MATHEMATICS 1st YEAR

UNIT # 08

MATHEMATICAL INDUCTION & BINOMIAL THEOREM

Muhammad Salman Sherazi
M.Phil (Math)

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Sherazi Mathematics

اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Mathematical Induction

A method of testing formulas, theorems, statements and propositions is called Mathematical Induction Method.
* This method is particularly used in series sum.

Principle of Mathematical Induction

If a proposition $S(n)$ satisfies the following two conditions.

C-1 $S(n)$ is true for $n=1$

C-2 $S(n)$ is true for $n=K$

$\Rightarrow S(n)$ is true for $n=K+1$

Then $S(n)$ is true for all positive integral values of n.

Example 1. Use mathematical induction to prove that $3+6+9+\ldots+3n = \frac{3n(n+1)}{2}$ for any positive integer n.

Solution:–

$$3+6+9+\ldots+3n = \frac{3n(n+1)}{2} \longrightarrow (i)$$

For $n=1$

$$3(1) = \frac{3(1)(1+1)}{2} \Rightarrow 3 = \frac{3(2)}{2}$$

$$\Rightarrow 3 = 3$$

so (i) is true for $n=1$
C-1 is satisfied

suppose (i) is true for $n=K$

i.e., $3+6+9+\ldots+3K = \frac{3K(K+1)}{2} \longrightarrow (ii)$

we shall prove that (i) is true for $n=K+1$ i.e.,

$$3+6+9+\ldots+3K+3(K+1) = \frac{3(K+1)(K+1+1)}{2}$$

$$\Rightarrow 3+6+9+\ldots+3K+3(K+1) = \frac{3(K+1)(K+2)}{2}$$

$$L.H.S = 3+6+9+\ldots+3K+3(K+1)$$

$$= \frac{3K(K+1)}{2} + 3(K+1) \quad \text{by (ii)}$$

$$= \frac{3K(K+1)+2\times3(K+1)}{2}$$

$$= \frac{3(K+1)(K+2)}{2} = R.H.S$$

$\Rightarrow$ (i) is true for $n=K+1$. C-2 is satisfied for each +ive integer n.

Example 2. Use mathematical Induction to prove that for any positive integer n,

$$1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution:–

$$1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6} \longrightarrow (i)$$

For $n=1$

$$1^2 = \frac{1(1+1)(2(1)+1)}{6} \Rightarrow 1 = \frac{2(3)}{6}$$

$$\Rightarrow 1 = 1$$

so (i) is true for $n=1$, C-1 is satisfied. suppose (i) is true for $n=K$ i.e.,

$$1^2+2^2+3^2+\ldots+K^2 = \frac{K(K+1)(2K+1)}{6} \longrightarrow (ii)$$

we shall prove that (i) is true for $n=K+1$ i.e.,

$$1^2+2^2+3^2+\ldots+K^2+(K+1)^2 = \frac{(K+1)(K+1+1)(2(K+1)+1)}{6}$$

$$1^2+2^2+3^2+\ldots+K^2+(K+1)^2 = \frac{(K+1)(K+2)(2K+2+1)}{6}$$

$$1^2+2^2+3^2+\ldots+K^2+(K+1)^2 = \frac{(K+1)(K+2)(2K+3)}{6}$$

$$L.H.S = 1^2+2^2+3^2+\ldots+K^2+(K+1)^2$$

$$= \frac{K(K+1)(2K+1)}{6} + (K+1)^2$$

$$= \frac{K(K+1)(2K+1)+6(K+1)^2}{6}$$

$$= \frac{(K+1)\{K(2K+1)+6(K+1)\}}{6}$$

$$= \frac{(K+1)\{2K^2+K+6K+6\}}{6}$$

$$= \frac{(K+1)\{2K^2+7K+6\}}{6}$$

$$= \frac{(K+1)\{2K^2+4K+3K+6\}}{6}$$

$$= \frac{(K+1)\{2K(K+2)+3(K+2)\}}{6}$$

$$= \frac{(K+1)\ (K+2)(2K+3)}{6} = R.H.S$$

$\rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for each +ive integer n.

Example 3. Show that $\frac{n^3+2n}{3}$ represents an integer $\forall\ n \in N$

Solution:-

Let $S(n) = \frac{n^3+2n}{3} \longrightarrow (i)$

For $n=1$

$$S(1) = \frac{(1)^3+2(1)}{3} = \frac{1+2}{3} = \frac{3}{3} = 1$$

$\rightarrow S(1) = 1$ (Which is an integer)

so (i) is an integer for $n=1$

C-1 is satisfied.

suppose (i) is true for $n=K$

i.e., $S(K) = \frac{K^3+2K}{3}$ is an integer. $\longrightarrow (ii)$

we shall prove that (i) is true for $n=K+1$ i.e.,

$S(K+1) = \frac{(K+1)^3+2(K+1)}{3}$ is an integer

$$= \frac{K^3+1+3K^2+3K+2K+2}{3}$$

$$= \frac{K^3+2K+3K^2+3K+3}{3}$$

$$= \frac{K^3+2K}{3} + \frac{3(K^2+K+1)}{3}$$

$$= \frac{K^3+2K}{3} + (K^2+K+1)$$

$\because \frac{K^3+2K}{3}$ is an integer as we assumed in (ii)

Also $(K^2+K+1) \in Z \ \forall\ K \in Z^+$ (or $K \in N$)

= integer + integer

= integer

$\rightarrow$ (i) is true for all +ive integer n. C-2 is satisfied.

$\rightarrow$ (i) is an integer $\forall\ n \in N$

Example 4. Use mathematical induction to prove that

$$3+3.5+3.5^2+\dots+3.5^n = \frac{3(5^{n+1}-1)}{4}$$

whenever n is non-negative integer.

Solution:-

Let

$$3+3.5+3.5^2+\dots+3.5^n = \frac{3(5^{n+1}-1)}{4} \longrightarrow (i)$$

For $n=0$

$$3.5^0 = \frac{3(5^{0+1}-1)}{4} \rightarrow 3.1 = \frac{3(4)}{4}$$

$\rightarrow 3 = 3$

$\rightarrow$ (i) is true for $n=0$

C-1 is satisfied.

Suppose (i) is true for $n=K$

i.e.,

$$3+3.5+3.5^2+\dots+3.5^K = \frac{3(5^{K+1}-1)}{4} \longrightarrow (ii)$$

We shall prove that (i) is true for $n=K+1$ i.e.,

$$3+3.5+\dots+3.5^K+3.5^{K+1} = \frac{3(5^{(K+1)+1}-1)}{4}$$

$$3+3.5+\dots+3.5^K+3.5^{K+1} = \frac{3(5^{K+2}-1)}{4}$$

$$L.H.S = 3+3.5+3.5^2+\dots+3.5^K+3.5^{K+1}$$

$$= \frac{3(5^{K+1}-1)}{4} + 3.5^{K+1} \quad \text{By (ii)}$$

$$= \frac{3(5^{K+1}-1)+3.5^{K+1}.4}{4}$$

$$= \frac{3[(5^{K+1}-1+5^{K+1}.4)]}{4}$$

$$= \frac{3[(5^{k+1}+4\cdot 5^{k+1})-1]}{4}$$

$$= \frac{3[5^{k+1}(1+4)-1]}{4}$$

$$= \frac{3[5^{k+1}\cdot 5-1]}{4} = \frac{3(5^{k+2}-1)}{4}$$

$$= R\cdot H\cdot S$$

$\Rightarrow$ (i) is true for $n = k+1$

C-2 is satisfied. Hence (i) is true for all non-negative integer.

Principle of Extended Mathematical Induction

Sometimes we want to prove formulas or results which are true for all integer n greater than or equa to some integer i (i-e., $n \geq i$ where $i \neq 1$)

In such cases we check formula $n = i$. This is known as extended mathematical induction

Example 5. Show that

$1+3+5+\cdots\cdots+(2n+5) = (n+3)^2$

for integral values of $n \geq -2$

Solution:-

$$1+3+5+\cdots\cdots+(2n+5) = (n+3)^2 \longrightarrow \text{(i)}$$

For $n = -2$

$2(-2)+5 = (-2+3)^2$

$\Rightarrow -4+5 = (1)^2 \Rightarrow 1 = 1$

$\Rightarrow$ (i) is true for $n = -2$

C-1 is satisfied.

suppose (i) is true for $n = k$ where $k \geq -2$ i-e.,

$$1+3+5+\cdots\cdots+(2k+5) = (k+3)^2 \longrightarrow \text{(ii)}$$

we shall prove that (i) is satisfied for $n = k+1$ i-e.,

$$1+3+5+\cdots\cdots+(2k+5)+(2(k+1)+5) = (k+1+3)^2$$

$$1+3+5+\cdots\cdots+(2k+5)+(2(k+1)+5) = (k+4)^2$$

$L\cdot H\cdot S = 1+3+5+\cdots\cdots+(2k+5)+(2(k+1)+5)$

$= (k+3)^2 + (2k+7)$ by (ii)

$= k^2+9+6k+2k+7$

$= k^2+8k+16 = (k+4)^2 = R\cdot H\cdot S$

$\Rightarrow$ (i) is true for $n = k+1$

C-2 is satisfied.

Hence (i) is true for all integers $n \geq -2$

Example 6. Show that the inequality $4^n > 3^n+4$ is true, for integral values of $n \geq 2$

Solution:-

$$4^n > 3^n+4 \quad \forall\ n \geq 2 \longrightarrow \text{(i)}$$

Put $n = 2$

$\Rightarrow 4^2 > 3^2+4 \Rightarrow 16 > 9+4$

$\Rightarrow 16 > 13$

$\Rightarrow$ (i) is true for $n = 2$

C-1 is satisfied.

suppose (i) is true for $n = k$ i-e.,

$$4^k > 3^k+4 \quad \forall\ k \geq 2 \longrightarrow \text{(ii)}$$

we shall prove that (i) is true for $n = k+1$ i-e.,

$$4^{k+1} > 3^{k+1}+4$$

To prove this multiplying (ii) by 4 on both sides we get

$4\cdot 4^k > 4(3^k+4)$

$\Rightarrow 4^{k+1} > 4\cdot 3^k + 16$

$\Rightarrow 4^{k+1} > (3+1)3^k+16$

$\Rightarrow 4^{k+1} > 3\cdot 3^k+3^k+4+12$

$\Rightarrow 4^{k+1} > 3^{k+1}+4+(3^k+12)$

$\Rightarrow 4^{k+1} > 3^{k+1}+4 \quad (\because 3^k+12 > 0\ \forall\ k \geq 2)$

$\Rightarrow$ (i) is true for $n = k+1$

C-2 is satisfied. Hence (i) is true for integers $n \geq 2$

Exercise 8.1

Use mathematical induction to prove the formula for every every integer n.

Q1. $1+5+9+\dots+(4n-3)=n(2n-1)$

Solution:-

$1+5+9+\dots\dots+(4n-3)=n(2n-1) \longrightarrow (i)$

For $n=1$

$4(1)-3=1(2(1)-1)$

$\Rightarrow 1=1$

$\Rightarrow$ (i) is true for $n=1$

C-1 is satisfied. Suppose (i) is true for $n=K$ i-e.,

$1+5+9+\dots+(4K-3)=K(2K-1) \longrightarrow (ii)$

we shall prove that (i) is true for $n=K+1$ i-e.,

$1+5+9+\dots+(4K-3)+(4(K+1)-3)=(K+1)[2(K+1)-1]$

$1+5+9+\dots+(4K-3)+(4K+1)=(K+1)(2K+1)$

$L.H.S = 1+5+9+\dots+(4K-3)+(4K+1)$

$= K(2K-1)+(4K+1)$ By (ii)

$= 2K^2-K+4K+1$

$= 2K^2+3K+1$

$= 2K^2+2K+K+1$

$= 2K(K+1)+1(K+1)$

$= (K+1)(2K+1) = R.H.S$

$\Rightarrow$ (i) is true for $n=K+1$

C-2 is satisfied. Hence (i) is true for all integers n.

Q2. $1+3+5+\dots\dots+(2n-1)=n^2$

Solution:-

$1+3+5+\dots\dots+(2n-1)=n^2 \longrightarrow (i)$

For $n=1$

$1=(1)^2 \Rightarrow 1=1$

$\Rightarrow$ (i) is true for $n=K$ i-e.,

$1+3+5+\dots\dots+(2K-1)=K^2 \longrightarrow (ii)$

we shall prove that (i) is true for $n=K+1$ i-e.,

$1+3+5+\dots+(2K-1)+(2(K+1)-1)=(K+1)^2$

$L.H.S = 1+3+5+\dots+(2K-1)+2(K+1)-1$

$= K^2+2K+2-1$ By (ii)

$= K^2+2K+1$

$= (K+1)^2 = R.H.S$

$\Rightarrow$ (i) is true for $n=K+1$.

C-2 is satisfied. Hence (i) is true for all integers n.

Q3. $1+4+7+\dots+(3n-2)=\dfrac{n(3n-1)}{2}$

Solution:-

$1+4+7+\dots+(3n-2)=\dfrac{n(3n-1)}{2} \longrightarrow (i)$

For $n=1$

$1=\dfrac{1(3(1)-1)}{2} \Rightarrow 1=1$

$\Rightarrow$ (i) is true for $n=1$. C-1 is satisfied. Suppose (i) is true for $n=K$ i-e.,

$1+4+7+\dots\dots+(3K-2)=\dfrac{K(3K-1)}{2} \longrightarrow (ii)$

we shall prove that (i) is true for $n=K+1$

$1+4+7+\dots+(3K-2)+(3(K+1)-2)=\dfrac{(K+1)(3(K+1)-1)}{2}$

$1+4+7+\dots+(3K-2)+(3K+1)=\dfrac{(K+1)(3K+2)}{2}$

$L.H.S = 1+4+7+\dots+(3K-2)+(3K+1)$

$= \dfrac{K(3K-1)}{2}+3K+1$

$= \dfrac{K(3K-1)+6K+2}{2}$

$= \dfrac{3K^2-K+6K+2}{2}$

$= \dfrac{3K^2+5K+2}{2} = \dfrac{3K^2+3K+2K+2}{2}$

$= \dfrac{3K(K+1)+2(K+1)}{2} = \dfrac{(K+1)(3K+2)}{2}$

$= R.H.S$

$\Rightarrow$ (i) is true for $n=K+1$

C-2 is satisfied. Hence (i) is true for all integers n.

Q4. $1+2+4+\dots+2^{n-1} = 2^n - 1$

Solution:-

$1+2+4+\dots+2^{n-1} = 2^n-1 \longrightarrow (i)$

For $n=1$

$1 = 2^1-1 \rightarrow 1=1$

$\rightarrow$ (i) is true for $n=1$, C-1 is satisfied. Suppose (i) is true for $n=K$ i-e.,

$1+2+4+\dots+2^{K-1} = 2^K - 1 \longrightarrow (ii)$

we shall prove that (i) is true for $n=K+1$ i-e.,

$1+2+4+\dots+2^{K-1}+2^{(K+1)-1} = 2^{K+1}-1$

$\rightarrow 1+2+4+\dots+2^{K-1}+2^K = 2^{K+1}-1$

$L.H.S = 1+2+4+\dots+2^{K-1}+2^K$

$= 2^K-1+2^K$ By (ii)

$= 2.2^K-1$

$= 2^{K+1}-1 = R.H.S$

$\rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for all integers n.

Q5. $1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^{n-1}} = 2\left[1-\frac{1}{2^n}\right]$

Solution:- Let

$1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^{n-1}} = 2\left[1-\frac{1}{2^n}\right] \longrightarrow (i)$

For $n=1$

$\frac{1}{2^{1-1}} = 2\left[1-\frac{1}{2^1}\right] \rightarrow \frac{1}{2^0} = 2\left(\frac{2-1}{2}\right)$

$\rightarrow 1 = 2\left(\frac{1}{2}\right) \rightarrow 1=1$

$\rightarrow$ (i) is true for $n=1$ C-1 is satisfied. Suppose (i) is true for $n=K$ i-e.,

$1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^{K-1}} = 2\left[1-\frac{1}{2^K}\right] \longrightarrow (ii)$

we shall prove that (i) is true for $n=K+1$ i-e.,

$1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^{K-1}}+\frac{1}{2^{K+1-1}} = 2\left[1-\frac{1}{2^{K+1}}\right]$

$1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^{K-1}}+\frac{1}{2^K} = 2\left[\frac{2^{K+1}-1}{2^{K+1}}\right]$

$L.H.S = 1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^{K-1}}+\frac{1}{2^K}$

$= 2\left[1-\frac{1}{2^K}\right]+\frac{1}{2^K}$ By (ii)

$= 2\left[\frac{2^K-1}{2^K}\right]+\frac{1}{2^K}$

$= \frac{2^{K+1}-2+1}{2^K} = \frac{2^{K+1}-1}{2^K}$

$= \frac{2(2^{K+1}-1)}{2.2^K}$ ($\div$ and $\times$ by 2)

$= 2\left[\frac{2^{K+1}-1}{2^{K+1}}\right] = R.H.S$

$\rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for all n.

Q6. $2+4+6+\dots+2n = n(n+1)$

Solution:-

$2+4+6+\dots+2n = n(n+1) \longrightarrow (i)$

For $n=1$

$2(1) = 1(1+1) \rightarrow 2=2$

$\rightarrow$ (i) is true for $n=1$, C-1 is satisfied. Suppose (i) is true for $n=K$ i-e.,

$2+4+6+\dots+2K = K(K+1) \longrightarrow (ii)$

we shall prove that (i) is true for $n=K+1$ i-e.,

$2+4+6+\dots+2K+2(K+1) = (K+1)(K+1+1)$

$2+4+6+\dots+2K+2(K+1) = (K+1)(K+2)$

$L.H.S = 2+4+6+\dots+2K+2(K+1)$

$= K(K+1)+2(K+1)$ By (ii)

$= (K+1)(K+2) = R.H.S$

$\rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for all integers n.

Q7. $2+6+18+\dots\dots+2\times3^{n-1} = 3^n-1$

Solution:-

$2+6+18+\dots\dots+2\times3^{n-1} = 3^n-1 \longrightarrow (i)$

For $n=1$

$2\times3^{1-1} = 3^1-1 \Rightarrow 2=2$

$\Rightarrow$ (i) is true for $n=1$, C-1 is satisfied. Suppose (i) is true for $n=k$ i-e.,

$2+6+18+\dots+2\times3^{k-1} = 3^k-1 \longrightarrow (ii)$

we shall prove that (i) is true for $n=k+1$ i-e.,

$2+6+18+\dots\dots+2\times3^{k-1}+2\times3^{k+1-1} = 3^{k+1}-1$

$2+6+18+\dots\dots+2\times3^{k-1}+2\times3^{k} = 3^{k+1}-1$

$L.H.S = 2+6+18+\dots\dots+2\times3^{k-1}+2\times3^{k}$

$= 3^k-1+2\times3^k$ By (ii)

$= 3^k+2\cdot3^k-1$

$= 3^k(1+2)-1 = 3^k\cdot3-1$

$= 3^{k+1}-1 = R.H.S$

$\Rightarrow$ (i) is true for $n=k+1$, C-2 is satisfied. Hence (i) is true for all integer n.

Q8. $1\times3+2\times5+3\times7+\dots\dots+n\times(2n+1) = \dfrac{n(n+1)(4n+5)}{6}$

Solution:-

$1\times3+2\times5+\dots\dots+n\times(2n+1) = \dfrac{n(n+1)(4n+5)}{6} \longrightarrow (i)$

For $n=1$

$1\times(2(1)+1) = \dfrac{1(1+1)(4(1)+5)}{6}$

$\Rightarrow 3 = \dfrac{2(9)}{6} \Rightarrow 3=3$

$\Rightarrow$ (i) is true for $n=k$, C-1 is satisfied. Suppose (i) is true for $n=k$ i-e.,

$1\times3+2\times5+\dots\dots+k\times(2k+1) = \dfrac{k(k+1)(4k+5)}{6} \longrightarrow (ii)$

we shall prove that (i) is true for $n=k+1$ i-e.,

$1\times3+2\times5+\dots\dots+k(2k+1)+(k+1)(2(k+1)+1) = \dfrac{(k+1)(k+1+1)(4(k+1)+5)}{6}$

$1\times3+2\times5+\dots\dots+k(2k+1)+(k+1)(2k+3) = \dfrac{(k+1)(k+2)(4k+9)}{6}$

$L.H.S = 1\times3+2\times5+\dots\dots+k(2k+1)+(k+1)(2k+3)$

$= \dfrac{k(k+1)(4k+5)}{6}+(k+1)(2k+3)$ By (ii)

$= \dfrac{k(k+1)(4k+5)+6(k+1)(2k+3)}{6}$

$= \dfrac{(k+1)\{4k^2+5k+12k+18\}}{6}$

$= \dfrac{(k+1)\{4k^2+17k+18\}}{6}$

$= \dfrac{(k+1)\{4k^2+8k+9k+18\}}{6}$

$= \dfrac{(k+1)\{4k(k+2)+9(k+2)\}}{6}$

$= \dfrac{(k+1)(k+2)(4k+9)}{6} = R.H.S$

$\Rightarrow$ (i) is true for $n=k+1$, C-2 is satisfied. Hence (i) is true for all integers 'n'.

Q9. $1\times2+2\times3+3\times4+\dots\dots+n\times(n+1) = \dfrac{n(n+1)(n+2)}{3}$

Solution:-

$1\times2+2\times3+\dots\dots+n\times(n+1) = \dfrac{n(n+1)(n+2)}{3} \longrightarrow (i)$

For $n=1$

$1\times(1+1) = \dfrac{1(1+1)(1+2)}{3}$

$\Rightarrow 2=2$

$\Rightarrow$ (i) is true for $n=1$, C-1 is satisfied. Suppose (i) is true for $n=k$ i-e.,

$1\times2+2\times3+\dots\dots+k\times(k+1) = \dfrac{k(k+1)(k+2)}{3} \longrightarrow (ii)$

we shall prove that (i) is true for $n=k+1$ i-e.,

$1\times2+2\times3+\dots\dots+k\times(k+1)+(k+1)\times(k+1+1) = \dfrac{(k+1)(k+1+1)(k+1+2)}{3}$

$$1\times2+2\times3+\dots+K\times(K+1)+(K+1)\times(K+2) = \frac{(K+1)(K+2)(K+3)}{3}$$

$$L.H.S = 1\times2+2\times3+\dots+K\times(K+1)+(K+1)\times(K+2)$$

$$= \frac{K(K+1)(K+2)}{3}+(K+1)\times(K+2) \quad \text{By (ii)}$$

$$= \frac{K(K+1)(K+2)+3(K+1)(K+2)}{3}$$

$$= \frac{(K+1)(K+2)\{K+3\}}{3}$$

$$= \frac{(K+1)(K+2)(K+3)}{3} = R.H.S$$

$\rightarrow$ (i) is true for $n=K+1$

C-2 is satisfied. Hence (i) is true for all integers n.

Q10. $1\times2+3\times4+5\times6+\dots+(2n-1)\times2n = \frac{n(n+1)(4n-1)}{3}$

Solution:-

$$1\times2+3\times4+\dots+(2n-1)\times2n = \frac{n(n+1)(4n-1)}{3} \longrightarrow (i)$$

For $n=1$

$$1\times2 = \frac{1(1+1)(4(1)-1)}{3} \rightarrow 2 = \frac{2(3)}{3}$$

$\rightarrow 2=2 \rightarrow$ (i) is true for $n=1$, C-1 is satisfied. Suppose (i) is true for $n=K$ i-e.,

$$1\times2+3\times4+\dots+(2K-1)\times2K = \frac{K(K+1)(4K-1)}{3} \longrightarrow (ii)$$

We shall prove that (i) is true for $n=K+1$ i-e.,

$$1\times2+3\times4+\dots+(2K-1)\times2K+(2(K+1)-1)\times2(K+1) = \frac{(K+1)(K+1+1)(4(K+1)-1)}{3}$$

$$1\times2+3\times4+\dots+(2K-1)\times2K+(2K+1)\times2(K+1) = \frac{(K+1)(K+2)(4K+3)}{3}$$

$$L.H.S = 1\times2+3\times4+\dots+(2K-1)\times2K+(2K+1)\times2(K+1)$$

$$= \frac{K(K+1)(4K-1)}{3}+(2K+1)\times2(K+1) \quad \text{By (ii)}$$

$$= \frac{K(K+1)(4K-1)+6(2K+1)(K+1)}{3}$$

$$= \frac{(K+1)\{4K^2-K+12K+6\}}{3}$$

$$= \frac{(K+1)\{4K^2+11K+6\}}{3}$$

$$= \frac{(K+1)\{4K^2+8K+3K+6\}}{3}$$

$$= \frac{(K+1)\{4K(K+2)+3(K+2)\}}{3}$$

$$= \frac{(K+1)(K+2)(4K+3)}{3} = R.H.S$$

$\rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for all integers n.

Q11. $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\dots+\frac{1}{n(n+1)} = 1-\frac{1}{n+1}$

Solution:-

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\dots+\frac{1}{n(n+1)} = 1-\frac{1}{n+1} \longrightarrow (i)$$

For $n=1$

$$\frac{1}{1(1+1)} = 1-\frac{1}{1+1} \rightarrow \frac{1}{2} = 1-\frac{1}{2}$$

$\rightarrow \frac{1}{2}=\frac{1}{2} \rightarrow$ (i) is true for $n=1$, C-1 is satisfied.

Suppose (i) is true for $n=K$ i-e.,

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\dots+\frac{1}{K(K+1)} = 1-\frac{1}{K+1} \longrightarrow (ii)$$

We shall prove that (i) is true for $n=K+1$ i-e.,

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\dots+\frac{1}{K(K+1)}+\frac{1}{(K+1)(K+1+1)} = 1-\frac{1}{K+1+1}$$

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\dots+\frac{1}{K(K+1)}+\frac{1}{(K+1)(K+2)} = 1-\frac{1}{K+2}$$

$$L.H.S = \frac{1}{1\times2}+\frac{1}{2\times3}+\cdots+\frac{1}{K(K+1)}+\frac{1}{(K+1)(K+2)}$$

$$= 1-\frac{1}{K+1}+\frac{1}{(K+1)(K+2)} \quad \text{By (ii)}$$

$$= \frac{K+1-1}{K+1}+\frac{1}{(K+1)(K+2)}$$

$$= \frac{K}{K+1}+\frac{1}{(K+1)(K+2)}$$

$$= \frac{K(K+2)+1}{(K+1)(K+2)}$$

$$= \frac{K^2+2K+1}{(K+1)(K+2)} = \frac{(K+1)^2}{(K+1)(K+2)}$$

$$= \frac{K+1}{K+2} = \frac{K+1+1-1}{K+2}$$

$$= \frac{K+2-1}{K+2} = \frac{K+2}{K+2}-\frac{1}{K+2}$$

$$= 1-\frac{1}{K+2} = R.H.S$$

$\Rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for all integers n.

Q12. $\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

Solution:-

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\cdots+\frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1} \longrightarrow (i)$$

For $n=1$

$$\frac{1}{[2(1)-1][2(1)+1]} = \frac{1}{2(1)+1} \Rightarrow \frac{1}{(1)(3)} = \frac{1}{3}$$

$\Rightarrow \frac{1}{3} = \frac{1}{3} \Rightarrow$ (i) is true

for $n=1$, C-1 is satisfied.

Suppose (i) is true for $n=K$ i-e.,

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\cdots+\frac{1}{(2K-1)(2K+1)} = \frac{K}{2K+1} \longrightarrow (ii)$$

We shall prove that (i) is true for $n=K+1$ i-e.,

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\cdots+\frac{1}{(2K-1)(2K+1)}+\frac{1}{(2(K+1)-1)(2(K+1)+1)} = \frac{K+1}{2(K+1)+1}$$

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\cdots+\frac{1}{(2K-1)(2K+1)}+\frac{1}{(2K+1)(2K+3)} = \frac{K+1}{2K+3}$$

$$L.H.S = \frac{1}{1\times3}+\frac{1}{3\times5}+\cdots+\frac{1}{(2K-1)(2K+1)}+\frac{1}{(2K+1)(2K+3)}$$

$$= \frac{K}{2K+1}+\frac{1}{(2K+1)(2K+3)} \quad \text{By (ii)}$$

$$= \frac{K(2K+3)+1}{(2K+1)(2K+3)}$$

$$= \frac{2K^2+3K+1}{(2K+1)(2K+3)} = \frac{2K^2+2K+K+1}{(2K+1)(2K+3)}$$

$$= \frac{2K(K+1)+1(K+1)}{(2K+1)(2K+3)} = \frac{(2K+1)(K+1)}{(2K+1)(2K+3)}$$

$$= \frac{K+1}{2K+3} = R.H.S$$

$\Rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for all integers n.

Q13. $\frac{1}{2\times5}+\frac{1}{5\times8}+\frac{1}{8\times11}+\cdots+\frac{1}{(3n-1)(3n+2)} = \frac{n}{2(3n+2)}$

Solution:-

$$\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3n-1)(3n+2)} = \frac{n}{2(3n+2)} \longrightarrow (i)$$

For $n=1$

$$\frac{1}{(3(1)-1)(3(1)+2)} = \frac{1}{2(3(1)+2)}$$

$$\Rightarrow \frac{1}{(2)(5)} = \frac{1}{2(5)}$$

$$\Rightarrow \frac{1}{10} = \frac{1}{10}$$

$\Rightarrow$ (i) is true for $n=1$, C-1 is satisfied. Suppose (i) is true for $n=K$ i.e.,

$$\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3K-1)(3K+2)}=\frac{K}{2(3K+2)} \rightarrow (ii)$$

We shall prove that (i) is true for $n=K+1$ i.e.,

$$\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3K-1)(3K+2)}+\frac{1}{[3(K+1)-1][3(K+1)+2]}=\frac{K+1}{2[3(K+1)+2]}$$

$$\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3K-1)(3K+2)}+\frac{1}{(3K+2)(3K+5)}=\frac{K+1}{2(3K+5)}$$

$$L.H.S=\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3(K-1)(3K+2)}+\frac{1}{(3K+2)(3K+5)}$$

$$=\frac{K}{2(3K+2)}+\frac{1}{(3K+2)(3K+5)} \quad \text{By (ii)}$$

$$=\frac{K(3K+5)+2}{2(3K+2)(3K+5)}$$

$$=\frac{3K^2+5K+2}{2(3K+2)(3K+5)}$$

$$=\frac{3K^2+3K+2K+2}{2(3K+2)(3K+5)}$$

$$=\frac{3K(K+1)+2(K+1)}{2(3K+2)(3K+5)}$$

$$=\frac{(K+1)(3K+2)}{2(3K+2)(3K+5)}=\frac{K+1}{2(3K+5)}$$

$$=R.H.S$$

$\Rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for all integers n.

Q14. $r+r^2+r^3+\cdots+r^n=\dfrac{r(1-r^n)}{1-r}, (r\neq1)$

Solution:-

$$r+r^2+r^3+\cdots+r^n=\frac{r(1-r^n)}{1-r}$$

For $n=1$

$$\Rightarrow r^1=\frac{r(1-r^1)}{1-r}$$

$$\Rightarrow r=\frac{r(1-r)}{1-r} \Rightarrow r=r$$

$\Rightarrow$ (i) is true for $n=1$, C-1 is satisfied. Suppose (i) is true for $n=K$ i.e.,

$$r+r^2+r^3+\cdots+r^K=\frac{r(1-r^K)}{1-r} \rightarrow (ii)$$

We shall prove that (i) is true for $n=K+1$ i.e.,

$$r+r^2+r^3+\cdots+r^K+r^{K+1}=\frac{r(1-r^{K+1})}{1-r}$$

$$L.H.S=r+r^2+r^3+\cdots+r^K+r^{K+1}$$

$$=\frac{r(1-r^K)}{1-r}+r^{K+1} \quad \text{By (ii)}$$

$$=\frac{r(1-r^K)+r^{K+1}(1-r)}{1-r}$$

$$=\frac{r-r\cdot r^K+r^{K+1}-r^{K+1}\cdot r}{1-r}$$

$$=\frac{r-r^{K+1}+r^{K+1}-r^{K+1}\cdot r}{1-r}$$

$$=\frac{r-r^{K+1}\cdot r}{1-r}=\frac{r(1-r^{K+1})}{1-r}=R.H.S$$

$\Rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for all integers n.

Q15. $a+(a+d)+(a+2d)+\cdots+[a+(n-1)d]=\dfrac{n}{2}[2a+(n-1)d]$

Solution:-

$$a+(a+d)+(a+2d)+\cdots+[a+(n-1)d]=\frac{n}{2}[2a+(n-1)d] \rightarrow (i)$$

For $n=1$

$$a=\frac{1}{2}(2a+(1-1)d)$$

$$\Rightarrow a=\frac{1}{2}(2a) \Rightarrow a=a$$

$\Rightarrow$ (i) is true for $n=1$, C-1 is satisfied. Suppose (i) is true for $n=K$ i.e.,

$$a+(a+d)+\cdots+[a+(K-1)d] = \frac{K}{2}[2a+(K-1)d] \longrightarrow (ii)$$

We shall prove that (i) is true for $n=K+1$ i.e.,

$$a+(a+d)+\cdots+[a+(K-1)d]+[a+(K+1-1)d] = \frac{K+1}{2}[2a+(K+1-1)d]$$

$$a+(a+d)+\cdots+[a+(K-1)d]+[a+Kd] = \frac{K+1}{2}[2a+Kd]$$

$$L.H.S = a+(a+d)+\cdots+[a+(K-1)d]+[a+Kd]$$

$$= \frac{K}{2}[2a+(K-1)d]+[a+Kd] \quad \text{By (ii)}$$

$$= \frac{K[2a+(K-1)d]+2[a+Kd]}{2}$$

$$= \frac{2aK+K(K-1)d+2a+2Kd}{2}$$

$$= \frac{2aK+K^2d-Kd+2a+2Kd}{2}$$

$$= \frac{K^2d+Kd+2aK+2a}{2}$$

$$= \frac{Kd(K+1)+2a(K+1)}{2}$$

$$= \frac{(K+1)\{Kd+2a\}}{2} = \frac{K+1}{2}[2a+Kd] = R.H.S$$

$\Rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for all integers n.

Q16. $1\lfloor 1+2\lfloor 2+3\lfloor 3+\cdots+n\lfloor n=\lfloor n+1-1$

Solution:-

$$1\lfloor 1+2\lfloor 2+3\lfloor 3+\cdots+n\lfloor n = \lfloor n+1 -1 \longrightarrow (i)$$

For $n=1$

$1\lfloor 1 = \lfloor 1+1 -1$ $\qquad \because \lfloor 1 = 1! = 1$

$\Rightarrow 1(1) = \lfloor 2 -1$ $\qquad \lfloor 2 = 2! = 2\cdot 1 = 2$

$\Rightarrow 1 = 1$

$\Rightarrow$ (i) is true for $n=1$, C-1 is satisfied.

Suppose (i) is true for $n=K$ i.e.,

$$1\lfloor 1+2\lfloor 2+3\lfloor 3+\cdots+K\lfloor K = \lfloor K+1 -1 \longrightarrow (ii)$$

We shall prove that (i) is true for $n=K+1$ i.e.,

$$1\lfloor 1+2\lfloor 2+3\lfloor 3+\cdots+K\lfloor K+(K+1)\lfloor K+1 = \lfloor K+1+1 -1$$

$$1\lfloor 1+2\lfloor 2+3\lfloor 3+\cdots+K\lfloor K+(K+1)\lfloor K+1 = \lfloor K+2 -1$$

$$L.H.S = 1\lfloor 1+2\lfloor 2+3\lfloor 3+\cdots+K\lfloor K+(K+1)\lfloor K+1$$

$$= \lfloor K+1 -1 +(K+1)\lfloor K+1 \quad \text{By (ii)}$$

$$= \lfloor K+1 +(K+1)\lfloor K+1 -1$$

$$= \lfloor K+1\,(1+K+1)-1$$

$$= \lfloor K+1\,(K+2)-1 = \lfloor K+2 -1 = R.H.S$$

$\Rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for all integers n.

Q17. $a_n = a_1+(n-1)d$ when $a_1, a_1+d, a_1+2d, \ldots$ form an A.P

Solution:- $a_n = a_1+(n-1)d \longrightarrow (i)$

For $n=1$

$\Rightarrow a_1 = a_1+(1-1)d \Rightarrow a_1 = a_1$

$\Rightarrow$ (i) is true for $n=1$, C-1 is satisfied. Suppose (i) is true for $n=K$ i.e.,

$$a_K = a_1+(K-1)d \longrightarrow (ii)$$

We shall prove that (i) is true for $n=K+1$ i.e.,

$$a_{K+1} = a_1+(K+1-1)d$$

$$a_{K+1} = a_1+Kd$$

$L.H.S = a_{K+1}$ $\qquad \because a_{n+1} = a_n+d$

$= a_K+d$ $\qquad (a_2 = a_1+d)$

$= a_1+(K-1)d+d$

$= a_1+(K-1+1)d$

$= a_1+Kd = R.H.S$

$\Rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for all integers n.

Q18. $a_n = a_1 r^{n-1}$ when $a_1, a_1r, a_1r^2, \ldots$ form a G.P

Solution:-

$a_n = a_1 r^{n-1} \longrightarrow (i)$

For $n = 1$

$\rightarrow a_1 = a_1 r^{1-1}$ $\quad \because r^0 = 1$

$\rightarrow a_1 = a_1$

$\rightarrow$ (i) is true for $n=1$, C-1 is satisfied. Suppose (i) is true for $n = k$ i-e.,

$a_k = a_1 r^{k-1} \longrightarrow (ii)$

We shall prove that (i) is true for $n = k+1$ i-e.,

$a_{k+1} = a_1 r^{k+1-1}$

$\rightarrow a_{k+1} = a_1 r^k$

L.H.S $= a_{k+1}$

$= a_k \cdot r$

$= (a_1 r^{k-1}) \cdot r$ By (ii) $(\because a_2 = a_1 r)$

$= a_1 r^{k-1+1} = a_1 r^k =$ R.H.S

$\rightarrow$ (i) is true for $n = k+1$, C-2 is satisfied. Hence (i) is true for all integers n.

Q19. $1^2+3^2+5^2+\ldots+(2n-1)^2 = \dfrac{n(4n^2-1)}{3}$

Solution:-

$1^2+3^2+5^2+\ldots+(2n-1)^2 = \dfrac{n(4n^2-1)}{3} \longrightarrow (i)$

For $n=1$

$(2(1)-1)^2 = \dfrac{1(4(1)^2-1)}{3}$

$\rightarrow (1)^2 = \dfrac{1(3)}{3} \rightarrow 1 = 1$

$\rightarrow$ (i) is true for $n=1$, C-1 is satisfied. Suppose (i) is true for $n = k$, i-e.,

$1^2+3^2+5^2+\ldots+(2k-1)^2 = \dfrac{k(4k^2-1)}{3} \longrightarrow (ii)$

We shall prove that (i) is true for $n = k+1$ i-e.,

$1^2+3^2+5^2+\ldots+(2k-1)^2+[2(k+1)-1]^2 = \dfrac{(k+1)[4(k+1)^2-1]}{3}$

$1^2+3^2+5^2+\ldots+(2k-1)^2+(2k+1)^2 = \dfrac{(k+1)[4(k^2+1+2k)-1]}{3}$

$1^2+3^2+5^2+\ldots+(2k-1)^2+(2k+1)^2 = \dfrac{(k+1)(4k^2+8k+4-1)}{3}$

$1^2+3^2+5^2+\ldots+(2k-1)^2+(2k+1)^2 = \dfrac{(k+1)(4k^2+8k+3)}{3}$

$1^2+3^2+5^2+\ldots+(2k-1)^2+(2k+1)^2 = \dfrac{(k+1)(4k^2+2k+6k+3)}{3}$

$1^2+3^2+5^2+\ldots+(2k-1)^2+(2k+1)^2 = \dfrac{(k+1)\{2k(2k+1)+3(2k+1)\}}{3}$

$1^2+3^2+5^2+\ldots+(2k-1)^2+(2k+1)^2 = \dfrac{(k+1)(2k+1)(2k+3)}{3}$

L.H.S $= 1^2+3^2+5^2+\ldots+(2k-1)^2+(2k+1)^2$

$= \dfrac{k(4k^2-1)}{3} + (2k+1)^2$

$= \dfrac{k(2k-1)(2k+1)}{3} + (2k+1)^2$

$= (2k+1)\left\{\dfrac{k(2k-1)}{3} + 2k+1\right\}$

$= (2k+1)\left\{\dfrac{2k^2-k+6k+3}{3}\right\}$

$= (2k+1)\left\{\dfrac{2k^2+5k+3}{3}\right\}$

$= (2k+1)\left\{\dfrac{2k^2+2k+3k+3}{3}\right\}$

$= (2k+1)\left\{\dfrac{2k(k+1)+3(k+1)}{3}\right\}$

$= \dfrac{(2k+1)(k+1)(2k+3)}{3}$

$= \dfrac{(k+1)(2k+1)(2k+3)}{3} =$ R.H.S

$\rightarrow$ (i) is true for $n=k+1$, C-2 is satisfied. Hence (i) is true for all integers n.

Q20. $\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\dots+\binom{n+2}{3}=\binom{n+3}{4}$

Solution:-

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\dots+\binom{n+2}{3}=\binom{n+3}{4} \longrightarrow (i)$$

For $n=1$

$$\binom{1+2}{3}=\binom{1+3}{4} \Rightarrow \binom{3}{3}=\binom{4}{4}$$

$$\Rightarrow 1=1 \qquad \because \binom{n}{n}=1$$

$\Rightarrow$ (i) is true for $n=1$, C-1 is satisfied. Suppose (i) is true for $n=K$ i-e.,

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\dots+\binom{K+2}{3}=\binom{K+3}{4} \longrightarrow (ii)$$

We shall prove that (i) is true for $n=K+1$ i-e.,

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\dots+\binom{K+2}{3}+\binom{K+1+2}{3}=\binom{K+1+3}{4}$$

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\dots+\binom{K+2}{3}+\binom{K+3}{3}=\binom{K+4}{4}$$

$$L.H.S=\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\dots+\binom{K+2}{3}+\binom{K+3}{3}$$

$$=\binom{K+3}{4}+\binom{K+3}{3} \quad \text{By (ii)}$$

$$=\binom{K+3+1}{4} \qquad \because \binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$$

$$=\binom{K+4}{4}=R.H.S$$

$\Rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for all +ive integers n.

Q21. Prove by mathematical induction that for all positive integral values of n

i) n^2+n is divisible by 2

Solution:-

n^2+n is divisible by 2 $\longrightarrow$ (i)

For $n=1$

$\Rightarrow n^2+n=(1)^2+1=2$ which is divisible by 2. C-1 is satisfied.

Suppose (i) is true for $n=K$. i-e., K^2+K is divisible by 2

$$\Rightarrow K^2+K=2Q \longrightarrow (ii)$$

We shall prove that (i) is true for $n=K+1$ i-e., $(K+1)^2+(K+1)$ is divisible by 2.

Now

$$(K+1)^2+K+1 = K^2+2K+1+K+1$$
$$=K^2+2K+K+2$$
$$=(K^2+K)+2(K+1)$$
$$=2Q+2(K+1) \quad \text{By (ii)}$$
$$=2\{Q+K+1\}$$

which is clearly divisible by 2, C-2 is satisfied. Hence (i) is true for all positive values of n.

ii) 5^n-2^n is divisible by 3

Solution:-

5^n-2^n is divisible by 3 $\longrightarrow$ (i)

$$5^n-2^n = 5^1-2^1=5-2=3$$

which is divisible by 3, C-1 is satisfied. Suppose (i) is true for $n=K$ i-e.,

5^K-2^K is divisible by 3

$$\Rightarrow 5^K-2^K=3Q \longrightarrow (ii)$$

We shall prove that (i) is true for $n=K+1$ i-e.,

$5^{K+1}-2^{K+1}$ is divisible by 3

Now

$$5^{K+1}-2^{K+1} = 5^K\cdot5^1-2^K\cdot2^1$$
$$=5^K(3+2)-2^K\cdot2$$
$$=3\cdot5^K+2\cdot5^K-2\cdot2^K$$
$$=3\cdot5^K+2(5^K-2^K)$$
$$=3\cdot5^K+2(3Q) \quad \text{By (ii)}$$
$$=3(5^K+2Q)$$

which is clearly divisible by 3, C-2 is satisfied. Hence (i) is satisfied for all integers n.

iii) $5^n - 1$ is divisible by 4

Solution:-

$5^n - 1$ is divisible by $4 \longrightarrow (i)$

For $n = 1$

$5^n - 1 = 5^1 - 1 = 4$ which is divisible by 4 ∴ C-1 is satisfied

Suppose (i) is true for $n = K$ i-e.,

$5^K - 1$ is divisible by 4

$\Rightarrow 5^K - 1 = 4Q \longrightarrow (ii)$

We shall prove that (i) is true for $n = K+1$ i-e.,

$5^{K+1} - 1$ is divisible by 4

Now $5^{K+1} - 1 = 5^K \cdot 5 - 1$

$= 5^K(4+1) - 1$

$= 4 \cdot 5^K + 1 \cdot 5^K - 1$

$= 4 \cdot 5^K + (5^K - 1)$

$= 4 \cdot 5^K + 4Q$ by (ii)

$= 4(5^K + Q)$

which is clearly divisible by 4. C-2 is satisfied. Hence (i) is true for all +ive integers n.

iv) $8 \times 10^n - 2$ is divisible by 6

Solution:-

$8 \times 10^n - 2$ is divisible by $6 \longrightarrow (i)$

For $n = 1$

$8 \times 10^n - 2 = 8 \times 10^1 - 2 = 80 - 2 = 78 = 6 \times 13$ which is divisible by 6, C-1 is satisfied. Suppose (i) is true for $n = K$ i-e.,

$8 \times 10^K - 2$ is divisible by 6

$\Rightarrow 8 \times 10^K - 2 = 6Q \longrightarrow (ii)$

We shall prove that (i) is true for $n = K+1$ i-e.,

$8 \times 10^{K+1} - 2$ is divisible by 6

Now $8 \times 10^{K+1} - 2 = 8 \times 10^K \cdot 10 - 2$

$= 80 \times 10^K - 2$

$= (72+8) \times 10^K - 2$

$= 72 \times 10^K + (8 \times 10^K - 2)$

$= 6 \times 12 \times 10^K + 6Q$ By (ii)

$= 6\{12 \times 10^K + Q\}$

which is clearly divisible by 6 C-2 is satisfied. Hence (i) is true for all +ive integers n.

v) $n^3 - n$ is divisible by 6

Solution:-

$n^3 - n$ is divisible by $6 \longrightarrow (i)$

For $n = 1$

$n^3 - n = (1)^3 - 1 = 1 - 1 = 0$ which is divisible by 6, C-1 is satisfied. Suppose (i) is true for $n = K$ i-e.,

$K^3 - K$ is divisible by 6

$\Rightarrow K^3 - K = 6Q \longrightarrow (ii)$

We shall prove that (i) is true for $n = K+1$ i-e.,

$(K+1)^3 - (K+1)$ is divisible by 6

Now
$(K+1)^3 - (K+1) = K^3 + 1 + 3K^2 + 3K - K - 1$

$= (K^3 - K) + 3K(K+1)$

$= 6Q + 3K(K+1)$ By (ii)

$= 6Q + 3(2P)$ ∵ $K(K+1)$ is an even integer

$= 6Q + 6P$

$= 6(Q+P)$

which is clearly divisible by 6 C-2 is satisfied. Hence (i) is true for all integers n.

Q22. $\frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n} = \frac{1}{2}\left[1 - \frac{1}{3^n}\right]$

Solution:-

$\frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n} = \frac{1}{2}\left[1 - \frac{1}{3^n}\right] \longrightarrow (i)$

For $n = 1$

$\frac{1}{3^1} = \frac{1}{2}\left[1 - \frac{1}{3^1}\right]$

$\frac{1}{3} = \frac{1}{2}\left(1 - \frac{1}{3}\right) \Rightarrow \frac{1}{3} = \frac{1}{2}\left(\frac{3-1}{3}\right)$

$\Rightarrow \frac{1}{3} = \frac{1}{2}\left(\frac{2}{3}\right) \Rightarrow \frac{1}{3} = \frac{1}{3}$

$\Rightarrow$ (i) is true for $n = 1$, C-1 is satisfied. Suppose (i) is true for $n = k$ i.e.,

$$\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^k} = \frac{1}{2}\left[1 - \frac{1}{3^k}\right]$$

$$\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^k} = \frac{1}{2}\left[\frac{3^k - 1}{3^k}\right]$$

$$\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^k} = \frac{3^k - 1}{2 \cdot 3^k} \longrightarrow \text{(ii)}$$

we shall prove that (i) is true for $n = k+1$ i.e.,

$$\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^k} + \frac{1}{3^{k+1}} = \frac{1}{2}\left(1 - \frac{1}{3^{k+1}}\right)$$

$$\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^k} + \frac{1}{3^{k+1}} = \frac{3^{k+1} - 1}{2 \cdot 3^{k+1}}$$

$$\text{L.H.S} = \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^k} + \frac{1}{3^{k+1}}$$

$$= \frac{3^k - 1}{2 \cdot 3^k} + \frac{1}{3^{k+1}} \quad \text{By (ii)}$$

$$= \frac{3^k - 1}{2 \cdot 3^k} + \frac{1}{3^k \cdot 3}$$

$$= \frac{3(3^k - 1) + 2}{2 \cdot 3^k \cdot 3} = \frac{3^{k+1} - 3 + 2}{2 \cdot 3^{k+1}}$$

$$= \frac{3^{k+1} - 1}{2 \cdot 3^{k+1}} = \text{R.H.S}$$

(i) is true for $n = k+1$. C-2 is satisfied. Hence (i) is true for all +ive integers n.

Q23. $1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1} \cdot n^2 = \frac{(-1)^{n-1} \cdot n(n+1)}{2}$

Solution:-

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1} \cdot n^2 = \frac{(-1)^{n-1} \cdot n(n+1)}{2}$$

For $n = 1$

$$(-1)^{1-1} \cdot (1)^2 = \frac{(-1)^{1-1} \cdot (1)(1+1)}{2}$$

$$1 = \frac{2}{2} \Rightarrow 1 = 1$$

(i) is true for $n = 1$, C-1 is satisfied. Suppose (i) is true for $n = k$ i.e.,

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{k-1} \cdot k^2 = \frac{(-1)^{k-1} \cdot k(k+1)}{2} \longrightarrow \text{(ii)}$$

we shall prove that (i) is true for $n = k+1$ i.e.,

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{k-1} \cdot k^2 + (-1)^{k+1-1} \cdot (k+1)^2 = \frac{(-1)^{k+1-1} \cdot (k+1)(k+1+1)}{2}$$

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{k-1} \cdot k^2 + (-1)^k (k+1)^2 = \frac{(-1)^k \cdot (k+1)(k+2)}{2}$$

$$\text{L.H.S} = 1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{k-1} \cdot k^2 + (-1)^k (k+1)^2$$

$$= \frac{(-1)^{k-1} \cdot k(k+1)}{2} + (-1)^k (k+1)^2 \quad \text{By (ii)}$$

$$= \frac{(-1)^k \cdot (-1)^{-1} \cdot k(k+1) + 2(-1)^k (k+1)^2}{2}$$

$$= \frac{(-1)^k (k+1)\{(-1)^{-1} \cdot k + 2(k+1)\}}{2}$$

$$= \frac{(-1)^k (k+1)\{(-1)k + 2k + 2\}}{2}$$

$$= \frac{(-1)^k (k+1)\{-k + 2k + 2\}}{2}$$

$$= \frac{(-1)^k (k+1)(k+2)}{2} = \text{R.H.S}$$

(i) is true for $n = k+1$. C-2 is satisfied. Hence (i) is true for all +ive integers n.

Q24. $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2[2n^2 - 1]$

Solution:-

$$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2[2n^2 - 1]$$

For $n = 1$

$$(2(1) - 1)^3 = (1)^2(2(1)^2 - 1)$$

$$\Rightarrow (1)^3 = 1(1) \Rightarrow 1 = 1$$

$\Rightarrow$ (i) is true for $n = 1$, C-1 is satisfied, Suppose (i) is true for $n = k$ i.e.,

$$1^3+3^3+5^3+\dots+(2k-1)^3 = k^2[2k^2-1] \longrightarrow (ii)$$

We shall prove that (i) is true for $n = k+1$ i.e.,

$$1^3+3^3+5^3+\dots+(2k-1)^3+[2(k+1)-1]^3 = (k+1)^2[2(k+1)^2-1]$$

$$1^3+3^3+5^3+\dots+(2k-1)^3+(2k+1)^3 = (k+1)^2(2(k^2+1+2k)-1)$$

$$1^3+3^3+5^3+\dots+(2k-1)^3+(2k+1)^3 = (k+1)^2(2k^2+2+4k-1)$$

$$1^3+3^3+5^3+\dots+(2k-1)^3+(2k+1)^3 = (k+1)^2(2k^2+4k+1)$$

$$L.H.S = 1^3+3^3+5^3+\dots+(2k-1)^3+(2k+1)^3$$

$$= k^2[2k^2-1]+(2k+1)^3$$

$$= 2k^4-k^2+(2k)^3+(1)^3+3(2k)^2(1)+3(2k)(1)^2$$

$$= 2k^4-k^2+8k^3+1+12k^2+6k$$

$$= 2k^4+8k^3+1+11k^2+6k$$

$$= 2k^4+2k^3+6k^3+6k^2+5k^2+5k+k+1$$

$$= 2k^3(k+1)+6k^2(k+1)+5k(k+1)+(k+1)$$

$$= (k+1)\{2k^3+6k^2+5k+1\}$$

$$= (k+1)\{2k^3+2k^2+4k^2+4k+k+1\}$$

$$= (k+1)\{2k^2(k+1)+4k(k+1)+(k+1)\}$$

$$= (k+1)\{(k+1)(2k^2+4k+1)\}$$

$$= (k+1)^2\{2k^2+4k+1\} = R.H.S$$

$\Rightarrow$ (i) is true for $n = k+1$.
Hence (i) is true for all +ive integers n.

Q25. $x+1$ is a factor of $x^{2n}-1$; $(x \neq -1)$

Solution:-

$x+1$ is a factor of $x^{2n}-1$; $(x\neq -1) \longrightarrow (i)$

For $n = 1$

$$x^{2n}-1 = x^{2(1)}-1 = x^2-1 = (x-1)(x+1)$$

$\Rightarrow (x+1)$ is clearly factor of x^2-1.
C-1 is satisfied. Suppose (i) is true for $n = k$ i.e.,

$x+1$ is a factor of $x^{2k}-1$

$$\Rightarrow \quad x^{2k}-1 = Q(x+1) \longrightarrow (ii)$$

We shall prove that (i) is true for $n = k+1$ i.e.,

$(x+1)$ is a factor of $x^{2(k+1)}-1$

Now

$$x^{2(k+1)}-1 = x^{2k+2}-1$$

$$= x^{2k}\cdot x^2-1$$

$$= x^{2k}\cdot x^2-x^2+x^2-1$$

$$= x^2(x^{2k}-1)+(x^2-1)$$

$$= x^2[Q(x+1)]+(x+1)(x-1) \quad \text{By (ii)}$$

$$= (x+1)\{x^2Q+(x-1)\}$$

$(x+1)$ is factor of $x^{2(k+1)}-1$, C-2 is satisfied. Hence (i) is true for all +ive integers n.

Q26. $x-y$ is a factor of x^n-y^n; $(x\neq y)$

Solution:-

$x-y$ is a factor of x^n-y^n ; $(x\neq y) \longrightarrow (i)$

For $n = 1$

$$x^n-y^n = x^1-y^1 = (x-y)$$

Clearly $(x-y)$ is a factor of $x-y$
C-1 is satisfied. Suppose (i) is true for $n = k$ i.e.,

$x-y$ is a factor of x^k-y^k ; $(x\neq y)$

$$\Rightarrow \quad x^k-y^k = Q(x-y) \longrightarrow (ii)$$

We shall prove that (i) is true for $n = k+1$ i.e.,

$$x^{k+1}-y^{k+1} = x^k\cdot x-y^k\cdot y$$

$$= x^k\cdot x-xy^k+xy^k-y^k\cdot y$$

$$= x(x^k-y^k)+y^k(x-y)$$

$$= x\cdot Q(x-y)+y^k(x-y) \quad \text{By (ii)}$$

$$= (x-y)\{xQ+y^k\}$$

$\Rightarrow (x-y)$ is a factor of $x^{K+1}-y^{K+1}$. C-2 is satisfied. Hence (i) is true for all positive integers n.

Q27. $x+y$ is a factor of $x^{2n-1}-y^{2n-1}$ $(x\neq -y)$

Solution:-

$x+y$ is a factor of $x^{2n-1}+y^{2n-1}$ $(x\neq -y) \longrightarrow (i)$

For $n=1$

$$x^{2n-1}+y^{2n-1} = x^{2(1)-1}+y^{2(1)-1} = x+y$$

clearly $x+y$ is factor of $x+y$ C-1 is satisfied. Suppose (i) is true for $n=K$ i-e.,

$x+y$ is a factor of $x^{2K-1}+y^{2K+1}$

$$\Rightarrow x^{2K-1}+y^{2K+1} = (x+y)Q \longrightarrow (ii)$$

We shall prove that (i) is true for $n=K+1$ i-e.,

$x+y$ is factor of $x^{2(K+1)-1}+y^{2(K+1)-1}$

Now

$$x^{2(K+1)-1}+y^{2(K+1)-1} = x^{2K+2-1}+y^{2K+2-1}$$

$$= x^{2K-1+2}+y^{2K-1+2}$$

$$= x^{2K-1}\cdot x^{2}+y^{2K-1}\cdot y^{2}$$

$$= x^{2K-1}\cdot x^2+x^2y^{2K-1}-x^2y^{2K-1}+y^{2K-1}\cdot y^2$$

$$= x^2(x^{2K-1}+y^{2K-1})-y^{2K-1}(x^2-y^2)$$

$$= x^2Q(x+y)-y^{2K-1}(x+y)(x-y) \quad \text{By (ii)}$$

$$= (x+y)\{x^2Q-y^{2K-1}(x-y)\}$$

$(x+y)$ is factor of $x^{2(K+1)-1}+y^{2(K+1)-1}$ C-2 is satisfied. Hence (i) is true for all +ive integers n.

Q28. Use mathematical induction to show that $1+2+2^2+\dots\dots+2^n = 2^{n+1}-1$ for all non-negative integers n.

Solution:-

$$1+2+2^2+\dots+2^n = 2^{n+1}-1 \longrightarrow (i)$$

For $n=0$

$$2^0 = 2^{0+1}-1 \Rightarrow 1 = 2^1-1$$

$$\Rightarrow 1=1 \Rightarrow (i) \text{ is true}$$

for $n=0$, C-1 is satisfied. Suppose (i) is true for $n=K$ i-e.,

$$1+2+2^2+\dots\dots+2^K = 2^{K+1}-1 \longrightarrow (ii)$$

We shall prove that (i) is true for $n=K+1$ i-e.,

$$1+2+2^2+\dots\dots+2^K+2^{K+1} = 2^{K+1+1}-1$$

$$1+2+2^2+\dots\dots+2^K+2^{K+1} = 2^{K+2}-1$$

$$\text{L.H.S} = 1+2+2^2+\dots\dots+2^K+2^{K+1}$$

$$= 2^{K+1}-1+2^{K+1} \quad \text{By (ii)}$$

$$= 2\cdot 2^{K+1}-1$$

$$= 2^{K+1+1}-1 = 2^{K+2}-1 = \text{R.H.S}$$

$\Rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for all +ive integers n.

Q29. If A and B are square matrices and $AB=BA$, then show by mathematical induction $AB^n = B^nA$ for any +ive integer n.

Solution:-

$AB^n = B^nA$ for any +ive integer $n \longrightarrow (i)$

For $n=1$

$$\Rightarrow AB^1 = B^1A$$

$$\Rightarrow AB = BA \quad \because AB=BA \text{ (given)}$$

$\Rightarrow$ (i) is true for $n=1$, C-1 is satisfied. Suppose (i) is true for $n=K$ i-e.,

$$AB^K = B^KA \longrightarrow (ii)$$

We shall prove that (i) is true for $n = K+1$ i.e.,

$$AB^{K+1} = B^{K+1}A$$

$$L.H.S = AB^{K+1}$$
$$= AB^K \cdot B$$
$$= (AB^K)\cdot B$$
$$= (B^K A)\cdot B \quad \text{By (ii)}$$
$$= B^K(AB)$$
$$= B^K(BA) \quad (\because AB = BA)$$
$$= (B^K \cdot B)A$$
$$= B^{K+1}A = R.H.S$$

$\Rightarrow$ (i) is true for $n = K+1$, C-2 is satisfied. Hence (i) is true for all +ive integers n.

Q30. Prove by the principle of mathematical induction that n^2-1 is divisible by 8 when n is an odd positive integer.

Solution:-

n^2-1 is divisible by 8, when n is odd positive integer $\longrightarrow$ (i)

For $n = 1$

$$n^2-1 = (1)^2-1 = 1-1 = 0$$

Clearly divisible by 8. C-1 is true satisfied. Suppose (i) is true for $n = K$ (K being odd) i.e.,

K^2-1 is divisible by 8

$$\Rightarrow K^2-1 = 8Q$$

We shall prove that (i) is true for $n = K+2$ i.e.,

$(K+2)^2-1$ is divisible by 8

Now,

$$(K+2)^2-1 = K^2+4K+4-1$$
$$= K^2-1+4K+4$$
$$= (K^2-1)+4(K+1)$$
$$= 8Q+4(K+1) \quad \text{By (ii)}$$
$$= 8Q+4(2P) \quad \left(\text{Let } 2P = K+1 \ \because K+1 \text{ is even}\right)$$
$$= 8Q+8P$$
$$= 8(Q+P)$$

$\Rightarrow$ (i) is true for $n = K+2$, C-2 is satisfied. Hence (i) is true for all positive odd integers n.

Q31. Use the principle of mathematical induction to prove that $\ln x^n = n\ln x$ for any integer $n \geq 0$ if x is positive number.

Solution:-

$$\ln x^n = n\ln x \quad \forall n \geq 0 \longrightarrow \text{(i)}$$

For $n = 0$

$$\Rightarrow \ln x^0 = 0(\ln x)$$
$$\Rightarrow \ln(1) = 0(\ln x)$$
$$\Rightarrow 0 = 0$$

$\Rightarrow$ (i) is true for $n = 0$, C-1 is satisfied. Suppose (i) is true for $n = K$ i.e.,

$$\ln x^K = K\ln x \longrightarrow \text{(ii)}$$

We shall prove that (i) is true for $n = K+1$ i.e.,

$$\ln x^{K+1} = (K+1)\ln x$$

$$L.H.S = \ln x^{K+1}$$
$$= \ln(x^K \cdot x)$$
$$= \ln x^K + \ln x$$
$$= K\ln x + \ln x$$
$$= \ln x(K+1)$$
$$= (K+1)\ln x = R.H.S$$

$\Rightarrow$ (i) is true for $n = K+1$, C-2 is satisfied. Hence (i) is true for all +ive integers n.

Use the principle of extended mathematical induction to prove that:

Q32. $n! > 2^n - 1$ for integral values of $n \geq 4$

Solution:-

$n! > 2^n - 1 \quad \forall\, n \geq 4 \longrightarrow (i)$

For $n = 4$

$4! > 2^4 - 1 \Rightarrow 24 > 16 - 1$

$\Rightarrow 24 > 15$

$\Rightarrow$ (i) is true for $n = 4$. C-1 is satisfied. Suppose (i) is true for $n = K$ i.e.,

$K! > 2^K - 1 \quad \forall\, K \geq 4 \longrightarrow (ii)$

We shall prove that (i) is true for $n = K+1$ i.e.,

$(K+1)! > 2^{K+1} - 1$

Now 'x' eq (ii) by $(K+1)$ we get

$(K+1)K! > (K+1)(2^K - 1)$

$\Rightarrow (K+1)! > K.2^K - K + 2^K - 1$

$\Rightarrow (K+1)! > K.2^K - 1 + 2^K - K$

$\Rightarrow (K+1)! > K.2^K - 1 \qquad \because 2^K - K > 0 \ \forall\, K \geq 4$

$\Rightarrow (K+1)! > 2.2^K - 1 \qquad \because K > 2$

$\Rightarrow (K+1)! > 2^{K+1} - 1$

$\Rightarrow$ (i) is true for $n = K+1$, C-2 is satisfied. Hence (i) is true for all $n \geq 4$.

Q33. $n^2 > n + 3$ for integral values of $n \geq 3$

Solution:-

$n^2 > n + 3 \quad \forall\, n \geq 3 \longrightarrow (i)$

For $n = 3$

$(3)^2 > 3 + 3 \Rightarrow 9 > 6$

$\Rightarrow$ (i) is true for $n = 3$, C-1 is satisfied. Suppose (i) is true for $n = K$ i.e.,

$K^2 > K + 3 \quad \forall\, K \geq 3 \longrightarrow (ii)$

We shall prove that (i) is true for $n = K+1$ i.e.,

$(K+1)^2 > K+1 + 3$

$\Rightarrow (K+1)^2 > K + 4$

Now adding $2K+1$ both sides of (ii)

$2K+1 + K^2 > 2K+1 + K+3$

$\Rightarrow (K+1)^2 > K+4 + 2K$

$\Rightarrow (K+1)^2 > K+4 \qquad \text{As } 2K > 0$

$\Rightarrow$ (i) is true for $n = K+1$, C-2 is satisfied. Hence (i) is true for all $n \geq 3$.

Q34. $4^n > 3^n + 2^{n-1}$ for integral values of $n \geq 2$

Solution:-

$4^n > 3^n + 2^{n-1} \quad \forall\, n \geq 2 \longrightarrow (i)$

For $n = 2$

$4^2 > 3^2 + 2^{2-1}$

$\Rightarrow 16 > 9 + 2^1$

$\Rightarrow 16 > 11$

$\Rightarrow$ (i) is true for $n = 2$, C-1 is satisfied. Suppose (i) is true for $n = K$ i.e.,

$K^n > 3^n + 2^{K-1} \quad \forall\, K \geq 2 \longrightarrow (ii)$

We shall prove that (i) is true for $n = K+1$ i.e.,

$4^{K+1} > 3^{K+1} + 2^{K+1-1}$

$\Rightarrow 4^{K+1} > 3^{K+1} + 2^K$

Now 'x' (ii) by 4

$4.4^K > 4(3^K + 2^{K-1})$

$\Rightarrow 4^{K+1} > 4.3^K + 4.2^{K-1}$

$\Rightarrow 4^{K+1} > (3+1).3^K + (2+2).2^{K-1}$

$\Rightarrow 4^{K+1} > 3.3^K + 3^K + 2.2^{K-1} + 2.2^{K-1}$

$\Rightarrow 4^{K+1} > 3^{K+1} + 3^K + 2^{K-1+1} + 2^{K-1+1}$

$\Rightarrow 4^{K+1} > 3^{K+1} + 3^K + 2^K + 2^K$

$\Rightarrow 4^{K+1} > 3^{K+1} + 2^K + (3^K + 2^K)$

$\Rightarrow 4^{K+1} > 3^{K+1} + 2^K$ As $3^K + 2^K > 0$ $\forall\ K \geq 2$

$\Rightarrow$ (i) is true for $n = K+1$, C-2 is satisfied. Hence (i) is true for all $n \geq 2$

Q35. $3^n < n!$ for integral values of $n > 6$

Solution:-

$3^n < n! \quad \forall\ n > 6 \longrightarrow$ (i)

For $n = 7$

$3^7 < 7! \Rightarrow 2187 < 5040$

$\Rightarrow$ (i) is true for $n = 7$, C-1 is satisfied. Suppose (i) is true for $n = K$ i-e.,

$3^K < K! \quad \forall\ K > 6 \longrightarrow$ (ii)

We shall prove that (i) is true for $n = K+1$ i-e.,

$3^{K+1} < (K+1)!$

Now 'x' (i) by 3, we get

$3.3^K < 3K!$ As $3 < K+1$ $\forall\ K > 6$

$\Rightarrow 3^{K+1} < (K+1)K!$

$\Rightarrow 3^{K+1} < (K+1)!$

$\Rightarrow$ (i) is true for $n = K+1$, C-2 is satisfied. Hence (i) is true for all $n > 6$

Q36. $n! > n^2$ for integral values of $n \geq 4$

Solution:-

$n! > n^2 \quad \forall\ n \geq 4 \longrightarrow$ (i)

For $n = 4$

$4! > (4)^2$

$\Rightarrow 24 > 16$

$\Rightarrow$ (i) is true for $n = 4$, C-1 is satisfied. Suppose (i) is true for $n = K$ i-e.,

$K! > K^2 \quad \forall\ K \geq 4 \longrightarrow$ (ii)

We shall prove that (i) is true for $n = K+1$ i-e.,

$(K+1)! > (K+1)^2$

Now 'x' (ii) by $(K+1)$, we get

$(K+1)K! > (K+1)K^2$

$\Rightarrow (K+1)! > (K+1)(K+1)$ As $K^2 > K+1$ $\forall\ K \geq 4$

$\Rightarrow (K+1)! > (K+1)^2$

$\Rightarrow$ (i) is true for $n = K+1$, C-2 is satisfied. Hence (i) is true for all integer $n \geq 4$

Q37. $3+5+7+\cdots+(2n+5) = (n+2)(n+4)$ for integral values of $n \geq -1$

Solution:-

$3+5+7+\cdots+(2n+5) = (n+2)(n+4) \quad \forall\ n \geq -1 \longrightarrow$ (i)

For $n = -1$

$2(-1)+5 = (-1+2)(-1+4)$

$\Rightarrow -2+5 = (1)(3)$

$\Rightarrow 3 = 3$

$\Rightarrow$ (i) is true for $n = -1$, C-1 is satisfied.

Suppose (i) is true for $n=K$

i-e.,

$3+5+7+\cdots+(2K+5)=(K+2)(K+4) \longrightarrow (ii)$

we shall prove that (i) is true for $n=K+1$ i-e.,

$3+5+7+\cdots+(2K+5)+(2(K+1)+5)$
$= (K+1+2)(K+1+4)$

$3+5+7+\cdots+(2K+5)+(2K+7)=(K+3)(K+5)$

$L.H.S = 3+5+7+\cdots+(2K+5)+(2K+7)$

$= (K+2)(K+4)+(2K+7)$ By (ii)

$= K^2+4K+2K+8+2K+7$

$= K^2+8K+15$

$= K^2+5K+3K+15$

$= K(K+5)+3(K+5)$

$= (K+5)(K+3)$

$= (K+3)(K+5) = R.H.S$

$\Rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for integral values of $n \geq -1$

Q38. $1+nx \leq (1+x)^n$ for $n \geq 2$ and $x > -1$

Solution:-

$1+nx \leq (1+x)^n$ for $n \geq 2, x>-1$

or $(1+x)^n \geq (1+nx$ for $n \geq 2, x>-1 \longrightarrow (i)$

For $n=2$

$(1+x)^2 \geq 1+2x$

$1+x^2+2x \geq 1+2x$

$\Rightarrow$ (i) is true for $n=2$, C-1 is satisfied. Suppose (i) is true for $n=K$ i-e.,

$(1+x)^K \geq 1+Kx \quad , K \geq 2, x>-1 \longrightarrow (ii)$

we shall prove that (i) is true for $n=K+1$ i-e.,

$(1+x)^{K+1} \geq 1+(K+1)x$

Now 'x' (ii) by $(1+x)$, we get

$(1+x).(1+x)^K \geq (1+x)(1+Kx)$

$(1+x)^{K+1} \geq 1+Kx+x+Kx^2$

$(1+x)^{K+1} \geq 1+(K+1)x+Kx^2$

$(1+x)^{K+1} \geq 1+(K+1)x$ As $Kx^2>0$ $\forall K \geq 0$

$\Rightarrow$ (i) is true for $n=K+1$, C-2 is satisfied. Hence (i) is true for all $n \geq 2$ and $x>-1$

"Binomial Theorem"

Statement:- Let "a" and "x" be two real numbers and "n" be a natural number. then

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x^{n-1} + \binom{n}{2}a^{n-2}x^2 + \cdots + \binom{n}{r-1}a^{n-(r-1)}\cdot x^{r-1} + \binom{n}{r}a^{n-r}x^r + \cdots + \binom{n}{n-1}ax^{n-1} + \binom{n}{n}x^n$$

Proof:- We prove it by Mathematical induction method. Consider

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x^{n-1} + \binom{n}{2}a^{n-2}x^2 + \cdots + \binom{n}{r-1}a^{n-(r-1)}\cdot x^{r-1} + \binom{n}{r}a^{n-r}x^r + \cdots + \binom{n}{n-1}ax^{n-1} + \binom{n}{n}x^n \longrightarrow (1)$$

For $n = 1$

$$(a+x)^1 = \binom{1}{0}a^1 + \binom{1}{1}a^{1-1}x^1$$

$$\Rightarrow a+x = 1a + 1a^0x \qquad \because \binom{1}{0} = \binom{1}{1} = 1$$

$$\Rightarrow a+x = a+x \qquad a^0 = 1$$

$\Rightarrow$ (1) is true for $n=1$. C-1 is satisfied.

Suppose (1) is true for $n = k$ i.e.,

$$(a+x)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}x^{n-1} + \binom{k}{2}a^{k-2}x^2 + \cdots + \binom{k}{r-1}a^{k-(r-1)}\cdot x^{r-1} + \binom{k}{r}a^{k-r}x^r + \cdots + \binom{k}{k-1}ax^{k-1} + \binom{k}{k}x^k \longrightarrow (2)$$

We shall prove that (1) is true for $n = k+1$.

For this multiplying eq (2) by $(a+x)$, we get

$$(a+x)(a+x)^k = (a+x)\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}x^{n-1} + \binom{k}{2}a^{k-2}x^2 + \cdots + \binom{k}{r-1}a^{k-(r-1)} + \binom{k}{r}a^{k-r}\cdot x^r + \cdots + \binom{k}{k-1}ar^{k-1} + \binom{k}{k}x^k\right]$$

$$\Rightarrow (a+x)^{k+1} = a\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}x^{n-1} + \binom{k}{2}a^{k-2}x^2 + \cdots + \binom{k}{r-1}a^{k-(r-1)} + \binom{k}{r}a^{k-r}\cdot x^r + \cdots + \binom{k}{k-1}ar^{k-1} + \binom{k}{k}x^k\right]$$

$$+ x\left[\binom{k}{0}a^k + \binom{k}{1}a^{k-1}x^{n-1} + \binom{k}{2}a^{k-2}x^2 + \cdots + \binom{k}{r-1}a^{k-(r-1)} + \binom{k}{r}a^{k-r}\cdot x^r + \cdots + \binom{k}{k-1}ar^{k-1} + \binom{k}{k}x^k\right]$$

$$(a+x)^{k+1} = \binom{k}{0}a^{k+1} + \binom{k}{1}a^kx^1 + \binom{k}{2}a^{k-1}x^2 + \cdots + \binom{k}{r-1}a^{k-r+2}x^{r-1} + \binom{k}{r}a^{k-r+1}x^r + \cdots + \binom{k}{k-1}a^2x^{k-1} + \binom{k}{k}ax^k$$
$$+ \binom{k}{0}a^kx^1 + \binom{k}{1}a^{k-1}x^2 + \binom{k}{2}a^{k-2}x^3 + \cdots + \binom{k}{r-1}a^{k-r+1}x^r + \binom{k}{r}a^{k-r}x^{r+1} + \cdots + \binom{k}{k-1}ax^k + \binom{k}{k}x^{k+1}$$

As we know that $\binom{n}{0}=1 \Rightarrow \binom{k}{0}=1$, $\binom{k+1}{0}=1$ $\Rightarrow \binom{k}{0}=\binom{k+1}{0}$
$\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r} \Rightarrow \binom{k}{1}+\binom{k}{0}=\binom{k+1}{1}$, $\binom{k}{2}+\binom{k}{1}=\binom{k+1}{2}$
$\binom{k}{r}+\binom{k}{r-1}=\binom{k+1}{r}$ and $\binom{k}{k}+\binom{k}{k-1}=\binom{k+1}{k}$
$\binom{n}{n}=1 \Rightarrow \binom{k}{k}=1$ also $\binom{k+1}{k+1}=1$ so $\binom{k}{k}=\binom{k+1}{k+1}$

Putting values we have

$$(a+x)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^{k}x^{1} + \binom{k+1}{2}a^{k-1}x^{2} + \dots + \binom{k+1}{r}a^{k-r+1}x^{r} + \dots + \binom{k+1}{k+1}x^{k+1}$$

$$(a+x)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^{k+1-1}x^{1} + \binom{k+1}{2}a^{k+1-2}x^{2} + \dots + \binom{k+1}{r}a^{k+1-r}x^{r} + \dots + \binom{k+1}{k+1}x^{k+1}$$

$\rightarrow$ (1) is true for $n = k+1$, C-2 is satisfied

Hence (1) is true for all natural numbers n.

Binomial Expression

A polynomial consisting of two terms is called Binomial or Binomial expression e.g., $x-2y^2$, $a+b$, $3x+5y$ etc.

* The expansion of $(a+b)^n$ for small values of n can be obtained by direct calculation such as

$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

Here

i) Right sides of these equations are called Binomial expansion.

ii) The exponents 1, 2, 3, 4 are called indices (plural of index)

To expand any binomial expression for higher values of n. We use a rule of expansion named as Binomial theorem.

Remember,

i) In the expansion of $(a+x)^n$ there are $n+1$ terms. (i-e., one term more than the exponent)

ii) The exponent of "a" decreases from n to zero. While exponent of 'x' increases from zero to n.

iii) In the expansion of $(a+x)^n$ the sum of exponents of "a" and "x" is equal to n.

iv) In the expansion of $(a+x)^n$ the term $\binom{n}{r}a^{n-r}x^{r}$ is called the $(r+1)$th term. i-e.,

$$T_{r+1} = \binom{n}{r}a^{n-r}\cdot x^{r}$$

It is also called general term. The successive terms can be obtained by putting

$$r = 0, 1, 2, 3, \dots, n$$

v) In binomial expansion, the coefficients from the begining and end are same. i-e., $\binom{n}{0} = \binom{n}{n}$

vi) $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{r}, \dots, \binom{n}{n}$ are called Binomial Coefficients.

Remember,

$^{n}C_{r} = \binom{n}{r}$

Example 1. Expand $\left(\frac{a}{2}-\frac{2}{a}\right)^6$ also find its general term.

Solution:- $\left(\frac{a}{2}-\frac{2}{a}\right)^6 = \left(\frac{a}{2}+\left(-\frac{2}{a}\right)\right)^6$

$$= \binom{6}{0}\left(\frac{a}{2}\right)^6 + \binom{6}{1}\left(\frac{a}{2}\right)^5\left(-\frac{2}{a}\right)^1 + \binom{6}{2}\left(\frac{a}{2}\right)^4\left(-\frac{2}{a}\right)^2 + \binom{6}{3}\left(\frac{a}{2}\right)^3\left(-\frac{2}{a}\right)^3 + \binom{6}{4}\left(\frac{a}{2}\right)^2\left(-\frac{2}{a}\right)^4 + \binom{6}{5}\left(\frac{a}{2}\right)^1\left(-\frac{2}{a}\right)^5 + \binom{6}{6}\left(-\frac{2}{a}\right)^6$$

$$= 1\cdot\frac{a^6}{64} + 6\frac{a^5}{32}\left(-\frac{2}{a}\right) + 15\frac{a^4}{16}\left(\frac{4}{a^2}\right) + 20\frac{a^3}{8}\left(-\frac{8}{a^3}\right) + 15\frac{a^2}{4}\left(\frac{16}{a^4}\right) + 6\frac{a}{2}\left(-\frac{32}{a^5}\right) + \frac{64}{a^6}$$

$$= \frac{a^6}{64} - \frac{3}{8}a^4 + \frac{15}{4}a^2 - 20 + \frac{60}{a^2} - \frac{96}{a^4} + \frac{64}{a^6}$$

As general term $= T_{r+1} = \binom{n}{r}a^{n-r}\cdot x^r$

so $T_{r+1} = \binom{6}{r}\left(\frac{a}{2}\right)^{6-r}\left(-\frac{2}{a}\right)^r$

$$\Rightarrow T_{r+1} = \binom{6}{r}\left(\frac{a}{2}\right)^{6-r}\cdot\left(\frac{2}{a}\right)^r(-1)^r$$

$$= (-1)^r\binom{6}{r}\left(\frac{a}{2}\right)^{6-r}\left(\frac{a}{2}\right)^{-r}$$

$$= (-1)^r\binom{6}{r}\left(\frac{a}{2}\right)^{6-r-r}$$

$$= (-1)^r\binom{6}{r}\left(\frac{a}{2}\right)^{6-2r}$$

Example 2. Evaluate $(9.9)^5$

Solution:-

$(9.9)^5 = (10-0.1)^5 = (10+(-0.1))^5$

Now

$$(10+(-0.1))^5 = \binom{5}{0}10^5 + \binom{5}{1}10^4(-0.1)^1 + \binom{5}{2}10^3(-0.1)^2 + \binom{5}{3}10^2(-0.1)^3 + \binom{5}{4}10^1(-0.1)^4 + \binom{5}{5}(-0.1)^5$$

$$= 1(100000) - 5(10000)(0.1) + 10(1000)(0.01) - 10(100)(0.001) + 5(10)(0.0001) - 1(0.00001)$$

$$= 100000 - 5000 + 100 - 1 + 0.005 - 0.00001$$

$$= 100100.005 - 5001.00001$$

$$= 95099.00499 = 95099.005$$

Example 3. Find the specified term in the expansion of $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11}$;

i) the term involving x^5

Solution:-

Let T_{r+1} be the required term then

$$T_{r+1} = \binom{n}{r}a^{n-r}\cdot x^r$$

$$= \binom{11}{r}\left(\frac{3}{2}x\right)^{11-r}\left(-\frac{1}{3x}\right)^r$$

$$= \binom{11}{r}\left(\frac{3}{2}\right)^{11-r}\cdot x^{11-r}\cdot\left(-\frac{1}{3}\right)^r\left(\frac{1}{x}\right)^r$$

$$= \binom{11}{r}\left(\frac{3}{2}\right)^{11-r}\left(-\frac{1}{3}\right)^r\cdot x^{11-r}(x)^{-r}$$

$$= \binom{11}{r}\left(\frac{3}{2}\right)^{11-r}\left(-\frac{1}{3}\right)^r\cdot x^{11-2r}$$

For required result put

$11-2r=5$

$\Rightarrow -2r = 5-11$

$\Rightarrow -2r = -6 \Rightarrow r = 3$

then $T_{r+1} = \binom{11}{r}\left(\frac{3}{2}\right)^{11-r}\left(-\frac{1}{3}\right)^r\cdot x^{11-2r}$

$$\Rightarrow T_{3+1} = \binom{11}{3}\left(\frac{3}{2}\right)^{11-3}\left(-\frac{1}{3}\right)^3\cdot x^{11-2(3)}$$

$$\Rightarrow T_4 = 165\left(\frac{3}{2}\right)^8\left(-\frac{1}{27}\right)x^5$$

$$\Rightarrow T_4 = 165\left(\frac{6561}{256}\right)\left(-\frac{1}{27}\right)x^5$$

$$\Rightarrow T_4 = -\frac{40095}{256}x^5$$

ii) the fifth term

Solution:- $T_5 = ?$

$\because T_{r+1} = \binom{11}{r}\left(\frac{3}{2}\right)^{11-r}\left(-\frac{1}{3}\right)^r\cdot x^{11-2r}$

Put $r=4$ we get

$$T_{4+1} = \binom{11}{4}\left(\frac{3}{2}\right)^{11-4}\cdot\left(-\frac{1}{3}\right)^4 x^{11-2(4)}$$

$$T_5 = \binom{11}{4}\left(\frac{3}{2}\right)^7\cdot\left(\frac{1}{81}\right)x^3$$

$$= 330\left(\frac{2187}{128}\right)\left(\frac{1}{81}\right)x^3$$

$$\Rightarrow T_5 = \left(\frac{165}{64}\right)(27)x^3$$

$$\Rightarrow T_5 = \frac{4455}{64}x^3$$

iii) the sixth term from the end

Solution:-

For this reverse the order in the given binomial

i.e., $\left(\frac{3}{2}x-\frac{1}{3x}\right)^{11} = \left(-\frac{1}{3x}+\frac{3}{2}x\right)^{11}$

$\because T_{r+1} = {}^nC_r\, a^{n-r}b^r$

Read an Important Note on Page # 33

$$T_{r+1} = 11_{C_r}\left(-\frac{1}{3x}\right)^{11-r}\left(\frac{3}{2}x\right)^r$$

Put r = 5 for 6th term

$$T_{5+1} = 11_{C_5}\left(-\frac{1}{3x}\right)^{11-5}\left(\frac{3}{2}x\right)^5$$

$$T_{5+1} = (462)\left(-\frac{1}{3x}\right)^6\left(\frac{3}{2}x\right)^5$$

$$T_6 = (462)\left(\frac{1}{3^6x^6}\right)\left(\frac{3^5x^5}{2^5}\right)$$

$$T_6 = (462)\left(\frac{1}{3x}\right)\left(\frac{1}{32}\right) = \frac{462}{96x} = \frac{77}{16x}$$

iv) coefficient of term involving x^{-1}

Solution:-

$$\because T_{r+1} = n_{C_r}a^{n-r}b^r$$

$$T_{r+1} = 11_{C_r}\left(\frac{3}{2}x\right)^{11-r}\left(-\frac{1}{3x}\right)^r$$

$$T_{r+1} = (-1)^r 11_{C_r}\left(\frac{3^{11-r}}{2^{11-r}}\right)(x^{11-r})\left(\frac{1}{3^r x^r}\right)$$

$$T_{r+1} = (-1)^r 11_{C_r}\left(\frac{3^{11-2r}}{2^{11-r}}\right)(x^{11-2r})$$

for the term involving x^{-1} put $11-2r=-1$ so $r=6$

$$T_{6+1} = (-1)^6 11_{C_6}\left(\frac{3^{11-2(6)}}{2^{11-6}}\right)(x^{11-2(6)})$$

$$T_7 = (462)\left(\frac{3^{-1}}{2^5}\right)(x^{-1}) = \frac{462}{32\times 3}x^{-1} = \frac{77}{16}x^{-1}$$

The middle term in the expansion of $(a+x)^n$

For $(a+x)^n$

Total number of terms is $n+1$

Case I: (n is even)

If n is even, then total number of terms $= n+1$ (odd)

Now, middle term $= \frac{n}{2}+1 = \frac{n+2}{2}$

e.g., in the expansion of $(a+x)^6$.

Here $n=6$ ($\because n$ is even)

Total terms $= 6+1 = 7$

Middle term $= \frac{6}{2}+1 = 3+1 = 4$

so 4th term will be its middle terms.

Case II: (n is odd)

If n is odd, then total number of terms $= n+1$ (Even)

Now in this case there will be two middle terms $\left(\frac{n+1}{2}\right)$ and $\left(\frac{n+3}{2}\right)$

e.g., in the expansion of $(a+x)^5$

$n = 5$ ($\because n$ is odd), total terms $= 5+1 = 6$

Middle term $= \frac{5+1}{2} = 3$

Middle term $= \frac{5+3}{2} = \frac{8}{2} = 4$

so 3rd and 4th terms will be middle terms.

Example 4. Find the following in the expansion of $\left(\frac{x}{2}+\frac{2}{x^2}\right)^{12}$;

i) the term independent of x

ii) the middle term

Solution:- i)

Let T_{r+1} be the required term

$$\because T_{r+1} = \binom{n}{r}a^{n-r}\cdot x^r$$

$$= \binom{12}{r}\left(\frac{x}{2}\right)^{12-r}\cdot\left(\frac{2}{x^2}\right)^r$$

$$= \binom{12}{r}\left(\frac{x}{2}\right)^{12-r}\cdot\left(\frac{2}{x}\right)^r\cdot\left(\frac{1}{x}\right)^r$$

$$= \binom{12}{r}\left(\frac{x}{2}\right)^{12-r}\left(\frac{x}{2}\right)^{-r}\left(\frac{x}{1}\right)^{-r}$$

$$= \binom{12}{r}\left(\frac{x}{2}\right)^{12-2r}\cdot x^{-r}$$

$$\Rightarrow T_{r+1} = \binom{12}{r}\left(\frac{1}{2}\right)^{12-2r}\cdot x^{12-3r} \longrightarrow (1)$$

For term independent of x

Put $12-3r=0 \Rightarrow 12 = 3r$

$\Rightarrow r = 4$

so (1) becomes as

$$T_{4+1} = \binom{12}{4}\left(\frac{1}{2}\right)^{12-2(4)} x^{12-3(4)}$$

$$= 495\cdot\left(\frac{1}{2}\right)^4\cdot x^0$$

$$\Rightarrow T_5 = \frac{495}{16}$$

ii)
Here $n = 12$ (even)

Middle term $= \frac{n+2}{2} = \frac{12+2}{2} = \frac{14}{2} = 7$

so, 7th term will be its middle term

$\because T_{r+1} = \binom{12}{r}\left(\frac{1}{2}\right)^{12-2r} \cdot x^{12-3r}$ (By 1)

Put $r = 6$

$\rightarrow T_{6+1} = \binom{12}{6}\left(\frac{1}{2}\right)^{12-2(6)} \cdot x^{12-3(6)}$

$= 924\left(\frac{1}{2}\right)^0 \cdot x^{-6}$ $\quad \because \left(\frac{1}{2}\right)^0 = 1$

$T_7 = \frac{924}{x^6}$ required term

binomial expansion of $(a+x)^n$

i) we know that

$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x^1 + \binom{n}{2}a^{n-2} \cdot x^2 + \binom{n}{3}a^{n-3} \cdot x^3 + \ldots\ldots + \binom{n}{n}x^n \longrightarrow (i)$

By Putting $a=1$ in (i)

$(1+x)^n = \binom{1}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots + x^n$

ii) By putting $a=1$ and replace x by $-x$ in (i)

$(1-x)^n = \binom{n}{0} + \binom{n}{1}(-x)^1 + \binom{n}{2}(-x)^2 + \binom{n}{3}(-x)^3 + \ldots + \binom{n}{n}(-x)^n$

$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots + (-1)^n x^n$

iii) To find sum of Binomial coefficients

By putting $a=1$, $x=1$ in (i)

$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots\ldots + \binom{n}{n}$

$2^n =$ sum of Binomial coefficients

iv) sum of even coefficients equals to sum of odd coefficients

$\because (a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x^1 + \binom{n}{2}a^{n-2}x^2 + \binom{n}{3}a^{n-3}x^3 + \ldots\ldots + \binom{n}{n}x^n$

By putting $a=1$, $x=-1$

$(1-1)^n = \binom{n}{0} + \binom{n}{1}(-1)^1 + \binom{n}{2}(-1)^2 + \binom{n}{3}(-1)^3 + \binom{n}{4}(-1)^4 + \ldots\ldots + \binom{n}{n-1}(-1)^{n-1} + \binom{n}{n}(-1)^n$

$\rightarrow \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} + \ldots\ldots + \binom{n}{n-1}(-1)^{n-1} + \binom{n}{n}(-1)^n = 0$

If n is odd positive integer

$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots\ldots + \binom{n}{n-1} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots\ldots + \binom{n}{n}$

If n is even positive integer

$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots\ldots \binom{n}{n} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots\ldots + \binom{n}{n-1}$

we conclude that

sum of even coefficients = sum of odd coefficients

Example 5. Show that:

$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots\ldots + n\binom{n}{n} = n \cdot 2^{n-1}$

Solution:-

$L.H.S = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots\ldots + n\binom{n}{n}$

$= n + 2\frac{n(n-1)}{2!} + 3\frac{n(n-1)(n-2)}{3!} + \ldots\ldots + n \cdot 1$

$= n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \ldots\ldots + 1\right\}$

in combination form

$= n\left\{\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \ldots\ldots + \binom{n-1}{n-1}\right\}$

we know that

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots\ldots + \binom{n}{n} = 2^n$

Similarly

$\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \ldots\ldots + \binom{n-1}{n-1} = 2^{n-1}$

Now

$L.H.S = n \cdot 2^{n-1} = R.H.S$

Exercise 8.2

Q1. Using binomial theorem, expand the following:

i) $(a+2b)^5$

Solution:-

$$(a+2b)^5 = \binom{5}{0}a^5 + \binom{5}{1}a^4(2b)^1 + \binom{5}{2}a^3(2b)^2 + \binom{5}{3}a^2(2b)^3 + \binom{5}{4}a(2b)^4 + \binom{5}{5}(2b)^5$$

$$= 1.a^5 + 5a^4(2b) + 10a^3(4b^2) + 10a^2(8b^3) + 5a(16b^4) + 1.(32b^5)$$

$$= a^5 + 10a^4b + 40a^3b^2 + 80a^2b^3 + 80ab^4 + 32b^5$$

ii) $\left(\frac{x}{2} - \frac{2}{x^2}\right)^6$

Solution:-

$$\left(\frac{x}{2} - \frac{2}{x^2}\right)^6 = \binom{6}{0}\left(\frac{x}{2}\right)^6 + \binom{6}{1}\left(\frac{x}{2}\right)^5\left(\frac{-2}{x^2}\right) + \binom{6}{2}\left(\frac{x}{2}\right)^4\left(\frac{-2}{x^2}\right)^2 + \binom{6}{3}\left(\frac{x}{2}\right)^3\left(\frac{-2}{x^2}\right)^3 + \binom{6}{4}\left(\frac{x}{2}\right)^2\left(\frac{-2}{x^2}\right)^4 + \binom{6}{5}\left(\frac{x}{2}\right)\left(\frac{-2}{x^2}\right)^5 + \binom{6}{6}\left(\frac{-2}{x^2}\right)^6$$

$$= 1.\frac{x^6}{64} - 6\frac{x^5}{32}.\frac{2}{x^2} + 15\frac{x^4}{16}.\frac{4}{x^4} - 20.\frac{x^3}{8}.\frac{8}{x^6} + 15\frac{x^2}{4}.\frac{16}{x^8} - 6.\frac{x}{2}.\frac{32}{x^{10}} + 1.\frac{64}{x^{12}}$$

$$= \frac{x^6}{64} - \frac{3}{8}x^3 + \frac{15}{4} - \frac{20}{x^3} + \frac{60}{x^6} - \frac{96}{x^9} + \frac{64}{x^{12}}$$

iii) $\left(3a - \frac{x}{3a}\right)^4$

Solution:-

$$\left(3a - \frac{x}{3a}\right)^4 = \binom{4}{0}(3a)^4 + \binom{4}{1}(3a)^3\left(-\frac{x}{3a}\right)^1 + \binom{4}{2}(3a)^2\left(-\frac{x}{3a}\right)^2 + \binom{4}{3}(3a)\left(-\frac{x}{3a}\right)^3 + \binom{4}{4}\left(-\frac{x}{3a}\right)^4$$

$$= 1(81a^4) - 4(27a^3)\left(\frac{x}{3a}\right) + 6(9a^2)\left(\frac{x^2}{9a^2}\right) - 4(3a)\left(\frac{x^3}{27a^3}\right) + \frac{x^4}{81a^4}$$

$$= 81a^4 - 36a^2x + 6x^2 - \frac{4x^3}{9a^2} + \frac{x^4}{81a^4}$$

iv) $\left(2a - \frac{x}{a}\right)^7$

Solution:-

$$\left(2a - \frac{x}{a}\right)^7 = \binom{7}{0}(2a)^7 + \binom{7}{1}(2a)^6\left(-\frac{x}{a}\right) + \binom{7}{2}(2a)^5\left(-\frac{x}{a}\right)^2 + \binom{7}{3}(2a)^4\left(-\frac{x}{a}\right)^3 + \binom{7}{4}(2a)^3\left(-\frac{x}{a}\right)^4 + \binom{7}{5}(2a)^2\left(-\frac{x}{a}\right)^5 + \binom{7}{6}(2a)\left(-\frac{x}{a}\right)^6 + \binom{7}{7}\left(-\frac{x}{a}\right)^7$$

$$= 1(128a^7) - 7(64a^6)\left(\frac{x}{a}\right) + 21(32a^5)\left(\frac{x^2}{a^2}\right) - 35(16a^4)\left(\frac{x^3}{a^3}\right) + 35(8a^3)\left(\frac{x^4}{a^4}\right) - 21(4a^2)\left(\frac{x^5}{a^5}\right) + 7(2a)\left(\frac{x^6}{a^6}\right) - \frac{x^7}{a^7}$$

$$= 128a^7 - 448a^5x + 672a^3x^2 - 560ax^3 + 280\frac{x^4}{a} - 84\frac{x^5}{a^3} + 14\frac{x^6}{a^5} - \frac{x^7}{a^7}$$

v) $\left(\frac{x}{2y} - \frac{2y}{x}\right)^8$

Solution:-

$$\left(\frac{x}{2y} - \frac{2y}{x}\right)^8 = \binom{8}{0}\left(\frac{x}{2y}\right)^8 + \binom{8}{1}\left(\frac{x}{2y}\right)^7\left(-\frac{2y}{x}\right) + \binom{8}{2}\left(\frac{x}{2y}\right)^6\left(-\frac{2y}{x}\right)^2 + \binom{8}{3}\left(\frac{x}{2y}\right)^5\left(\frac{-2y}{x}\right)^3 + \binom{8}{4}\left(\frac{x}{2y}\right)^4\left(-\frac{2y}{x}\right)^4 + \binom{8}{5}\left(\frac{x}{2y}\right)^3\left(-\frac{2y}{x}\right)^5 + \binom{8}{6}\left(\frac{x}{2y}\right)^2\left(\frac{-2y}{x}\right)^6 + \binom{8}{7}\left(\frac{x}{2y}\right)\left(\frac{-2y}{x}\right)^7 + \binom{8}{8}\left(-\frac{2y}{x}\right)^8$$

$$= 1\left(\frac{x^8}{256y^8}\right) - 8\left(\frac{x^7}{128y^7}\right)\left(\frac{2y}{x}\right) + 28\left(\frac{x^6}{64y^6}\right)\left(\frac{4y^2}{x^2}\right) - 56\left(\frac{x^5}{32y^5}\right)\left(\frac{8y^3}{x^3}\right) + 70\left(\frac{x^4}{16y^4}\right)\left(\frac{16y^4}{x^4}\right) - 56\left(\frac{x^3}{8y^3}\right)\left(\frac{32y^5}{x^5}\right) + 28\left(\frac{x^2}{4y^2}\right)\left(\frac{64y^6}{x^6}\right) - 8\left(\frac{x}{2y}\right)\left(\frac{128y^7}{x^7}\right) + \frac{256y^8}{x^8}$$

$$= \frac{x^8}{256y^8} - \frac{x^6}{8y^6} + \frac{7x^4}{4y^4} - \frac{14x^2}{y^2} + 70 - \frac{224y^2}{x^2} + \frac{448y^4}{x^4} - \frac{512y^6}{x^6} + \frac{256y^8}{x^8}$$

vi) $\left(\sqrt{\frac{a}{x}}-\sqrt{\frac{x}{a}}\right)^6$

Solution:-

$$\left(\sqrt{\tfrac{a}{x}}-\sqrt{\tfrac{x}{a}}\right)^6=\binom{6}{0}\left(\sqrt{\tfrac{a}{x}}\right)^6+\binom{6}{1}\left(\sqrt{\tfrac{a}{x}}\right)^5\left(-\sqrt{\tfrac{x}{a}}\right)^1$$
$$+\binom{6}{2}\left(\sqrt{\tfrac{a}{x}}\right)^4\left(-\sqrt{\tfrac{x}{a}}\right)^2+\binom{6}{3}\left(\sqrt{\tfrac{a}{x}}\right)^3\left(-\sqrt{\tfrac{x}{a}}\right)^3$$
$$+\binom{6}{4}\left(\sqrt{\tfrac{a}{x}}\right)^2\left(-\sqrt{\tfrac{x}{a}}\right)^4+\binom{6}{5}\left(\sqrt{\tfrac{a}{x}}\right)^1\left(-\sqrt{\tfrac{x}{a}}\right)^5+\binom{6}{6}\left(\sqrt{\tfrac{x}{a}}\right)^6$$
$$=1\cdot\left(\tfrac{a}{x}\right)^{6/2}-6\left(\tfrac{a}{x}\right)^{5/2}\left(\tfrac{x}{a}\right)^{1/2}+15\left(\tfrac{a}{x}\right)^{4/2}\left(\tfrac{x}{a}\right)^{2/2}$$
$$-20\left(\tfrac{a}{x}\right)^{3/2}\left(\tfrac{x}{a}\right)^{3/2}+15\left(\tfrac{a}{x}\right)^{2/2}\left(\tfrac{x}{a}\right)^{4/2}$$
$$-6\left(\tfrac{a}{x}\right)^{1/2}\left(\tfrac{x}{a}\right)^{5/2}+\left(\tfrac{x}{a}\right)^{6/2}$$
$$=\left(\tfrac{a}{x}\right)^3-6\left(\tfrac{a}{x}\right)^{5/2}\left(\tfrac{a}{x}\right)^{-1/2}+15\left(\tfrac{a}{x}\right)^{4/2}\left(\tfrac{a}{x}\right)^{-2/2}$$
$$-20\left(\tfrac{a}{x}\right)^{3/2}\left(\tfrac{a}{x}\right)^{-3/2}+15\left(\tfrac{a}{x}\right)^{2/2}\left(\tfrac{a}{x}\right)^{-4/2}-6\left(\tfrac{a}{x}\right)^{1/2}\left(\tfrac{a}{x}\right)^{-5/2}$$
$$+\left(\tfrac{a}{x}\right)^{-6/2}$$
$$=\left(\tfrac{a}{x}\right)^3-6\left(\tfrac{a}{x}\right)^{4/2}+15\left(\tfrac{a}{x}\right)^{2/2}-20+15\left(\tfrac{a}{x}\right)^{-2/2}$$
$$-6\left(\tfrac{a}{x}\right)^{-4/2}+\left(\tfrac{a}{x}\right)^{-6/2}$$
$$=\frac{a^3}{x^3}-\frac{6a^2}{x^2}+\frac{15a}{x}-20+15\left(\tfrac{a}{x}\right)^{-1}-6\left(\tfrac{a}{x}\right)^{-2}$$
$$+\left(\tfrac{a}{x}\right)^{-3}$$
$$=\frac{a^3}{x^3}-\frac{6a^2}{x^2}+\frac{15a}{x}-20+\frac{15x}{a}-\frac{6x^2}{a^2}+\frac{x^3}{a^3}$$

Q2. Calculate the following by means of binomial theorem:

i) $(0.97)^3$

Solution:- $(0.97)^3=(1-0.03)^3$

$$=\binom{3}{0}(1)^3+\binom{3}{1}(1)^2(-0.03)^1+\binom{3}{2}(1)^1(-0.03)^2$$
$$+\binom{3}{3}(-0.03)^3$$
$$=(1)(1)+3(1)(-0.03)+3(1)(0.0009)+1(-0.000027)$$
$$=1-0.09+0.0027-0.000027$$
$$=1.0027-0.090027=0.912673$$

ii) $(2.02)^4$

Solution:- $(2.02)^4=(2+0.02)^4$

$$=\binom{4}{0}(2)^4+\binom{4}{1}(2)^3(0.02)^1+\binom{4}{2}(2)^2(0.02)^2$$
$$+\binom{4}{3}(2)^1(0.02)^3+\binom{4}{4}(0.02)^4$$
$$=1(16)+4(8)(0.02)+6(4)(0.0004)+4(2)(0.000008)$$
$$+1(0.00000016)$$
$$=16+0.64+0.0096+0.000064+0.00000016$$
$$=16.64966416$$

iii) $(9.98)^4$

Solution:- $(9.98)^4=(10-0.02)^4$

$$=\binom{4}{0}(10)^4+\binom{4}{1}(10)^3(-0.02)^1+\binom{4}{2}(10)^2(-0.02)^2$$
$$+\binom{4}{3}(10)^1(-0.02)^3+\binom{4}{4}(-0.02)^4$$
$$=1(10000)-4(1000)(0.02)+6(100)(0.0004)$$
$$-4(10)(0.000008)+1(0.00000016)$$
$$=10000-80+0.240-0.00032+0.00000016$$
$$=10000.24000016-80.00032$$
$$=9920.23968016$$

iv) $(21)^5$

Solution:- $(21)^5=(20+1)^5$

$$=\binom{5}{0}(20)^5+\binom{5}{1}(20)^4(1)^1+\binom{5}{2}(20)^3(1)^2$$
$$+\binom{5}{3}(20)^2(1)^3+\binom{5}{4}(20)(1)^4+\binom{5}{5}(1)^5$$
$$=1(3200000)+5(160000)+10(8000)+10(400)$$
$$+5(20)+1$$
$$=3200000+800000+80000+4000+100+1$$
$$=4084101$$

Q3. Expand and simplify the following:

i) $(a+\sqrt{2}x)^4+(a-\sqrt{2}x)^4$

Solution:- SEE ON NEXT PAGE

Remember that

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

$(a-b)^4 = a^4-4a^3b+6a^2b^2-4ab^3+b^4$

$$(a+\sqrt{2}x)^4 = \binom{4}{0}a^4+\binom{4}{1}a^3(\sqrt{2}x)^1+\binom{4}{2}a^2(\sqrt{2}x)^2 + \binom{4}{3}a^1(\sqrt{2}x)^3+\binom{4}{4}(\sqrt{2}x)^4$$

$$= 1.a^4+4a^3(\sqrt{2}x)+6a^2(2x^2)+4a(2\sqrt{2}x^3) + 1(4x^4)$$

$$\rightarrow (a+\sqrt{2}x)^4 = a^4+4\sqrt{2}a^3x+12a^2x^2+8\sqrt{2}ax^3 + 4x^4 \longrightarrow (i)$$

Replace $\sqrt{2}$ by $-\sqrt{2}$, we get

$$(a-\sqrt{2}x)^4 = a^4-4\sqrt{2}a^3x+12a^2x^2-8\sqrt{2}ax^3+4x^4 \longrightarrow (ii)$$

Adding (i) and (ii)

$$(a+\sqrt{2}x)^4+(a-\sqrt{2}x)^4 = 2\{a^4+12a^2x^2+4x^4\}$$

ii) $(2+\sqrt{3})^5+(2-\sqrt{3})^5$

Solution:-

$$(2+\sqrt{3})^5 = \binom{5}{0}(2)^5+\binom{5}{1}(2)^4(\sqrt{3})^1+\binom{5}{2}(2)^3(\sqrt{3})^2 + \binom{5}{3}(2)^2(\sqrt{3})^3+\binom{5}{4}(2)^1(\sqrt{3})^4+\binom{5}{5}(\sqrt{3})^5$$

$$= 1(32)+5(16)(\sqrt{3})+10(8)(3)+10(4)(3\sqrt{3}) + 5(2)(9)+1(9\sqrt{3})$$

$$\rightarrow (2+\sqrt{3})^5 = 32+80\sqrt{3}+240+120\sqrt{3}+90+9\sqrt{3} \longrightarrow (i)$$

Replace $\sqrt{3}$ by $-\sqrt{3}$, we get

$$(2-\sqrt{3})^5 = 32-80\sqrt{3}+240-120\sqrt{3}+90-9\sqrt{3} \longrightarrow (ii)$$

Adding (i) and (ii)

$$(2+\sqrt{3})^5+(2-\sqrt{3})^5 = 2\{32+240+90\} = 2(362) = 724$$

iii) $(2+i)^5-(2-i)^5$

Solution:-

$$(2+i)^5 = \binom{5}{0}(2)^5+\binom{5}{1}(2)^4(i)^1+\binom{5}{2}(2)^3(i)^2 + \binom{5}{3}(2)^2(i)^3+\binom{5}{4}(2)^1(i)^4+\binom{5}{5}(i)^5$$

$$\rightarrow (2+i)^5 = 32+80i+80i^2+40i^3+10i^4+i^5 \longrightarrow (i)$$

Replace i by $-i$, we get

$$(2-i)^5 = 32-80i+80i^2-40i^3+10i^4-i^5 \longrightarrow (ii)$$

Subtracting (ii) from (i)

$$(2+i)^5-(2-i)^5 = 2\{80i+40i^3+i^5\}$$

$$= 2i\{80+40i^2+i^4\}$$

$$= 2i\{80+40(-1)+1\} \quad \because i=\sqrt{-1}$$

$$= 2i(81-40) \quad \rightarrow i^2=-1$$

$$= 2i(41) = 82i \quad i^4=(i^2)^2=(-1)^2=1$$

iv) $(x+\sqrt{x^2-1})^3+(x-\sqrt{x^2-1})^3$

Solution:-

$$(x+\sqrt{x^2-1})^3 = \binom{3}{0}x^3+\binom{3}{1}x^2(\sqrt{x^2-1})^1 + \binom{3}{2}x(\sqrt{x^2-1})^2+\binom{3}{3}(\sqrt{x^2-1})^3$$

$$(x+\sqrt{x^2-1})^3 = 1.x^3+3x^2\sqrt{x^2-1}+3x(x^2-1) + (\sqrt{x^2-1})^3$$

$$= x^3+3x^2\sqrt{x^2-1}+3x(x^2-1)+(\sqrt{x^2-1})^3 \longrightarrow (i)$$

Replace $\sqrt{x^2-1}$ by $-\sqrt{x^2-1}$, we get

$$(x-\sqrt{x^2-1})^3 = x^3-3x^2\sqrt{x^2-1}+3x(x^2-1) - (\sqrt{x^2-1})^3 \longrightarrow (ii)$$

Adding (i) and (ii)

$$(x+\sqrt{x^2-1})^3+(x-\sqrt{x^2-1})^3 = 2\{x^3+3x(x^2-1)\}$$

$$= 2\{x^3+3x^3-3x\}$$

$$= 2(4x^3-3x) = 2x(4x^2-3).$$

Q4. Expand the following in ascending power of x:

i) $(2+x-x^2)^4$

Solution:– $(2+x-x^2)^4$

$$=\binom{4}{0}(2+x)^4+\binom{4}{1}(2+x)^3(-x^2)^1+\binom{4}{2}(2+x)^2(-x^2)^2+\binom{4}{3}(2+x)^1(-x^2)^3+\binom{4}{4}(-x^2)^4$$

$$=1(2+x)^4-4x^2(2+x)^3+6x^4(2+x)^2-4x^6(2+x)+1.x^8$$

$$=2^4+4(2)^3(x)+6(2)^2x^2+4(2)^1(x^3)+x^4-4x^2(2^3+3.2^2.x+3.2.x^2+x^3)+6x^4(4+4x+x^2)-8x^6-4x^7+x^8$$

$$=16+32x+24x^2+8x^3+x^4-32x^2-48x^3-24x^4-4x^5+24x^4+24x^5+6x^6-8x^6-4x^7+x^8$$

$$=16+32x+(24-32)x^2+(8-48)x^3+(1-24+24)x^4+(-4+24)x^5+(6-8)x^6-4x^7+x^8$$

$$=16+32x-8x^2-40x^3+x^4+20x^5-2x^6-4x^7+x^8$$

ii) $(1-x+x^2)^4$

Solution:– $(1-x+x^2)^4$

$$=\binom{4}{0}(1-x)^4+\binom{4}{1}(1-x)^3(x^2)^1+\binom{4}{2}(1-x^2)^2(x^2)^2+\binom{4}{3}(1-x)^1(x^2)^3+\binom{4}{4}(x^2)^4$$

$$=(1-x)^4+4x^2(1-x^3)+6x^4(1-x)^2+4x^6(1-x)+x^8$$

$$=1-4x+6x^2-4x^3+x^4+4x^2(1-3x+3x^2-x^3)+6x^4(1-2x+x^2)+4x^6-4x^7+x^8$$

$$=1-4x+6x^2-4x^3+x^4+4x^2-12x^3+12x^4-4x^5+6x^4-12x^5+6x^6+4x^6-4x^7+x^8$$

$$=1-4x+(6+4)x^2+(-4-12)x^3+(1+12+6)x^4+(-4-12)x^5+(6+4)x^6-4x^7+x^8$$

$$=1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$$

iii) $(1-x-x^2)^4$

Solution:– $(1-x-x^2)^4$

$$=\binom{4}{0}(1-x)^4+\binom{4}{1}(1-x)^3(-x^2)^1+\binom{4}{2}(1-x)^2(-x^2)^2+\binom{4}{3}(1-x)^1(-x^2)^3+\binom{4}{4}(-x^2)^4$$

$$=(1-x)^4-4x^2(1-x)^3+6x^4(1-x)^2-4x^6(1-x)+x^8$$

$$=(1-x)^4-4x^2(1-x)^3+6x^4(1-2x+x^2)-4x^6+4x^7+x^8$$

$$=1-4x+6x^2-4x^3+x^4-4x^2(1-3x+3x^2-x^3)+6x^4(1-2x+x^2)-4x^6+4x^7+x^8$$

$$=1-4x+6x^2-4x^3+x^4-4x^2+12x^3-12x^4+4x^5+6x^4-12x^5+6x^6-4x^6+4x^7+x^8$$

$$=1-4x+(6-4)x^2+(-4+12)x^3+(1-12+6)x^4+(4-12)x^5+(6-4)x^6+4x^7+x^8$$

$$=1-4x+2x^2+8x^3+5x^4-8x^5+2x^6+4x^7+x^8$$

Q5. Expand the following in descending powers of x:

i) $(x^2+x-1)^3$

Solution:– $(x^2+x-1)^3$

$$=\binom{3}{0}(x^2)^3+\binom{3}{1}(x^2)^2(x-1)^1+\binom{3}{2}(x^2)^1(x-1)^2+\binom{3}{3}(x-1)^3$$

$$=x^6+3x^4(x-1)+3x^2(x^2+1-2x)+(x^3-1-3x^2+3x)$$

$$=x^6+3x^5-3x^4+3x^4+3x^2-6x^3+x^3-1-3x^2+3x$$

$$=x^6+3x^5+(-3+3)x^4+(-6+1)x^3+(3-3)x^2+3x-1$$

$$=x^6+3x^5-5x^3+3x-1$$

ii) $\left(x-1-\frac{1}{x}\right)^3$

Solution:– $\left(x-1-\frac{1}{x}\right)^3$

$$=\binom{3}{0}(x-1)^3+\binom{3}{1}(x-1)^2\left(-\frac{1}{x}\right)^1+\binom{3}{2}(x-1)\left(-\frac{1}{x}\right)^2+\binom{3}{3}\left(-\frac{1}{x}\right)^3$$

$$=x^3-1-3x^2+3x-3(x^2+1-2x)\left(\frac{1}{x}\right)+3(x-1)\left(\frac{1}{x^2}\right)-\left(\frac{1}{x^3}\right)$$

$= x^3 - 3x^2 + 3x - 1 - 3x + 6 - \frac{3}{x} + \frac{3}{x} - \frac{3}{x^2} - \frac{1}{x^3}$

$= x^3 - 3x^2 + (3-3)x - 1 + 6 + (-3+3)\frac{1}{x} - \frac{3}{x^2} - \frac{1}{x^3}$

$= x^3 - 3x^2 + 5 - \frac{3}{x^2} - \frac{1}{x^3}$

Q6. Find the term involving:

i) x^4 in the expansion of $(3-2x)^7$

Solution:-

Let T_{r+1} be the required term

$\because T_{r+1} = \binom{n}{r} a^{n-r}$

$\rightarrow T_{r+1} = \binom{7}{r}(3)^{7-r}(-2x)^r$

$T_{r+1} = \binom{7}{r}(3)^{7-r}\cdot(-2)^r\cdot x^r$

For required result put $r=4$

$\rightarrow T_{4+1} = \binom{7}{4}(3)^{7-4}(-2)^4 x^4$

$\rightarrow T_5 = 35(27)(16)x^4$

$\rightarrow T_5 = 15120x^4$

ii) x^{-2} in the expansion of $\left(x - \frac{2}{x^2}\right)^{13}$

Solution:-

Let T_{r+1} be the required term

$\because T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$\rightarrow T_{r+1} = \binom{13}{r} x^{13-r}\left(\frac{-2}{x^2}\right)^r$

$= \binom{13}{r} x^{13-r}\cdot(-2)^r\cdot\left(\frac{1}{x^2}\right)^r$

$T_{r+1} = \binom{13}{r} x^{13-r}(-2)^r\cdot(x^{-2})^r$

$T_{r+1} = \binom{13}{r}(-2)^r\cdot x^{13-r-2r}$

$\rightarrow T_{r+1} = \binom{13}{r}(-2)^r\cdot x^{13-3r}$

For required result $13-3r=-2$

$\rightarrow 13+2=3r \rightarrow 15=3r$

$\rightarrow r=5$

$\rightarrow T_{5+1} = \binom{13}{5}(-2)^5\cdot x^{13-3(5)}$

$T_6 = 1287(-32)x^{-2}$

$\rightarrow T_6 = -41184x^{-2}$

iii) a^4 in the expansion of $\left(\frac{2}{x} - a\right)^9$.

Solution:-

Let T_{r+1} be the required term

$\because T_{r+1} = \binom{n}{r} a^{n-r}\cdot b^r$

$\rightarrow T_{r+1} = \binom{9}{r}\left(\frac{2}{x}\right)^{9-r}(-a)^r$

$T_{r+1} = \binom{9}{r} 2^{9-r}\cdot\left(\frac{1}{x}\right)^{9-r}(-a)^r$

For required result put $r=4$

$\rightarrow T_{4+1} = \binom{9}{4} 2^{9-4}\left(\frac{1}{x}\right)^{9-4}(-a)^4$

$= 126(2)^5\cdot\left(\frac{1}{x}\right)^5 a^4$

$\rightarrow T_5 = 126(32)\cdot\frac{1}{x^5}a^4$

$\rightarrow T_5 = \frac{4032a^4}{x^5}$

iv) y^3 in the expansion of $(x-\sqrt{y})^{11}$

Solution:-

Let T_{r+1} be the required term

$\because T_{r+1} = \binom{n}{r} a^{n-r}\cdot b^r$

$\rightarrow T_{r+1} = \binom{11}{r} x^{11-r}(-\sqrt{y})^r$

$T_{r+1} = \binom{11}{r} x^{11-r}(-1)^r y^{r/2}$

For required result put $\frac{r}{2}=3$

$\rightarrow r=6$

$T_{6+1} = \binom{11}{6} x^{11-6}(-1)^6 y^{6/2}$

$\rightarrow T_7 = 462x^5(1)y^3 = 462x^5y^3$

Q7. Find the coefficient of;

i) x^5 in the expansion of $\left(x^2 - \frac{3}{2x}\right)^{10}$

Solution:-

$\because T_{r+1} = \binom{n}{r} a^{n-r}\cdot b^r$

$\rightarrow T_{r+1} = \binom{10}{r}(x^2)^{10-r}\cdot\left(\frac{-3}{2x}\right)^r$

$= \binom{10}{r}(x^{20-2r})\cdot(-\frac{3}{2})^r\cdot(\frac{1}{x})^r$

$= \binom{10}{r}(-\frac{3}{2})^r\cdot x^{20-2r}\cdot x^{-r}$

$= \binom{10}{r}(-\frac{3}{2})^r\cdot x^{20-3r}$

For required result put $20-3r=5$

$\Rightarrow -3r = 5-20$

$\Rightarrow -3r=-15 \Rightarrow r=5$

$\Rightarrow T_{5+1} = \binom{10}{5}(-\frac{3}{2})^5 x^{20-3(5)}$

$T_6 = 252(-\frac{243}{32})x^5$

$T_6 = \frac{63(-243)x^5}{8}$

$\Rightarrow T_6 = \frac{-15309}{8}x^5$

Thus coefficient of x^5 is $\frac{-15309}{8}$.

ii) x^n in the expansion of $(x^2-\frac{1}{x})^{2n}$

Solution:-

$\because T_{r+1} = \binom{n}{r}a^{n-r}\cdot b^r$

$\Rightarrow T_{r+1} = \binom{2n}{r}(x^2)^{2n-r}(-\frac{1}{x})^r$

$T_{r+1} = \binom{2n}{r}(-1)^r\cdot x^{4n-2r}\cdot x^{-r}$

$\Rightarrow T_{r+1} = \binom{2n}{r}(-1)^r\cdot x^{4n-3r}$

For required result put $4n-3r=n$

$\Rightarrow -3r = n-4n$

$\Rightarrow -3r=-3n \Rightarrow r=n$

$\Rightarrow T_{n+1} = \binom{2n}{n}(-1)^n\cdot x^{4n-3n}$

$= \frac{2n!}{n!(2n-n)!}(-1)^n\cdot x^n$

$= \frac{2n!}{n!\,n!}(-1)^n\cdot x^n$

$T_{n+1} = (-1)^n\cdot\frac{2n!}{(n!)^2}x^n$

Thus coefficient of x^n is $(-1)^n\frac{2n!}{(n!)^2}$

Q8. Find 6th term in the expansion of $(x^2-\frac{3}{2x})^{10}$

Solution:-

$\because T_{r+1} = \binom{n}{r}a^{n-r}\cdot b^r$

$\Rightarrow T_{r+1} = \binom{10}{r}(x^2)^{10-r}\cdot(\frac{-3}{2x})^r$

For 6th term, put $r=5$

$\Rightarrow T_{5+1} = \binom{10}{5}(x^2)^{10-5}\cdot(\frac{-3}{2x})^5$

$= 252(x^2)^5\cdot(-\frac{3}{2})^5\cdot(\frac{1}{x})^5$

$= 252\,x^{10}\cdot(-\frac{243}{32})x^{-5}$

$T_6 = \frac{252(-243)}{32}x^{10-5}$

$\Rightarrow T_6 = \frac{63(-243)}{8}x^5 = \frac{-15309}{8}x^5$

Q9. Find the term independent of x in the following expansions.

i) $(x-\frac{2}{x})^{10}$

Solution:-

Let T_{r+1} be the required term.

$\because T_{r+1} = \binom{n}{r}a^{n-r}\cdot b^r$

$\Rightarrow T_{r+1} = \binom{10}{r}x^{10-r}(-\frac{2}{x})^r$

$= \binom{10}{r}(-2)^r\cdot x^{10-r}\cdot(\frac{1}{x})^r$

$T_{r+1} = \binom{10}{r}(-2)^r\cdot x^{10-r}\cdot x^{-r}$

$\Rightarrow T_{r+1} = \binom{10}{r}(-2)^r\cdot x^{10-2r}$

For required result put

$10-2r=0 \Rightarrow 2r=10 \Rightarrow r=5$

$\Rightarrow T_{5+1} = \binom{10}{5}(-2)^5\cdot x^{10-2(5)}$

$= 252(-32)x^0$

$T_6 = -8064(1) = -8064$

ii) $(\sqrt{x}+\frac{1}{2x^2})^{10}$

Solution:-

Let T_{r+1} be the required term

$\because T_{r+1} = \binom{n}{r}a^{n-r}\cdot b^r$

$\Rightarrow T_{r+1} = \binom{10}{r}(x^{1/2})^{10-r}(\frac{1}{2x^2})^{10}$

$$= \binom{10}{r} x^{\frac{10-r}{2}} \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{x^2}\right)^r$$

$$= \binom{10}{r}\left(\frac{1}{2}\right)^r \cdot x^{\frac{10-r}{2}} \cdot x^{-2r}$$

$$= \binom{10}{r}\left(\frac{1}{2}\right)^r \cdot x^{\frac{10-r}{2}-2r}$$

$$= \binom{10}{r}\left(\frac{1}{2}\right)^r x^{\frac{10-r-4r}{2}}$$

$$T_{r+1} = \binom{10}{r}\left(\frac{1}{2}\right)^r x^{\frac{10-5r}{2}}$$

For required result put

$$\frac{10-5r}{2} = 0 \Rightarrow 10-5r = 0$$

$$10 = 5r \Rightarrow r = 2$$

$$\Rightarrow T_{2+1} = \binom{10}{2}\left(\frac{1}{2}\right)^2 \cdot x^{\frac{10-5(2)}{2}}$$

$$= 45\left(\frac{1}{4}\right)x^0$$

$$\Rightarrow T_3 = \frac{45}{4}(1) = \frac{45}{4}$$

iii) $(1+x^2)^3\left(1+\frac{1}{x^2}\right)^4$

Solution:-

$$(1+x^2)^3 \cdot \left(1+\frac{1}{x^2}\right)^4 = (1+x^2)^3\left(\frac{x^2+1}{x^2}\right)^4$$

$$= (1+x^2)^3 \frac{(1+x^2)^4}{(x^2)^4}$$

$$= \frac{(1+x^2)^7}{x^8} = \left[\frac{1+x^2}{x^{8/7}}\right]^7$$

$$= \left[\frac{1+x^2}{x^{8/7}}\right]^7 = \left(x^{-8/7}(1+x^2)\right)^7$$

$$= \left(x^{-8/7} + x^{-8/7+2}\right)^7$$

$$= \left(x^{-8/7} + x^{6/7}\right)^7$$

Let T_{r+1} be required term

$$\because T_{r+1} = \binom{n}{r} a^{n-r} \cdot b^r$$

$$\Rightarrow T_{r+1} = \binom{7}{r}\left(x^{-8/7}\right)^{7-r}\left(x^{6/7}\right)^r$$

$$= \binom{7}{r} x^{\frac{-56+8r}{7}} \cdot x^{6r/7}$$

$$= \binom{7}{r} x^{\frac{-56+8r+6r}{7}}$$

$$= \binom{7}{r} x^{\frac{-56+14r}{7}}$$

For required result, put

$$-56+14r = 0$$

$$\Rightarrow 14r = 56 \Rightarrow r = 4$$

$$T_{4+1} = \binom{7}{4} x^{\frac{-56+14(4)}{7}}$$

$$= 35\, x^{\frac{-56+56}{7}} = 35x^0$$

$$\Rightarrow T_5 = 35(1) = 35$$

Q10. Determine the middle term in the following expansions:

i) $\left(\frac{1}{x} - \frac{x^2}{2}\right)^{12}$

Solution:-

Here $n = 12$ (i.e., even)

Middle term $= \frac{n+2}{2} = \frac{12+2}{2} = \frac{14}{2} = 7$

so, 7th term will be middle term

$$\because T_{r+1} = \binom{n}{r} a^{n-r} \cdot b^r$$

$$\Rightarrow T_{r+1} = \binom{12}{r}\left(\frac{1}{x}\right)^{12-r} \cdot \left(-\frac{x^2}{2}\right)^r$$

Put $r = 6$

$$\Rightarrow T_{6+1} = \binom{12}{6}\left(\frac{1}{x}\right)^{12-6} \cdot \left(-\frac{x^2}{2}\right)^6$$

$$T_7 = 924\left(\frac{1}{x}\right)^6 \cdot \left(-\frac{1}{2}\right)^6 (x^2)^6$$

$$T_7 = 924\left(\frac{1}{64}\right) x^{-6} \cdot x^{12}$$

$$\Rightarrow T_7 = \frac{231}{16} x^6$$

ii) $\left(\frac{3}{2}x - \frac{1}{3x}\right)^{11}$

Solution:-

Here $n = 11$ (i.e., odd)

Middle term $= \frac{n+1}{2} = \frac{11+1}{2} = \frac{12}{2} = 6$

Also middle term $= \frac{n+3}{2} = \frac{11+3}{2} = \frac{14}{2} = 7$

so 6th and 7th terms will be the middle terms

As $T_{r+1} = \binom{n}{r} a^{n-r} \cdot b^r$

$$T_{r+1} = \binom{11}{r}\left(\frac{3}{2}x\right)^{11-r} \cdot \left(-\frac{1}{3x}\right)^r$$

For 6th term put $r=5$

$$T_{5+1} = \binom{11}{5}\left(\frac{3}{2}x\right)^{11-5}\cdot\left(-\frac{1}{3x}\right)^{5}$$

$$T_6 = 462\left(\frac{3x}{2}\right)^6\left(-\frac{1}{3}\right)^5\cdot\left(\frac{1}{x}\right)^5$$

$$T_6 = 462\cdot\left(\frac{3}{2}\right)^6 x^6(-1)^5\cdot\left(\frac{1}{3}\right)^5\cdot x^{-5}$$

$$T_6 = -\frac{462}{64}\cdot 3^1 x$$

$$\rightarrow T_6 = -\frac{462(3)}{64}x = -\frac{231(3)}{32}x$$

$$\rightarrow T_6 = -\frac{693}{32}x$$

For 7th term put $r=6$

$$T_{6+1} = \binom{11}{6}\left(\frac{3}{2}x\right)^{11-6}\cdot\left(\frac{-1}{3x}\right)^6$$

$$= 462\cdot\frac{3^5}{2^5}x^5(-1)^6\cdot\frac{1}{3^6x^6}$$

$$= \frac{462}{32(3x)} = \frac{231}{48x} = \frac{77}{16x}$$

iii) $\left(2x-\frac{1}{2x}\right)^{2m+1}$

Solution:-

Here $n=2m+1$ (i.e., n is odd)

$$\text{Middle term} = \frac{n+1}{2}$$

$$= \frac{2m+1+1}{2}$$

$$= \frac{2m+2}{2}$$

$$= \frac{2(m+1)}{2}$$

$$= m+1$$

Also

$$\text{middle term} = \frac{n+3}{2}$$

$$= \frac{2m+1+3}{2} = \frac{2m+4}{2} = \frac{2(m+2)}{2}$$

$$= m+2$$

So $(m+1)$th and $(m+2)$th terms will be middle terms

$$\because T_{r+1} = \binom{2m+1}{r}(2x)^{2m+1-r}\cdot\left(\frac{-1}{2x}\right)^r$$

For $(m+1)$th term put $r=m$

$$\rightarrow T_{m+1} = \binom{2m+1}{m}(2x)^{2m+1-m}\cdot\left(\frac{-1}{2x}\right)^m$$

$$= \frac{(2m+1)!}{m!(2m+1-m)!}(-1)^m(2x)^{m+1}\cdot(2x)^{-m}$$

$$=(-1)^m\frac{(2m+1)!}{m!(m+1)!}(2x)^1$$

$$T_{m+1} = 2(-1)^m\frac{(2m+1)!}{m!(m+1)!}x$$

For $(m+2)$th term put $r=m+1$

$$T_{m+1+1} = \binom{2m+1}{m+1}(2x)^{2m+1-m-1}\cdot\left(\frac{-1}{2x}\right)^{m+1}$$

$$T_{m+2} = \frac{(2m+1)!}{(m+1)!(2m+1-m-1)!}(2x)^{2m+1-m-1}\cdot(-1)^{m+1}\left(\frac{1}{2x}\right)^{m+1}$$

$$=(-1)^{m+1}\frac{(2m+1)!}{(m+1)!\,m!}(2x)^m\cdot(2x)^{-m-1}$$

$$= (-1)^{m+1}\frac{(2m+1)!}{(m+1)!m!}(2x)^{-1}$$

$$T_{m+2} = (-1)^{m+1}\frac{(2m+1)!}{(m+1)!\,m!}\cdot\frac{1}{2x}$$

$$T_{m+2} = \frac{1}{2}(-1)^{m+1}\frac{(2m+1)!}{m!(m+1)!}\cdot\frac{1}{x}$$

Important note

$$\because (a+b)^3 = a^3+3a^2b+\boxed{3ab^2}+b^3$$

and $(b+a)^3 = b^3+\boxed{3ab^2}+3a^2b+a^3$

Now $3a^2b$ = 3rd term from the end in the expansion of $(a+b)^3$

$3a^2b$ = 3rd term from the begining in the expansion of $(b+a)^3$

so, we conclude that Required term from the end in the expansion of $(a+b)^n$ is equal to required term from begining in the expansion of $(b+a)^n$.

Q11. Find $(2n+1)$th term from the end in the expansion of $\left(x-\frac{1}{2x}\right)^{3n}$

Solution:-

For this reverse the order in the given binomial.
i.e.,

$$\left(x-\frac{1}{2x}\right)^{3n}=\left(-\frac{1}{2x}+x\right)^{3n}$$

Here $a=-\frac{1}{2x}$, $b=x$, $n=3n$

$\because T_{r+1}=\binom{n}{r}a^{n-r}b^{r}$

$\rightarrow T_{r+1}=\binom{3n}{r}\left(-\frac{1}{2x}\right)^{3n-r}(x)^{r}$

For required result put $r=2n$

$$T_{2n+1}=\binom{3n}{2n}\left(-\frac{1}{2x}\right)^{3n-2n}\cdot x^{2n}$$

$$=\frac{3n!}{2n!(3n-2n)!}\left(-\frac{1}{2x}\right)^{n}\cdot x^{2n}$$

$$T_{2n+1}=\frac{3n!}{2n!\ n!}(-1)^{n}\cdot\left(\frac{1}{2}\right)^{n}x^{-n}\cdot x^{2n}$$

$$T_{2n+1}=\frac{(-1)^{n}\,3n!}{2^{n}\,(2n)!\,n!}x^{n}$$

Q12. Show that the middle term of $(1+x)^{2n}$ is $\frac{1.3.5\ldots(2n-1)}{n!}2^{n}x^{n}$

Solution:-

Here $n=2n$ (even)

Total terms $=2n+1$

Middle term $=\frac{2n+2}{2}=n+1$ $\left(\because \frac{n+2}{2}\right)$ using formula

$\rightarrow T_{r+1}=\binom{2n}{r}(1)^{2n-r}\cdot x^{r}$

For middle term put $r=n$

$$T_{n+1}=\binom{2n}{n}(1)^{2n-n}\cdot x^{n}$$

$$T_{n+1}=\frac{2n!}{n!(2n-n)!}(1)^{n}\cdot x^{n}$$

$$=\frac{2n!}{n!\ n!}x^{n}$$

$$=\frac{2n(2n-1)(2n-2)(2n-3)(2n-4)\ldots4.3.2.1}{n!\ n!}x^{n}$$

$$=\frac{[2n(2n-2)(2n-4)\ldots4.2][(2n-1)(2n-3)\ldots3.1]}{n!\ n!}x^{n}$$

$$=\frac{2^{n}[n(n-1)(n-2)\ldots2.1][(2n-1)(2n-3)\ldots3.1]}{n!\ n!}x^{n}$$

$$=\frac{2^{n}\cancel{n!}(2n-1)(2n-3)\ldots3.1}{\cancel{n!}\,n!}x^{n}$$

$$=\frac{1.3.5\ldots(2n-3)(2n-1)}{n!}x^{n}2^{n}$$

$$\rightarrow T_{n+1}=\frac{1.3.5\ldots(2n-3)(2n-1)}{n!}2^{n}x^{n}$$

Hence proved

Q13. Show that

$$\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots+\binom{n}{n-1}=2^{n-1}$$

Solution:-

We know that

$$(1+x)^{n}=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^{2}+\binom{n}{3}x^{3}+\cdots+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^{n}\longrightarrow(i)$$

By putting $x=-1$ in (i) we get

$$(1-1)^{n}=\binom{n}{0}+\binom{n}{1}(-1)+\binom{n}{2}(-1)^{2}+\binom{n}{3}(-1)^{3}+\binom{n}{4}(-1)^{4}+\binom{n}{5}(-1)^{5}+\cdots+\binom{n}{n-1}(-1)^{n-1}+\binom{n}{n}(-1)^{n}$$

If n is even then $n-1$ will be odd so

$$0=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\binom{n}{4}-\binom{n}{5}+\cdots-\binom{n}{n-1}+\binom{n}{n}$$

$$\rightarrow\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots+\binom{n}{n-1}=\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots+\binom{n}{n}\longrightarrow(ii)$$

By putting $x=1$ in (i) we get

$$(1+1)^{n}=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\binom{n}{4}+\binom{n}{5}+\cdots+\binom{n}{n-1}+\binom{n}{n}$$

$$2^{n}=\left\{\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots+\binom{n}{n-1}\right\}+\left\{\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots+\binom{n}{n-1}\right\}$$

$$2^{n}=2\left\{\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots+\binom{n}{n-1}\right\}$$

$$\rightarrow\frac{2^{n}}{2}=\left\{\binom{n}{1}+\binom{n}{3}+\binom{n}{3}+\cdots+\binom{n}{n-1}\right\}$$

$\rightarrow 2^{n-1} = \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots+\binom{n}{n-1}$

Hence $\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots+\binom{n}{n-1} = 2^{n-1}$

Q14. Show that

$$\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+\cdots+\frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$$

Solution:-

$$L.H.S = \binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+\cdots+\frac{1}{n+1}\binom{n}{n}$$

$$= \frac{n!}{0!(n-0)!}+\frac{1}{2}\frac{n!}{1!(n-1)!}+\frac{1}{3}\frac{n!}{2!(n-2)!}+\frac{1}{4}\frac{n!}{3!(n-3)!}+\cdots+\frac{1}{n+1}\cdot\frac{n!}{n!(n-n)!}$$

$$= n!\left\{\frac{1}{n!}+\frac{1}{2!(n-1)!}+\frac{1}{3!(n-2)!}+\frac{1}{4!(n-3)!}+\cdots+\frac{1}{(n+1)!\,0!}\right\}$$

Multiplying and dividing by $(n+1)$

$$= \frac{(n+1)n!}{n+1}\left\{\frac{1}{1!\,n!}+\frac{1}{2!(n-1)!}+\frac{1}{3!(n-2)!}+\frac{1}{4!(n-3)!}+\cdots+\frac{1}{(n+1)!\,0!}\right\}$$

$$= \frac{(n+1)!}{n+1}\left\{\frac{1}{1!\,n!}+\frac{1}{2!(n-1)!}+\frac{1}{3!(n-2)!}+\frac{1}{4!(n-3)!}+\cdots+\frac{1}{(n+1)!\,0!}\right\}$$

$$= \frac{1}{n+1}\left\{\frac{(n+1)!}{1!\,n!}+\frac{(n+1)!}{2!(n-1)!}+\frac{(n+1)!}{3!(n-2)!}+\frac{(n+1)!}{4!(n-3)!}+\cdots+\frac{(n+1)!}{(n+1)!\,0!}\right\}$$

$$= \frac{1}{n+1}\left\{\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\binom{n+1}{4}+\cdots+\binom{n+1}{n+1}\right\}$$

By adding and subtracting $\binom{n+1}{0}$, we get

$$= \frac{1}{n+1}\left\{\binom{n+1}{0}+\binom{n+1}{2}+\binom{n+1}{3}+\binom{n+1}{4}+\cdots+\binom{n+1}{n+1}-\binom{n+1}{0}\right\} \rightarrow (i)$$

we know that

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\binom{n}{4}+\cdots+\binom{n}{n} = 2^n$$

Similarly

$$\binom{n+1}{0}+\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\binom{n+1}{4}+\cdots+\binom{n+1}{n+1} = 2^{n+1}$$

Also $\binom{n}{0}=1$ Similarly $\binom{n+1}{0}=1$

Putting all values in equation (i)

$$L.H.S = \frac{1}{n+1}\left\{2^{n+1}-1\right\}$$

$$= \frac{2^{n+1}-1}{n+1} = R.H.S$$

The Binomial Theorem when the index 'n' is a negative integer or a fraction

When n is negative integer or a fraction, then

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3 + \cdots\cdots \infty \text{ provided } |x|<1$$

This is called Binomial series.

Note:- 1) In Binomial series

First term $= T_1 = 1$

Second term $= T_2 = nx$

Third term $= T_3 = \frac{n(n-1)}{2!}x^2$

Fourth term $= T_4 = \frac{n(n-1)(n-2)}{3!}x^3$

Similarly

General Term $= T_{r+1}$

$$= \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)}{r!}x^r$$

2) The symbols $\binom{n}{0}\binom{n}{1}\binom{n}{2}$ etc are meaningless when n is negative or a fraction

Example 1. Find the general term in the expansion of $(1+x)^{-3}$ when $|x|<1$

Solution:- $n=-3$

$$\because T_{r+1} = \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)}{r!}x^r$$

$$\Rightarrow T_{r+1} = \frac{-3(-3-1)(-3-2)\cdots\cdots(-3-r+1)}{r!}x^r$$

$$= \frac{(-3)(-4)(-5)\cdots\cdots(-2-r)}{r!}x^r$$

$$= (-1)^r\frac{3\cdot4\cdot5\cdots\cdots(2+r)}{r!}x^r$$

$$= (-1)^r\frac{1\cdot2\cdot3\cdot4\cdot5\cdots\cdots(r+2)}{1\cdot2\cdot r!}x^r$$

$$= (-1)^r\frac{(r+2)!}{2\cdot r!}x^r$$

$$= (-1)^r\frac{(r+2)(r+1)r!}{2\cdot r!}x^r$$

$$= (-1)^r\frac{(r+1)(r+2)}{2}x^r$$

Summation of infinit series

To find the sum of a series, the given series is compared with

$$1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$

In this way we find values of n and x. then putting values of n and x in $(1+x)^n$, we get the required sum.

Example 2. Expand $(1-2x)^{1/3}$ to four terms and apply it to evaluate $(.8)^{1/3}$ correct to three places of decimal

Solution:-

Using Binomial Expansion

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$

$$(1-2x)^{1/3} = 1+\frac{1}{3}(-2x)+\frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}(-2x)^2+\frac{(\frac{1}{3})(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!}(-2x)^3 - \cdots$$

$$= 1-\frac{2}{3}x+\frac{\frac{1}{3}(-\frac{2}{3})}{2\cdot1}4x^2+\frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{3\cdot2\cdot1}(-8x^3) - \cdots$$

$$= 1-\frac{2}{3}x-\frac{1\cdot2\cdot4}{2\cdot3\cdot3}x^2-\frac{1\cdot2\cdot5\cdot8}{3\cdot3\cdot3\cdot3\cdot2}x^3-\cdots$$

$$= 1-\frac{2}{3}x-\frac{4}{9}x^2-\frac{40}{81}x^3-\cdots$$

Put $x=0.1$ we get

$$[1-2(0.1)]^{1/3} = 1-\frac{2}{3}(0.1)-\frac{4}{9}(0.1)^2-\frac{40}{81}(0.1)^3-\cdots$$

$$(1-0.2)^{1/3} = 1-\frac{0.2}{3}-\frac{0.04}{9}-\frac{0.04}{81}-\cdots$$

$$(0.8)^{1/3} \approx 1-0.06666-0.00444-0.00049$$

$$\approx 1-0.07159 \approx 0.92841$$

$$\Rightarrow (0.8)^{1/3} \approx 0.92841$$

Note:- The expansion of $(1-2x)^{1/3}$ is valid if $|2x|<1 \Rightarrow 2|x|<1$

$\Rightarrow |x|<\frac{1}{2}$

Example 3. Expand $(8-5x)^{-2/3}$ to four terms.

Solution:-

$$(8-5x)^{-2/3} = \left[8\left(1-\frac{5}{8}x\right)\right]^{-2/3} = 8^{-2/3}\left(1-\frac{5}{8}x\right)^{-2/3}$$

$$=(2^3)^{-2/3}\left(1-\frac{5}{8}x\right)^{-2/3} = 2^{-2}\left(1-\frac{5}{8}x\right)^{-2/3}$$

$$=\frac{1}{4}\left\{1+\left(-\frac{2}{3}\right)\left(-\frac{5}{8}x\right)+\frac{(-\frac{2}{3})(-\frac{2}{3}-1)}{2!}\left(-\frac{5}{8}x\right)^2 + \frac{(-\frac{2}{3})(-\frac{2}{3}-1)(-\frac{2}{3}-2)}{3!}\left(-\frac{5}{8}x\right)^3+\cdots\right\}$$

$$=\frac{1}{4}\left\{1+\frac{5}{12}x+\frac{1}{2}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(\frac{25}{64}x^2\right) + \frac{1}{6}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{125}{512}x^3\right)+\cdots\right\}$$

$$=\frac{1}{4}\left\{1+\frac{5}{12}x+\frac{125}{576}x^2+\frac{625}{5184}x^3+\cdots\right\}$$

Note:- The expansion of $(8-5x)^{-2/3}$ is valid if $\left|\frac{5}{8}x\right|<1 \Rightarrow \frac{5}{8}|x|<1$

$\Rightarrow |x|<\frac{8}{5}$

Example 4. Evaluate $\sqrt[3]{30}$ correct to three places of decimal.

Solution:-

$$\sqrt[3]{30}=(30)^{1/3}=(27+3)^{1/3}=\left[27\left(1+\frac{3}{27}\right)\right]^{1/3}$$

$$=(27)^{1/3}\left[\left(1+\frac{1}{9}\right)^{1/3}\right] = (3^3)^{1/3}\left(1+\frac{1}{9}\right)^{1/3}$$

$$=3\left\{1+\frac{1}{3}\left(\frac{1}{9}\right)+\frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}\left(\frac{1}{9}\right)^2+\frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!}\left(\frac{1}{9}\right)^3+\cdots\right\}$$

$$=3\left\{1+\frac{1}{27}+\frac{1}{2}\cdot\frac{1}{3}\left(-\frac{2}{3}\right)\left(\frac{1}{81}\right)+\frac{1}{6}\cdot\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(\frac{1}{729}\right)+\cdots\right\}$$

$$=3\left\{1+\frac{1}{27}-\frac{1}{729}+\frac{5}{59049}+\cdots\right\}$$

$$=3\{1+0.03704-0.001372+0.000085+\cdots\}$$

$$=3\{1.035753\}$$

$$=3.107259 = 3.107$$

Example 5. Find the coefficient of x^n in the expansion of

$$\frac{1-x}{(1+x)^2}$$

Solution:-

$$\frac{1-x}{(1+x)^2} = (1-x)(1+x)^{-2}$$

$$=(1-x)\left\{1+(-2)x+\frac{(-2)(-3)}{2!}x^2+\frac{(-2)(-3)(-4)}{3!}x^3+\cdots\right\}$$

$$=(1-x)\{1+(-1)^1 2x+(-1)^2 3x^2+(-1)^3 4x^3+\cdots\}$$

Following the above pattern, we have

$$=(1-x)\{1+(-1)^1 2x+(-1)^2 3x^2+(-1)^3 4x^3+\cdots + (-1)^{n-1}\cdot nx^{n-1} + (-1)^n(n+1)x^n+\cdots\}$$

The terms involving x^n in the expansion of $(1-x)(1+x)^{-2}$ are

$$1(-1)^n(n+1)x^n = (-1)^n(n+1)x^n \quad \text{and}$$

$$(-x)(-1)^{n-1}\cdot nx^{n-1} = (-1)(-1)^{n-1}nx^n = (-1)^n nx^n$$

Therefore coefficient of x^n

$$=(-1)^n(n+1)+(-1)^n\cdot n$$

$$=(-1)^n\{n+1+n\} = (-1)^n(2n+1)$$

Example 6. If x is so small that its cube and higher power can be neglected, show that

$$\sqrt{\frac{1-x}{1+x}} \approx 1-x+\frac{1}{2}x^2$$

Solution:-

$$\text{L.H.S} = \sqrt{\frac{1-x}{1+x}} = \frac{(1-x)^{1/2}}{(1+x)^{1/2}}$$

$$=(1-x)^{1/2}(1+x)^{-1/2}$$

$$=\left[1+\frac{1}{2}(-x)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2+\cdots\right]\left[1+\left(-\frac{1}{2}\right)x+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}x^2+\cdots\right]$$

$$=\left[1-\frac{1}{2}x+\frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^2+\cdots\right]\left[1-\frac{1}{2}x+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^2+\cdots\right]$$

$$=\left(1-\frac{1}{2}x-\frac{1}{8}x^2+\cdots\right)\left(1-\frac{1}{2}x+\frac{3}{2}x^2+\cdots\right)$$

$$=1\left(1-\frac{1}{2}x+\frac{3}{2}x^2\right)-\frac{1}{2}x\left(1-\frac{1}{2}x\right)-\frac{1}{8}x^2(1)+\cdots$$

$$=1-\frac{1}{2}x+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{4}x^2-\frac{1}{8}x^2+\cdots$$

$= 1-\left(\frac{1}{2}+\frac{1}{2}\right)x+\left(\frac{3}{8}+\frac{1}{4}-\frac{1}{8}\right)x^2+\dots$

$= 1-1x+\frac{1}{2}x^2+\dots \approx 1-x+\frac{1}{2}x^2 = R.H.S$

$\Rightarrow \sqrt{\frac{1-x}{1+x}} \approx 1-x+\frac{1}{2}x^2$

Hence proved

Example 7. If m and n are nearly equal, show that

$$\left(\frac{5m-2n}{3n}\right)^{1/3} \approx \frac{m}{m+2n}+\frac{n+m}{3n}$$

Solution:– $L.H.S = \left(\frac{5m-2n}{3n}\right)^{1/3}$

Let $m = n+h$

where h is so small that its squares and higher powers be neglected. then

$L.H.S = \left(\frac{5(n+h)-2n}{3n}\right)^{1/3} = \left(\frac{5n+5h-2n}{3n}\right)^{1/3}$

$= \left(\frac{3n+5h}{3n}\right)^{1/3} = \left(\frac{3n}{3n}+\frac{5h}{3n}\right)^{1/3}$ $\quad \because (1+x)^n = 1+nx$

$= \left(1+\frac{5}{3}\frac{h}{n}\right)^{1/3} = \left(1+\frac{5}{3n}h\right)^{1/3} = 1+\frac{1}{3}\left(\frac{5}{3n}h\right)$

$= 1+\frac{5h}{9n}$

$\Rightarrow L.H.S = 1+\frac{5h}{9n} \longrightarrow (i)$

$R.H.S = \frac{m}{m+2n}+\frac{n+m}{3n}$

$= \frac{n+h}{n+h+2n}+\frac{n+n+h}{3n}$ $\quad \because m = n+h$

$= \frac{n+h}{3n+h}+\frac{2n+h}{3n}$

$= (n+h)\left\{\frac{1}{3n+h}\right\}+\left\{\frac{2n}{3n}+\frac{h}{3n}\right\}$

$= (n+h)\left\{\frac{1}{3n\left(1+\frac{h}{3n}\right)}\right\}+\left(\frac{2}{3}+\frac{h}{3n}\right)$

$= \frac{n+h}{3n}\left(1+\frac{h}{3n}\right)^{-1}+\left(\frac{2}{3}+\frac{h}{3n}\right)$

$= \frac{n+h}{3n}\left(1+(-1)\frac{h}{3n}+\dots\right)+\frac{2}{3}+\frac{h}{3n}$

$= \left(\frac{n}{3n}+\frac{h}{3n}\right)\left(1-\frac{h}{3n}+\dots\right)+\frac{2}{3}+\frac{h}{3n}$

$= \left(\frac{1}{3}+\frac{h}{3n}\right)\left(1-\frac{h}{3n}+\dots\right)+\frac{2}{3}+\frac{h}{3n}$

$= \frac{1}{3}-\frac{h}{9n}+\frac{h}{3n}+\frac{2}{3}+\frac{h}{3n}$

$= \left(\frac{1}{3}+\frac{2}{3}\right)+\left(-\frac{1}{9}+\frac{1}{3}+\frac{1}{3}\right)\frac{h}{n}$

$= \frac{1+2}{3}+\frac{-1+3+3}{9}\cdot\frac{h}{n}$

$= 1+\frac{5h}{9n}$

$\Rightarrow R.H.S = 1+\frac{5h}{9n} \longrightarrow (ii)$

From (i) and (ii)

$\left(\frac{5m-2n}{3n}\right)^{1/3} \approx 1+\frac{5h}{9n}$

Hence proved

Example 8. Identify the series: $1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\dots$ as a binomial expansion and find its sum.

Solution:– Consider

$y = 1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\dots \longrightarrow (i)$

Let the series be identical with

$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots \longrightarrow (ii)$

By comparing (i) and (ii) we get

$y = (1+x)^n \longrightarrow (iii)$, $\quad nx = \frac{1}{3}$

$\Rightarrow x = \frac{1}{3n}$

and $\frac{n(n-1)}{2!}x^2 = \frac{1.3}{3.6}$

$\Rightarrow n(n-1)x^2 = \frac{1}{3}$

$\Rightarrow n(n-1)\left(\frac{1}{3n}\right)^2 = \frac{1}{3}$ $\quad \left(\because x = \frac{1}{3n}\right)$

$\Rightarrow n(n-1)\frac{1}{9n^2} = \frac{1}{3}$

$\Rightarrow \frac{n-1}{n} = \frac{9}{3} \Rightarrow n-1 = 3n$

$\Rightarrow -1 = 3n-n \Rightarrow -1 = 2n$

$\Rightarrow n = -\frac{1}{2}$

Now $x = \frac{1}{3\left(-\frac{1}{2}\right)} \Rightarrow x = \frac{1}{-\frac{3}{2}} = -\frac{2}{3}$

Putting values of x and n in (iii)

$y = \left(1+\left(-\frac{2}{3}\right)\right)^{-1/2} = \left(1-\frac{2}{3}\right)^{-1/2} = \left(\frac{1}{3}\right)^{-1/2}$

$\rightarrow y = \left(\frac{3}{1}\right)^{1/2} \rightarrow y = \sqrt{3}$

Hence sum of series is $= \sqrt{3}$

Example 9. For

$$y = \frac{1}{2}\left(\frac{4}{9}\right) + \frac{1.3}{2^2 2!}\left(\frac{4}{9}\right)^2 + \frac{1.3.5}{2^3 3!}\left(\frac{4}{9}\right)^3 + \cdots$$

show that $5y^2 + 10y - 4 = 0$

Solution:- Consider

$$y = \frac{1}{2}\left(\frac{4}{9}\right) + \frac{1.3}{2^2 2!}\left(\frac{4}{9}\right)^2 + \frac{1.3.5}{2^3 . 3!}\left(\frac{4}{9}\right)^3 + \cdots$$

$$\rightarrow y = \frac{2}{9} + \frac{3}{8} \cdot \frac{16}{81} + \frac{1.3.5}{2^3 . 3!}\left(\frac{4}{9}\right)^3 + \cdots$$

$$\rightarrow y = \frac{2}{9} + \frac{2}{27} + \frac{1.3.5}{2^3 . 3!}\left(\frac{4}{9}\right)^3 + \cdots$$

Adding 1 on both sides

$$1 + y = 1 + \frac{2}{9} + \frac{2}{27} + \frac{1.3.5}{2^3 3!}\left(\frac{4}{9}\right)^3 + \cdots \longrightarrow (i)$$

Let the series be identical with

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \longrightarrow (ii)$$

Comparing (i) and (ii)

$$1 + y = (1+x)^n \longrightarrow (iii)$$

$$nx = \frac{2}{9} \rightarrow x = \frac{2}{9n}$$

and $\frac{n(n-1)}{2!}x^2 = \frac{2}{27}$

$$\rightarrow \frac{n(n-1)}{2!}\left(\frac{2}{9n}\right)^2 = \frac{2}{27} \quad \left(\because x = \frac{2}{9n}\right)$$

$$\rightarrow n(n-1) . \frac{4}{81n^2} = \frac{4}{27}$$

$$\rightarrow \frac{n-1}{n} = \frac{4}{27} \times \frac{81}{4}$$

$$\rightarrow \frac{n-1}{n} = 3 \rightarrow n - 1 = 3n$$

$$\rightarrow n - 3n = 1 \rightarrow -2n = 1$$

$$\rightarrow n = -\frac{1}{2} \quad \text{Now } x = \frac{2}{9\left(-\frac{1}{2}\right)} = \frac{-4}{9}$$

Putting values of x and n in (iii)

$$(iii) \rightarrow 1 + y = \left(1 + \left(-\frac{4}{9}\right)\right)^{-1/2}$$

$$1 + y = \left(1 - \frac{4}{9}\right)^{-1/2} \rightarrow 1 + y = \left(\frac{5}{9}\right)^{-1/2}$$

$$1 + y = \left(\frac{9}{5}\right)^{1/2}$$

$$\rightarrow (1+y)^2 = \frac{9}{5} \quad \text{(By squaring)}$$

$$\rightarrow 1 + y^2 + 2y = \frac{9}{5}$$

$$\rightarrow 5 + 5y^2 + 10y = 9$$

$$\rightarrow 5y^2 + 10y + 5 - 9 = 0$$

$$\rightarrow 5y^2 + 10y - 4 = 0$$

Hence proved

Exercise 8.3

Q1. Expand the following upto 4 terms, taking the values of x such that the expansion in each case is valid.

i) $(1-x)^{1/2}$

Solution:-

$$(1-x)^{1/2} = 1 + \left(\frac{1}{2}\right)(-x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^3 + \cdots$$

$$= 1 - \frac{1}{2}x + \frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^2 - \frac{1}{6}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)x^3$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \cdots$$

Note:- The expansion of $(1-x)^{1/2}$ is valid if $|x| < 1$

ii) $(1+2x)^{-1}$

Solution:-

$$(1+2x)^{-1} = 1 + (-1)(2x) + \frac{(-1)(-2)}{2!}(2x)^2 + \frac{(-1)(-2)(-3)}{3!}(2x)^3 + \cdots$$

$$= 1 - 2x + 4x^2 - 8x^3 + \cdots$$

Note:- The expansion of $(1+2x)^{-1}$ is valid if $|2x| < 1 \rightarrow 2|x| < 1 \rightarrow |x| < \frac{1}{2}$

iii) $(1+x)^{-1/3}$

Solution:-

$$(1+x)^{-1/3} = 1 + \left(-\frac{1}{3}\right)x + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}x^3 + \cdots$$

$$= 1 - \frac{1}{3}x + \frac{1}{2}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)x^2 + \frac{1}{6}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)x^3 + \cdots$$

$$= 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \cdots$$

Note:- The expansion $(1+x)^{-1/3}$ is valid if $|x| < 1$

iv) $(4-3x)^{\frac{1}{2}}$

Solution:-

$(4-3x)^{\frac{1}{2}} = \left[4\left(1-\frac{3}{4}x\right)\right]^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1-\frac{3}{4}x\right)^{\frac{1}{2}} = 2\left(1-\frac{3}{4}x\right)^{\frac{1}{2}}$

$= 2\left\{1+\frac{1}{2}\left(-\frac{3}{4}x\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{3}{4}x\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{3}{4}x\right)^3+\cdots\right\}$

$= 2\left\{1-\frac{3}{8}x+\frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{9}{16}x^2\right)+\frac{1}{6}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{27}{64}x^3\right)+\cdots\right\}$

$= 2\left\{1-\frac{3}{8}x-\frac{9}{128}x^2-\frac{27}{1024}x^3-\cdots\right\}$

$= 2-\frac{3}{4}x-\frac{9}{64}x^2-\frac{27}{512}x^3-\cdots$

Note:- The expansion of $(4-3x)^{\frac{1}{2}}$ is valid if $\left|\frac{3}{4}x\right|<1$

$\rightarrow \frac{3}{4}|x|<1 \rightarrow |x|<\frac{4}{3}$

v) $(8-2x)^{-1}$

Solution:-

$(8-2x)^{-1} = \left[8\left(1-\frac{2}{8}x\right)\right]^{-1} = 8^{-1}\left(1-\frac{1}{4}x\right)^{-1} = \frac{1}{8}\left(1-\frac{1}{4}x\right)^{-1}$

$= \frac{1}{8}\left\{1+(-1)\left(-\frac{1}{4}x\right)+\frac{(-1)(-2)}{2!}\left(-\frac{1}{4}x\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(-\frac{1}{4}x\right)^3+\cdots\right\}$

$= \frac{1}{8}\left\{1+\frac{1}{4}x+\frac{1}{16}x^2+\frac{1}{64}x^3+\cdots\right\}$

$= \frac{1}{8}+\frac{1}{32}x+\frac{1}{128}x^2+\frac{1}{512}x^3+\cdots$

Note:- The expansion of $(8-2x)^{-1}$ is valid if $\left|\frac{1}{4}x\right|<1 \rightarrow \frac{1}{4}|x|<1$

$\rightarrow |x|<4$

vi) $(2-3x)^{-2}$

Solution:-

$(2-3x)^{-2} = \left[2\left(1-\frac{3}{2}x\right)\right]^{-2} = 2^{-2}\left(1-\frac{3}{2}x\right)^{-2} = \frac{1}{4}\left(1-\frac{3}{2}x\right)^{-2}$

$= \frac{1}{4}\left\{1+(-2)\left(-\frac{3}{2}x\right)+\frac{(-2)(-3)}{2!}\left(-\frac{3}{2}x\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(-\frac{3}{2}x\right)^3+\cdots\right\}$

$= \frac{1}{4}\left\{1+3x+\frac{27}{4}x^2+\frac{27}{2}x^3+\cdots\right\}$

$= \frac{1}{4}+\frac{3}{4}x+\frac{27}{16}x^2+\frac{27}{8}x^3+\cdots$

Note:- The expansion of $(2-3x)^{-2}$ is valid if $\left|\frac{3}{2}x\right|<1 \rightarrow \frac{3}{2}|x|<1$

$\rightarrow |x|<\frac{2}{3}$

vii) $\dfrac{(1-x)^{-1}}{(1+x)^2}$

Solution:-

$\dfrac{(1-x)^{-1}}{(1+x)^2} = (1-x)^{-1}(1+x)^{-2}$

$$= \left\{1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \frac{(-1)(-2)(-3)}{3!}(-x)^3 + \cdots\right\}\left\{1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \cdots\right\}$$

$$= \{1 + x + x^2 + x^3 + \cdots\}\{1 - 2x + 3x^2 - 4x^3 + \cdots\}$$

$$= 1(1 - 2x + 3x^2 - 4x^3 + \cdots) + x(1 - 2x + 3x^2 \cdots) + x^2(1 - 2x + \cdots) + x^3(1 - \cdots) + \cdots$$

$$= 1 - 2x + 3x^2 - 4x^3 + x - 2x^2 + 3x^3 + x^2 - 2x^3 + x^3 + \cdots$$

$$= 1 - 2x + x + 3x^2 - 2x^2 + x^2 - 4x^3 + 3x^3 - 2x^3 + x^3$$

$$= 1 - x + 2x^2 - 2x^3 + \cdots$$

Note:- The expansion of $(1-x)^{-1}$ and $(1+x)^{-2}$ are valid if $|x| < 1$

Thus expansion of $(1-x)^{-1}(1+x)^{-2}$ is valid if $|x|<1$

viii) $\dfrac{\sqrt{1+2x}}{1-x}$

Solution:-

$$\frac{\sqrt{1+2x}}{1-x} = (1+2x)^{1/2}(1-x)^{-1}$$

$$= \left\{1 + \tfrac{1}{2}(2x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(2x)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(2x)^3 + \cdots\right\}$$
$$\times \left\{1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \frac{(-1)(-2)(-3)}{3!}(-x)^3 + \cdots\right\}$$

$$= \left\{1 + x + \tfrac{1}{2}(\tfrac{1}{2})(-\tfrac{1}{2})(4x^2) + \tfrac{1}{6}(\tfrac{1}{2})(-\tfrac{1}{2})(-\tfrac{3}{2})(8x^3) + \cdots\right\}\{1 + x + x^2 + x^3 + \cdots\}$$

$$= \{1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 + \cdots\} \times \{1 + x + x^2 + x^3 + \cdots\}$$

$$= 1(1 + x + x^2 + x^3 + \cdots) + x(1 + x + x^2 + \cdots) - \tfrac{1}{2}x^2(1 + x + \cdots) + \tfrac{1}{2}(1 + \cdots) + \cdots$$

$$= 1 + x + x^2 + x^3 + x + x^2 + x^3 - \tfrac{1}{2}x^2 - \tfrac{1}{2}x^3 + \tfrac{1}{2} + \cdots$$

$$= 1 + x + x + x^2 + x^2 - \tfrac{1}{2}x^2 + x^3 + x^3 + \cdots$$

$$= 1 + 2x + (2 - \tfrac{1}{2})x^2 + 2x^3 + \cdots = 1 + 2x + \tfrac{3}{2}x^2 + 2x^3 + \cdots$$

Note:- The expansion of $(1+2x)^{1/2}$ is valid if $|2x| < 1 \rightarrow 2|x| < 1 \rightarrow |x| < \frac{1}{2}$

and the expansion of $(1-x)^{-1}$ is valid if $|x| < \frac{1}{2}$

Thus expansion of $\dfrac{\sqrt{1+2x}}{1-x}$ is valid if $|x| < \frac{1}{2}$

ix) $\dfrac{(4+2x)^{1/2}}{2-x}$

Solution:-

$$\frac{(4+2x)^{1/2}}{2-x} = \left[4(1 + \tfrac{2}{4}x)\right]^{1/2}(2-x)^{-1} = 4^{1/2}(1 + \tfrac{1}{2}x)^{1/2}\left[2(1 - \tfrac{1}{2}x)\right]^{-1}$$

$$= 2(1 + \tfrac{1}{2}x)^{1/2} \cdot 2^{-1}(1 - \tfrac{1}{2}x)^{-1} = (1 + \tfrac{1}{2}x)^{1/2}(1 - \tfrac{1}{2}x)^{-1}$$

$$= \left\{1 + \tfrac{1}{2}(\tfrac{1}{2}x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(\tfrac{1}{2}x)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(\tfrac{1}{2}x)^3 + \cdots\right\}\left\{1 + (-1)(-\tfrac{1}{2}x) + \frac{(-1)(-2)}{2!}(-\tfrac{1}{2}x)^2 + \frac{(-1)(-2)(-3)}{3!}(-\tfrac{1}{2}x)^3 + \cdots\right\}$$

$$= \left\{1+\frac{1}{4}x+\frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{4}x^2\right)+\frac{1}{6}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{8}x^3\right)+\cdots\right\}\left\{1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\cdots\right\}$$

$$= \left\{1+\frac{1}{4}x-\frac{1}{32}x^2+\frac{1}{128}x^3+\cdots\right\}\left\{1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\cdots\right\}$$

$$= 1\left(1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\cdots\right)+\frac{1}{4}x\left(1+\frac{1}{2}x+\frac{1}{4}x^2+\cdots\right)-\frac{1}{32}x^2\left(1+\frac{1}{2}x+\cdots\right)+\frac{1}{128}x^3(1+\cdots)+\cdots$$

$$= 1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\frac{1}{4}x+\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{1}{32}x^2-\frac{1}{64}x^3+\frac{1}{128}x^3+\cdots$$

$$= 1+\left(\frac{1}{2}+\frac{1}{4}\right)x+\left(\frac{1}{4}+\frac{1}{8}-\frac{1}{32}\right)x^2+\left(\frac{1}{8}+\frac{1}{16}-\frac{1}{64}+\frac{1}{128}\right)x^3+\cdots$$

$$= 1+\frac{2+1}{4}x+\frac{8+4-1}{32}x^2+\frac{16+8-2+1}{128}x^3+\cdots$$

$$= 1+\frac{3}{4}x+\frac{11}{32}x^2+\frac{23}{128}x^3+\cdots$$

Note:- The expansion $(4+2x)^{1/2}$ is valid if $\left|\frac{1}{2}x\right|<1 \rightarrow \frac{1}{2}|x|<1 \rightarrow |x|<2$ and the expansion of $(2-x)^{-1}$ is valid if $\left|\frac{1}{2}x\right|<1 \rightarrow \frac{1}{2}|x|<1 \rightarrow |x|<2$

Thus expansion of $\frac{(4+2x)^{1/2}}{2-x}$ is valid if $|x|<2$

x) $(1+x-2x^2)^{1/2}$

Solution:-

$$\left[1+(x-2x^2)\right]^{1/2} = 1+\frac{1}{2}(x-2x^2)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(x-2x^2)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(x-2x^2)^3+\cdots$$

$$= 1+\frac{1}{2}x-x^2+\frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left[x^2+4x^4-4x^3\right]+\frac{1}{6}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left\{x^3-3x^2(2x^2)+3x(2x^2)^2-(2x^2)^3\right\}+\cdots$$

$$= 1+\frac{1}{2}x-x^2-\frac{1}{8}(x^2+4x^4-4x^3)+\frac{1}{16}(x^3-6x^4+12x^5-8x^6)+\cdots$$

$$= 1+\frac{1}{2}x-x^2-\frac{1}{8}x^2-\frac{1}{2}x^4+\frac{1}{2}x^3+\frac{1}{16}x^3-\frac{3}{8}x^4+\frac{3}{4}x^5-\frac{1}{2}x^6+\cdots$$

$$= 1+\frac{1}{2}x+\left(-1-\frac{1}{8}\right)x^2+\left(\frac{1}{2}+\frac{1}{16}\right)x^3+\left(-\frac{1}{2}-\frac{3}{8}\right)x^4+\frac{3}{4}x^5-\frac{1}{2}x^6+\cdots$$

$$= 1+\frac{1}{2}x-\frac{9}{8}x^2+\left(\frac{8+1}{16}\right)x^3+\left(\frac{-4-3}{8}\right)x^4+\frac{3}{4}x^5-\frac{1}{2}x^6+\cdots$$

$$= 1+\frac{1}{2}x-\frac{9}{8}x^2+\frac{9}{16}x^3-\frac{7}{8}x^4+\frac{3}{4}x^5-\frac{1}{2}x^6+\cdots$$

$$= 1+\frac{1}{2}x-\frac{9}{8}x^2+\frac{9}{16}x^3-\cdots$$

Note:- The expansion of $\left[1+(x-2x^2)\right]^{1/2}$ is valid if

$$|x-2x^2|<1$$

$\rightarrow x-2x^2<1$ or $-(x-2x^2)<1$

$\rightarrow x-2x^2-1<0$ or $-x+2x^2-1<0$

$-2x^2+x-1<0$ or $2x^2-x-1<0$

Solving $-2x^2+x-1=0$ Solving $2x^2-x-1=0$

$\rightarrow x=\frac{-1\pm\sqrt{(1)^2-4(-2)(-1)}}{2(-2)}$ or $2x^2-2x+x-1=0$

$\Rightarrow x = \dfrac{-1 \pm \sqrt{1-8}}{-4}$ or $2x(x-1)+1(x-1)=0$

$\Rightarrow x = \dfrac{-1 \pm \sqrt{-7}}{-4}$ (rejected being complex)

$\Rightarrow (x-1)(2x+1)=0$

$x-1=0,\ 2x+1=0$

$x=1,\ x=-\frac{1}{2}$

Thus expansion of $[1+(x-2x^2)]^{\frac{1}{2}}$ is valid if $-\frac{1}{2}<x<1$

xi) $(1-2x+3x^2)^{\frac{1}{2}}$

Solution:-

$$[1+(3x^2-2x)]^{\frac{1}{2}} = 1+\frac{1}{2}(3x^2-2x)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(3x^2-2x)^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(3x^2-2x)^3+\ldots\ldots$$

$$=1+\frac{3}{2}x^2-x+\frac{1}{2}(\frac{1}{2})(-\frac{1}{2})(9x^4-12x^3+4x^2)+\frac{1}{6}(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})[(3x^2)^3-3(3x^2)^2(2x)+3(3x^2)(2x)^2-(2x)^3]+\ldots\ldots$$

$$=1+\frac{3}{2}x^2-x-\frac{1}{8}(9x^4-12x^3+4x^2)+\frac{1}{16}(27x^6-54x^5+36x^4-8x^3)+\ldots\ldots$$

$$=1+\frac{3}{2}x^2-x-\frac{9}{8}x^4+\frac{3}{2}x^3-\frac{1}{2}x^2+\frac{27}{16}x^6-\frac{27}{8}x^5+\frac{9}{4}x^4-\frac{1}{2}x^3+\ldots\ldots$$

$$=1-x+(\frac{3}{2}-\frac{1}{2})x^2+(\frac{3}{2}-\frac{1}{2})x^3+(\frac{9}{4}-\frac{9}{8})x^4-\frac{27}{8}x^5+\frac{27}{16}x^6+\ldots\ldots$$

$$=1-x+\frac{3-1}{2}x^2+\frac{3-1}{2}x^3+\frac{18-9}{8}x^4-\frac{27}{8}x^5+\frac{27}{16}x^6+\ldots\ldots$$

$$=1-x+x^2+x^3+\frac{9}{8}x^4-\frac{27}{8}x^5+\frac{27}{16}x^6+\ldots\ldots$$

$$=1-x+x^2+x^3+\ldots\ldots$$

Note:- The expansion $[1+(3x^2-2x)]^{\frac{1}{2}}$ is valid if $|3x^2-2x|<1$

$3x^2-2x<1$ or $-(3x^2-2x)<1$

$3x^2-2x-1<0$ or $-3x^2+2x-1<0$

Solving $3x^2-2x-1=0$ or $3x^2-2x+1<0$

$3x^2-3x+x-1=0$ Solving $3x^2-2x+1=0$

$3x(x-1)+1(x-1)=0$

$x=\dfrac{-(-2)\pm\sqrt{(-2)^2-(4)(3)(1)}}{2(3)}$

$(x-1)(3x+1)=0$

$x-1=0,\ 3x+1=0$

$x=\dfrac{2\pm\sqrt{4-12}}{6}$

$x=1,\ x=-\frac{1}{3}$

$\Rightarrow x=\dfrac{2\pm\sqrt{-8}}{6}$ (rejected being complex)

Thus expansion of $[1+(3x^2-2x)]^{\frac{1}{2}}$ is valid if $-\frac{1}{3}<x<1$.

Q2. Using Binomial theorem find the value of the following to three places of decimals.

i) $\sqrt{99}$

Solution:-

$\sqrt{99} = (99)^{1/2} = (100-1)^{1/2}$

$= \left[100\left(1-\frac{1}{100}\right)\right]^{1/2} = (100)^{1/2}(1-0.01)^{1/2}$

$= 10\left\{1+\frac{1}{2}(-0.01)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(-0.01)^2+\dots\right\}$

$= 10\left\{1-0.005+\frac{1}{2}(\frac{1}{2})(-\frac{1}{2})(0.0001)+\dots\right\}$

$= 10\{1-0.005-0.0000125-\dots\}$

$= 10(1-0.0050125)$

$= 10(0.9949) = 9.95$

ii) $(.98)^{1/2}$

Solution:-

$(0.98)^{1/2} = (1-0.02)^{1/2}$

$= 1+\frac{1}{2}(-0.02)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(-0.02)^2+\dots$

$= 1-0.01+\frac{1}{2}(\frac{1}{2})(-\frac{1}{2})(0.00040)+\dots$

$= 1-0.01-\frac{1}{8}(0.00040)+\dots$

$= 1-0.01-0.000050+\dots$

$= 1-0.010050 = 0.989 = 0.99$

iii) $(1.03)^{1/3}$

Solution:-

$(1.03)^{1/3} = (1+0.03)^{1/3}$

$= 1+\frac{1}{3}(0.03)+\frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}(0.03)^2+\dots$

$= 1+0.010+\frac{1}{3}(\frac{1}{2})(-\frac{2}{3})(0.0009)+\dots$

$= 1+0.010-\frac{1}{9}(0.0009)+\dots$

$= 1+0.010-0.0001+\dots$

$= 1.0099$

iv) $\sqrt[3]{65}$

Solution:-

$\sqrt[3]{65} = (65)^{1/3} = (64+1)^{1/3}$

$= \left[64\left(1+\frac{1}{64}\right)\right]^{1/3} = (64)^{1/3}\left(1+\frac{1}{64}\right)^{1/3}$

$= (4^3)^{1/3}\left(1+\frac{1}{64}\right)^{1/3}$

$= 4\left(1+\frac{1}{64}\right)^{1/3}$

$= 4\left\{1+\frac{1}{3}\left(\frac{1}{64}\right)+\frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}\left(\frac{1}{64}\right)^2+\dots\right\}$

$= 4\left\{1+\frac{1}{192}+\frac{1}{2}(\frac{1}{3})(-\frac{2}{3})(\frac{1}{4096})+\dots\right\}$

$= 4\left\{1+0.0052-\frac{1}{36864}+\dots\right\}$

$= 4\{1+0.0052-0.000027+\dots\}$

$= 4(1.0051) = 4.02$

v) $\sqrt[4]{17}$

Solution:-

$\sqrt[4]{17} = (17)^{1/4} = (16+1)^{1/4}$

$= \left[16\left(1+\frac{1}{16}\right)\right]^{1/4} = (16)^{1/4}\left(1+\frac{1}{16}\right)^{1/4}$

$= (2^4)^{1/4}\left(1+\frac{1}{16}\right)^{1/4} = 2\left(1+\frac{1}{16}\right)^{1/4}$

$= 2\left\{1+\frac{1}{4}\left(\frac{1}{16}\right)+\frac{\frac{1}{4}(\frac{1}{4}-1)}{2!}\left(\frac{1}{16}\right)^2+\dots\right\}$

$= 2\left\{1+\frac{1}{64}+\frac{1}{2}(\frac{1}{4})(-\frac{3}{4})(\frac{1}{256})+\dots\right\}$

$= 2\left\{1+0.015-\frac{3}{8192}+\dots\right\}$

$= 2\{1+0.015-0.00036+\dots\}$

$= 2(1.014) = 2.029 = 2.03$

vi) $\sqrt[5]{31}$

Solution:-

$\sqrt[5]{31} = (31)^{1/5} = (32-1)^{1/5} = \left[32\left(1-\frac{1}{32}\right)\right]^{1/5}$

$= (32)^{1/5}\left[1-\frac{1}{32}\right]^{1/5} = (2^5)^{1/5}\left(1-\frac{1}{32}\right)^{1/5}$

$= 2\left(1-\frac{1}{32}\right)^{1/5}$

$= 2\left\{1+(\frac{1}{5})(-\frac{1}{32})+\frac{\frac{1}{5}(\frac{1}{5}-1)}{2!}\left(-\frac{1}{32}\right)^2+\dots\right\}$

$= 2\left\{1-\frac{1}{160}+\frac{1}{2}(\frac{1}{5})(-\frac{4}{5})(\frac{1}{1024})+\dots\right\}$

$= 2\left\{1-0.006-\frac{1}{12800}+\dots\right\}$

$= 2\{1-0.006-0.00078+\dots\}$

$= 2(0.993) = 1.987$

vii) $\frac{1}{\sqrt[3]{998}}$

Solution:-

$$\frac{1}{\sqrt[3]{998}} = \frac{1}{(998)^{1/3}} = (998)^{-1/3}$$

$$= (1000-2)^{-1/3} = \left[1000\left(1-\frac{2}{1000}\right)\right]^{-1/3}$$

$$= (10^3)^{-1/3}\left(1-\frac{1}{500}\right)^{-1/3} = 10^{-1}\left(1-\frac{1}{500}\right)^{-1/3}$$

$$= \frac{1}{10}\left\{1+\left(-\frac{1}{3}\right)\left(-\frac{1}{500}\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}\left(-\frac{1}{500}\right)^2+\cdots\right\}$$

$$= \frac{1}{10}\left\{1+\frac{1}{1500}+\frac{1}{2}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(\frac{1}{250000}\right)+\cdots\right\}$$

$$= \frac{1}{10}\{1+0.0006+\cdots\} = \frac{1}{10}(1.0006)$$

$$= 0.100$$

viii) $\frac{1}{\sqrt[5]{252}}$

Solution:-

$$\frac{1}{\sqrt[5]{252}} = \frac{1}{(252)^{1/5}} = (252)^{-1/5}$$

$$= (243+9)^{-1/5} = \left[243\left(1+\frac{9}{243}\right)\right]^{-1/5}$$

$$= (3^5)^{-1/5}\left(1+\frac{9}{243}\right)^{-1/5} = 3^{-1}\left(1+\frac{1}{27}\right)^{-1/5}$$

$$= \frac{1}{3}\left\{1+\left(-\frac{1}{5}\right)\left(\frac{1}{27}\right)+\frac{\left(-\frac{1}{5}\right)\left(-\frac{1}{5}-1\right)}{2!}\left(\frac{1}{27}\right)^2+\cdots\right\}$$

$$= \frac{1}{3}\left\{1-\frac{1}{135}+\frac{1}{2}\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)\left(\frac{1}{729}\right)+\cdots\right\}$$

$$= \frac{1}{3}\{1-0.007+0.000016+\cdots\}$$

$$= \frac{1}{3}(0.993) = 0.331$$

ix) $\frac{\sqrt{7}}{\sqrt{8}}$

Solution:-

$$\frac{\sqrt{7}}{\sqrt{8}} = \sqrt{\frac{7}{8}} = \left(1-\frac{1}{8}\right)^{1/2}$$

$$= 1+\frac{1}{2}\left(-\frac{1}{8}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{8}\right)^2+\cdots$$

$$= 1-\frac{1}{16}+\frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{64}\right)+\cdots$$

$$= 1-0.062-\frac{1}{512}+\cdots$$

$$= 1-0.062-0.0019+\cdots$$

$$= 0.936$$

x) $(.998)^{-1/3}$

Solution:-

$$(.998)^{-1/3} = (1-0.002)^{-1/3}$$

$$= 1+\left(-\frac{1}{3}\right)(-0.002)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}(-0.002)^2+\cdots$$

$$= 1+0.00066+\frac{1}{2}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)(0.000004)+\cdots$$

$$= 1+0.00066+\cdots$$

$$= 1.00066 = 1.001$$

xi) $\frac{1}{\sqrt[6]{486}}$

Solution:-

$$\frac{1}{\sqrt[6]{486}} = \frac{1}{(486)^{1/6}} = (486)^{-1/6}$$

$$= (729-243)^{-1/6} = \left[729\left(1-\frac{243}{729}\right)\right]^{-1/6}$$

$$= (3^6)^{-1/6}\left(1-\frac{1}{3}\right)^{-1/6}$$

$$= 3^{-1}\left(1-\frac{1}{3}\right)^{-1/6} = \frac{1}{3}\left(1-\frac{1}{3}\right)^{-1/6}$$

$$= \frac{1}{3}\left\{1+\left(-\frac{1}{6}\right)\left(-\frac{1}{3}\right)+\frac{\left(-\frac{1}{6}\right)\left(-\frac{1}{6}-1\right)}{2!}\left(-\frac{1}{3}\right)^2+\cdots\right\}$$

$$= \frac{1}{3}\left\{1+\frac{1}{18}+\frac{1}{2}\left(-\frac{1}{6}\right)\left(-\frac{7}{6}\right)\left(\frac{1}{9}\right)+\cdots\right\}$$

$$= \frac{1}{3}\{1+0.05+0.01+\cdots\}$$

$$= \frac{1}{3}\{1.06\} = 0.3536$$

xii) $(1280)^{1/4}$

Solution:-

$$(1280)^{1/4} = (1296-16)^{1/4}$$

$$= \left[1296\left(1-\frac{16}{1296}\right)\right]^{1/4} = (6^4)^{1/4}\left(1-\frac{1}{81}\right)^{1/4}$$

$$= 6\left\{1+\frac{1}{4}\left(-\frac{1}{81}\right)+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\left(-\frac{1}{81}\right)^2+\cdots\right\}$$

$$= 6\left\{1-\frac{1}{324}+\frac{1}{2}\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)\left(\frac{1}{6561}\right)+\cdots\right\}$$

$$= 6\{1-0.003-\cdots\}$$

$$= 6(0.997) = 5.98$$

Q3. Find the coefficient of x^n in the expansion of

i) $\dfrac{1+x^2}{(1+x)^2}$

Solution:-

$$\frac{1+x^2}{(1+x)^2} = (1+x^2)(1+x)^{-2}$$

$$=(1+x^2)\left\{1+(-2)x+\frac{(-2)(-3)}{2!}x^2+\frac{(-2)(-3)(-4)}{3!}x^3+\cdots\right\}$$

$$=(1+x^2)\{1+(-1)^1 2x+(-1)^2 3x^2+(-1)^3\cdot 4x^3+\cdots\}$$

Following the above pattern, we have

$$=(1+x^2)\{1+(-1)^1 2x+(-1)^2 3x^2+(-1)^3\cdot 4x^3+\cdots + (-1)^{n-2}(n-1)x^{n-2}+(-1)^{n-1}nx^{n-1}+(-1)^n(n+1)x^n+\cdots\}$$

The terms involving x^n in the expression of $(1+x^2)(1+x)^{-2}$ are

$1(-1)^n(n+1)x^n = (-1)^n(n+1)x^n$ and

$(x^2)(-1)^{n-2}(n-1)x^{n-2} = (-1)^{n-2}(n-1)x^n$

Therefore coefficients of x^n

$$=(-1)^n(n+1)+(-1)^{n-2}(n-1)$$

$$=(-1)^n\{n+1+(-1)^{-2}(n-1)\}$$

$$=(-1)^n\{n+1+n-1\} = (-1)^n(2n)$$

ii) $\dfrac{(1+x)^2}{(1-x)^2}$

Solution:-

$$\frac{(1+x)^2}{(1-x)^2} = (1+x)^2(1-x)^{-2}$$

$$=(1+x)^2\left\{1+(-2)(-x)+\frac{(-2)(-3)}{2!}(-x)^2+\frac{(-2)(-3)(-4)}{3!}(-x)^3+\cdots\right\}$$

$$=(1+x)^2\{1+2x+3x^2+4x^3+\cdots\}$$

Following the above pattern we have

$$=(1+x^2+2x)\{1+2x+3x^2+4x^3+\cdots+(n-1)x^{n-2}+nx^{n-1}+(n+1)x^n+\cdots\}$$

The terms involving x^n in the expansion of $(1+x)^2(1-x)^{-2}$ are

$1(n+1)x^n = (n+1)x^n$

$(2x)nx^{n-1} = 2nx^n$ and

$\therefore x^2(n-1)x^{n-2} = (n-1)x^n$

Therefore coefficients of x^n

$$= n+1+2n+n-1 = 4n$$

iii) $\dfrac{(1+x)^3}{(1-x)^2}$

Solution:-

$$\frac{(1+x)^3}{(1-x)^2} = (1+x)^3(1-x)^{-2}$$

$$=(1+3x+3x^2+x^3)\left\{1+(-2)(-x)+\frac{(-2)(-3)}{2!}(-x)^2+\frac{(-2)(-3)(-4)}{3!}(-x)^3+\cdots\right\}$$

$$=(1+3x+3x^2+x^3)\{1+2x+3x^2+4x^3+\cdots\}$$

Following the above pattern we have

$$=(1+3x+3x^2+x^3)\{1+2x+3x^2+4x^3+\cdots+(n-2)x^{n-3}+(n-1)x^{n-2}+nx^{n-1}+(n+1)x^n+\cdots\}$$

The terms involving x^n are

$1\cdot(n+1)x^n = (n+1)x^n$

$3x\cdot nx^{n-1} = 3nx^n$

$3x^2(n-1)x^{n-2} = 3(n-1)x^n$

and $x^3(n-2)x^{n-3} = (n-2)x^n$

$\therefore$ Therefore coefficient of x^n

$$= n+1+3n+3(n-1)+n-2$$

$$= n+1+3n+3n-3+n-2$$

$$= 8n-4 = 4(2n-1)$$

iv) $\dfrac{(1+x)^2}{(1-x)^3}$

Solution:-

$$\frac{(1+x)^2}{(1-x)^3} = (1+x)^2(1-x)^{-3}$$

$$=(1+x^2+2x)\left\{1+(-3)(-x)+\frac{(-3)(-4)}{2!}(-x)^2+\frac{(-3)(-4)(-5)}{3!}(-x)^3+\cdots\right\}$$

$$=(1+x^2+2x)\left\{1+3x+\frac{3\times 4}{2}x^2+\frac{4\times 5}{2}x^3+\cdots\right\}$$

Following the above pattern we have

$$= (1+2x+x^2)\left\{1 + \frac{2\times 3}{2}x + \frac{3\times 4}{2}x^2 + \frac{4\times 5}{2}x^3 \cdots + \frac{(n-1)}{2}x^{n-2} + \frac{n(n+1)}{2}x^{n-1} + \frac{(n+1)(n+2)}{2}x^n + \cdots\right.$$

The term involving x^n are

$$1.\frac{(n+1)(n+2)}{2}x^n = \frac{(n+1)(n+2)}{2}x^n$$

$$2x.\frac{n(n+1)}{2}x^{n-1} = \frac{2n(n+1)}{2}x^n$$

and $x^2\dfrac{(n-1)}{2}x^{n-2} = \dfrac{(n-1)n}{2}x^n$

Therefore coefficient of x^n

$$= \frac{(n+1)(n+2)}{2} + \frac{2n(n+1)}{2} + \frac{(n-1)n}{2}$$

$$= \frac{n^2+2n+n+2+2n^2+2n+n^2-n}{2}$$

$$= \frac{4n^2+4n+2}{2} = \frac{2(2n^2+2n+1)}{2}$$

$$= 2n^2+2n+1$$

v) $(1-x+x^2-x^3+\cdots)^2$

Solution:-

we know that

$$(1+x)^{-1} = 1+(-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots$$

$$\Rightarrow (1+x)^{-1} = 1-x+x^2-x^3+\cdots$$

squaring both sides

$$\Rightarrow \left[(1+x)^{-1}\right]^2 = (1-x+x^2-x^3+\cdots)^2$$

$$\Rightarrow (1-x+x^2-x^3+\cdots)^2 = (1+x)^{-2}$$

$$= 1+(-2)x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \cdots$$

$$= 1+(-1)^1 2x + (-1)^2 3x^2 + (-1)^3 4x^3 + \cdots$$

Following the above pattern we have

$$= 1+(-1)^1 2x + (-1)^2 3x^2 + (-1)^3 4x^3 + \cdots + (-1)^n(n+1)x^n + \cdots$$

The coefficient of x^n

$$= (-1)^n(n+1)$$

Q4. If x is so small that its square and higher powers can be neglected, then show that

i) $\dfrac{1-x}{\sqrt{1+x}} \approx 1 - \dfrac{3}{2}x$

Solution:-

$$L.H.S = \frac{1-x}{\sqrt{1+x}}$$

$$= (1-x)(1+x)^{-1/2}$$

$$= (1-x)\left\{1+\left(-\tfrac{1}{2}\right)x + \cdots\right\}$$

$$= (1-x)\left(1-\tfrac{1}{2}x + \cdots\right)$$

$$= 1-\tfrac{1}{2}x - x + \cdots$$

$$= 1-\left(\tfrac{1}{2}+1\right)x + \cdots$$

$$= 1-\tfrac{3}{2}x + \cdots$$

$$\approx 1-\tfrac{3}{2}x \approx R.H.S$$

Hence proved

ii) $\dfrac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1+\dfrac{3}{2}x$

Solution:-

$$L.H.S = \frac{\sqrt{1+2x}}{\sqrt{1-x}}$$

$$= (1+2x)^{1/2}(1-x)^{-1/2}$$

$$= \left\{1+\tfrac{1}{2}(2x)+\cdots\right\}\left\{1+\left(-\tfrac{1}{2}\right)(-x)+\cdots\right\}$$

$$= (1+x)\left(1+\tfrac{1}{2}x\right)+\cdots$$

$$= 1+\tfrac{1}{2}x+x+\cdots$$

$$= 1+\left(\tfrac{1}{2}+1\right)x+\cdots$$

$$= 1+\tfrac{3}{2}x+\cdots$$

$$\therefore \approx 1+\tfrac{3}{2}x \approx R.H.S$$

Hence proved

iii) $\dfrac{(9+7x)^{1/2}-(16+3x)^{1/4}}{4+5x} \approx \dfrac{1}{4} - \dfrac{17}{384}x$

Solution:-

$$L.H.S = \frac{(9+7x)^{1/2}-(16+3x)^{1/4}}{4+5x}$$

$$= \frac{\left[9\left(1+\tfrac{7}{9}x\right)\right]^{1/2} - \left[16\left(1+\tfrac{3}{16}x\right)\right]^{1/4}}{4+5x}$$

$$= \frac{\left[3^2\left(1+\frac{7}{9}x\right)\right]^{1/2} - \left[2^4\left(1+\frac{3}{16}x\right)\right]^{1/4}}{4\left(1+\frac{5}{4}x\right)}$$

$$= \frac{1}{4}\left[3\left(1+\frac{7}{9}x\right)^{1/2} - 2\left(1+\frac{3}{16}x\right)^{1/4}\right]\left(1+\frac{5}{4}x\right)^{-1}$$

$$= \frac{1}{4}\left[3\left\{1+\frac{1}{2}\left(\frac{7}{9}x\right)+\cdots\right\} - 2\left\{1+\frac{1}{4}\left(\frac{3}{16}x\right)+\cdots\right\}\right] \times \left(1+(-1)\left(\frac{5}{4}x\right)+\cdots\right)$$

$$= \frac{1}{4}\left[3\left(1+\frac{7}{18}x+\cdots\right) - 2\left(1+\frac{3}{64}x+\cdots\right)\right]\left(1-\frac{5}{4}x+\cdots\right)$$

$$= \frac{1}{4}\left[3+\frac{7}{6}x-2-\frac{3}{32}x+\cdots\right]\left(1-\frac{5}{4}x+\cdots\right)$$

$$= \frac{1}{4}\left[3-2+\left(\frac{7}{6}-\frac{3}{32}\right)x+\cdots\right]\left(1-\frac{5}{4}x+\cdots\right)$$

$$= \frac{1}{4}\left(1+\frac{112-9}{96}x+\cdots\right)\left(1-\frac{5}{4}x+\cdots\right)$$

$$= \frac{1}{4}\left(1+\frac{103}{96}x+\cdots\right)\left(1-\frac{5}{4}x+\cdots\right)$$

$$= \frac{1}{4}\left(1-\frac{5}{4}x+\frac{103}{96}x-\cdots\right)$$

$$= \frac{1}{4}\left(1+\left(\frac{-120+103}{96}\right)x-\cdots\right)$$

$$= \frac{1}{4}\left(1-\frac{17}{96}x-\cdots\right)$$

$$\approx \frac{1}{4}-\frac{1}{4}\left(\frac{17}{96}x\right)-\cdots$$

$$\approx \frac{1}{4}-\frac{17}{384}x \approx R.H.S$$

Hence proved

iv) $\dfrac{\sqrt{4+x}}{(1-x)^3} \approx 2+\dfrac{25}{4}x$

Solution:-

$$L.H.S = \frac{\sqrt{4+x}}{(1-x)^3} = (4+x)^{1/2}(1-x)^{-3}$$

$$= \left[4\left(1+\frac{1}{4}x\right)\right]^{1/2}(1-x)^{-3}$$

$$= 2\left(1+\frac{1}{4}x\right)^{1/2}(1-x)^{-3}$$

$$= 2\left[1+\frac{1}{2}\left(\frac{1}{4}x\right)+\cdots\right]\left[1+(-3)(-x)+\cdots\right]$$

$$= 2\left(1+\frac{1}{8}x+\cdots\right)\left(1+3x+\cdots\right)$$

$$= 2\left(1+3x+\frac{1}{8}x+\cdots\right)$$

$$= 2\left(1+\left(3+\frac{1}{8}\right)x+\cdots\right)$$

$$= 2\left(1+\frac{25}{8}x+\cdots\right)$$

$$= 2+\frac{25}{4}x+\cdots$$

$$\approx 2+\frac{25}{4}x \approx R.H.S$$

Hence proved

v) $\dfrac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} \approx 4\left(1-\dfrac{5}{6}x\right)$

Solution:-

$$L.H.S = \frac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}}$$

$$= \frac{(1+x)^{1/2}\left[4\left(1-\frac{3}{4}x\right)\right]^{3/2}}{\left[8\left(1+\frac{5}{8}x\right)\right]^{1/3}} = \frac{(1+x)^{1/2}\left[2^2\left(1-\frac{3}{4}x\right)\right]^{3/2}}{\left[2^3\left(1+\frac{5}{8}x\right)\right]^{1/3}}$$

$$= \frac{(1+x)^{1/2}\cdot 2^3\left(1-\frac{3}{4}x\right)^{3/2}}{2\left(1+\frac{5}{8}x\right)^{1/3}}$$

$$= \frac{8}{2}(1+x)^{1/2}\left(1-\frac{3}{4}x\right)^{3/2}\left(1+\frac{5}{8}x\right)^{-1/3}$$

$$= 4\left\{1+\frac{1}{2}x+\cdots\right\}\left\{1+\frac{3}{2}\left(-\frac{3}{4}x\right)+\cdots\right\}\left\{1+\left(-\frac{1}{3}\right)\left(\frac{5}{8}x\right)+\cdots\right\}$$

$$= 4\left(1+\frac{1}{2}x+\cdots\right)\left(1-\frac{9}{8}x+\cdots\right)\left(1-\frac{5}{24}x+\cdots\right)$$

$$= 4\left(1-\frac{9}{8}x+\frac{1}{2}x+\cdots\right)\left(1-\frac{5}{24}x+\cdots\right)$$

$$= 4\left(1+\left(-\frac{9}{8}+\frac{1}{2}\right)x+\cdots\right)\left(1-\frac{5}{24}x+\cdots\right)$$

$$= 4\left(1-\frac{5}{8}x+\cdots\right)\left(1-\frac{5}{24}x+\cdots\right)$$

$$= 4\left(1-\frac{5}{24}x-\frac{5}{8}x+\cdots\right)$$

$$= 4\left(1-\left(\frac{5}{24}+\frac{5}{8}\right)x+\cdots\right)$$

$$= 4\left(1-\frac{5+15}{24}x+\cdots\right) = 4\left(1-\frac{20}{24}x+\cdots\right)$$

$$= 4\left(1-\frac{5}{6}x+\cdots\right)$$

$$\approx 4\left(1-\frac{5}{6}x\right) \approx R.H.S$$

Hence proved

vi) $\dfrac{(1-x)^{1/2}(9-4x)^{1/2}}{(8+3x)^{1/3}} \approx \dfrac{3}{2}-\dfrac{61}{48}x$

Solution:-

$$L.H.S = \frac{(1-x)^{1/2}(9-4x)^{1/2}}{(8+3x)^{1/3}}$$

$$= \frac{(1-x)^{1/2}\left[9\left(1-\frac{4}{9}x\right)\right]^{1/2}}{\left[8\left(1+\frac{3}{8}x\right)\right]^{1/3}}$$

$$= \frac{(1-x)^{\frac{1}{2}}\left[3^2\left(1-\frac{4}{9}x\right)\right]^{\frac{1}{2}}}{\left[2^3\left(1+\frac{3}{8}x\right)\right]^{\frac{1}{3}}}$$

$$= \frac{(1-x)^{\frac{1}{2}} \cdot 3\left(1-\frac{4}{9}x\right)^{\frac{1}{2}}}{2\left(1+\frac{3}{8}x\right)^{\frac{1}{3}}}$$

$$= \frac{3}{2}(1-x)^{\frac{1}{2}}\left(1-\frac{4}{9}x\right)^{\frac{1}{2}}\left(1+\frac{3}{8}x\right)^{-\frac{1}{3}}$$

$$= \frac{3}{2}\left\{1-\frac{1}{2}x+\cdots\right\}\left\{1+\frac{1}{2}\left(-\frac{4}{9}x\right)+\cdots\right\}\left\{1+\left(-\frac{1}{3}\right)\left(\frac{3}{8}x\right)+\cdots\right\}$$

$$= \frac{3}{2}\left\{1-\frac{1}{2}x+\cdots\right\}\left\{1-\frac{2}{9}x+\cdots\right\}\left\{1-\frac{1}{8}x+\cdots\right\}$$

$$= \frac{3}{2}\left\{1-\frac{2}{9}x-\frac{1}{2}x+\cdots\right\}\left\{1-\frac{1}{8}x+\cdots\right\}$$

$$= \frac{3}{2}\left(1-\left(\frac{2}{9}+\frac{1}{2}\right)x+\cdots\right)\left(1-\frac{1}{8}x+\cdots\right)$$

$$= \frac{3}{2}\left(1-\frac{13}{18}x+\cdots\right)\left(1-\frac{1}{8}x+\cdots\right)$$

$$= \frac{3}{2}\left(1-\frac{1}{8}x-\frac{13}{18}x+\cdots\right)$$

$$= \frac{3}{2}\left(1-\left(\frac{1}{8}+\frac{13}{18}\right)x+\cdots\right)$$

$$= \frac{3}{2}\left(1-\frac{9+52}{72}x+\cdots\right)$$

$$= \frac{3}{2}\left(1-\frac{61}{72}x+\cdots\right)$$

$$= \frac{3}{2}-\frac{3}{2}\left(\frac{61}{72}x\right)+\cdots$$

$$\approx \frac{3}{2}-\frac{61}{48}x \approx R.H.S$$

Hence proved

vii) $\dfrac{\sqrt{4-x}+(8-x)^{\frac{1}{3}}}{(8-x)^{\frac{1}{3}}} \approx 2-\frac{1}{12}x$

Solution:-

$$L.H.S = \frac{\sqrt{4-x}+(8-x)^{\frac{1}{3}}}{(8-x)^{\frac{1}{3}}}$$

$$= \frac{(4-x)^{\frac{1}{2}}}{(8-x)^{\frac{1}{3}}} + \frac{(8-x)^{\frac{1}{3}}}{(8-x)^{\frac{1}{3}}}$$

$$= 1 + \frac{\left[4\left(1-\frac{1}{4}x\right)\right]^{\frac{1}{2}}}{\left[8\left(1-\frac{1}{8}x\right)\right]^{\frac{1}{3}}}$$

$$= 1 + \frac{\left[2^2\left(1-\frac{1}{4}x\right)\right]^{\frac{1}{2}}}{\left[2^3\left(1-\frac{1}{8}x\right)\right]^{\frac{1}{3}}}$$

$$= 1 + \frac{2\left(1-\frac{1}{4}x\right)^{\frac{1}{2}}}{2\left(1-\frac{1}{8}x\right)^{\frac{1}{3}}}$$

$$= 1 + \left(1-\frac{1}{4}x\right)^{\frac{1}{2}}\left(1-\frac{1}{8}x\right)^{-\frac{1}{3}}$$

$$= 1+\left(1+\frac{1}{2}\left(-\frac{1}{4}x\right)+\cdots\right)\left(1+\left(-\frac{1}{3}\right)\left(-\frac{1}{8}x\right)+\cdots\right)$$

$$= 1+\left(1-\frac{1}{8}x+\cdots\right)\left(1+\frac{1}{24}x+\cdots\right)$$

$$= 1+\left(1+\frac{1}{24}x-\frac{1}{8}x+\cdots\right)$$

$$= 1+\left(1+\left(\frac{1}{24}-\frac{1}{8}\right)x+\cdots\right)$$

$$= 1+\left(1+\left(\frac{1-3}{24}\right)x+\cdots\right)$$

$$= 1+1-\frac{2}{24}x+\cdots$$

$$\approx 2-\frac{1}{12}x \approx R.H.S$$

Hence proved

Q5. If x is so small that its cube and higher power can be neglected, then show that

i) $\sqrt{1-x-2x^2} \approx 1-\frac{1}{2}x-\frac{9}{8}x^2$

Solution:-

$$L.H.S = \sqrt{1-x-2x^2}$$

$$= (1-x-2x^2)^{\frac{1}{2}}$$

$$= \left[1-(x+2x^2)\right]^{\frac{1}{2}}$$

$$= 1+\frac{1}{2}\left\{-(x+2x^2)\right\}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left\{-(x+2x^2)\right\}^2+\cdots$$

$$= 1-\frac{1}{2}x-x^2+\frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(x+2x^2)^2+\cdots$$

$$= 1-\frac{1}{2}x-x^2-\frac{1}{8}(x^2+4x^3+4x^4)+\cdots$$

$$= 1-\frac{1}{2}x-x^2-\frac{1}{8}x^2-\cdots$$

$$= 1-\frac{1}{2}x-\left(1+\frac{1}{8}\right)x^2-\cdots$$

$$= 1-\frac{1}{2}x-\frac{9}{8}x^2-\cdots$$

$$\approx 1-\frac{1}{2}x-\frac{9}{8}x^2 \approx R.H.S$$

Hence proved

ii) $\sqrt{\dfrac{1+x}{1-x}} \approx 1+x+\dfrac{1}{2}x^2$

Solution:-

$$L.H.S = \sqrt{\frac{1+x}{1-x}} = \frac{\sqrt{1+x}}{\sqrt{1-x}}$$

$$= \frac{(1+x)^{1/2}}{(1-x)^{1/2}} = (1+x)^{1/2}(1-x)^{-1/2}$$

$$= \left\{1+\frac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2+\cdots\right\}\left\{1+(-\tfrac{1}{2})(-x)+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}(-x)^2+\cdots\right\}$$

$$= \left\{1+\tfrac{1}{2}x+\tfrac{1}{2}(\tfrac{1}{2})(-\tfrac{1}{2}x^2)+\cdots\right\}\left\{1+\tfrac{1}{2}x+\tfrac{1}{2}(-\tfrac{1}{2})(-\tfrac{3}{2})x^2+\cdots\right\}$$

$$= \left\{1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\cdots\right\}\left\{1+\tfrac{1}{2}x+\tfrac{3}{8}x^2+\cdots\right\}$$

$$= 1+\tfrac{1}{2}x+\tfrac{3}{8}x^2+\tfrac{1}{2}x+\tfrac{1}{4}x^2-\tfrac{1}{8}x^2+\cdots$$

$$= 1+(\tfrac{1}{2}+\tfrac{1}{2})x+(\tfrac{3}{8}+\tfrac{1}{4}-\tfrac{1}{8})x^2+\cdots$$

$$= 1+x+\left(\frac{3+2-1}{8}\right)x^2+\cdots = 1+x+\frac{4}{8}x^2+\cdots$$

$$= 1+x+\tfrac{1}{2}x^2+\cdots \approx 1+x+\tfrac{1}{2}x^2 \approx R.H.S$$

Hence proved.

Q6. If x is very nearly equal to 1, then prove that
$$px^p-qx^q \approx (p-q)x^{p+q}$$

Solution:-

$$L.H.S = px^p-qx^q$$

Let $x=1+h$ where h is so small that its squares and higher power can be neglected, so

$$L.H.S = p(1+h)^p-q(1+h)^q$$

$$= p\{1+ph+\cdots\}-q\{1+qh+\cdots\}$$

$$= \{p+p^2h+\cdots\}-\{q+q^2h+\cdots\}$$

$$= (p-q)+(p^2h-q^2h)+\cdots$$

$$= (p-q)+(p^2-q^2)h+\cdots$$

$$= (p-q)+(p-q)(p+q)h+\cdots$$

$$= (p-q)\{1+(p+q)h+\cdots\}$$

$$\approx (p-q)\{1+(p+q)h\}$$

$$\approx (p-q)(1+h)^{p+q} \qquad \because (1+x)^n = 1+nx$$

$$\approx (p-q)x^{p+q} \qquad \because x=1+h$$

$$\approx R.H.S$$

Hence proved

Q7. If $p-q$ is small when compared with p and q, show that
$$\frac{(2n+1)p+(2n-1)q}{(2n-1)p+(2n+1)q} \approx \left(\frac{p+q}{2q}\right)^{1/n}$$

Solution:-

$$L.H.S = \frac{(2n+1)p+(2n-1)q}{(2n-1)p+(2n+1)q}$$

Let $p-q=h \rightarrow p=q+h$ where h is so small that its squares and higher powers can be neglected, so

$$L.H.S = \frac{(2n+1)(q+h)+(2n-1)q}{(2n-1)(q+h)+(2n+1)q}$$

$$= \frac{(2n+1)q+(2n+1)h+(2n-1)q}{(2n-1)q+(2n-1)h+(2n+1)q}$$

$$= \frac{(2n+1+2n-1)q+(2n+1)h}{(2n-1+2n+1)q+(2n-1)h}$$

$$= \frac{4nq+(2n+1)h}{4nq+(2n-1)h}$$

$$= \frac{4nq\{1+(\frac{2n+1}{4nq})h\}}{4nq\{1+(\frac{2n-1}{4nq})h\}}$$

$$= \left\{1+\left(\frac{2n+1}{4nq}\right)h\right\}\left\{1+\left(\frac{2n-1}{4nq}\right)h\right\}^{-1}$$

$$= \left\{1+\left(\frac{2n-1}{4nq}\right)h\right\}\left\{1-\left(\frac{2n-1}{4nq}\right)h\right\}$$

$$= 1-\left(\frac{2n-1}{4nq}\right)h+\left(\frac{2n+1}{4nq}\right)h \ldots\ldots$$

$$= 1+\frac{h}{4nq}\{-2n+1+2n+1\}\ldots\ldots$$

$$= 1+\frac{2h}{4nq}\ldots\ldots = 1+\frac{h}{2nq}\ldots\ldots$$

$$= 1+\frac{1}{n}\left(\frac{h}{2q}\right)\ldots\ldots \approx \left\{1+\frac{h}{2q}\right\}^{1/n}$$

$$\approx \left\{1+\frac{p-q}{2q}\right\}^{1/n} \qquad \because h = p-q$$

$$\approx \left\{\frac{2q+p-q}{2q}\right\}^{1/n} \approx \left(\frac{p+q}{2q}\right)^{1/n}$$

$\approx R.H.S$ Hence proved

Q8. Show that $\left[\frac{n}{2(n+N)}\right]^{1/2} \approx \frac{8n}{9n-N} - \frac{n+N}{4n}$ where n and N are nearly equal.

Solution:-

$$L.H.S = \left[\frac{n}{2(n+N)}\right]^{1/2}$$

Let $\because N = n+h$ where h is so small that its squares and higher power can be neglected, so

$$L.H.S = \left[\frac{n}{2(n+N)}\right]^{1/2}$$

$$= \left[\frac{n}{2(n+n+h)}\right]^{1/2} = \left[\frac{n}{2(2n+h)}\right]^{1/2}$$

$$= \left[\frac{n}{4n+2h}\right]^{1/2} = \left[\frac{n}{4n\left(1+\frac{2h}{4n}\right)}\right]^{1/2}$$

$$= \left[\frac{1}{2^2\left(1+\frac{2h}{4n}\right)}\right]^{1/2} =$$

$$= \frac{1}{2\left(1+\frac{h}{2n}\right)^{1/2}} = \frac{1}{2}\left(1+\frac{h}{2n}\right)^{-1/2}$$

$$= \frac{1}{2}\left\{1+\left(-\frac{1}{2}\right)\left(\frac{h}{2n}\right)+\ldots\ldots\right\}$$

$$= \frac{1}{2}\left\{1-\frac{h}{4n}+\ldots\ldots\right\}$$

$$= \frac{1}{2}-\frac{h}{8n}+\ldots\ldots$$

$$\approx \frac{1}{2}-\frac{h}{8n} \longrightarrow (i)$$

$$R.H.S = \frac{8n}{9n-N} - \frac{n+N}{4n}$$

$\because N = n+h$

$$= \frac{8n}{9n-(n+h)} - \frac{n+n+h}{4n}$$

$$= \frac{8n}{9n-n-h} - \frac{2n+h}{4n}$$

$$= \frac{8n}{8n-h} - \frac{2n+h}{4n}$$

$$= \frac{8n}{8n\left(1-\frac{h}{8n}\right)} - \frac{2n\left(1+\frac{h}{2n}\right)}{4n}$$

$$= \frac{1}{\left(1-\frac{h}{8n}\right)} - \frac{1+\frac{h}{2n}}{2}$$

$$= \left(1-\frac{h}{8n}\right)^{-1} - \frac{1}{2}\left(1+\frac{h}{2n}\right)$$

$$= -\frac{1}{2}-\frac{h}{4n}+\left(1-\frac{h}{8n}\right)^{-1}$$

$$= -\frac{1}{2}-\frac{h}{4n}+1+(-1)\left(-\frac{h}{8n}\right)+\ldots\ldots$$

$$= -\frac{1}{2}-\frac{h}{4n}+1+\frac{h}{8n}+\ldots\ldots$$

$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{8}-\frac{1}{4}\right)\frac{h}{n}+\ldots\ldots$$

$$= \frac{1}{2}+\left(\frac{1-2}{8}\right)\frac{h}{n}+\ldots$$

$$\approx \frac{1}{2}-\frac{1}{8}\frac{h}{n} \approx \frac{1}{2}-\frac{h}{8n} \longrightarrow (ii)$$

By (i) and (ii)

$L.H.S = R.H.S$

Hence proved

Q9. Identify the following series as binomial expansion and find the sum in each case.

i) $1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1.3}{2!\,4}\left(\frac{1}{4}\right)^2-\frac{1.3.5}{3!\,8}\left(\frac{1}{4}\right)^3+\ldots$

Solution:- Consider

$$y = 1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1.3}{2!\,4}\left(\frac{1}{4}\right)^2-\frac{1.3.5}{3!\,8}\left(\frac{1}{4}\right)^3+\ldots \longrightarrow (i)$$

Let the series be identical with

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots\ldots \longrightarrow (ii)$$

By comparing (i) and (ii)

$$y = (1+x)^n \longrightarrow (iii)$$

Now $nx = -\frac{1}{2}\left(\frac{1}{4}\right) \Rightarrow nx = -\frac{1}{8}$

$$\Rightarrow x = -\frac{1}{8n}$$

and $\frac{n(n-1)}{2!}x^2 = \frac{1.3}{2!4}\left(\frac{1}{4}\right)^2$

$\rightarrow \frac{n(n-1)}{2!}\left(-\frac{1}{8n}\right)^2 = \frac{1.3}{2!4}\left(\frac{1}{4}\right)^2 \quad \because x = -\frac{1}{8n}$

$\rightarrow \frac{n(n-1)}{2} \cdot \frac{1}{64n^2} = \frac{3}{128} \rightarrow n(n-1) \cdot \frac{1}{64n^2} = \frac{3}{64}$

$\rightarrow \frac{n-1}{n} = 3$

$\rightarrow n-1 = 3n \rightarrow -1 = 3n-n$

$\rightarrow -1 = 2n \rightarrow n = -\frac{1}{2}$

so $x = \frac{-1}{8(-\frac{1}{2})} \rightarrow x = \frac{1}{4}$

Putting values of x and n in (iii)

$y = \left(1+\frac{1}{4}\right)^{-1/2} = \left(\frac{5}{4}\right)^{-1/2} = \left(\frac{4}{5}\right)^{1/2} = \frac{2}{\sqrt{5}}$

$\rightarrow$ sum of series is $\frac{2}{\sqrt{5}}$.

ii) $1 - \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1.3}{2.4}\left(\frac{1}{2}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{2}\right)^3 + \cdots$

Solution:- Consider

$y = 1 - \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1.3}{2.4}\left(\frac{1}{2}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{2}\right)^3 + \cdots \longrightarrow (i)$

Let the series be identical with

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \longrightarrow (ii)$

By comparing (i) and (ii)

$y = (1+x)^n \longrightarrow (iii)$

Now
$nx = -\frac{1}{2}\left(\frac{1}{2}\right) \rightarrow nx = -\frac{1}{4} \rightarrow x = -\frac{1}{4n}$

and $\frac{n(n-1)}{2!}x^2 = \frac{1.3}{2.4}\left(\frac{1}{2}\right)^2$

$\rightarrow \frac{n(n-1)}{2}\left(-\frac{1}{4n}\right)^2 = \frac{3}{8}\left(\frac{1}{4}\right)$

$n(n-1) \cdot \frac{1}{16n^2} = \frac{3}{16}$

$\rightarrow \frac{n-1}{n} = 3 \rightarrow n-1 = 3n$

$n - 3n = 1 \rightarrow -2n = 1$

$\rightarrow n = \frac{-1}{2}$ so $x = \frac{-1}{4(-\frac{1}{2})} = \frac{1}{2}$

Putting values of x and n in (iii)

$y = \left(1+\frac{1}{2}\right)^{-1/2} = \left(\frac{3}{2}\right)^{-1/2} = \left(\frac{2}{3}\right)^{1/2}$

$y = \sqrt{\frac{2}{3}}$

$\rightarrow$ sum of series is $\sqrt{\frac{2}{3}}$.

iii) $1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \cdots$

Solution:- Consider

$y = 1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \cdots \longrightarrow (i)$

Let the series be identical with

$\therefore (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \longrightarrow (ii)$

By comparing (i) and (ii)

$y = (1+x)^n$

$\rightarrow nx = \frac{3}{4} \rightarrow x = \frac{3}{4n}$

and $\frac{n(n-1)}{2!}x^2 = \frac{3.5}{4.8}$

$\rightarrow \frac{n(n-1)}{2}\left(\frac{3}{4n}\right)^2 = \frac{15}{32}$

$\rightarrow n(n-1) \cdot \frac{9}{16n^2} = \frac{15}{16}$

$\rightarrow \frac{n-1}{n} = \frac{15}{9}$

or $9n - 9 = 15n \rightarrow -9 = 15n - 9n$

$\rightarrow -9 = 6n$

$\rightarrow n = -\frac{9}{6} \rightarrow n = -\frac{3}{2}$

so $x = \frac{3}{4(-\frac{3}{2})} = -\frac{1}{2}$

Putting values of x and n in (iii)

$y = \left(1-\frac{1}{2}\right)^{-3/2} = \left(\frac{1}{2}\right)^{-3/2}$

$= (2)^{3/2} = \left[2^{1/2}\right]^3 = (\sqrt{2})^3$

$\rightarrow y = 2\sqrt{2}$

$\rightarrow$ sum of series is $2\sqrt{2}$

iv) $1-\frac{1}{2}\cdot\frac{1}{3}+\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{3}\right)^3+\cdots$

Solution:- Consider

$$y=1-\frac{1}{2}\cdot\frac{1}{3}+\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2-\frac{1.3.5}{2.4.6}\left(\frac{1}{3}\right)^3+\cdots \longrightarrow (i)$$

Let the series be identical with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\cdots \longrightarrow (ii)$$

By comparing (i) and (ii),

$$y=(1+x)^n \longrightarrow (iii)$$

Now

$$nx=-\frac{1}{2}\cdot\frac{1}{3} \rightarrow nx=\frac{-1}{6} \rightarrow x=\frac{-1}{6n}$$

and $\frac{n(n-1)}{2!}x^2=\frac{1.3}{2.4}\left(\frac{1}{3}\right)^2$

$$\rightarrow \frac{n(n-1)}{2!}\left(-\frac{1}{6n}\right)^2=\frac{3}{8}\left(\frac{1}{9}\right)$$

$$n(n-1)\cdot\frac{1}{36n^2}=\frac{3}{63}$$

$$\rightarrow \frac{n-1}{n}=\frac{1}{12}\times 36 \rightarrow \frac{n-1}{n}=3$$

$$\rightarrow n-1=3n \rightarrow n-3n=1$$

$$\rightarrow -2n=1 \rightarrow n=-\frac{1}{2}$$

so $x=-\frac{1}{6\left(-\frac{1}{2}\right)} \rightarrow x=\frac{1}{3}$

Putting values of x and n in (iii)

$$y=\left(1+\frac{1}{3}\right)^{-\frac{1}{2}}=\left(\frac{4}{3}\right)^{-\frac{1}{2}}=\left(\frac{3}{4}\right)^{\frac{1}{2}}$$

$$\rightarrow y=\frac{\sqrt{3}}{2}$$

$\rightarrow$ sum of series is $\frac{\sqrt{3}}{2}$

Q10. Use binomial theorem to show that

$$1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\cdots=\sqrt{2}$$

Solution:- Consider

$$y=1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\cdots \longrightarrow (i)$$

Let the series be identical with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\cdots \longrightarrow (ii)$$

By comparing (i) and (ii)

$$y=(1+x)^n \longrightarrow (iii)$$

Now

$$nx=\frac{1}{4} \rightarrow x=\frac{1}{4n}$$

and $\frac{n(n-1)}{2!}x^2=\frac{1.3}{4.8}$

$$\rightarrow \frac{n(n-1)}{2}\left(\frac{1}{4n}\right)^2=\frac{3}{32}$$

$$\rightarrow \frac{n(n-1)}{2}\cdot\frac{1}{16n^2}=\frac{3}{32}$$

$$\frac{n-1}{n}=3 \rightarrow n-1=3n$$

$$-1=3n-n \rightarrow -1=2n$$

$$\rightarrow n=-\frac{1}{2} \quad \text{so} \quad x=\frac{1}{4\left(-\frac{1}{2}\right)}=-\frac{1}{2}$$

Putting values of x and n in (iii)

$$y=\left(1-\frac{1}{2}\right)^{-\frac{1}{2}} \rightarrow y=\left(\frac{1}{2}\right)^{-\frac{1}{2}}$$

$$\rightarrow y=(2)^{\frac{1}{2}} \rightarrow y=\sqrt{2}$$

Hence

$$1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\cdots=\sqrt{2}$$

Q11. If $y=\frac{1}{3}+\frac{1.3}{2!}\left(\frac{1}{3}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3+\cdots$ then prove that $y^2+2y-2=0$

Solution:- Here

$$y=\frac{1}{3}+\frac{1.3}{2!}\left(\frac{1}{3}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3+\cdots \longrightarrow (i)$$

adding 1 on both sides

$$1+y=1+\frac{1}{3}+\frac{1.3}{2!}\left(\frac{1}{3}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3+\cdots$$

Let the series be identical with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\cdots \longrightarrow (ii)$$

By comparing (i) and (ii)

$$1+y=(1+x)^n \longrightarrow (iii)$$

Now

$$nx=\frac{1}{3} \rightarrow x=\frac{1}{3n}$$

and $\frac{n(n-1)}{2!}x^2=\frac{1.3}{2!}\left(\frac{1}{3}\right)^2$

$$\rightarrow n(n-1)x^2=\frac{1}{3}$$

$\rightarrow n(n-1)\left(\frac{1}{3n}\right)^2 = \frac{1}{3}$ $\quad \because x = \frac{1}{3n}$

$n(n-1)\cdot\frac{1}{9n^2} = \frac{1}{3}$

$\rightarrow \frac{n-1}{n} = 3 \rightarrow n-1 = 3n$

$\rightarrow n-3n = 1 \rightarrow -2n = 1$

$\rightarrow n = -\frac{1}{2}$ so $x = \frac{1}{3(-\frac{1}{2})} = -\frac{2}{3}$

Putting values of x and n in (iii)

$1+y = (1-\frac{2}{3})^{-1/2} \rightarrow 1+y = \left(\frac{1}{3}\right)^{-1/2}$

$\rightarrow 1+y = (3)^{1/2} \rightarrow 1+y = \sqrt{3}$

squaring both sides

$(1+y)^2 = (\sqrt{3})^2 \rightarrow 1+y^2+2y = 3$

$\rightarrow y^2+2y+1-3 = 0$

$\rightarrow y^2+2y-2 = 0$ Hence proved.

Q12. If $2y = \frac{1}{2^2} + \frac{1.3}{2!}\cdot\frac{1}{2^4} + \frac{1.3.5}{3!}\cdot\frac{1}{2^6}+\dots$

Prove that $4y^2+4y-1 = 0$

Solution:- Here

$2y = \frac{1}{2^2} + \frac{1.3}{2!}\cdot\frac{1}{2^4} + \frac{1.3.5}{3!}\cdot\frac{1}{2^6}+\dots$

Adding 1 on both sides

$1+2y = 1+\frac{1}{2^2} + \frac{1.3}{2!}\cdot\frac{1}{2^4} + \frac{1.3.5}{3!}\cdot\frac{1}{2^6}+\dots \rightarrow (i)$

Let the series be identical with

$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\dots \rightarrow (ii)$

By comparing (i) and (ii)

$1+2y = (1+x)^n \rightarrow (iii)$

$nx = \frac{1}{2^2} \rightarrow nx = \frac{1}{4}$

$\rightarrow x = \frac{1}{4n}$

and $\frac{n(n-1)}{2!}x^2 = \frac{1.3}{2!}\cdot\frac{1}{2^4}$

$\rightarrow n(n-1)\left(\frac{1}{4n}\right)^2 = \frac{3}{16}$

$n(n-1)\cdot\frac{1}{16n^2} = \frac{3}{16}$

$\frac{n-1}{n} = 3 \rightarrow n-1 = 3n$

$\rightarrow n-3n = 1 \rightarrow -2n = 1$

$\rightarrow n = -\frac{1}{2}$ so $x = \frac{1}{4(-\frac{1}{2})}$

$\rightarrow x = -\frac{1}{2}$

Putting values of x and n in (iii)

$1+2y = (1-\frac{1}{2})^{-1/2} \rightarrow 1+2y = \left(\frac{1}{2}\right)^{-1/2}$

$\rightarrow 1+2y = (2)^{1/2}$

$\rightarrow 1+2y = \sqrt{2}$ (squaring)

$(1+2y)^2 = (\sqrt{2})^2$

$\rightarrow 1+4y^2+4y = 2$

$\rightarrow 4y^2+4y+1-2 = 0$

$\rightarrow 4y^2+4y-1 = 0$

Hence proved.

Q13. If $y = \frac{2}{5} + \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3+\dots$

show that $y^2+2y-4 = 0$

Solution:- Here

$y = \frac{2}{5} + \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3+\dots$

Adding 1 on both sides

$1+y = 1+\frac{2}{5} + \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3+\dots \rightarrow (i)$

Let the series identical with

$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\dots \rightarrow (ii)$

By comparing (i) and (ii)

$1+y = (1+x)^n \rightarrow (iii)$

$nx = \frac{2}{5} \rightarrow x = \frac{2}{5n}$ and

$\frac{n(n-1)}{2!}x^2 = \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 \rightarrow n(n-1)\left(\frac{2}{5n}\right)^2 = \frac{12}{25}$

$\rightarrow n(n-1)\cdot\frac{4}{25n^2} = \frac{12}{25} \rightarrow \frac{n-1}{n} = 3$

$n-1 = 3n \rightarrow -1 = 3n-n = 2n$

$\rightarrow n = -\frac{1}{2}$ so $x = \frac{2}{5(-\frac{1}{2})} = -\frac{4}{5}$

So (iii) $1+y = (1-\frac{4}{5})^{-1/2} = \left(\frac{1}{5}\right)^{-1/2}$

$\rightarrow 1+y = \sqrt{5} \rightarrow (1+y)^2 = 5$

$1+2y+y^2 = 5 \rightarrow y^2+2y+1-5 = 0$

$\rightarrow y^2+2y-4 = 0$ Hence proved.