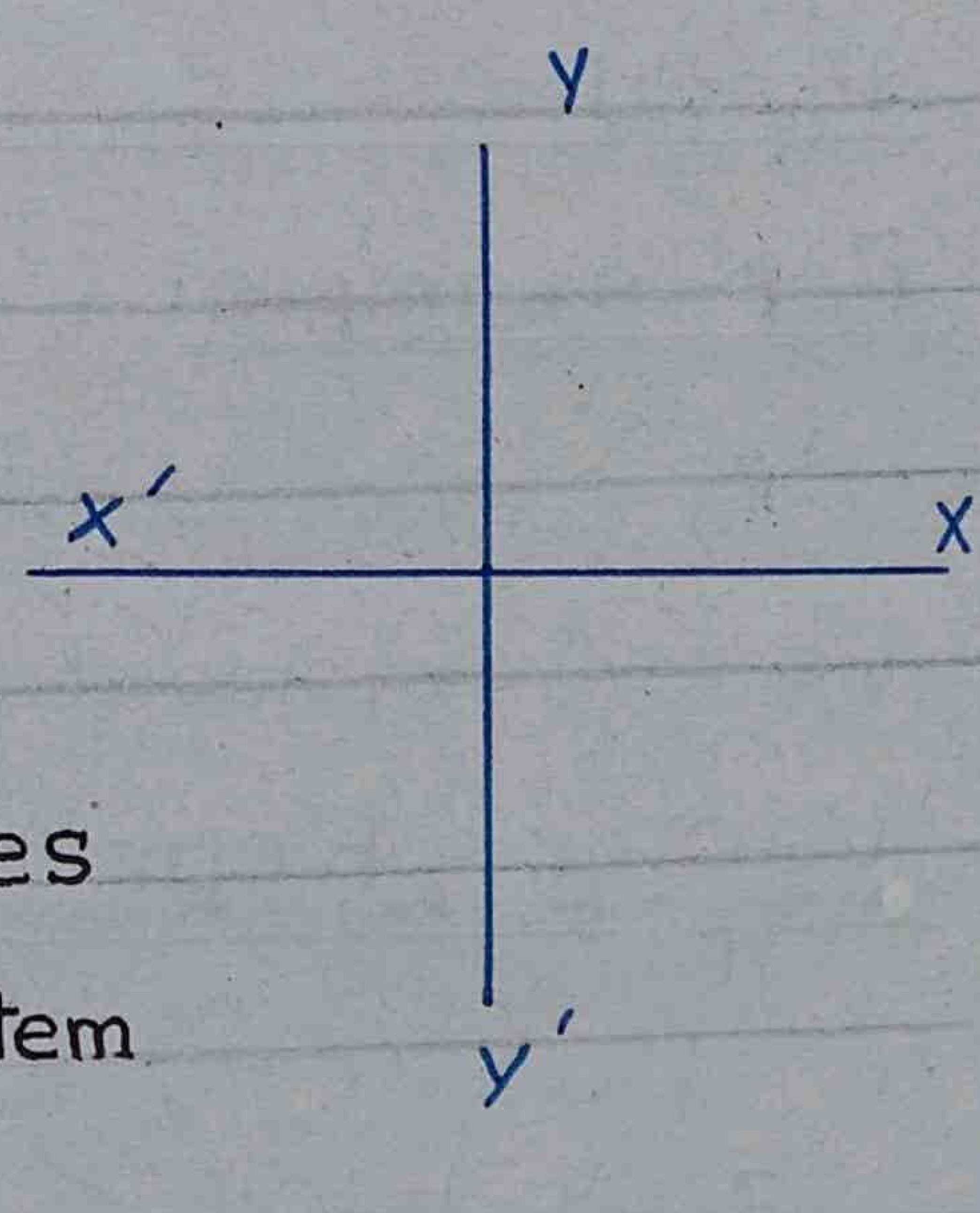


Fig (a)

rectangular coordinate system. Fig (a).

Normally x-axis is taken as horizontal axis and y-axis as vertical axis.

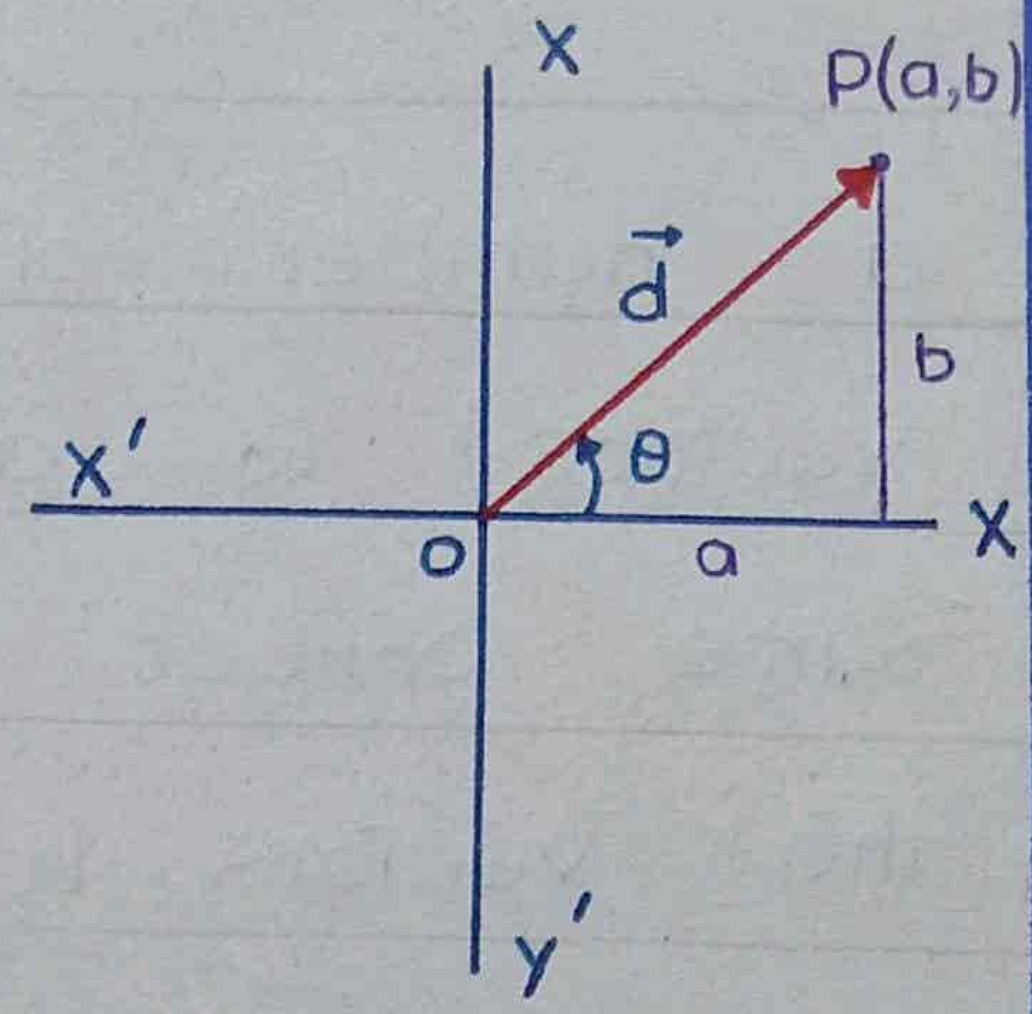


Fig (b)

<u>The direction of a vector is given by the angle θ which it makes with positive x-axis in anti-clockwise direction.</u> Fig (b).

Here the coordinates of the P are (a, b) in XY-plane.

If a vector is in space, z-axis is also required. Fig (c).

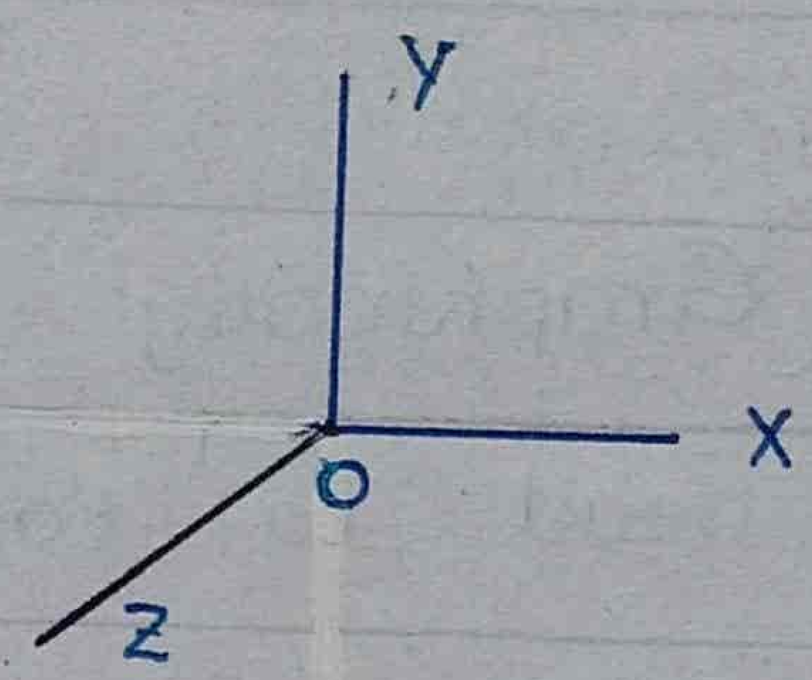


Fig (c)

z-axis is perpendicular to both X-axis and Y-axis.

In space any vector $\vec{d}$ makes angles α with x-axis, β with y-axis and γ with z-axis.

In space the point P of the vector has three coordinates (a, b, c). Fig (d).

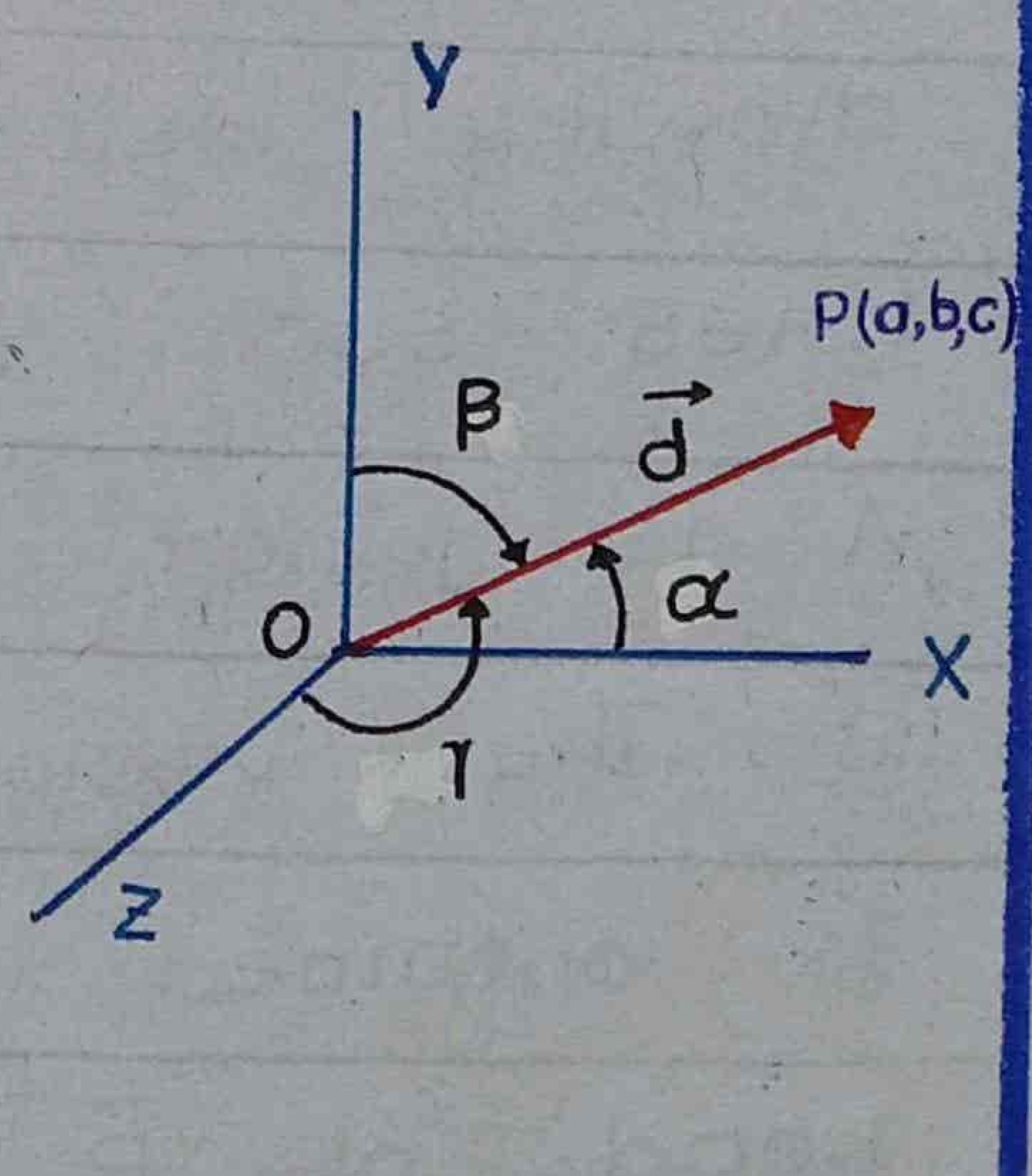


Fig (d)

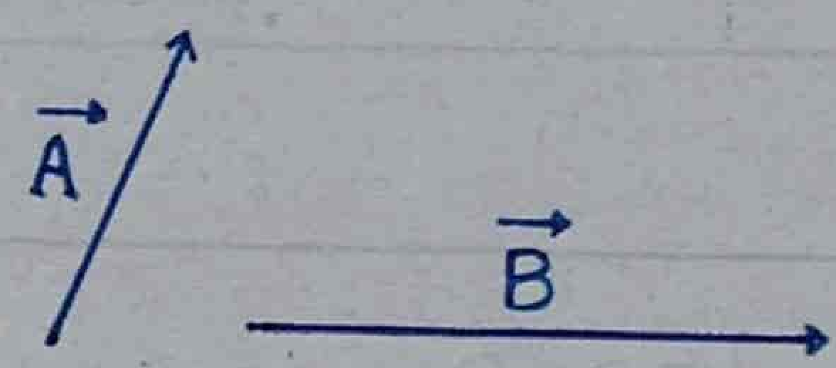
Resultant Vector

The resultant vector of a number of similar vectors (such as force vectors) is a single vector which has the same effect as the combined effect of all the vectors. It is usually denoted by $\vec{R}$.

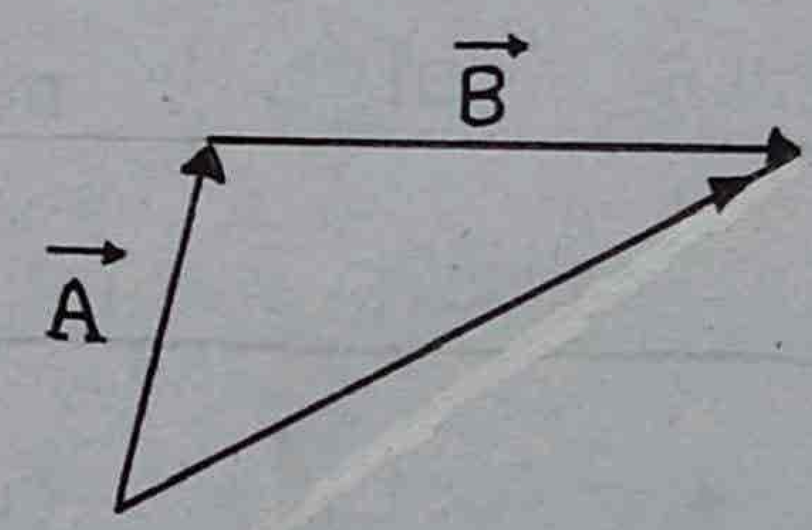
$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \cdots\cdots$$

Addition of vectors by Graphical Method

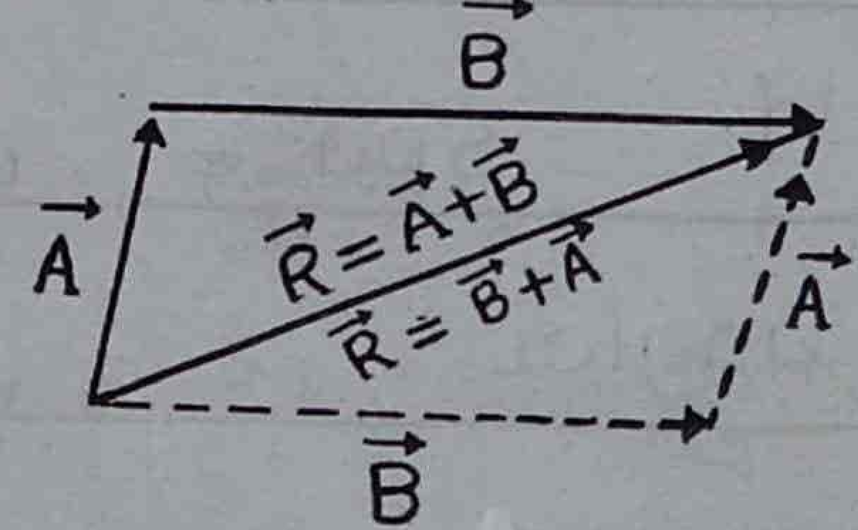
Head to tail Rule:



Graphically vectors are added by head to tail rule as follows.



Suppose two vectors $\vec{A}$ and $\vec{B}$ are given. The sum of $\vec{A}$ and $\vec{B}$ is obtained by drawing their representative lines such that the head of $\vec{A}$ is joined with the tail of $\vec{B}$. The vector sum $\vec{R} = \vec{A} + \vec{B}$ is obtained by joining the tail of $\vec{A}$ to head of $\vec{B}$. From Fig.



It is clear that

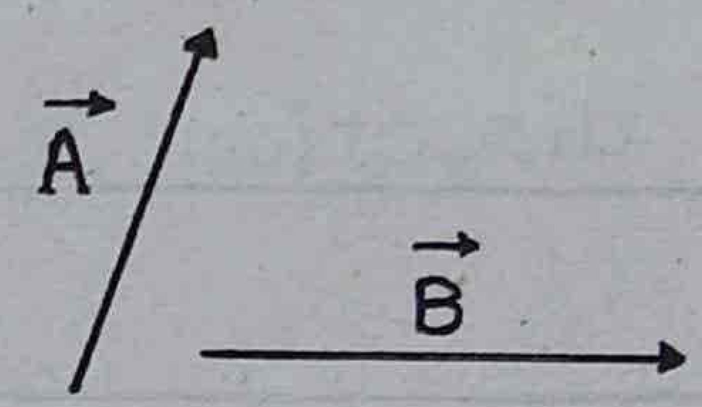
$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

So

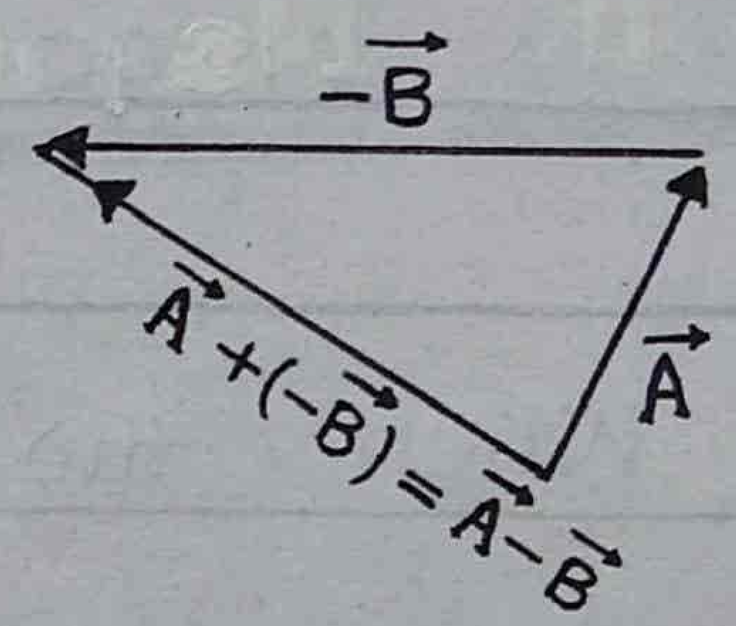
vector addition is commutative.

This rule can be used to find the sum of any number of vectors.

Vector Subtraction:



Subtraction of a vector is equivalent to the addition of the same vector with its direction reversed.



Consider two vectors $\vec{A}$ and $\vec{B}$, Suppose $\vec{B}$ is to be subtracted from $\vec{A}$. Reverse the direction of $\vec{B}$ and add it to vector $\vec{A}$. This is shown in Fig.

$$\vec{A} + (-\vec{B}) = \vec{A} - \vec{B}$$

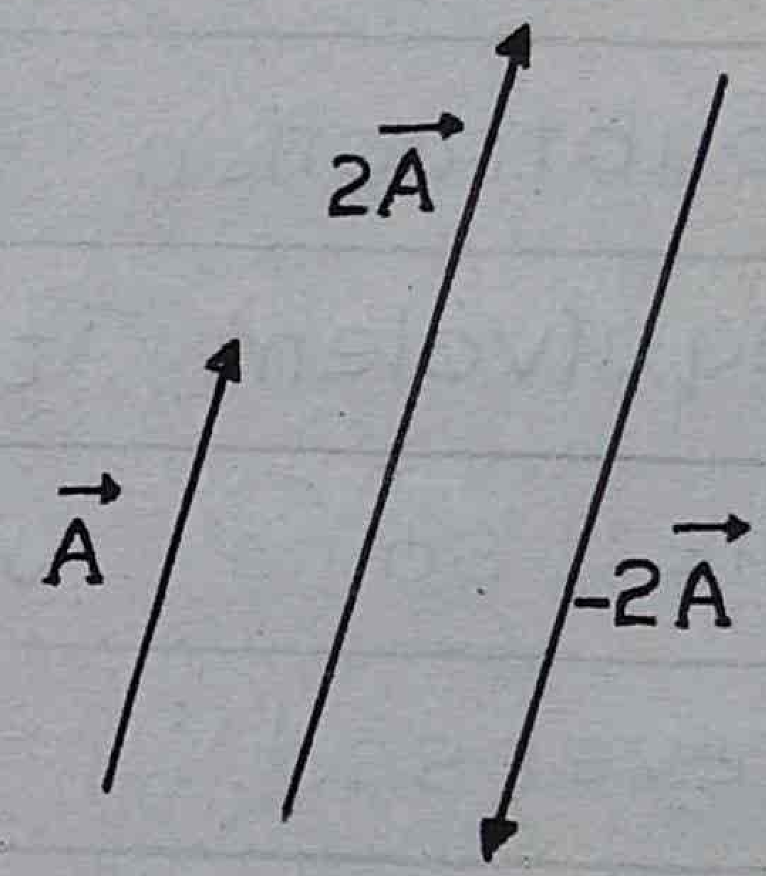
Multiplication of a vector by a scalar

When a vector is multiplied by a number $(n \neq 0)$ its magnitude is changed and its direction may or may not change.

(i) Positive Number:

When a vector $\vec{A}$ is multiplied by a positive number "n", the new vector is $n\vec{A}$. Its magnitude is n times the magnitude of $\vec{A}$, and its direction remains the same.

(ii) Negative Number:



When the vector is multiplied by a negative number "$-n$". The new vector is $-n\vec{A}$. Its magnitude is n times the magnitude of $\vec{A}$, but its direction is reversed.

(iii) When a vector is multiplied by a physical quantity (scalar) having dimensions. Then the product will be a new physical quantity whose dimensions will be the same as the product of two quantites which are multiplied.

Example:

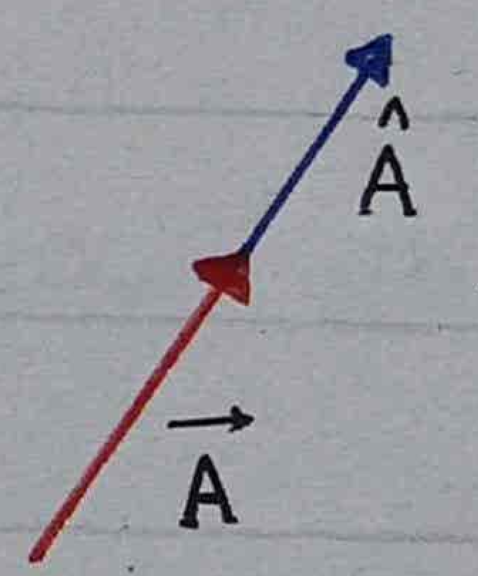
When velocity $\vec{v}$ is multiplied by a scalar; mass m, the product is a new physical quantity $m\vec{v}$ which is called momentum. Momentum has same dimensions as the product of dimensions of mass and velocity.

Unit Vector:

"A unit vector in a given direction is a vector with magnitude one in that direction."

It is used to represent the direction of a vector.

A unit vector in the direction of $\vec{A}$ is written as $\hat{A}$.

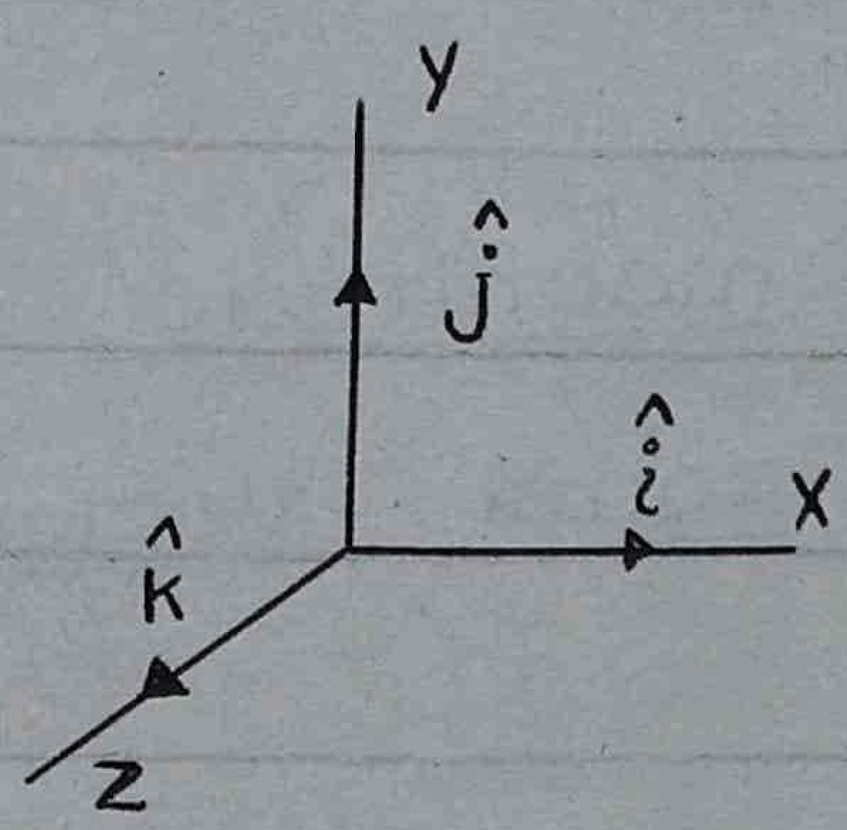


$$\vec{A} = A\hat{A}$$

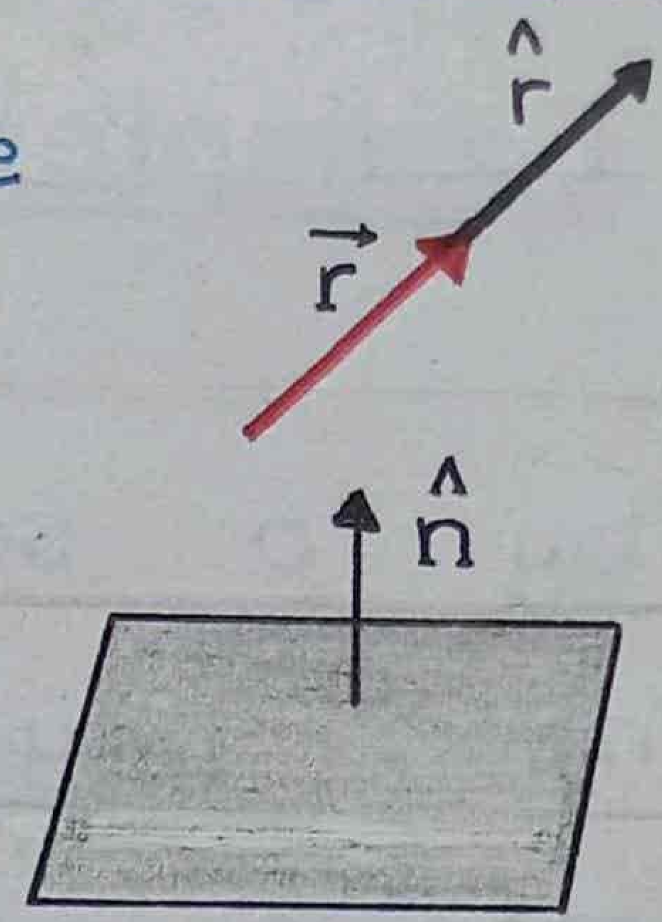
$$\boxed{\hat{A} = \frac{\vec{A}}{A}}$$

Note:

A unit vector is obtained by dividing the vector with its magnitude.



The directions of cartesian coordinate axes i.e. x-axis, y-axis, z-axis are represented by unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$ respectively.



Usually $\hat{r}$ gives the direction of $\vec{r}$, and $\hat{n}$ gives the direction of normal drawn on a specified surface.

Null Vector:

"A vector having zero magnitude and arbitrary direction is called a null vector." It is denoted by $\vec{O}$.

$$\vec{A} + (-\vec{A}) = \vec{A} - \vec{A} = \vec{O}$$

In case of vector product

$$\vec{A} \times \vec{A} = \vec{O}$$

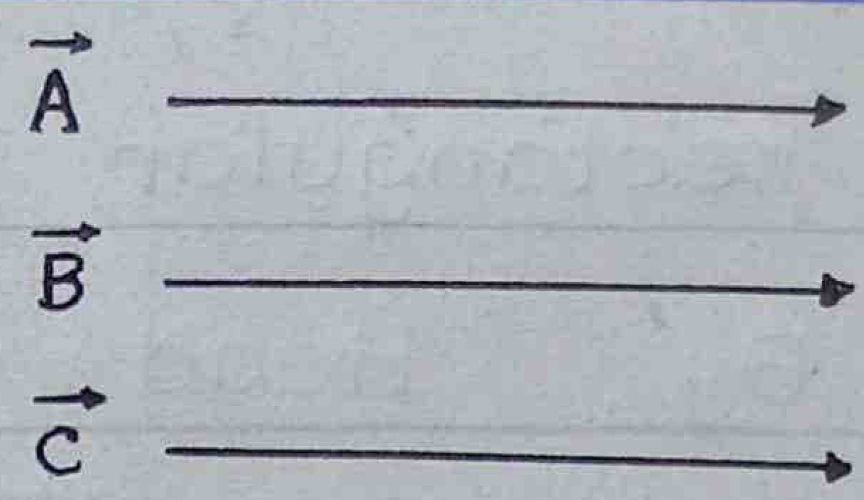
Equal Vector:

"Vectors having the same magnitude and same direction are called equal vectors."

The position of their initial points may be different.

In Fig.

$\vec{A}$, $\vec{B}$, $\vec{C}$ are equal vectors.



Component of a Vector:

"Effective value of a vector in a certain given direction is called its component."

A vector is a resultant of its components.

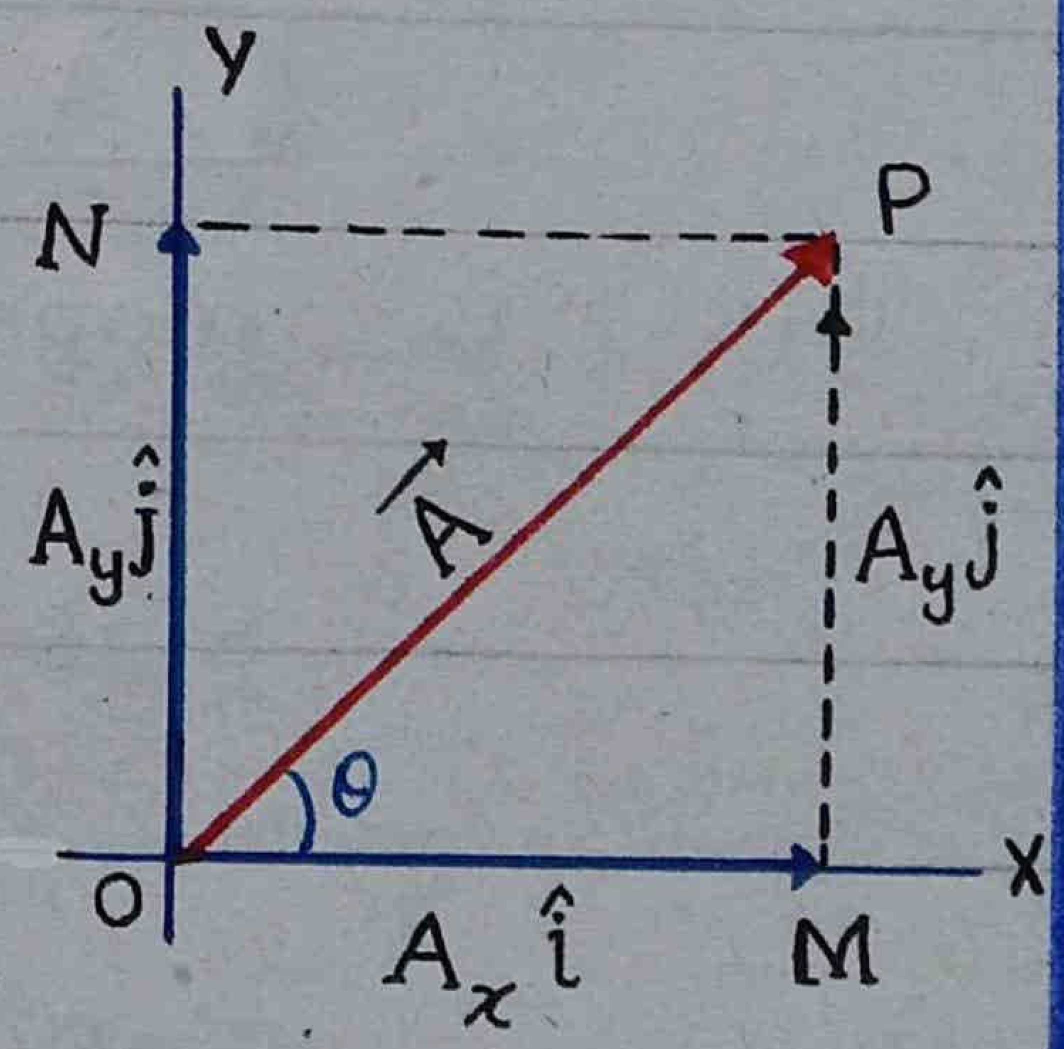
Rectangular Components of a Vector:

"The components of a vector which are mutually perpendicular to each other are called rectangular components of a vector."

Resolution of a vector into its rectangular components

Explanation:

Consider a vector $\vec{A} = \overrightarrow{OP}$ makes an angle θ with x-axis.



Draw projections from point P on x-axis and y-axis.

$\overrightarrow{OM} = A_x\hat{i}$ and $ON = A_y\hat{j}$ are two rectangular components.

By Head to tail rule

$$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$$

$$\vec{A} = \vec{A_x} + \vec{A_y}$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

$\vec{A_x}$ = Horizontal component of $\vec{A}$

or

$\vec{A_x}$ = Projection of $\vec{A}$ along x-axis

$\vec{A_y}$ = Vertical component of $\vec{A}$

or

$\vec{A_y}$ = Projection of $\vec{A}$ along y-axis.

In right angled triangle ΔOMP

$$\cos\theta = \frac{OM}{OP}$$

$$\cos\theta = \frac{A_x}{A}$$

$$\boxed{A_x = A\cos\theta}$$

This is magnitude of horizontal component of $\vec{A}$.

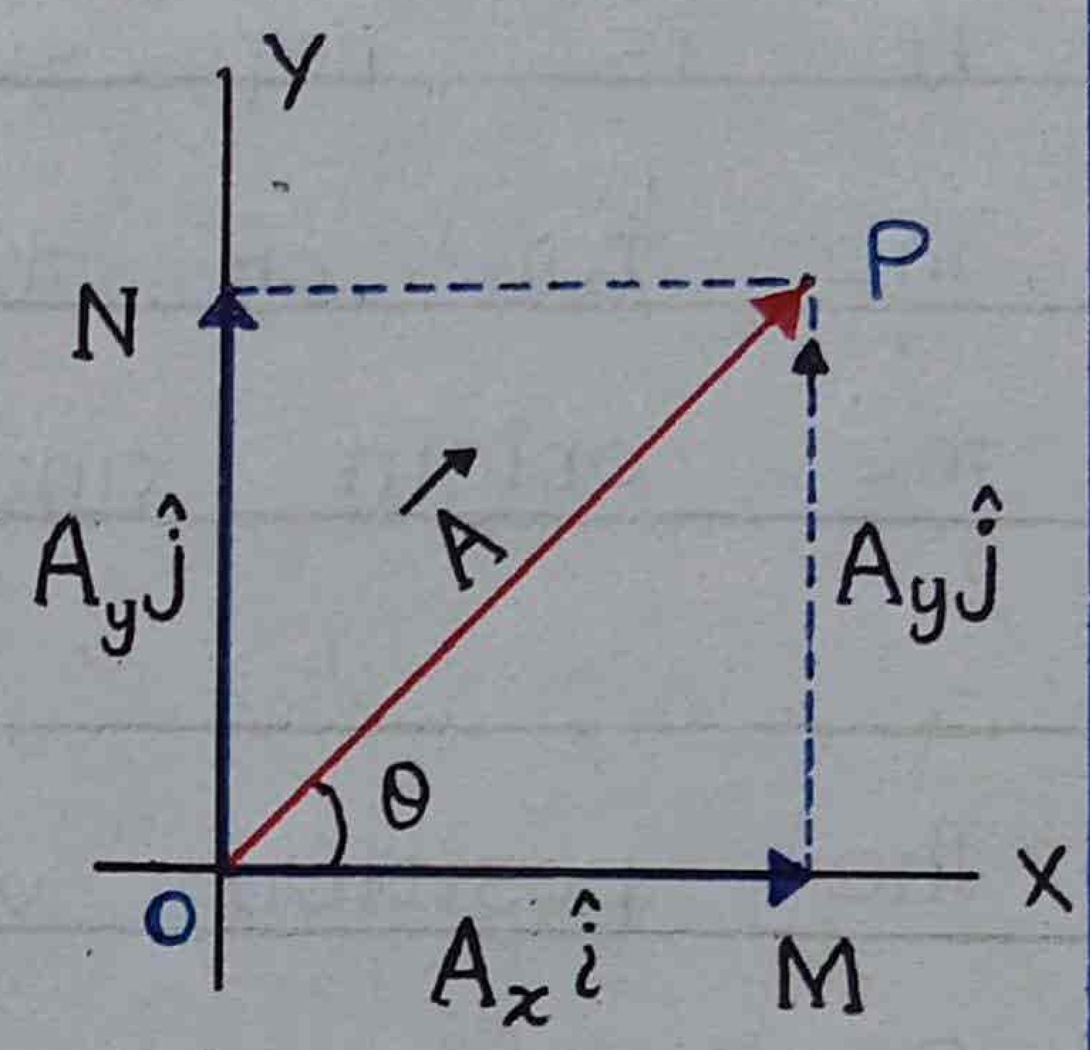
$$\sin\theta = \frac{MP}{OP}$$

$$\sin\theta = \frac{A_y}{A}$$

$\boxed{A_y = A\sin\theta}$ This is the magnitude of vertical component of $\vec{A}$.

Determination of a vector from its Rectangular Components:

When the rectangular components of a vector are given, the vector can be found as follows.



Magnitude:

In ΔOMP use Pythagorean Theorem.

$$OP^2 = OM^2 + MP^2$$

$$A^2 = A_x^2 + A_y^2$$

$$\boxed{A = \sqrt{A_x^2 + A_y^2}}$$

This gives the magnitude of $\vec{A}$.

Direction:

In the ΔOMP

$$\tan\theta = \frac{MP}{OM}$$

$$\tan\theta = \frac{A_y}{A_x}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)}$$

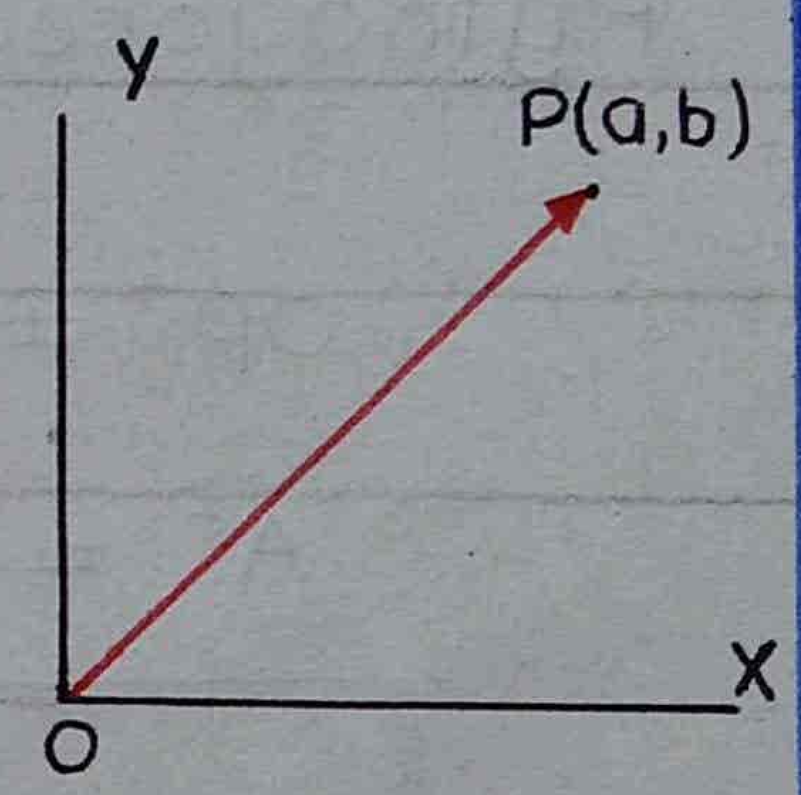
This gives the direction of $\vec{A}$.

Position Vector

"A vector which gives the location of particle (or a point P) with respect to origin is called the position vector."

It is represented by $\vec{r}$.

The tail of the position vector $\vec{r}$ lies at the origin and its head at the point P(a,b).



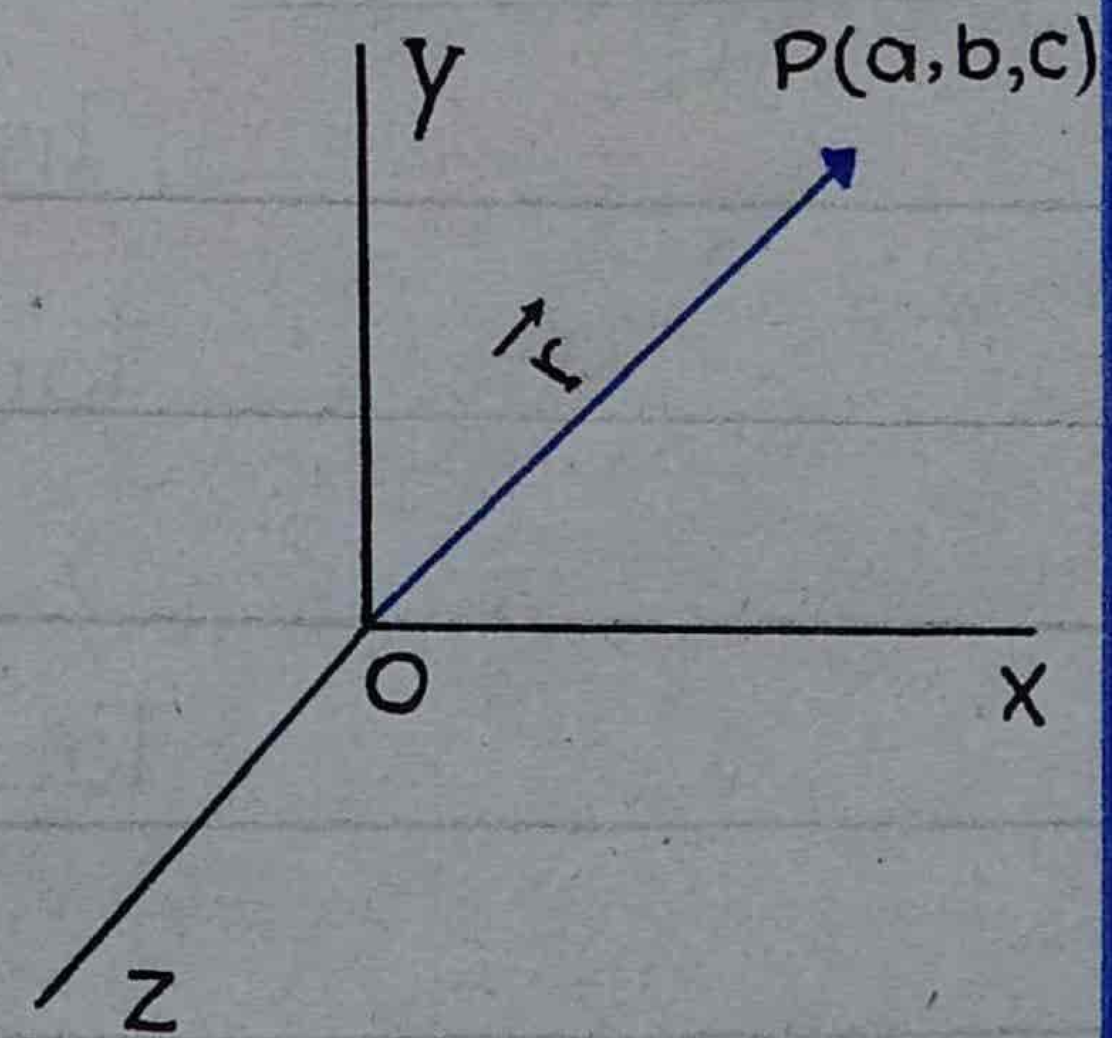
The position vector of a point P(a,b) in XY-plane is

$$\boxed{\vec{r} = a\hat{i} + b\hat{j}}$$

or

$$\vec{r} = x\hat{i} + y\hat{j}$$

The position vector of a point P(a,b,c) in three dimensional space is



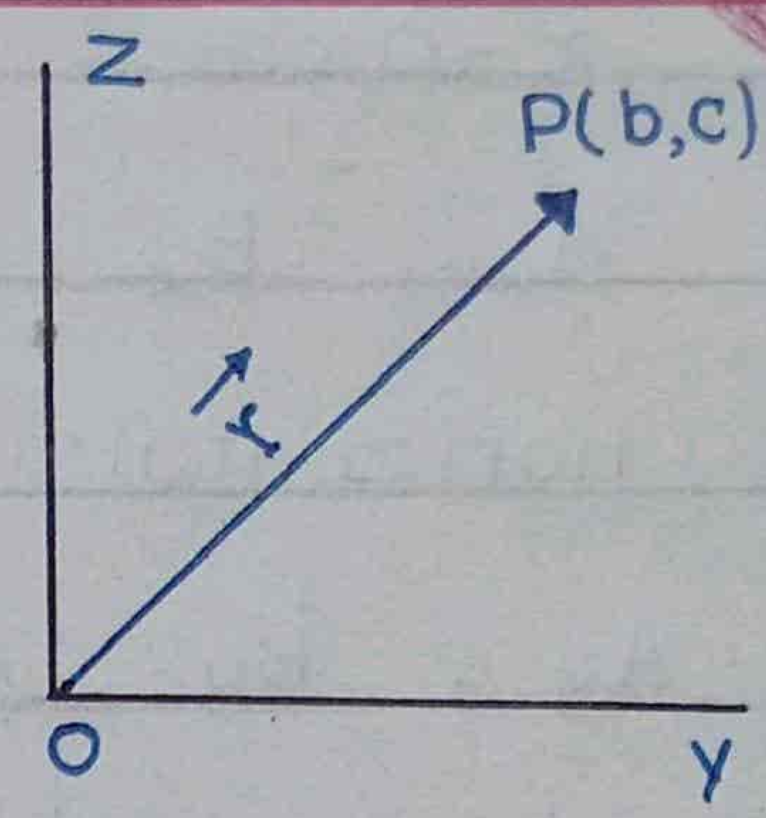
$$\boxed{\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}}$$

or

$$r = x\hat{i} + y\hat{j} + z\hat{k}$$

Just For Understand

The position vector in yz-plane is

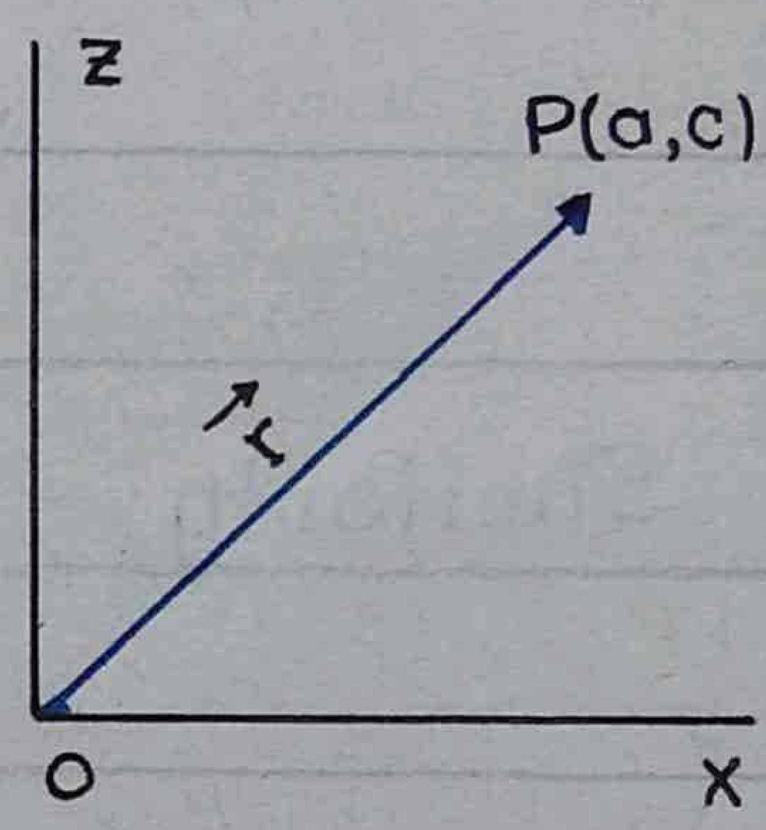


$$\boxed{\vec{r} = b\hat{j} + c\hat{k}}$$

or

$$r = y\hat{j} + z\hat{k}$$

The position vector in xz-plane is

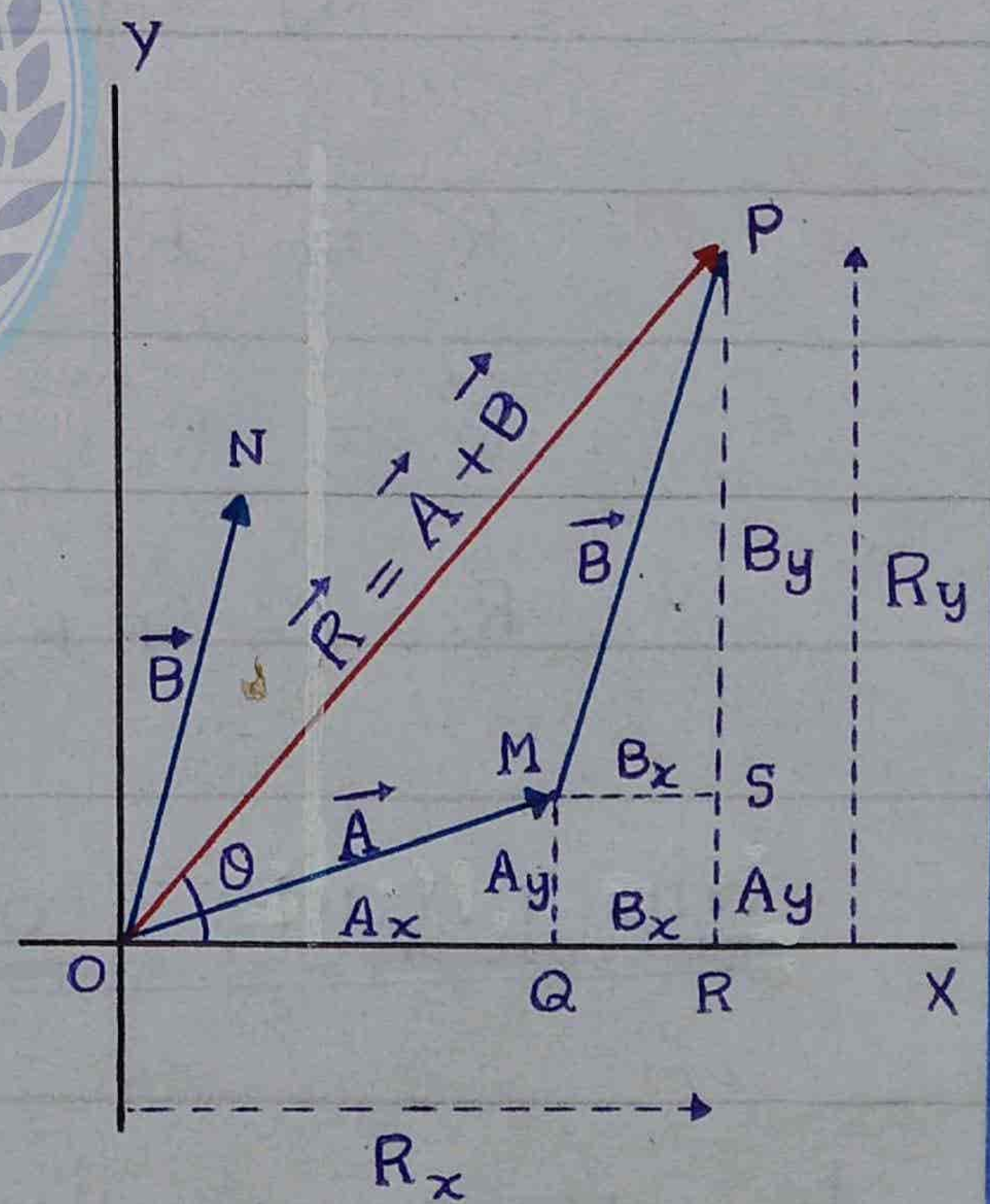


$$\boxed{\vec{r} = a\hat{i} + c\hat{k}}$$

$$\vec{r} = x\hat{i} + z\hat{k}$$

2.2 Vector Addition by Rectangular Components

Consider two vectors $\vec{A} = \overrightarrow{OM}$ and $\vec{B} = \overrightarrow{ON}$. In figure $\vec{A}$ and $\vec{B}$ are added by head to tail rule. $\vec{R}$ is the resultant vector.



$$\vec{R} = \vec{A} + \vec{B}$$

$\vec{R}$ makes an angle θ with

x-axis.

A_x, B_x, R_x are the magnitudes of horizontal components of $\vec{A}$, $\vec{B}$, $\vec{R}$.

A_y, B_y, R_y are the magnitudes of vertical components of $\vec{A}$, $\vec{B}$, $\vec{R}$.

In Figure

$$OR = OQ + QR$$

$$R_x = A_x + B_x$$

$$\vec{R_x} = R_x \hat{i}$$

$$\vec{R_x} = (A_x + B_x)\hat{i}$$

Similarly

$$RP = RS + SP$$

$$R_y = A_y + B_y$$

$$\vec{R_y} = R_y \hat{j}$$

$$\vec{R_y} = (A_y + B_y)\hat{j}$$

$$\vec{R} = \vec{R_x} + \vec{R_y}$$

$$\vec{R} = R_x \hat{i} + R_y \hat{j}$$

$$\vec{R} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$

Magnitude of the resultant vector

In right angled triangle ΔORP,

Using Pythagorean Theorem

$$OP^2 = OR^2 + RP^2$$

$$R^2 = R_x^2 + R_y^2$$

$$\boxed{R = \sqrt{R_x^2 + R_y^2}}$$

$$\boxed{R = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}}$$

Direction of resultant vector

In triangle ΔORP

$$\tan\theta = \frac{RP}{OR}$$

$$\tan\theta = \frac{R_y}{R_x}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{A_y + B_y}{A_x + B_x}\right)}$$

For any number of coplanar vectors A, B, C, ...

Magnitude:

$$\boxed{R = \sqrt{(A_x + B_x + C_x + \cdots\cdots)^2 + (A_y + B_y + C_y + \cdots\cdots)^2}}$$

Direction:

$$\boxed{\theta = \tan^{-1}\left(\frac{A_y + B_y + C_y + \cdots\cdots}{A_x + B_x + C_x + \cdots\cdots}\right)}$$

Main steps for addition of vectors by Rectangular Components:

1. Find x and y components of all the vectors.
2. Find x-component of the resultant vector as

$$R_x = A_x + B_x + C_x + \cdots\cdots$$

3. Find y-component of the resultant vector as

$$R_y = A_y + B_y + C_y + \cdots\cdots$$

4. Find the magnitude of the resultant vector $\vec{R}$ as

$$R = \sqrt{R_x^2 + R_y^2}$$

5. Find the direction of the resultant vector $\vec{R}$ as

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

$\vec{R}$ makes the angle θ with positive x-axis.

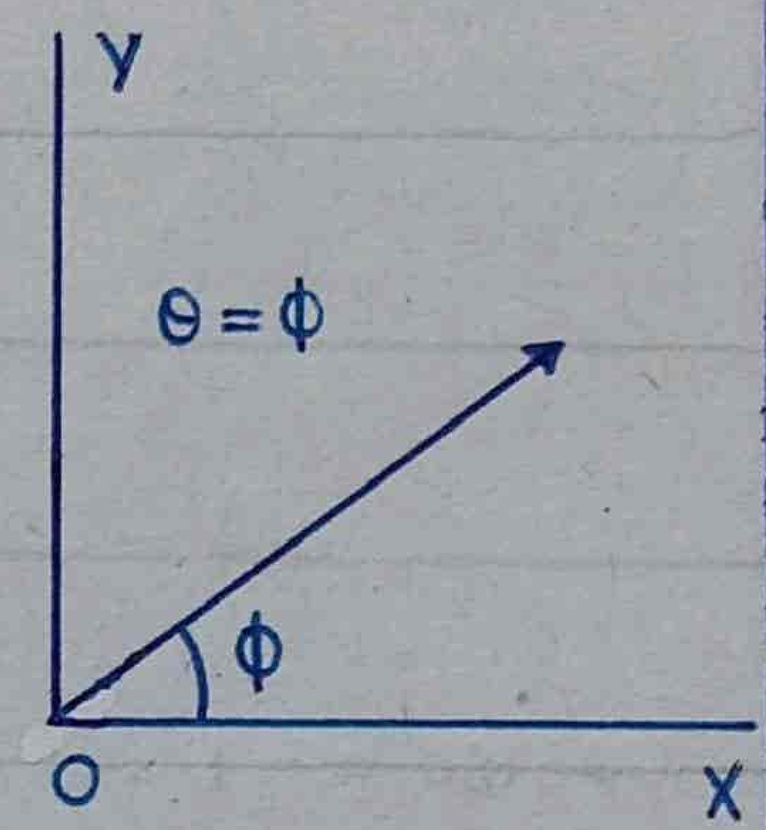
By the signs of R_x and R_y, we can find the quadrant in which $\vec{R}$ lies.

Ignoring the signs of R_x and R_y, find

$$\phi = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

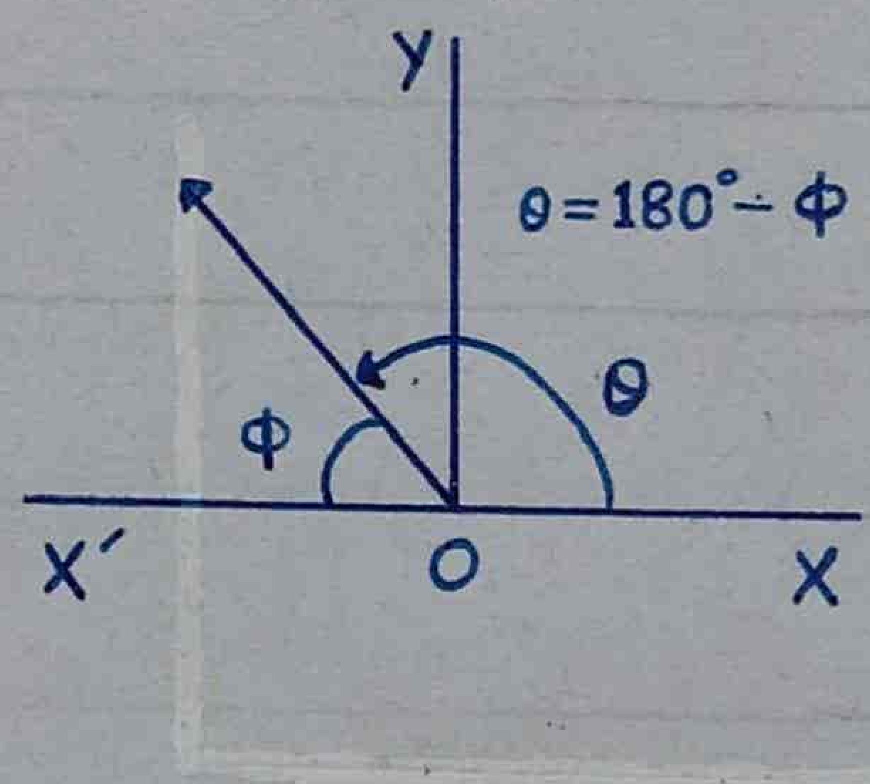
θ is found as follows:

1st quadrant



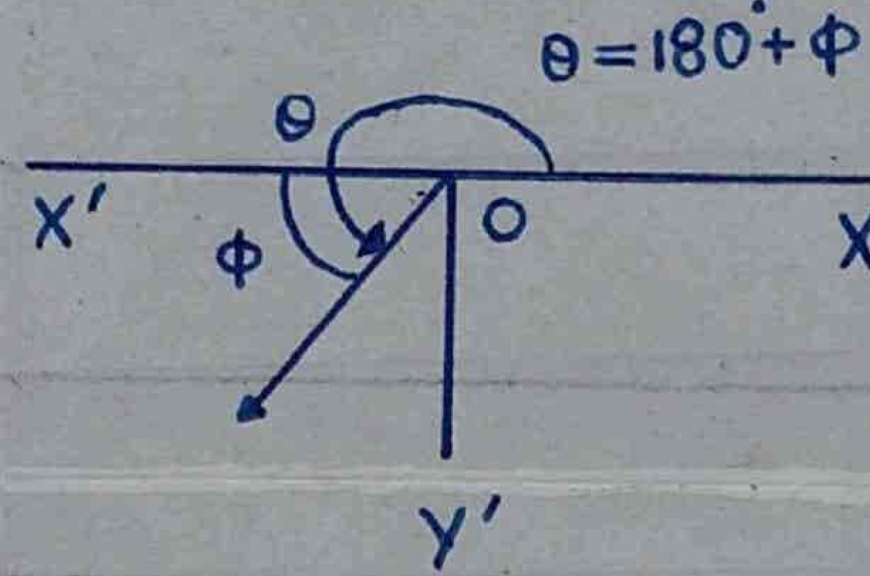
a) If both R_x and R_y components are positive, then then the resultant lies in the first quadrant and its direction is $\theta = \phi$

2nd quadrant



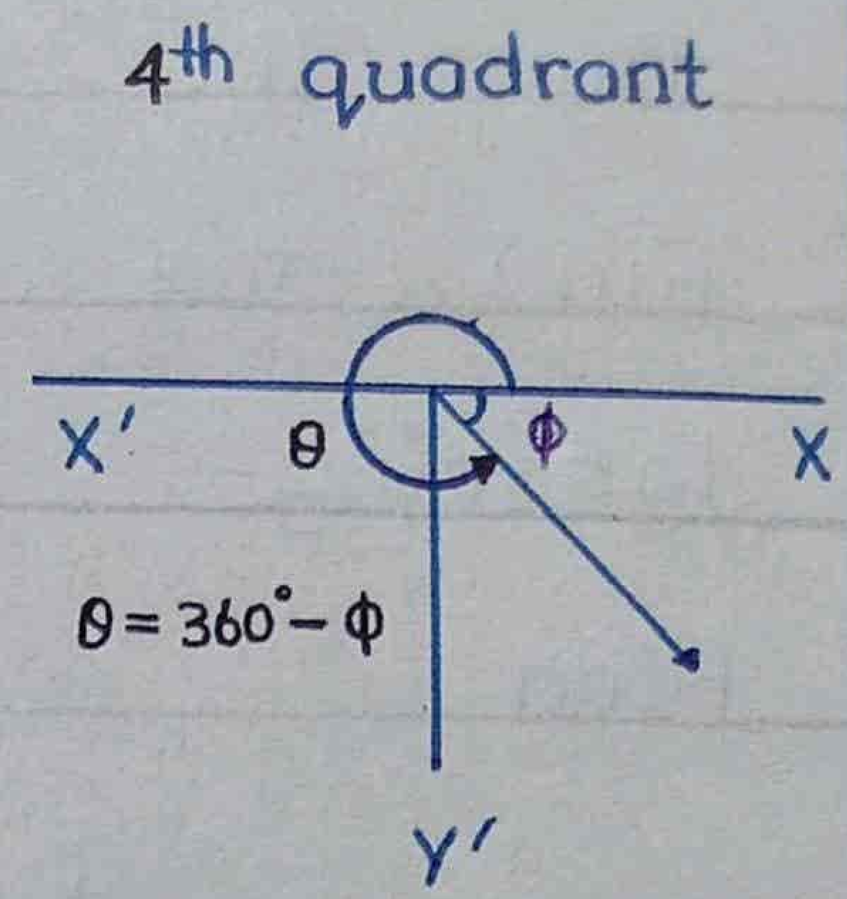
b) If R_x is $-ve$ and R_y components is $+ve$, then the resultant lies in 2^{nd} quadrant and its direction is $\theta = 180° - \phi$.

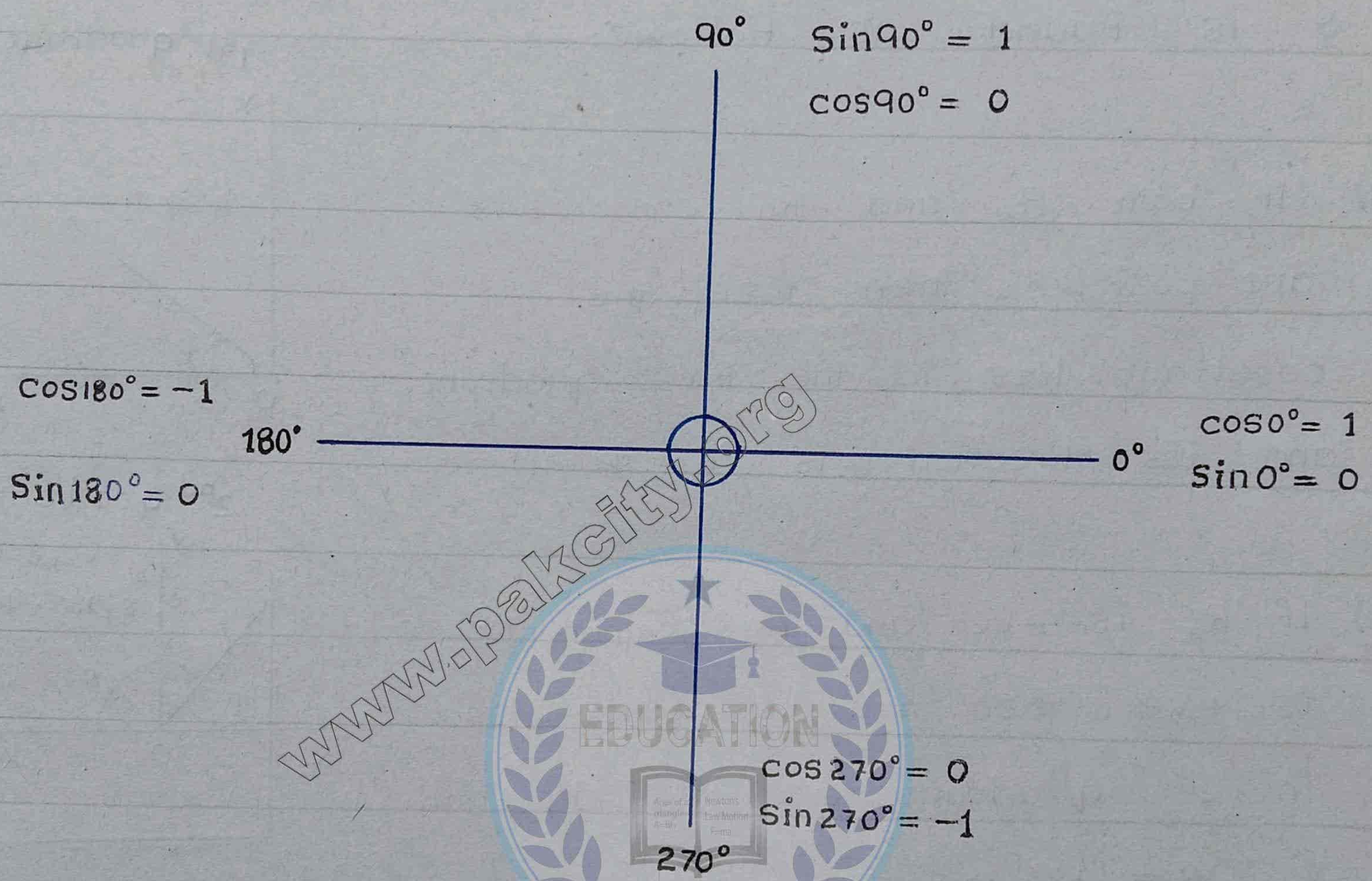
3rd quadrant



c) If both R_x and R_y components are $-ve$, the resultant lies in the 3^{rd} quadrant and its direction is $\theta = 180° + \phi$.

d) If R_x is positive and R_y is negative, the resultant lies in the 4^{th} quadrant and its direction is $\theta = 360° - \phi$.





	$30°$	$60°$	$45°$
Sin	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{2} = 0.866$	$\frac{1}{\sqrt{2}} = 0.707$
Cos	$\frac{\sqrt{3}}{2} = 0.866$	$\frac{1}{2} = 0.5$	$\frac{1}{\sqrt{2}} = 0.707$

Product of two vectors

When two vectors multiplied, their product may be a scalar quantity or a vector quantity. So, there are two types of vector multiplications.

Scalar Product (Dot Product)

"When the product of two vectors is a scalar quantity, it is called a scalar product or dot product."

It is defined as

$$\boxed{\vec{A} \cdot \vec{B} = AB\cos\theta}$$

Here A and B are the magnitudes of the vectors $\vec{A}$ and $\vec{B}$, θ is the angle between them.

Physical Interpretation of the Dot Product

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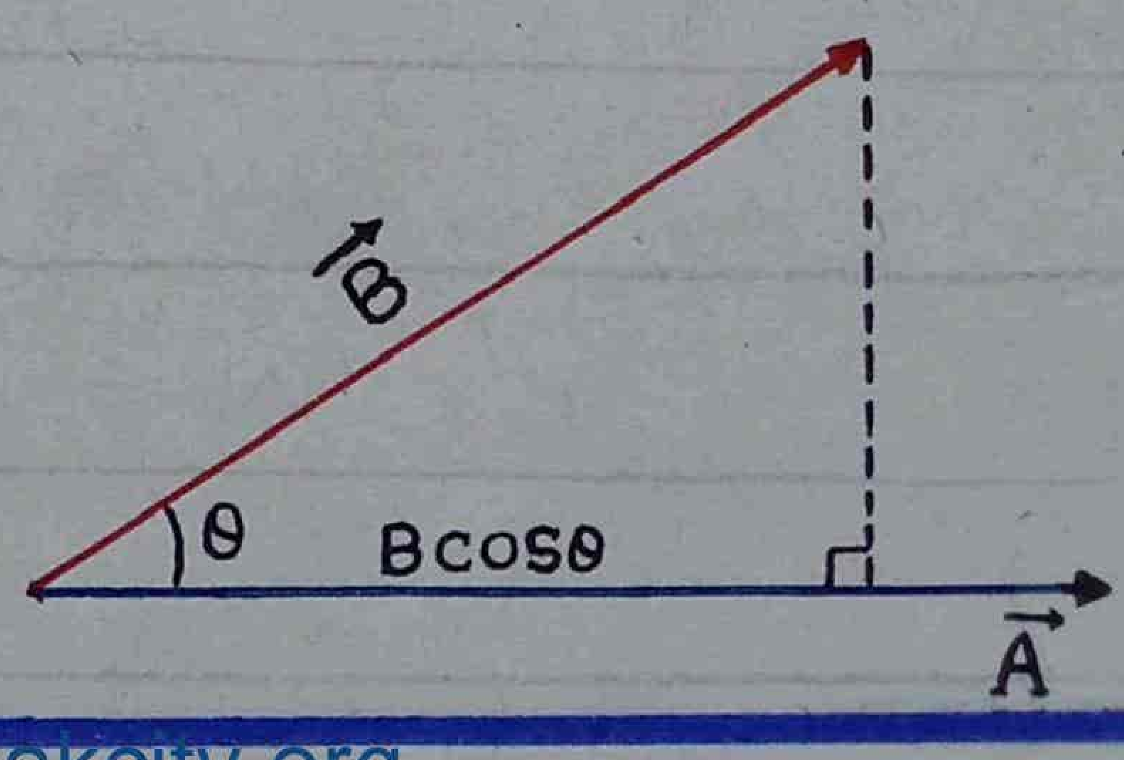
$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

$$\vec{A} \cdot \vec{B} = A(B\cos\theta)$$



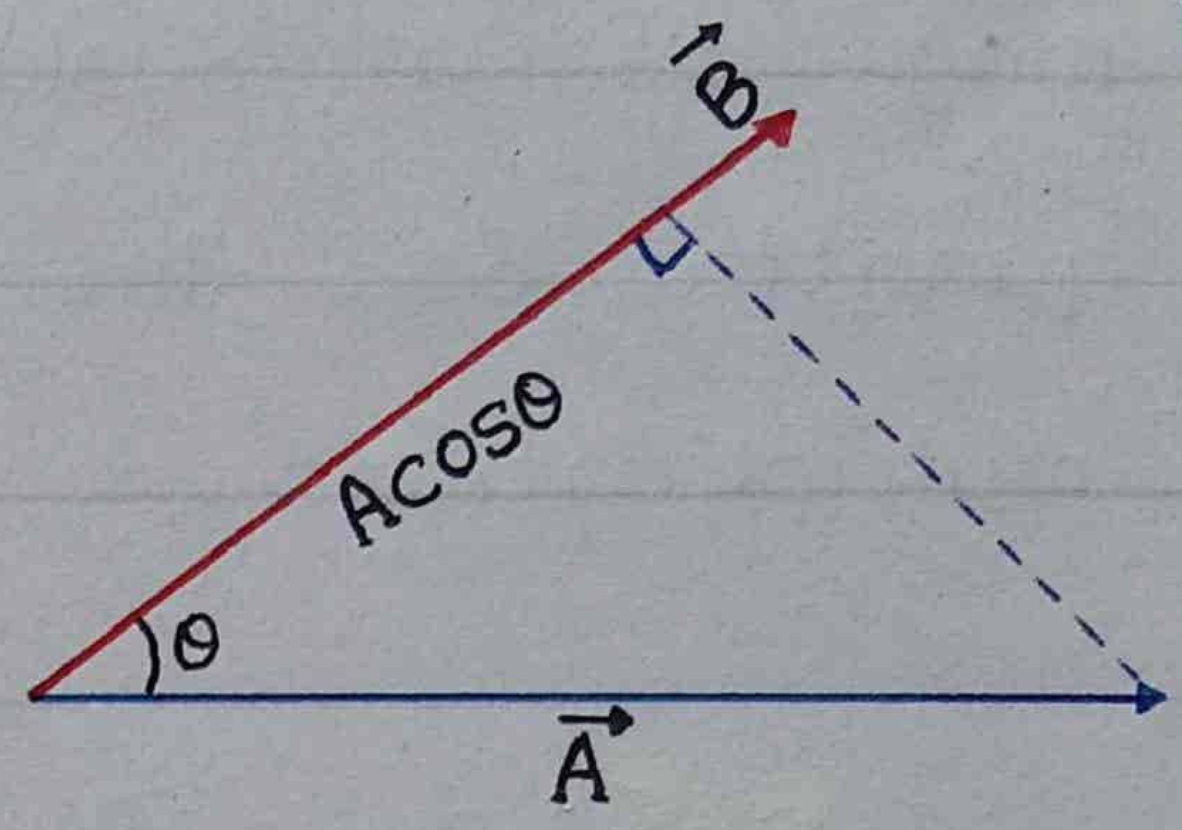
$$\vec{A}\cdot\vec{B} = A(\text{ projection of } \vec{B} \text{ on } \vec{A})$$

$$\vec{A}\cdot\vec{B} = A(\text{ magnitude of component of } \vec{B} \text{ along } \vec{A})$$

$$\vec{B}\cdot\vec{A} = BA\cos\theta$$

$$\vec{B}\cdot\vec{A} = B(A\cos\theta)$$

$$\vec{B}\cdot\vec{A} = B(\text{Projection of } \vec{A} \text{ on } \vec{B})$$



$$\vec{B}\cdot\vec{A} = B(\text{magnitude of component of } \vec{A} \text{ along } \vec{B})$$

Conclusion:

"The dot product of two vectors is the product of magnitude of one vector and the magnitude of the component of the other vector along the direction of the first vector."

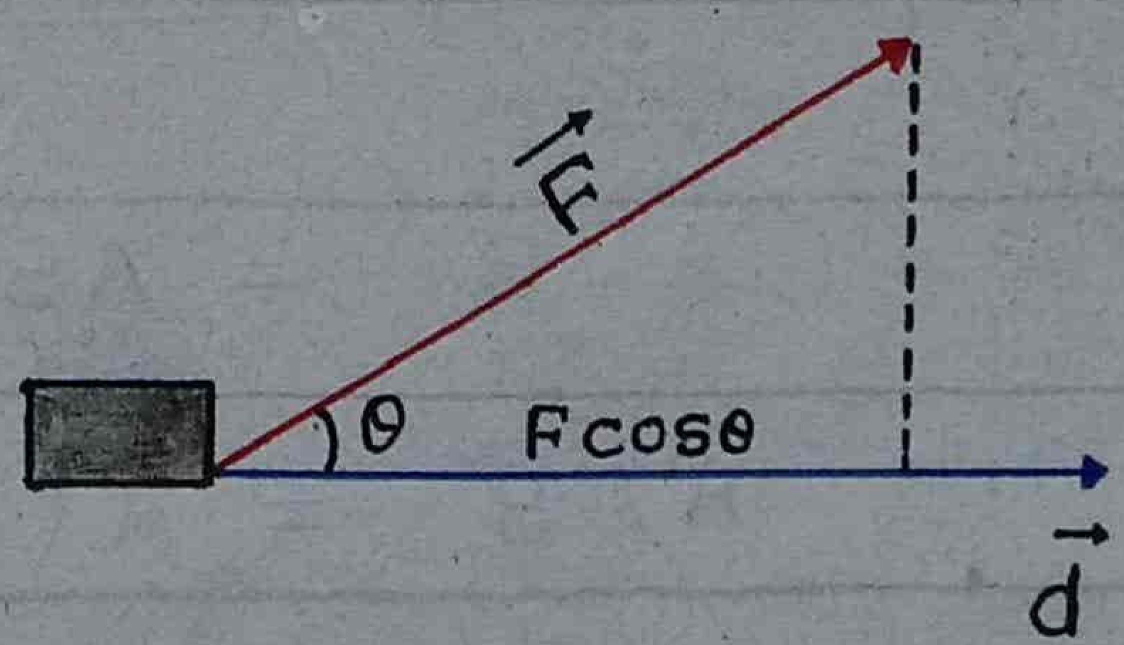
Examples:

(i) "Work is the scalar product of force $\vec{F}$ and displacement $\vec{d}$."

$$W = \vec{F}\cdot\vec{d}$$

$$W = Fd\cos\theta$$

$$W = (F\cos\theta)(d)$$



$W = $ (Effective component of force in the direction of motion)(distance moved)

(ii) $$P = \vec{F} \cdot \vec{v}$$

"<u>Power is the scalar product of force and velocity.</u>"

Characteristics of Scalar Product

1. Scalar Product is Commutative:

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

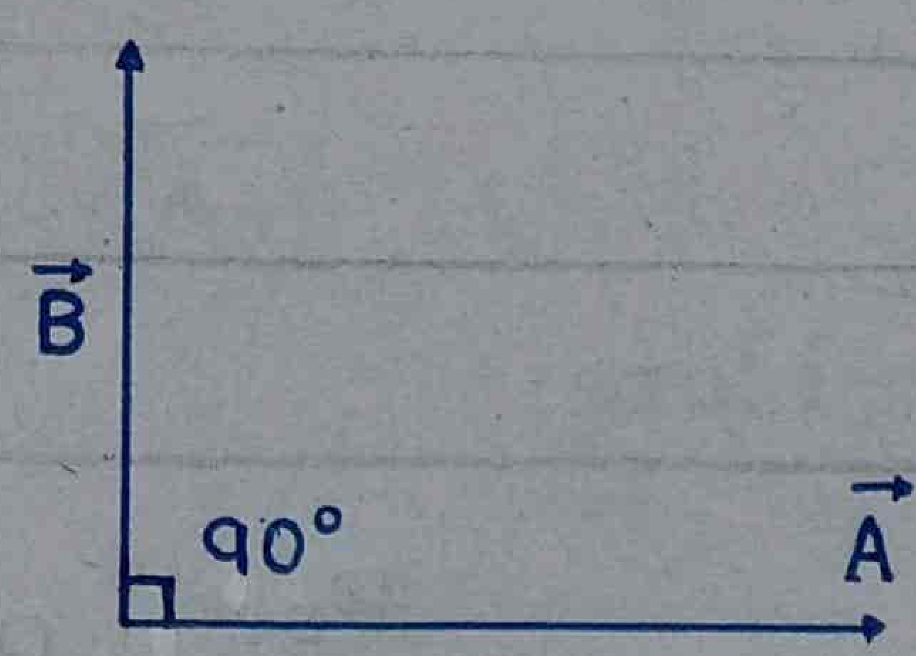
Proof:

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

$$\vec{B} \cdot \vec{A} = BA\cos\theta = AB\cos\theta$$

So $$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

2. Scalar Product of two mutually perpendicular vectors is zero:



$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

$$\vec{A} \cdot \vec{B} = AB\cos 90°$$

$$\vec{A} \cdot \vec{B} = AB(0)$$

$$\vec{A} \cdot \vec{B} = 0$$

For unit vectors:

As $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular.

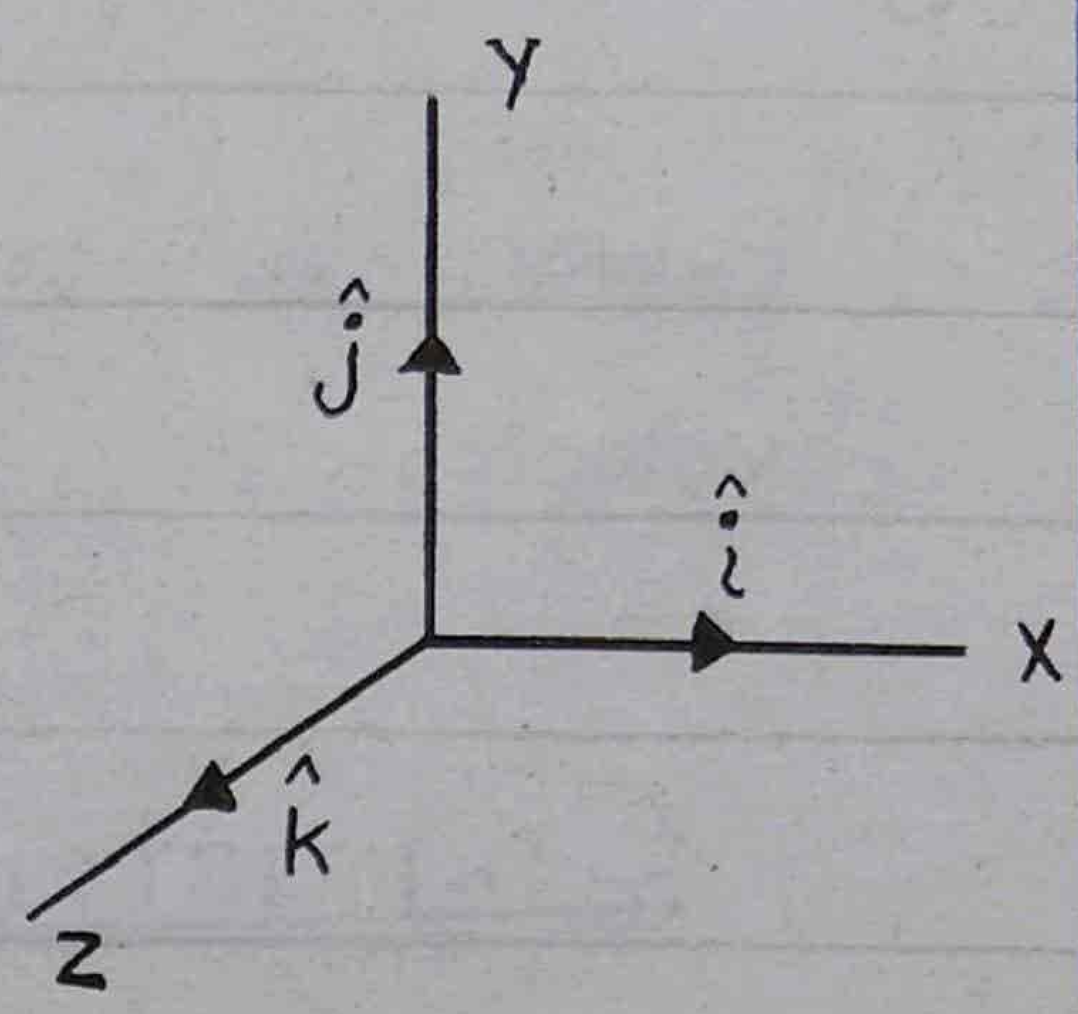
$$\hat{i} \cdot \hat{j} = (1)(1)\cos 90^\circ$$

$$\hat{i} \cdot \hat{j} = (1)(1)(0)$$

$$\hat{i} \cdot \hat{j} = 0$$

$$\hat{j} \cdot \hat{k} = 0$$

$$\hat{k} \cdot \hat{i} = 0$$



$$\boxed{\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0}$$

3. Scalar Product of two parallel vectors is equal to the product of their magnitudes:

(i) Angle between two parallel vectors is $\theta = 0^\circ$

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

$$\vec{A} \cdot \vec{B} = AB\cos 0^\circ$$

$$\vec{A} \cdot \vec{B} = AB(1)$$

$$\vec{A} \cdot \vec{B} = AB \quad \text{(Maximum value)}$$

Note:

For parallel vectors scalar product has maximum value.

(ii) For anti-parallel vectors $\theta = 180^\circ$

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$
$$\vec{A}\cdot\vec{B} = AB\cos 180^\circ$$
$$\vec{A}\cdot\vec{B} = AB(-1)$$
$$\vec{A}\cdot\vec{B} = -AB$$

$\hat{n}\cdot\hat{n} = 1$

$\hat{r}\cdot\hat{r} = 1$

$\hat{A}\cdot\hat{A} = 1$

For unit vector:

$$\hat{i}\cdot\hat{i} = (1)(1)\cos 0^\circ$$
$$\hat{i}\cdot\hat{i} = (1)(1)(1)$$
$$\hat{i}\cdot\hat{i} = 1$$
$$\hat{j}\cdot\hat{j} = 1$$
$$\hat{k}\cdot\hat{k} = 1$$

$$\boxed{\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1}$$

4. Self product of a vector $\vec{A}$ is equal to square of its magnitude:

$$\vec{A}\cdot\vec{A} = AA\cos\theta$$
$$\vec{A}\cdot\vec{A} = A^2\cos 0^\circ$$
$$\vec{A}\cdot\vec{A} = A^2(1)$$
$$\vec{A}\cdot\vec{A} = A^2$$

$\vec{F}\cdot\vec{F} = F^2$

$\vec{V}\cdot\vec{V} = V^2$

5- Scalar Product in terms of Rectangular Components:

As $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$\vec{A}\cdot\vec{B} = A_xB_x\,\hat{i}\cdot\hat{i} + A_xB_y\,\hat{i}\cdot\hat{j} + A_xB_z\,\hat{i}\cdot\hat{k} + A_yB_x\,\hat{j}\cdot\hat{i} + A_yB_y\,\hat{j}\cdot\hat{j} + A_yB_z\,\hat{j}\cdot\hat{k} + A_zB_x\,\hat{k}\cdot\hat{i} + A_zB_y\,\hat{k}\cdot\hat{j} + A_zB_z\,\hat{k}\cdot\hat{k}$$

As $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$; $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

$$\vec{A}\cdot\vec{B} = A_xB_x(1) + A_xB_y(0) + A_xB(0) + A_yB_x(0) + A_yB_y(1) + A_yB_z(0) + A_zB_x(0) + A_zB_y(0) + A_zB_z(1)$$

$$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

6- Angle θ between two vectors $\vec{A}$ and $\vec{B}$ can be Found by Scalar Product as Follows:

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB}$$

$\because \vec{A}\cdot\vec{A} = A_xA_x + A_yA_y + A_zA_z$

$A^2 = A_x^2 + A_y^2 + A_z^2$

$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

$B = \sqrt{B_x^2 + B_y^2 + B_z^2}$

$$\cos\theta = \frac{(\vec{A}\cdot\vec{B})}{AB}$$

$$\theta = \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{AB}\right)$$

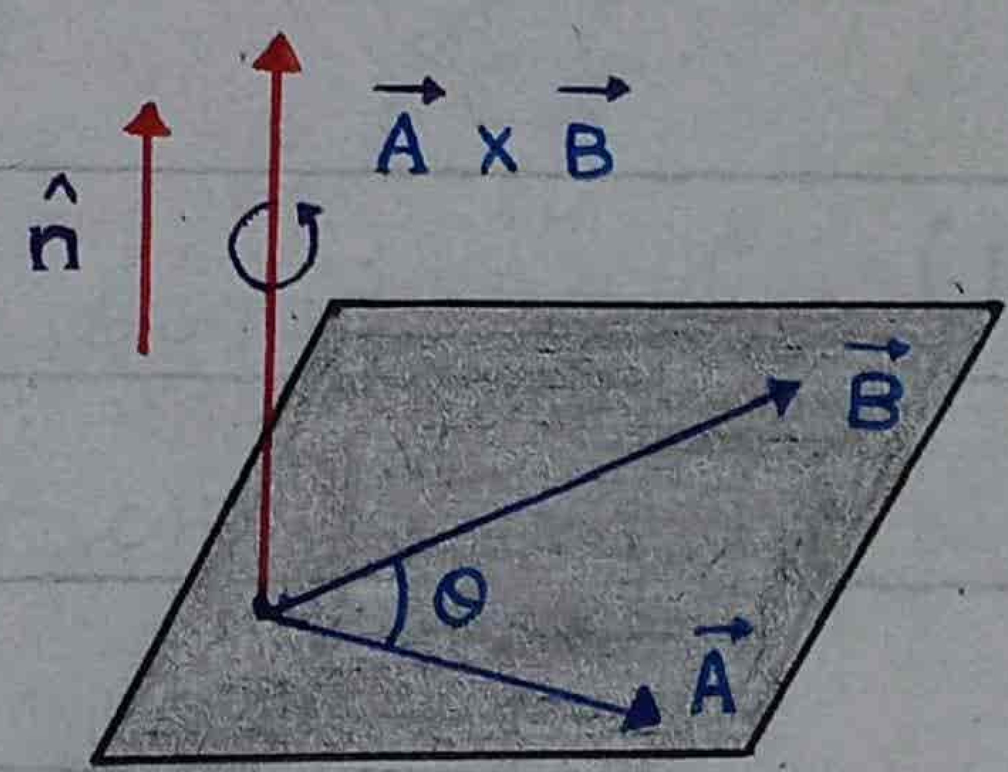
$$\theta = \cos^{-1}\left(\frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\ \sqrt{B_x^2 + B_y^2 + B_z^2}}\right)$$

Vector or Cross Product

"When the product of two vectors is a vector quantity, it is called a vector product or cross product."

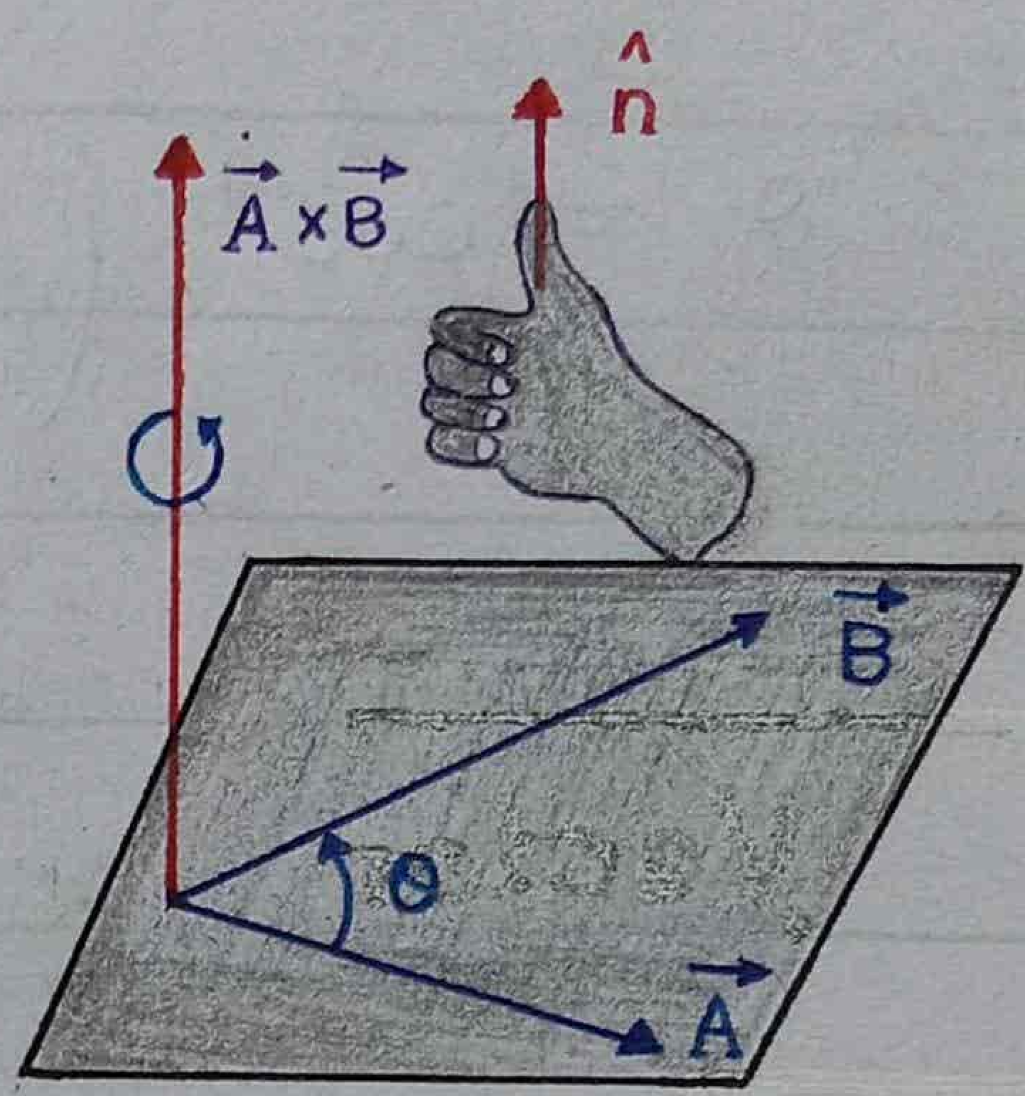
$$\boxed{\vec{A} \times \vec{B} = AB\sin\theta\ \hat{n}}$$

Here A and B are the magnitudes of $\vec{A}$ and $\vec{B}$, θ is the angle between $\vec{A}$ and $\vec{B}$. $\hat{n}$ is a unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ as shown in Fig.



Right Hand Rule:

The direction of $\hat{n}$ or $\vec{A} \times \vec{B}$ is found by the right hand rule.



"Place the tails of the vectors $\vec{A}$ and $\vec{B}$ to form a plane of $\vec{A}$ and $\vec{B}$. The direction of $\hat{n}$ or vector product $\vec{A} \times \vec{B}$ is perpendicular to this plane.

Rotate $\vec{A}$ towords $\vec{B}$ through the smaller angle. Now rotate fingers of right hand along the direction of rotation, keeping the thumb erect. (سیدھا) The thumb gives the direction of $\vec{A} \times \vec{B}$ or $\hat{n}$."

Examples:

(i) Torque is the vector product of position vector $\vec{r}$ and force $\vec{F}$.

$$\tau = \vec{r} \times \vec{F}$$

(ii) The force $\vec{F}$ on a charge q moving with velocity $\vec{v}$ in a magnetic field strength $\vec{B}$ is

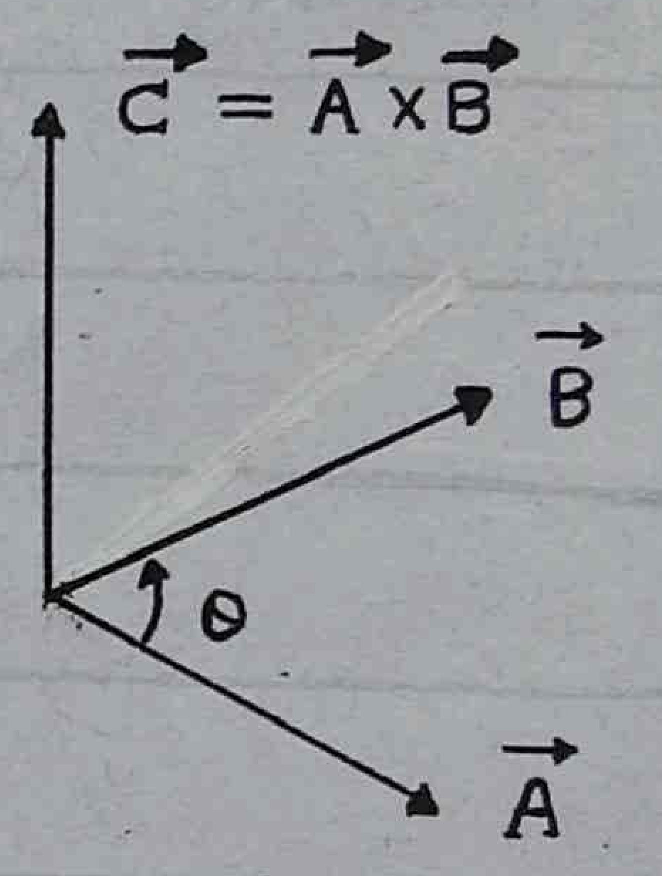
$$F = q(\vec{v} \times \vec{B})$$

(iii) $\vec{v} = \vec{\omega} \times \vec{r}$ (iv) $\vec{a} = \vec{\alpha} \times \vec{r}$

Characteristics of Cross Product

1- Vector Product is not Commutative:

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$



Proof:

$$\vec{A} \times \vec{B} = \vec{C} \quad \text{—— (i)}$$

$$\vec{B} \times \vec{A} = -\vec{C}$$

Mutipling by −1, we have

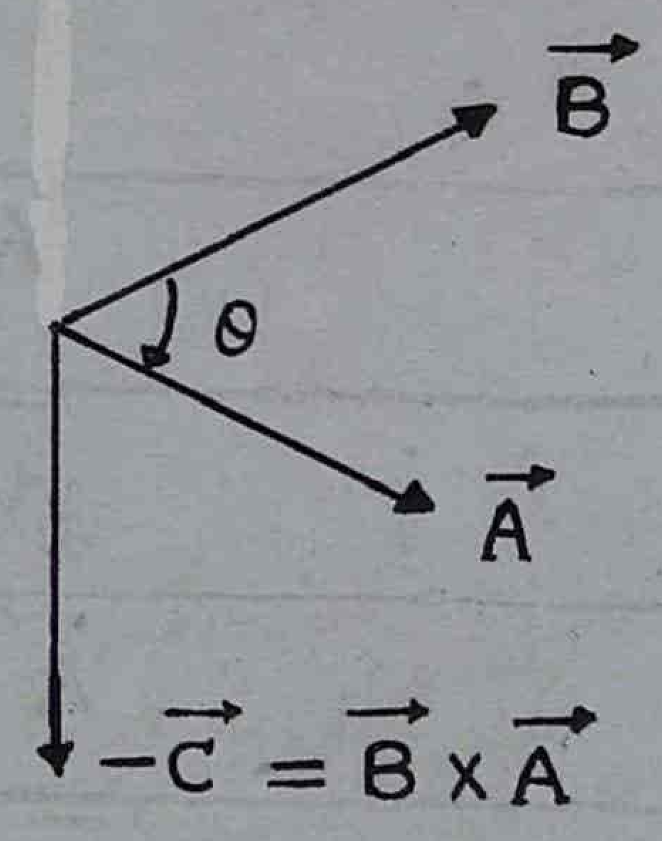
$$-\vec{B} \times \vec{A} = \vec{C} \quad \text{—— (ii)}$$

By equation (i) and (ii)

$$\vec{A} \times \vec{B} = -B \times A$$

So,

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$



2- Cross Product of Perpendicular vectors has maximum value:

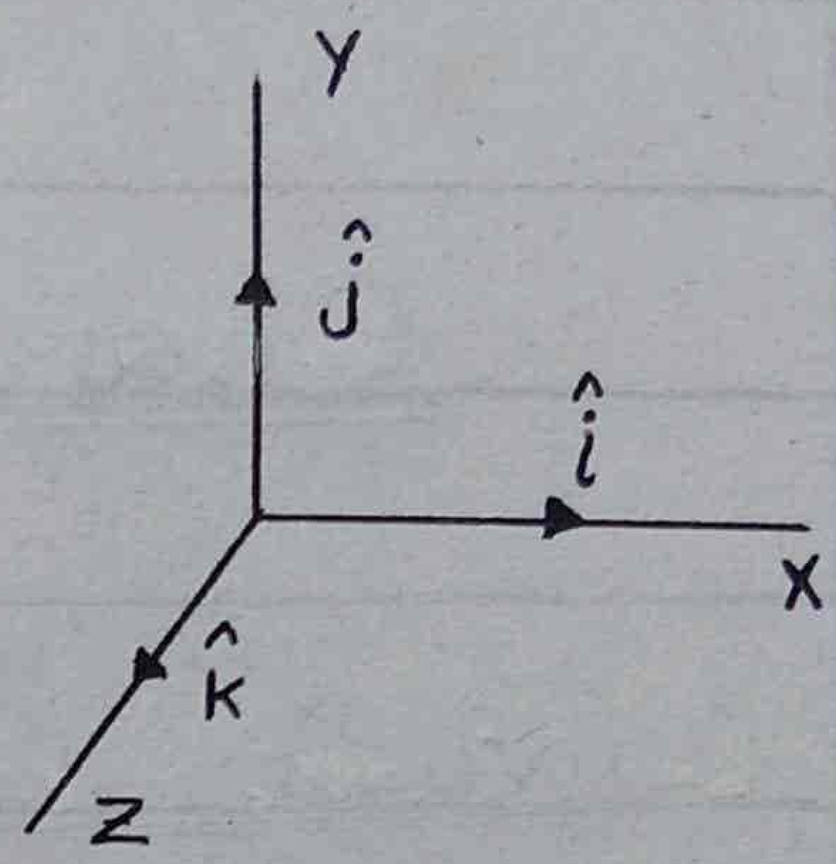
$$\vec{A} \times \vec{B} = AB\sin\theta \cdot \hat{n}$$

$$\vec{A} \times \vec{B} = AB\sin 90^\circ \cdot \hat{n}$$

$$\vec{A} \times \vec{B} = AB(1) \cdot \hat{n}$$

$$\vec{A} \times \vec{B} = AB \cdot \hat{n}$$ (Maximum value)

For unit vectors:



$$\hat{i} \times \hat{j} = (1)(1)\sin 90^\circ \hat{k}$$

$$\hat{i} \times \hat{j} = (1)(1)(1) \cdot \hat{k}$$

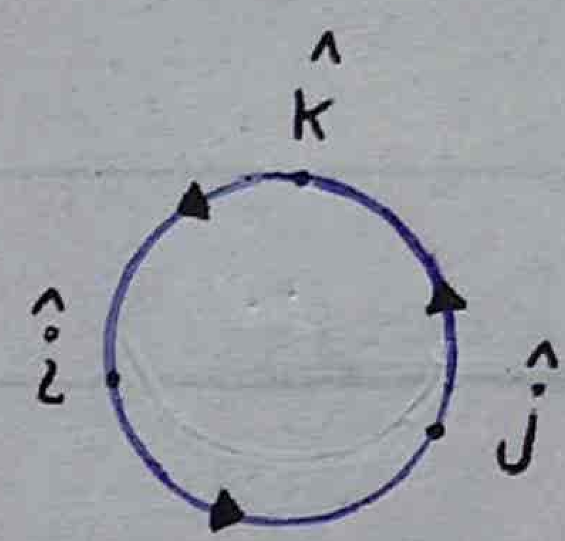
$$\hat{i} \times \hat{j} = \hat{k}$$

Similarly

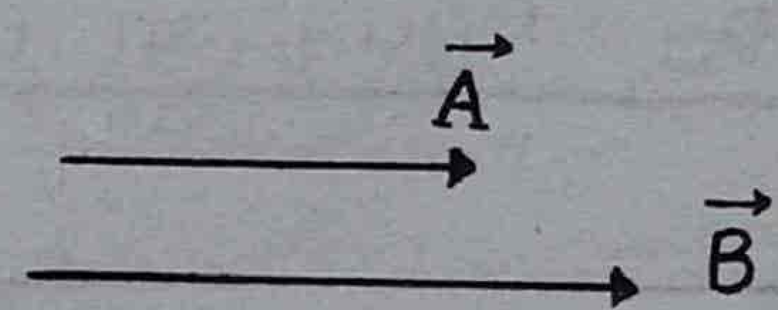
$\hat{j} \times \hat{k} = \hat{i}$; $\hat{j} \times \hat{i} = -\hat{k}$

$\hat{k} \times \hat{i} = \hat{j}$; $\hat{k} \times \hat{j} = -\hat{i}$

$\hat{i} \times \hat{k} = -\hat{j}$



3- Vector Product of Parallel Vectors is a Null Vector:

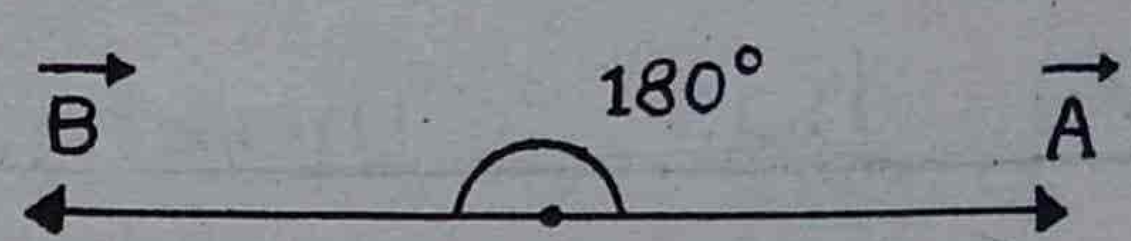


$$\vec{A} \times \vec{B} = AB\sin\theta \cdot \hat{n}$$

$$\vec{A} \times \vec{B} = AB\sin 0^\circ \cdot \hat{n}$$

$$\vec{A} \times \vec{B} = AB(0) \cdot \hat{n}$$

$$\vec{A} \times \vec{B} = 0$$



$$\vec{A} \times \vec{B} = AB\sin 180^\circ \cdot \hat{n}$$

$$\vec{A} \times \vec{B} = AB(0) \cdot \hat{n}$$

$$\vec{A} \times \vec{B} = 0$$

For unit vectors:

$$\hat{i} \times \hat{i} = (1)(1)\sin\theta \cdot \hat{n}$$

$$\hat{i} \times \hat{i} = (1)(1)\sin 0^\circ \cdot \hat{n}$$

$$\hat{i} \times \hat{i} = (1)(1)(0) \cdot \hat{n}$$

$$\hat{i} \times \hat{i} = 0$$

$$\hat{j} \times \hat{j} = 0 \quad ; \quad \hat{k} \times \hat{k} = 0$$

$$\boxed{\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0}$$

4- Vector Product in terms of Rectangular Components:

$$\vec{A} \times \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$\begin{aligned}\vec{A} \times \vec{B} &= A_xB_x(\hat{i} \times \hat{i}) + A_xB_y(\hat{i} \times \hat{j}) + A_xB_z(\hat{i} \times \hat{k}) \\ &+ A_yB_x(\hat{j} \times \hat{i}) + A_yB_y(\hat{j} \times \hat{j}) + A_yB_z(\hat{j} \times \hat{k}) \\ &+ A_zB_x(\hat{k} \times \hat{i}) + A_zB_y(\hat{k} \times \hat{j}) + A_zB_z(\hat{k} \times \hat{k})\end{aligned}$$

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$\hat{i} \times \hat{j} = \hat{k} \qquad \hat{j} \times \hat{i} = -\hat{k}$$

$$\hat{j} \times \hat{k} = \hat{i} \qquad \hat{k} \times \hat{j} = -\hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j} \qquad \hat{i} \times \hat{k} = -\hat{j}$$

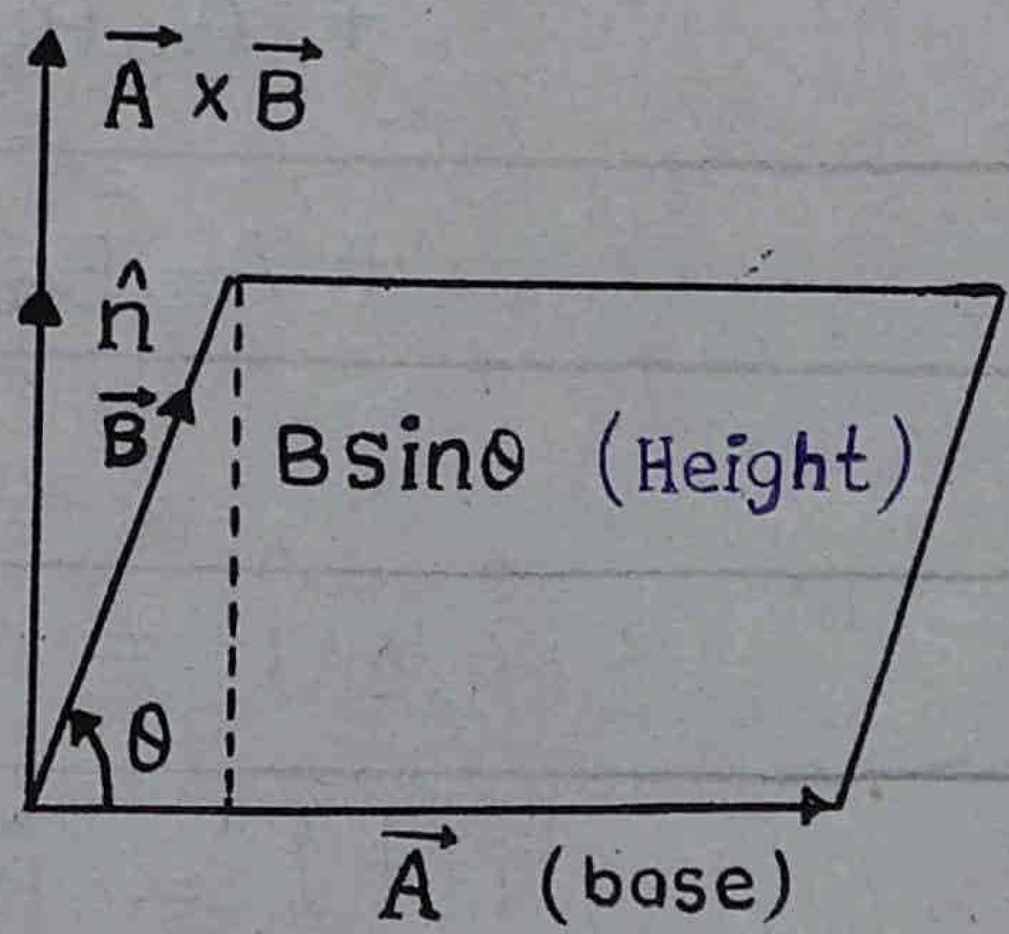
$$\vec{A} \times \vec{B} = A_xB_x(0) + A_xB_y(\hat{k}) + A_xB_z(-\hat{j})$$
$$+ A_yB_x(-\hat{k}) + A_yB_y(0) + A_yB_z(\hat{i})$$
$$+ A_zB_x(\hat{j}) + A_zB_y(-\hat{i}) + A_zB_z(0)$$

$$\boxed{\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}}$$

This result in "Determinant Form" is

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

5- Magnitude of $\vec{A} \times \vec{B}$ is equal to the area of the parallelogram Formed with two adjacent sides $\vec{A}$ and $\vec{B}$.



$$\vec{A} \times \vec{B} = AB\sin\theta \, \hat{n}$$
$$|\vec{A} \times \vec{B}| = AB\sin\theta$$
$$|\vec{A} \times \vec{B}| = A(B\sin\theta)$$
$$|\vec{A} \times \vec{B}| = (\text{base})(\text{height})$$

$|\vec{A} \times \vec{B}|$ = Area of the parallelogram with sides $\vec{A}$ and $\vec{B}$.

Torque or Moment of Force

1- "The product of force and moment arm is called torque."

$$\boxed{\tau = F\,l}$$

Moment Arm l:

"Perpendicular distance between the line of action of force and the axis of rotation is called moment arm."

Note that

When line of action of force posses through the axis of rototion, moment arm is zero ($l = 0$), the torque is zero.

$$\tau = F\,l = F(0) = 0$$

2- "Torque is the vector product of position vector $\vec{r}$ and force $\vec{F}$."

$$\boxed{\tau = \vec{r} \times \vec{F}}$$

$$\tau = rF\sin\theta \cdot \hat{n}$$

Torque is a vector quantity. Its direction is perpendicular to both $\vec{r}$ and $\vec{F}$ and is found by the right hand rule of vector product.

Unit:

SI unit of torque is Nm.

$\tau = Fl$
$\tau = mal$
$\tau = kg \cdot \frac{m}{s^2} \cdot m$
$\tau = kgm^2s^{-2}$
$\tau = [ML^2T^{-2}]$

Dimensions:

$$[ML^2T^{-2}]$$

Torque on a Rigid Body:

Consider a force $\vec{F}$ acts on a rigid body at a point P as shown in fig (a).

$\vec{r}$ = position vector of point P relative to the point of rotation O.

Resolve $\vec{F}$ into rectangular components:

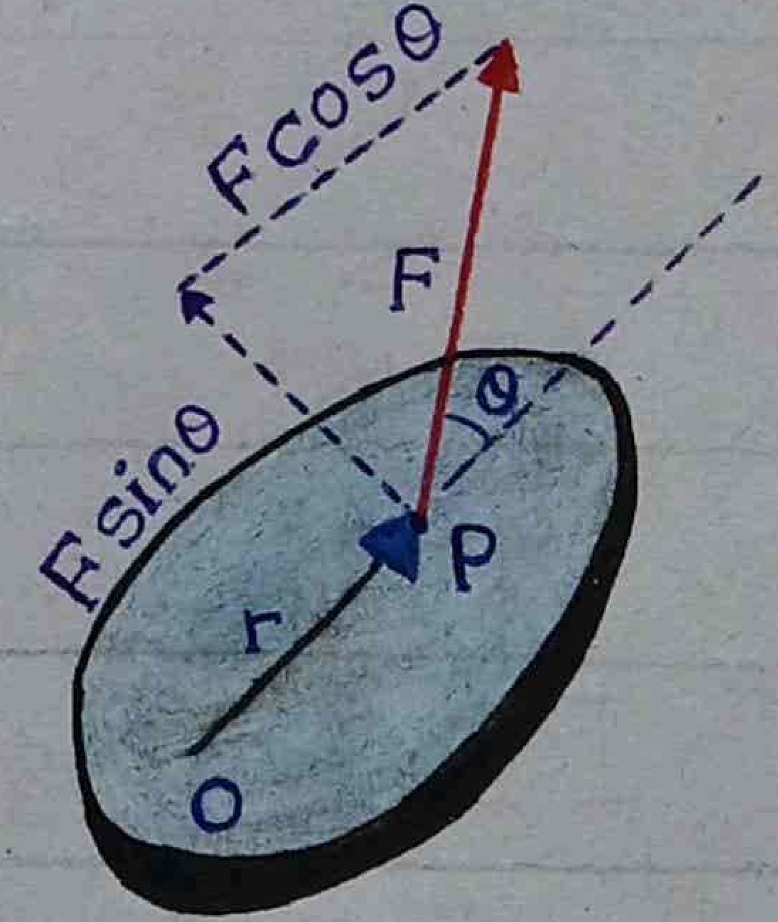


Fig (a)

$F\sin\theta$ = It is perpendicular to $\vec{r}$.

$F\cos\theta$ = It is parallel to $\vec{r}$.

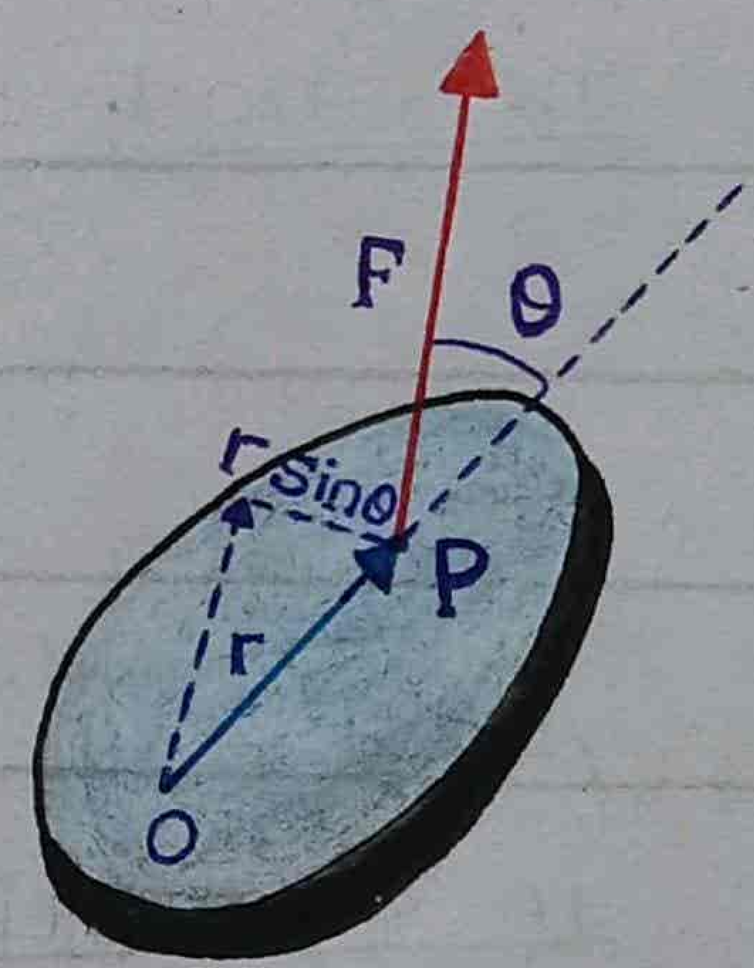
As the line of action of force '$F\cos\theta$'

passes through O, so it produces no torque. Hence, Torque is due to $F\sin\theta$ only.

$$\tau = (F\sin\theta)(r)$$

$$\tau = rF\sin\theta \quad \text{——————} \quad (I)$$

Alternately, the moment arm l is equal to the magnitude of component of $\vec{r}$ perpendicular to the line of action of force. Fig (b).



Fig(b)

$$\tau = (r\sin\theta)(F)$$

$$\tau = rF\sin\theta \quad \text{——————} \quad (II)$$

Equations (I) and (II) can be written as '<u>Vector Product</u>'

$$\tau = \vec{r} \times \vec{F}$$

$$\tau = rF\sin\theta \cdot \hat{n}$$

$\hat{n}$ is a unit vector normal to the plane containing $\vec{r}$ and $\vec{F}$.

<u>Magnitude:</u>

$$\tau = rF\sin\theta$$

Dependance:

Torque depends on

(i) r (ii) F (iii) θ

Note:

i) Torque produces angular acceleration 'α' in the body.

$$\tau \propto \alpha$$

$$\tau = I\alpha$$

ii) Force produces linear acceleration 'a' in the body.

$$F \propto a$$

$$F = ma$$

So, Torque is the counter part of Force in rotational motion.

Note:

i) If $\theta = 0°$ or $180°$

$$\tau = rF\sin 0°$$

$$\tau = rF(0)$$

$$\tau = 0$$

ii) If $\theta = 90°$

$$\tau = rF\sin 90°$$

$$\tau = rF(1)$$

$$\tau = rF$$ Torque is maximum.

Equilibrium

"When a body is at rest or moving with uniform velocity, it is in equilibrium."

Its linear acceleration "a" and angular acceleration "α" are both zero.

$$\vec{a} = 0 \quad ; \quad \vec{\alpha} = 0$$

$\vec{a} = 0$ means net Force zero $\Sigma \vec{F} = 0$

$\vec{\alpha} = 0$ means net torque zero $\Sigma \vec{\tau} = 0$

Examples:

(i) A book lying on a table.

(ii) A car moving with uniform linear velocity.

(iii) A fan rotating with constant angular velocity.

Types of Equilibrium

Static Equilibrium:

"When a body is at rest, it is in static equilibrium."

Dynamic Equilibrium:

"When a body is moving with uniform linear velocity or rotating with constant angular velocity, it is in dynamic equilibrium."

Dynamic Equilibrium has two types:

i) **Translational Equilibrium:**

"When a body is moving with uniform linear velocity, it is in translational equilibrium. It has zero linear acceleration."

$$a = 0$$

ii) **Rotational Equilibrium:**

"When a body is rotating with uniform angular velocity, It is in rotational equilibrium. It has zero angular acceleration."

$$\alpha = 0$$

Conditions of Equilibrium

First condition of Equilibrium:

" Vector sum of all the forces acting on a body must be equal to zero."

$$\boxed{\sum \vec{F} = 0}$$

It means that sum of upward forces is equal to sum of downward forces, sum of rightward forces is equal to sum of leftward forces.

In case of coplanar forces, this condition is expressed in terms of x-component and y-component of the forces.

$\sum \vec{F}_x = 0$ Sum of x-directed forces is zero.

$\sum \vec{F}_y = 0$ Sum of y-directed forces is zero.

Second condition of Equilibrium:

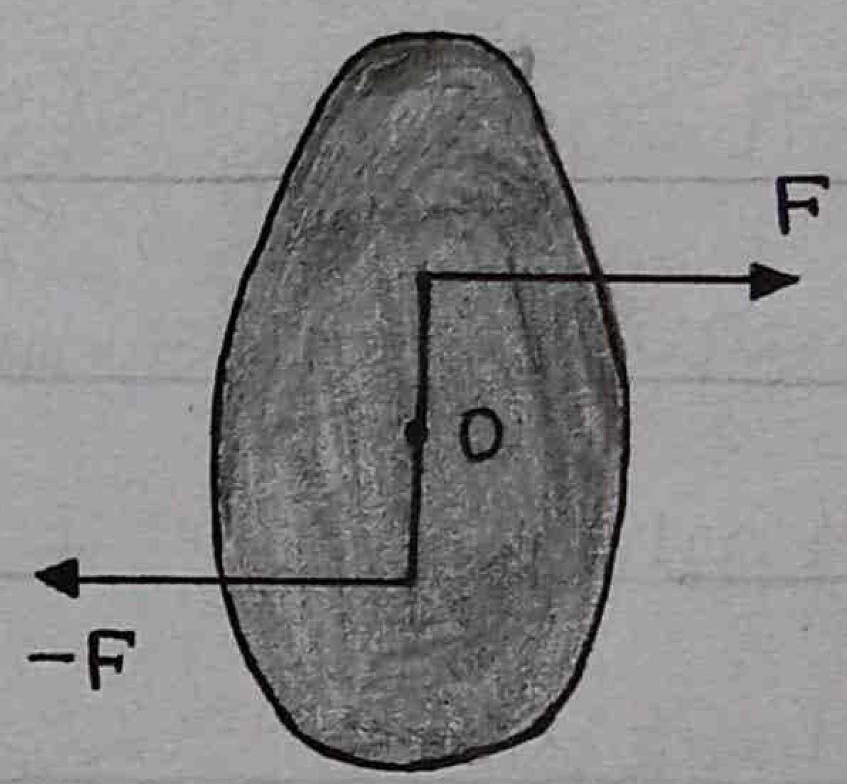
" Vector sum of all the torques acting

on a body about any arbitrary axis must be equal to zero."

$$\boxed{\Sigma \vec{\tau} = 0}$$

Sum of anticlockwise torques = Sum of clockwise torques

Explanation:



Suppose two equal and opposite forces $\vec{F}$ and $-\vec{F}$ acting on a body. Although the first condition is satisfied, yet the body will be rotate in clockwise direction. Hence torque is produced and the body is not in equilibrium. It means that in addition to the first condition of equilibrium, a second condition is also required, which is explained above.

Note:

"For a body to be in complete equilibrium both 1^{st} and 2^{nd} conditions of equilibrium must be satified at the same time."

i) $\Sigma \vec{F} = 0$ ii) $\Sigma \vec{\tau} = 0$

(i)

"When 1^{st} condition is satisfied, there is no linear acceleration $(a = 0)$ and body will be in translational Equilibrium."

(ii)

"When 2^{nd} condition is satisfied, there is no angular acceleration $(\alpha = 0)$ and the body will be in Rotational Equilibrium."

QUESTIONS

2.1 Define the terms (i) unit vector (ii) Position vector and (iii) Components of a vector.

2.2 The vector sum of three vectors gives a zero resultant. What can be the orientation of the vectors?

2.3 Vector **A** lies in the xy plane. For what orientation will both of its rectangular components be negative ? For what orientation will its components have opposite signs?

2.4 If one of the rectangular components of a vector is not zero, can its magnitude be zero ? Explain.

2.5 Can a vector have a component greater than the vector's magnitude?

2.6 Can the magnitude of a vector have a negative value?

2.7 If **A** + **B** = **0**, What can you say about the components of the two vectors?

2.8 Under what circumstances would a vector have components that are equal in magnitude?

2.9 Is it possible to add a vector quantity to a scalar quantity? Explain.

2.10 Can you add zero to a null vector?

2.11 Two vectors have unequal magnitudes. Can their sum be zero? Explain.

2.12 Show that the sum and difference of two perpendicular vectors of equal lengths are also perpendicular and of the same length.

2.13 How would the two vectors of the same magnitude have to be oriented, if they were to be combined to give a resultant equal to a vector of the same magnitude?

2.14 The two vectors to be combined have magnitudes 60 N and 35 N. Pick the correct answer from those given below and tell why is it the only one of the three that is correct.

i) 100 N ii) 70 N iii) 20 N

2.15 Suppose the sides of a closed polygon represent vector arranged head to tail. What is the sum of these vectors?

2.16 Identify the correct answer.

i) Two ships X and Y are travelling in different directions at equal speeds. The actual direction of motion of X is due north but to an observer on Y, the apparent direction of motion of X is north-east. The actual direction of motion of Y as observed from the shore will be

(A) East (B) West (C) south-east (D) south-west

ii) A horizontal force F is applied to a small object P of mass *m* at rest on a smooth plane inclined at an angle θ to the horizontal as shown in Fig. 2.22. The magnitude of the resultant force acting up and along the surface of the plane, on the object is

a) $F\cos\theta - mg\sin\theta$

b) $F\sin\theta - mg\cos\theta$

c) $F\cos\theta + mg\cos\theta$

d) $F\sin\theta + mg\sin\theta$

e) $mg\tan\theta$

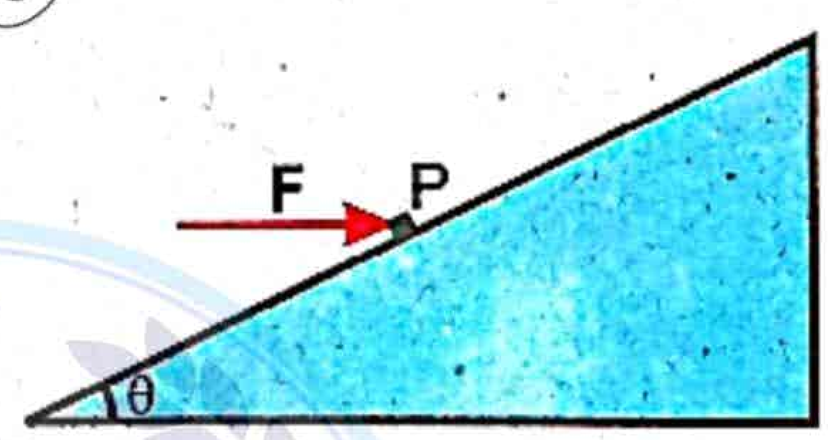


Fig. 2.21

2.17 If all the components of the vectors, $\mathbf{A_1}$ and $\mathbf{A_2}$ were reversed, how would this alter $\mathbf{A_1} \times \mathbf{A_2}$?

2.18 Name the three different conditions that could make $\mathbf{A_1} \times \mathbf{A_2} = 0$.

2.19 Identify true or false statements and explain the reason.

a) A body in equilibrium implies that it is not moving nor rotating.

b) If coplanar forces acting on a body form a closed polygon, then the body is said to be in equilibrium.

2.20 A picture is suspended from a wall by two strings. Show by diagram the configuration of the strings for which the tension in the strings will be minimum.

2.21 Can a body rotate about its centre of gravity under the action of its weight?

Answers of Questions

Q.-1:

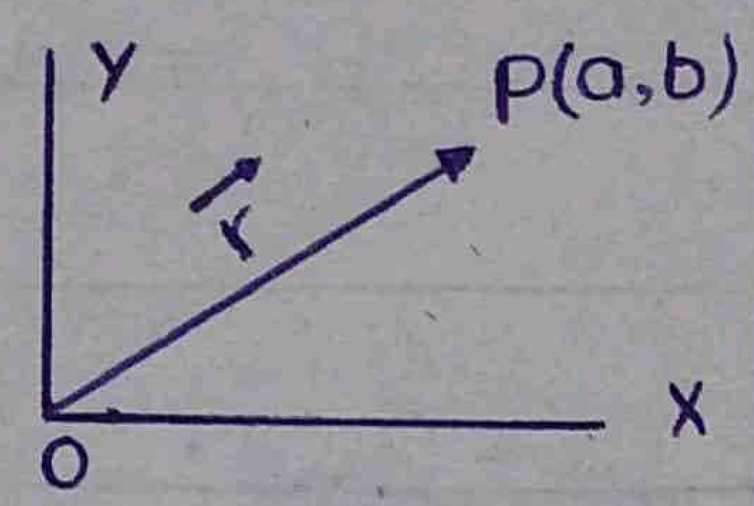
Unit Vector

A vector whose magnitude is one is called a unit vector. It gives the direction of a vector. A unit vector $\hat{A}$ is obtained by dividing the vector $\vec{A}$ with its magnitude A.

$$\vec{A} = A\hat{A} \Rightarrow \boxed{\hat{A} = \frac{\vec{A}}{A}}$$

Position Vector

Position vector gives the location of a particle with respect to origin. The position vector of a point P(a,b) in XY-plane is given by



$$\boxed{\vec{r} = a\hat{i} + b\hat{j}}$$

The position vector of a point P(a,b,c) in space is

$$\boxed{r = a\hat{i} + b\hat{j} + c\hat{k}}$$

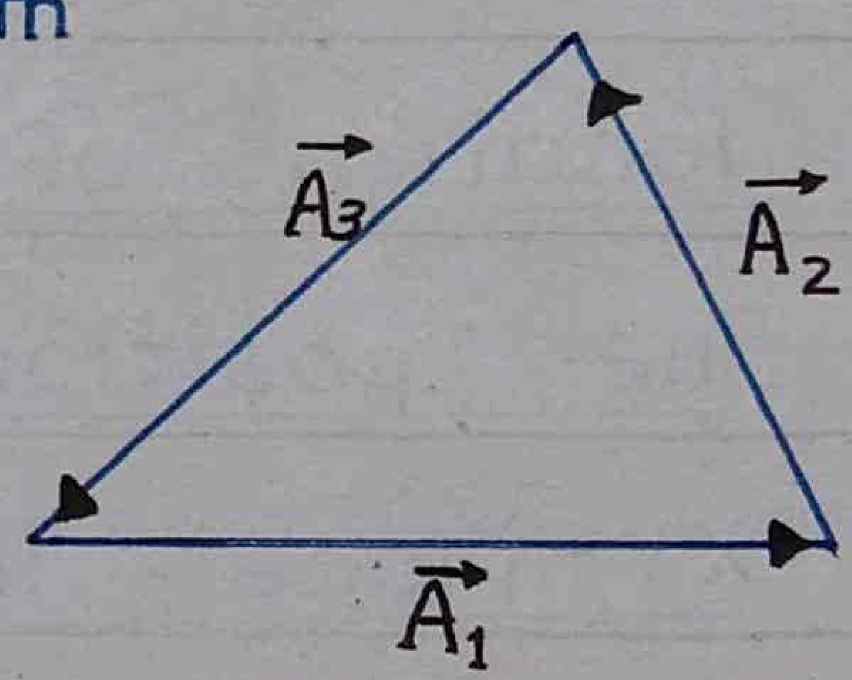
Component of a vector

The component of a vector is its effective value in a given direction. A vector is considered to be the resultant of its component vectors along specified directions.

Q-2.2:

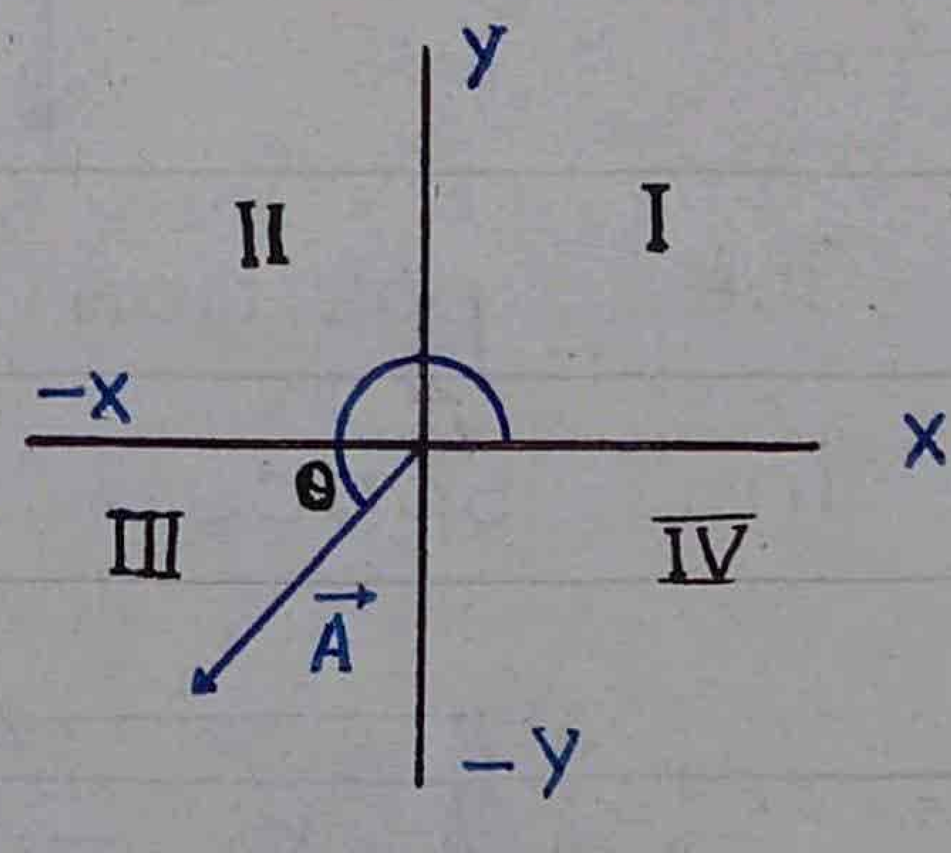
If the three vectors are represented by the three sides of a triangle taken in order, then the vector sum of the vectors will be zero as shown in figure.
Tail of the first vector meets with the head of the last vector.
So resultant is zero.

$$\vec{R} = \vec{A_1} + \vec{A_2} + \vec{A_3} = \vec{O}$$



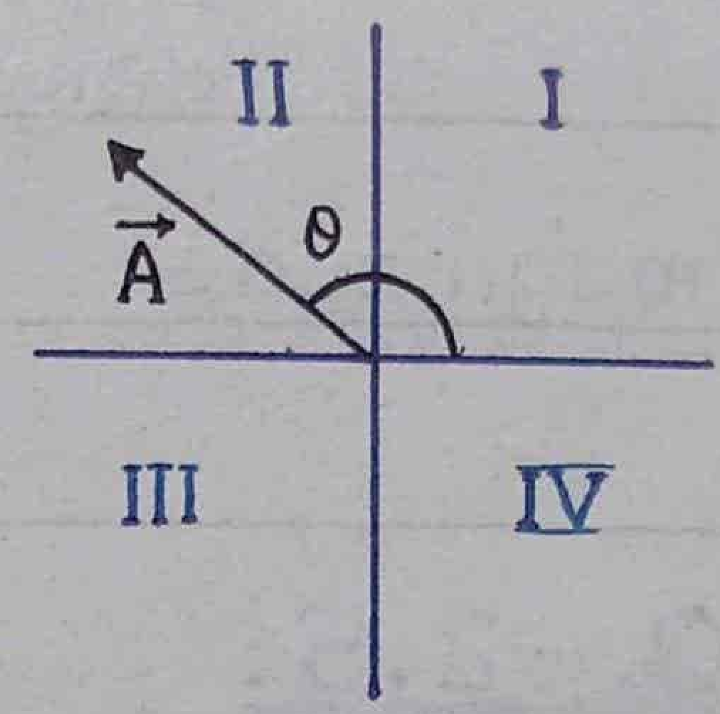
Q-2.3:

i) If a vector lies in III-quadrant of XY-plane, then both of its components are negative. When

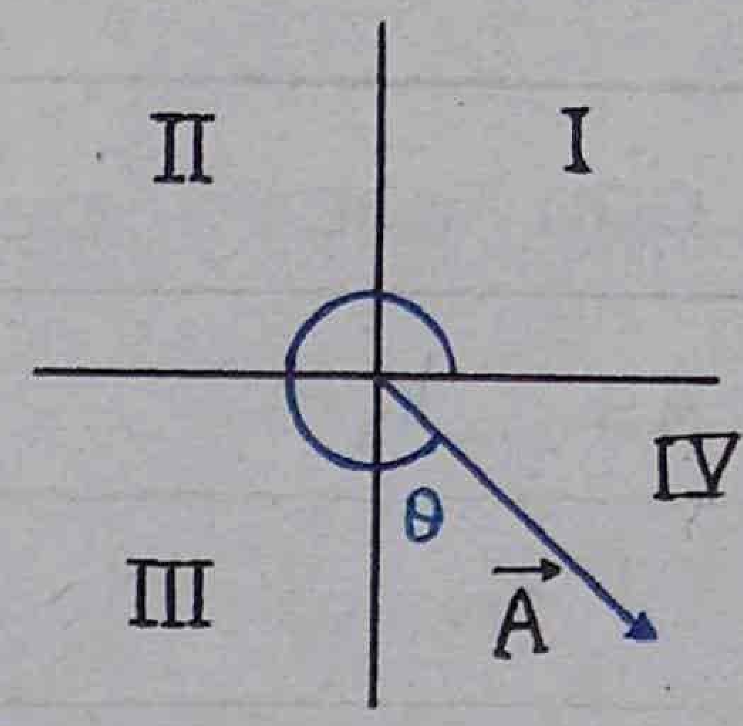


$$180 < \theta < 270$$

ii) When a vector $\vec{A}$ lies in the second quadrant $(90^\circ < \theta < 180^\circ)$, Its x-component is negative while Y-component is positive.



(iii) When a vector $\vec{A}$ lies in IV-quadrant, Its x-component is positive and Y-component is negative.



Q-2.4: No.

If one of the components of a vector is not zero, its magnitude cannot be zero.

The magnitude of a vector $A = A_x\hat{i} + A_y\hat{j}$ is

$$A = \sqrt{A_x^2 + A_y^2}$$

i) If $A_x = 0$ $\quad A = \sqrt{0 + A_y^2} = A_y \neq 0$

ii) If $A_y = 0$ $\quad A = \sqrt{A_x^2 + 0} = A_x \neq 0$

Q-2.5: No.

A vector cannot have a rectangular component greater then the vectors magnitude. The maximum value of the magnitude of a component can be equal to the magnitude of the vector itself.

Q-2.6: No.

The magnitude of a vector can not be negative.

As

$$A = \sqrt{A_x^2 + A_y^2}$$

The square of real quantities always give a positive value. Hence the magnitude of a vector has a positive value.

Q-2.7:

As $\vec{A} + \vec{B} = 0$

$$\vec{A} = -\vec{B}$$

In rectangular components

$$(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) = -(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

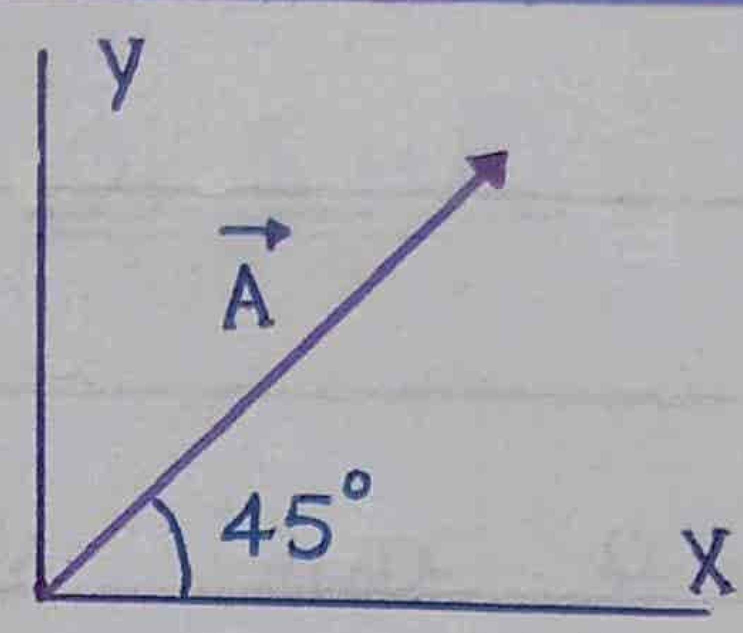
Comparing the co-efficients of $\hat{i}$, $\hat{j}$, and $\hat{k}$

$$A_x = -B_x$$

$$A_y = -B_y$$

$$A_z = -B_z$$

Hence the magnitudes of the components are equal, but they are in opposite in direction.



Q-2.8: In the first quadrant when the vector makes an angle of 45° with x-axis, then its rectangular components are equal in magnitude.

Because

$$A_x = A\cos\theta = A\cos 45^\circ = A \times \frac{1}{\sqrt{2}}$$

$$A_y = A\sin\theta = A\sin 45^\circ = A \times \frac{1}{\sqrt{2}}$$

In second, third and fourth quadrants at angles

$$180 - 45^\circ = 135^\circ$$

$$180 + 45^\circ = 225^\circ$$

$$360 - 45^\circ = 315^\circ$$

respectively, the rectangular components are equal in magnitudes.

Q-2.9:

No.

It is not possible to add a vector quantity to a scalar quantity, because they are different physical quantities. A scalar has magnitude only, whereas a vector has both magnitude and direction.

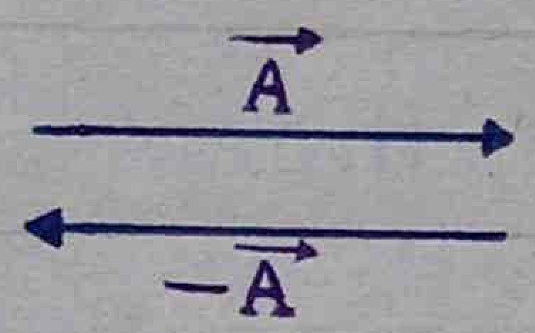
Q-2.10: No.

Zero cannot be added to a null vector. Because, zero is a scalar whereas null vector is a vector. A scalar cannot be added to a vector.

Q-2.11:

No. Their sum cannot be zero.

The sum of two vectors is zero, if they have same magnitudes but opposite direction. In this case, according to the head to tail rule, the head of second vector lies on the tail of the first vector, giving zero resultant.

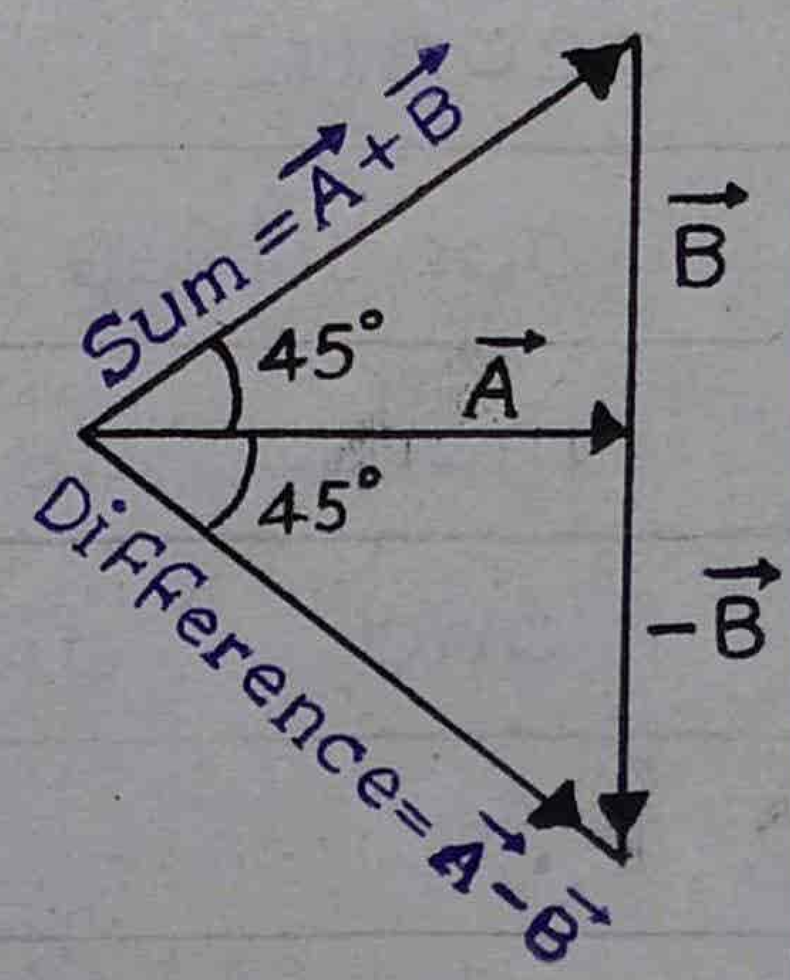


Q-2.12:

Consider $\vec{A}$ and $\vec{B}$ are two perpendicular vectors having same lengths.

$$\text{Sum} = \vec{A} + \vec{B}$$

$$\text{Difference} = \vec{A} - \vec{B}$$



The angle between sum and difference $= 45^\circ + 45^\circ = 90^\circ$.

So, sum $(\vec{A} + \vec{B})$ and

difference $(\vec{A}-\vec{B})$ are perpendicular.

From Figure

$$|\vec{A}+\vec{B}| = \sqrt{A^2+B^2}$$

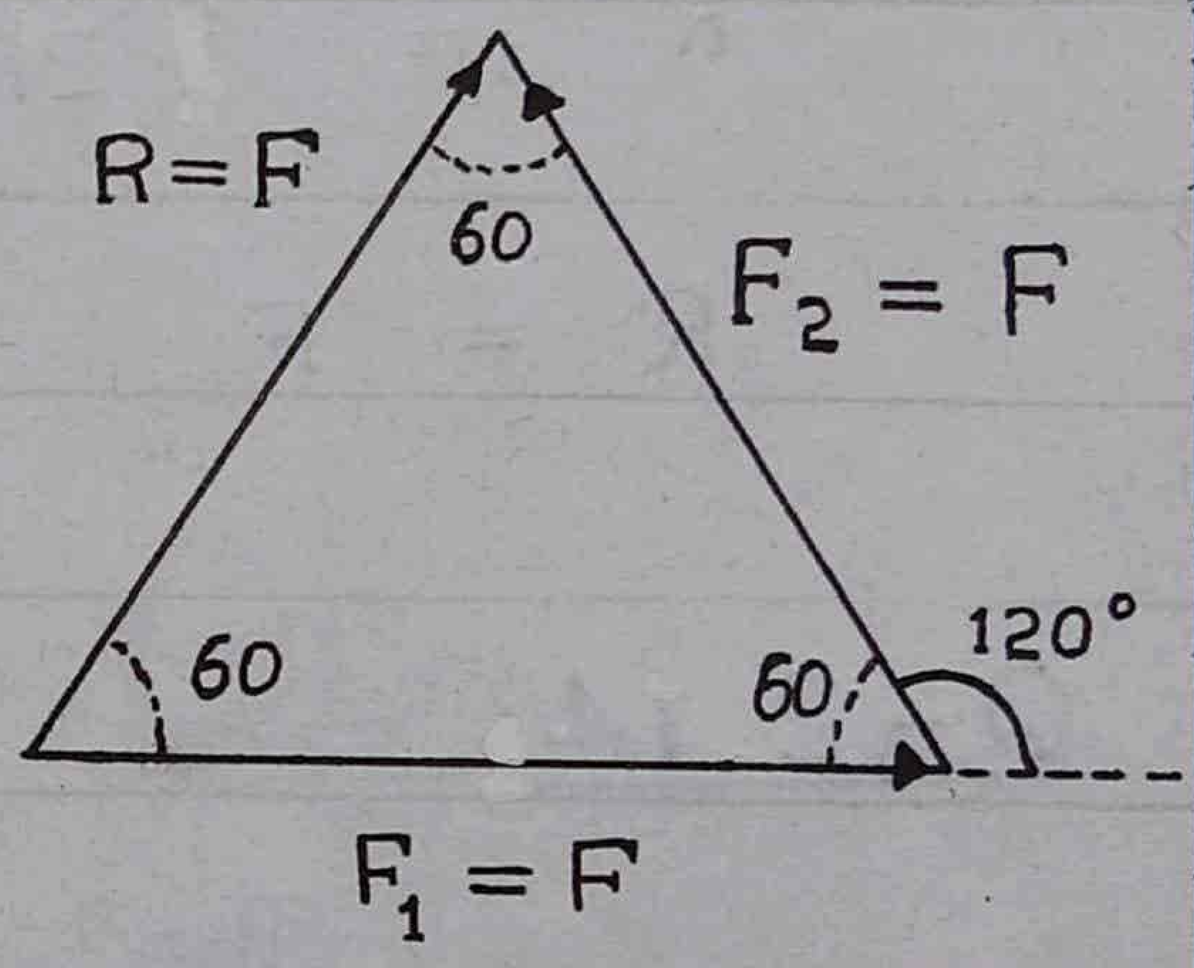
$$|\vec{A}-\vec{B}| = \sqrt{A^2+(-B)^2}$$

$$|\vec{A}-\vec{B}| = \sqrt{A^2+B^2}$$

$$|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$$

Hence sum and difference have same lengths.

Q-2.13:



When the two vectors are represented by the two equal sides of an equilateral triangle, then the magnitude of their resultant vector is also same.

In this case the angle between the two vectors is 120° as shown in fig.

This can be proved as follows. Consider two vectors $\vec{F_1}$ and $\vec{F}$ having same magnitudes.

Their resultant is

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$$

As $F_2 = F_1 = F$

$$R = \sqrt{F^2 + F^2 + 2FF\cos\theta}$$

$$R = \sqrt{2F^2 + 2F^2\cos\theta}$$

IF $\theta = 120^\circ$

$$R = \sqrt{2F^2 + 2F^2\cos 120^\circ} \qquad \therefore \cos 120^\circ = -\frac{1}{2}$$

$$R = \sqrt{2F^2 + \cancel{2}F^2\left(-\frac{1}{\cancel{2}}\right)}$$

$$R = \sqrt{2F^2 - F^2} = \sqrt{F^2}$$

$$R = F$$

Q-2.14:

The correct answer is (ii) 70N

The resultant of two vectors is maximum when they are parallel i.e, in the same direction.

$60 + 35 = 95N$ (Max. value)

$60 - 35 = 25N$ (Min. value)

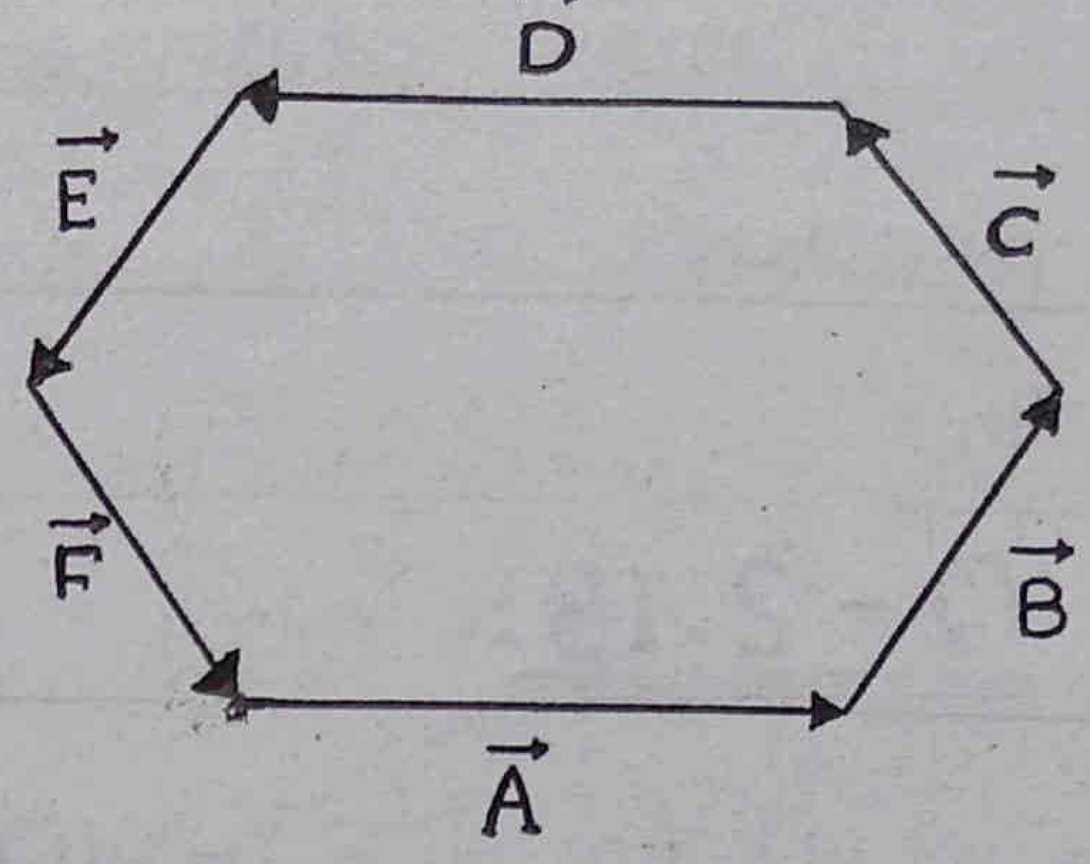
The resultant of two vectors is minimum. When they are in opposite direction.

So,

the resultant 100N and 20N are not possible.

Q-2.15:

The sum of these vectors is zero.



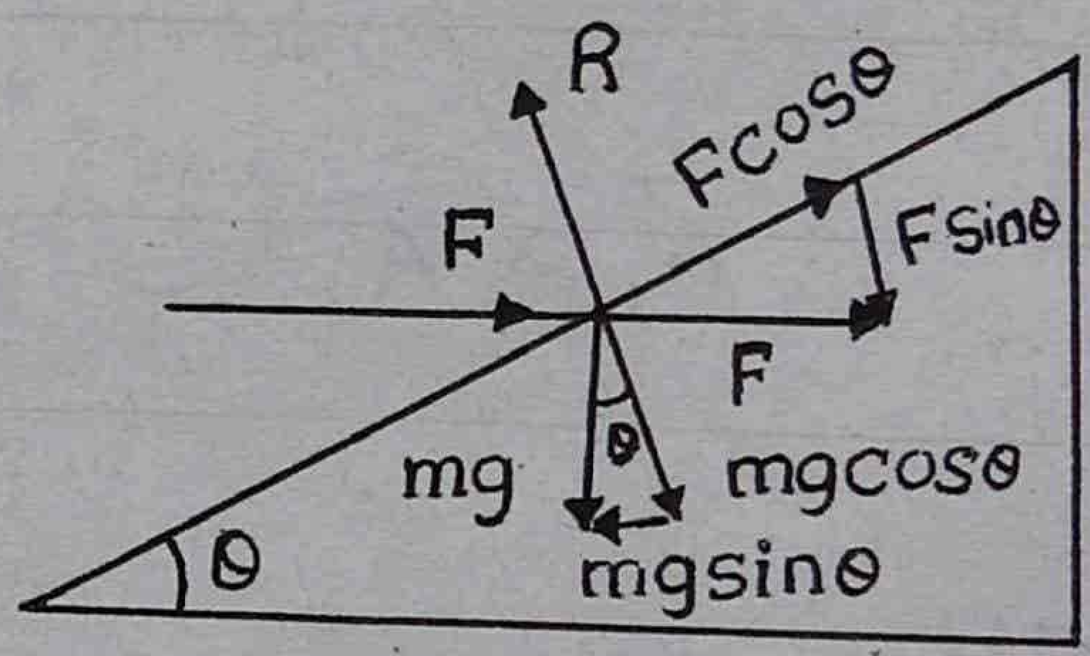
This is so, because the head of the last vector lies at the tail of the first vector.

According to the head to tail rule.

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E} + \vec{F} = \vec{0}$$

Q-2.16:

(ii) Resolve the weight 'mg' and $\vec{F}$ into rectangular components. The magnitude of the resultant force acting up and along the surface of the plane is



$$= F\cos\theta - mg\sin\theta$$

Q-2.17:

When components of vectors $\vec{A}_1$ and $\vec{A}_2$ are reversed, the given vectors become $-\vec{A}_1$ and $-\vec{A}_2$.

$$(-\vec{A}_1) \times (-\vec{A}_2) = \vec{A}_1 \times \vec{A}_2$$

So,

no change occurs in the vector product $\vec{A_1} \times \vec{A_2}$.

Q-2.18:

Conditions

(i) If $\vec{A_1}$ is a null vector ; $\vec{A_1} = \vec{0}$

$$\vec{A_1} \times \vec{A_2} = \vec{0} \times \vec{A_2} = \vec{0}$$

(ii) If $\vec{A_2}$ is a null vector ; $\vec{A_2} = \vec{0}$

$$\vec{A_1} \times \vec{A_2} = \vec{A_1} \times \vec{0} = \vec{0}$$

(iii) If $\vec{A_1}$ and $\vec{A_2}$ are parallel $(\theta = 0°)$ or Anti-parallel $(\theta = 180°)$

$$\vec{A_1} \times \vec{A_2} = A_1 A_2 \sin\theta \, \hat{n} = A_1 A_2 \sin 0° \, \hat{n}$$
$$= A_1 A_2 (0) \hat{n} = 0$$

$$\vec{A_1} \times \vec{A_2} = A_1 A_2 \sin\theta \cdot \hat{n} = A_1 A_2 \sin 180° \cdot \hat{n}$$
$$= A_1 A_2 (0) \hat{n}$$
$$\vec{A_1} \times \vec{A_2} = 0$$

Q-2.19:

(a) False

A body in equilibrium may be at rest, or moving with constant linear velocity $\vec{v}$ or rotating with constant angular velocity $\vec{\omega}$.

(b) False

If coplanar forces acting on a body form a closed polygon, their vector sum is zero.

$$\Sigma \vec{F} = 0$$

First condion of equilibrium is satisfied but the second condition of equilibrium is not satisfied.

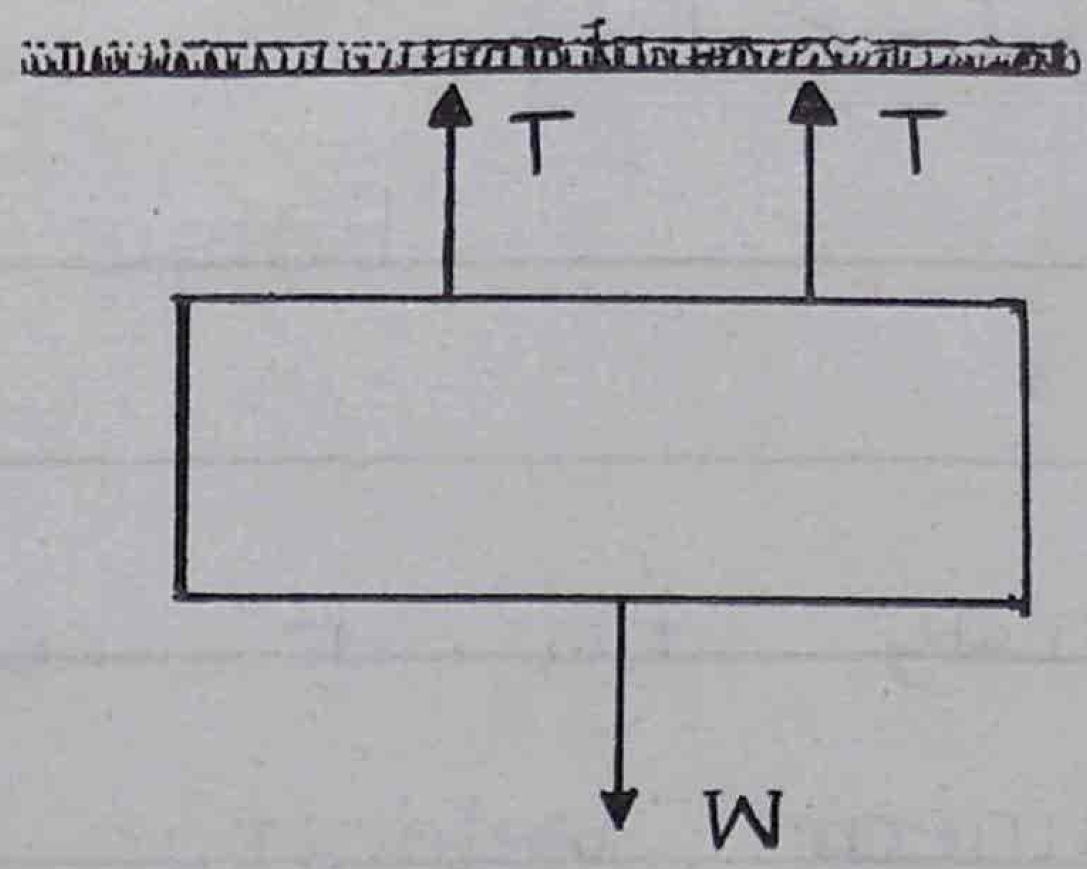
Hence the body is not in equilibrium.

For a body to be in complete equilibrium. both the first and the second conditions of equilibrium must be satisfied at the same time.

Q-2.20:

The strings should be vertical

For minimum tension.



$$\Sigma F_y = 0$$

$$T + T - W = 0$$

$$2T = W$$

$$T = \frac{W}{2} \quad (T \text{ is minimum})$$

In this situation, Tension in each string is equal to half the weight of the picture.

Q-2.21:

No.

As the weight acts on the centre of gravity.

So,

the moment arm

$$\boxed{l = 0}$$

As

$$\tau = Fl$$

$$\tau = F(0)$$

$$\boxed{\tau = 0}$$

No turning effect is produced. The body will not rotate.