MATHEMATICS 1st YEAR

UNIT

05

PARTIAL FRACTIONS

Muhammad Salman Sherazi
M.Phil (Math)
pakcity.org

Contents

Sherazi Mathematics

اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Rational fraction:-

The Quotient of two polynomials $\frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$ with no common factor is called "Rational fraction". e.g., $\frac{x^2+1}{x^2-1}$, $\frac{x^4}{x^2-1}$, $\frac{2x^3-3}{(x-1)(x^2+1)}$

Proper Rational fraction:-

A rational fraction $\frac{P(x)}{Q(x)}$ is called a proper rational fraction if the degree of polynomial $P(x)$ is less than the degree of polynomial $Q(x)$. e.g., $\frac{2x-5}{x^2+4}$, $\frac{3}{x+1}$

Improper Rational fraction:-

A rational fraction $\frac{P(x)}{Q(x)}$ is called an improper rational fraction if the degree of the polynomial $P(x)$ is greater than or equal to the degree of polynomial $Q(x)$. e.g., $\frac{3x^2+1}{x-1}$, $\frac{x^4}{x^2-1}$

Partial Fraction:-

To express a single rational fraction as a sum of two or more single rational fractions is called Partial fraction.

Partial Fraction Resolution:-

Expressing a rational fraction as a sum of partial fractions is called partial fraction resolution.

Conditional equation:-

It is an equation which is true for particular values of the variable e.g., $2x = 3$ is true only if $x = \frac{3}{2}$

* For simplicity, a conditional equation is called an equation.

Identity:- It is an equation which holds good for all values of variable e.g.,

$$(a+b)x = ax + bx$$

The symbol "=" be used both for equation and identity.

Resolution of a Rational Fraction $\frac{P(x)}{Q(x)}$ into Partial Fractions

Following are the main points of resolving a rational fraction $\frac{P(x)}{Q(x)}$ into partial fractions:

i) The degree of $P(x)$ must be less than that of $Q(x)$. If not, divide and work with the remainder theorem.

ii) Clear the given equation of fractions.

iii) Equate the coefficients of like terms (powers of x)

iv) Solve the resulting equations for the coefficients.

Case I

Resolution of $\frac{P(x)}{Q(x)}$ into partial fractions when $Q(x)$ has only repeated linear factors:-

The polynomial $Q(x)$ may be written as:

$Q(x) = (x-a_1)(x-a_2)\dots(x-a_n)$, where

$a_1 \neq a_2 \neq \dots \neq a_n$

$\therefore \frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \dots + \frac{A_n}{x-a_n}$ is an identity. where $A_1, A_2, \dots A_n$ are numbers to be found.

Example 1.

Resolve $\frac{7x+25}{(x+3)(x+4)}$ into partial fractions.

Solution:-

Suppose

$$\frac{7x+25}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4} \longrightarrow (i)$$

'x' by $(x+3)(x+4)$ we get

$\Rightarrow 7x+25 = A(x+4)+B(x+3) \longrightarrow (ii)$

Put $x+3=0 \Rightarrow x=-3$ in (ii)

$7(-3)+25 = A(-3+4)+B(-3+3)$

$-21+25 = A(1) \Rightarrow A=4$

Put $x+4=0 \Rightarrow x=-4$ in (ii)

$7(-4)+25 = A(-4+4)+B(-4+3)$

$-28+25 = B(-1) \Rightarrow -B=-3$

$B=3$

Put values in (i)

$$\frac{7x+25}{(x+3)(x+4)} = \frac{4}{x+3} + \frac{3}{x+4}$$

Note:- * If $Q(x)$ has a factor which can be factorized, first of all factorize it.

* We use partial fraction when the fraction $\frac{P(x)}{Q(x)}$ is proper rational fraction.

* If we are given improper fraction (division is possible) then first of all divide the fraction and make it proper fraction. after this, use partial fraction.

Example 2.

Resolve $\frac{x^2-10x+13}{(x-1)(x^2-5x+6)}$ into partial fractions.

Solution:-

$$\frac{x^2-10x+13}{(x-1)(x^2-5x+6)}$$

$$= \frac{x^2-10x+13}{(x-1)(x-2)(x-3)}$$

$\because x^2-5x+6$
$= x^2-3x-2x+6$
$= x(x-3)-2(x-3)$
$= (x-3)(x-2)$

suppose

$$\frac{x^2-10x+13}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \longrightarrow (i)$$

'x' by $(x-1)(x-2)(x-3)$.

$\Rightarrow x^2-10x+13 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \longrightarrow (ii)$

Put $x-1=0 \Rightarrow x=1$ in (ii)

$(1)^2-10(1)+13 = A(1-2)(1-3)$

$\Rightarrow 1-10+13 = A(-1)(-2)$

$4 = 2A \Rightarrow A=2$

Put $x-2=0 \Rightarrow x=2$ in (ii)

$(2)^2-10(2)+13 = B(2-1)(2-3)$
$= B(1)(-1)$

$4-20+13 = -B \Rightarrow -3=-B$

$B=3$

Put $x-3=0 \Rightarrow x=3$ in (ii)

$(3)^2-10(3)+13 = C(3-1)(3-2)$

$9-30+13 = C(2)(1)$

$-8 = 2C \Rightarrow C=-4$

Put values in (i)

$$\frac{x^2-10x+13}{(x-1)(x-2)(x-3)} = \frac{2}{x-1} + \frac{3}{x-2} - \frac{4}{x-3}$$

Example 3.

Resolve $\frac{2x^3+x^2-x-3}{x(2x+3)(x-1)}$ into partial fractions.

Solution:-

$$\frac{2x^3+x^2-x-3}{x(2x+3)(x-1)} \quad (\text{Improper})$$

$$= \frac{2x^3+x^2-x-3}{x(2x^2-2x+3x-3)} = \frac{2x^3+x^2-x-3}{x(2x^2+x-3)}$$

$$= \frac{2x^3+x^2-x-3}{2x^3+x^2-3x}$$

$$\begin{array}{r|l} & 1 \\ 2x^3+x^2-3x & 2x^3+x^2-x-3 \\ & \underline{\pm 2x^3 \pm x^2 \mp 3x} \\ & 2x-3 \end{array}$$

so

$$\frac{2x^3+x^2-x-3}{2x^3+x^2-3x} = 1 + \frac{2x-3}{2x^3+x^2-3x} \quad (\text{Proper})$$

or

$$\frac{2x^3+x^2-x-3}{x(2x+3)(x-1)} = 1 + \frac{2x-3}{x(2x+3)(x-1)}$$

Suppose

$$\frac{2x-3}{x(2x+3)(x-1)} = \frac{A}{x} + \frac{B}{2x+3} + \frac{C}{x-1} \longrightarrow (i)$$

'x' eq (i) by $x(2x+3)(x-1)$

$\Rightarrow 2x-3 = A(2x+3)(x-1) + Bx(x-1) + Cx(2x+3) \longrightarrow (ii)$

Put $x=0$ in (ii)

$2(0)-3 = A(2(0)+3)(0-1)$

$-3 = A(3)(-1) \rightarrow -3=-3A$

or $A=1$

Put $2x+3=0 \rightarrow x=-\frac{3}{2}$ in (ii)

$2(-\frac{3}{2})-3 = B(-\frac{3}{2})(-\frac{3}{2}-1)$

$-3-3 = B(-\frac{3}{2})(\frac{-3-2}{2})$

$\rightarrow -6 = B(-\frac{3}{2})(-\frac{5}{2}) \rightarrow -6 = B(\frac{15}{4})$

$B = -\frac{6\times 4}{15} = \frac{-24}{15} \rightarrow B = \frac{-24}{15} = \frac{-8}{5}$

Put $x-1=0 \rightarrow x=1$ in (ii)

$2(1)-3 = C(1)(2(1)+3)$

$\rightarrow 2-3 = C(5) \rightarrow -1=5C$

or $C=-\frac{1}{5}$

Put values in (i)

$$\frac{2x-3}{x(2x+3)(x-1)} = \frac{1}{x} + \frac{-8/5}{2x+3} + \frac{-1/5}{x-1}$$

So
$$\frac{2x^3+x^2-x-3}{x(2x+3)(x-1)} = 1+\frac{1}{x} - \frac{8}{5(2x+3)} - \frac{1}{5(x-1)}$$

Exercise 5.1

Resolve the following into Partial Fractions:

Q1. $\frac{1}{x^2-1}$

Solution:- $\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)}$

Suppose

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \longrightarrow (i)$$

'x' by $(x-1)(x+1)$, we get

$$1 = A(x+1) + B(x-1) \longrightarrow (ii)$$

Put $x+1=0 \rightarrow x=-1$ in (ii)

$1 = B(-1-1) \rightarrow 1=-2B$

or $B=-\frac{1}{2}$

Put $x-1=0 \rightarrow x=1$ in (ii)

$1 = A(1+1) \rightarrow 1=2A$

or $A=\frac{1}{2}$

Put values in (i)

$$\frac{1}{(x-1)(x+1)} = \frac{1/2}{x-1} + \frac{-1/2}{x+1}$$

or
$$\frac{1}{x^2-1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

Q2. $\frac{x^2+1}{(x+1)(x-1)}$

Solution:- $\frac{x^2+1}{(x+1)(x-1)} = \frac{x^2+1}{x^2-1}$ (improper)

$$\begin{array}{r|l} & 1 \\ \hline x^2-1 & x^2+1 \\ & \underline{-x^2 \mp 1} \\ & 2 \end{array}$$

So
$$\frac{x^2+1}{x^2-1} = 1 + \frac{2}{x^2-1} \text{ (proper)}$$

Now take $\frac{2}{x^2-1} = \frac{2}{(x-1)(x+1)}$

Suppose
$$\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \longrightarrow (i)$$

'x' by $(x-1)(x+1)$, we get

$$2 = A(x+1) + B(x-1) \longrightarrow (ii)$$

Put $x+1=0 \rightarrow x=-1$ in (ii)

$2 = B(-1-1) \rightarrow 2=B(-2) \rightarrow B=-1$

Put $x-1=0 \rightarrow x=1$ in (ii)

$2 = A(1+1) \rightarrow 2=2A \rightarrow A=1$

Put values in (i)

$$\frac{2}{(x-1)(x+1)} = \frac{1}{x-1} + \frac{-1}{x+1}$$

So
$$\frac{x^2+1}{(x+1)(x-1)} = 1+\frac{1}{x-1} - \frac{1}{x+1}$$

Q3. $\frac{2x+1}{(x-1)(x+2)(x+3)}$

Solution:-

Suppose
$$\frac{2x+1}{(x-1)(x+2)(x+3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+3} \longrightarrow (i)$$

'x' by $(x-1)(x+2)(x+3)$, we get

$$(2x+1) = A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2) \longrightarrow (ii)$$

Put $x-1=0 \rightarrow x=1$ in (ii)

$$2(1)+1 = A(1+2)(1+3)$$

$$\cancel{3} = A(\cancel{3})(4) \rightarrow 4A = 1$$

$$\rightarrow A = \frac{1}{4}$$

Put $x+2=0 \rightarrow x=-2$ in (ii)

$$2(-2)+1 = B(-2-1)(-2+3)$$

$$-4+1 = B(-3)(1) \rightarrow -\cancel{3} = B(-\cancel{3})(1)$$

$$\rightarrow B=1$$

Put $x+3=0 \rightarrow x=-3$ in (ii)

$$2(-3)+1 = C(-3-1)(-3+2)$$

$$-6+1 = C(-4)(-1) \rightarrow -5 = 4C$$

$$C = -\frac{5}{4}$$

Put values in (i)

$$\frac{2x+1}{(x-1)(x+2)(x+3)} = \frac{1}{4(x-1)} + \frac{1}{x+2} - \frac{5}{4(x+3)}$$

Q4. $\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)}$

Solution:-

$$\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)} = \frac{3x^2-4x-5}{(x-2)(x+2)(x+5)}$$

$$\because x^2+7x+10 = x^2+5x+2x+10 = x(x+5)+2(x+5) = (x+2)(x+5)$$

Suppose

$$\frac{3x^2-4x-5}{(x-2)(x+2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x+5} \longrightarrow (i)$$

'x' by $(x-2)(x+2)(x+5)$, we get

$$\rightarrow 3x^2-4x-5 = A(x+2)(x+5) + B(x-2)(x+5) + C(x-2)(x+2) \longrightarrow (ii)$$

Put $x-2=0 \rightarrow x=2$ in (ii)

$$3(2)^2-4(2)-5 = A(2+2)(2+5)$$

$$3(4)-8-5 = A(4)(7)$$

$$12-8-5 = 28A \rightarrow 28A = -1$$

$$A = \frac{-1}{28}$$

Put $x+2=0 \rightarrow x=-2$ in (ii)

$$3(-2)^2-4(-2)-5 = B(-2-2)(-2+5)$$

$$3(4)+8-5 = B(-4)(3)$$

$$12+8-5 = -12B \rightarrow 15 = -12B$$

$$B = -\frac{15}{12} \rightarrow B = -\frac{5}{4}$$

Put $x+5=0 \rightarrow x=-5$ in (ii)

$$3(-5)^2-4(-5)-5 = C(-5-2)(-5+2)$$

$$3(25)+20-5 = C(-7)(-3)$$

$$75+20-5 = 21C \rightarrow 90 = 21C$$

$$C = \frac{90}{21} \rightarrow C = \frac{30}{7}$$

Put values in (i)

$$\frac{3x^2-4x-5}{(x-2)(x+2)(x+5)} = \frac{-1}{28(x-2)} - \frac{15}{12(x+2)} + \frac{30}{7(x+5)}$$

Q5. $\frac{1}{(x-1)(2x-1)(3x-1)}$

Solution:- Suppose

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{A}{x-1} + \frac{B}{2x-1} + \frac{C}{3x-1} \longrightarrow (i)$$

'x' by $(x-1)(2x-1)(3x-1)$, we get

$$1 = A(2x-1)(3x-1) + B(x-1)(3x-1) + C(x-1)(2x-1) \longrightarrow (ii)$$

Put $x-1=0 \rightarrow x=1$ in (ii)

$$1 = A(2(1)-1)(3(1)-1) \rightarrow 1 = A(2-1)(3-1)$$

$$1 = A(1)(2) \rightarrow 1 = 2A \rightarrow A = \frac{1}{2}$$

Put $2x-1=0 \rightarrow x = \frac{1}{2}$ in (ii)

$$1 = B\left(\frac{1}{2}-1\right)\left(3\left(\frac{1}{2}\right)-1\right) \rightarrow 1 = B\left(-\frac{1}{2}\right)\left(\frac{3-2}{2}\right)$$

$$1 = B\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) \rightarrow 1 = -\frac{1}{4}B$$

$$\rightarrow B = -4$$

Put $3x-1=0 \rightarrow x = \frac{1}{3}$ in (ii)

$$1 = C\left(\frac{1}{3}-1\right)\left(2\left(\frac{1}{3}\right)-1\right) \rightarrow 1 = C\left(\frac{1-3}{3}\right)\left(\frac{2-3}{3}\right)$$

$$1 = C\left(-\frac{2}{3}\right)\left(-\frac{1}{3}\right) \rightarrow 1 = C\left(\frac{2}{9}\right)$$

$$C = \frac{9}{2}$$ put values in (i)

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{1}{2(x-1)} - \frac{4}{2x-1} + \frac{9}{2(3x-1)}$$

Q6. $\dfrac{x}{(x-a)(x-b)(x-c)}$

Solution:-

suppose

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \longrightarrow (i)$$

'x' by $(x-a)(x-b)(x-c)$, we get

$$\rightarrow x = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b) \longrightarrow (ii)$$

Put $x-a=0 \rightarrow x=a$ in (ii)

$$a = A(a-b)(a-c)$$

$$\rightarrow A = \frac{a}{(a-b)(a-c)}$$

Put $x-b=0 \rightarrow x=b$ in (ii)

$$b = B(b-a)(b-c)$$

$$\rightarrow B = \frac{b}{(b-a)(b-c)}$$

Put $x-c=0 \rightarrow x=c$ in (ii)

$$c = C(c-a)(c-b)$$

$$\rightarrow C = \frac{c}{(c-a)(c-b)}$$

Put values in (i)

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{a}{(a-b)(a-c)(x-a)} + \frac{b}{(b-a)(b-c)(x-b)} + \frac{c}{(c-a)(c-b)(x-c)}$$

Q7. $\dfrac{6x^3+5x^2-7}{2x^2-x-1}$

Solution:- $\dfrac{6x^3+5x^2-7}{2x^2-x-1}$ (improper)

$$\begin{array}{r|l} & 3x+4 \\ \hline 2x^2-x-1 & 6x^3+5x^2-7 \\ & \underline{\mp 6x^3 \mp 3x^2 \pm 3x} \\ & 8x^2+3x-7 \\ & \underline{\mp 8x^2 \mp 4x \pm 4} \\ & 7x-3 \end{array}$$

so

$$\frac{6x^3+5x^2-7}{2x^2-x-1} = 3x+4 + \frac{7x-3}{2x^2-x-1} \text{ (Proper)} \longrightarrow (i)$$

Now

$$\frac{7x-3}{2x^2-x-1} = \frac{7x-3}{2x^2-2x+x-1}$$

$$= \frac{7x-3}{2x(x-1)+1(x-1)} = \frac{7x-3}{(2x+1)(x-1)}$$

$$\frac{7x-3}{(2x+1)(x-1)} = \frac{A}{x-1} + \frac{B}{2x+1} \longrightarrow (ii)$$

'x' by $(2x+1)(x-1)$, we get

$$7x-3 = A(2x+1) + B(x-1) \longrightarrow (iii)$$

Put $2x+1=0 \rightarrow x=-\frac{1}{2}$ in (iii)

$$7\left(-\tfrac{1}{2}\right)-3 = B\left(-\tfrac{1}{2}-1\right)$$

$$-\frac{7}{2}-3 = -\frac{3}{2}B \rightarrow -\frac{3}{\cancel{2}}B = \frac{-7-6}{\cancel{2}}$$

$$\rightarrow B = \frac{-13}{-3} \text{ or } B = \frac{13}{3}$$

Put $x-1=0 \rightarrow x=1$ in (iii)

$$7(1)-3 = A(2(1)+1)$$

$$4 = 3A \rightarrow A = \frac{4}{3}$$

Put values in (ii)

$$\frac{7x-3}{(x-1)(2x+1)} = \frac{4}{3(x-1)} + \frac{13}{3(2x-1)}$$

Hence

$$\frac{6x^3+5x^2-7}{2x^2-x-1} = 3x+4 + \frac{4}{3(x-1)} + \frac{13}{3(2x-1)}$$

Q8. $\dfrac{2x^3+x^2-5x+3}{2x^3+x^2-3x}$

Solution:- $\dfrac{2x^3+x^2-5x+3}{2x^3+x^2-3x}$ (improper)

$$\begin{array}{r|l} & 1 \\ \hline 2x^3+x^2-3x & 2x^3+x^2-5x+3 \\ & \underline{\mp 2x^3 \mp x^2 \pm 3x} \\ & -2x+3 \end{array}$$

so

$$\frac{2x^3+x^2-5x+3}{2x^3+x^2-3x} = 1 + \frac{-2x+3}{2x^3+x^2-3x} \longrightarrow (i)$$

Now

$$\frac{-2x+3}{2x^3+x^2-3x} = \frac{3-2x}{x(2x^2+x-3)}$$

$$= \frac{3-2x}{x[2x^2+3x-2x-3]}$$

$$= \frac{3-2x}{x[x(2x+3)-1(2x+3)]}$$

$$= \frac{3-2x}{x(2x+3)(x-1)}$$

Now

$$\frac{3-2x}{x(2x+3)(x-1)} = \frac{A}{x} + \frac{B}{2x+3} + \frac{C}{x-1} \longrightarrow (ii)$$

'x' by $x(2x+3)(x-1)$, we get

$$3-2x = A(2x+3)(x-1) + Bx(x-1) + C(x)(2x+3) \longrightarrow (iii)$$

Put $x=0$ in (iii)

$$3 = A(2(0)+3)(0-1)$$

$$\cancel{3} = A(\cancel{3})(-1) \rightarrow 1 = A(-1)$$

$$A = -1$$

Put $2x+3=0 \rightarrow x = -\frac{3}{2}$ in (iii)

$$3 - \cancel{2}\left(\frac{-3}{\cancel{2}}\right) = B\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)$$

$$\rightarrow 6 = B\left(-\frac{3}{2}\right)\left(\frac{-3-2}{2}\right)$$

$$6 = B\left(-\frac{3}{2}\right)\left(\frac{-5}{2}\right) \rightarrow 6 = \frac{15}{4}B$$

$$B = \frac{6\times 4}{15} \rightarrow B = \frac{2\times 4}{5} = \frac{8}{5}$$

Put $x-1=0 \rightarrow x=1$ in (iii)

$$3-2(1) = C(1)(2(1)+3) \rightarrow 1 = 5C$$

$$\rightarrow C = 1/5$$

Put values in (ii)

$$\frac{3-2x}{x(2x+3)(x-1)} = \frac{-1}{x} + \frac{8}{5(2x+3)} + \frac{1}{5(x-1)}$$

Q9. $\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}$

Solution:-

$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)} \quad \text{(improper)}$$

$$= \frac{(x-1)(x^2-5x-3x+15)}{(x-2)(x^2-6x-4x+24)}$$

$$= \frac{(x-1)(x^2-8x+15)}{(x-2)(x^2-10x+24)}$$

$$= \frac{x^3-8x^2+15x-x^2+8x-15}{x^3-10x^2+24x-2x^2+20x-48}$$

$$= \frac{x^3-9x^2+23x-15}{x^3-12x^2+44x-48}$$

$$\begin{array}{r|l} & \quad 1 \\ x^3-12x^2+44x-48 & x^3-9x^2+23x-15 \\ & \underline{\underset{-}{x^3} \underset{+}{-12x^2} \underset{-}{+44x} \underset{+}{-48}} \\ & 3x^2-21x+33 \end{array}$$

So

$$\frac{x^3-9x^2+23x-15}{x^3-12x^2+44x-48} = 1 + \frac{3x^2-21x+33}{(x-2)(x-4)(x-6)} \longrightarrow (i)$$

Now suppose

$$\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)} = \frac{A}{x-2} + \frac{B}{x-4} + \frac{C}{x-6} \longrightarrow (ii)$$

'x' by $(x-2)(x-4)(x-6)$, we get

$$3x^2-21x+33 = A(x-4)(x-6) + B(x-2)(x-6) + C(x-2)(x-4) \longrightarrow (iii)$$

Put $x-2=0 \rightarrow x=2$ in (iii)

$$3(2)^2-21(2)+33 = A(2-4)(2-6)$$

$$3(4)-42+33 = A(-2)(-4)$$

$$12-42+33 = 8A \rightarrow A = \frac{3}{8}$$

Put $x-4=0 \rightarrow x=4$ in (iii)

$$3(4)^2-21(4)+33 = B(4-2)(4-6)$$

$$3(16)-84+33 = B(2)(-2)$$

$$\rightarrow -4B = 48-84+33$$

$$\rightarrow -4B = -3 \rightarrow B = \frac{3}{4}$$

Put $x-6=0 \rightarrow x=6$ in (iii)

$$3(6)^2-21(6)+33 = C(6-2)(6-4)$$

$$3(36)-126+33 = C(4)(2)$$

$$108-126+33 = 8C$$

$$\rightarrow 8C = 15 \text{ or } C = \frac{15}{8}$$

Put values in (ii)

$$\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)} = \frac{3}{8(x-2)} + \frac{3}{4(x-4)} + \frac{15}{8(x-6)}$$

Hence

$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)} = 1 + \frac{3}{8(x-2)} + \frac{3}{4(x-4)} + \frac{15}{8(x-6)}$$

Q10. $\frac{1}{(1-ax)(1-bx)(1-cx)}$

Solution:-

suppose

$$\frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{A}{1-ax} + \frac{B}{1-bx} + \frac{C}{1-cx} \longrightarrow (i)$$

'x' by $(1-ax)(1-bx)(1-cx)$, we get

$$1 = A(1-bx)(1-cx) + B(1-ax)(1-cx) + C(1-ax)(1-bx) \longrightarrow (ii)$$

Put $1-ax=0 \rightarrow ax=1 \rightarrow x=\frac{1}{a}$ in (ii)

$$1 = A\left(1-\frac{b}{a}\right)\left(1-\frac{c}{a}\right) \rightarrow 1 = A\left(\frac{a-b}{a}\right)\left(\frac{a-c}{a}\right)$$

$$\rightarrow 1 = A\frac{(a-b)(a-c)}{a^2} \rightarrow \frac{a^2}{(a-b)(a-c)} = A$$

$$\rightarrow A = \frac{a^2}{(a-b)(a-c)}$$

Put $1-bx=0 \rightarrow x=\frac{1}{b}$ in (ii)

$$1 = B\left(1-\frac{a}{b}\right)\left(1-\frac{c}{b}\right) \rightarrow 1 = B\left(\frac{b-a}{b}\right)\left(\frac{b-c}{b}\right)$$

$$1 = B\left(\frac{(b-a)(b-c)}{b^2}\right) \rightarrow \frac{b^2}{(b-a)(b-c)} = B$$

$$\rightarrow B = \frac{b^2}{(b-a)(b-c)}$$

Put $1-cx=0 \rightarrow x=\frac{1}{c}$ in (ii)

$$1 = C\left(1-\frac{a}{c}\right)\left(1-\frac{b}{c}\right)$$

$$\rightarrow 1 = C\left(\frac{c-a}{c}\right)\left(\frac{c-b}{c}\right) \rightarrow 1 = C\left(\frac{(c-a)(c-b)}{c^2}\right)$$

$$\rightarrow c^2 = C(c-a)(c-b)$$

or $C = \frac{c^2}{(c-a)(c-b)}$

Put values in (i)

$$\frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{a^2}{(a-b)(a-c)(1-ax)} + \frac{b^2}{(b-a)(b-c)(1-bx)} + \frac{c^2}{(c-a)(c-b)(1-cx)}$$

Q11. $\frac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)}$

Solution:-

$$\frac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)}$$

Replace x^2 by y

$$= \frac{y+a^2}{(y+b^2)(y+c^2)(y+d^2)}$$

Suppose

$$\frac{y+a^2}{(y+b^2)(y+c^2)(y+d^2)} = \frac{A}{y+b^2} + \frac{B}{y+c^2} + \frac{C}{y+d^2} \longrightarrow (i)$$

'x' by $(y+b^2)(y+c^2)(y+d^2)$

$$y+a^2 = A(y+c^2)(y+d^2) + B(y+a^2)(y+d^2) + C(y+a^2)(y+c^2) \longrightarrow (ii)$$

Put $y+b^2=0 \rightarrow y=-b^2$ in (ii)

$$-b^2+a^2 = A(-b^2+c^2)(-b^2+d^2)$$

$$\rightarrow A = \frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)}$$

Put $y+c^2=0 \rightarrow y=-c^2$ in (ii)

$$-c^2+a^2 = B(-c^2+a^2)(-c^2+d^2)$$

$$\rightarrow B = \frac{a^2-c^2}{(a^2-c^2)(d^2-c^2)}$$

Put $y+d^2=0 \rightarrow y=-d^2$ in (ii)

$$-d^2+a^2 = C(-d^2+a^2)(-d^2+c^2)$$

$$\rightarrow C = \frac{a^2-d^2}{(a^2-d^2)(c^2-d^2)}$$

Put values in (i)

$$\frac{y+a^2}{(y+b^2)(y+c^2)(y+d^2)} = \frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)(y+b^2)} + \frac{a^2-c^2}{(b^2-c^2)(d^2-c^2)(y+c^2)} + \frac{a^2-d^2}{(b^2-d^2)(c^2-d^2)}$$

Thus

$$\frac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)} = \frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)(x^2+b^2)} + \frac{a^2-c^2}{(b^2-c^2)(d^2-c^2)(x^2+c^2)} + \frac{a^2-d^2}{(b^2-c^2)(d^2-c^2)(x^2+c^2)}$$

Case II

When $Q(x)$ has repeated linear factors: If $Q(x)$ has a factor $(x-a)^n$, $n \geq 2$ and n is +ive integer, then $\frac{P(x)}{Q(x)}$ may be written as the following identity:

$$\therefore \frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)^2} + \cdots + \frac{A_n}{(x-a_n)^n}$$

where $A_1, A_2, \ldots, A_n$ are numbers to found.

Example 1. Resolve $\frac{x^2+x-1}{(x+2)^3}$ into partial fractions.

Solution:- suppose

$$\frac{x^2+x-1}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} \rightarrow (i)$$

'x' by $(x+2)^3$, we get

$$\rightarrow x^2+x-1 = A(x+2)^2 + B(x+2) + C \rightarrow (ii)$$

Put $x+2=0 \rightarrow x=-2$ in (ii)

$(-2)^2+(-2)-1 = C \rightarrow C = 4-2-1$

$\rightarrow C = 1$

From (ii)

$x^2+x-1 = A(x^2+4x+4) + Bx + 2B + C$

$x^2+x-1 = Ax^2+4Ax+4A+Bx+2B+C$

Equating the coefficients of x^2 and x

$1 = A$ (For x^2)

$1 = 4A+B$ (For x)

$1 = 4(1)+B \quad \because A=1$

$B = 1-4 \rightarrow B=-3$

Put values in (i)

$$\frac{x^2+x-1}{(x+2)^3} = \frac{1}{x+2} - \frac{3}{(x+2)^2} + \frac{1}{(x+2)^3}$$

Example 2. Resolve $\frac{1}{(x+1)^2(x^2-1)}$ into partial fractions:

Solution:-

$$\frac{1}{(x+1)^2(x^2-1)} = \frac{1}{(x+1)^2(x-1)(x+1)}$$

$$= \frac{1}{(x+1)^3(x-1)}$$

suppose

$$\frac{1}{(x+1)^3(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x-1} \rightarrow (i)$$

'x' by $(x+1)^3(x-1)$, we get

$$1 = A(x+1)^2(x-1) + B(x+1)(x-1) + C(x-1) + D(x+1)^3 \rightarrow (ii)$$

Put $x+1=0 \rightarrow x=-1$ in (ii)

$1 = C(-1-1) \rightarrow 1=-2C$

or $C = -\frac{1}{2}$

Put $x-1=0 \rightarrow x=1$ in (ii)

$1 = D(1+1)^3 \rightarrow 1 = D(2)^3$

$1 = 8D$ or $D = \frac{1}{8}$

From (ii)

$1 = A(x^2+1+2x)(x-1) + B(x^2-1) + Cx - C + D(x^3+1+3x^2+3x)$

$1 = A[x^3-x^2+x-1+2x^2-2x] + Bx^2 - B + Cx - C + Dx^3 + D + 3Dx^2 + 3Dx$

$1 = A(x^3+x^2-x-1) + Bx^2 - B + Cx - C + Dx^3 + D + 3Dx^2 + 3Dx$

$1 = Ax^3 + Ax^2 - Ax - A + Bx^2 - B + Cx - C + Dx^3 + D + 3Dx^2 + 3Dx$

Equating coefficients of x^3 and x^2

$0 = A+D$ (For x^3)

$A = -D \rightarrow A = -\frac{1}{8}$

$0 = A+B+3D$ (For x^2)

$0 = -\frac{1}{8} + B + 3(\frac{1}{8})$

$0 = -\frac{1}{8} + B + \frac{3}{8} \rightarrow B = \frac{1}{8} - \frac{3}{8} = -\frac{2}{8}$

or $B = -\frac{1}{4}$

Put values in (i)

$$\frac{1}{(x+1)^3(x-1)} = \frac{-1}{8(x+1)} - \frac{1}{4(x+1)^2} - \frac{1}{2(x+1)^3} + \frac{1}{8(x-1)}$$

Exercise 5.2

Resolve the following into Partial Fractions:

Q1. $\dfrac{2x^2-3x+4}{(x-1)^3}$

Solution:- suppose

$$\frac{2x^2-3x+4}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} \longrightarrow (i)$$

'x' by $(x-1)^3$, we get

$$2x^2-3x+4 = A(x-1)^2 + B(x-1) + C \longrightarrow (ii)$$

Put $x-1=0 \rightarrow x=1$ in (ii)

$2(1)^2-3(1)+4 = C \rightarrow 2-3+4 = C$

$C = 3$

From (ii)

$2x^2-3x+4 = Ax^2-2Ax+A+Bx-B+C$

Equating coefficients

x^2; $2 = A$

x; $-3 = -2A+B \rightarrow -3 = -2(2)+B$

$-3 = -4+B \rightarrow B = 4-3$

$B = 1$

(i) becomes as

$$\frac{2x^2-3x+4}{(x-1)^3} = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{3}{(x-1)^3}$$

Q2. $\dfrac{5x^2-2x+3}{(x+2)^3}$

Solution:- suppose

$$\frac{5x^2-2x+3}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} \longrightarrow (i)$$

'x' by $(x+2)^3$, we get

$$5x^2-2x+3 = A(x+2)^2 + B(x+2) + C \longrightarrow (ii)$$

Put $x+2=0 \rightarrow x=-2$ in (ii)

$5(-2)^2-2(-2)+3 = C$

$5(4)+4+3 = C \rightarrow C = 20+7$

$C = 27$

From (ii)

$5x^2-2x+3 = A(x^2+4x+4)+Bx+2B+C$

$5x^2-2x+3 = Ax^2+4Ax+4A+Bx+2B+C$

Equating coefficients

x^2; $5 = A$

x; $-2 = 4A+B \rightarrow -2 = 4(5)+B$

$B = -2-20 \rightarrow B = -22$

(i) becomes as

$$\frac{5x^2-2x+3}{(x+2)^3} = \frac{5}{x+2} - \frac{22}{(x+2)^2} + \frac{27}{(x+3)^3}$$

Q3. $\dfrac{4x}{(x+1)^2(x-1)}$

Solution:- suppose

$$\frac{4x}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} \rightarrow (i)$$

'x' by $(x+1)^2(x-1)$, we get

$$4x = A(x+1)(x-1) + B(x-1) + C(x+1)^2 \longrightarrow (ii)$$

Put $x+1=0 \rightarrow x=-1$ in (ii)

$4(-1) = B(-1-1) \rightarrow -4 = -2B$

$B = 2$

Put $x-1=0 \rightarrow x=1$ in (ii)

$4(1) = C(1+1)^2 \rightarrow 4 = C(2)^2$

$4 = 4C \rightarrow C = 1$

From (ii)

$4x = A(x^2-1) + Bx - B + C(x^2+1+2x)$

$4x = Ax^2 - A + Bx - B + Cx^2 + C + 2Cx$

comparing coefficients,

x^2; $0 = A+C \rightarrow 0 = A+1$

$A = -1$

(i) becomes as

$$\frac{4x}{(x+1)^2(x-1)} = \frac{-1}{x+1} + \frac{2}{(x+1)^2} + \frac{1}{x-1}$$

Q4. $\dfrac{9}{(x+2)^2(x-1)}$

Solution:- Suppose

$$\frac{9}{(x+2)^2(x-1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1} \longrightarrow (i)$$

'x' by $(x+2)^2(x-1)$ we get

$9 = A(x+2)(x-1) + B(x-1) + C(x+2)^2 \rightarrow (ii)$

Put $x-1=0 \rightarrow x=1$ in (ii)

$9 = C(1+2)^2 \rightarrow 9 = C(3)^2$

$\rightarrow 9 = 9C \rightarrow C = 1$

Put $x+2=0 \rightarrow x=-2$ in (ii)

$9 = B(-2-1) \rightarrow 9 = -3B$

$B = -3$

From (ii)

$9 = A(x^2-x+2x-2) + Bx - B + C(x^2+4+4x)$

$9 = Ax^2 + Ax - 2A + Bx - B + Cx^2 + 4C + 4Cx$

Equating coefficients

x^2; $0 = A + C \rightarrow 0 = A + 1$

$A = -1$

(i) becomes as

$$\frac{9}{(x+2)^2(x-1)} = \frac{-1}{x+2} - \frac{3}{(x+2)^2} + \frac{1}{x-1}$$

Q5. $\frac{1}{(x-3)^2(x+1)}$

Solution:- suppose

$$\frac{1}{(x-3)^2(x+1)} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+1} \rightarrow (i)$$

'x' by $(x-3)^2(x+1)$, we get

$\rightarrow 1 = A(x-3)(x+1) + B(x+1) + C(x-3)^2 \rightarrow (ii)$

Put $x-3=0 \rightarrow x=3$ in (ii)

$1 = B(3+1) \rightarrow 1 = 4B$

$\rightarrow B = \frac{1}{4}$

Put $x+1=0 \rightarrow x=-1$ in (ii)

$1 = C(-1-3)^2 \rightarrow 1 = C(-4)^2$

$1 = 16C \rightarrow C = \frac{1}{16}$

From (ii)

$1 = A(x^2+x-3x-3) + Bx + B + C(x^2+9-9x)$

$1 = Ax^2 - 2Ax - 3A + Bx + B + Cx^2 + 9C - 9Cx$

Equating coefficient

x^2; $0 = A + C \rightarrow 0 = A + \frac{1}{16}$

$\rightarrow A = -\frac{1}{16}$

(i) becomes as

$$\frac{1}{(x-3)^2(x+1)} = \frac{-1}{16(x-3)} + \frac{1}{4(x-3)^2} + \frac{1}{16(x+1)}$$

Q6. $\frac{x^2}{(x-2)(x-1)^2}$

Solution:- suppose

$$\frac{x^2}{(x-2)(x-1)^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \rightarrow (i)$$

'x' by $(x-2)(x-1)^2$, we get

$x^2 = A(x-1)^2 + B(x-2)(x-1) + C(x-2) \rightarrow (ii)$

Put $x-2=0 \rightarrow x=2$ in (ii)

$(2)^2 = A(2-1)^2 \rightarrow 4 = A$

Put $x-1=0 \rightarrow x=1$ in (ii)

$(1)^2 = C(1-2) \rightarrow 1 = -C \rightarrow C = -1$

From (ii)

$x^2 = A(x^2+1-2x) + B(x^2-x-2x+2) + Cx - 2C$

$x^2 = Ax^2 + A - 2Ax + Bx^2 - 3Bx + 2B + Cx - 2C$

Equating coefficients

x^2; $1 = A + B \rightarrow 1 = 4 + B$

$B = 1 - 4 \rightarrow B = -3$

(i) becomes as

$$\frac{x^2}{(x-2)(x-1)^2} = \frac{4}{x-2} - \frac{3}{x-1} - \frac{1}{(x-1)^2}$$

Q7. $\frac{1}{(x-1)^2(x+1)}$

Solution:- suppose

$$\frac{1}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \rightarrow (i)$$

'x' by $(x-1)^2(x+1)$, we get

$1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2 \rightarrow (ii)$

Put $x-1=0 \rightarrow x=1$ in (ii)

$1 = B(1+1) \rightarrow 1 = 2B \rightarrow B = \frac{1}{2}$

Put $x+1=0 \rightarrow x=-1$ in (ii)

$1 = C(-1-1)^2 \rightarrow 1 = C(-2)^2$

$1 = 4C \rightarrow C = \frac{1}{4}$

From (ii)

$1 = A(x^2-1) + Bx + B + C(x^2-2x+1)$

$1 = Ax^2 - A + Bx + B + Cx^2 - 2Cx + C$

Equating coefficients

x^2; $0 = A + C \rightarrow 0 = A + \frac{1}{4}$

$A = -\frac{1}{4}$

(i) becomes as

$$\frac{1}{(x-1)^2(x+1)} = \frac{-1}{4(x-1)} + \frac{1}{2(x-1)^2} + \frac{1}{4(x+1)}$$

Q8. $\frac{x^2}{(x-1)^3(x+1)}$

Solution:- Suppose

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1} \longrightarrow (i)$$

'x' by $(x-1)^3(x+1)$ we get

$x^2 = A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1)$
$+ D(x-1)^3 \longrightarrow (ii)$

Put $x-1=0 \rightarrow x=1$ in (ii)

$(1)^2 = C(1+1) \rightarrow 1 = 2C \rightarrow C = \frac{1}{2}$

Put $x+1=0 \rightarrow x=-1$ in (ii)

$(-1)^2 = D(-1-1)^3 \rightarrow 1 = D(-2)^3$

$1 = -8D \rightarrow D = -\frac{1}{8}$

From (ii)

$x^2 = A(x^2+1-2x)(x+1) + B(x^2-1) + Cx + C$
$+ D(x^3 - 1 - 3x^2 + 3x)$

$x^2 = A(x^3+x^2+x+1-2x^2-2x) + Bx^2 - B$
$+ Cx + C + Dx^3 - D - 3Dx^2 + 3Dx$

$x^2 = A(x^3 - x^2 - x + 1) + Bx^2 - B + Cx$
$+ C + Dx^3 - D - 3Dx^2 + 3Dx$

$x^2 = Ax^3 - Ax^2 - Ax + A + Bx^2 - B + Cx$
$+ C + Dx^3 - D - 3Dx^2 + 3Dx$

Equating coefficients

x^3; $0 = A + D \rightarrow 0 = A - \frac{1}{8}$

$A = \frac{1}{8}$

x^2; $1 = -A + B - 3D$

$1 = -\frac{1}{8} + B - 3\left(-\frac{1}{8}\right)$

$\rightarrow 1 = -\frac{1}{8} + B + \frac{3}{8}$

$B = 1 + \frac{1}{8} - \frac{3}{8}$

$B = \frac{8+1-3}{8} = \frac{6}{8} = \frac{3}{4}$

$\rightarrow B = \frac{3}{4}$

(i) becomes as

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{1}{8(x-1)} + \frac{3}{4(x-1)^2} + \frac{1}{2(x-1)^3} - \frac{1}{8(x+1)}$$

Q9. $\frac{x-1}{(x-2)(x+1)^3}$

Solution:- Suppose

$$\frac{x-1}{(x-2)(x+1)^3} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3} \longrightarrow (i)$$

'x' by $(x-2)(x+1)^3$, we get

$x-1 = A(x+1)^3 + B(x-2)(x+1)^2 + C(x-2)(x+1)$
$+ D(x-2) \longrightarrow (ii)$

Put $x-2=0 \rightarrow x=2$ in (ii)

$2-1 = A(2+1)^3 \rightarrow 1 = A(3)^3 \rightarrow 1 = 27A$

$A = \frac{1}{27}$

Put $x+1=0 \rightarrow x=-1$ in (ii)

$-1-1 = D(-1-2) \rightarrow -2 = -3D$

$D = \frac{2}{3}$

From (ii)

$x-1 = A(x^3+1+3x^2+3x) + B(x-2)(x^2+1+2x)$
$+ C(x^2+x-2x-2) + Dx - 2D$

$x-1 = Ax^3 + A + 3Ax^2 + 3Ax + B(x^3+x+2x^2$
$-2x^2-2-4x)$
$+ Cx^2 - Cx - 2C + Dx - 2D$

$x-1 = Ax^3 + A + 3Ax^2 + 3Ax + Bx^3 - 3Bx - 2B$
$+ Cx^2 - Cx - 2C + Dx - 2D$

Equating coefficients

x^3; $0 = A + B \rightarrow 0 = \frac{1}{27} + B$

$B = -\frac{1}{27}$

x^2; $0 = 3A + C \rightarrow 0 = 3\left(\frac{1}{27}\right) + C$

$C = -\frac{1}{9}$

so (i) becomes as

$$\frac{x-1}{(x-2)(x+1)^3} = \frac{1}{27(x-2)} - \frac{1}{27(x+1)} - \frac{1}{9(x+1)^2} + \frac{2}{3(x+1)^3}$$

Q10. $\frac{4x^3}{(x^2-1)(x+1)^2}$

Solution:-

$$\frac{4x^3}{(x^2-1)(x+1)^2} = \frac{4x^3}{(x-1)(x+1)(x+1)^2}$$

$$= \frac{4x^3}{(x-1)(x+1)^3}$$

suppose

$$\frac{4x^3}{(x-1)(x+1)^3} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3} \longrightarrow (i)$$

'x' by $(x-1)(x+1)^3$, we get

$$4x^3 = A(x+1)^3 + B(x-1)(x+1)^2 + C(x-1)(x+1) + D(x-1) \longrightarrow (ii)$$

Put $x-1=0 \rightarrow x = 1$ in (ii)

$4(1)^3 = A(1+1)^3$

$4 = A(2)^3 \rightarrow 8A = 4 \rightarrow A = \frac{1}{2}$

Put $x+1=0 \rightarrow x = -1$ in (ii)

$4(-1)^3 = D(-1-1) \rightarrow -4 = -2D$

$D = 2$

From (ii)

$$4x^3 = A(x^3+1+3x^2+3x) + B(x-1)(x^2+1+2x) + C(x^2-1) + Dx - D$$

$$4x^3 = Ax^3 + A + 3Ax^2 + 3Ax + B(x^3+x+2x^2 - x^2-1-2x) + Cx^2 - C + Dx - D$$

$$4x^3 = Ax^3 + A + 3Ax^2 + 3Ax + Bx^3 + Bx^2 - Bx - B + Cx^2 - C + Dx - D$$

Equating coefficients

x^3; $4 = A + B \rightarrow 4 = \frac{1}{2} + B$

$B = 4 - \frac{1}{2} = \frac{8-1}{2} = \frac{7}{2} \rightarrow B = \frac{7}{2}$

x^2; $0 = 3A + B + C$

$0 = 3\left(\frac{1}{2}\right) + \frac{7}{2} + C$

$C = -\frac{3}{2} - \frac{7}{2} = -\frac{10}{2} = -5 \rightarrow C = -5$

so (i) becomes as

$$\frac{4x^3}{(x-1)(x+1)^3} = \frac{1}{2(x-1)} + \frac{7}{2(x+1)} - \frac{5}{(x+1)^2} + \frac{2}{(x+1)^3}$$

or

$$\frac{4x^3}{(x^2-1)(x+1)^2} = \frac{1}{2(x-1)} + \frac{7}{2(x+1)} - \frac{5}{(x+1)^2} + \frac{2}{(x+1)^3}$$

Q11. $\frac{2x+1}{(x+3)(x-1)(x+2)^2}$

Solution:-

suppose

$$\frac{2x+1}{(x+3)(x-1)(x+2)^2} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x+2)} + \frac{D}{(x+2)^2} \longrightarrow (i)$$

'x' by $(x+3)(x-1)(x+2)^2$, we get

$$\rightarrow 2x+1 = A(x-1)(x+2)^2 + B(x+3)(x+2)^2 + C(x+3)(x-1)(x+2) + D(x+3)(x-1) \longrightarrow (ii)$$

Put $x+3=0 \rightarrow x = -3$ in (ii)

$2(-3)+1 = A(-3-1)(-3+2)^2$

$-6+1 = A(-4)(-1)^2 \rightarrow -5 = A(-4)(1)$

$-5 = -4A \rightarrow A = \frac{5}{4}$

Put $x-1=0 \rightarrow x = 1$ in (ii)

$2(1)+1 = B(1+3)(1+2)^2$

$3 = B(4)(9) \rightarrow 3 = 36B$

$B = \frac{3}{36} \rightarrow B = \frac{1}{12}$

Put $x+2=0 \rightarrow x = -2$ in (ii)

$2(-2)+1 = D(-2+3)(-2-1)$

$-4+1 = D(1)(-3) \rightarrow -3 = -3D$

$\rightarrow D = 1$

From (ii)

$$2x+1 = A(x-1)(x^2+4+4x) + B(x+3)(x^2+4+4x) + C(x+3)(x^2+2x-x-2) + D(x^2-x+3x-3)$$

$$2x+1 = A(x^3+\cancel{4x}+4x^2-x^2-4-\cancel{4x}) + B(x^3+4x+4x^2+3x^2+12+12x) + C(x+3)(x^2+x-2) + D(x^2+2x-3)$$

$$\Rightarrow 2x+1 = A(x^3+3x^2-4) + B(x^3+7x^2+16x+12) + C(x^3+x^2-2x+3x^2+3x-6) + Dx^2+2Dx-3D$$

$$2x+1 = Ax^3+3Ax^2-4A+Bx^3+7Bx^2+16Bx+12B + Cx^3+4Cx^2+Cx-6C+Dx^2+2Dx-3D$$

Equating coefficients

x^3; $0 = A+B+C$

$$0 = \frac{5}{4}+\frac{1}{12}+C$$

$$\Rightarrow C = -\frac{5}{4}-\frac{1}{12} = \frac{-15-1}{12} = -\frac{16}{12}$$

$$C = -\frac{4}{3}$$

so (i) becomes as

$$\frac{2x+1}{(x+3)(x-1)(x+2)^2} = \frac{5}{4(x+3)} + \frac{1}{12(x-1)} - \frac{4}{3(x-1)} + \frac{1}{(x+2)^2}$$

Q12. $\dfrac{2x^4}{(x-3)(x+2)^2}$

Solution:-

$$\frac{2x^4}{(x-3)(x+2)^2} \quad \text{(Improper)}$$

$$= \frac{2x^4}{(x-3)(x^2+4x+4)} = \frac{2x^4}{x^3+4x^2+4x-3x^2-12x-12}$$

$$= \frac{2x^4}{x^3+x^2-8x-12}$$

$$\begin{array}{r|l} & 2x-2 \\ \hline x^3+x^2-8x-12 & 2x^4 \\ & \underline{\mp 2x^4 \pm 2x^3 \mp 16x^2 \mp 24x} \\ & -2x^3+16x^2+24x \\ & \underline{\mp 2x^3 \mp 2x^2 \pm 16x \pm 24} \\ & 18x^2+8x-24 \end{array}$$

So

$$\frac{2x^4}{x^3+x^2-8x-12} = 2x-2+\frac{18x^2+8x-24}{x^3+x^2-8x-12} \quad \text{(Proper)}$$

Now suppose

$$\frac{18x^2+8x-24}{(x-3)(x+2)^2} = \frac{A}{x-3}+\frac{B}{x+2}+\frac{C}{(x+2)^2} \rightarrow (i)$$

'x' by $(x-3)(x+2)^2$, we get

$$18x^2+8x-24 = A(x+2)^2+B(x-3)(x+2)+C(x-3) \rightarrow (ii)$$

Put $x-3=0 \Rightarrow x=3$ in (ii)

$$18(3)^2+8(3)-24 = A(3+2)^2$$

$$162+24-24 = 25A \Rightarrow A = \frac{162}{25}$$

Put $x+2=0 \Rightarrow x=-2$ in (ii)

$$18(-2)^2+8(-2)-24 = C(-2-3)$$

$$72-16-24 = -5C$$

$$32 = -5C \Rightarrow C = \frac{-32}{5}$$

From (ii)

$$18x^2+8x-24 = A(x^2+4+4x)+B(x^2+2x-3x-6) + Cx-3C$$

$$18x^2+8x-24 = Ax^2+4A+4Ax+Bx^2-Bx-6B + Cx-3C$$

Equating coefficients

x^2; $18 = A+B \Rightarrow 18 = \frac{162}{25}+B$

$$B = 18-\frac{162}{25} = \frac{450-162}{25}$$

$$\Rightarrow B = \frac{288}{25}$$

so (i) becomes

$$\frac{18x^2+8x-24}{(x-3)(x+2)^2} = \frac{162}{25(x-3)}+\frac{288}{25(x+2)} - \frac{32}{5(x+2)^2}$$

Hence

$$\frac{2x^4}{(x-3)(x+2)^2} = 2x-2+\frac{162}{25(x-3)}+\frac{288}{25(x+2)}-\frac{32}{5(x+2)^2}$$

Case III

when $Q(x)$ contains non-repeated irreducible quadratic factor :-

If $Q(x)$ contains non-repeated irreducible quadratic factor then $\frac{P(x)}{Q(x)}$ may be written as the identity having partial fractions of the form $\frac{Ax+B}{ax^2+bx+c}$ where A and B are numbers to be found.

Irreducible Quadratic factor:- A quadratic factor is irreducible if it cannot be written as the product of two linear factors with real coefficients e.g., x^2+x+1 and x^2+3

Example 1. Resolve $\frac{3x-11}{(x^2+1)(x+3)}$ into partial fractions.

Solution:- $\frac{3x-11}{(x^2+1)(x+3)}$

suppose

$$\frac{3x-11}{(x^2+1)(x+3)} = \frac{Ax+B}{x^2+1}+\frac{C}{x+3} \longrightarrow (i)$$

'x' by $(x^2+1)(x+3)$, we get

$$\rightarrow 3x-11 = (Ax+B)(x+3)+C(x^2+1) \longrightarrow (ii)$$

Put $x+3=0 \rightarrow x=-3$ in (ii)

$$3(-3)-11 = C((-3)^2+1)$$

$$-9-11 = C(9+1) \rightarrow -20 = 10C$$

$$C=-2$$

From (ii)

$$3x-11 = Ax^2+3Ax+Bx+3B+Cx^2+C$$

Equating coefficients

x^2; $0 = A+C \rightarrow 0 = A-2$

$A=2$

x; $3 = 3A+B \rightarrow 3 = 3(2)+B$

$3-6=B \rightarrow B=-3$

So (i) becomes

$$\frac{3x-11}{(x^2+1)(x+3)} = \frac{2x-3}{x^2+1}-\frac{2}{x+3}$$

Example 2. Resolve $\frac{4x^2+8x}{x^4+2x^2+9}$ into partial fractions.

Solution:- $\frac{4x^2+8x}{x^4+2x^2+9}$

Here $x^4+2x^2+9 = x^4+2x^2+9+4x^2-4x^2$

$= x^4+6x^2+9-4x^2$

$=(x^2)^2+2(3)(x^2)+(3)^2-(2x)^2$

$=(x^2+3)^2-(2x)^2$

$=(x^2+3-2x)(x^2+3-2x)$

$\rightarrow x^4+2x^2+9 = (x^2+2x+3)(x^2-2x+3)$

Now

$$\frac{4x^2+8x}{x^4+2x^2+9} = \frac{4x^2+8x}{(x^2+2x+3)(x^2-2x+3)}$$

Suppose

$$\frac{4x^2+8x}{(x^2-2x+3)(x^2+2x+3)} = \frac{Ax+B}{x^2+2x+3}+\frac{Cx+D}{x^2-2x+3} \longrightarrow (i)$$

'x' by $(x^2-2x+3)(x^2+2x+3)$, we get

$$4x^2+8x = (Ax+B)(x^2-2x+3)+(Cx+D)(x^2+2x+3)$$

$$4x^2+8x = Ax^3-2Ax^2+3Ax+Bx^2-2Bx+3B+Cx^3+2Cx^2+3Cx+Dx^2+2Dx+3D$$

Comparing coefficients

x^3; $0 = A+C \longrightarrow (ii)$

x^2; $4 = -2A+B+2C+D \longrightarrow (iii)$

x; $8 = 3A-2B+3C+2D \longrightarrow (iv)$

cons. term; $0 = 3B+3D \rightarrow B+D=0$ (v)

Put $B+D=0$ in (iii)

$-2A+2C+0=4$

$\Rightarrow 2C-2A=4$

or $C-A=2$ (÷ by 2) ⟶ (vi)

By (ii)+(vi) $\Rightarrow 2C=2 \Rightarrow C=1$

so (ii) $\Rightarrow A+1=0 \Rightarrow A=-1$

By (iv) $\Rightarrow 8=3(-1)-2B+3(1)+2D$

$8=-3-2B+3+2D$

$8=-2B+2D \Rightarrow -B+D=4$

$D-B=4 \longrightarrow$ (vii)

By (v)+(vii) $\Rightarrow 2D=4 \Rightarrow D=2$

so (v) $\Rightarrow B+2=0 \Rightarrow B=-2$

so (i) becomes

$$\frac{4x^2+8x}{(x^2+2x+3)(x^2-2x+3)} = \frac{(-1)x-2}{x^2+2x+3} + \frac{(1)x+2}{x^2-2x+3}$$

$$= \frac{-x-2}{x^2+2x+3} + \frac{x+2}{x^2-2x+3}$$

Hence

$$\frac{4x^2+8x}{x^4+2x^2+9} = \frac{-x-2}{x^2+2x+3} + \frac{x+2}{x^2-2x+3}$$

Exercise 5.3

Resolve the following into Partial Fractions:

Q1. $\frac{9x-7}{(x^2+1)(x+3)}$

Solution:- suppose

$$\frac{9x-7}{(x^2+1)(x+3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+3} \longrightarrow (i)$$

'x' by $(x^2+1)(x+3)$, we get

$\Rightarrow 9x-7 = (Ax+B)(x+3)+C(x^2+1) \longrightarrow$ (ii)

Put $x+3=0 \Rightarrow x=-3$ in (ii)

$9(-3)-7 = C((-3)^2+1)$

$-27-7 = C(9+1) \Rightarrow C = \frac{-34}{10} = \frac{-17}{5}$

$\Rightarrow C = -\frac{17}{5}$

From (ii)

$9x-7 = Ax^2+3Ax+Bx+3B+Cx^2+C$

Equating coefficients

x^2; $0 = A+C \Rightarrow 0 = A-\frac{17}{5}$

$\Rightarrow A = \frac{17}{5}$

x; $9 = 3A+B \Rightarrow 9 = 3(\frac{17}{5})+B$

$\Rightarrow 9 = \frac{51}{5}+B \Rightarrow 9-\frac{51}{5} = B$

$\Rightarrow B = \frac{45-51}{5} = -\frac{6}{5} \Rightarrow B = \frac{-6}{5}$

so (i) becomes as

$$\frac{9x-7}{(x^2+1)(x+3)} = \frac{\frac{17x}{5}-\frac{6}{5}}{x^2+1} + \frac{-17/5}{x+3}$$

or
$$\frac{9x-7}{(x^2+1)(x+3)} = \frac{17x-6}{5(x^2+1)} - \frac{17}{5(x+3)}$$

Q2. $\frac{1}{(x^2+1)(x+1)}$

Solution:- suppose

$$\frac{1}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} \longrightarrow (i)$$

'x' by $(x^2+1)(x+1)$, we get

$1 = (Ax+B)(x+1)+C(x^2+1) \longrightarrow$ (ii)

Put $x+1=0 \Rightarrow x=-1$ in (ii)

$1 = C((-1)^2+1) \Rightarrow 1 = C(1+1)$

$\Rightarrow C = \frac{1}{2}$

From (ii)

$1 = Ax^2+Ax+Bx+B+Cx^2+C$

Equating coefficients

x^2; $A+C=0 \Rightarrow A+\frac{1}{2}=0$

$A = -\frac{1}{2}$

x; $0 = A+B \Rightarrow 0 = -\frac{1}{2}+B$

$\Rightarrow B = \frac{1}{2}$

so (i) becomes

$$\frac{1}{(x^2+1)(x+1)} = \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1} + \frac{\frac{1}{2}}{x+1}$$

or

$$\frac{1}{(x^2+1)(x+1)} = \frac{-x+1}{2(x^2+1)} + \frac{1}{2(x+1)}$$

Q3. $\dfrac{3x+7}{(x^2+4)(x+3)}$

Solution:- suppose

$$\frac{3x+7}{(x^2+4)(x+3)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+3} \longrightarrow (i)$$

'x' by $(x^2+4)(x+3)$, we get

$$3x+7 = (Ax+B)(x+3) + C(x^2+4) \longrightarrow (ii)$$

Put $x+3=0 \rightarrow x=-3$ in (ii)

$$3(-3)+7 = C((-3)^2+4)$$

$$-9+7 = C(9+4) \rightarrow -2 = 13C$$

$$\text{or } C = \frac{-2}{13}$$

From (ii)

$$3x+7 = Ax^2+3Ax+Bx+3B+Cx^2+4C$$

Equating coefficients

x^2; $0 = A+C \rightarrow 0 = A-\frac{2}{13}$

$\rightarrow A = \frac{2}{13}$

x; $3 = 3A+B$

$\rightarrow 3 = 3\left(\frac{2}{13}\right)+B \rightarrow B = 3-\frac{6}{13}$

$\rightarrow B = \frac{39-6}{13} \rightarrow B = \frac{33}{13}$

So (i) becomes as

$$\frac{3x+7}{(x^2+4)(x+3)} = \frac{\frac{2}{13}x+\frac{33}{13}}{x^2+4} + \frac{-12/13}{x+3}$$

$$\frac{3x+7}{(x^2+4)(x+3)} = \frac{2x+33}{13(x^2+4)} - \frac{12}{13(x+3)}$$

Q4. $\dfrac{x^2+15}{(x^2+2x+5)(x-1)}$

Solution:- Suppose

$$\frac{x^2+15}{(x^2+2x+5)(x-1)} = \frac{Ax+B}{x^2+2x+5} + \frac{C}{x-1} \longrightarrow (i)$$

'x' by $(x^2+2x+5)(x-1)$, we get

$$x^2+15 = (Ax+B)(x-1) + C(x^2+2x+5) \longrightarrow (ii)$$

Put $x-1=0 \rightarrow x=1$ in (ii)

$$(1)^2+15 = C((1)^2+2(1)+5)$$

$$16 = C(1+2+5) \rightarrow 16 = 8C$$

$$C = 2$$

From (ii)

$$x^2+15 = Ax^2-Ax+Bx-B+Cx^2+2Cx+5C$$

Equating coefficients

x^2; $1 = A+C \rightarrow 1 = A+2$

$A = 1-2 \rightarrow A = -1$

x; $0 = -A+B+2C$

$0 = -(-1)+B+2(2)$

$\rightarrow 0 = 1+B+4 \rightarrow 0 = B+5$

or $B = -5$

so (i) becomes as

$$\frac{x^2+15}{(x^2+2x+5)(x-1)} = \frac{(-1)x+(-5)}{x^2+2x+5} + \frac{2}{x-1}$$

$$\frac{x^2+15}{(x^2+2x+5)(x-1)} = \frac{-x-5}{x^2+2x+5} + \frac{2}{x-1}$$

Q5. $\dfrac{x^2}{(x^2+4)(x+2)}$

Solution:- Suppose

$$\frac{x^2}{(x^2+4)(x+2)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+2} \longrightarrow (i)$$

'x' by $(x^2+4)(x+2)$, we get

$$\rightarrow x^2 = (Ax+B)(x+2) + C(x^2+4) \longrightarrow (ii)$$

Put $x+2=0 \rightarrow x=-2$ in (ii)

$$(-2)^2 = C((-2)^2+4) \rightarrow 4 = C(4+4)$$

$$4 = 8C \rightarrow \frac{1}{2} = C$$

From (ii)

$$x^2 = Ax^2+2Ax+Bx+2B+Cx^2+4C$$

Equating coefficients

x^2; $1 = A+C \rightarrow 1 = A+\frac{1}{2}$

$1-\frac{1}{2} = A \rightarrow A = \frac{1}{2}$

x; $0 = 2A+B$

$0 = 2\left(\frac{1}{2}\right) + B \rightarrow B = -1$

so (i) becomes as

$$\frac{x^2}{(x^2+4)(x+2)} = \frac{\frac{1}{2}x + (-1)}{x^2+4} + \frac{\frac{1}{2}}{x+2}$$

$$\frac{x^2}{(x^2+4)(x+2)} = \frac{\frac{1}{2}x - 1}{x^2+4} + \frac{1}{2(x+2)}$$

$$\frac{x^2}{(x^2+4)(x+2)} = \frac{x-2}{2(x^2+4)} + \frac{1}{2(x+2)}$$

Q6. $\frac{x^2+1}{x^3+1}$

Solution:- $\frac{x^2+1}{x^3+1}$

$$= \frac{x^2+1}{(x+1)(x^2-x+1)}$$

suppose

$$\frac{x^2+1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \rightarrow (i)$$

'x' by $(x+1)(x^2-x+1)$, we get

$$\rightarrow x^2+1 = A(x^2-x+1) + (Bx+C)(x+1) \rightarrow (ii)$$

Put $x+1=0 \rightarrow x=-1$ in (ii)

$(-1)^2+1 = A[(-1)^2-(-1)+1]$

$\rightarrow 1+1 = A(1+1+1) \rightarrow 2 = 3A$

$\rightarrow A = \frac{2}{3}$

from (ii)

$x^2+1 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$

Equating coefficients

x^2; $1 = A + B \rightarrow 1 = \frac{2}{3} + B$

$\rightarrow 1 - \frac{2}{3} = B \rightarrow B = \frac{1}{3}$

x; $0 = -A + B + C$

$0 = -\frac{2}{3} + \frac{1}{3} + C \rightarrow 0 = -\frac{1}{3} + C$

$C = \frac{1}{3}$

so (i) becomes as

$$\frac{x^2+1}{(x+1)(x^2-x+1)} = \frac{2}{3(x+1)} + \frac{\frac{1}{3}x + \frac{1}{3}}{x^2-x+1}$$

or $$\frac{x^2+1}{x^3+1} = \frac{2}{3(x+1)} + \frac{x+1}{3(x^2-x+1)}$$

Q7. $\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}$

Solution:- Suppose

$$\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)} = \frac{Ax+B}{x^2+3} + \frac{C}{x+1} + \frac{D}{x-1} \rightarrow (i)$$

'x' by $(x^2+3)(x+1)(x-1)$, we get

$$x^2+2x+2 = (Ax+B)(x+1)(x-1) + C(x^2+3)(x-1) + D(x^2+3)(x+1) \rightarrow (ii)$$

Put $x+1=0 \rightarrow x=-1$ in (ii)

$(-1)^2+2(-1)+2 = C[(-1)^2+3](-1-1)$

$1-2+2 = C(4)(-2) \rightarrow 1 = -8C$

$C = -\frac{1}{8}$

Put $x-1=0 \rightarrow x=1$ in (ii)

$(1)^2+2(1)+2 = D((1)^2+3)(1+1)$

$1+2+2 = D(1+3)(2)$

$\rightarrow 5 = 8D \rightarrow D = \frac{5}{8}$

From (ii)

$x^2+2x+2 = (Ax+B)(x^2-1) + C(x^3-x^2+3x-3) + D(x^3+x^2+3x+3)$

$x^2+2x+2 = Ax^3 - Ax + Bx^2 - B + Cx^3 - Cx^2 + 3Cx - 3C + Dx^3 + Dx^2 + 3Dx + 3D$

Equating coefficients

x^3; $0 = A + C + D$

$0 = A - \frac{1}{8} + \frac{5}{8} \rightarrow 0 = A + \frac{4}{8}$

$0 = A + \frac{1}{2}$ or $A = -\frac{1}{2}$

x^2; $1 = B - C + D$

$1 = B - \left(-\frac{1}{8}\right) + \frac{5}{8}$

$1 = B + \frac{1}{8} + \frac{5}{8} \rightarrow 1 = B + \frac{6}{8}$

$\rightarrow 1 = B + \frac{3}{4} \rightarrow B = 1 - \frac{3}{4}$

$B = \frac{1}{4}$

so (i) becomes as

$$\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)} = \frac{-\frac{1}{2}x+\frac{1}{4}}{x^2+3} + \frac{-\frac{1}{8}}{x+1} + \frac{5/8}{x-1}$$

or

$$\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)} = \frac{\frac{1}{4}(-2x+1)}{x^2+3} - \frac{1}{8(x+1)} + \frac{5}{8(x-1)}$$

or

$$\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)} = \frac{-2x+1}{4(x^2+3)} - \frac{1}{8(x+1)} + \frac{5}{8(x-1)}$$

Q8. $\frac{1}{(x-1)^2(x^2+2)}$

Solution:- Suppose

$$\frac{1}{(x-1)^2(x^2+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2} \longrightarrow (i)$$

'x' by $(x-1)^2(x^2+2)$, we get

$$1 = A(x-1)(x^2+2) + B(x^2+2) + (Cx+D)(x-1)^2 \longrightarrow (ii)$$

Put $x-1=0 \Rightarrow x=1$ in (ii)

$$1 = B(1^2+2) \Rightarrow 1 = 3B \Rightarrow B = \frac{1}{3}$$

From (ii)

$$1 = A(x^3+2x-x^2-2) + Bx^2 + 2B + (Cx+D)(x^2+1-2x)$$

$$1 = Ax^3+2Ax-Ax^2-2A+Bx^2+2B + Cx^3+Cx-2Cx^2+Dx^2+D-2Dx$$

Equating coefficients

$x^3;\quad 0 = A + C \longrightarrow (iii)$

$x^2;\quad 0 = B - A - 2C + D$

$\Rightarrow B + D - A - 2C = 0 \longrightarrow (iv)$

$x;\quad 0 = 2A + C - 2D$

$\Rightarrow 2A + C - 2D = 0 \longrightarrow (v)$

Put $B = \frac{1}{3}$ in (iv)

$\frac{1}{3} + D - A - 2C = 0$

$D - A - 2C = -\frac{1}{3}$

$\Rightarrow 2D - 2A - 4C = -\frac{2}{3} \longrightarrow (vi)$ ('x' by 2)

By (v) + (vi)

$-\cancel{2D} + \cancel{2A} + C = 0$

$\cancel{2D} - \cancel{2A} - 4C = -\frac{2}{3}$

$-3C = -\frac{2}{3} \Rightarrow C = \frac{2}{9}$

So (iii) $0 = A + \frac{2}{9} \Rightarrow A = -\frac{2}{9}$

Now (v) $2\left(-\frac{2}{9}\right) + \frac{2}{9} - 2D = 0$

$(\div \text{ by } 2) \Rightarrow -\frac{2}{9} + \frac{1}{9} - D = 0$

$-\frac{1}{9} = D$ or $D = -\frac{1}{9}$

so (i) becomes

$$\frac{1}{(x-1)^2(x^2+2)} = \frac{-2}{9(x-1)} + \frac{1}{3(x-1)^2} + \frac{2x-1}{9(x^2+2)}$$

Q9. $\frac{x^4}{1-x^4}$

Solution:- $\frac{x^4}{1-x^4}$ (improper)

$$\begin{array}{r|l} & -1 \\ 1-x^4 & x^4 \\ & x^4 - 1 \\ & \underline{-\ \ +} \\ & 1 \end{array}$$

$$\frac{x^4}{1-x^4} = -1 + \frac{1}{1-x^4} \text{ (proper)}$$

$$= -1 + \frac{1}{(1-x^2)(1+x^2)}$$

$$\frac{x^4}{1-x^4} = -1 + \frac{1}{(1-x)(1+x)(1+x^2)}$$

Suppose

$$\frac{1}{(1-x)(1+x)(1+x^2)} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2} \longrightarrow (1)$$

'x' by $(1-x)(1+x)(1+x^2)$, we get

$$1 = A(1+x)(1+x^2) + B(1-x)(1+x^2) + (Cx+D)(1-x)(1+x) \longrightarrow (ii)$$

Put $1-x=0 \rightarrow x=1$ in (ii)

$1 = A(1+1)(1+(1)^2) \rightarrow 1 = A(2)(2)$

$A = \frac{1}{4}$

Put $1+x=0 \rightarrow x=-1$ in (ii)

$1 = B(1-(-1))(1+(-1)^2)$

$1 = B(2)(2) \rightarrow B = \frac{1}{4}$

From (ii)

$1 = A(1+x^2+x+x^3)+B(1+x^2-x-x^3)$
$+(Cx+D)(1-x^2)$

$1 = A+Ax^2+Ax+Ax^3+B+Bx^2-Bx-Bx^3$
$+Cx-Cx^3+D-Dx^2$

Equating coefficients

$x^3;\ 0 = A-B-C$

$0 = \frac{1}{4}-\frac{1}{4}-C \rightarrow 0 = -C$

or $C=0$

$x^2;\ 0 = A+B-D \rightarrow 0 = \frac{1}{4}+\frac{1}{4}-D$

$0 = \frac{2}{4}-D \rightarrow D = \frac{1}{2}$

So (i) becomes

$$\frac{1}{(1-x)(1+x)(1+x^2)} = \frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{0x+\frac{1}{2}}{1+x^2}$$

Hence

$$\frac{x^4}{1-x^4} = \frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1+x^2)}$$

Q10. $\frac{x^2-2x+3}{x^4+x^2+1}$

Solution:-

$$\frac{x^2-2x+3}{x^4+x^2+1+x^2-x^2} = \frac{x^2-2x+3}{x^4+2x^2+1-x^2}$$

$$= \frac{x^2-2x+3}{(x^2)^2+2x^2+(1)^2-(x)^2} = \frac{x^2-2x+3}{(x^2+1)^2-(x)^2}$$

$$= \frac{x^2-2x+3}{(x^2+1+x)(x^2+1-x)}$$

Suppose

$$\frac{x^2-2x+3}{(x^2+x+1)(x^2-x+1)} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1} \rightarrow (i)$$

'x' by $(x^2+x+1)(x^2-x+1)$, we get

$x^2-2x+3 = (Ax+B)(x^2-x+1)+(Cx+D)(x^2+x+1)$

$x^2-2x+3 = Ax^3-Ax^2+Ax+Bx^2-Bx+B$
$+Cx^3+Cx^2+Cx+Dx^2+Dx+D$

Equating coefficients

$x^3;\ 0 = A+C \rightarrow$ (ii)

$x^2;\ 1 = -A+B+C+D \rightarrow$ (iii)

$x;\ -2 = A-B+C+D \rightarrow$ (iv)

Cons. term; $3 = B+D \rightarrow$ (v)

Put $A+C=0$ in (iv)

$-B+D = -2 \rightarrow$ (vi)

By (v)+(vi) $\rightarrow 2D = 1 \rightarrow D = \frac{1}{2}$

Now (v) $\rightarrow 3 = B+\frac{1}{2} \rightarrow B = 3-\frac{1}{2}$

$B = \frac{5}{2}$

Put $B+D=3$ in (iii) $3+C-A=1$

$\rightarrow C-A=-2 \rightarrow$ (vii)

By (ii)+(vii) $\rightarrow 2C=-2 \rightarrow C=-1$

So (ii) $\rightarrow 0 = A-1 \rightarrow A=1$

So (i) becomes as

$$\frac{x^2-2x+3}{(x^2+x+1)(x^2-x+1)} = \frac{(1)x+\frac{5}{2}}{x^2+x+1} + \frac{(-1)x+\frac{1}{2}}{x^2-x+1}$$

or

$$\frac{x^2-2x+3}{x^4+x^2+1} = \frac{x+\frac{5}{2}}{x^2+x+1} + \frac{-x+\frac{1}{2}}{x^2-x+1}$$

$$= \frac{2x+5}{2(x^2+x+1)} + \frac{-2x+1}{x^2-x+1}$$

$$= \frac{2x+5}{2(x^2+x+1)} - \frac{2x-1}{x^2-x+1}$$

Case IV

When $Q(x)$ has repeated irreducible quadratic factors

If $Q(x)$ contains a repeated irreducible quadratic factors $(ax^2+bx+c)^n$, $n \geq 2$ and n is a +ive integer, then $\frac{P(x)}{Q(x)}$ may be written as the following identity:

$$\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$ where

$A_1, A_2, B_1, B_2, \ldots A_n, B_n$ are nos. to be found.

Example 1. Resolve $\frac{4x^2}{(x^2+1)^2(x-1)}$ into partial fractions.

Solution:- suppose

$$\frac{4x^2}{(x^2+1)^2(x-1)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x-1} \longrightarrow (i)$$

'x' by $(x^2+1)^2(x-1)$, we get

$$4x^2 = (Ax+B)(x^2+1)(x-1) + (Cx+D)(x-1) + E(x^2+1)^2 \longrightarrow (ii)$$

Put $x-1=0 \rightarrow x=1$ in (ii)

$$4(1)^2 = E\left((1)^2+1\right)^2 \rightarrow 4 = E(2)^2$$

$$4 = E(4) \rightarrow E = 1$$

From (ii)

$$4x^2 = (Ax+B)(x^3-x^2+x-1) + Cx^2 - Cx + Dx - D + E(x^4+1+2x^2)$$

$$4x^2 = Ax^4 - Ax^3 + Ax^2 - Ax + Bx^3 - Bx^2 + Bx - B + Cx^2 - Cx + Dx - D + Ex^4 + E + 2Ex^2$$

Equating coefficients

x^4; $0 = A+E \rightarrow 0 = A+1$

$A = -1$

x^3; $0 = -A+B \rightarrow 0 = -(-1)+B$

$0 = 1+B \rightarrow B = -1$

x^2; $4 = A-B+C+2E$

$4 = -1-(-1)+C+2(1)$

$4 = \cancel{-1}+\cancel{1}+C+2$

$\rightarrow 4-2 = C \rightarrow C = 2$

x; $0 = -A+B-C+D$

$0 = -(-1)-1-2+D$

$0 = 1-1-2+D \rightarrow D = 2$

so (i) becomes as

$$\frac{4x^2}{(x^2+1)^2(x-1)} = \frac{-x-1}{x^2+1} + \frac{2x+2}{(x^2+1)^2}$$

Exercise 5.4

Resolve into Partial Fractions:

Q1. $\frac{x^3+2x+2}{(x^2+x+1)^2}$

Solution:- Suppose

$$\frac{x^3+2x+2}{(x^2+x+1)^2} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2} \longrightarrow (i)$$

'x' by $(x^2+x+1)^2$, we get

$$x^3+2x+2 = (Ax+B)(x^2+x+1) + Cx + D$$

$$x^3+2x+2 = Ax^3 + Ax^2 + Ax + Bx^2 + Bx + B + Cx + D$$

Equating coefficients

x^3; $1 = A$, x^2; $0 = A+B$

$0 = 1+B$, $B = -1$

x; $2 = A+B+C$

$2 = 1+(-1)+C \rightarrow C = 2$

cons. term; $2 = B+D \rightarrow 2 = -1+D$

$D = 3$

so (i) becomes as

$$\frac{x^3+2x+2}{(x^2+x+1)^2} = \frac{x-1}{x^2+x+1} + \frac{2x+3}{(x^2+x+1)^2}$$

Q2. $\dfrac{x^2}{(x^2+1)^2(x-1)}$

Solution:- suppose

$$\frac{x^2}{(x^2+1)^2(x-1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x-1} \longrightarrow (i)$$

'x' by $(x^2+1)^2(x-1)$, we get

$$x^2 = (Ax+B)(x^2+1)(x-1) + (Cx+D)(x-1) + E(x^2+1)^2 \longrightarrow (ii)$$

$$x^2 = (Ax+B)(x^3-x^2+x-1) + Cx^2 - Cx + Dx - D + E(x^4+1+2x^2)$$

$$x^2 = Ax^4 - Ax^3 + Ax^2 - Ax + Bx^3 - Bx^2 + Bx - B + Cx^2 - Cx + Dx - D + Ex^4 + E + 2Ex^2 \longrightarrow (iii)$$

Put $x-1=0 \rightarrow x=1$ in (ii)

$$(1)^2 = E((1)^2+1)^2 \rightarrow 1 = E(2)^2$$

$$E = \frac{1}{4}$$

Comparing coefficients of eq (iii)

x^4; $0 = A+E \rightarrow 0 = A + \frac{1}{4}$

$A = -\frac{1}{4}$

x^3; $0 = B - A \rightarrow 0 = B - (-\frac{1}{4})$

$\frac{1}{4} + B = 0 \rightarrow B = -\frac{1}{4}$

x^2; $1 = A - B + C + 2E$

$1 = -\frac{1}{4} - (-\frac{1}{4}) + C + 2(\frac{1}{4})$

$1 = \cancel{-\frac{1}{4}} + \cancel{\frac{1}{4}} + C + \frac{1}{2}$

$\rightarrow C = 1 - \frac{1}{2} \rightarrow C = \frac{1}{2}$

x; $0 = B - A - C + D$

$0 = -\frac{1}{4} - (-\frac{1}{4}) - \frac{1}{2} + D$

$0 = -\frac{1}{4} + \frac{1}{4} - \frac{1}{2} + D$

$\rightarrow 0 = -\frac{1}{2} + D \rightarrow D = \frac{1}{2}$

So (i) becomes

$$\frac{x^2}{(x^2+1)^2(x-1)} = \frac{-\frac{1}{4}x - \frac{1}{4}}{x^2+1} + \frac{\frac{1}{2}x + \frac{1}{2}}{(x^2+1)^2} + \frac{\frac{1}{4}}{x-1}$$

$$\frac{x^2}{(x^2+1)^2(x-1)} = \frac{-(x+1)}{4(x^2+1)} + \frac{x+1}{2(x^2+1)^2} + \frac{1}{4(x-1)}$$

Q3. $\dfrac{2x-5}{(x^2+2)^2(x-2)}$

Solution:- suppose

$$\frac{2x-5}{(x^2+2)^2(x-2)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} + \frac{E}{x-2} \longrightarrow (i)$$

'x' by $(x^2+2)^2(x-2)$, we get

$$2x-5 = (Ax+B)(x-2)(x^2+2) + (Cx+D)(x-2) + E(x^2+2)^2 \longrightarrow (ii)$$

Put $x-2=0 \rightarrow x=2$ in (ii)

$$2(2)-5 = E((2)^2+2)^2 \rightarrow 4-5 = E(4+2)^2$$

$$-1 = E(36) \rightarrow E = -\frac{1}{36}$$

from (ii)

$$2x-5 = (Ax+B)(x^3+2x-2x^2-4) + Cx^2 - 2Cx + Dx - 2D + E(x^4+4+4x^2)$$

$$2x-5 = Ax^4 + 2Ax^2 - 2x^3A - 4Ax + Bx^3 + 2Bx - 2Bx^2 - 4B + Cx^2 - 2Cx + Dx - 2D + Ex^4 + 4E + 4Ex^2$$

Equating coefficients

x^4; $0 = A+E \rightarrow 0 = A - \frac{1}{36}$

$A = \frac{1}{36}$

x^3; $0 = -2A + B \rightarrow 0 = -2(\frac{1}{36}) + B$

$0 = -\frac{1}{18} + B \rightarrow B = \frac{1}{18}$

x^2; $0 = 2A - 2B + C + 4E$

$0 = 2(\frac{1}{36}) - 2(\frac{1}{18}) + C + 4(-\frac{1}{36})$

$0 = \frac{1}{18} - \frac{1}{9} + C - \frac{1}{9}$

$\rightarrow \frac{1}{9} + \frac{1}{9} - \frac{1}{18} = C$

$\frac{2+2-1}{18} = C \rightarrow C = \frac{3}{18} = \frac{1}{6}$

x; $2 = 2B - 4A - 2C + D$

$2 = 2(\frac{1}{18}) - 4(\frac{1}{36}) - 2(\frac{1}{6}) + D$

$2 = \frac{1}{9} - \frac{1}{9} - \frac{1}{3} + D$

$D = 2 + \frac{1}{3} \Rightarrow D = \frac{7}{3}$

so (i) becomes as

$$\frac{2x-5}{(x^2+2)^2(x-2)} = \frac{\frac{1}{36}x + \frac{1}{18}}{x^2+2} + \frac{\frac{1}{6}x + \frac{7}{3}}{(x^2+2)^2} + \frac{-\frac{1}{36}}{x-2}$$

$$\frac{2x-5}{(x^2+2)^2(x-2)} = \frac{x+2}{36(x^2+2)} + \frac{x+14}{6(x^2+2)^2} - \frac{1}{36(x-2)}$$

Q4. $\frac{8x^2}{(x^2+1)^2(1-x^2)}$

Solution:-

$$\frac{8x^2}{(x^2+1)^2(1-x^2)} = \frac{8x^2}{(x^2+1)^2(1-x)(1+x)}$$

Suppose

$$\frac{8x^2}{(x^2+1)^2(1-x)(1+x)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{1-x} + \frac{F}{1+x} \longrightarrow (i)$$

'x' by $(x^2+1)^2(1-x)(1+x)$, we get

$$8x^2 = (Ax+B)(x^2+1)(1-x)(1+x) + (Cx+D)(1-x)(1+x) + E(x^2+1)^2(1+x) + F(x^2+1)^2(1-x) \longrightarrow (ii)$$

Put $1-x=0 \Rightarrow x=1$ in (ii)

$8(1)^2 = E(1^2+1)^2(1+1) \Rightarrow 8 = E(2)^2(2)$

$8 = E(8) \Rightarrow E = 1$

Put $1+x=0 \Rightarrow x=-1$ in (ii)

$8(-1)^2 = F((-1)^2+1)^2(1-(-1))$

$8 = F(2)^2(2) \Rightarrow 8 = 8F \Rightarrow F = 1$

From (ii)

$$8x^2 = (Ax^3+Ax+Bx^2+B)(1-x^2) + (Cx+D)(1-x^2) + E(x^4+1+2x^2)(1+x) + F(x^4+1+2x^2)(1-x)$$

$$8x^2 = Ax^3+Ax+Bx^2+B-Ax^5-Ax^3-Bx^4-Bx^2+Cx-Cx^3+D-Dx^2+(Ex^4+E+2Ex^2)(1+x)+(Fx^4+F+2Fx^2)(1-x)$$

$$8x^2 = \cancel{Ax^3}+Ax+Bx^2+B-Ax^5-\cancel{Ax^3}-Bx^4-Bx^2+Cx-Cx^3+D-Dx^2+Ex^4+E+2Ex^2+Ex^5+Ex+2Ex^3+Fx^4+F+2Fx^2-Fx^5-Fx-2Fx^3$$

Equating coefficients

x^5; $0 = -A+E-F$

$0 = -A+1-1 \Rightarrow 0 = -A$

$A = 0$

x^4; $0 = -B+E+F$

$0 = -B+1+1 \Rightarrow B = 2$

x^3; $0 = -C+2E-2F$

$= -C+2(1)-2(1)$

$0 = -C+2-2 \Rightarrow C = 0$

x^2; $8 = 2E+2F \Rightarrow 4 = E+F$

x; $0 = A+C+E-F$

const. term; $0 = B+D+E+F$

$0 = 2+D+1+1 \Rightarrow 0 = 4+D$

$D = -4$

so (i) becomes as

$$\frac{8x^2}{(x^2+1)^2(1-x)(1+x)} = \frac{(0)x+2}{x^2+1} + \frac{(0)x-4}{(x^2+1)^2} + \frac{1}{1-x} + \frac{1}{1+x}$$

or

$$\frac{8x^2}{(x^2+1)^2(1-x^2)} = \frac{2}{x^2+1} \cdot \frac{-4}{(x^2+1)^2} + \frac{1}{1-x} + \frac{1}{1+x}$$

Q5. $\dfrac{4x^4+3x^3+6x^2+5x}{(x-1)(x^2+x+1)^2}$

Solution:- suppose

$$\frac{4x^4+3x^3+6x^2+5x}{(x-1)(x^2+x+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2} \longrightarrow (i)$$

'x' by $(x-1)(x^2+x+1)^2$, we get

$$4x^4+3x^3+6x^2+5x = A(x^2+x+1)^2 + (Bx+C)(x-1)(x^2+x+1) + (Dx+E)(x-1) \longrightarrow (ii)$$

Put $x-1=0 \rightarrow x=1$ in (ii)

$$4(1)^4+3(1)^3+6(1)^2+5(1) = A[(1)^2+1+1]^2$$

$$4+3+6+5 = A(3)^2 \rightarrow 18 = 9A \rightarrow A=2$$

From (ii)

$$4x^4+3x^3+6x^2+5x = A(x^4+2x^3+3x^2+2x+1) + (Bx+C)(x^3-1) + Dx^2-Dx+Ex-E$$

$$4x^4+3x^3+6x^2+5x = Ax^4+2Ax^3+3Ax^2+2Ax+A+Bx^4-Bx+Cx^3-C+Dx^2-Dx+Ex-E$$

Equating coefficients

x^4; $4=A+B \rightarrow 4=2+B \rightarrow B=2$

x^3; $3=2A+C \rightarrow 3=2(2)+C$

$3=4+C \rightarrow 3-4=C \rightarrow C=-1$

x^2; $6=3A+D \rightarrow 6=3(2)+D \rightarrow D=0$

x; $5=2A-B-D+E$

$5=2(2)-2-0+E \rightarrow 5=4-2+E$

$5=2+E \rightarrow 5-2=E \rightarrow E=3$

So (i) becomes as

$$\frac{4x^4+3x^3+6x^2+5x}{(x-1)(x^2+x+1)^2} = \frac{2}{x-1} + \frac{2x-1}{x^2+x+1} + \frac{3}{(x^2+x+1)^2}$$

Q6. $\dfrac{2x^4-3x^3-4x}{(x^2+2)^2(x+1)^2}$

Solution:- suppose

$$\frac{2x^4-3x^3-4x}{(x^2+2)^2(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{(x^2+2)^2} \longrightarrow (i)$$

'x' by $(x^2+2)^2(x+1)^2$, we get

$$2x^4-3x^3-4x = A(x+1)(x^2+2)^2 + B(x^2+2)^2 + (Cx+D)(x^2+2)(x+1)^2 + (Ex+F)(x+1)^2 \longrightarrow (ii)$$

Put $x+1=0 \rightarrow x=-1$ in (ii)

$$2(-1)^4-3(-1)^3-4(-1) = B((-1)^2+2)^2$$

$$2+3+4 = B(1+2)^2 \rightarrow 9=9B$$

$B=1$

From (ii)

$$2x^4-3x^3-4x = A(x+1)(x^4+4x^2+4) + B(x^4+4+4x^2) + C(x+D)(x^2+2)(x^2+1+2x) + (Ex+F)(x^2+1+2x)$$

$$2x^4-3x^3-4x = A(x^5+4x^3+4x+x^4+4x^2+4) + B(x^4+4+4x^2) + (Cx+D)(x^4+2x^3+x^2+2x^2+4x+2) + Ex^3+Ex+2Ex^2+Fx^2+F+2Fx$$

$$2x^4-3x^3-4x = Ax^5+4Ax^3+4Ax+Ax^4+4Ax^2+4A + Bx^4+4B+4Bx^2+Cx^5+2Cx^4+3Cx^3+4Cx^2+2Cx+Dx^4+2Dx^3+3Dx^2+4Dx+2D+Ex^3+Ex+2Ex^2+Fx^2+F+2Fx$$

Equating coefficients

x^5; $0=A+C \longrightarrow (iii)$

x^4; $2=A+B+2C+D \longrightarrow (iv)$

x^3; $-3=4A+3C+2D+E \longrightarrow (v)$

x^2; $0=4A+4B+4C+3D+2E+F \longrightarrow (vi)$

x; $-4=4A+2C+4D+E+2F \longrightarrow (vii)$

cons. term; $0 = 4A + 4B + 2D + F \longrightarrow$ (viii)

Now

(iii) $\Rightarrow C = -A$ put values of B and C in (iv)

(iv) $\Rightarrow 2 = A + 1 + 2(-A) + D \Rightarrow 2 = A + 1 - 2A + D$

$2 - 1 = -A + D \Rightarrow 1 = -A + D$

or $D = 1 + A \longrightarrow$ (ix)

Put values of C and D in (v)

(v) $\Rightarrow -3 = 4A + 3(-A) + 2(1 + A) + E$

$-3 = 4A - 3A + 2 + 2A + E \Rightarrow -3 = 3A + 2 + E$

or $E = -5 - 3A \longrightarrow$ (x)

Put values of B, C, D and E in (vi)

(vi) $\Rightarrow 0 = 4A + 4(1) + 4(-A) + 3(1 + A) + 2(-5 - 3A) + F$

$0 = 4A + 4 - 4A + 3 + 3A - 10 - 6A + F$

$0 = -3 - 3A + F \Rightarrow F = 3 + 3A \longrightarrow$ (xi)

Put values of C, D and E in (vii)

(vii) $\Rightarrow -4 = 4A + 2(-A) + 4(1 + A) + (-5 - 3A) + 2F$

$-4 = 4A - 2A + 4 + 4A - 5 - 3A + 2F$

$-4 = 3A - 1 + 2F \Rightarrow -4 + 1 = 3A + 2F$

$3A + 2F = -3 \longrightarrow$ (xii)

By (xi) + (xii) $\Rightarrow$

$$\begin{array}{r} 3 = -3A + F \\ -3 = 3A + 2F \\ \hline 0 = \quad 3F \end{array} \Rightarrow F = 0 \quad \text{so}$$

(xi) $\Rightarrow 3 = -3A + 0 \Rightarrow A = -1$,

(iii) $\Rightarrow 0 = -1 + C \Rightarrow C = 1$, (ix) $\Rightarrow 1 = -(-1) + D$

$1 = 1 + D \Rightarrow D = 0$

(x) $\Rightarrow -5 = 3(-1) + E \Rightarrow -5 = -3 + E$

$\Rightarrow E = -2$

so (i) becomes as

$$\frac{2x^4 - 3x^3 - 4x}{(x^2+2)^2(x+1)^2} = \frac{-1}{x+1} + \frac{1}{(x+1)^2} + \frac{x+0}{x^2+2} + \frac{-2x+0}{(x^2+2)^2}$$

$$= \frac{-1}{x+1} + \frac{1}{(x+1)^2} + \frac{x}{x^2+2} - \frac{2x}{(x^2+2)^2}$$