MATHEMATICS 1st YEAR

UNIT # 07

PERMUTATION, COMBINATION & PROBABALITY

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Contents

Sherazi Mathematics

اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Factorial:- The factorial of positive integer n is "the product of n and all smaller positive integers."

factorial notation:-
Let n be a positive integer. then the product $n(n-1)(n-2)\ldots 3.2.1$ is denoted by $n!$ or $\underline{|n}$ and read as n factorial.

that is
$$n! = n(n-1)(n-2)\ldots 3.2.1$$

Examples:-
$1! = 1$, $2! = 2.1 = 2 \rightarrow 2! = 2.1!$
$3! = 3.2.1 = 6 \rightarrow 3! = 3.2!$
$4! = 4.3.2.1 = 24 \rightarrow 4! = 4.3!$
$5! = 5.4.3.2.1 = 120 \rightarrow 5! = 5.4!$
$6! = 6.5.4.3.2.1 = 720 \rightarrow 6! = 6.5!$

Thus for positive integer n,
$$n! = n(n-1)!$$

Prove that $0! = 1$
Proof:- We know that
$$n! = n(n-1)!$$
Put $n = 1$
$$\rightarrow 1! = 1(1-1)! \rightarrow 1 = 0!$$
Thus $0! = 1$ proved

Example 1. Evaluate $\frac{8!}{6!}$
Solution:-
$$\frac{8!}{6!} = \frac{8.7.\cancel{6!}}{\cancel{6!}} = 8.7 = 56$$

Example 2. Write $8.7.6.5$ in the factorial form
Solution:-
$$8.7.6.5 = \frac{8.7.6.5.4.3.2.1}{4.3.2.1} = \frac{8!}{4!}$$

Example 3. Evaluate $\frac{9!}{6!\,3!}$
Solution:-
$$\frac{9!}{6!\,3!} = \frac{9.8.7.6!}{6!(3.2.1)}$$
$$= \frac{9.8.7}{3.2.1} = 3.4.7 = 84$$

Exercise 7.1

Q1. Evaluate each of the following.
Solution:-

i) $4! = 4.3.2.1 = 24$

ii) $6! = 6.5.4.3.2.1 = 720$

iii) $\frac{8!}{7!} = \frac{8.7!}{7!} = 8$

iv) $\frac{10!}{7!} = \frac{10.9.8.7!}{7!} = 720$

v) $\frac{11!}{4!\,7!} = \frac{11.10.9.8.7!}{(4.3.2.1)\,7!}$
$$= \frac{11.10.9.8}{4.3.2.1} = 11.10.3 = 330$$

vi) $\frac{6!}{3!\,3!} = \frac{6.5.4.3!}{3!\,3!} = \frac{6.5.4}{3.2.1}$
$$= 5.4 = 20$$

vii) $\frac{8!}{4!\,2!} = \frac{8.7.6.5.4!}{4!\,(2.1)} = \frac{8.7.6.5}{2.1}$
$$= 8.7.3.5 = 840$$

viii) $\frac{11!}{2!\,4!\,5!} = \frac{11.10.9.8.7.6.5!}{2.1.4.3.2.1\;5!}$
$$= \frac{11.10.9.8.7.6}{2.4.3.2} = 6930$$

ix) $\frac{9!}{2!(9-2)!} = \frac{9.8.7!}{(2.1)\,7!} = \frac{9.8}{2} = 36$

x) $\frac{15!}{15!(15-15)!} = \frac{15!}{15!\,0!} = \frac{1!}{0!} = \frac{1}{1} = 1$

xi) $\frac{3!}{0!} = \frac{3.2.1}{1} = 6$

xii) $4!\,0!\,1! = (4.3.2.1)(1)(1) = 24$

Q2. Write each of the following in the factorial form:
Solution:-

i) $6.5.4$ ('x' and $\div$ by $3!$)
$$= \frac{6.5.4.3!}{3!} = \frac{6!}{3!}$$

ii) $12.11.10$ ('x' and $\div$ by $9!$)
$$= \frac{12.11.10.9!}{9!} = \frac{12!}{9!}$$

iii) $20.19.18.17$ ($\times$ and $\div$ by $16!$)

$= \dfrac{20.19.18.17.16!}{16!} = \dfrac{20!}{16!}$

iv) $\dfrac{10.9}{2.1}$ ($\times$ and $\div$ by $8!$)

$= \dfrac{10.9.8!}{2.1.8!} = \dfrac{10!}{2!8!}$

v) $\dfrac{8.7.6}{3.2.1}$ ($\times$ and $\div$ by $5!$)

$= \dfrac{8.7.6.5!}{3.2.1.5!} = \dfrac{8!}{3!5!}$

vi) $\dfrac{52.51.50.49}{4.3.2.1}$ ($\times$ and $\div$ by $52!$)

$= \dfrac{52.51.50.49.48!}{4!48!} = \dfrac{52!}{4!48!}$

vii) $n(n-1)(n-2)$ ($\times$ and $\div$ by $(n-3)!$)

$= \dfrac{n(n-1)(n-2)(n-3)!}{(n-3)!}$

$= \dfrac{n!}{(n-3)!}$

viii) $(n+2)(n+1)(n)$ ($\times$ and $\div$ by $(n-1)!$)

$= \dfrac{(n+2)(n+1)(n)(n-1)!}{(n-1)!}$

$= \dfrac{(n+2)!}{(n-1)!}$

ix) $\dfrac{(n+1)(n)(n-1)}{3.2.1}$ ($\times$ and $\div$ by $(n-2)!$)

$= \dfrac{(n+1)(n)(n-1)(n-2)!}{3.2.1\ (n-2)!}$

$= \dfrac{(n+1)!}{3!\ (n-2)!}$

x) $n(n-1)(n-2)\dots(n-r+1)$

$= \dfrac{n(n-1)(n-2)\dots(n-r+1)(n-r)!}{(n-r)!}$ ($\times$ and $\div$ by $(n-r)!$)

$= \dfrac{n!}{(n-r)!}$

Permutation:-

A permutation of n different objects taken $r\,(\leq n)$ at a time is an arrangement of the r objects. Generally it is denoted by ${}^nP_r$ or $P(n,r)$. and defined as $\boxed{{}^nP_r = \dfrac{n!}{(n-r)!}}$

Fundamental Principle of Counting:

"Suppose A and B are two events. The first event A can occure in p different ways. After A has occurred, B can occure in q different ways. The number of ways that two events can occure is the product $p.q$"

Prove that

$${}^nP_r = n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}$$

Proof:- As there are n different objects to fill up r places. So, the first place can be filled in n ways. $\because$ repetitions are not allowed, the second place can be filled in $(n-1)$ ways, the third place filled in $(n-2)$ ways and so on. The rth place has $n-(r-1) = n-r+1$ choices to be filled in. Therefore, by the fundamental principle of counting, r places can be filled by n different objects in $n(n-1)(n-2)\dots(n-r+1)$ ways

$\therefore\ {}^nP_r = n(n-1)(n-2)\dots(n-r+1)$

$= \dfrac{n(n-1)(n-2)\dots(n-r+1)(n-r)(n-r-1)\dots3.2.1}{(n-r)(n-r-1)\dots3.2.1}$

$\rightarrow {}^nP_r = \dfrac{n!}{(n-r)!}$

Hence proved.

Corollary: If $r=n$ then

$${}^nP_n = \frac{n!}{(n-n)!} \quad \because {}^nP_r = \frac{n!}{(n-r)!}$$

$$= \frac{n!}{0!} = \frac{n!}{1}$$

$$\boxed{\Rightarrow {}^nP_n = n!}$$

Example 1. How many different 4-digit numbers can be formed out of the digits 1, 2, 3, 4, 5, 6, when no digit is repeated?

Solution:-

Here $n=6$, $r=4$

Total 4-digit numbers $= {}^6P_4$

$$= \frac{6!}{(6-4)!} = \frac{6!}{2!}$$

$$= \frac{6.5.4.3.2!}{2!} = 360$$

Example 2. How many signals can be made with 4-different flags when any number of them are to be used at a time?

Solution:- Here $n=4$, $r=1,2,3,4$

No. of signal using 1 flag $= {}^4P_1 = 4$

No. of signal using 2 flag $= {}^4P_2 = 12$

No. of signal using 3 flag $= {}^4P_3 = 24$

No. of signal using 4 flag $= {}^4P_4 = 24$

Total signals $= 4+12+24+24 = 64$

Example 3. In how many ways can a set of 4 different mathematics books and 5 different physics books be placed on a shelf with a space for 9 books, if all books on the same subject are kept together?

Solution:-

No. of arrangements of Maths books $= {}^4P_4 = 24$

No. of arrangements of physics books $= {}^5P_5 = 120$

No. of arrangements of subject wise books $= {}^2P_2 = 2$

Total arrangements $= 24 \times 120 \times 2 = 5760$

Exercise 7.2

Q1. Evaluate the following.

i) ${}^{20}P_3$ $\quad \because {}^nP_r = \frac{n!}{(n-r)!}$

Solution:-

$${}^{20}P_3 = \frac{20!}{(20-3)!} = \frac{20!}{17!}$$

$$= \frac{20.19.18.17!}{17!} = 20.19.18 = 6840$$

ii) ${}^{16}P_4$

Solution:-

$${}^{16}P_4 = \frac{16!}{(16-4)!} = \frac{16!}{12!} = \frac{16.15.14.13.12!}{12!}$$

$$= 43680$$

iii) ${}^{12}P_5$

Solution:-

$${}^{12}P_5 = \frac{12!}{(12-5)!} = \frac{12!}{7!} = \frac{12.11.10.9.8.7!}{7!}$$

$$= 95040$$

iv) ${}^{10}P_7$

Solution:-

$${}^{10}P_7 = \frac{10!}{(10-7)!} = \frac{10!}{3!} = \frac{10.9.8.7.6.5.4.3!}{3!}$$

$$= 604800$$

v) ${}^9P_8$

Solution:-

$${}^9P_8 = \frac{9!}{(9-8)!} = \frac{9!}{1!} = \frac{9.8.7.6.5.4.3.2.1}{1}$$

$$= 362880$$

Q2. Find the value of n when

i) ${}^nP_2 = 30$

Solution:-

$${}^nP_2 = 30$$

$$\Rightarrow \frac{n!}{(n-2)!} = 30$$

$\Rightarrow \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 30$

$\Rightarrow n(n-1) = 30$

$\Rightarrow n(n-1) = 6\cdot 5$

$\Rightarrow n = 6$

ii) ${}^{11}P_n = 11\cdot 10\cdot 9$

Solution:-

${}^{11}P_n = 11\cdot 10\cdot 9$

$\Rightarrow \frac{11!}{(11-n)!} = \frac{11\cdot 10\cdot 9\cdot 8!}{8!}$

$\Rightarrow \frac{11!}{(11-n)!} = \frac{11!}{8!}$

$\Rightarrow 11-n = 8$

$\Rightarrow n = 11-8 \Rightarrow n = 3$

iii) ${}^nP_4 : {}^{n-1}P_3 = 9:1$

Solution:-

${}^nP_4 : {}^{n-1}P_3 = 9:1$

$\Rightarrow \frac{{}^nP_4}{{}^{n-1}P_3} = \frac{9}{1}$

$\Rightarrow \frac{\frac{n!}{(n-4)!}}{\frac{(n-1)!}{(n-1-3)!}} = 9$

$\Rightarrow \frac{n\,\cancel{(n-1)!}}{\cancel{(n-4)!}} \times \frac{\cancel{(n-4)!}}{\cancel{(n-1)!}} = 9$

$\Rightarrow n = 9$

Q3. Prove from the first principle that:

i) ${}^nP_r = n\cdot {}^{n-1}P_{r-1}$

Solution:-

$R.H.S = n\cdot {}^{n-1}P_{r-1}$

$= n\cdot \frac{(n-1)!}{[n-1-(r-1)]!}$

$= \frac{n(n-1)!}{(n-1-r+1)!} = \frac{n!}{(n-r)!}$

$= {}^nP_r = L.H.S$

Hence proved

ii) ${}^nP_r = {}^{n-1}P_r + r\cdot {}^{n-1}P_{r-1}$

Solution:-

$R.H.S = {}^{n-1}P_r + r\cdot {}^{n-1}P_{r-1}$

$= \frac{(n-1)!}{(n-1-r)!} + r\cdot \frac{(n-1)!}{(n-1-r+1)!}$

$= \frac{(n-1)!}{(n-1-r)!} + r\cdot \frac{(n-1)!}{(n-r)!}$

$= \frac{(n-1)!}{(n-1-r)!} + r\cdot \frac{(n-1)!}{(n-r)(n-r-1)!}$

$= \frac{(n-1)!}{(n-r-1)!}\left\{1 + \frac{r}{n-r}\right\}$

$= \frac{(n-1)!}{(n-r-1)!}\left\{\frac{n-r+r}{n-r}\right\}$

$= \frac{n(n-1)!}{(n-r)(n-r-1)!} = \frac{n!}{(n-r)!}$

$= {}^nP_r = L.H.S$

Hence proved

Q4. How many signals can be given by 5 flags of different colours, using 3 flags at a time?

Solution:-

Here $n=5$, $r=3$

Number of signals $= {}^5P_3$

$= \frac{5!}{(5-3)!} = \frac{5!}{2!}$

$= \frac{5\cdot 4\cdot 3\cdot 2!}{2!} = 60$

Q5. How many signals can be given by 6 flags of different colours when any number of flags can be used at a time?

Solution:-

$n = 6$, $r = 1,2,3,4,5,6$

${}^6P_1 = \frac{6!}{(6-1)!} = \frac{6\cdot 5!}{5!} = 6$

${}^6P_2 = \frac{6!}{(6-2)!} = \frac{6\cdot 5\cdot 4!}{4!} = 30$

$$^6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6\cdot5\cdot4\cdot3!}{3!} = 120$$

$$^6P_4 = \frac{6!}{(6-4)!} = \frac{6\cdot5\cdot4\cdot3\cdot2!}{2!} = 360$$

$$^6P_5 = \frac{6!}{(6-5)!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1!}{1!} = 720$$

$$^6P_6 = \frac{6!}{(6-6)!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{0!} = 720$$

Number of signals

$= 6+30+120+360+720+720$

$= 1956$

Q6. How many words can be formed from the letters of the following words using all letters when no letter is to be repeated:

i) PLANE

Solution:- $n = 5$, $r = 5$

No. of words $= {}^5P_5 = \frac{5!}{(5-5)!}$

$= \frac{5!}{0!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{1} = 120$

ii) OBJECT

Solution:- $n=6$, $r=6$

No. of words $= {}^6P_6 = \frac{6!}{(6-6)!}$

$= \frac{6!}{0!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{1} = 720$

iii) FASTING

Solution:- $n=7$, $r=7$

No. of words $= {}^7P_7 = \frac{7!}{(7-7)!}$

$= \frac{7!}{0!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{1} = 5040$

Q7. How many 3-digit numbers can be formed by using each one of the digits 2, 3, 5, 7, 9 only once?

Solution:- $n=5$, $r=3$

Total 3-digit numbers $= {}^5P_3$

$= \frac{5!}{(5-3)!} = \frac{5!}{2!}$

$= \frac{5\cdot4\cdot3\cdot2!}{2} = 60$

Q8. Find the numbers greater than 23000 that can be formed from the digits 1,2,3,5,6 without repeating any digit.

Solution:-

Numbers greater than 23000 will be as;

2	3	*	*	*	$= {}^3P_3 = 3! = 6$
2	5	*	*	*	$= {}^3P_3 = 3! = 6$
2	6	*	*	*	$= {}^3P_3 = 3! = 6$
3	*	*	*	*	$= {}^4P_4 = 4! = 24$
5	*	*	*	*	$= {}^4P_4 = 4! = 24$
6	*	*	*	*	$= {}^4P_4 = 4! = 24$

so the numbers greater than 23,000 are:

= 6+6+6+24+24+24= 90

Q9. Find the number of 5-digit numbers that can be formed from the digits 1, 2, 4, 6, 8 (when no digit is repeated), but

i) the digits 2 and 8 are next to each other;

Solution:-

Let (2,8) as one digit

Now the digits are 1, 28, 4,6 or 1, 82, 4, 6 so

Permutations containing (28) = $^4P_4$ = 4! =24

Permutations containing (82) = $^4P_4$ = 4! = 24

Total permutations in which 2 and 8 are next to each other = 24+24 =48

ii) the digit 2 and 8 are not next to each other.

Solution:-

Number of total permutation

$= {}^5P_5 = 5! = 120$

Numbers in which 2 and 8 are together = 48
so Numbers in which 2 and 8 are not next to each other = 120-48= 72

Q10. How many 6-digit numbers can be formed, without repeating any digit from the digits 0, 1, 2, 3, 4, 5? In how many of them will 0 be at the tens place?

Solution:- When 0 is placed at extreme left position, then 6-digit number can't be formed.

hence, 6-digit number will be as;

1	*	*	*	*	*	$= {}^5P_5 = 5! = 120$
2	*	*	*	*	*	$= {}^5P_5 = 5! = 120$
3	*	*	*	*	*	$= {}^5P_5 = 5! = 120$
4	*	*	*	*	*	$= {}^5P_5 = 5! = 120$
5	*	*	*	*	*	$= {}^5P_5 = 5! = 120$

so total 6-digit numbers are= 120 X 5= 600

Numbers having 0 at ten's place (0 is fixed at ten's place)

*	*	*	*	0	*	$= {}^5P_5 = 5! = 120$

Q11. How many 5-digit multiples of 5 can be formed from the digits 2, 3, 5, 7, 9 when no digit is repeated.

Solution:-

5-digit number multiples of 5 will be as;

*	*	*	*	5	(fix 5 in its unit place)

$= {}^4P_4 = 4! = 24$

Q12. In how many ways can 8 books including 2 on English be arranged on a shelf in such a way that the English books are never together?

Solution:-

Suppose E_1, E_2 are two English books and $B_1, B_2, B_3, B_4, B_5, B_6$, remaining books

Total $= {}^8P_8 = \frac{8!}{(8-8)!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{0!} = 40320$

English books are together

Case I :- $B_1, B_2, B_3, B_4, B_5, B_6, \boxed{E_1E_2}$

$= {}^7P_7 = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{(7-7)!} = 5040$

Case II :- $B_1, B_2, B_3, B_4, B_5, B_6, \boxed{E_2E_1}$

$= {}^7P_7 = 5040$

English books together $= 5040+5040$
$= 10080$

English books not together
= total − together
$= 40320 - 10080 = 30240$

Q13. Find the number of arrangements of 3 books on English and 5 books on Urdu for placing them on a shelf such that the books on the same subjects are together.

Solution:-

E_1, E_2, E_3 are English and U_1, U_2, U_3, U_4, U_5 are urdu books.

Case I :- $U_1, U_2, U_3, U_4, U_5 \times E_1, E_2, E_3$

$= {}^5P_5 \times {}^3P_3 = \frac{5!}{(5-5)!} \times \frac{3!}{(3-3)!}$

$= \frac{5\cdot4\cdot3\cdot2\cdot1}{0!} \times \frac{3\cdot2\cdot1}{0!} = 120 \times 6 = 720$

Case II :- $E_1, E_2, E_3 \times U_1, U_2, U_3, U_4, U_5$

$= {}^3P_3 \times {}^5P_5 = \frac{3!}{(3-3)!} \times \frac{5!}{(5-5)!}$

$= \frac{3\cdot2\cdot1}{0!} \times \frac{5\cdot4\cdot3\cdot2\cdot1}{0!} = 6 \times 120 = 720$

Total arrangements $= 720 + 720 = 1440$

Q14. In how many ways can 5 boys and 4 girls be seated on a bench so that the girls and the boys occupy alternate seats?

Solution:-

B represents Boys and G girls

Total no. of ways

$B_1 G_1 B_2 G_2 B_3 G_3 B_4 G_4 B_5$

$= {}^5P_5 \times {}^4P_4 = \frac{5\cdot4\cdot3\cdot2\cdot1}{(5-5)!} \times \frac{4\cdot3\cdot2\cdot1}{(4-4)!}$

$= 120 \times 24 = 2880$

Theorem:- The number of permutations of n objects taken all at a time when n_1 of them are alike of one kind, n_2 are alike of second kind and n_3 are alike of third kind are given by

$$\binom{n}{n_1, n_2, n_3} = \frac{n!}{n_1!\, n_2!\, n_3!}$$

Proof:- We know that

Arrangment of n_1 like objects $= {}^{n_1}P_{n_1} = n_1!$

Arrangment of n_2 like objects $= {}^{n_2}P_{n_2} = n_2!$

Arrangment of n_3 like objects $= {}^{n_3}P_{n_3} = n_3!$

Let x be the required no. of permutations of n objects.

then

total permutations $= x\, n_1!\, n_2!\, n_3!$

But

no. of permutations of n objects $= n!$

Therefore

$x \cdot n_1!\, n_2!\, n_3! = n!$

$\rightarrow x = \frac{n!}{n_1!\, n_2!\, n_3!}$

$\rightarrow x = \binom{n}{n_1, n_2, n_3}$

Example 1. In how many ways can be letters of the word MISSISSIPPI be arranged when all the letters are to be used?

Solution:-

Total letters $= n = 11$

M repeated $= 1$

I repeated $= 4$

S repeated $= 4$

P repeated $= 2$

Total arrangements $= \binom{11}{4, 4, 2, 1}$

$= \frac{11!}{4!\,4!\,2!\,1!}$

$= \frac{11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4!}{4!\,(4\cdot3\cdot2\cdot1)(2\cdot1)(1)} = 34650$

Circular Permutation

The permutation of things which can be represented by the points on a circle is called circular permutation.

Example 2. In how many ways can 5 persons be seated at a round table?

Solution:- $n = 5$

$$\boxed{\text{for Round table formula} = (n-1)!}$$

The number of arrangements

$= (5-1)! = 4! = 24.$

Example 3. In how many ways can a necklace of 8 beads of different colours be made?

Solution:- $n = 8$

$$\boxed{\text{Circular Permutation} = \frac{(n-1)!}{2}}$$

Number of ways $= \frac{(n-1)!}{2}$

$= \frac{(8-1)!}{2}$

$= \frac{7!}{2} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2!}{2!} = 2520$

Exercise 7.3

Q1. How many arrangements of the letters of the following words, taken at together can be made:

i) PAKPATTAN

Solution:-

Total letters $= n = 9$

P repeated = 2

A repeated = 3

T repeated = 2

K repeated = 1

N repeated = 1

Req. number of permutations

$$= \binom{9}{3,2,2,1,1}$$

$$= \frac{9!}{3!2!2!1!1!} = 15120$$

ii) PAKISTAN

Solution:-

Total letters = 2

P, K, I, S, T, N come only once

Req. number of Permutations $= \binom{8}{2,1,1,1,1,1,1}$

$$= \frac{8!}{2!} = 20160$$

iii) MATHEMATICS

Solution:-

Total letters = 11

M repeated = 2

A repeated = 2

T repeated = 2

H, E, I, C, S come only once.

Req. number of Permutations

$$= \binom{11}{2,2,2,1,1,1,1,1}$$

$$= \frac{11!}{2!2!2!} = 4989600$$

iv) ASSASSINATION

Solution:-

Total letters = 13

A repeated = 3

S repeated = 4

I repeated = 2

N repeated = 2

T repeated = 1

O repeated = 1

Req. number of permutations

$$= \binom{13}{3,4,2,2,1,1}$$

$$= \frac{13!}{3!4!2!2!} = 43243200$$

Q2. How many permutation of the letters of the word PANAMA can be made, if P is to be the first letter in each arrangement?

Solution:- PANAMA

If P is first letter $\boxed{P}$ ANAMA, $n=5$

A repeated = 3

N repeated = 1

M repeated = 1

Total arrangements $= \frac{5!}{3!1!1!} = \frac{5\cdot4\cdot3!}{3!\cdot1\cdot1} = 20$

Q3. How many arrangements of the letters of the word ATTACKED can be made, if each arrangement begins with C and ends with K?

Solution:- ATTACKED

If C is first and K is last letter then $\boxed{C}$ ATTAED $\boxed{K}$, $n=6$

A repeated = 2

T repeated = 2

E repeated = 1

D repeated = 1

Total arrangements $= \frac{6!}{2!2!1!1!}$

$= \frac{6\cdot5\cdot4\cdot3\cdot2!}{2!\,2\cdot1\cdot1\cdot1} = \frac{360}{2} = 180$

Q4. How many numbers greater than 1000,000 can be formed from the digits 0, 2, 2, 2, 3, 4, 4?

Solution:- When we fix 0 at extreme left position as ;

0	*	*	*	*	*

then number greater than 10,00000 can't be formed. Hence for required resul extreme left place will be filled by digits other than 0.

arrangements having 2 as;

2	*	*	*	*	*

$= \frac{6!}{1!\,2!\,1!\,2!} = \frac{6.5.4.3.2.1}{1.2.1.2} = 180$

arrangements having 3 as;

3	*	*	*	*	*

$= \frac{6!}{1!\,3!\,2!} = \frac{6.5.4.3!}{1\times3!\times2} = 60$

arrangements having 4 as;

4	*	*	*	*	*

$= \frac{6!}{1!\,3!\,1!\,1!} = \frac{6.5.4.3.2.1}{1\times3!\times1\times1} = 120$

so total arrangements (numbers greater than 10,00000) are = 180+60+120= 360

Q5. How many 6-digit numbers can be formed from the digits 2, 2, 3, 3, 4, 4? How many of them will lie between 400,000 and 430,000?

Solution:- 2, 2, 3, 3, 4, 4 $\quad n = 6$

2 repeated = 2

3 repeated = 2

4 repeated = 2

Total arrangements

$= \frac{6!}{2!\,2!\,2!} = \frac{6\cdot5\cdot4\cdot3\cdot2!}{2!\cdot2\cdot1\cdot2\cdot1} = 90$

For numbers between 400,000 and 430,000 fix 42 at first place then numbers

$\boxed{42}$, 2, 3, 3, 4 $\quad n = 4$

Now arrangements

$= \frac{4!}{1!\,2!\,1!} = \frac{4\cdot3\cdot2!}{1\cdot2!\cdot1} = 12$

Q6. 11 members of a club form 4 committees of 3, 4, 2, 2 members so that no member is a member of more than one committee. Find the number of committees.

Solution:- $n = 11$

4 commitees have 3, 4, 2, 2 members

No. of committes $= \frac{11!}{3!\,4!\,2!\,2!}$

$= \frac{11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4!}{3\cdot2\cdot1\cdot4!\cdot2\cdot1\cdot2\cdot1} = 69300$

Q7. The D.C.Os of 11 districts meet to discuss the law and order situation in their districts. In how many ways can they be seated at a round table, when two particular D.C.Os insist on sitting together?

Solution:- Total D.C.O's = 11

Let D1 , D2 be the two particular D.C.O's insisting of sitting together.

If D1D2 are sitting together then the number of permutation = (10-1)! =9! = 362880

If D2D1 are sitting together then the number of permutation = (10-1)! =9! = 362880

Total number of arrangements are = 362880+362880
= 725760

Q8. The Governor of the punjab calls a meeting of 12 officers. In how many ways can they be seated at a round table?

Solution:- $n = 12$

for Round table formula $= (n-1)!$

No. of ways when one chair is fixed for Chairperson

$= (12-1)! = 11! = 39916800$

Q9. Fatima invites 14 people to a dinner. There are 9 males and 5 females who are seated at two different tables so that guests of one sex sit at one round table and the guest of other sex at the second table. Find the number of ways in which all guests are seated.

Solution:- Here is independent arrangements due to separate tables

Male = 9 and Female = 5

Number of ways

$= \begin{pmatrix}\text{arrangements}\\ \text{of males}\end{pmatrix}\begin{pmatrix}\text{arrangements}\\ \text{of females}\end{pmatrix}$

$= (9-1)! \times (5-1)!$

$= 8! \times 4! = (40320)(24)$

$= 967680$

(∵ for round table formula $= (n-1)!$)

Q10. Find the number of ways in which 5 men and 5 women can be seated at a round in such a way no person of the same sex sit together.

Solution:-

Circular permutation of 5 men and 5 women sitting alternately = (5-1)! X 5!

= 4! X 5!
=24 X 120
=2880

Fixing men or women alternately.
Here we have fix 1 men. so using formula (n-1)!

Q11. In how many ways can 4 keys be arranged on a circular key ring?

Solution:- $n = 4$

Circular Permutation $= \frac{(n-1)!}{2}$

No. of ways

$= \frac{(4-1)!}{2} = \frac{3!}{2}$

$= \frac{3.2.1}{2} = 3$

Q12. How many necklaces can be made from 6 beads of different colours?

Solution:- $n = 6$

Number of ways $= \frac{(n-1)!}{2} = \frac{(6-1)!}{2} = \frac{5!}{2}$

$= \frac{120}{2} = 60$

Combinations

When selection of objects is done neglecting its order, this is called combination.

★ The number of combinations taken of n different-objects taken 'r' at a time is denoted by ^nC_r or $\binom{n}{r}$ or $C(n,r)$ and defined as

$^nC_r = \frac{^nP_r}{r!}$

or $\boxed{^nC_r = \frac{n!}{r!(n-r)!}}$ $\quad ∵ {}^nP_r = \frac{n!}{(n-r)!}$

Prove that $^nC_r = \frac{n!}{r!(n-r)!}$

Proof:- There are ^nC_r combinations of n different objects taken r at a time. Each combination consists of r different objects among which can be permuted themselves in $r!$ ways. So, each combination will give rise to $r!$ permutation. Thus there will be ${}^nC_r \times r!$ permutation of n different objects taken r at a time ${}^nC_r \times r! = {}^nP_r$

$$\Rightarrow {}^nC_r \times r! = \frac{n!}{(n-r)!}$$

$$\Rightarrow {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

Hence proved.

Corollary:

i) If $r=n$, then ${}^nC_n = \frac{n!}{n!\,(n-n)!}$

$$= \frac{n!}{n!\;0!} = 1$$

ii) If $r=0$, then ${}^nC_0 = \frac{n!}{0!\,(n-0)!}$

$$= \frac{n!}{0!\;n!} = 1$$

Complementary Combination

Combination (selection) of " r " objects from " n " different objects is denoted by ${}^nC_r$. Selecting (remaining) " n-r " objects from " n " is complementry combiniton denoted by ${}^nC_{n-r}$

Prove that $\boxed{{}^nC_r = {}^nC_{n-r}}$

Proof:-

R.H.S $= {}^nC_{n-r}$

$$= \frac{n!}{(n-r)!\,[n-(n-r)]!}$$

$$= \frac{n!}{(n-r)!\,(n-n+r)!} = \frac{n!}{(n-r)!\,r!}$$

$$= \frac{n!}{r!\,(n-r)!} = {}^nC_r = \text{L.H.S}$$

Hence proved

Example 1. If ${}^nC_8 = {}^nC_{12}$, find n

Solution:-

we know that

$${}^nC_r = {}^nC_{n-r}$$

$$\Rightarrow {}^nC_8 = {}^nC_{n-8} \longrightarrow (i)$$

But ${}^nC_8 = {}^nC_{12} \longrightarrow (ii)$

By (i) and (ii)

$${}^nC_{n-8} = {}^nC_{12}$$

$$\Rightarrow n-8 = 12$$

$$\Rightarrow n = 12+8$$

$$\Rightarrow n = 20$$

Example 2. Find the number of the diagonals of a 6-sided figure.

Solution:-

No. of diagonal $= {}^6C_2 - 6$

$$= \frac{6!}{2!\,(6-2)!} - 6$$

$$= \frac{6\cdot5\cdot4!}{2!\;4!} - 6$$

$$= 15 - 6 = 9$$

Example 3. Prove that:

$${}^{n-1}C_r + {}^{n-1}C_{r-1} = {}^nC_r$$

Solution:-

L.H.S $= {}^{n-1}C_r + {}^{n-1}C_{r-1}$

$$= \frac{(n-1)!}{r!\,(n-1-r)!} + \frac{(n-1)!}{(r-1)!\,(n-1-r+1)!}$$

$$= \frac{(n-1)!}{r!\,(n-1-r)!} + \frac{(n-1)!}{(r-1)!\,(n-r)!}$$

$$= \frac{(n-1)!}{r\,(r-1)!\,(n-1-r)!} + \frac{(n-1)!}{(r-1)!\,(n-r)(n-r-1)!}$$

$$= \frac{(n-1)!}{(r-1)!\,(n-1-r)!}\left\{\frac{1}{r} + \frac{1}{n-r}\right\}$$

$$= \frac{(n-1)!}{(r-1)!\,(n-1-r)!}\left\{\frac{n-r+r}{r(n-r)}\right\}$$

$$= \frac{(n-1)!}{(r-1)!\,(n-1-r)!}\left(\frac{n}{r(n-r)}\right)$$

$$= \frac{n\,(n-1)!}{r\,(r-1)!\,(n-r)(n-1-r)!}$$

$$= \frac{n!}{r!(n-r)!} = {}^nC_r = R.H.S$$

Hence proved

Exercise 7.4

Q1. Evaluate the following.

i) ${}^{12}C_3$

Solution:-

$${}^{12}C_3 = \frac{12!}{3!(12-3)!} = \frac{12!}{3!\,9!} = \frac{12.11.10.\cancel{9!}}{3!\,\cancel{9!}}$$

$$= \frac{12.11.10}{3.2.1} = 220$$

ii) ${}^{20}C_{17}$

Solution:-

$${}^{20}C_{17} = \frac{20!}{17!(20-17)!} = \frac{20!}{17!\,3!}$$

$$= \frac{20\cdot19\cdot18\cdot17!}{17!\,(3\cdot2\cdot1)} = 1140$$

iii) ${}^nC_4$

Solution:-

$${}^nC_4 = \frac{n!}{4!(n-4)!}$$

$$= \frac{n(n-1)(n-2)(n-3)(n-4)!}{4!\,(n-4)!}$$

$$= \frac{n(n-1)(n-2)(n-3)}{4!}$$

Q2. Find the value of n, when

i) ${}^nC_5 = {}^nC_4$

Solution:-

$${}^nC_5 = {}^nC_4 \longrightarrow (i)$$

we know that

$${}^nC_r = {}^nC_{n-r} \Rightarrow {}^nC_5 = {}^nC_{n-5}$$

so (i) becomes

$$\Rightarrow {}^nC_{n-5} = {}^nC_4$$

$$\Rightarrow n-5 = 4$$

$$\Rightarrow n = 9$$

ii) ${}^nC_{10} = \frac{12\times11}{2!}$

Solution:-

$${}^nC_{10} = \frac{12\times11}{2!} \quad (\times \text{ and } \div \text{ by } 10!)$$

$$\Rightarrow {}^nC_{10} = \frac{12\times11\times10!}{2!\,10!}$$

$$\Rightarrow {}^nC_{10} = \frac{12!}{2!\,10!} = \frac{12!}{10!\,(12-10)!}$$

$$\Rightarrow {}^nC_{10} = {}^{12}C_{10}$$

$$\Rightarrow n = 12$$

iii) ${}^nC_{12} = {}^nC_6$

Solution:-

$$\because {}^nC_r = {}^nC_{n-r}$$

$$\Rightarrow {}^nC_{12} = {}^nC_{n-12}$$

$$\Rightarrow {}^nC_6 = {}^nC_{n-12} \quad (\because {}^nC_{12} = {}^nC_6)$$

$$\Rightarrow 6 = n-12$$

$$\Rightarrow n = 18$$

Q3. Find the values of n and r, when

i) ${}^nC_r = 35$ and ${}^nP_r = 210$

Solution:-

we know that

$${}^nC_r = \frac{{}^nP_r}{r!}$$

$$\Rightarrow r! = \frac{{}^nP_r}{{}^nC_r} = \frac{210}{35} = 6$$

$$\Rightarrow r! = 3.2.1 \Rightarrow r! = 3!$$

$$\Rightarrow r = 3$$

Now ${}^nP_r = 210$

$$\Rightarrow \frac{n!}{(n-r)!} = 210$$

$$\Rightarrow \frac{n!}{(n-3)!} = 210 \quad (\because r=3)$$

$$\Rightarrow \frac{n(n-1)(n-2)(n-3)!}{(n-3)!} = 210$$

$$n(n-1)(n-2) = 7\cdot6\cdot5$$

$$\Rightarrow n = 7$$

ii) ${}^{n-1}C_{r-1} : {}^{n}C_r : {}^{n+1}C_{r+1} = 3:6:11$

Solution:-

$$\frac{{}^{n-1}C_{r-1}}{{}^{n}C_r} = \frac{3}{6} \quad \text{and} \quad \frac{{}^{n}C_r}{{}^{n+1}C_{r+1}} = \frac{6}{11}$$

$$\Rightarrow \frac{\frac{(n-1)!}{(r-1)!(n-1-r+1)!}}{\frac{n!}{r!(n-r)!}} = \frac{1}{2}$$

$$\Rightarrow \frac{(n-1)!}{(r-1)!\cancel{(n-r)}!} \times \frac{r!\cancel{(n-r)}!}{n!} = \frac{1}{2}$$

$$\Rightarrow \frac{(n-1)!\,r!}{(r-1)!\,n!} = \frac{1}{2}$$

$$\Rightarrow \frac{\cancel{(n-1)!}\ r\cancel{(r-1)!}}{\cancel{(r-1)!}\ n\cancel{(n-1)!}} = \frac{1}{2}$$

$$\Rightarrow \frac{r}{n} = \frac{1}{2} \Rightarrow 2r = n \longrightarrow (i)$$

Now
$$\frac{{}^{n}C_r}{{}^{n+1}C_{r+1}} = \frac{6}{11}$$

$$\Rightarrow \frac{\frac{n!}{r!(n-r)!}}{\frac{(n+1)!}{(r+1)!(n+1-r-1)!}} = \frac{6}{11}$$

$$\Rightarrow \frac{\frac{n!}{r!(n-r)!}}{\frac{(n+1)!}{(r+1)!(n-r)!}} = \frac{6}{11}$$

$$\Rightarrow \frac{n!}{r!\cancel{(n-r)}!} \times \frac{(r+1)!\cancel{(n-r)}!}{(n+1)!} = \frac{6}{11}$$

$$\Rightarrow \frac{n!\,(r+1)!}{r!\,(n+1)!} = \frac{6}{11}$$

$$\Rightarrow \frac{\cancel{n!}\,(r+1)\cancel{r!}}{\cancel{r!}\,(n+1)\cancel{n!}} = \frac{6}{11}$$

$$\Rightarrow \frac{r+1}{n+1} = \frac{6}{11}$$

or $11r + 11 = 6n + 6 \longrightarrow (ii)$

Put $n = 2r$ in (ii)

$\Rightarrow 11r + 11 = 6(2r) + 6$

$\Rightarrow 11r + 11 = 12r + 6$

$\Rightarrow 12r + 6 - 11r - 11 = 0$

$\Rightarrow r - 5 = 0$

$\Rightarrow r = 5$

so (i) $\Rightarrow n = 2(5)$

$\Rightarrow n = 10$

Q4. How many (a) diagonals and (b) triangles can be formed by joining vertices of the polygon having:

i) 5 sides   ii) 8 sides

iii) 12 sides

Solution:-

For diagonals $= {}^{n}C_2 - n$ **(Excluding n-sides)**

For Triangles $= {}^{n}C_3$ **(made by n-sided polygon)**

i) 5 sides

$n = 5$

No. of diagonals $= {}^{5}C_2 - 5$

$$= \frac{5!}{2!(5-2)!} - 5$$

$$= \frac{5!}{2!\ 3!} - 5 = \frac{5.4.3!}{2!\ 3!} - 5$$

$$= 10 - 5 = 5$$

No. of triangles $= {}^{5}C_3$

$$= \frac{5!}{3!(5-3)!} = \frac{5!}{3!\ 2!}$$

$$= \frac{5.4.3!}{3!.2.1} = 10$$

ii) 8 sides

$n = 8$

No. of diagonals $= {}^{8}C_2 - 8$

$$= \frac{8!}{2!(8-2)!} - 8 = \frac{8.7.6!}{(2.1)\,6!} - 8$$

$$= 28 - 8 = 20$$

No. of triangles $= {}^{8}C_3 = \frac{8!}{3!\,(8-3)!}$

$= \frac{8!}{3!\ 5!} = \frac{8.7.6.5!}{3!\ 5!}$

$= \frac{8.7.6}{3.2.1} = 42$

iii) 12 sides

$n = 12$

No. of diagonals $= {}^{12}C_2 - 12$

$= \frac{12!}{2!\,(12-2)!} - 12 = \frac{12.11.10!}{(2.1)\ 10!} - 12$

$= 66 - 12 = 54$

No. of triangles $= {}^{12}C_3 = \frac{12!}{3!\,(12-3)!}$

$= \frac{12.11.10.9!}{(3.2.1)\ 9!} = 220$

Q5. The members of a club are 12 boys and 8 girls. In how many ways can committee of 3 boys and 2 girls be formed?

Solution:-

Boys: $n = 12$, $r = 3$

Girls: $n = 8$, $r = 2$

No. of ways $= {}^{12}C_3 \times {}^{8}C_2$

$= \frac{12!}{3!\,(12-3)!} \times \frac{8!}{2!\,(8-2)!}$

$= \frac{12.11.10.9!}{(3.2.1)\ 9!} \times \frac{8.7.6!}{(2.1)\ 6!}$

$= 220 \times 28 = 6160$

Q6. How many committee of 5 members can be chosen from a group of 8 persons when each committee must include 2 particular persons?

Solution:-

$n = 8$, $r = 5$

For 2 particular persons

$n = 6$, $r = 3$

No. of committees $= {}^{6}C_3$

$= \frac{6!}{3!\,(6-3)!}$

$= \frac{6.5.4.\cancel{3!}}{3!\ \cancel{3!}} = \frac{6.5.4}{6}$

$= 20$

Q7. In how many ways can a hockey team of 11 players be selected out of 15 players? How many of them will include a particular player?

Solution:-

$n = 15$, $r = 11$

No. of ways hockey team is selected $= {}^{15}C_{11} = \frac{15!}{11!\,(15-11)!}$

$= \frac{15!}{11!\ 4!} = \frac{15.14.13.12.11!}{11!\ (4.3.2.1)}$

$= 1365$

If we include one particular player then $n = 14$, $r = 10$

No. of ways $= {}^{14}C_{10}$

$= \frac{14!}{10!\,(14-10)!} = \frac{14!}{10!\ 4!}$

$= \frac{14.13.12.11.10!}{10!\ (4.3.2.1)} = 1001$

Q8. Show that: ${}^{16}C_{11} + {}^{16}C_{10} = {}^{17}C_{11}$

Solution:-

L.H.S $= {}^{16}C_{11} + {}^{16}C_{10}$

$= {}^{17-1}C_{11} + {}^{17-1}C_{10}$

$= {}^{17}C_{11}$

= R.H.S

Since

${}^{n-1}C_r + {}^{n-1}C_{r-1} = {}^{n}C_r$

Hence Proved

Q9. There are 8 men and 10 women members of a club. How many committee can be formed, having

i) 4 women

Solution:-

Men = 8 and women = 10

No. of committees = $^{10}C_4 \times {}^8C_3$

$$= \frac{10!}{4!\,(10-4)!} \times \frac{8!}{3!\,(8-3)!}$$

$$= \frac{10.9.8.7.6!}{4!\;6!} \times \frac{8.7.6.5!}{3!\;5!}$$

$$= 210 \times 56 = 11760$$

ii) at the most 4 women

Solution:-

It means women are less than or equal to 4, which implies

$$= {}^{10}C_1 \times {}^8C_6 + {}^{10}C_2 \times {}^8C_5 + {}^{10}C_3 \times {}^8C_4 + {}^{10}C_4 \times {}^8C_3 + {}^8C_7$$

$$= (10)(28) + (45)(56) + (120)(70) + (210)(56) + 8$$

$$= 280 + 2520 + 8400 + 11760 + 8$$

$$= 22968$$

iii) Atleast 4 women?

Solution:-

It means women are greater than or equal to 4, which implies

$$= {}^{10}C_4 \times {}^8C_3 + {}^{10}C_5 \times {}^8C_2 + {}^{10}C_6 \times {}^8C_1 + {}^{10}C_7$$

$$= (210)\times(56) + (252)(28) + (210)(8) + 120$$

$$= 11760 + 7056 + 1680 + 120$$

$$= 20616$$

Q10. Prove that

$$^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

Solution:-

$$L.H.S = {}^nC_r + {}^nC_{r-1}$$

$$= \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-(r-1))!\,(r-1)!}$$

$$= \frac{n!}{(n-r)!\,r(r-1)!} + \frac{n!}{(n-r+1)!\,(r-1)!}$$

$$= \frac{n!}{(n-r)!\,r(r-1)!} + \frac{n!}{(n-r+1)(n-r)!\,(r-1)!}$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left(\frac{n-r+1+r}{r(n-r+1)}\right)$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left(\frac{n+1}{r(n-r+1)}\right)$$

$$= \frac{(n+1)\,n!}{r(r-1)!\,(n-r+1)(n-r)!}$$

$$= \frac{(n+1)!}{r!\,(n-r)!} = {}^{n+1}C_r = R.H.S$$

Hence proved

Probability:-

Probability is the numerical evaluation of a chance that a particular event would occur.

OR Measurement of uncertainty.

Sample Space:- The set S consisting of all possible outcome of a given experiment is called a sample space.

Event:- The particular outcome of an experiment is called an event.

* An event is a subset of the sample space.

* Sample space is denoted by S.
* Events are usually denoted by Capital letters A, B, C, ...

Mutually Exclusive (Disjoint) Events:- Two events A and B are said to be mutually exclusive if and only if they cannot both occur at the same time. e.g., In tossing a coin, the sample space $S = \{H, T\}$. Now if event $A = \{H\}$ and event $B = \{T\}$, then A and B are mutually exclusive events.

Equally Likely Events:- Two events A and B are said to be equally likely if each one of them has equal number of chances of occurrence. e.g., when a coin is tossed, we get either head H or tail T. chances of occurence of head is 1/2, while chances of occurence of tail is also 1/2. Thus, the two events head and tail are equally likely events.

Note:- i) Let E be an event. then probability of E is denoted by P(E) and defined as;

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{Number of favorable outcomes}}{\text{Number of possible outcomes}}$$

ii) Probability of an event must be a number lying between 0 and 1.
i.e., $0 \leq P(E) \leq 1$

iii) If $P(E) = 1$ then E is called certain event (i.e., Event E will must occure)

iv) If $P(E) = 0$ then E is called impossible event (i.e., Event E could not occur)

Example 1. A die is rolled. What is the probability that the dots on the top are greater than 4?

Solution:-
The sample space is
$S = \{1, 2, 3, 4, 5, 6\} \rightarrow n(S) = 6$
The event E that the dots on the top of the die are greater than 4
i.e., $E = \{5, 6\}$
$\rightarrow n(E) = 2$
$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$

Example 2. what is the probability that a slip of numbers divisible by 4 are picked from the slips bearing numbers 1, 2, 3, ... 10?

Solution:-
The sample space is
$S = \{1, 2, 3, \ldots, 10\} \rightarrow n(S) = 10$
Let E be the event of picking slip with number divisible by 4
$E = \{4, 8\} \rightarrow n(E) = 2$
$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{2}{10} = \frac{1}{5}$

"Probability that an Event does not occur"

Suppose $n(S) = N$ and $n(E) = R$
then $P(E) = \frac{n(E)}{n(S)} = \frac{R}{N}$
Let $\bar{E}$ denotes the non-occurrence of event E. then $n(\bar{E}) = N - R$

$\rightarrow P(\bar{E}) = \frac{n(\bar{E})}{n(S)} \rightarrow P(\bar{E}) = \frac{N-R}{N}$

$\rightarrow P(\bar{E}) = \frac{N}{N} - \frac{R}{N}$

$\rightarrow P(\bar{E}) = 1 - \frac{R}{N} \rightarrow P(\bar{E}) = 1 - P(E)$

Exercise 7.5

For the following experiments, find the probability in each case:

Q1.

Experiment:
From a box containing orange-flavoured sweets, Bilal takes out one sweet without looking.

i) the sweet is orange-flavoured
ii) the sweet is lemon-flavoured

Solution:- S={orange flavoured}, n(S)=1

i) 'A' represent sweet is orange

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{1} = 1$$

ii) 'B' represent sweet is lemon

$$P(B) = \frac{n(B)}{n(S)} = \frac{0}{1} = 0$$

Q2.

Experiment:
Pakistan and India play a cricket match. The result is:

Events Happening:
i) Pakistan wins
ii) India does not lose.

Solution:-
S={Win, Lose, Tie}, So n(S)=3

i)
'A' represents "Pakistan wins"
i.e., A={Win}, So n(A)=1

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{3}$$

ii) India does not lose
'B' represent 'india does not lose' i.e., B={Win, Tie}, So $n(B) = 2$

$$P(B) = \frac{n(B)}{n(S)} = \frac{2}{3}$$

Q3.

Experiment.
There are 5 green and 3 red balls in a box, one ball is taken out,

Events happening:
i) the ball is green
ii) the ball is red

Solution:-

i) Green balls = 5
$n(S) = 8$

'A' represents "green balls" $n(A)=5$

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{8}$$

ii) Red balls = 3, $n(S)=8$

'B' represents "Red balls" $n(B)=3$

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{8}$$

Q4.

Experiment:
A fair coin is tossed three times. It shows

Events Happening: i) One tail
ii) alleast one head

Solution:-

i) Sample space of coins 3 times

S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

'A' represents "one tail"

A = {THH, HTH, HHT}, n(A)=3

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$$

ii) 'B' represents "at least one head"

B = {HHH, HHT, HTH, THH, TTH, THT, HTT}, n(B)=8

$$P(B) = \frac{n(B)}{n(S)} = \frac{7}{8}$$

Q5. Experiment:

A die is rolled. The top shows

Events Happening: i) 3 or 4 dots

ii) dots less than 5

Solution:-

i) $S = \{1,2,3,4,5,6\}$, $n(S) = 6$

'A' represents "3 or 4 dots"

$A = \{3,4\}$, $n(A) = 2$

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

ii) 'B' represents "dots less than 5"

$B = \{1,2,3,4\}$, $n(B) = 4$

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{6} = \frac{2}{3}$$

Q6. Experiment:

From a box containing slips numbered 1,2,3,....,5 one slip is picked up

Events Happeining:

i) the number on the slip is a prime number

ii) the number on the slip is a multiple of 3.

Solution:-

i) $S = \{1,2,3,4,5\}$, $n(S) = 5$

'A' represents "Prime number"

$A = \{2,3,5\}$, $n(A) = 3$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{5}$$

ii) B represents "Multiple of 3"

$B = \{3\}$, $n(B) = 1$

$$P(B) = \frac{n(B)}{n(S)} = \frac{1}{3}$$

Q7. Experiment:

Two dice, one red and the other blue, are rolled simultaneously. The number of dots on the tops are added. The total of the two scores is :

Events Happening:

i) 5 ii) 7 iii) 11

Solution:-

i) 'A' represents "sum of dots 5"

$A = \{(1,4),(2,3),(3,2),(4,1)\}$

$n(A) = 4$

$$\rightarrow P(A) = \frac{n(A)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

ii) B represents "sum of dots 7"

$B = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$$n(B) = 6 \rightarrow P(B) = \frac{n(B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

iii) C represents "sum of dots is 11"

$C = \{(5,6),(6,5)\}$, $n(C) = 2$

$$\rightarrow P(C) = \frac{n(C)}{n(S)} = \frac{2}{36} = \frac{1}{18}$$

Q8. Experiment:

A bag contains 40 balls out of which 5 are green, 15 are black and the remaining are yellow. A ball is taken out of the bag.

Events happening:

i) The ball is black

ii) The ball is green

iii) The ball is not green

Solution:-

i) $n(S) = 40$

Green balls = 15

Black balls = 15

Yellow balls = 20

'A' represents "black balls"

$$n(A) = 15 \rightarrow P(A) = \frac{n(A)}{n(S)}$$

$$\rightarrow P(A) = \frac{15}{40} = \frac{3}{8}$$

ii) B represents "Green balls"

$n(B) = 5$

$$P(B) = \frac{n(B)}{n(S)} = \frac{5}{40} = \frac{1}{8}$$

iii) 'C' represents "Balls is not green" $n(C) = 35$

$$P(C) = \frac{n(C)}{n(S)} = \frac{35}{40} = \frac{7}{8}$$

Q9. Experiment:
One chit of 30 containing the names of 30 students of a class of 18 boys and 12 girls is taken out at random, for nomination as the monitor of the class. Events Happening:

i) The monitor is a boy

ii) The monitor is a girl

Solution:-

Boys = 18 , Girls = 12

$S = \{1, 2, 3, \ldots, 30\}$, $n(S) = 30$

i) 'A' "monitor is a boy"

$\therefore$ $n(A) = 18$

$$P(A) = \frac{n(A)}{n(S)} = \frac{18}{30} = \frac{3}{5}$$

ii) 'B' "The monitor is a girl"

$n(B) = 12$

$$\rightarrow P(B) = \frac{n(B)}{n(S)} = \frac{12}{30} = \frac{2}{5}$$

Q10. Experiment:
A coin is tossed four times. The tops show

Events Happening:

i) all heads ii) 2 heads and 2 tails

Solution:-

Coins Tossed 4 times

$n(S) = 2^n = 2^4 = 16$

S = {HHHH, HHHT, HHTH, HTHH, THHH, TTHH, HTHT, THTH, HHTT, THHT, HTTH, TTTH, TTHT, THTT, HTTT, TTTT}

i) 'A' represents "all heads"

A = {HHHH}

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{16}$$

ii) 'B' represents "2 heads 2 tails"

B = {HHTT, TTHH, THHT, HTHT, THTH, HTTH}

$\rightarrow n(B) = 6$

$$P(B) = \frac{n(B)}{n(S)} = \frac{6}{16} = \frac{3}{8}$$

Estimating Probability and Tally Marks

Example 1. The table given below shows the result of rolling a die 100 times. Find the probability in which odd numbers occur.

Event	Tally Marks	Frequency
1	~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~	25
2	~~||||~~ ~~||||~~ |||	13
3	~~||||~~ ~~||||~~ ||||	14
4	~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~ ||||	24
5	~~||||~~ |||	8
6	~~||||~~ ~~||||~~ ~~||||~~ |	16

Solution:-

$$\text{Required probability} = \frac{25+14+8}{100}$$

$$= \frac{47}{100} = \frac{1}{2} \text{(approx)}$$

Example 2. The number of rainy days in Murree during the month of July for the past ten years are:

20, 20, 22, 22, 23, 21, 24, 20, 22, 21

Estimate the probability of the rain falling on a particular day of July. Hence find the number of days in which picinic programme can be made by a group of students who whish to spend 20 days in Murree.

Solution:- Let E be the event that rain falls on a particulor day of a July.

$$P(E) = \frac{20+20+22+22+23+21+24+20+22+21}{31 \times 10}$$

$$= \frac{215}{310} = 0.7 \text{ (approx)}$$

No. of days of raining in 20 days of July $= 20 \times 0.7 = 14$

$\therefore$ The number of days fit for picnic $= 20 - 14 = 6$

Exercise 7.6

Q1. A fair coin is tossed 30 times, the result of which is tabulated below. Study the table and answer the questions given below the table:

Event	Tally Marks	Frequency
Head	~~||||~~ ~~||||~~ ||||	14
Tail	~~||||~~ ~~||||~~ ~~||||~~ |	16

i) How many times does 'head appear'?

ii) How many times does 'Tail appear?

iii) Estimate the probability of the appearance of head?

iv) Estimate the probability of the appearance of tail?

Solution:-

i) Head appear = 14

ii) Tail appear = 16

iii) $P(\text{Head}) = \frac{n(\text{Head})}{n(S)} = \frac{14}{30} = \frac{7}{15}$

iv) $P(\text{Tail}) = \frac{n(\text{Tail})}{n(S)} = \frac{16}{30} = \frac{8}{15}$

Q2. A die is tossed 100 times. The result is tabulated below. Study the table and answer the questions given below the table:

Event	Tally Marks	Frequency
1	~~||||~~ ~~||||~~ ||||	14
2	~~||||~~ ~~||||~~ ~~||||~~ ||	17
3	~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~	20
4	~~||||~~ ~~||||~~ ~~||||~~ |||	18
5	~~||||~~ ~~||||~~ ~~||||~~	15
6	~~||||~~ ~~||||~~ ~~||||~~ |	16

i) How many times do 3 dots appear?

ii) How many times do 5 dots appear?

iii) How many times does an even number of dots appear?

iv) How many times does a prime number of dots appear?

v) Find the probability of each one of the above cases?

Solution:-

i) 3 dots appear $= n(3) = 20$

ii) 5 dots appear $= n(5) = 15$

iii) Even dots $= n(\text{Even}) = 17+18+16 = 51$

iv) Prime dots $= n(\text{Prime}) = 17+20+15 = 52$

$P(3) = \frac{n(3)}{n(S)} = \frac{20}{100} = \frac{1}{5}$

$P(5) = \frac{n(5)}{n(S)} = \frac{15}{100} = \frac{3}{20}$

$P(\text{Prime}) = \frac{n(\text{Prime})}{n(S)} = \frac{52}{100} = \frac{13}{25}$

and $P(\text{Even}) = \frac{n(\text{Even})}{n(S)} = \frac{51}{100}$

Q4. The eggs supplied by a poultry farm during a week, broke during transit as follows: 1%, 2%, $1\frac{1}{2}$%, $\frac{1}{2}$%, 1%, 2%, 1%

Find the probability of the eggs that broke in a day. Calculate the number of eggs that will be broken in transiting the following number of eggs:

i) 7,000 ii) 8,400 iii) 10,500

Solution:-

Eggs broken in one day

$= \frac{9}{100} \times \frac{1}{7} = \frac{9}{700}$

i) 7,000

Eggs are 7,000 then broken

eggs $= 7000 \times \frac{9}{700} = 90$

ii) 8,400

Eggs are 8400 then broken

eggs $= 8400 \times \frac{9}{700} = 108$

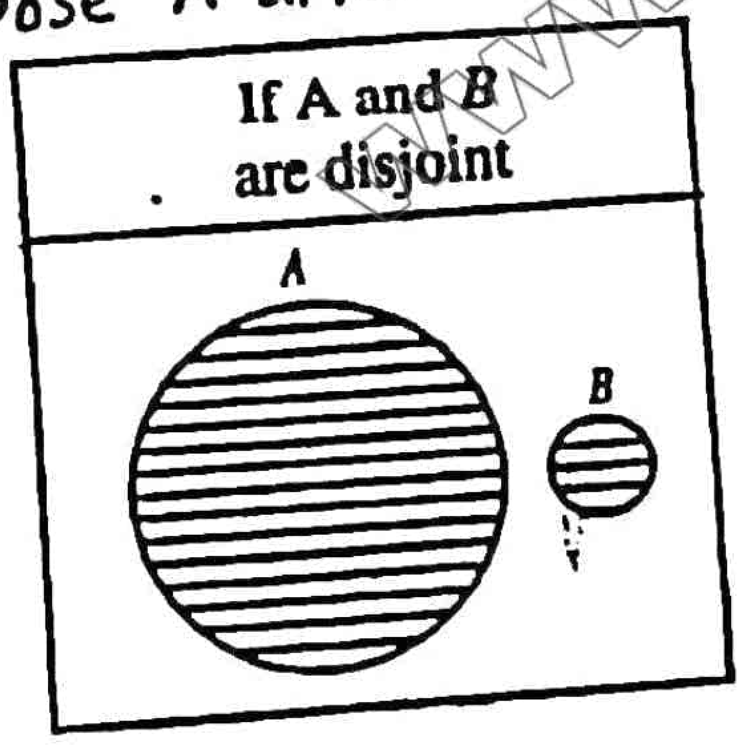
iii) 10,500

Eggs are 10500 then broken

eggs $= 10500 \times \frac{9}{700} = 135$

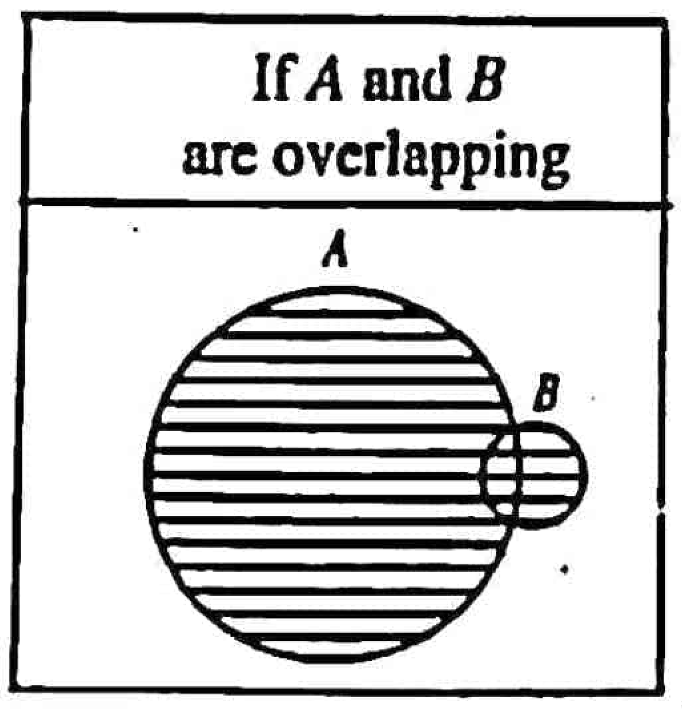
Addition of Probabilities

Suppose A and B be two events



$A \cup B$

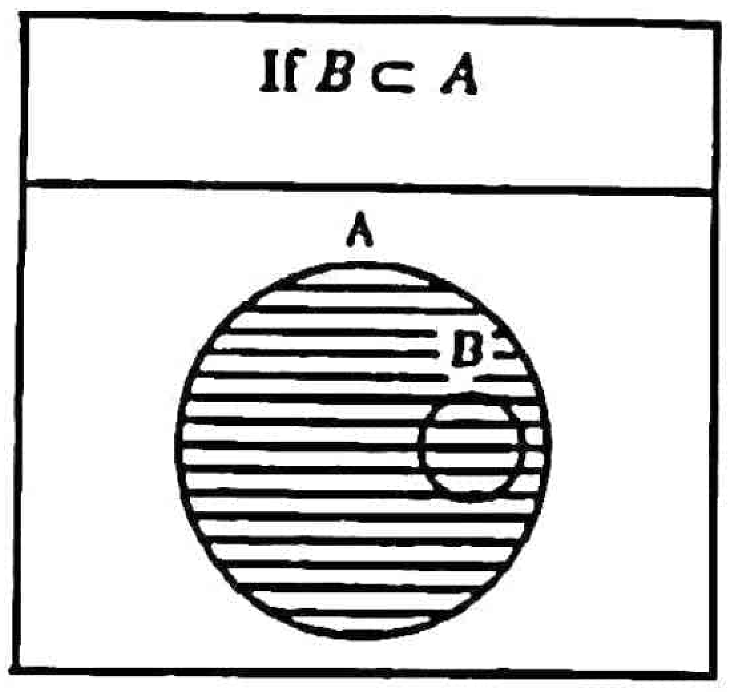
then $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$



$A \cup B$

then $P(A \text{ or } B) = P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$



$A \cup B$

then $P(A \text{ or } B) = P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

Example 1. There are 20 chits marked 1, 2, 3, ..., 20 in a bag. Find the probability of picking a chit, the number written on which is a multiple of 4 or a multiple of 7.

Solution:-

$\because S = \{1, 2, 3, \ldots, 20\}$

$\rightarrow n(S) = 20$

Let A be the event that a chit picked is multiple of 4

$\rightarrow A = \{4, 8, 12, 16, 20\} \rightarrow n(A) = 5$

so $P(A) = \frac{n(A)}{n(S)} = \frac{5}{20}$

Let B denotes the event that a chit picked is multiple of 7,

then $B = \{7, 14\} \rightarrow n(B) = 2$

$\rightarrow P(B) = \frac{n(B)}{n(S)} = \frac{2}{20}$

$\because A \cap B = \{\ \}$, so A and B are disjoint events

$\rightarrow P(A \cup B) = P(A) + P(B)$

$P(A \cup B) = \frac{5}{20} + \frac{2}{20} = \frac{7}{20}$

Example 2. A die is thrown. Find the probability that the dots on the top are prime numbers or odd numbers.

Solution:-

$\because S = \{1,2,3,4,5,6\} \rightarrow n(S) = 6$

Let A denotes the dots on the top are prime numbers

then $A = \{2,3,5\} \rightarrow n(A) = 3$

$\rightarrow P(A) = \dfrac{n(A)}{n(S)} = \dfrac{3}{6}$

Let B denotes the dots on the top are odd numbers

then $B = \{1,3,5\} \rightarrow n(B) = 3$

so $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{3}{6}$

$\because A \cap B = \{3,5\}$, so A and B are overlapping events

$\because n(A \cap B) = 2 \rightarrow P(A \cap B) = \dfrac{n(A \cap B)}{n(S)} = \dfrac{2}{6}$

P(dots on the top are prime numbers or odd number) $= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= \dfrac{3}{6} + \dfrac{3}{6} - \dfrac{2}{6} = \dfrac{3+3-2}{6}$

$P(A \cup B) = \dfrac{4}{6} = \dfrac{2}{3} \rightarrow P(A \cup B) = \dfrac{2}{3}$

Exercise 7.7

Q1. If sample space is $S = \{1,2,3,\ldots,9\}$, Event $A = \{2,4,6,8\}$ and event $B = \{1,3,5\}$ Find $P(A \cup B)$

Solution:-

$\because n(S) = 9,\ n(A) = 4,\ n(B) = 3$

$\rightarrow P(A) = \dfrac{n(A)}{n(S)} = \dfrac{4}{9}$

Also $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{3}{9}$

$\because A \cap B = \{\ \}$, so A and B are disjoint events

$P(A \cup B) = P(A) + P(B)$

$= \dfrac{4}{9} + \dfrac{3}{9} = \dfrac{7}{9}$

Q2. A box contains 10 red, 30 white and 20 black marble. A marble is drawn at random. Find the probability that it is either red or white.

Solution:-

Red = 10, White = 30, black = 20

$n(S) = 60$

A represents 'Red' and B represents 'White'

$n(A) = 10,\ n(B) = 30$

$P(A) = \dfrac{n(A)}{n(S)} = \dfrac{10}{60} = \dfrac{1}{6}$

$P(B) = \dfrac{n(B)}{n(S)} = \dfrac{30}{60} = \dfrac{1}{2}$

$\because A \cap B = \{\ \}$ so A and B are disjoint events.

So $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$

$= \dfrac{1}{6} + \dfrac{1}{2} = \dfrac{1+3}{6} = \dfrac{4}{6}$

$\rightarrow P(A \cup B) = \dfrac{2}{3}$

Q3. A natural number is chosen out of the first fifty natural numbers. What is the probability that the chosen number is a multiple of 3 or of 5?

Solution:-

$S = \{1,2,3,\ldots\ldots,50\}$

$n(S) = 50$

A represents "Multiple of 3"

$A = \{3,6,9,12,15,18,21,24,27,30,33,36,39,42,45,48\}$

$n(A) = 16$

$\rightarrow P(A) = \frac{n(A)}{n(S)} = 16/50$

'B' represents "Multiple of 5"

$B = \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50\}$

$n(B) = 10 \rightarrow P(B) = \frac{n(B)}{n(S)} = \frac{10}{50}$

$\because A \cap B = \{15, 30, 45\}$

$\rightarrow$ A and B are overlapping events.

so $n(A \cap B) = 3$

$\rightarrow P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{3}{50}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{16}{50} + \frac{10}{50} - \frac{3}{50}$

$\rightarrow P(A \cup B) = \frac{16 + 10 - 3}{50} = \frac{23}{50}$

Q4. A card is drawn from a deck of 52 playing cards. What is the probability that it is a diamond card or an ace?

Solution:-

$n(S) =$ Total cards $= 52$

Diamond cards $= n(A) = 13$

Ace cards $= n(B) = 4$

$\because n(A \cap B) = 1$ ($\because$ a card of diamond is also an Ace)

$\rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$

$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$

$\rightarrow P(A \cup B) = \frac{13 + 4 - 1}{52} = \frac{16}{52} = \frac{4}{13}$

Q5. A die is thrown twice. What is the probability that the sum of the number of dots shown is 3 or 11?

Solution:-

$n(S) = 36$

A represents "sum of dots is 3"

$\rightarrow A = \{(1,2), (2,1)\}$

$n(A) = 2 \rightarrow P(A) = \frac{2}{36}$

B represents "sum of dots is 11"

$\rightarrow B = \{(5,6), (6,5)\}$

$n(B) = 2 \rightarrow P(B) = \frac{n(B)}{n(S)} = \frac{2}{36}$

$\because A \cap B = \phi$ (A and B are disjoint events)

$\rightarrow P(A \cup B) = P(A) + P(B)$

$P(A \cup B) = \frac{2}{36} + \frac{2}{36} = \frac{4}{36} = \frac{1}{9}$

Q6. Two dice are thrown. What is the probability that the sum of the number of dots appearing on them is 4 or 6?

Solution:-

$n(S) = 36$

A represents sum is 4

B represents sum is 6.

$A = \{(1,3), (2,2), (3,1)\}$, $n(A) = 3$

$B = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$, $n(B) = 5$

$P(A) = \frac{n(A)}{n(S)} = \frac{3}{36}$

$P(B) = \frac{n(B)}{n(S)} = \frac{5}{36}$

$\because A \cap B = \phi$ (A and B are disjoint events)

$\rightarrow P(A \cup B) = P(A) + P(B)$

$= \frac{3}{36} + \frac{5}{36} = \frac{8}{36}$

$P(A \cup B) = \frac{2}{9}$

Q7. Two dice are thrown simutaneously. If the event A is that the sum of the number of dots shown is an odd number and the event B is that the number of dots shown on at least on die is 3. Find $P(A \cup B)$.

Solution:-

$n(S) = 36$

A represents "sum is odd"

$\rightarrow A = \{(1,2),(1,4),(1,6),(2,1),(2,3)$
$(2,5),(3,2),(3,4),(3,6),(4,1),(4,3)$
$(4,5),(5,2),(5,4),(5,6),(6,1),(6,3)$
$(6,5)\}$

$n(A) = 18 \rightarrow P(A) = \frac{n(A)}{n(S)} = \frac{18}{36}$

B represents "one dice is 3"

$\rightarrow B = \{(1,3),(2,3),(3,1),(3,2),(3,3)$
$(3,4),(3,5),(3,6),(4,3),(5,3)$
$(6,3)\}$

$n(B) = 11 \rightarrow P(B) = \frac{n(B)}{n(S)} = \frac{11}{36}$

$A \cap B = \{(2,3),(3,2),(3,4),(3,6)$
$(4,3),(6,3)\}$, $n(A \cap B) = 6$

$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{18}{36} + \frac{11}{36} - \frac{6}{36}$

$P(A \cup B) = \frac{23}{36}$

Q8. There are 10 girls and 20 boys in a class. Half of the boys and half of the girls have blue eyes. Find the probability that one student chosen as monitor is either a girl or has blue eyes.

Solution:-

Girls $= 10$, Boys $= 20$, $n(S) = 20$

A represent girls $\rightarrow n(A) = 10$

B represent students have blue eyes $\rightarrow n(B) = 15$

and $(A \cap B) = 5$

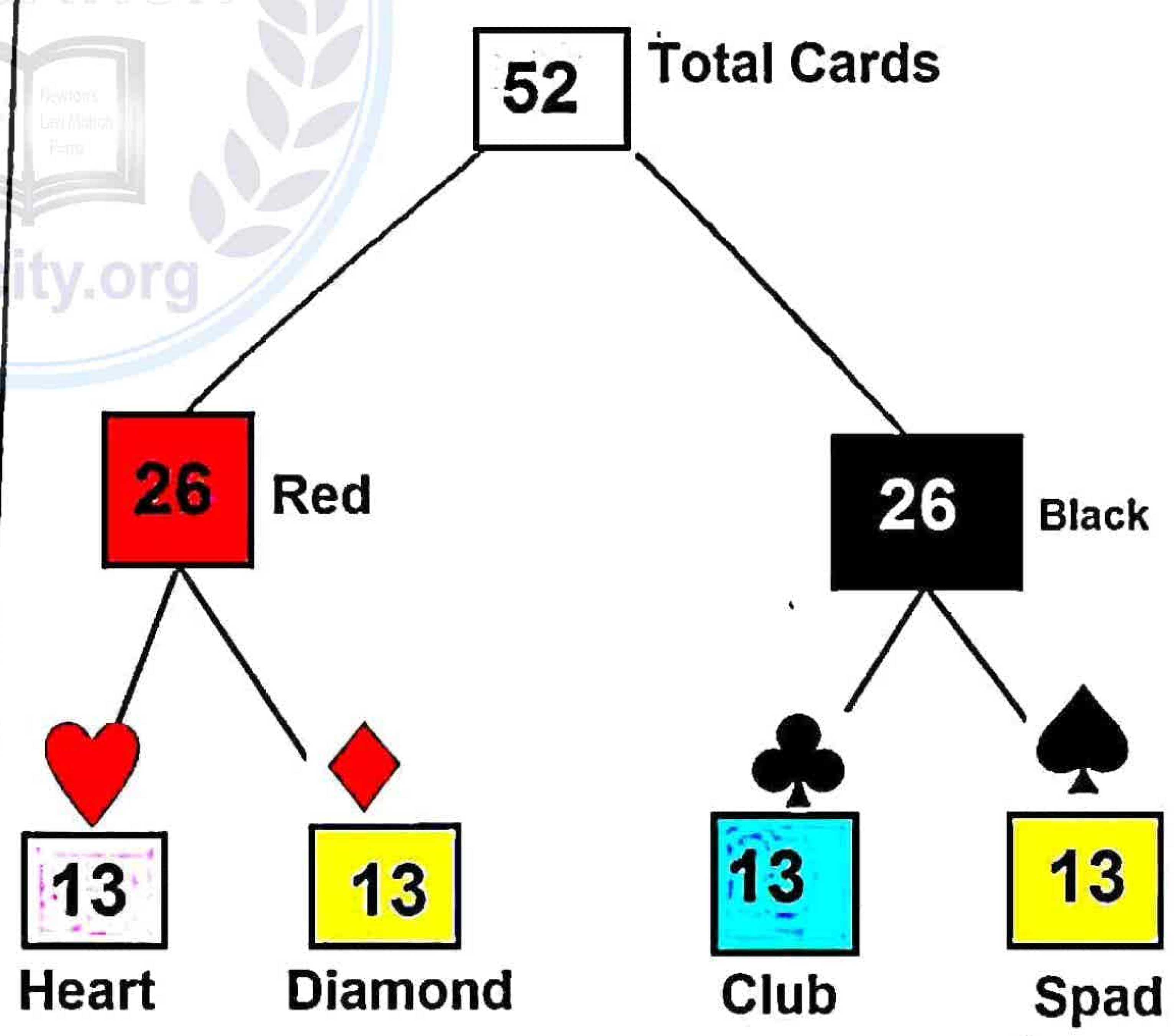
$P(A \cup B) = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$

$= \frac{10}{30} + \frac{15}{30} - \frac{5}{30}$

$\rightarrow P(A \cup B) = \frac{20}{30} = \frac{2}{3}$

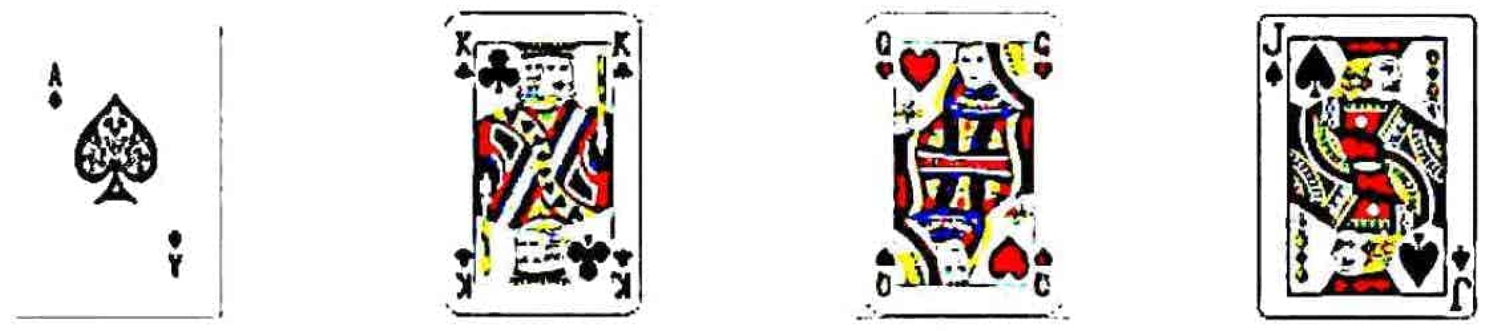
Playing Cards

Need to remember



Each type (Heart, Diamond, Club, Spad) consists of

1- Ace 1- King 1- Queen 1- Jack



Multiplication of Probabilities

Dependent Event:-

Two events are said to be dependent, if the occurrence of any one of them affect the occurrence of the other.

Independent Events:-

Two events are said to be independent, if the occurrence of any one of them does not affect the occurrence of the other.

Theorem:- If A and B are independent events, the probability that both of them occur is equal to the probability of the occurrence of A multiplied by the probability of the occurrence of B. Symbolically, it is denoted by

$$P(A\cap B) = P(A)\cdot P(B)$$

Proof:-

Let event A belong to the sample space S_1 such that

$n(S_1) = n_1$ and $n(A) = m_1$

$\rightarrow P(A) = \dfrac{m_1}{n_1}$

Let event B belong to the sample space S_2 such that

$n(S_2) = n_2$ and $n(B) = m_2$

$\rightarrow P(B) = \dfrac{m_2}{n_2}$

Favorable cases of $A\cap B = m_1 m_2$

Possible cases of $A\cap B = n_1 n_2$

$$P(A\cap B) = \frac{\text{favorable cases}}{\text{possible cases}}$$

$$= \frac{m_1 m_2}{n_1 n_2}$$

$$= \frac{m_1}{n_1}\cdot\frac{m_2}{n_2}$$

$\rightarrow P(A\cap B) = P(A)\cdot P(B)$

Example 1. The probabilities that a man and his will be alive in the next 20 years are 0.8 and 0.75 respectively. Find the probability that both of them will be alive in the next 20 years.

Solution:-

$P(A) = 0.8$; $P(B) = 0.75$

$P(A\cap B) = P(A)\cdot P(B)$

$P(A\cap B) = 0.8 \times 0.75 = 0.6$

Example 2. Two dice are thrown. E_1 is the event that the sum of their dots is an odd numbers and E_2 is the event that 1 is the dot on the top of the first die. Show that $P(E_1\cap E_2) = P(E_1)\cdot P(E_2)$

Solution:- The sample space is

$$S = \left\{\begin{matrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{matrix}\right\}$$

$n(S) = 36$ $(\because 6^2 = 36)$

then

$$E_1 = \{(1,2), (1,4), (1,6), (2,3), (2,5), (3,4), (3,6), (4,3), (4,5), (5,6), (2,1), (4,1), (6,1), (3,2), (5,2), (6,3), (5,4), (6,5)\}$$

$\rightarrow n(E_1) = 18$

$\rightarrow P(E_1) = \dfrac{n(E_1)}{n(S)} = \dfrac{18}{36} = \dfrac{1}{2}$

Now

$E_2 = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\}$

$\rightarrow n(E_2) = 6 \quad \rightarrow P(E_2) = \frac{n(E_2)}{n(S)} = \frac{6}{36}$

$\rightarrow P(E_2) = \frac{1}{6}$

$\because P(E_1).P(E_2) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12} \ldots\ldots (i)$

$E_1 \cap E_2 = \{(1,2),(1,4),(1,6)\}$

$n(E_1 \cap E_2) = 3$

$P(E_1 \cap E_2) = \frac{n(E_1 \cap E_2)}{n(S)} = \frac{3}{36}$

$\rightarrow P(E_1 \cap E_2) = \frac{1}{12} \ldots\ldots (ii)$

By (i) and (ii)

$P(E_1 \cap E_2) = P(E_1).P(E_2)$

Hence proved

Exercise 7.8

Q1. The probability that a person A will be alive 15 year hence is $\frac{5}{7}$ and the probability that another person B will be alive 15 years hence is $\frac{7}{9}$. Find the probability that both will be alive 15 year hence.

Solution:-

$P(A) = \frac{5}{7}, \quad P(B) = \frac{7}{9}$

$P(A \cap B) = P(A).P(B) = \frac{5}{7} \cdot \frac{7}{9} = \frac{5}{9}$

Q2. A die is rolled twice: Event E_1 is the appearance of even number of dots and event E_2 is the appearance of more than 4 dots. Prove that:

$P(E_1 \cap E_2) = P(E_1).P(E_2)$

Solution:-

$S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)$
$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)$
$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)$
$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)$
$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)$
$(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

$\rightarrow n(S) = 36$

$E_1 = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$

$\rightarrow n(E_1) = 9$

$\rightarrow P(E_1) = \frac{n(E_1)}{n(S)} = \frac{9}{36} = \frac{1}{4}$

$E_2 = \{(5,5),(5,6),(6,5),(6,6)\}$

$n(E_2) = 4 \quad \rightarrow P(E_2) = \frac{n(E_2)}{n(S)}$

$= \frac{4}{36} = \frac{1}{9}$

$P(E_1).P(E_2) = \frac{1}{4} \times \frac{1}{9} = \frac{1}{36} \ldots\ldots (i)$

$E_1 \cap E_2 = \{(6,6)\}$

$\rightarrow n(E_1 \cap E_2) = 1$

$P(E_1 \cap E_2) = \frac{n(E_1 \cap E_2)}{n(S)} = \frac{1}{36} \ldots\ldots (ii)$

From (i) and (ii)

$P(E_1 \cap E_2) = P(E_1).P(E_2)$

Hence proved

Q3. Determine the probability of getting 2 heads in two successive tosses of a balanced coin.

Solution:-

$S = \{HH, HT, TH, TT\}, n(S) = 4$

Let A represents 2 heads

$A = \{HH\}, \quad n(A) = 1$

$\rightarrow P(A) = \frac{n(A)}{n(S)} = \frac{1}{4}$

Q4. Two coins are tossed twice each. Find the probability that the head appears on the first toss and the same faces appear on the two tosses.

Solution:-

$S = \{HH, HT, TH, TT\}$, $n(S) = 4$

Let A represents "head appears first"

$A = \{HH, HT\}$; $n(A) = 2$

B represents "same faces"

$B = \{HH, TT\}$, $n(B) = 2$

$\rightarrow P(A \cap B) = P(A) \cdot P(B)$

$= \dfrac{n(A)}{n(S)} \cdot \dfrac{n(B)}{n(S)} = \dfrac{2}{4} \cdot \dfrac{2}{4} = \dfrac{1}{4}$

Q5. Two cards are drawn from a deck of 52 playing cards. If one card is drawn and replaced before the drawing the second card, find the probability that both the cards are aces.

Solution:-

$n(S) = 52$

Let A represent "aces"

B represent "aces"

$\rightarrow n(A) = 4$, $n(B) = 4$

$P(A \cap B) = P(A) \cdot P(B)$

$= \dfrac{n(A)}{n(S)} \cdot \dfrac{n(B)}{n(S)} = \dfrac{4}{52} \cdot \dfrac{4}{52}$

$P(A \cap B) = \dfrac{1}{169}$

Q6. Two cards from a deck of 52 playing cards are drawn in such a way that the card is replaced after the first draw. Find the probabilities in the following cases:

i) first card is king and the second is a queen.

Solution:-

Let A represents "King"

$n(S) = 52$, $n(A) = 4$

B represents "queen"

$n(B) = 4$

$P(A \cap B) = P(A) \cdot P(B)$

$= \dfrac{n(A)}{n(S)} \cdot \dfrac{n(B)}{n(S)}$

$= \dfrac{4}{52} \cdot \dfrac{4}{52} = \dfrac{1}{169}$

ii) both the cards are faced cards i.e., King, queen, jack

Solution:-

A represent face cards

B represent face cards

$n(A) = 12$, $n(B) = 12$

$n(S) = 52$

$P(A \cap B) = P(A) \cdot P(B)$

$= \dfrac{n(A)}{n(S)} \cdot \dfrac{n(B)}{n(S)}$

$= \dfrac{12}{52} \cdot \dfrac{12}{52} = \dfrac{9}{169}$

Q7. Two dice are thrown twice. What is probability that the sum of the dots shown in the first throw is 7 and that of second throw is 11?

Solution:-

$n(S) = 36$

Let A represents "sum is 7"

B represents "sum is 11"

$A = \{(1,6),(2,5),(3,4),(4,3),(5,2)$
$(6,1)\}$

$n(A) = 6, \quad P(A) = \frac{n(A)}{n(S)} = \frac{6}{36}$

$B = \{(5,6),(6,5)\}$

$n(B) = 2, \quad P(B) = \frac{n(B)}{n(S)} = \frac{2}{36}$

$P(A \cap B) = P(A) \cdot P(B)$

$= \frac{6}{36} \cdot \frac{2}{36} = \frac{1}{108}$

Q8. Find the probability that the sum of dots appearing in two successive throws of two dice is every time 7.

Solution:- $n(S) = 36$

Let A represents "sum is 7"

B represents "sum is 7"

$A = B = \{(1,6),(2,5),(3,4),(4,3)$
$(5,2),(6,1)\}$

$\rightarrow P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$

$P(B) = \frac{n(B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$

$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{6} \cdot \frac{1}{6}$

$= \frac{1}{36}$

Q9. A fair die is thrown twice. Find the probability that a prime number of dots appear in the first throw and the number of dots in the second throw is less than 5.

Solution:-

$S = \{1,2,3,4,5,6\} \rightarrow n(S) = 6$

Let A represents "prime number of dots appear in the first throw"

$A = \{2,3,5\} \rightarrow n(A) = 3$

$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

Let B represents "dots appear in second throw are less than 5"

$B = \{1,2,3,4\} \rightarrow n(B) = 4$

$P(B) = \frac{n(B)}{n(S)} = \frac{4}{6} = \frac{2}{3}$

$\rightarrow P(A \cap B) = P(A) \cdot P(B) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$

Q10. A bag contains 8 red, 5 white and 7 black balls. 3 balls are drawn from the bag. What is the probability that the first ball is red, the second ball is white and the third ball is black, when every time the ball is replaced?

Solution:- Total balls $= 8+5+7 = 20$

$n(S) = 20$

Red $= 8$, White $= 5$, Black $= 7$

$n(\text{Red}) = 8$, $n(\text{White}) = 20$, $n(\text{Black}) = 7$

$P(\text{Red, White, Black}) = P(\text{red}) \cdot P(\text{white}) \cdot P(\text{black})$

$= \left(\frac{8}{20}\right)\left(\frac{5}{20}\right)\left(\frac{7}{20}\right)$

$= \frac{280}{8000} = \frac{7}{200}$