MATHEMATICS 1st YEAR

UNIT

04

QUADRATIC EQUATIONS

Muhammad Salman Sherazi
M.Phil (Math)

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Sherazi Mathematics

اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Quadratic equation:-

An equation containing one or more terms in which the variable is raised to maximum positive power two. In general $ax^2+bx+c=0$, where $a\neq 0$ is called quadratic equation. e.g.,

$x^2-7x+10=0$; $a=1, b=-7, c=10$

$3x^2-x=0$; $a=3, b=-1, c=0$

Solution of Quadratic Equation

We solve Quadratic equation by following three methods.

i) by factorization

ii) by completing square

iii) by quadratic formula

By Factorization

Example 1. Solve the equation $x^2-7x+10=0$ by factorization

Solution:- $x^2-7x+10=0$

$\rightarrow x^2-2x-5x+10=0$

$\rightarrow x(x-2)-5(x-2)=0$

$\rightarrow (x-2)(x-5)=0$

$\rightarrow x-2=0$ or $x-5=0$

$x=2$ or $x=5$

S.S = $\{2, 5\}$

Note:- The solutions of an equation are also called its roots

$\therefore$ 2, 5 are roots of $x^2-7x+10=0$

By Completing Square

Example 2. Solve the equation $x^2+4x-437=0$ by completing square.

Solution:- $x^2+4x-437=0$

$\rightarrow x^2+4x=437$

adding $(4/2)^2=(2)^2$ on both sides

$\rightarrow x^2+4x+(2)^2=437+(2)^2$

$\rightarrow (x+2)^2=437+4$

$\rightarrow (x+2)^2=441$

$\rightarrow x+2=\pm\sqrt{441}$

$x+2=\pm 21$

$\rightarrow x+2=21$, $x+2=-21$

$x=21-2$, $x=-21-2$

$x=19$, $x=-23$

S.S = $\{19, -23\}$

By Quadratic Formula

Derivation of the Quadratic Formula

Quadratic equation in standard form is;

$$ax^2+bx+c=0$$

Dividing both sides by a

$\rightarrow \frac{ax^2}{a}+\frac{bx}{a}+\frac{c}{a}=0$

$\rightarrow x^2+\frac{b}{a}x=-\frac{c}{a}$

adding $\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2$ on both sides

$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$

$\rightarrow \left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$

$\rightarrow \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$

$\rightarrow x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$

$\rightarrow x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$

$\rightarrow x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$

$\rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

This is called "Quadratic formula".

* Quadratic equation is also known as "second degree polynomial".
* In quadratic equation $ax^2+bx+c=0$, $a, b, c \in R$

Example 3. Solve the equation $6x^2+x-15=0$ by using quadratic formula.

Solution:- $6x^2+x-15=0$

Here $a=6$, $b=1$, $c=-15$

using Quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x=\frac{-1\pm\sqrt{(1)^2-4(6)(-15)}}{2(6)}$$

$$\Rightarrow x=\frac{-1\pm\sqrt{1+361}}{12}$$

$$\Rightarrow x=\frac{-1\pm19}{12}$$

$x=\frac{-1+19}{12}$ or $x=\frac{-1-19}{12}$

$x=\frac{18}{12}$ or $x=\frac{-20}{12}$

$x=\frac{3}{2}$ or $x=\frac{-5}{3}$

$S.S=\{3/2, -5/3\}$

Example 4. Solve the $8x^2-14x-15=0$ by using quadratic formula.

Solution:- $8x^2-14x-15=0$

Here $a=8$, $b=-14$, $c=-15$

using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(-14)\pm\sqrt{(-14)^2-4(8)(-15)}}{2(8)}$$

$$x=\frac{14\pm\sqrt{196+480}}{16}$$

$$\Rightarrow x=\frac{14\pm\sqrt{676}}{16}$$

$$\Rightarrow x=\frac{14\pm26}{16}$$

$x=\frac{14+26}{16}$ or $x=\frac{14-26}{16}$

$x=\frac{40}{16}=5/2$ or $x=\frac{-12}{16}=\frac{-3}{4}$

$S.S=\{\frac{5}{2}, -\frac{3}{4}\}$

Exercise 4.1

Solve the following equations by factorization.

Q1. $3x^2+4x+1=0$

Solution:- $3x^2+4x+1=0$

$\Rightarrow 3x^2+3x+x+1=0$

$\Rightarrow 3x(x+1)+1(x+1)=0$

$\Rightarrow (x+1)(3x+1)=0$

$x+1=0$ or $3x+1=0$

$x=-1$ or $x=-\frac{1}{3}$

$S.S=\{-1, -1/3\}$

Q2. $x^2+7x+12=0$

Solution:- $x^2+7x+12=0$

$\Rightarrow x^2+4x+3x+12=0$

$\Rightarrow x(x+4)+3(x+4)=0$

$\Rightarrow (x+4)(x+3)=0$

$\Rightarrow x+4=0$ or $x+3=0$

$x=-4$ or $x=-3$ $S.S=\{-4,-3\}$

Q3. $9x^2-12x-5=0$

Solution:- $9x^2-12x-5=0$

$\Rightarrow 9x^2+3x-15x-5=0$

$\Rightarrow 3x(3x+1)-5(3x+1)=0$

$\Rightarrow (3x+1)(3x-5)=0$

$3x+1=0$ or $3x-5=0$

$x=-\frac{1}{3}$ or $x=\frac{5}{3}$

$S.S=\{-\frac{1}{3}, \frac{5}{3}\}$

Q4. $x^2-x=2$

Solution:- $x^2-x=2$

$\Rightarrow x^2-x-2=0$

$\Rightarrow x^2+x-2x-2=0$

$\Rightarrow x(x+1)-2(x+1)=0$

$\Rightarrow (x+1)(x-2)=0$

$x+1=0$ or $x-2=0$

$\Rightarrow x = -1, \quad x = 2$

$S.S = \{2, -1\}$

Q5. $x(x+7) = (2x-1)(x+4)$

Solution:- $x(x+7) = (2x-1)(x+4)$

$\Rightarrow x^2 + 7x = 2x^2 + 8x - x - 4$

$\Rightarrow x^2 + 7x = 2x^2 + 7x - 4$

$\Rightarrow 2x^2 + 7x - 4 - x^2 - 7x = 0$

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2 \quad \Rightarrow x = 2 \text{ or } x = -2$

$S.S = \{2, -2\}$

Q6. $\frac{x}{x+1} + \frac{x+1}{x} = \frac{5}{2}$

Solution:- $\frac{x}{x+1} + \frac{x+1}{x} = \frac{5}{2}$

$\Rightarrow 2x(x+1)\cdot\frac{x}{x+1} + 2x(x+1)\cdot\frac{x+1}{x} = 2x(x+1)\cdot\frac{5}{2}$

$\Rightarrow 2x^2 + 2(x+1)^2 = 5x(x+1)$

$\Rightarrow 2x^2 + 2(x^2+1+2x) = 5x^2 + 5x$

$\Rightarrow 2x^2 + 2x^2 + 2 + 4x = 5x^2 + 5x$

$\Rightarrow 4x^2 + 4x + 2 = 5x^2 + 5x$

$\Rightarrow 5x^2 + 5x - 4x^2 - 4x - 2 = 0$

$\Rightarrow x^2 + x - 2 = 0$

$\Rightarrow x^2 + 2x - x - 2 = 0$

$x(x+2) - 1(x+2) = 0$

$(x+2)(x-1) = 0$

$\Rightarrow x + 2 = 0 \text{ or } x - 1 = 0$

$x = -2 \text{ or } x = 1$

$S.S = \{-2, 1\}$

Q7. $\frac{1}{x+1} + \frac{2}{x+2} = \frac{7}{x+5}$

Solution:-

$(x+1)(x+2)(x+5)\cdot\frac{1}{x+1} + (x+1)(x+2)(x+5)\cdot\frac{2}{x+2} = (x+1)(x+2)(x+5)\cdot\frac{7}{x+5}$

$(x+2)(x+5) + 2(x+1)(x+5) = 7(x+1)(x+2)$

$\Rightarrow x^2 + 5x + 2x + 10 + 2(x^2 + 5x + x + 5) = 7(x^2 + 2x + x + 2)$

$\Rightarrow x^2 + 7x + 10 + 2x^2 + 12x + 10 = 7x^2 + 21x + 14$

$\Rightarrow 3x^2 + 19x + 20 = 7x^2 + 21x + 14$

$\Rightarrow 7x^2 + 21x + 14 - 3x^2 - 19x - 20 = 0$

$\Rightarrow 4x^2 + 2x - 6 = 0$

$\Rightarrow 2x^2 + x - 3 = 0 \quad (\div \text{ by } 2)$

$\Rightarrow 2x^2 - 2x + 3x - 3 = 0$

$2x(x-1) + 3(x-1) = 0$

$(x-1)(2x+3) = 0$

$\Rightarrow x - 1 = 0 \text{ or } 2x + 3 = 0$

$x = 1 \text{ or } x = \frac{-3}{2}$

$S.S = \{1, \frac{-3}{2}\}$

Q8. $\frac{a}{ax-1} + \frac{b}{bx-1} = a+b; \; x \neq \frac{1}{a}, \frac{1}{b}$

Solution:- $\frac{a}{ax-1} + \frac{b}{bx-1} = a+b$

$\Rightarrow \frac{a}{ax-1} - b + \frac{b}{bx-1} - a = 0$

$\Rightarrow \frac{a - b(ax-1)}{ax-1} + \frac{b - a(bx-1)}{bx-1} = 0$

$\Rightarrow \frac{a - abx + b}{ax-1} + \frac{b - abx + a}{bx-1} = 0$

$(a - abx + b)\left[\frac{1}{ax-1} + \frac{1}{bx-1}\right] = 0$

$(a - abx + b)\left(\frac{bx-1+ax-1}{(ax-1)(bx-1)}\right) = 0$

$(a - abx + b)(ax + bx - 2) = 0$

$\Rightarrow a - abx + b = 0 \text{ or } ax + bx - 2 = 0$

$\Rightarrow abx = a + b \text{ or } x(a+b) - 2 = 0$

$\Rightarrow x = \frac{a+b}{ab} \text{ or } x(a+b) = 2$

$x = \frac{a+b}{ab} \text{ or } x = \frac{2}{a+b}$

$S.S = \left\{\frac{a+b}{ab}, \frac{2}{a+b}\right\}$

Solve the following equations by completing the square:

Q9. $x^2-2x-899=0$

Solution:- $x^2-2x-899=0$

$\Rightarrow x^2-2x=899$

adding $\left(\frac{2}{2}\right)^2=(1)^2$ on both sides

$\Rightarrow x^2-2x+(1)^2=899+(-1)^2$

$\Rightarrow (x-1)^2=899+1$

$(x-1)^2=900$

$\Rightarrow x-1=\pm 30$

$\Rightarrow x-1=30 \quad , \quad x-1=-30$

$x=30+1 \quad , \quad x=-30+1$

$x=31 \quad , \quad x=-29$

S.S $=\{31,-29\}$

Q10. $x^2+4x-1085=0$

Solution:- $x^2+4x-1085=0$

$\Rightarrow x^2+4x=1085$

Adding $\left(\frac{4}{2}\right)^2=(2)^2$ on both sides

$x^2+4x+(2)^2=1085+(2)^2$

$\Rightarrow (x+2)^2=1085+4$

$\Rightarrow (x+2)^2=1089$

$x+2=\pm 33$

$\Rightarrow x+2=33$ or $x+2=-33$

$\Rightarrow x=33-2$ or $x=-33-2$

$\Rightarrow x=31$ or $x=-35$

S.S $=\{31,-35\}$

Q11. $x^2+6x-567=0$

Solution:- $x^2+6x-567=0$

$\Rightarrow x^2+6x=567$

Adding $\left(\frac{6}{2}\right)^2=(3)^2$ on both sides

$\Rightarrow x^2+6x+(3)^2=567+(3)^2$

$\Rightarrow (x+3)^2=567+9$

$(x+3)^2=576$

$\Rightarrow x+3=\pm 24$

$x+3=24$ or $x+3=-24$

$x=24-3$ or $x=-24-3$

$x=21$ or $x=-27$

S.S $=\{21,-27\}$

Q12. $x^2-3x-648=0$

Solution:- $x^2-3x-648=0$

$\Rightarrow x^2-3x=648$

adding $\left(\frac{3}{2}\right)^2$ on both sides

$\Rightarrow x^2-3x+\left(\frac{3}{2}\right)^2=648+\left(\frac{3}{2}\right)^2$

$\Rightarrow \left(x-\frac{3}{2}\right)^2=648+\frac{9}{4}$

$\left(x-\frac{3}{2}\right)^2=\frac{2592+9}{4}$

$\Rightarrow \left(x-\frac{3}{2}\right)^2=\frac{2601}{4}$

$\Rightarrow x-\frac{3}{2}=\pm\frac{51}{2}$

$\Rightarrow x-\frac{3}{2}=\frac{51}{2}$ or $x-\frac{3}{2}=-\frac{51}{2}$

$\Rightarrow x=\frac{3}{2}+\frac{51}{2}$ or $x=\frac{3}{2}-\frac{51}{2}$

$x=\frac{54}{2}=27$ or $x=\frac{-48}{2}=-24$

S.S $=\{27,-24\}$

Q13. $x^2-x-1806=0$

Solution:- $x^2-x-1806=0$

$\Rightarrow x^2-x=1806$

adding $\left(\frac{1}{2}\right)^2$ on both sides

$x^2-x+\left(\frac{1}{2}\right)^2=1806+\left(\frac{1}{2}\right)^2$

$\Rightarrow \left(x-\frac{1}{2}\right)^2=1806+\frac{1}{4}$

$\left(x-\frac{1}{2}\right)^2=\frac{7224+1}{4}$

$\left(x-\frac{1}{2}\right)^2=\frac{7225}{4}$

$\Rightarrow x-\frac{1}{2}=\pm\frac{85}{2}$

$\Rightarrow x=\frac{1}{2}\pm\frac{85}{2}$

$x=\frac{1}{2}+\frac{85}{2}$ or $x=\frac{1}{2}-\frac{85}{2}$

$x=\frac{86}{2}=43$ or $x=-\frac{84}{2}=-42$

S.S $=\{43,-42\}$

Q14. $2x^2+12x-110=0$

Solution:- $2x^2+12x-110=0$

$\rightarrow x^2+6x-55=0 \quad (\div \text{ by } 2)$

$\rightarrow x^2+6x=55$

adding $\left(\frac{6}{2}\right)^2=(3)^2$ on both sides

$x^2+6x+(3)^2=55+(3)^2$

$\rightarrow (x+3)^2=55+9$

$(x+3)^2=64$

$\rightarrow x+3=\pm 8$

$\rightarrow x+3=8 \quad \text{or} \quad x+3=-8$

$x=8-3 \quad \text{or} \quad x=-8-3$

$\rightarrow x=5 \quad \text{or} \quad x=-11$

$S.S=\{5,-11\}$

Find roots of the following equations by using quadratic formula:

Q15. $5x^2-13x+6=0$

Solution:- $5x^2-13x+6=0$

Here $a=5$, $b=-13$, $c=6$

using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-(-13)\pm\sqrt{(-13)^2-4(5)(6)}}{2(5)}$

$x=\frac{13\pm\sqrt{169-120}}{10}$

$\rightarrow x=\frac{13\pm\sqrt{49}}{10}$

$x=\frac{13\pm 7}{10}$

$\rightarrow x=\frac{13+7}{10} \quad \text{or} \quad x=\frac{13-7}{10}$

$x=\frac{20}{10}, \quad \text{or} \quad x=\frac{6}{10}=\frac{3}{5}$

$x=2, \quad x=\frac{3}{5}$

$S.S=\{2,\frac{3}{5}\}$

Q16. $4x^2+7x-1=0$

Solution:- $4x^2+7x-1=0$

Here $a=4$, $b=7$, $c=-1$

using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-7\pm\sqrt{(7)^2-4(4)(-1)}}{2(4)}$

$\rightarrow x=\frac{-7\pm\sqrt{49+16}}{8}=\frac{-7\pm\sqrt{65}}{8}$

$\rightarrow x=\frac{-7\pm\sqrt{65}}{8}$

$S.S=\left\{\frac{-7\pm\sqrt{65}}{8}\right\}$

Q17. $15x^2+2ax-a^2=0$

Solution:- $15x^2+2ax-a^2=0$

Here $a=15$, $b=2a$, $c=-a^2$

using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-2a\pm\sqrt{(2a)^2-4(15)(-a^2)}}{2(15)}$

$=\frac{-2a\pm\sqrt{4a^2+60a^2}}{30}$

$x=\frac{-2a\pm\sqrt{64a^2}}{30}=\frac{-2a\pm 8a}{30}$

$x=\frac{-2a+8a}{30} \quad \text{or} \quad x=\frac{-2a-8a}{30}$

$x=\frac{6a}{30} \quad \text{or} \quad x=-\frac{10a}{30}$

$\rightarrow x=\frac{a}{5} \quad \text{or} \quad x=-\frac{a}{3}$

$S.S=\left\{\frac{a}{5},-\frac{a}{3}\right\}$

Q18. $16x^2+8x+1=0$

Solution:- $16x^2+8x+1=0$

Here $a=16$, $b=8$, $c=1$

using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\rightarrow x=\frac{-8\pm\sqrt{(8)^2-4(16)(1)}}{2(16)}$

$$\rightarrow x = \frac{-8 \pm \sqrt{64-64}}{32} = \frac{-8 \pm 0}{32}$$

$$\rightarrow x = \frac{-8}{32} \quad \rightarrow x = -\frac{1}{4}$$

$$S.S = \left\{-\frac{1}{4}\right\}$$

Q19. $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$

Solution:-

$$(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$$

$$\rightarrow x^2-bx-ax+ab+x^2-cx-bx+bc+x^2-ax-cx+ac=0$$

$$3x^2-2ax-2bx-2cx+ab+bc+ca=0$$

$$\rightarrow 3x^2-2(a+b+c)x+(ab+bc+ca)=0$$

Here $A=3,\ B=-2(a+b+c),\ C=ab+bc+ca$

using $x = \dfrac{-B \pm \sqrt{B^2-4AC}}{2A}$

$$x = \frac{-(-2(a+b+c)) \pm \sqrt{[-2(a+b+c)]^2-4(3)(ab+bc+ca)}}{2(3)}$$

$$x = \frac{2(a+b+c) \pm \sqrt{4(a^2+b^2+c^2+2ab+2bc+2ca)-4(3ab+3bc+3ca)}}{6}$$

$$x = \frac{2(a+b+c) \pm 2\sqrt{a^2+b^2+c^2+2ab+2bc+2ca-3ab-3bc-3ca}}{6}$$

$$x = \frac{a+b+c \pm \sqrt{a^2+b^2+c^2-ab-bc-ca}}{3}$$

$$S.S = \left\{\frac{a+b+c \pm \sqrt{a^2+b^2+c^2-ab-bc-ca}}{3}\right\}$$

Q20. $(a+b)x^2+(a+2b+c)x+b+c=0$

Solution:- $(a+b)x^2+(a+2b+c)x+b+c=0$

Here $A=a+b,\ B=a+2b+c,\ C=b+c$

using $x = \dfrac{-B \pm \sqrt{B^2-4AC}}{2A}$

$$x = \frac{-(a+2b+c) \pm \sqrt{(a+2b+c)^2-4(a+b)(b+c)}}{2(a+b)}$$

$$x = \frac{-(a+2b+c) \pm \sqrt{a^2+4b^2+c^2+4ab+4bc+2ca-4(ab+ac+b^2+bc)}}{2(a+b)}$$

$$x = \frac{-(a+2b+c) \pm \sqrt{a^2+4b^2+c^2+4ab+4bc+2ca-4ab-4ac-4b^2-4bc}}{2(a+b)}$$

$$x = \frac{-(a+2b+c) \pm \sqrt{a^2+c^2-2ac}}{2(a+b)}$$

$$x = \frac{-(a+2b+c) \pm \sqrt{(a-c)^2}}{2(a+b)}$$

$$x = \frac{-(a+2b+c) \pm (a-c)}{2(a+b)}$$

$$x = \frac{-(a+2b+c)+a-c}{2(a+b)} \quad \text{or} \quad x = \frac{-(a+2b+c)-(a-c)}{2(a+b)}$$

$$x = \frac{-a-2b-c+a-c}{2(a+b)} \quad \text{or} \quad x = \frac{-a-2b-c-a+c}{2(a+b)}$$

$$x = \frac{-2b-2c}{2(a+b)} \quad \text{or} \quad x = \frac{-2b-2a}{2(a+b)}$$

$$x = \frac{-2(b+c)}{2(a+b)} \quad \text{or} \quad x = \frac{-2(a+b)}{2(a+b)}$$

$$x = -\frac{(b+c)}{a+b} \quad \text{or} \quad x = -1$$

$$S.S = \left\{-\frac{(b+c)}{a+b},\ -1\right\}$$

Solution of Equations Reducible to the Quadratic Equation

Type I:- The equations of the form $ax^{2n}+bx^n+c=0,\ a \neq 0$

$$\rightarrow a(x^n)^2+bx^n+c=0$$

Put $x^n = y$ it becomes as

$ay^2+by+c=0$ (Quadratic Eq. in y)

Questions related to type I

Example #1, Q 1,2,3,4,5

Example 1. Solve the equation

$$x^{1/2}-x^{1/4}-6=0$$

Solution:- $x^{1/2}-x^{1/4}-6=0$

$$\rightarrow (x^{1/4})^2-x^{1/4}-6=0$$

Put $x^{1/4}=y$ then

$$y^2-y-6=0$$

$$\rightarrow y^2-3y+2y-6=0$$

$\Rightarrow y(y+2)-3(y+2)=0$

$\Rightarrow (y+2)(y-3)=0$

$y+2=0$ or $y-3=0$

$y=-2$ or $y=3$

$\Rightarrow x^{1/4}=-2$ or, $x^{1/4}=3$ $(\because y=x^{1/4})$

$\Rightarrow (x^{1/4})^4=(-2)^4$ or $(x^{1/4})^4=(3)^4$

$x=16$ or $x=81$

$S.S=\{16, 81\}$

Type II:- The equation of the form: $(x+a)(x+b)(x+c)(x+d)=k$

where $a+b=c+d$

Questions related to type II

Example 2, Q 6, 7, 8, 9, 10, 11, 12, 13

Example 2. Solve

$(x-7)(x-3)(x+1)(x+5)-1680=0$

Solution:-

$(x-7)(x-3)(x+1)(x+5)-1680=0$

$\Rightarrow (x-7)(x-3)(x+1)(x+5)=1680$

Re-arranging it

$(x+1)(x-3)\cdot(x+5)(x-7)=1680$

$\Rightarrow (x^2-3x+x-3)(x^2-7x+5x-35)=1680$

$(x^2-2x-3)(x^2-2x-35)=1680$

put $x^2-2x=y$

$\Rightarrow (y-3)(y-35)=1680$

$\Rightarrow y^2-35y-3y+105=1680$

$y^2-38y+105-1680=0$

$\Rightarrow y^2-38y-1575=0$

using $y=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\dfrac{-(-38)\pm\sqrt{(-38)^2-4(1)(-1575)}}{2(1)}$

$y=\dfrac{38\pm\sqrt{1444+6300}}{2}$

$y=\dfrac{38\pm\sqrt{7744}}{2}=\dfrac{38\pm 88}{2}$

$\Rightarrow y=\dfrac{38+88}{2}$ or $y=\dfrac{38-88}{2}$

$y=\dfrac{126}{2}$, $y=-\dfrac{50}{2}$

$y=63$, $y=-25$

$\Rightarrow x^2-2x=63$, $x^2-2x=-25$

$x^2-2x-63=0$, $x^2-2x+25=0$

$x^2+7x-9x-63=0$, using $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x(x+7)-9(x+7)=0$, $x=\dfrac{-(-2)\pm\sqrt{(-2)^2-4(1)(25)}}{2(1)}$

$(x+7)(x-9)=0$, $x=\dfrac{2\pm\sqrt{4-100}}{2}$

$x+7=0,\ x-9=0$, $x=\dfrac{2\pm\sqrt{-96}}{2}$

$x=-7,\ x=9$, $x=\dfrac{2\pm\sqrt{16\times6\times(-1)}}{2}$

$x=\dfrac{2\pm4\sqrt{6}\sqrt{-1}}{2}$

$x=\dfrac{2(1\pm2\sqrt{6}i)}{2}$ $\because \sqrt{-1}=i$

$\Rightarrow x=1\pm2\sqrt{6}i$

$S.S=\{-7, 9, 1\pm2\sqrt{6}i\}$

Type III:- Exponential Equations

Equations, in which the variable occurs in exponent, are called exponential equations.

Questions related to Type III

Example 3, 4 Q 14, 15, 16, 17

Example 3. Solve the equation

$2^{2x}-3.2^{x+2}+32=0$

Solution:-

$2^{2x}-3.2^{x+2}+32=0$

$\Rightarrow 2^{2x}-3.2^x.2^2+32=0$

$\Rightarrow (2^x)^2-3.2^x.4+32=0$

$\Rightarrow (2^x)^2-12.2^x+32=0$

Put $2^x=y$

$\Rightarrow y^2-12y+32=0$

$y^2-8y-4y+32=0$

$\Rightarrow y(y-8)-4(y-8)=0$

$(y-8)(y-4)=0$

$y-8=0$ or $y-4=0$

$y=8$ or $y=4$

$\Rightarrow 2^x=8$ or $2^x=4$

$2^x=2^3$, $2^x=2^2$

$\Rightarrow x=3$ or $x=2$

S.S = $\{2, 3\}$

Example 4. Solve the equation:

$$4^{1+x}+4^{1-x}=10$$

Solution:- $4^{1+x}+4^{1-x}=10$

$\Rightarrow 4\cdot 4^x+4\cdot 4^{-x}=10$

$\Rightarrow 4\cdot 4^x+4\cdot\frac{1}{4^x}=10$

put $4^x=y$

$\Rightarrow 4y+\frac{4}{y}=10$

$\Rightarrow 4y^2+4=10y$

$\Rightarrow 4y^2-10y+4=0$

$\Rightarrow 2y^2-5y+2=0$ ($\div$ by 2)

$2y^2-y-4y+2=0$

$y(2y-1)-2(2y-1)=0$

$(2y-1)(y-2)=0$

$2y-1=0$ or $y-2=0$

$\Rightarrow y=\frac{1}{2}$ or $y=2$

$\Rightarrow 4^x=\frac{1}{2}$ or $4^x=2$

$(2^2)^x=2^{-1}$ or $(2^2)^x=2^1$

$\Rightarrow 2^{2x}=2^{-1}$ or $2^{2x}=2^1$

$\Rightarrow 2x=-1$ or $2x=1$

$x=-\frac{1}{2}$ or $x=\frac{1}{2}$

S.S = $\{-\frac{1}{2}, \frac{1}{2}\}$

Type IV:- Reciprocal Equations:-

An equation, which remains unchanged when x is replaced by $\frac{1}{x}$, is called a reciprocal equation.

Questions related to Type IV

Example 5, Q 18, 19, 20, 21, 22, 23, 24

Example 5. Solve the equation

$$x^4-3x^3+4x^2-3x+1=0$$

Solution:-

$x^4-3x^3+4x^2-3x+1=0$

($\div$ by x^2)

$\Rightarrow \frac{x^4}{x^2}-\frac{3x^3}{x^2}+\frac{4x^2}{x^2}-\frac{3x}{x^2}+\frac{1}{x^2}=\frac{0}{x^2}$

$\Rightarrow x^2-3x+4-\frac{3}{x}+\frac{1}{x^2}=0$

$\Rightarrow x^2+\frac{1}{x^2}-3x-\frac{3}{x}+4=0$

$\Rightarrow x^2+\frac{1}{x^2}-3\left(x+\frac{1}{x}\right)+4=0$

put $x+\frac{1}{x}=y$

$\Rightarrow \left(x+\frac{1}{x}\right)^2=y^2 \Rightarrow x^2+\frac{1}{x^2}+2=y^2$

$\Rightarrow x^2+\frac{1}{x^2}=y^2-2$

so $y^2-2-3y+4=0$

$y^2-3y+2=0$

$\Rightarrow y^2-y-2y+2=0$

$y(y-1)-2(y-1)=0$

$(y-1)(y-2)=0$

$\Rightarrow y-1=0$ or $y-2=0$

$y=1$ or $y=2$

$\Rightarrow x+\frac{1}{x}=1$

$\Rightarrow x^2+1=x$

$\Rightarrow x^2-x+1=0$

By Quadratic formula

$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2}$

$=\frac{1\pm\sqrt{1-4}}{2}$

$x=\frac{1\pm\sqrt{-3}}{2}$

$\Rightarrow x+\frac{1}{x}=2$

$\Rightarrow x^2+1=2x$

$x^2-2x+1=0$

$(x-1)^2=0$

$x-1=0$

$x=1$

S.S = $\{1, \frac{1\pm\sqrt{-3}}{2}\}$

Exercise 4.2

Solve the following equations:

Q1. $x^4 - 6x^2 + 8 = 0$

Solution:- $x^4 - 6x^2 + 8 = 0$

$\rightarrow (x^2)^2 - 6x^2 + 8 = 0$

Put $x^2 = y$

$\rightarrow y^2 - 6y + 8 = 0$

$\rightarrow y^2 - 2y - 4y + 8 = 0$

$\rightarrow y(y-2) - 4(y-2) = 0$

$(y-2)(y-4) = 0$

$\rightarrow y - 2 = 0$ or $y - 4 = 0$

$y = 2$ or $y = 4$

$\rightarrow x^2 = 2$ or $x^2 = 4$

$\rightarrow x = \pm\sqrt{2}$ or $x = \pm 2$

$S.S = \{\pm\sqrt{2}, \pm 2\}$

Q2. $x^{-2} - 10 = 3x^{-1}$

Solution:- $x^{-2} - 10 = 3x^{-1}$

$\rightarrow x^{-2} - 3x^{-1} - 10 = 0$

$\rightarrow (x^{-1})^2 - 3x^{-1} - 10 = 0$

Put $x^{-1} = y$

$\rightarrow y^2 - 3y - 10 = 0$

$y^2 - 5y + 2y - 10 = 0$

$y(y-5) + 2(y-5) = 0$

$(y-5)(y+2) = 0$

$\rightarrow y - 5 = 0$ or $y + 2 = 0$

$y = 5$ or $y = -2$

$\rightarrow x^{-1} = 5$ or $x^{-1} = -2$

$\frac{1}{x} = 5$ or $\frac{1}{x} = -2$

$\rightarrow x = \frac{1}{5}$ or $x = -\frac{1}{2}$

$S.S = \{\frac{1}{5}, -\frac{1}{2}\}$

Q3. $x^6 - 9x^3 + 8 = 0$

Solution:- $x^6 - 9x^3 + 8 = 0$

$\rightarrow (x^3)^2 - 9x^3 + 8 = 0$

Put $x^3 = y$

$\rightarrow y^2 - 9y + 8 = 0$

$y^2 - 8y - y + 8 = 0$

$\rightarrow y(y-8) - 1(y-8) = 0$

$\rightarrow (y-8)(y-1) = 0$

$y - 8 = 0$ or $y - 1 = 0$

$y = 8$ or $y = 1$

$\rightarrow x^3 = 8$ or $x^3 = 1$

$\rightarrow x^3 - 8 = 0$ or $x^3 - 1 = 0$

$(x)^3 - (2)^3 = 0$ or $(x)^3 - (1)^3 = 0$

$(x-2)(x^2+2x+4) = 0$ or $(x-1)(x^2+x+1) = 0$

$x - 2 = 0$, $x^2 + 2x + 4 = 0$ $\quad$ $x - 1 = 0$, $x^2 + x + 1 = 0$

$\rightarrow x = 2$, $\quad$ $x = 1$

$x^2 + 2x + 4 = 0$ $\quad$ $x^2 + x + 1 = 0$

using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\rightarrow x = \frac{-2 \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$, $\quad x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$

$= \frac{-2 \pm \sqrt{4 - 16}}{2}$, $\quad x = \frac{-1 \pm \sqrt{1 - 4}}{2}$

$= \frac{-2 \pm \sqrt{-12}}{2}$, $\quad x = \frac{-1 \pm \sqrt{-3}}{2}$

$= \frac{-2 \pm \sqrt{4 \times (-3)}}{2}$

$= \frac{-2 \pm 2\sqrt{-3}}{2}$

$x = \frac{2(-1 \pm \sqrt{-3})}{2}$

$x = -1 \pm \sqrt{-3}$

$S.S = \{1, 2, \frac{-1 \pm \sqrt{-3}}{2}, -1 \pm \sqrt{-3}\}$

Q4. $8x^6 - 19x^3 - 27 = 0$

Solution:- $8x^6 - 19x^3 - 27 = 0$

$8(x^3)^2 - 19x^3 - 27 = 0$

Put $x^3 = y$

$8y^2 - 19y - 27 = 0$

$8y^2 + 8y - 27y - 27 = 0$

$\rightarrow 8y(y+1) - 27(y+1) = 0$

$(8y-27)(y+1) = 0$

$8y - 27 = 0$ or $y + 1 = 0$

$\rightarrow 8y = 27$ or $y = -1$

$\rightarrow y = \frac{27}{8}$ or $y = -1$

$\rightarrow x^3 = \frac{27}{8}$ or $x^3 = -1$

$\rightarrow 8x^3 = 27$ or $x^3 + 1 = 0$

$(2x)^3 - (3)^3 = 0$ or $(x)^3 + (1)^3 = 0$

$(2x-3)(4x^2+6x+9) = 0$, $(x+1)(x^2-x+1) = 0$

$\left\{ \because a^3+b^3 = (a+b)(a^2-ab+b^2) \right.$
$\left. a^3-b^3 = (a-b)(a^2+ab+b^2) \right.$

$2x-3=0$, $4x^2+6x+9=0$, $x+1=0$, $x^2-x+1=0$

$\rightarrow x = \frac{3}{2}$ $x=-1$, $x^2-x+1=0$

$4x^2+6x+9=0$, $x^2-x+1=0$

using $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{-6 \pm \sqrt{(6)^2-4(4)(9)}}{2(4)}$, $x = \frac{-(-1) \pm \sqrt{(-1)^2-4(1)(1)}}{2(1)}$

$= \frac{-6 \pm \sqrt{36-144}}{8}$, $x = \frac{1 \pm \sqrt{1-4}}{2}$

$= \frac{-6 \pm \sqrt{-108}}{8}$, $x = \frac{1 \pm \sqrt{-3}}{2}$

$= \frac{-6 \pm \sqrt{36 \times (-3)}}{8}$

$= \frac{-6 \pm 6\sqrt{-3}}{8}$

$x = \frac{6(-1 \pm \sqrt{-3})}{8}$

$\rightarrow x = \frac{3(-1 \pm \sqrt{-3})}{4}$

$S.S = \left\{ -1, \frac{3}{4}, \frac{3(-1 \pm \sqrt{-3})}{4}, \frac{1 \pm \sqrt{-3}}{2} \right\}$

Q5. $x^{2/5} + 8 = 6x^{1/5}$

Solution:- $x^{2/5} + 8 = 6x^{1/5}$

$\rightarrow (x^{1/5})^2 - 6x^{1/5} + 8 = 0$

Put $x^{1/5} = y$

$\rightarrow y^2 - 6y + 8 = 0$

$\rightarrow y^2 - 4y - 2y + 8 = 0$

$\rightarrow y(y-4) - 2(y-4) = 0$

$(y-4)(y-2) = 0$

$y-4 = 0$, $y-2 = 0$

$y = 4$, $y = 2$

$\rightarrow x^{1/5} = 4$, $x^{1/5} = 2$

$\rightarrow (x^{1/5})^5 = (4)^5$, $(x^{1/5})^5 = (2)^5$

$\rightarrow x = 1024$, $x = 32$

$S.S = \{32, 1024\}$

Q6. $(x+1)(x+2)(x+3)(x+4) = 24$

Solution:- $(x+1)(x+2)(x+3)(x+4) = 24$

Rearranging it

$(x+1)(x+4)\ (x+2)(x+3) = 24$

$\rightarrow (x^2+4x+x+4)\ (x^2+3x+2x+6) = 24$

$(x^2+5x+4)(x^2+5x+6) = 24$

Put $x^2+5x = y$

$\rightarrow (y+4)(y+6) = 24$

$\rightarrow y^2 + 6y + 4y + 24 = 24$

$y^2 + 10y + 24 - 24 = 0$

$y^2 + 10y = 0$

$y(y+10) = 0$

$\rightarrow y = 0$ or $y + 10 = 0$

$\rightarrow x^2 + 5x = 0$ or $x^2 + 5x + 10 = 0$

$x(x+5) = 0$ using $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x=0$, $x+5=0$, $x = \frac{-5 \pm \sqrt{(5)^2-4(1)(10)}}{2(1)}$

$x=0$, $x=-5$

$x = \frac{-5 \pm \sqrt{25-40}}{2}$

$x = \frac{-5 \pm \sqrt{-15}}{2}$

$S.S = \left\{ 0, -5, \frac{-5 \pm \sqrt{-15}}{2} \right\}$

Q7. $(x-1)(x+5)(x+8)(x+2)-880=0$

Solution:- $(x-1)(x+5)(x+8)(x+2)-880=0$

Re-arranging it

$(x-1)(x+8)\,(x+2)(x+5) = 880$

$(x^2+8x-x-8)(x^2+5x+2x+10) = 880$

$\rightarrow (x^2+7x-8)(x^2+7x+10) = 880$

Put $x^2+7x = y$

$\rightarrow (y-8)(y+10) = 880$

$y^2+10y-8y-80 = 880$

$\rightarrow y^2+2y-80-880=0$

$\rightarrow y^2+2y-960=0$

$y^2+32y-30y-960=0$

$y(y+32)-30(y+32)=0$

$(y+32)(y-30)=0$

$y+32=0$ or $y-30=0$

$y=-32$ or $y=30$

$\rightarrow x^2+7x=-32$ or $x^2+7x=30$

$x^2+7x+32=0$, $x^2+7x-30=0$

using $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, $x^2-3x+10x-30=0$

$x = \frac{-7\pm\sqrt{(7)^2-4(1)(32)}}{2(1)}$, $x(x-3)+10(x-3)=0$

$(x-3)(x+10)=0$

$x = \frac{-7\pm\sqrt{49-128}}{2}$, $x-3=0,\ x+10=0$

$x=3,\ x=-10$

$x = \frac{-7\pm\sqrt{-79}}{2}$

$S.S = \{3, -10, \frac{-7\pm\sqrt{-79}}{2}\}$

Q8. $(x-5)(x-7)(x+6)(x+4)-504=0$

Solution:- $(x-5)(x-7)(x+6)(x+4)-504=0$

Re-arranging it,

$(x-5)(x+4)\,(x-7)(x+6) = 504$

$\rightarrow (x^2+4x-5x-20)(x^2+6x-7x-42)=504$

$(x^2-x-20)(x^2-x-42)=504$

Put $x^2-x=y$

$\rightarrow (y-20)(y-42) = 504$

$\rightarrow y^2-42y-20y+840-504=0$

$y^2-62y+336=0$

$\rightarrow y^2-6y-56y+336=0$

$y(y-6)-56(y-6)=0$

$(y-6)(y-56)=0$

$y-6=0$, $y-56=0$

$y=6$, $y=56$

$x^2-x=6$, $x^2-x=56$

$x^2-x-6=0$, $x^2-x-56=0$

$x^2-3x+2x-6=0$, $x^2-8x+7x-56=0$

$x(x-3)+2(x-3)=0$, $x(x-8)+7(x-8)=0$

$(x-3)(x+2)=0$, $(x-8)(x+7)=0$

$x-3=0,\ x+2=0$, $x-8=0,\ x+7=0$

$x=3,\ x=-2$, $x=8,\ x=-7$

$S.S=\{3,-2,8,-7\}$

Q9. $(x-1)(x-2)(x-8)(x+5)+360=0$

Solution:- $(x-1)(x-2)(x-8)(x+5)+360=0$

$\rightarrow (x^2-2x-x+2)(x^2+5x-8x-40) = -360$

$\rightarrow (x^2-3x+2)(x^2-3x-40) = -360$

Put $x^2-3x=y$

$\rightarrow (y+2)(y-40) = -360$

$y^2-40y+2y-80+360=0$

$y^2-38y+280=0$

$y^2-10y-28y+280=0$

$y(y-10)-28(y-10)=0$

$(y-10)(y-28)=0$

$\rightarrow y-10=0$, $y-28=0$

$y=10$, $y=28$

$x^2-3x=10$, $x^2-3x=28$

$x^2-3x-10=0$, $x^2-3x-28=0$

$x^2+2x-5x-10=0$, $x^2+4x-7x-28=0$

$x(x+2)-5(x+2)=0$, $x(x+4)-7(x+4)=0$

$(x+2)(x-5)=0$, $(x+4)(x-7)=0$

$x+2=0,\ x-5=0$, $x+4=0,\ x-7=0$

$x=-2,\ x=5$, $x=-4,\ x=7$

$S.S=\{-2, 5, -4, 7\}$

Q10. $(x+1)(2x+3)(2x+5)(x+3)=945$

Solution:- $(x+1)(2x+3)(2x+5)(x+3)=945$

Re-arranging it,

$(x+1)(x+3)\cdot(2x+3)(2x+5)=945$

$\rightarrow (x^2+3x+x+3)(4x^2+10x+6x+15)=945$

$\rightarrow (x^2+4x+3)(4x^2+16x+15)=945$

$\rightarrow (x^2+4x+3)[4(x^2+4x)+15]=945$

Put $x^2+4x=y$

$\rightarrow (y+3)(4y+15)=945$

$4y^2+15y+12y+45-945=0$

$\rightarrow 4y^2+27y-900=0$

$\rightarrow 4y^2-48y+75y-900=0$

$\rightarrow 4y(y-12)+75(y-12)=0$

$\rightarrow (y-12)(4y+75)=0$

$\rightarrow y-12=0$ or $4y+75=0$

$\rightarrow y=12$ or $y=-\frac{75}{4}$

$\rightarrow x^2+4x=12$ or $x^2+4x=-\frac{75}{4}$

$x^2+4x-12=0$

$x^2+6x-2x-12=0$

$x(x+6)-2(x+6)=0$

$(x+6)(x-2)=0$

$x+6=0,\ x-2=0$

$x=-6$, $x=2$

$\rightarrow 4x^2+16x=-75$

$\rightarrow 4x^2+16x+75=0$

using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\rightarrow x=\frac{-16\pm\sqrt{(16)^2-4(4)(75)}}{2(4)}$

$x=\frac{-16\pm\sqrt{256-1200}}{8}$

$x=\frac{-16\pm\sqrt{-944}}{8}$

$\rightarrow x=\frac{-16\pm\sqrt{16\times(-59)}}{8}$

$=\frac{-16\pm4\sqrt{-59}}{8}$

$=\frac{4(-4\pm\sqrt{59}\sqrt{-1})}{8}$

$x=\frac{-4\pm\sqrt{59}\,i}{2}$

$S.S=\{-6, 2, \frac{-4\pm\sqrt{59}\,i}{2}\}$

Q11. $(2x-7)(x^2-9)(2x+5)-91=0$

Solution:- $(2x-7)(x^2-9)(2x+5)-91=0$

$\rightarrow (2x-7)(x-3)(x+3)(2x+5)-91=0$

Re-arranging it

$(2x-7)(x+3)(x-3)(2x+5)-91=0$

$(2x^2+6x-7x-21)(2x^2+5x-6x-15)-91=0$

$(2x^2-x-21)(2x^2-x-15)-91=0$

Put $2x^2-x=y$

$(y-21)(y-15)-91=0$

$\rightarrow y^2-15y-21y+315-91=0$

$\rightarrow y^2-36y+224=0$

$y^2-8y-28y+224=0$

$y(y-8)-28(y-8)=0$

$(y-8)(y-28)=0$

$\rightarrow y-8=0$, $y-28=0$

$\rightarrow y=8$, $y=28$

$2x^2-x=8$, $2x^2-x=28$

$\rightarrow 2x^2-x-8=0$

using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(2)(-8)}}{2(2)}$

$x=\frac{1\pm\sqrt{1+64}}{4}$

$x=\frac{1\pm\sqrt{65}}{4}$

$2x^2-x-28=0$

$2x^2-8x+7x-28=0$

$2x(x-4)+7(x-4)=0$

$(x-4)(2x+7)=0$

$x-4=0,\ 2x+7=0$

$x=4,\ x=-\frac{7}{2}$

$S.S=\{4, -\frac{7}{2}, \frac{1\pm\sqrt{65}}{4}\}$

Q12. $(x^2+6x+8)(x^2+14x+48)=105$

Solution:- $(x^2+6x+8)(x^2+14x+48)=105$

$\rightarrow [x^2+2x+4x+8][x^2+6x+8x+48]=105$

$[x(x+2)+4(x+2)][x(x+6)+8(x+6)]=105$

$\rightarrow (x+2)(x+4)(x+6)(x+8)=105$

Re-arranging it

$\rightarrow (x+2)(x+8)(x+4)(x+6)=105$

$(x^2+2x+8x+16)(x^2+6x+4x+24)=105$

$(x^2+10x+16)(x^2+10x+24)=105$

Put $x^2+10x=y$

$\rightarrow (y+16)(y+24)=105$

$\rightarrow y^2+24y+16y+384-105=0$

$y^2+40y+279=0$

$\rightarrow y^2+9y+31y+279=0$

$y(y+9)+31(y+9)=0$

$(y+9)(y+31)=0$

$y+9=0$, $y+31=0$

$y=-9$, $y=-31$

$x^2+10x=-9$, $x^2+10x=-31$

$x^2+10x+9=0$, $x^2+10x+31=0$

$x^2+x+9x+9=0$, using $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x(x+1)+9(x+1)=0$, $x=\dfrac{-10\pm\sqrt{(10)^2-4(1)(31)}}{2(1)}$

$(x+1)(x+9)=0$, $x=\dfrac{-10\pm\sqrt{100-124}}{2}$

$x+1=0$, $x+9=0$

$x=-1$, $x=-9$, $x=\dfrac{-10\pm\sqrt{-24}}{2}$

$x=\dfrac{-10\pm\sqrt{-6\times 4}}{2}$

$x=\dfrac{-10\pm 2\sqrt{-6}}{2}$

$x=\dfrac{2(-5\pm\sqrt{6}\,i)}{2}$

$x=-5\pm\sqrt{6}\,i$

$S.S=\{-1,-9,-5\pm\sqrt{6}\,i\}$

Q13. $(x^2+6x-27)(x^2-2x-35)=385$

Solution:- $(x^2+6x-27)(x^2-2x-35)=385$

$\rightarrow (x^2-3x+9x-27)(x^2+5x-7x-35)=385$

$[x(x-3)+9(x-3)][x(x+5)-7(x+5)]=385$

$(x-3)(x+9)(x+5)(x-7)=385$

Re-arranging it

$(x-3)(x+5)(x+9)(x-7)=385$

$(x^2+5x-3x-15)(x^2-7x+9x-63)=385$

$(x^2+2x-15)(x^2+2x-63)-385=0$

Put $x^2+2x=y$

$\rightarrow (y-15)(y-63)-385=0$

$y^2-63y-15y+945-385=0$

$\rightarrow y^2-78y+560=0$

$y^2-8y-70y+560=0$

$y(y-8)-70(y-8)=0$

$(y-8)(y-70)=0$

$y-8=0$, $y-70=0$

$y=8$, $y=70$

$x^2+2x-8=0$, $x^2+2x-70=0$

$x^2+4x-2x-8=0$, using $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x(x+4)-2(x+4)=0$, $x=\dfrac{-2\pm\sqrt{(2)^2-4(1)(-70)}}{2(1)}$

$(x+4)(x-2)=0$

$x+4=0$, $x-2=0$, $x=\dfrac{-2\pm\sqrt{4+280}}{2}$

$x=-4$, $x=2$

$=\dfrac{-2\pm\sqrt{284}}{2}$

$=\dfrac{-2\pm\sqrt{71\times 4}}{2}$

$=\dfrac{-2\pm 2\sqrt{71}}{2}$

$=\dfrac{2(-1\pm\sqrt{71})}{2}$

$x=-1\pm\sqrt{71}$

$S.S=\{-4,2,-1\pm\sqrt{71}\}$

Q14. $4 \cdot 2^{2x+1} - 9 \cdot 2^{x} + 1 = 0$

Solution:- $4 \cdot 2^{2x+1} - 9 \cdot 2^{x} + 1 = 0$

$\rightarrow 4 \cdot 2^{2x} \cdot 2 - 9 \cdot 2^{x} + 1 = 0$

$\rightarrow 8 (2^{x})^{2} - 9 \cdot 2^{x} + 1 = 0$

Put $2^{x} = y$

$\rightarrow 8y^{2} - 9y + 1 = 0$

$\rightarrow 8y^{2} - 8y - y + 1 = 0$

$\rightarrow 8y(y-1) - 1(y-1) = 0$

$\rightarrow (y-1)(8y-1) = 0$

$\rightarrow y - 1 = 0 \quad , \quad 8y - 1 = 0$

$y = 1 \quad , \quad y = \frac{1}{8} = \frac{1}{2^{3}}$

$\rightarrow 2^{x} = 1 \quad , \quad 2^{x} = 2^{-3}$

$\rightarrow 2^{x} = 2^{0} \quad , \quad 2^{x} = 2^{-3}$

$\rightarrow x = 0, \quad x = -3$

$S \cdot S = \{0, -3\}$

Q15. $2^{x} + 2^{-x+6} - 20 = 0$

Solution:- $2^{x} + 2^{-x+6} - 20 = 0$

$\rightarrow 2^{x} + 2^{-x} \cdot 2^{6} - 20 = 0$

$\rightarrow 2^{x} + \frac{64}{2^{x}} - 20 = 0$

Put $2^{x} = y$

$\rightarrow y + \frac{64}{y} - 20 = 0$

$\rightarrow y^{2} + 64 - 20y = 0$

$\rightarrow y^{2} - 20y + 64 = 0$

$\rightarrow y^{2} - 4y - 16y + 64 = 0$

$y(y-4) - 16(y-4) = 0$

$(y-4)(y-16) = 0$

$y - 4 = 0 \quad , \quad y - 16 = 0$

$y = 4 \quad , \quad y = 16$

$\rightarrow 2^{x} = 2^{2} \quad , \quad 2^{x} = 2^{4}$

$\rightarrow x = 2 \quad , \quad x = 4$

$S \cdot S = \{2, 4\}$

Q16. $4^{x} - 3 \cdot 2^{x+3} + 128 = 0$

Solution:- $4^{x} - 3 \cdot 2^{x+3} + 128 = 0$

$\rightarrow (2^{2})^{x} - 3 \cdot 2^{x} \cdot 2^{3} + 128 = 0$

$\rightarrow (2^{x})^{2} - 3 \cdot 2^{x} \cdot 8 + 128 = 0$

$\rightarrow (2^{x})^{2} - 24 \cdot 2^{x} + 128 = 0$

Put $2^{x} = y$

$y^{2} - 24y + 128 = 0$

$\rightarrow y^{2} - 8y - 16y + 128 = 0$

$y(y-8) - 16(y-8) = 0$

$(y-8)(y-16) = 0$

$\rightarrow y - 8 = 0, \quad y - 16 = 0$

$y = 8 \quad , \quad y = 16$

$\rightarrow 2^{x} = 2^{3} \quad , \quad 2^{x} = 2^{4}$

$\rightarrow x = 3, \quad x = 4$

$S \cdot S = \{3, 4\}$

Q17. $3^{2x-1} - 12 \cdot 3^{x} + 81 = 0$

Solution:- $3^{2x-1} - 12 \cdot 3^{x} + 81 = 0$

$\rightarrow 3^{2x} \cdot 3^{-1} - 12 \cdot 3^{x} + 81 = 0$

$\rightarrow (3^{x})^{2} \cdot \frac{1}{3} - 12 \cdot 3^{x} + 81 = 0$

$\rightarrow (3^{x})^{2} - 36 \cdot 3^{x} + 243 = 0$ ('x' by 3)

Put $3^{x} = y$

$\rightarrow y^{2} - 36y + 243 = 0$

$y^{2} - 9y - 27y + 243 = 0$

$y(y-9) - 27(y-9) = 0$

$(y-9)(y-27) = 0$

$\rightarrow y - 9 = 0, \quad y - 27 = 0$

$\rightarrow y = 9 \quad , \quad y = 27$

$\rightarrow 3^{x} = 3^{2} \quad , \quad 3^{x} = 3^{3}$

$\rightarrow x = 2, \quad x = 3$

$S \cdot S = \{2, 3\}$

Q18. $(x+\frac{1}{x})^2+3(x+\frac{1}{x})-4=0$

Solution:- $(x+\frac{1}{x})^2+3(x+\frac{1}{x})-4=0$

Put $x+\frac{1}{x}=y$

$\rightarrow y^2+3y-4=0$

$\rightarrow y^2+4y-y-4=0$

$y(y+4)-1(y+4)=0$

$(y+4)(y-1)=0$

$\rightarrow y+4=0 \quad , \quad y-1=0$

$y=-4 \quad , \quad y=1$

$\rightarrow x+\frac{1}{x}=-4 \qquad x+\frac{1}{x}=1$

$\rightarrow x^2+1=-4x \qquad x^2+1=x$

$\rightarrow x^2+4x+1=0 \qquad \rightarrow x^2-x+1=0$

using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-4\pm\sqrt{(4)^2-4(1)(1)}}{2(1)} \quad , \quad x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}$

$x=\frac{-4\pm\sqrt{16-4}}{2} \quad , \quad x=\frac{1\pm\sqrt{1-4}}{2}$

$x=\frac{-4\pm\sqrt{12}}{2} \quad , \quad x=\frac{1\pm\sqrt{-3}}{2}$

$x=\frac{-4\pm\sqrt{4\times3}}{2} \quad , \quad x=\frac{1\pm\sqrt{3}\sqrt{-1}}{2}$

$x=\frac{-4\pm2\sqrt{3}}{2} \quad , \quad x=\frac{1\pm\sqrt{3}i}{2}$

$x=\frac{2(-2\pm\sqrt{3})}{2}$

$x=-2\pm\sqrt{3}$

$S.S=\{-2\pm\sqrt{3}, \frac{1\pm\sqrt{3}i}{2}\}$

Q19. $x^2+x-4+\frac{1}{x}+\frac{1}{x^2}=0$

Solution:- $x^2+x-4+\frac{1}{x}+\frac{1}{x^2}=0$

$\rightarrow x^2+\frac{1}{x^2}+x+\frac{1}{x}-4=0$

Put $x+\frac{1}{x}=y$

$(x+\frac{1}{x})^2=y^2$, squaring

$\rightarrow x^2+\frac{1}{x^2}+2=y^2$

$\rightarrow x^2+\frac{1}{x^2}=y^2-2$ so

$\rightarrow y^2-2+y-4=0$

$\rightarrow y^2+y-6=0$

$y^2+3y-2y-6=0$

$y(y+3)-2(y+3)=0$

$\rightarrow (y+3)(y-2)=0$

$y+3=0 \quad , \quad y-2=0$

$y=-3 \quad , \quad y=2$

$x+\frac{1}{x}=-3 \quad , \quad x+\frac{1}{x}=2$

$x^2+1=-3x \quad , \quad x^2+1=2x$

$x^2+3x+1=0 \quad , \quad x^2-2x+1=0$

using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad (x-1)^2=0$

$x=\frac{-3\pm\sqrt{(3)^2-4(1)(1)}}{2(1)} \qquad x-1=0$

$\rightarrow x=1$

$x=\frac{-3\pm\sqrt{9-4}}{2}=\frac{-3\pm\sqrt{5}}{2}$

$S.S=\{1, \frac{-3\pm\sqrt{5}}{2}\}$

Q20. $(x-\frac{1}{x})^2+3(x+\frac{1}{x})=0$

Solution:- $(x-\frac{1}{x})^2+3(x+\frac{1}{x})=0$

$\rightarrow x^2+\frac{1}{x^2}-2+3(x+\frac{1}{x})=0$

Put $x+\frac{1}{x}=y$

$\rightarrow (x+\frac{1}{x})^2=y^2$, squaring

$x^2+\frac{1}{x^2}+2=y^2$

$\rightarrow x^2+\frac{1}{x^2}=y^2-2$ so

$y^2-2-2+3y=0$

$\rightarrow y^2+3y-4=0$

$y^2-y+4y-4=0$

$y(y-1)+4(y-1)=0$

$(y-1)(y+4)=0$

$y-1=0 \quad , \quad y+4=0$

$y=1 \quad , \quad y=-4$

$x+\frac{1}{x}=1$, $x+\frac{1}{x}=-4$

$x^2+1=x$, $x^2+1=-4x$

$x^2-x+1=0$, $x^2+4x+1=0$

using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}$, $x=\frac{-4\pm\sqrt{(4)^2-4(1)(1)}}{2(1)}$

$x=\frac{1\pm\sqrt{1-4}}{2}$, $x=\frac{-4\pm\sqrt{16-4}}{2}$

$x=\frac{1\pm\sqrt{-3}}{2}$, $x=\frac{-4\pm\sqrt{12}}{2}$

$x=\frac{1\pm\sqrt{3}i}{2}$, $x=\frac{-4\pm\sqrt{4\times3}}{2}$

$x=\frac{-4\pm2\sqrt{3}}{2}$

$x=\frac{2(-2\pm\sqrt{3})}{2}$

$x=-2\pm\sqrt{3}$

$S.S=\left\{\frac{1\pm\sqrt{3}i}{2}, -2\pm\sqrt{3}\right\}$

Q21. $2x^4-3x^3-x^2-3x+2=0$

Solution:- $2x^4-3x^3-x^2-3x+2=0$

($\div$ by x^2)

$\rightarrow \frac{2x^4}{x^2}-\frac{3x^3}{x^2}-\frac{x^2}{x^2}-\frac{3x}{x^2}+\frac{2}{x^2}=\frac{0}{x^2}$

$\rightarrow 2x^2-3x-1-\frac{3}{x}+\frac{2}{x^2}=0$

$\rightarrow 2x^2+\frac{2}{x^2}-3x-\frac{3}{x}-1=0$

$2\left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)-1=0$

Put $x+\frac{1}{x}=y$

$\left(x+\frac{1}{x}\right)^2=y^2$

$\rightarrow x^2+\frac{1}{x^2}+2=y^2$

$\rightarrow x^2+\frac{1}{x^2}=y^2-2$

$\rightarrow 2(y^2-2)-3y-1=0$

$2y^2-4-3y-1=0$

$2y^2-3y-5=0$

$2y^2+2y-5y-5=0$

$2y(y+1)-5(y+1)=0$

$\rightarrow (y+1)(2y-5)=0$

$y+1=0$, $2y-5=0$

$y=-1$, $y=\frac{5}{2}$

$x+\frac{1}{x}=-1$, $x+\frac{1}{x}=\frac{5}{2}$

$x^2+1=-x$, $2x^2+2=5x$

$x^2+x+1=0$, $2x^2-5x+2=0$

using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-1\pm\sqrt{(1)^2-4(1)(1)}}{2(1)}$

$=\frac{-1\pm\sqrt{1-4}}{2}$

$x=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm\sqrt{3}i}{2}$

$2x^2-4x-x+2=0$

$2x(x-2)-1(x-2)=0$

$(x-2)(2x-1)=0$

$x-2=0$, $2x-1=0$

$x=2$, $x=\frac{1}{2}$

$S.S=\left\{\frac{-1\pm\sqrt{3}i}{2}, 2, \frac{1}{2}\right\}$

Q22. $2x^4+3x^3-4x^2-3x+2=0$

Solution:- $2x^4+3x^3-4x^2-3x+2=0$

($\div$ by x^2)

$\rightarrow \frac{2x^4}{x^2}+\frac{3x^3}{x^2}-\frac{4x^2}{x^2}-\frac{3x}{x^2}+\frac{2}{x^2}=0$

$\rightarrow 2x^2+3x-4-\frac{3}{x}+\frac{2}{x^2}=0$

$\rightarrow 2x^2+\frac{2}{x^2}+3x-\frac{3}{x}-4=0$

$\rightarrow 2\left(x^2+\frac{1}{x^2}\right)+3\left(x-\frac{1}{x}\right)-4=0$

Put $x-\frac{1}{x}=y$

$\rightarrow \left(x-\frac{1}{x}\right)^2=y^2$

$\rightarrow x^2+\frac{1}{x^2}-2=y^2$

$\rightarrow x^2+\frac{1}{x^2}=y^2+2$ so

$2(y^2+2)+3y-4=0$

$\rightarrow 2y^2+4+3y-4=0$

$2y^2+3y=0$

$y(2y+3)=0$

$y=0$, $2y+3=0$

$y=0$, $y=\frac{-3}{2}$

$x-\frac{1}{x}=0$, $x-\frac{1}{x}=\frac{-3}{2}$

$\Rightarrow x^2-1=0$, $2x^2-2=-3x$

$\Rightarrow (x-1)(x+1)=0$, $2x^2+3x-2=0$

$x-1=0,\ x+1=0$, $2x^2+4x-x-2=0$

$x=1,\ x=-1$, $2x(x+2)-1(x+2)=0$

$(x+2)(2x-1)=0$

$x+2=0,\ 2x-1=0$

$x=-2,\ x=\frac{1}{2}$

$S.S=\{1,-1,-2,\frac{1}{2}\}$

Q23. $6x^4-35x^3+62x^2-35x+6=0$

Solution:- $6x^4-35x^3+62x^2-35x+6=0$

($\div$ by x^2)

$$\frac{6x^4}{x^2}-\frac{35x^3}{x^2}+\frac{62x^2}{x^2}-\frac{35x}{x^2}+\frac{6}{x^2}=\frac{0}{x^2}$$

$$\Rightarrow 6x^2-35x+62-\frac{35}{x}+\frac{6}{x^2}=0$$

$$6x^2+\frac{6}{x^2}-35x-\frac{35}{x}+62=0$$

$$6\left(x^2+\frac{1}{x^2}\right)-35\left(x+\frac{1}{x}\right)+62=0$$

Put $x+\frac{1}{x}=y$

$$\left(x+\frac{1}{x}\right)^2=y^2$$

$$\Rightarrow x^2+\frac{1}{x^2}+2=y^2$$

$$\Rightarrow x^2+\frac{1}{x^2}=y^2-2 \quad \text{so}$$

$$6(y^2-2)-35y+62=0$$

$$6y^2-12-35y+62=0$$

$$6y^2-35y+50=0$$

$$\Rightarrow 6y^2-15y-20y+50=0$$

$$3y(2y-5)-10(2y-5)=0$$

$$(2y-5)(3y-10)=0$$

$2y-5=0$, $3y-10=0$

$y=\frac{5}{2}$, $y=\frac{10}{3}$

$x+\frac{1}{x}=\frac{5}{2}$, $x+\frac{1}{x}=\frac{10}{3}$

$2x^2+2=5x$, $3x^2+3=10x$

$\Rightarrow 2x^2-5x+2=0$, $3x^2-10x+3=0$

$2x^2-4x-x+2=0$, $3x^2-9x-x+3=0$

$2x(x-2)-1(x-2)=0$, $3x(x-3)-1(x-3)=0$

$(x-2)(2x-1)=0$, $(x-3)(3x-1)=0$

$x-2=0,\ 2x-1=0$, $x-3=0,\ 3x-1=0$

$x=2$, $x=\frac{1}{2}$ $x=3,\ x=\frac{1}{3}$

$S.S=\{2,\frac{1}{2},3,\frac{1}{3}\}$.

Q24. $x^4-6x^2+10-\frac{6}{x^2}+\frac{1}{x^4}=0$

Solution:- $x^4-6x^2+10-\frac{6}{x^2}+\frac{1}{x^4}=0$

$$\Rightarrow x^4+\frac{1}{x^4}-6x^2-\frac{6}{x^2}+10=0$$

$$x^4+\frac{1}{x^4}-6\left(x^2+\frac{1}{x^2}\right)+10=0$$

Put $x^2+\frac{1}{x^2}=y$

$$\left(x^2+\frac{1}{x^2}\right)^2=y^2$$

$$x^4+\frac{1}{x^4}+2=y^2$$

$$x^4+\frac{1}{x^4}=y^2-2 \quad \text{so}$$

$$y^2-2-6y+10=0$$

$$y^2-6y+8=0$$

$$y^2-4y-2y+8=0$$

$$y(y-4)-2(y-4)=0$$

$$(y-4)(y-2)=0$$

$y-4=0$, $y-2=0$

$y=4$, $y=2$

$x^2+\frac{1}{x^2}=4$, $x^2+\frac{1}{x^2}=2$

$x^4+1=4x^2$, $x^4+1=2x^2$

$x^4-4x^2+1=0$, $x^4-2x^2+1=0$

$(x^2)^2-4x^2+1=0$, $(x^2-1)^2=0$

Put $x^2=t$, $x^2-1=0$

$t^2-4t+1=0$, $x^2=1$

using $t=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, $\Rightarrow x=\pm 1$

$$t=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}$$

$$t=\frac{4\pm\sqrt{16-4}}{2}=\frac{4\pm\sqrt{12}}{2}=\frac{4\pm\sqrt{4\times 3}}{2}$$

$$t = \frac{4 \pm 2\sqrt{3}}{2} = \frac{2(2 \pm \sqrt{3})}{2}$$

$t = 2 \pm \sqrt{3}$

$\rightarrow x^2 = 2 \pm \sqrt{3}$

$\rightarrow x = \pm\sqrt{2 \pm \sqrt{3}}$

$\{1, -1, \pm\sqrt{2 \pm \sqrt{3}}\}$

Type V:- Redical equations

Equations involving redical expressions of the variable are called redical equations. e.g.,

$\sqrt{2x+8} + \sqrt{x+5} = 7$

$2x^2 + 2x - \sqrt{2x^2+2x} - 7 = 0$

Extraneous roots:-

A root that does not satisfy given equation is called an extraneous root.

i) The equations of the form:

$l(ax^2+bx) + m\sqrt{ax^2+bx+c} = 0$

Questions related to this type

Example 1, Q 1, 2, 10

Example 1. Solve the equation

$3x^2 + 15x - 2\sqrt{x^2+5x+1} = 2$

Solution:-

$3x^2 + 15x - 2\sqrt{x^2+5x+1} = 2$

$\rightarrow 3(x^2+5x) - 2\sqrt{x^2+5x+1} = 2$

Let $\sqrt{x^2+5x+1} = 2$

$\rightarrow x^2+5x+1 = 4$

$\rightarrow x^2+5x = 4-1$

$\rightarrow x^2+5x = 3$

$3(y^2-1) - 2y = 2$

$\rightarrow 3y^2 - 3 - 2y = 2$

$3y^2 - 3 - 2y - 2 = 0$

$\rightarrow 3y^2 - 2y - 5 = 0$

$3y^2 + 3y - 5y - 5 = 0$

$3y(y+1) - 5(y+1) = 0$

$(y+1)(3y-5) = 0$

$y+1 = 0, \quad 3y-5 = 0$

$y = -1, \quad y = \frac{5}{3}$

$\rightarrow \sqrt{x^2+5x+1} = -1, \quad \sqrt{x^2+5x+1} = \frac{5}{3}$

squaring, squaring

$\rightarrow x^2+5x+1 = 1 \qquad x^2+5x+1 = \frac{25}{9}$

$x^2+5x = 0 \qquad \rightarrow 9x^2+45x+9 = 25$

$x(x+5) = 0 \qquad 9x^2+45x+9-25 = 0$

$x = 0, \; x+5 = 0 \qquad 9x^2+45x-16 = 0$

$x = -5 \qquad 9x^2-3x+48x-16 = 0$

$3x(3x-1)+16(3x-1) = 0$

$(3x-1)(3x+16) = 0$

$3x-1 = 0, \; 3x+16 = 0$

$x = 1/3, \; x = -\frac{16}{3}$

On checking, 0 and -5 are extraneous roots so

$S.S = \{\frac{1}{3}, -\frac{16}{3}\}$

ii) The equations of the form:

$\sqrt{x+a} + \sqrt{x+b} = \sqrt{x+c}$

Questions related to this type

Example 2, Q 3, 4, 5

Example 2. Solve the equation:

$\sqrt{x+8} + \sqrt{x+3} = \sqrt{12x+13}$

Solution:- $\sqrt{x+8} + \sqrt{x+3} = \sqrt{12x+13}$

squaring both sides

$(\sqrt{x+8} + \sqrt{x+3})^2 = (\sqrt{12x+13})^2$

$(\sqrt{x+8})^2 + (\sqrt{x+3})^2 + 2(\sqrt{x+8})(\sqrt{x+3}) = 12x+13$

$x+8 + x+3 + 2\sqrt{(x+8)(x+3)} = 12x+13$

$\rightarrow 2x + 11 + 2\sqrt{x^2+3x+8x+24} = 12x+13$

$2\sqrt{x^2+11x+24} = 12x+13-2x-11$

$2\sqrt{x^2+11x+24} = 10x+2$

$\rightarrow \sqrt{x^2+11x+24} = 5x+1$

$\rightarrow (\sqrt{x^2+11x+24})^2 = (5x+1)^2$

$x^2+11x+24 = 25x^2+1+10x$

$\rightarrow 25x^2+10x+1-x^2-11x-24 = 0$

$24x^2 - x - 23 = 0$

$24x^2 - 24x + 23x - 23 = 0$

$24x(x-1) + 23(x-1) = 0$

$(x-1)(24x+23) = 0$

$x-1=0$, $24x+23=0$

$\rightarrow x=1$, $x=-\frac{23}{24}$

On checking $-\frac{23}{24}$ is extraneous root

so $\therefore S.S=\{1\}$

iii) The equations of the form:

$$\sqrt{ax^2+bx+c}+\sqrt{px^2+qx+r}=\sqrt{lx^2+mx+n}$$

where ax^2+bx+c, px^2+qx+r, lx^2+mx+n have common factor

Questions related to this type

Examp 3, Q 7, 8, 9

Example 3. Solve the equation:

$$\sqrt{x^2+4x-21}+\sqrt{x^2-x-6}=\sqrt{6x^2-5x-39}$$

Solution:-

$$\sqrt{x^2+4x-21}+\sqrt{x^2-x-6}=\sqrt{6x^2-5x-39}$$

$$\sqrt{x^2+7x-3x-21}+\sqrt{x^2+2x-3x-6}=\sqrt{6x^2-18x+13x-39}$$

$$\sqrt{x(x+7)-3(x+7)}+\sqrt{x(x+2)-3(x+2)}=\sqrt{6x(x-3)+13(x-3)}$$

$$\sqrt{(x+7)(x-3)}+\sqrt{(x+2)(x-3)}=\sqrt{(x-3)(6x+13)}$$

$$\sqrt{(x+7)(x-3)}+\sqrt{(x+2)(x-3)}-\sqrt{(x-3)(6x+13)}=0$$

$$\sqrt{x-3}\left\{\sqrt{x+7}+\sqrt{x+2}-\sqrt{6x+13}\right\}=0$$

$\sqrt{x-3}=0$, $\sqrt{x+7}+\sqrt{x+2}-\sqrt{6x+13}=0$

$\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$

$x-3=0$ squaring both sides

$x=3$ $\left(\sqrt{x+7}+\sqrt{x+2}\right)^2=\left(\sqrt{6x+13}\right)^2$

$$\rightarrow x+7+x+2+2(\sqrt{x+7})(\sqrt{x+2})=6x+13$$

$$2x+9+2\sqrt{(x+7)(x+2)}=6x+13$$

$$2\sqrt{x^2+2x+7x+14}=6x+13-2x-9$$

$$2\sqrt{x^2+9x+14}=4x+4$$

$$\rightarrow \sqrt{x^2+9x+14}=2x+2$$

$$\left(\sqrt{x^2+9x+14}\right)^2=(2x+2)^2$$

$$x^2+9x+14=4x^2+4+4x$$

$$4x^2+4x+4-x^2-9x-14=0$$

$$3x^2-x-10=0$$

$$\rightarrow 3x^2-6x+5x-10=0$$

$$3x(x-2)+5(x-2)=0$$

$(x-2)(3x+5)=0 \rightarrow x-2=0,\ 3x+5=0$

$x=2$, $x=-\frac{5}{3}$

on checking $-\frac{5}{3}$ is extraneous root

so $S.S=\{2, 3\}$

IV) The equations of the form:

$$\sqrt{ax^2+bx+c}+\sqrt{px^2+qx+r}=mx+n$$

where $(mx+n)$ is a factor of $(ax^2+bx+c)-(px^2+qx+r)$

Questions related to this type

Examp 4, Q 6, 11, 12

Example 4. Solve the equation:

$$\sqrt{3x^2-7x-30}-\sqrt{2x^2-7x-5}=x-5$$

Solution:-

$$\sqrt{3x^2-7x-30}-\sqrt{2x^2-7x-5}=x-5$$

Put $\sqrt{3x^2-7x-30}=a$

$\sqrt{2x^2-7x-5}=b$ so

$$a-b=x-5 \longrightarrow (i)$$

Now $a^2-b^2=\left(\sqrt{3x^2-7x-30}\right)^2-\left(\sqrt{2x^2-7x-5}\right)^2$

$$=3x^2-7x-30-(2x^2-7x-5)$$

$$=3x^2-7x-30-2x^2+7x+5$$

$$a^2-b^2=x^2-25$$

$$\rightarrow (a-b)(a+b)=(x-5)(x+5) \quad (ii)$$

By $\frac{(ii)}{(i)} \rightarrow \frac{(a-b)(a+b)}{a-b}=\frac{(x-5)(x+5)}{x-5}$

$$\rightarrow a+b=x+5 \longrightarrow (iii)$$

Now (i) + (iii) $\rightarrow 2a=2x$

$\rightarrow x=a$ put in (i)

(i) $\rightarrow a-b=a-5 \rightarrow -b=-5$

$\rightarrow b=5$

so $a=x$, $b=5$

$\sqrt{3x^2-7x-30}=x$, $\sqrt{2x^2-7x-5}=5$

$\rightarrow 3x^2-7x-30=x^2$, $2x^2-7x-5=25$

$3x^2-7x-30-x^2=0$, $2x^2-7x-5-25=0$

$\Rightarrow 2x^2-7x-30=0 \quad , \quad 2x^2-7x-30=0$

Both equations are same. so

By solving $2x^2-7x-30=0$

$\Rightarrow 2x^2-12x+5x-30=0$

$2x(x-6)+5(x-6)=0$

$(x-6)(2x+5)=0$

$\Rightarrow x-6=0, \quad 2x+5=0$

$x=6 \quad , \quad x=-\frac{5}{2}$

On checking $-\frac{5}{2}$ as extraneous root

so $S.S=\{6\}$

Exercise 4.3

Solve the following equations:

Q1. $3x^2+2x-\sqrt{3x^2+2x-1}=7$

Solution:-

$3x^2+2x-\sqrt{3x^2+2x-1}=7$

Put $\sqrt{3x^2+2x-1}=y$

$\Rightarrow (\sqrt{3x^2+2x-1})^2=y^2$ squaring

$\Rightarrow 3x^2+2x-1=y^2$

$\Rightarrow 3x^2+2x=y^2+1$ so

$y^2+1-y=7$

$\Rightarrow y^2-y+1-7=0$

$y^2-y-6=0$

$y^2-3y+2y-6=0$

$y(y-3)+2(y-3)=0$

$(y-3)(y+2)=0$

$y-3=0 \quad , \quad y+2=0$

$y=3 \quad , \quad y=-2$

$\Rightarrow \sqrt{3x^2+2x-1}=3 \quad , \quad \sqrt{3x^2+2x-1}=-2$

$3x^2+2x-1=9 \quad , \quad 3x^2+2x-1=4$

$3x^2+2x-1-9=0 \quad , \quad 3x^2+2x-1-4=0$

$3x^2+2x-10=0 \quad , \quad 3x^2+2x-5=0$

using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, $3x^2+5x-3x-5=0$

$\Rightarrow x=\frac{-2\pm\sqrt{(2)^2-4(3)(-10)}}{2(3)}$, $x(3x+5)-1(3x+5)=0$

$(3x+5)(x-1)=0$

$x=\frac{-2\pm\sqrt{4+120}}{6}$, $3x+5=0, \quad x-1=0$

$=\frac{-2\pm\sqrt{124}}{6}$, $x=-\frac{5}{3}, \quad x=1$

$=\frac{-2\pm\sqrt{4\times31}}{6}$

$=\frac{-2\pm2\sqrt{31}}{6}$

$x=\frac{2(-1\pm\sqrt{31})}{6}$

$x=\frac{-1\pm\sqrt{31}}{6}$

On checking 1 and $-\frac{5}{3}$ are extraneous roots. so

$S.S=\left\{\frac{-1\pm\sqrt{31}}{2}\right\}$

Q2. $x^2-\frac{x}{2}-7=x-3\sqrt{2x^2-3x+2}$

Solution:- $x^2-\frac{x}{2}-7=x-3\sqrt{2x^2-3x+2}$

$\Rightarrow 2x^2-x-14=2x-6\sqrt{2x^2-3x+2}$

$2x^2-x-2x-14+6\sqrt{2x^2-3x+2}=0$

$2x^2-3x-14+6\sqrt{2x^2-3x+2}=0$

Put $\sqrt{2x^2-3x+2}=y$

$\Rightarrow 2x^2-3x+2=y^2$

$\Rightarrow 2x^2-3x=y^2-2$ so

$y^2-2-14+6y=0$

$\Rightarrow y^2+6y-16=0$

$y^2+8y-2y-16=0$

$y(y+8)-2(y+8)=0$

$(y+8)(y-2)=0$

$y+8=0 \quad , \quad y-2=0$

$y=-8 \quad , \quad y=2$

$\Rightarrow \sqrt{2x^2-3x+2}=-8 \quad , \quad \sqrt{2x^2-3x+2}=2$

$\Rightarrow 2x^2-3x+2=64 \quad , \quad \Rightarrow 2x^2-3x+2=4$

$2x^2-3x+2-64=0 \quad , \quad 2x^2-3x+2-4=0$

$2x^2-3x-62=0 \quad , \quad 2x^2-3x-2=0$

using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, $2x^2-4x+x-2=0$

$x=\frac{-(-3)\pm\sqrt{(-3)^2-4(2)(-62)}}{2(2)}$, $2x(x-2)+1(x-2)=0$

$(x-2)(2x+1)=0$

$x=\frac{3\pm\sqrt{9+496}}{4}$, $x-2=0, \quad 2x+1=0$

$x=2, \quad x=-\frac{1}{2}$

$x=\frac{3\pm\sqrt{505}}{4}$ on checking $\frac{3\pm\sqrt{505}}{4}$

is extraneous root. so

$S.S = \{2, -\frac{1}{2}\}$

Q3. $\sqrt{2x+8} + \sqrt{x+5} = 7$

Solution:- $\sqrt{2x+8} + \sqrt{x+5} = 7$

squaring both sides

$(\sqrt{2x+8} + \sqrt{x+5})^2 = (7)^2$

$\rightarrow 2x+8 + x+5 + 2(\sqrt{2x+8})(\sqrt{x+5}) = 49$

$3x + 13 + 2\sqrt{2x^2+10x+8x+40} = 49$

$2\sqrt{2x^2+18x+40} = 49 - 3x - 13$

$2\sqrt{2x^2+18x+40} = 36 - 3x$ squaring

$\rightarrow (2\sqrt{2x^2+18x+40})^2 = (36-3x)^2$

$4(2x^2+18x+40) = 1296 + 9x^2 - 216x$

$8x^2 + 72x + 160 = 1296 + 9x^2 - 216x$

$\rightarrow 9x^2 - 8x^2 - 216x - 72x + 1296 - 160 = 0$

$\rightarrow x^2 - 288x + 1136 = 0$

$x^2 - 4x - 284x + 1136 = 0$

$x(x-4) - 284(x-4) = 0$

$(x-4)(x-284) = 0$

$\rightarrow x - 4 = 0, \quad x - 284 = 0$

$x = 4, \quad x = 284$

On checking 284 is extraneous root. so $S.S = \{4\}$

Q4. $\sqrt{3x+4} = 2 + \sqrt{2x-4}$

Solution:- $\sqrt{3x+4} = 2 + \sqrt{2x-4}$

$\rightarrow \sqrt{3x+4} - \sqrt{2x-4} = 2$

squaring both sides

$(\sqrt{3x+4} - \sqrt{2x-4})^2 = (2)^2$

$\rightarrow 3x+4 + 2x-4 - 2(\sqrt{3x+4})(\sqrt{2x-4}) = 4$

$\rightarrow 5x - 2\sqrt{(3x+4)(2x-4)} = 4$

$\rightarrow 5x - 2\sqrt{6x^2-12x+8x-16} = 4$

$-2\sqrt{6x^2-4x-16} = 4 - 5x$

squaring both sides

$\rightarrow (-2\sqrt{6x^2-4x-16})^2 = (4-5x)^2$

$\rightarrow 4(6x^2-4x-16) = 16 + 25x^2 - 40x$

$\rightarrow 24x^2 - 16x - 64 = 16 + 25x^2 - 40x$

$\rightarrow 25x^2 - 40x + 16 - 24x^2 + 16x + 64 = 0$

$\rightarrow x^2 - 24x + 80 = 0$

$x^2 - 4x - 20x + 80 = 0$

$x(x-4) - 20(x-4) = 0$

$(x-4)(x-20) = 0$

$x - 4 = 0, \quad x - 20 = 0$

$x = 4, \quad x = 20$

On checking no root is extraneo root so $S.S = \{4, 20\}$

Q5. $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

Solution:-

$\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

squaring both sides

$(\sqrt{x+7} + \sqrt{x+2})^2 = (\sqrt{6x+13})^2$

$\rightarrow x+7 + x+2 + 2(\sqrt{x+7})(\sqrt{x+2}) = 6x + 13$

$2x + 9 + 2\sqrt{(x+7)(x+2)} = 6x + 13$

$\rightarrow 2\sqrt{x^2+2x+7x+14} = 6x + 13 - 2x - 9$

$2\sqrt{x^2+9x+14} = 4x + 4$

$\rightarrow \sqrt{x^2+9x+14} = 2x + 2$ (÷ by 2)

squaring

$(\sqrt{x^2+9x+14})^2 = (2x+2)^2$

$\rightarrow x^2 + 9x + 14 = 4x^2 + 4 + 8x$

$\rightarrow 4x^2 + 4 + 8x - x^2 - 9x - 14 = 0$

$3x^2 - x - 10 = 0$

$3x^2 - 6x + 5x - 10 = 0$

$3x(x-2) + 5(x-2) = 0$

$(x-2)(3x+5) = 0$

$x-2=0$, $3x+5=0$

$x=2$, $x=-\frac{5}{3}$

On checking $-5/3$ is extraneous root. so $S.S=\{2\}$

Q6. $\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1$

Solution:- $\sqrt{x^2+x+1}-\sqrt{x^2+x-1}=1$

Put $\sqrt{x^2+x+1}=a$

$\sqrt{x^2+x-1}=b$ so

$a-b=1 \longrightarrow (i)$

Now $a^2-b^2=(\sqrt{x^2+x+1})^2-(\sqrt{x^2+x-1})^2$

$a^2-b^2=x^2+x+1-(x^2+x-1)$

$a^2-b^2=x^2+x+1-x^2-x+1$

$\rightarrow (a-b)(a+b)=2 \longrightarrow (ii)$

By $\frac{(ii)}{(i)} \rightarrow \frac{(a-b)(a+b)}{a-b}=\frac{2}{1}$

$\rightarrow a+b=2 \longrightarrow (iii)$

By $(i)+(iii) \rightarrow 2a=3$

$a=\frac{3}{2}$ put in (i)

$\frac{3}{2}-b=1 \rightarrow b=\frac{3}{2}-1=\frac{1}{2}$

Now

$a=\frac{3}{2}$, $b=\frac{1}{2}$

$\sqrt{x^2+x+1}=\frac{3}{2}$, $\sqrt{x^2+x-1}=\frac{1}{2}$

$\rightarrow x^2+x+1=\frac{9}{4}$, $x^2+x-1=\frac{1}{4}$

$\rightarrow 4x^2+4x+4=9$, $4x^2+4x-4=1$

$4x^2+4x+4-9=0$, $4x^2+4x-4-1=0$

$4x^2+4x-5=0$, $4x^2+4x-5=0$

Solving any one

$4x^2+4x-5=0$

using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-4\pm\sqrt{(4)^2-4(4)(-5)}}{2(4)}$

$=\frac{-4\pm\sqrt{16+80}}{8}$

$=\frac{-4\pm\sqrt{96}}{8}=\frac{-4\pm\sqrt{16\times6}}{8}$

$x=\frac{-4\pm4\sqrt{6}}{8}=\frac{4(-1\pm\sqrt{6})}{8}$

$x=\frac{-1\pm\sqrt{6}}{2}$

On checking no root is extraneous

so $S.S=\left\{\frac{-1\pm\sqrt{6}}{2}\right\}$

Q7. $\sqrt{x^2+2x-3}+\sqrt{x^2+7x-8}=\sqrt{5(x^2+3x-4)}$

Solution:-

$\sqrt{x^2+2x-3}+\sqrt{x^2+7x-8}=\sqrt{5(x^2+3x-4)}$

Solution:-

$\sqrt{x^2+2x-3}+\sqrt{x^2+7x-8}=\sqrt{5(x^2+3x-4)}$

$\sqrt{x^2-x+3x-3}+\sqrt{x^2-x+8x-8}=\sqrt{5(x^2-x+4x-4)}$

$\sqrt{x(x-1)+3(x-1)}+\sqrt{x(x-1)+8(x-1)}=\sqrt{5[x(x-1)+4(x-1)]}$

$\sqrt{(x-1)(x+3)}+\sqrt{(x-1)(x+8)}=\sqrt{5(x-1)(x+4)}$

$\sqrt{(x-1)(x+3)}+\sqrt{(x-1)(x+8)}-\sqrt{5(x-1)(x+4)}=0$

$\sqrt{(x-1)}\{\sqrt{x+3}+\sqrt{x+8}-\sqrt{5(x+4)}\}=0$

$\sqrt{x-1}=0$, $\sqrt{x+3}+\sqrt{x+8}-\sqrt{5(x+4)}$

$x-1=0$,

$x=1$ $\sqrt{x+3}+\sqrt{x+8}=\sqrt{5(x+4)}$

squaring both sides

$\rightarrow (\sqrt{x+3}+\sqrt{x+8})^2=(\sqrt{5(x+4)})^2$

$\rightarrow x+3+x+8+2(\sqrt{x+3})(\sqrt{x+8})=5(x+4)$

$2x+11+2\sqrt{(x+3)(x+8)}=5x+20$

$\rightarrow 2\sqrt{x^2+8x+3x+24}=5x+20-2x-11$

$2\sqrt{x^2+11x+24}=3x+9$

$(2\sqrt{x^2+11x+24})^2=(3x+9)^2$ squaring

$4(x^2+11x+24)=9x^2+81+54x$

$4x^2+44x+96=9x^2+81+54x$

$\rightarrow 9x^2+81+54x-4x^2-44x-96=0$

$\Rightarrow 5x^2+10x-15=0$

$\Rightarrow x^2+2x-3=0 \quad (\div \text{ by } 5)$

$x^2-x+3x-3=0$

$x(x-1)+3(x-1)=0$

$(x-1)(x+3)=0$

$x-1=0, \quad x+3=0$

$x=1, \quad x=-3$

On checking no root is extraneous root. so $S.S=\{1,-3\}$

Q8. $\sqrt{2x^2-5x-3}+3\sqrt{2x+1}=\sqrt{2x^2+25x+12}$

Solution:-

$\sqrt{2x^2-5x-3}+3\sqrt{2x+1}=\sqrt{2x^2+25x+12}$

$\sqrt{2x^2+x-6x-3}+3\sqrt{2x+1}=\sqrt{2x^2+x+24x+12}$

$\sqrt{x(2x+1)-3(2x+1)}+3\sqrt{2x+1}=\sqrt{x(2x+1)+12(2x+1)}$

$\sqrt{(2x+1)(x-3)}+3\sqrt{2x+1}=\sqrt{(x+12)(2x+1)}$

$\sqrt{(2x+1)(x-3)}+3\sqrt{2x+1}-\sqrt{(x+12)(2x+1)}=0$

$\sqrt{2x+1}\{\sqrt{x-3}+3-\sqrt{x+12}\}=0$

$\sqrt{2x+1}=0, \quad \sqrt{x-3}+3-\sqrt{x+12}=0$

$2x+1=0, \quad \sqrt{x-3}+3=\sqrt{x+12}$

$x=-\frac{1}{2}$ squaring both sides

$(\sqrt{x-3}+3)^2=(\sqrt{x+12})^2$

$\Rightarrow x-3+9+2(3)\sqrt{x-3}=x+12$

$\Rightarrow x+6+6\sqrt{x-3}=x+12$

$6\sqrt{x-3}=x+12-x-6$

$6\sqrt{x-3}=6$

$\Rightarrow (6\sqrt{x-3})^2=(6)^2$ squaring

$\Rightarrow 36(x-3)=36$

$\Rightarrow x-3=1 \Rightarrow x=1+3$

$\Rightarrow x=4$

On checking no root is extraneous

so $S.S=\{-\frac{1}{2},4\}$

Q9. $\sqrt{3x^2-5x+2}+\sqrt{6x^2-11x+5}=\sqrt{5x^2-9x+4}$

Solution:-

$\sqrt{3x^2-5x+2}+\sqrt{6x^2-11x+5}=\sqrt{5x^2-9x+4}$

$\sqrt{3x^2-3x-2x+2}+\sqrt{6x^2-6x-5x+5}=\sqrt{5x^2-5x-4x+4}$

$\sqrt{3x(x-1)-2(x-1)}+\sqrt{6x(x-1)-5(x-1)}=\sqrt{5x(x-1)-4(x-1)}$

$\sqrt{(x-1)(3x-2)}+\sqrt{(x-1)(6x-5)}=\sqrt{(x-1)(5x-4)}$

$\sqrt{(x-1)(3x-2)}+\sqrt{(x-1)(6x-5)}-\sqrt{(x-1)(5x-4)}=0$

$\sqrt{x-1}\{\sqrt{3x-2}+\sqrt{6x-5}-\sqrt{5x-4}\}=0$

$\sqrt{x-1}=0, \quad \sqrt{3x-2}+\sqrt{6x-5}-\sqrt{5x-4}=0$

$x-1=0, \quad \sqrt{3x-2}+\sqrt{6x-5}=\sqrt{5x-4}$

$x=1, \quad$ squaring

$(\sqrt{3x-2}+\sqrt{6x-5})^2=(\sqrt{5x-4})^2$

$\Rightarrow 3x-2+6x-5+2(\sqrt{3x-2})(\sqrt{6x-5})=5x-4$

$9x-7+2\sqrt{(3x-2)(6x-5)}=5x-4$

$2\sqrt{18x^2-15x-12x+10}=5x-4-9x+7$

$2\sqrt{18x^2-27x+10}=3-4x$

squaring

$(2\sqrt{18x^2-27x+10})^2=(3-4x)^2$

$4(18x^2-27x+10)=9+16x^2-24x$

$72x^2-108x+40=9+16x^2-24x$

$\Rightarrow 72x^2-108x+40-9-16x^2+24x=0$

$56x^2-84x+31=0$

using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-(-84)\pm\sqrt{(-84)^2-4(56)(31)}}{2(56)}$

$x=\frac{84\pm\sqrt{7056-6944}}{112}$

$x=\frac{84\pm\sqrt{112}}{112}=\frac{84\pm\sqrt{16\times7}}{112}$

$x=\frac{84\pm4\sqrt{7}}{112}=\frac{4(21\pm\sqrt{7})}{112}$

$\Rightarrow x = \frac{21 \pm \sqrt{7}}{28}$

On checking no root is extraneous

so $S.S = \left\{ \frac{21 \pm \sqrt{7}}{28} \right\}$

Q10. $(x+4)(x+1) = \sqrt{x^2+2x-15} + 3x + 31$

Solution:-

$(x+4)(x+1) = \sqrt{x^2+2x-15} + 3x + 31$

$\Rightarrow x^2 + x + 4x + 4 = \sqrt{x^2+2x-15} + 3x + 31$

$\Rightarrow x^2 + 5x + 4 - \sqrt{x^2+2x-15} - 3x - 31 = 0$

$x^2 + 2x - 27 - \sqrt{x^2+2x-15} = 0$

Put $\sqrt{x^2+2x-15} = y$

$\Rightarrow x^2 + 2x - 15 = y^2$

$x^2 + 2x = y^2 + 15$ so

$y^2 + 15 - 27 - y = 0$

$\Rightarrow y^2 - y - 12 = 0$

$y^2 - 4y + 3y - 12 = 0$

$y(y-4) + 3(y-4) = 0$

$(y-4)(y+3) = 0$

$y - 4 = 0$, $y + 3 = 0$

$y = 4$, $y = -3$

$\Rightarrow \sqrt{x^2+2x-15} = 4$, $\sqrt{x^2+2x-15} = -3$

$\Rightarrow x^2 + 2x - 15 = 16$, $x^2 + 2x - 15 = 9$

$x^2 + 2x - 15 - 16 = 0$, $x^2 + 2x - 15 - 9 = 0$

$x^2 + 2x - 31 = 0$, $x^2 + 2x - 24 = 0$

using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-31)}}{2(1)}$

$x = \frac{-2 \pm \sqrt{4 + 124}}{2}$

$x = \frac{-2 \pm \sqrt{128}}{2} = \frac{-2 \pm \sqrt{64 \times 2}}{2}$

$x = \frac{-2 \pm 8\sqrt{2}}{2} = \frac{2(-1 \pm 4\sqrt{2})}{2}$

$x = -1 \pm 4\sqrt{2}$

$x^2 - 4x + 6x - 24 = 0$

$x(x-4) + 6(x-4) = 0$

$(x-4)(x+6) = 0$

$x - 4 = 0$, $x + 6 = 0$

$x = 4$, $x = -6$

On checking 4 and -6 are extraneous

so $S.S = \{-1 \pm 4\sqrt{2}\}$

Q11. $\sqrt{3x^2-2x+9} - \sqrt{3x^2-2x-4} = 13$

Solution:-

$\sqrt{3x^2-2x+9} - \sqrt{3x^2-2x-4} = 13$

Put $\sqrt{3x^2-2x+9} = a$, $\sqrt{3x^2-2x-4} = b$

so $a + b = 13 \longrightarrow (i)$

Now $a^2 - b^2 = \left(\sqrt{3x^2-2x+9}\right)^2 - \left(\sqrt{3x^2-2x-4}\right)^2$

$= 3x^2 - 2x + 9 - 3x^2 + 2x + 4$

$a^2 - b^2 = 13$

$\Rightarrow (a-b)(a+b) = 13 \longrightarrow (ii)$

By $\frac{(ii)}{(i)} \Rightarrow \frac{(a-b)(a+b)}{a+b} = \frac{13}{13}$

$\Rightarrow a - b = 1 \longrightarrow (iii)$

By (i)+(iii) $\Rightarrow 2a = 14 \Rightarrow a = 7$ put in (i)

so (i) $\Rightarrow 7 + b = 13 \Rightarrow b = 6$ so

$a = 7$, $b = 6$

$\sqrt{3x^2-2x+9} = 7$, $\sqrt{3x^2-2x-4} = 6$

$3x^2 - 2x + 9 = 49$, $3x^2 - 2x - 4 = 36$

$3x^2 - 2x + 9 - 49 = 0$, $3x^2 - 2x - 4 - 36 = 0$

$3x^2 - 2x - 40 = 0$, $3x^2 - 2x - 40 = 0$

Solving any one so

$3x^2 - 2x - 40 = 0$

$3x^2 - 12x + 10x - 40 = 0$

$3x(x-4) + 10(x-4) = 0$

$(x-4)(3x+10) = 0$

$x - 4 = 0$, $3x + 10 = 0$

$x = 4$, $x = -\frac{10}{3}$

On checking no root is extraneous

so $S.S = \left\{4, -\frac{10}{3}\right\}$

Q12. $\sqrt{5x^2+7x+2}-\sqrt{4x^2+7x+18}=x-4$

Solution:-

$\sqrt{5x^2+7x+2}-\sqrt{4x^2+7x+18}=x-4$

Put $a=\sqrt{5x^2+7x+2}$ and

$b=\sqrt{4x^2+7x+18}$ so

$a-b=x-4 \longrightarrow (i)$

Now $a^2-b^2=\left(\sqrt{5x^2+7x+2}\right)^2-\left(\sqrt{4x^2+7x+18}\right)^2$

$a^2-b^2=5x^2+7x+2-(4x^2+7x+18)$

$a^2-b^2=5x^2+7x+2-4x^2-7x-18$

$a^2-b^2=x^2-16$

$(a-b)(a+b)=(x-4)(x+4) \longrightarrow (ii)$

By $\frac{(ii)}{(i)} \rightarrow \frac{(a-b)(a+b)}{a-b}=\frac{(x-4)(x+4)}{x-4}$

$\rightarrow a+b=x+4 \longrightarrow (iii)$

By (i) + (iii) $\rightarrow 2a=2x \rightarrow a=x$
Put in (i)

$x-b=x-4 \rightarrow b=4$ so

$a=x$, $b=4$

$\sqrt{5x^2+7x+2}=x$, $\sqrt{4x^2+7x+18}=4$

$\rightarrow 5x^2+7x+2=x^2$, $4x^2+7x+18=16$

$5x^2-x^2+7x+2=0$, $4x^2+7x+18-16=0$

$4x^2+7x+2=0$, $4x^2+7x+2=0$

Solving any one

$4x^2+7x+2=0$

using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\frac{-7\pm\sqrt{(7)^2-4(4)(2)}}{2(4)}$

$x=\frac{-7\pm\sqrt{49-32}}{8}=\frac{-7\pm\sqrt{17}}{8}$

On checking no root is extraneous. so

$S.S=\left\{\frac{-7\pm\sqrt{17}}{8}\right\}$

Three Cube Roots of Unity

Let x be cube root of unity. then

$x=\sqrt[3]{1} \rightarrow x=(1)^{1/3}$

$\rightarrow (x)^3=\left[(1)^{1/3}\right]^3 \rightarrow x^3=1$

$\rightarrow x^3-1=0 \rightarrow (x)^3-(1)^3=0$

$(x-1)(x^2+x+1)=0$ $\because a^3-b^3=(a-b)(a^2+ab+b^2)$

$x-1=0$, $x^2+x+1=0$

$x=1$, using $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-1\pm\sqrt{(1)^2-4(1)(1)}}{2(1)}$

$x=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}$

$\rightarrow x=\frac{-1\pm\sqrt{3}\sqrt{-1}}{2}=\frac{-1\pm\sqrt{3}i}{2}$ $\because \sqrt{-1}=i$

$x=\frac{-1+\sqrt{3}i}{2}$, $x=\frac{-1-\sqrt{3}i}{2}$

Hence three cube roots of unity are 1, $\frac{-1+\sqrt{3}i}{2}$, $\frac{-1-\sqrt{3}i}{2}$

Important note

"We know that the numbers containing i are called complex nos. so $\frac{-1+\sqrt{3}i}{2}$, $\frac{-1-\sqrt{3}i}{2}$ are complex or imaginary roots of cube roots of unity."

* Every non-zero real number has one real and two complex cube roots.
* Let $\frac{-1+\sqrt{3}i}{2}=\omega$, $\frac{-1-\sqrt{3}i}{2}=\omega^2$

so $1,\omega,\omega^2$ also called cube roots of unity.

Properties of Cube Roots of Unity

i) Each complex cube root of unity is square of the other.

Proof:- we know that complex cube roots of unity are $\frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}}{2}i$

Let $\omega = \frac{-1+\sqrt{3}i}{2}$

$\rightarrow \omega^2 = \left(\frac{-1+\sqrt{3}i}{2}\right)^2$ squaring

$\omega^2 = \frac{(-1)^2+(\sqrt{3}i)^2+2(-1)(\sqrt{3}i)}{4}$

$\omega^2 = \frac{1+3i^2-2\sqrt{3}i}{4} = \frac{1+3(-1)-2\sqrt{3}i}{4}$

$\omega^2 = \frac{1-3-2\sqrt{3}i}{4} = \frac{-2-2\sqrt{3}i}{2}$

$\omega^2 = 2\left(\frac{-1-\sqrt{3}i}{4}\right) = \frac{-1-\sqrt{3}i}{2}$

which is other root.

Hence proved

ii) The sum of all the three cube roots of unity is zero

i-e., $1+\omega+\omega^2 = 0$

Proof:- we know that three cube roots of unity are

$1,\ \omega = \frac{-1+\sqrt{3}i}{2}$ and $\omega^2 = \frac{-1-\sqrt{3}i}{2}$

Now

$1+\omega+\omega^2 = 1+\frac{-1+\sqrt{3}i}{2}+\frac{-1-\sqrt{3}i}{2}$

$= \frac{2-1+\sqrt{3}i-1-\sqrt{3}i}{2} = \frac{2-2}{2}$

$\rightarrow 1+\omega+\omega^2 = 0$ Hence proved

iii) Product of all the three cube roots of unity is unity

i-e., $\omega^3 = 1$

Proof:- we know that three cube roots of unity are

$1,\ \omega = \frac{-1+\sqrt{3}i}{2},\ \frac{-1-\sqrt{3}i}{2} = \omega^2$

so $(1)(\omega)(\omega^2) = 1\left(\frac{-1+\sqrt{3}i}{2}\right)\left(\frac{-1-\sqrt{3}i}{2}\right)$

$\rightarrow \omega^3 = \frac{(-1+\sqrt{3}i)(-1-\sqrt{3}i)}{4}$

$\omega^3 = \frac{(-1)^2-(\sqrt{3}i)^2}{4}$

$\omega^3 = \frac{1-3i^2}{4}$ $\because a^2-b^2 = (a-b)(a+b)$

$\omega^3 = \frac{1+3}{4} = \frac{4}{4}$ $i^2 = -1$

$\omega^3 = 1$

Hence proved

IV) For any $n \in Z$, ω^n is equivalent to one of the cube roots of unity.

Proof:-

$\omega^4 = \omega^3.\omega = (1)\omega = \omega$

$\omega^5 = \omega^3.\omega^2 = (1)\omega^2 = \omega^2$

$\omega^6 = \omega^3.\omega^3 = (1)(1) = 1$

$\omega^{15} = (\omega^3)^5 = (1)^5 = 1$

$\omega^{27} = (\omega^3)^9 = (1)^9 = 1$

$\omega^{11} = \omega^9.\omega = (\omega^3)^3.\omega = (1)^3.\omega = \omega$

$\omega^{-1} = \omega^{-3}.\omega^2 = (\omega^3)^{-1}.\omega^2 = (1)^{-1}.\omega^2 = \omega$

$\omega^{-5} = \omega^{-6}.\omega = (\omega^3)^{-2}.\omega = (1)^{-2}.\omega = \omega$

$\omega^{-12} = (\omega^3)^{-4} = (1)^{-4} = 1$ Hence proved.

Example 1. Prove that :

$(x+y)^3 = (x+y)(x+\omega y)(x+\omega^2 y)$

Solution:-

$R.H.S = (x+y)(x+\omega y)(x+\omega^2 y)$

$= (x+y)[x^2+\omega^2 xy+\omega xy+\omega^3 y^2]$

$= (x+y)\{x^2+xy(\omega^2+\omega)+\omega^3 y^2\}$

$= (x+y)(x^2+xy(-1)+(1)y^2)$

$= (x+y)(x^2-xy+y^2)$

$= x^3+y^3$ $\because 1+\omega+\omega^2 = 0$

$= L.H.S$ $\rightarrow \omega+\omega^2 = -1$

and $\omega^3 = 1$

Hence proved

Example 2. Prove that:

$(-1+\sqrt{-3})^4+(-1-\sqrt{-3})^4 = -16$

Solution:-

$L.H.S = (-1+\sqrt{-3})^4+(-1-\sqrt{-3})^4$

$= (-1+\sqrt{3}\sqrt{-1})^4+(-1-\sqrt{3}\sqrt{-1})^4$

$= (-1+\sqrt{3}i)^4+(-1-\sqrt{3}i)^4$ $\because \sqrt{-1} = i$

$= \left[2\left(\frac{-1+\sqrt{3}i}{2}\right)\right]^4+\left[2\left(\frac{-1-\sqrt{3}i}{2}\right)\right]^4$

$= (2\omega)^4 + (2\omega^2)^4$ $\quad \because \omega = \frac{-1+\sqrt{3}i}{2}$, $\omega^2 = \frac{-1-\sqrt{3}i}{2}$

$= 16\omega^4 + 16\omega^8$

$= 16(\omega^4 + \omega^8)$

$= 16[\omega^3.\omega + \omega^6.\omega^2]$

$= 16[(1)\omega + (\omega^3)^2.\omega^2]$

$= 16(\omega + (1)^2.\omega^2)$

$= 16(\omega + \omega^2)$ $\quad \because 1+\omega+\omega^2=0 \rightarrow \omega+\omega^2=-1$

$= 16(-1) = -16$

$=$ R.H.S

Hence proved

Four Fourth Roots of Unity

Let x be the fourth root of unity $\therefore x = \sqrt[4]{1} = (1)^{1/4}$

$\rightarrow x^4 = [(1)^{1/4}]^4 \rightarrow x^4 = 1$

$\rightarrow x^4 - 1 = 0 \rightarrow (x^2)^2 - (1)^2 = 0$

$(x^2-1)(x^2+1) = 0$

$\rightarrow x^2 - 1 = 0$, $\quad x^2 + 1 = 0$

$x^2 = 1$, $\quad x^2 = -1$

$x = \pm 1$, $\quad x = \pm\sqrt{-1} = \pm i$

so four fourth roots of unity are $1, -1, i, -i$

Properties of four Fourth Roots of Unity

i) Sum of four forth roots of unity is zero i-e.,

$1 + (-1) + i + (-i) = 0$

ii) The real fourth roots of unity are additive inverses of each other i-e., $1+(-1) = (-1)+1 = 0$

iii) Both the complex fourth roots of unity are conjugate of each other. i-e.,

conjugate of $i = -i$

conjugate of $-i = i$

iv) Product of all the fourth roots of unity is -1 i-e.,

$1\times(-1)\times i\times(-i)$

$= -1\times(-i^2) = i^2 = -1$

Exercise 4.4

Q1. Find the three cube roots of: 8, -8, 27, -27, 64.

Solution:- i) 8

Let x be cube root of 8

$\rightarrow x = \sqrt[3]{8} \rightarrow x = (8)^{1/3}$

$\rightarrow x^3 = [(8)^{1/3}]^3 \rightarrow x^3 = 8$

$x^3 - 8 = 0 \rightarrow (x)^3 - (2)^3 = 0$

$(x-2)(x^2+2x+4) = 0$ $\quad \because a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$x-2=0$, $\quad x^2+2x+4=0$

$x=2$ $\quad$ using $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x = \frac{-2\pm\sqrt{(2)^2-4(1)(4)}}{2(1)}$

$x = \frac{-2\pm\sqrt{4-16}}{2}$

$x = \frac{-2\pm\sqrt{-12}}{2}$

$x = \frac{-2\pm\sqrt{4\times(-3)}}{2}$

$\rightarrow x = \frac{-2\pm2\sqrt{-3}}{2} = 2\left(\frac{-1\pm\sqrt{3}\sqrt{-1}}{2}\right)$

$x = 2\left(\frac{-1\pm\sqrt{3}i}{2}\right)$

$\rightarrow x = 2\left(\frac{-1+\sqrt{3}i}{2}\right)$, $x = 2\left(\frac{-1-\sqrt{3}i}{2}\right)$

or $x = 2\omega$, $\quad x = 2\omega^2$

Hence cube roots of 8 are $2, 2\omega, 2\omega^2$

ii) -8

Let x be cube roots of -8. so

$x = \sqrt[3]{-8} \rightarrow x = (-8)^{1/3}$

$(x)^3 = [(-8)^{1/3}]^3 \rightarrow x^3 = -8$

$x^3 + 8 = 0 \rightarrow (x)^3 + (2)^3 = 0$

$(x+2)(x^2-2x+4) = 0$ $\quad a^3+b^3 = (a+b)(a^2-ab+b^2)$

$x+2=0$, $\quad x^2-2x+4=0$

$x=-2$, $\quad$ using $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{-(-2) \pm \sqrt{4 - 16}}{2}$$

$$x = \frac{-(-2) \pm \sqrt{-12}}{2} = \frac{-(-2) \pm \sqrt{4 \times (-3)}}{2}$$

$$x = \frac{-(-2) \pm 2\sqrt{-3}}{2} = -2\left(\frac{-1 \pm \sqrt{3}i}{2}\right)$$

$$x = -2\left(\frac{-1 + \sqrt{3}i}{2}\right), \quad x = -2\left(\frac{-1 - \sqrt{3}i}{2}\right)$$

$$x = -2\omega, \quad x = -2\omega^2$$

Hence cube roots of -8 are $-2, -2\omega, -2\omega^2$

iii) 27

Let x be cube roots of 27.

$$x = \sqrt[3]{27} \rightarrow x = (27)^{\frac{1}{3}}$$

$$\rightarrow (x)^3 = \left[(27)^{\frac{1}{3}}\right]^3 \rightarrow x^3 = 27$$

$$\rightarrow x^3 - 27 = 0 \rightarrow (x)^3 - (3)^3 = 0$$

$$(x-3)(x^2 + 3x + 9) = 0$$

$x - 3 = 0$, $x^2 + 3x + 9 = 0$

$x = 3$, using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{-3 \pm \sqrt{9 - 36}}{2}$$

$$x = \frac{-3 \pm \sqrt{-27}}{2} = \frac{-3 \pm \sqrt{9 \times (-3)}}{2}$$

$$x = \frac{-3 \pm 3\sqrt{-3}}{2} = 3\left(\frac{-1 \pm \sqrt{3}i}{2}\right)$$

$$x = 3\left(\frac{-1 + \sqrt{3}i}{2}\right), \quad x = 3\left(\frac{-1 - \sqrt{3}i}{2}\right)$$

$$x = 3\omega, \quad x = 3\omega^2$$

Hence cube roots of 27 are $3, 3\omega, 3\omega^2$

iv) -27

Let x be cube roots of -27.

$$x = \sqrt[3]{-27} \rightarrow x = (-27)^{\frac{1}{3}}$$

$$(x)^3 = \left[(-27)^{\frac{1}{3}}\right]^3 \rightarrow x^3 = -27$$

$$x^3 + 27 = 0$$

$$(x)^3 + (3)^3 = 0$$

$$(x+3)(x^2 + 9 - 3x) = 0$$

$x + 3 = 0$, $x^2 - 3x + 9 = 0$

$x = -3$, using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{-(-3) \pm \sqrt{9 - 36}}{2}$$

$$x = \frac{-(-3) \pm \sqrt{-27}}{2} = \frac{-(-3) \pm \sqrt{9 \times (-3)}}{2}$$

$$x = \frac{-(-3) \pm 3\sqrt{-3}}{2} = -3\left(\frac{-1 \pm \sqrt{3}i}{2}\right)$$

$$x = -3\left(\frac{-1 + \sqrt{3}i}{2}\right), \quad x = -3\left(\frac{-1 - \sqrt{3}i}{2}\right)$$

$$x = -3\omega, \quad x = -3\omega^2$$

Hence cube roots of -27 are $-3, -3\omega, -3\omega^2$

v) 64

Let x be cube root of 64.

$$\rightarrow x = \sqrt[3]{64} \rightarrow x = (64)^{\frac{1}{3}}$$

$$\rightarrow x^3 = \left[(64)^{\frac{1}{3}}\right]^3 \rightarrow x^3 = 64$$

$$\rightarrow x^3 - 64 = 0 \rightarrow (x)^3 - (4)^3 = 0$$

$$(x-4)(x^2 + 4x + 16) = 0$$

$x - 4 = 0$, $x^2 + 4x + 16 = 0$

$x = 4$, using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(16)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 - 64}}{2}$$

$$x = \frac{-4 \pm \sqrt{-48}}{2} = \frac{-4 \pm \sqrt{16 \times (-3)}}{2}$$

$$x = \frac{-4 \pm 4\sqrt{-3}}{2} = 4\left(\frac{-1 \pm \sqrt{3}i}{2}\right)$$

$$x = 4\left(\frac{-1 + \sqrt{3}i}{2}\right), \quad x = 4\left(\frac{-1 - \sqrt{3}i}{2}\right)$$

$$x = 4\omega, \quad x = 4\omega^2$$

Hence cube roots of 64 are $4, 4\omega, 4\omega^2$

Q2. Evaluate:

i) $(1+\omega-\omega^2)^8$

Solution:-

$(1+\omega-\omega^2)^8$

$= [(1+\omega)-\omega^2]^8$ $\quad \because 1+\omega+\omega^2=0 \Rightarrow 1+\omega=-\omega^2$

$= (-\omega^2-\omega^2)^8$

$= (-2\omega^2)^8 = (-2)^8(\omega^2)^8 = 256\omega^{16}$

$= 256\,\omega^{15}\cdot\omega = 256(\omega^3)^5\cdot\omega$

$= 256(1)^5\cdot\omega = 256\omega$ $\quad \because \omega^3=1$

ii) $(1+\omega-\omega^2)(1-\omega+\omega^2)$

Solution:-

$(1+\omega-\omega^2)(1-\omega+\omega^2)$

$= [(1+\omega)-\omega^2][(1+\omega^2)-\omega]$ $\quad \because 1+\omega+\omega^2=0,\ 1+\omega=-\omega^2$ and $1+\omega+\omega^2=0,\ 1+\omega^2=-\omega$

$= (-\omega^2-\omega^2)(-\omega-\omega)$

$= (-2\omega^2)(-2\omega)$

$= 4\omega^3 = 4(1) = 4$ $\quad \because \omega^3=1$

iii) $\omega^{28}+\omega^{29}+1$

Solution:-

$\omega^{28}+\omega^{29}+1$

$= \omega^{27}\cdot\omega + \omega^{27}\cdot\omega^2+1$

$= \omega^{27}(\omega+\omega^2)+1$ $\quad \because 1+\omega+\omega^2=0,\ \omega+\omega^2=-1$

$= (\omega^3)^9(-1)+1$

$= (1)^9(-1)+1 = 1(-1)+1$

$= -1+1 = 0$

iv) $\left(\frac{-1+\sqrt{-3}}{2}\right)^7+\left(\frac{-1-\sqrt{-3}}{2}\right)^7$

Solution:-

$\left(\frac{-1+\sqrt{-3}}{2}\right)^7+\left(\frac{-1-\sqrt{-3}}{2}\right)^7$

$= \left(\frac{-1+\sqrt{3}i}{2}\right)^7+\left(\frac{-1-\sqrt{3}i}{2}\right)^7$

$= \omega^7 + (\omega^2)^7 = \omega^7+\omega^{14}$

$= \omega^6\cdot\omega + \omega^{12}\cdot\omega^2$

$= (\omega^3)^2\cdot\omega + (\omega^3)^4\cdot\omega^2$

$= (1)^2\cdot\omega + (1)^4\cdot\omega^2$

$= \omega+\omega^2 = -1$

v) $(-1+\sqrt{-3})^5+(-1-\sqrt{-3})^5$

Solution:-

$(-1+\sqrt{-3})^5+(-1-\sqrt{-3})^5$

$= (-1+\sqrt{3}i)^5+(-1-\sqrt{3}i)^5$

$\because \frac{-1+\sqrt{3}i}{2}=\omega \Rightarrow -1+\sqrt{3}i=2\omega$

and $\frac{-1-\sqrt{3}i}{2}=\omega^2 \Rightarrow -1-\sqrt{3}i=2\omega^2$

$= (2\omega)^5+(2\omega^2)^5 = 32\omega^5+32\omega^{10}$

$= 32(\omega^5+\omega^{10}) = 32(\omega^3\cdot\omega^2+\omega^9\cdot\omega)$

$= 32((1)\omega^2+(\omega^3)^3\cdot\omega)$

$= 32(\omega^2+(1)^3\cdot\omega) = 32(\omega^2+\omega)$

$= 32(-1) = -32$ $\quad \because 1+\omega+\omega^2=0,\ \omega^3=1,\ \omega+\omega^2=-1$

Q3. Show that

i) $x^3-y^3=(x-y)(x-\omega y)(x-\omega^2 y)$

Solution:-

$R.H.S = (x-y)(x-\omega y)(x-\omega^2 y)$

$= (x-y)\{x^2-\omega^2xy-\omega xy+\omega^3y^2\}$

$= (x-y)(x^2-xy(\omega^2+\omega)+\omega^3y^2)$

$= (x-y)(x^2-xy(-1)+(1)y^2)$ $\quad \because 1+\omega+\omega^2=0,\ \omega+\omega^2=-1,\ \omega^3=1$

$= (x-y)(x^2+xy+y^2)$

$= x^3-y^3 = L.H.S$

Hence proved

ii) $x^3+y^3+z^3-3xyz = (x+y+z)(x+\omega y+\omega^2z)(x+\omega^2y+\omega z)$

Solution:-

$R.H.S = (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2y+\omega z)$

$= (x+y+z)(x^2+\omega^2xy+\omega xz+\omega xy+\omega^3y^2+\omega^2yz+\omega^2xz+\omega^4yz+\omega^3z^2)$

$= (x+y+z)(x^2+\omega^3y^2+\omega^3z^2+\omega xy+\omega^3\cdot\omega yz+\omega xz+\omega^2xy+\omega^2yz+\omega^2xz)$

$= (x+y+z)(x^2+y^2+z^2+\omega(xy+yz+zx)+\omega^2(xy+yz+zx))$

$= (x+y+z)(x^2+y^2+z^2+(\omega+\omega^2)(xy+yz+zx))$

$= (x+y+z)(x^2+y^2+z^2+(-1)(xy+yz+zx))$

$= (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

$= x^3+y^3+z^3-3xyz = L.H.S$

Hence proved

iii) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots 2n \text{ factors} = 1$

Solution:-

$L.H.S = (1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots 2n$ factors

$= (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)\ldots 2n$ factors

$\left\{ \begin{array}{l} \because \omega^4 = \omega^3\cdot\omega = (1)\omega = \omega \\ \omega^8 = \omega^6\cdot\omega^2 = (\omega^3)^2\cdot\omega^2 = (1)^2\cdot\omega^2 = \omega^2 \end{array}\right.$

$= (1+\omega^2+\omega+\omega^3)(1+\omega^2+\omega+\omega^3)\ldots n$ factors

$= (0+1)(0+1)\ldots\ldots n$ factors

$= (1)(1)\ldots\ldots n$ factors

$= 1 = R.H.S$ $\quad \because 1+\omega^2+\omega = 0,\ \omega^3 = 1$

Hence proved

Q4. If ω is a root of $x^2+x+1=0$. Show that its other root is ω^2 and prove that $\omega^3 = 1$

Solution:- $x^2+x+1=0 \longrightarrow (i)$

$\because \omega$ is root of (i) so put $x=\omega$ in (i)

$\omega^2+\omega+1=0 \longrightarrow (ii)$

To check ω^2, put $x=\omega^2$ in (i)

$(\omega^2)^2+\omega^2+1=0$

$\rightarrow \omega^4+\omega^2+1=0 \longrightarrow (iii)$

Now $L.H.S = \omega^4+\omega^2+1$

$= \omega^3\cdot\omega+\omega^2+1$

$= (1)\omega+\omega^2+1$

$= \omega+\omega^2+1$

$= 1+\omega+\omega^2 = 0 = R.H.S$

Thus ω^2 is root of (i)

Now by (iii) − (ii)

$\omega^4+\omega^2+1=0$

$\underline{\omega^2 \pm \omega \pm 1 = 0}$

$\omega^4-\omega = 0 \rightarrow \omega(\omega^3-1)=0$

$\omega \neq 0,\ \omega^3-1=0 \rightarrow \omega^3=1$

Hence proved.

Q5. Prove that complex cube roots of -1 are $\frac{1+\sqrt{3}i}{2}$ and $\frac{1-\sqrt{3}i}{2}$; and hence $\left(\frac{1+\sqrt{-3}}{2}\right)^9+\left(\frac{1-\sqrt{-3}}{2}\right)^9=-2$

Solution:-

Let x be cube roots of -1 so

$x = \sqrt[3]{-1} \rightarrow x = (-1)^{1/3}$

$x^3 = [(-1)^{1/3}]^3 \rightarrow x^3 = -1$

$x^3+1=0 \rightarrow (x)^3+(1)^3=0$

$(x+1)(x^2-x+1)=0$ $\quad \because a^3+b^3 = (a+b)(a^2-ab+b^2)$

$x+1=0,\quad x^2-x+1=0$

$x=-1,\quad$ using $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x = \frac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}$

$x = \frac{1\pm\sqrt{1-4}}{2} = \frac{1\pm\sqrt{-3}}{2}$

$x = \frac{1\pm\sqrt{3}i}{2} \rightarrow x = \frac{1+\sqrt{3}i}{2},\ x = \frac{1-\sqrt{3}i}{2}$

Hence complex cube roots of -1 are $\frac{1+\sqrt{3}i}{2}$ and $\frac{1-\sqrt{3}i}{2}$

Now

$L.H.S = \left(\frac{1+\sqrt{-3}}{2}\right)^9+\left(\frac{1-\sqrt{-3}}{2}\right)^9$

$= \left[-\left(\frac{-1+\sqrt{3}i}{2}\right)\right]^9+\left[-\left(\frac{-1-\sqrt{3}i}{2}\right)\right]^9$

$= (-\omega)^9+(-\omega^2)^9 = -\omega^9-\omega^{18}$

$= -(\omega^9+\omega^{18}) = -((\omega^3)^3+(\omega^3)^6)$

$= -((1)^3+(1)^6) = -(1+1) = -2 = R.H.S$

Hence proved.

Q6. If ω is a cube root of unity, form an equation whose roots are 2ω and $2\omega^2$.

Solution:-

Let $\alpha = 2\omega$, $\beta = 2\omega^2$

$S = \alpha + \beta = 2\omega + 2\omega^2 = 2(\omega + \omega^2)$

$= 2(-1) = -2$ $\quad \because 1+\omega+\omega^2=0$, $\omega+\omega^2=-1$

$P = \alpha\beta = (2\omega)(2\omega^2) = 4\omega^3$

$= 4(1) = 4$ $\quad \because \omega^3 = 1$

Required equation is

$x^2 - Sx + P = 0$

$\rightarrow x^2 - (-2)x + 4 = 0$

$\rightarrow x^2 + 2x + 4 = 0$

Q7. Find fourth roots of 16, 81, 625

Solution:- i) 16

Let x be fourth root of 16 so

$x = \sqrt[4]{16} \rightarrow x = (16)^{1/4}$

$\rightarrow x^4 = [(16)^{1/4}]^4 \rightarrow x^4 = 16$

$x^4 - 16 = 0 \rightarrow (x^2)^2 - (4)^2 = 0$

$(x^2-4)(x^2+4) = 0$

$x^2 - 4 = 0$, $\quad x^2 + 4 = 0$

$x^2 = 4$, $\quad x^2 = -4$

$x = \pm 2$, $\quad x = \pm\sqrt{-4}$

$\rightarrow x = \pm\sqrt{4}\sqrt{-1} = \pm 2i$

Hence fourth roots are $2, -2$, $-2i, 2i$

ii) 81

Let x be fourth root of 81 so

$x = \sqrt[4]{81} \rightarrow x = (81)^{1/4}$

$x^4 = [(81)^{1/4}]^4 \rightarrow x^4 = 81$

$x^4 - 81 = 0 \rightarrow (x^2)^2 - (9)^2 = 0$

$\rightarrow (x^2-9)(x^2+9) = 0$

$x^2 - 9 = 0$, $\quad x^2 + 9 = 0$

$x^2 = 9$, $\quad x^2 = -9$

$x = \pm 3$, $\quad x = \pm\sqrt{-9} = \pm\sqrt{9}\,i$

$x = \pm 3i$

Hence fourth roots are, $3, -3, 3i, -3i$

iii) 625

Let x be fourth root of 625 so

$x = \sqrt[4]{625} \rightarrow x = (625)^{1/4}$

$x^4 = [(625)^{1/4}]^4 \rightarrow x^4 = 625$

$x^4 - 625 = 0 \rightarrow (x^2)^2 - (25)^2 = 0$

$(x^2-25)(x^2+25) = 0$

$x^2 - 25 = 0$, $\quad x^2 + 25 = 0$

$x^2 = 25$, $\quad x^2 = -25$

$x = \pm 5$, $\quad x = \pm\sqrt{-25} = \pm\sqrt{25}\,i$

$x = \pm 5i$

Hence fourth roots are $5, -5, 5i, -5i$

Q8. Solve the following equations:

i) $2x^4 - 32 = 0$

Solution:-

$2x^4 - 32 = 0$

$\rightarrow 2(x^4 - 16) = 0$

$x^4 - 16 = 0$ $\quad (\div \text{ by } 2)$

$(x^2)^2 - (4)^2 = 0$

$(x^2-4)(x^2+4) = 0$

$x^2 = 4$, $\quad x^2 = -4$

$x = \pm 2$, $\quad x = \pm 2i$

$S.S = \{\pm 2, \pm 2i\}$

ii) $3y^5 - 243y = 0$

Solution:-

$3y^5 - 243y = 0$

$3y(y^4 - 81) = 0$

$3y = 0$, $\quad y^4 - 81 = 0$

$y = 0$, $\quad (y^2)^2 - (9)^2 = 0$

$(y^2-9)(y^2+9) = 0$

$\rightarrow y^2 - 9 = 0, \quad y^2 + 9 = 0$

$\rightarrow y^2 = 9, \quad y^2 = -9 \rightarrow y = \pm\sqrt{-9}$

$\rightarrow y = \pm 3, \quad y = \pm 3i$

$S.S = \{0, \pm 3, \pm 3i\}$

iii) $x^3 + x^2 + x + 1 = 0$

Solution:-

$x^3 + x + x + 1 = 0$

$\rightarrow x^2(x+1) + 1(x+1) = 0$

$\rightarrow (x+1)(x^2+1) = 0$

$\rightarrow x+1 = 0, \quad x^2 + 1 = 0$

$\rightarrow x = -1, \quad x^2 = -1$

$x = -1, \quad x = \pm\sqrt{-1} = \pm i$

$S.S = \{-1, \pm i\}$

iv) $5x^5 - 5x = 0$

Solution:-

$5x^5 - 5x = 0$

$\rightarrow 5x(x^4 - 1) = 0$

$\rightarrow 5x = 0, \quad x^4 - 1 = 0$

$x = 0, \quad (x^2)^2 - (1)^2 = 0$

$\rightarrow x^2 - 1 = 0, \quad x^2 + 1 = 0$

$x^2 = 1, \quad x^2 = -1$

$\rightarrow x = \pm 1, \quad x = \pm\sqrt{-1} = \pm i$

$S.S = \{0, \pm 1, \pm i\}$

Polynomial function

A polynomial in x is an expression of the form

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \; a_n \neq 0 \longrightarrow (i)$

where n is non-negative integer and $a_n, a_{n-1}, \dots a_1$ and a_0 are real nos.

* The highest power of polynomial in x are called degree of polynomial. so expression (i) is polynomial of degree n.

Note:- Consider a polynomial $P(x)$ is divided by $x-a$ and $Q(x)$ be quotient while R be remainder then

$$x-a \,\overline{)\, P(x)} \; \begin{matrix} Q(x) \rightarrow \text{quotient} \\ P(x) \rightarrow \text{dividend} \\ R \rightarrow \text{remainder} \end{matrix}$$

($x-a$ → divisor)

i-e., dividend = (divisor)(quotient) + Remainder

$\rightarrow P(x) = (x-a)(Q(x)) + R$

Theorems

Remainder theorem:-

"If a polynomial $f(x)$ of degree $n \geq 1$, n is non-negative integer is divided by $x-a$ till no x-term exists in the remainder, then $f(a)$ is the remainder"

Proof:- Let us divide a polynomial $f(x)$ by $x-a$, then we get a unique quotient $q(x)$ and a unique remainder R such that $f(x) = (x-a)q(x) + R \longrightarrow (i)$

Put $x = a$ in (i) we get

$\rightarrow f(a) = (a-a)q(a) + R = 0 \cdot q(a) + R$

$\rightarrow f(a) = R$ Hence

$f(a) = R = \text{remainder}$

> * Remainder obtained when $f(x)$ is divided by $x-a$ is same as the value of the polynomial $f(x)$ at $x=a$

Example 1. Find the remainder when the polynomial $x^3 + 4x^2 - 2x + 5$ is divided by $x-1$.

Solution:- Let $f(x) = x^3 + 4x^2 - 2x + 5$

Take $x - 1 = 0 \rightarrow x = 1$

Now $f(1) = (1)^3 + 4(1)^2 - 2(1) + 5$

$= 1 + 4 - 2 + 5 = 8$

$\rightarrow$ remainder $= 8$

Example 2. Find the numerical value of K if the polynomial $x^3 + Kx^2 - 7x + 6$ has a remainder of -4, when divided by $x+2$.

Solution:-

Let $f(x) = x^3 + Kx^2 - 7x + 6$

Take $x+2=0 \rightarrow x=-2$

$f(-2)=(-2)^3+k(-2)^2-7(-2)+6$

$f(-2) = -8+4k+14+6$

∵ remainder $= -4$ (given) so

$\rightarrow -4 = -8+4k+20$

$-4 = 12+4k$

$\rightarrow 4k = -4-12 \rightarrow 4k=-16$

$\rightarrow k=-4$

Factor theorem:— The polynomial $x-a$ is a factor of the polynomial $f(x)$ if and only if $f(a)=0$ **OR** $x-a$ is a factor of $f(x)$ if and only if $x=a$ is root of polynomial equation $f(x)=0$

Proof:- Suppose $g(x)$ is quotient and R is the remainder when a polynomial $f(x)$ is divided by $x-a$, then by Remainder theorem,

$f(x)=(x-a)g(x)+R \longrightarrow (i)$

∵ $x-a$ is factor of $f(x)$

$\rightarrow$ remainder $=R=0$

(i) $\rightarrow f(x)=(x-a)g(x)+0$

$f(x)=(x-a)g(x)$

Take $x=a$

$\rightarrow f(a)=(a-a)g(a)$

$f(a)=0$ Hence proved

Conversely, suppose $f(a)=0$

we prove $x-a$ is factor of $f(x)$.

By remainder theorem

$f(x)=(x-a)g(x)+R \longrightarrow (ii)$

∵ $f(a)=0$ so

$f(a)=(a-a)g(a)+R \rightarrow R=0$

(ii) $\rightarrow f(x)=(x-a)g(x)$

This shows that $x-a$ is factor of $f(x)$. Hence proved.

> + To determine if a given linear polynomial $x-a$ is a factor of $f(x)$, all we need to check whether $f(a)=0$

Example 3. Show that $x-2$ is a factor of x^4-13x^2+36

Solution:-

Let $f(x)=x^4-13x^2+36$

Take $x-2=0 \rightarrow x=2$

$\rightarrow f(2)=(2)^4-13(2)^2+36 = 16-52+36$

$= 52-52=0$

$\rightarrow x-2$ is factor of x^4-13x^2+36

Synthetic Division

This is shortcut method for long division of a polynomial $f(x)$ by polynomial of the form $x-a$.

Out line of the Method

i) Write down the coefficients of the dividend $f(x)$ from left to right in decreasing order of powers of x. Insert 0 for any missing term.

ii) To the left of the first line, write a of the divisor $(x-a)$.

iii) Use the following patterns to write the second and third lines:

Vertical pattern (↓) Add terms

Diagonal pattern (↗) Multiply by a

Example 4. Use synthetic division to find the quotient and the remainder when the polynomial x^4-10x^2-2x+4 is divided by $x+3$

Solution:-

Let $f(x)=x^4-10x^2-2x+4$

Take $x+3=0 \rightarrow x=-3$

Using synthetic division

−3	1	0	−10	−2	4
		−3	9	3	−3
	1	−3	−1	1	1

Quotient $= x^3-3x^2-x+1$

Remainder $= 1$

Example 6. If $(x-2)$ and $(x+2)$ are factors of x^4-13x^2+36. Using synthetic division, find the other two factors.

Solution:-

Let $f(x) = x^4 - 13x^2 + 36$

$= x^4 + 0x^3 - 13x^2 + 0x + 36$

Take $x-2=0 \rightarrow x=2$

and $x+2=0 \rightarrow x=-2$

using synthetic division

2	1	0	-13	0	36
		2	4	-18	-36
-2	1	2	-9	-18	0
		-2	0	18	
	1	0	-9	0	

Quotient $= x^2 - 9 = (x)^2 - (3)^2$

$= (x-3)(x+3)$

Hence other two factors are $x-3$ and $x+3$.

Example 6. If $x+1$ and $x-2$ are factors of x^3+px^2+qx+2. By use of synthetic division find the values of p and q.

Solution:-

Let $f(x) = x^3 + px^2 + qx + 2$

Take $x+1=0 \rightarrow x=-1$

and $x-2=0 \rightarrow x=2$

using synthetic division,

-1	1	p	q	2
		-1	-p+1	p-q-1
2	1	p-1	q-p+1	p-q+1
		2	2p+2	
	1	p+1	p+q+3	

$\because$ $x+1$ and $x-2$ are factors of $f(x)$. so

$p-q+1=0 \rightarrow (i)$

and $p+q+3=0 \rightarrow (ii)$

$2p+4=0$ By adding (i) and (ii)

$\rightarrow 2p=-4 \rightarrow p=-2$ put in (i)

(i) $\rightarrow -2-q+1=0 \rightarrow -q-1=0$

$\rightarrow -q=1 \rightarrow q=-1$

Example 7. By the use of synthetic division, solve the equation $x^4-5x^2+4=0$ if -1 and 2 are its roots.

Solution:-

Let $f(x) = x^4 - 5x^2 + 4$

$= x^4 + 0x^3 - 5x^2 + 0x + 4$

By synthetic division

-1	1	0	-5	0	4
		-1	1	4	-4
2	1	-1	-4	4	0
		2	2	-4	
	1	1	-2	0	

Now we solve

$x^2 + x - 2 = 0$

$\rightarrow x^2 + 2x - x - 2 = 0$

$x(x+2) - 1(x+2) = 0$

$\rightarrow (x+2)(x-1) = 0$

$x+2=0$, $x-1=0$

$x=-2$, $x=1$ $\quad S.S = \{-2, 1\}$

Exercise 4.5

Use the remainder theorem to find the remainder when the first polynomial is divided by the second polynomial:

Q1. x^2+3x+7 , $x+1$

Solution:-

Let $f(x) = x^2+3x+7$

Take $x+1=0 \rightarrow x=-1$

$\rightarrow f(-1) = (-1)^2 + 3(-1) + 7$

$f(-1) = 1-3+7 = 8-3 = 5 = R$

Q2. x^3-x^2+5x+4 , $x-2$

Solution:-

Let $f(x) = x^3 - x^2 + 5x + 4$

Take $x-2=0 \rightarrow x=2$

$\rightarrow f(2) = (2)^3 - (2)^2 + 5(2) + 4$

$f(2) = 8 - 4 + 10 + 4 = 18 = R$

Q3. $3x^4 + 4x^3 + x - 5,\ x+1$

Solution:-

Let $f(x) = 3x^4 + 4x^3 + x - 5$

Take $x + 1 = 0 \rightarrow x = -1$

$f(-1) = 3(-1)^4 + 4(-1)^3 + (-1) - 5$

$= 3(1) + 4(-1) - 1 - 5$

$f(-1) = 3 - 4 - 1 - 5 = 3 - 9 = -7 = R$

Q4. $x^3 - 2x^2 + 3x + 3,\ x - 3$

Solution:-

Let $f(x) = x^3 - 2x^2 + 3x + 3,$

Take $x - 3 = 0 \rightarrow x = 3$

$f(3) = (3)^3 - 2(3)^2 + 3(3) + 3$

$= 27 - 18 + 9 + 3 = 21$

Use the factor theorem to determine if the first polynomial is a factor of the second polynomial.

Q5. $x - 1,\ x^2 + 4x - 5$

Solution:-

Let $f(x) = x^2 + 4x - 5$

Take $x - 1 = 0 \rightarrow x = 1$

$\rightarrow f(1) = (1)^2 + 4(1) - 5$

$= 1 + 4 - 5 = 5 - 5 = 0$

Hence $x - 1$ is factor of $f(x)$.

Q6. $x - 2,\ x^3 + x^2 - 7x + 1$

Solution:-

Let $f(x) = x^3 + x^2 - 7x + 1$

Take $x - 2 = 0 \rightarrow x = 2$

$f(2) = (2)^3 + (2)^2 - 7(2) + 1$

$= 8 + 4 - 14 + 1 = 12 - 14 + 1$

$= -2 + 1 = -1 \neq 0$

Hence $x - 2$ is not factor of $f(x)$.

Q7. $w + 2,\ 2w^3 + w^2 - 4w + 7$

Solution:-

Let $f(w) = 2w^3 + w^2 - 4w + 7$

Take $w + 2 = 0 \rightarrow w = -2$

$f(-2) = 2(-2)^3 + (-2)^2 - 4(-2) + 7$

$= 2(-8) + 4 + 8 + 7$

$f(-2) = -16 + 19 = 3 \neq 0$

Hence $w + 2$ is not factor of $f(w)$.

Q8. $x - a,\ x^n - a^n$ where n is a positive integer.

Solution:-

Let $f(x) = x^n - a^n$

Take $x - a = 0 \rightarrow x = a$

$f(a) = a^n - a^n = 0$

Hence $x - a$ is factor of $f(x)$.

Q9. $x + a,\ x^n + a^n$, where n is an odd integer

Solution:-

Let $f(x) = x^n + a^n$

Take $x + a = 0 \rightarrow x = -a$

$f(-a) = (-a)^n + a^n = -a^n + a^n$ $\because n$ is odd

$f(-a) = 0$

Hence $x + a$ is factor of $f(x)$.

Q10. When $x^4 + 2x^3 + kx^2 + 3$ is divided by $x - 2$, the remainder is 1. Find the value of k.

Solution:-

Let $f(x) = x^4 + 2x^3 + kx^2 + 3$

Take $x - 2 = 0 \rightarrow x = 2$

$f(2) = (2)^4 + 2(2)^3 + k(2)^2 + 3$

$= 16 + 2(8) + k(4) + 3$

$= 16 + 16 + 4k + 3$

$f(2) = 35 + 4k$

$\because$ remainder is 1 so

$35 + 4k = 1 \rightarrow 4k = 1 - 35$

$\rightarrow k = \frac{-34}{4} \rightarrow k = \frac{-17}{2}$

Q11. When the polynomial x^3+2x^2+Kx+4 is divided by $x-2$, the remainder is 14. Find the value of K.

Solution:-

Let $f(x) = x^3+2x^2+Kx+4$

Take $x-2=0 \rightarrow x=2$

$f(2) = (2)^3+2(2)^2+K(2)+4$

$= 8+8+2K+4$

$f(2) = 20+2K$

$\because$ remainder is 14 so

$20+2K = 14 \rightarrow 2K = 14-20$

$\rightarrow 2K = -6 \rightarrow K = -3$

Use synthetic division to show that x is the solution of the polynomial and use the result to factorize the polynomial completely.

Q12. $x^3-7x+6=0, \quad x=2$

Solution:-

Let $f(x) = x^3-7x+6$

$f(x) = x^3+0x^2-7x+6$

using synthetic division

2	1	0	-7	6
		2	4	-6
	1	2	-3	0

Remainder is 0 so $x=2$ is solution. Also

$x^3-7x+6 = (x^2+2x-3)(x-2)$

$= (x^2+3x-x-3)(x-2)$

$= [x(x+3)-1(x+3)](x-2)$

$x^3-7x+6 = (x+3)(x-1)(x-2)$

Q13. $x^3-28x-48=0, \quad x=-4$

Solution:-

Let $f(x) = x^3-28x-48$

$= x^3+0x^2-28x-48$

using synthetic division

-4	1	0	-28	-48
		-4	16	48
	1	-4	-12	0

Remainder is 0 so $x=-4$ is solution. Also

$x^3-28x-48 = (x^2-4x-12)(x+4)$

$= (x^2-6x+2x-12)(x+4)$

$= [x(x-6)+2(x-6)](x+4)$

$= (x-6)(x+2)(x+4)$

Q14. $2x^4+7x^3-4x^2-27x-18$

$x=2, \quad x=-3$

Solution:-

$f(x) = 2x^4+7x^3-4x^2-27x-18$

using synthetic division

	2	7	-4	-27	-18
2		4	22	36	18
	2	11	18	9	0
-3		-6	-15	-9	
	2	5	3	0	

Remainder is 0 so $x=2$ and $x=-3$ are solutions. Also

$2x^4+7x^3-4x^2-27x-18$

$= (2x^2+5x+3)(x-2)(x+3)$

$= (2x^2+2x+3x+3)(x-2)(x+3)$

$= [2x(x+1)+3(x+1)](x-2)(x+3)$

$= (x+1)(2x+3)(x-2)(x+3)$

Q15. Use synthetic division to find the values of p and q if $x+1$ and $x-2$ are the factors of the polynomial x^3+px^2+qx+6

Solution:- Let

$f(x) = x^3+px^2+qx+6$

Take $x+1=0 \rightarrow x=-1$

and $x-2=0 \rightarrow x=2$

using synthetic division

-1	1	p	q	6
		-1	-p+1	-q+p-1
2	1	p-1	q-p+1	p-q+5
		2	2p+2	
	1	p+1	p+q+3	

$\because x+1$ and $x-2$ are factors of $f(x)$. So

$p-q+5=0 \rightarrow (i)$ adding (i) and (ii)

$p+q+3=0 \rightarrow (ii)$

$2p+8=0 \rightarrow 2p=-8 \rightarrow p=-\frac{8}{2}$

$p=-4$ put in (i)

$(i) \rightarrow -4-q+5=0 \rightarrow -q+1=0$

$-q=-1 \rightarrow q=1$

Q16. Find the values of a and b if -2 and 2 are the roots of the polynomial x^3-4x^2+ax+b.

Solution:-

Let $f(x)=x^3-4x^2+ax+b \rightarrow (i)$

Put $x=-2$ in (i)

$f(-2)=(-2)^3-4(-2)^2+a(-2)+b$

$=-8-4(4)-2a+b$

$f(-2)=-8-16-2a+b$

$f(-2)=-2a+b-24$

$\because -2$ is root of $f(x)$ so

$-2a+b-24=0 \rightarrow (ii)$

Put $x=2$ in (i)

$f(2)=(2)^3-4(2)^2+a(2)+b$

$=8-16+2a+b$

$f(2)=2a+b-8$

$\because 2$ is root of $f(x)$ so

$2a+b-8=0 \rightarrow (iii)$

By (ii) + (iii) $\rightarrow$

$-2a+b-24=0$

$2a+b-8=0$

$2b-32=0$

$\rightarrow 2b=32 \rightarrow b=16$ put in (iii)

$2a+16-8=0 \rightarrow 2a+8=0$

$2a=-8 \rightarrow a=-4$

Relations between the Roots and the Coefficients of a Quadratic Equation

Let α, β be the roots of

$ax^2+bx+c=0, \quad a\neq 0$

such that

$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$

Sum of Roots $=\alpha+\beta$

$\rightarrow S=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$

$S=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$

$S=\frac{-2b}{2a} \rightarrow \boxed{S=\frac{-b}{a}}$

Product of Roots $=\alpha\beta$

$\rightarrow P=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$

$=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{(2a)^2}$

$=\frac{b^2-(b^2-4ac)}{4a^2}=\frac{b^2-b^2+4ac}{4a^2}$

$\rightarrow P=\frac{4ac}{4a^2} \rightarrow \boxed{P=\frac{c}{a}}$

Thus

Sum of roots $= S = -\frac{b}{a} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$

Product of roots $= P = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

Example 1. If α, β are the roots of $ax^2+bx+c=0, a\neq 0$, find the values of i) $\alpha^2+\beta^2$ ii) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$ iii) $(\alpha-\beta)^2$

Solution:-

$\because \alpha, \beta$ are roots of $ax^2+bx+c=0$,

$\therefore \alpha+\beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$

i) $\alpha^2+\beta^2$

$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(-\frac{b}{a}\right)^2 - 2\frac{c}{a} = \frac{b^2}{a^2} - \frac{2c}{a}$

$\rightarrow \alpha^2+\beta^2 = \frac{b^2-2ac}{a^2}$

ii) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$

$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\alpha\beta}$

$= \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta}$

$= \frac{\left(-\frac{b}{a}\right)^3 - 3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)}{\frac{c}{a}}$

$= \frac{a}{c}\left[-\frac{b^3}{a^3} + \frac{3bc}{a^2}\right] = \frac{a}{c}\left(\frac{-b^3+3abc}{a^3}\right)$

$= \frac{-b^3+3abc}{a^2c}$

iii) $(\alpha-\beta)^2$

$(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$

$= \left(-\frac{b}{a}\right)^2 - 4\frac{c}{a} = \frac{b^2}{a^2} - \frac{4c}{a}$

$\rightarrow (\alpha-\beta)^2 = \frac{b^2-4ac}{a^2}$

Example 2. Find the condition that one root of $ax^2+bx+c=0, a\neq 0$ is square of the other.

Solution:-

$ax^2+bx+c=0$, $a\neq 0$

Let α be one root and α^2 be the other root. Now

Sum of roots $= \alpha+\alpha^2 = -\frac{b}{a} \rightarrow (i)$

Product of roots $= \alpha\cdot\alpha^2 = \frac{c}{a}$

$= \alpha^3 = \frac{c}{a} \rightarrow (ii)$

Taking cube on both sides of (i)

$(\alpha+\alpha^2)^3 = \left(-\frac{b}{a}\right)^3$

$\rightarrow \alpha^3 + (\alpha^2)^3 + 3\alpha\cdot\alpha^2(\alpha+\alpha^2) = -\frac{b^3}{a^3}$

$\alpha^3 + (\alpha^3)^2 + 3\alpha^3(\alpha+\alpha^2) = -\frac{b^3}{a^3}$

from (i) and (ii)

$\rightarrow \left(\frac{c}{a}\right) + \left(\frac{c}{a}\right)^2 + 3\frac{c}{a}\left(-\frac{b}{a}\right) = -\frac{b^3}{a^3}$

$\frac{c}{a} + \frac{c^2}{a^2} - \frac{3bc}{a^2} = -\frac{b^3}{a^3}$

$\rightarrow a^2c + ac^2 - 3abc = -b^3$ ('x' by a^3)

or $a^2c + ac^2 + b^3 = 3abc$

required condition

Formation of an Equation whose roots are Given

$\because$ the quadratic equation in standard form is

$ax^2+bx+c=0, a\neq 0 \rightarrow (i)$

Let α, β be the roots of (i)

then $S = \alpha+\beta = -\frac{b}{a}$

$P = \alpha\beta = \frac{c}{a}$

dividing eq (i) by a

$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = \frac{0}{a}$

$\rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

$\Rightarrow x^2 - (-\frac{b}{a})x + \frac{c}{a} = 0 \qquad \because -\frac{b}{a} = S,\ P = \frac{c}{a}$

$\Rightarrow x^2 - Sx + P = 0$

which is req. equation.

Example 3. If α, β are the roots of $ax^2+bx+c=0$, form the equation whose roots are double the roots of this equation.

Solution:- $ax^2+bx+c=0 \longrightarrow (i)$

$\because \alpha, \beta$ be the roots of eq (i)

then $\alpha+\beta = \frac{-b}{a}, \quad \alpha\beta = \frac{c}{a}$

Roots of required equation are $2\alpha, 2\beta$. Now

$S = 2\alpha + 2\beta = 2(\alpha+\beta) = 2(-\frac{b}{a})$

$= -\frac{2b}{a}$

$P = (2\alpha)(2\beta) = 4\alpha\beta = 4\frac{c}{a}$

$\therefore$ req. equation is

$x^2 - Sx + P = 0$

$\Rightarrow x^2 - (-\frac{2b}{a})x + 4\frac{c}{a} = 0$

$x^2 + \frac{2bx}{a} + \frac{4c}{a} = 0$

or $ax^2 + 2bx + 4c = 0$

Exercise 4.6

Q1. If α, β are the roots of $3x^2-2x+4=0$, find the values of

i) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ ii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

iii) $\alpha^4+\beta^4$ iv) $\alpha^3+\beta^3$ v) $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$

vi) $\alpha^2 - \beta^2$

Solution:-

$3x^2 - 2x + 4 = 0 \longrightarrow (i)$

$\because \alpha, \beta$ be roots of eq (i)

so $\alpha+\beta = -\frac{(-2)}{3} = \frac{2}{3}, \quad \alpha\beta = \frac{4}{3}$

i) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2+\alpha^2}{\alpha^2\beta^2} = \frac{\alpha^2+\beta^2}{(\alpha\beta)^2}$

$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$

$= \frac{(\frac{2}{3})^2 - 2(\frac{4}{3})}{(\frac{4}{3})^2} = \frac{\frac{4}{9} - \frac{8}{3}}{\frac{16}{9}}$

$= \frac{9}{16}(\frac{4}{9} - \frac{8}{3}) = \frac{9}{16}(\frac{4-24}{9})$

$= \frac{9}{16}(\frac{-20}{9}) = \frac{-20}{16} = \frac{-5}{4}$

ii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

$= \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$

$= \frac{(\frac{2}{3})^2 - 2(\frac{4}{3})}{\frac{4}{3}}$

$= \frac{3}{4}[\frac{4}{9} - \frac{8}{3}] = \frac{3}{4}(\frac{4-24}{9})$

$= \frac{3}{4}(\frac{-20}{9}) = -\frac{5}{3}$

iii) $\alpha^4+\beta^4$

$= (\alpha^2)^2 + (\beta^2)^2$

$= (\alpha^2)^2 + (\beta^2)^2 + 2\alpha^2\beta^2 - 2\alpha^2\beta^2$

$= (\alpha^2+\beta^2)^2 - 2(\alpha\beta)^2$

$= [(\alpha+\beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$

$= ((\frac{2}{3})^2 - 2(\frac{4}{3}))^2 - 2(\frac{4}{3})^2$

$= (\frac{4}{9} - \frac{8}{3})^2 - 2(\frac{16}{9})$

$= [\frac{4-24}{9}]^2 - \frac{32}{9} = (\frac{-20}{9})^2 - \frac{32}{9}$

$= \frac{400}{81} - \frac{32}{9} = \frac{400-288}{81} = \frac{112}{81}$

iv) $\alpha^3+\beta^3$ $\quad \because (a+b)^3 = a^3+b^3+3ab(a+b)$

$= (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= (\frac{2}{3})^3 - 3(\frac{4}{3})(\frac{2}{3})$

$= \frac{8}{27} - \frac{24}{9} = \frac{8-72}{27} = \frac{-64}{27}$

v) $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$

$= \frac{\beta^3+\alpha^3}{\alpha^3\beta^3}$

$$= \frac{\alpha^3+\beta^3}{(\alpha\beta)^3} \qquad \because (a+b)^3 = a^3+b^3+3ab(a+b)$$

$$= \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}$$

$$= \frac{(\frac{2}{3})^3 - 3(\frac{4}{3})(\frac{2}{3})}{(\frac{4}{3})^3}$$

$$= \frac{\frac{8}{27} - \frac{24}{9}}{\frac{64}{27}} = \frac{27}{64}\left(\frac{8}{27} - \frac{24}{9}\right)$$

$$= \frac{27}{64}\left(\frac{8-72}{27}\right) = \frac{27}{64}\left(-\frac{64}{27}\right) = -1$$

vi) $\alpha^2 - \beta^2$

$$= (\alpha+\beta)(\alpha-\beta) \qquad \because (a+b)^2 - (a-b)^2 = 4ab$$

$$= (\alpha+\beta)\sqrt{(\alpha-\beta)^2}$$

$$= (\alpha+\beta)\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$$

$$= (\frac{2}{3})\sqrt{(\frac{2}{3})^2 - 4(\frac{4}{3})}$$

$$= \frac{2}{3}\sqrt{\frac{4}{9} - \frac{16}{3}}$$

$$= \frac{2}{3}\sqrt{\frac{12-48}{9}} = \frac{2}{3}\frac{\sqrt{-44}}{3}$$

$$= \frac{2}{9}\sqrt{4\times(-11)} = \frac{2\times2}{9}\sqrt{-11} = \frac{4}{9}\sqrt{11}\,i$$

Q2. If α, β are roots of $x^2 - px - p - c = 0$, prove that $(1+\alpha)(1+\beta) = 1 - c$

Solution:-

$$x^2 - px - p - c = 0 \longrightarrow (i)$$

$\because \alpha, \beta$ be roots of eq (i) so

$$\alpha+\beta = -\frac{(-p)}{1} = \frac{p}{1} = p, \quad \alpha\beta =$$

Now

$$L.H.S = (1+\alpha)(1+\beta)$$

$$= 1+\beta+\alpha+\alpha\beta$$

$$= 1+\alpha+\beta+\alpha\beta = 1+p-p-c$$

$$= 1-c = R.H.S$$

Hence proved.

Q3. Find the condition that one root of $x^2+px+q=0$ is i) double the other

Solution:-

$$x^2+px+q=0 \quad (a=1,\ b=p,\ c=q)$$

According to the given condition

$\alpha = \alpha$ and $\beta = 2\alpha$ so

$$\alpha+\beta = -\frac{b}{a} \rightarrow \alpha+2\alpha = -\frac{p}{1}$$

$$\rightarrow 3\alpha = -p \rightarrow \alpha = -\frac{p}{3} \longrightarrow (i)$$

and $\alpha\beta = \frac{c}{a} \rightarrow (\alpha)(2\alpha) = \frac{q}{1}$

$$2\alpha^2 = q$$

$$\rightarrow 2(-\frac{p}{3})^2 = q \quad \text{by (i)}$$

$$2\cdot\frac{p^2}{9} = q \rightarrow 2p^2 = 9q$$

ii) square the other

Solution:-

According to given condition

$\alpha = \alpha$ and $\beta = \alpha^2$ then

$$\alpha+\beta = -\frac{b}{a} \rightarrow \alpha+\alpha^2 = \frac{-p}{1} = -p$$

$$\rightarrow \alpha+\alpha^2 = -p \longrightarrow (i)$$

and $\alpha\beta = \frac{c}{a} \rightarrow \alpha(\alpha^2) = \frac{q}{1}$

$$\rightarrow \alpha^3 = q \longrightarrow (ii)$$

Taking cube of eq (i) both sides

$$(\alpha+\alpha^2)^3 = (-p)^3$$

$$\alpha^3 + (\alpha^2)^3 + 3\alpha(\alpha^2)(\alpha+\alpha^2) = -p^3$$

$$\alpha^3 + (\alpha^3)^2 + 3\alpha^3(\alpha+\alpha^2) = -p^3$$

from (i) and (ii)

$$\rightarrow q + q^2 + 3q(-p) = -p^3$$

$$\rightarrow p^3 + q^2 + q - 3pq = 0$$

$$\text{or } p^3 + q^2 + q = 3pq$$

iii) Additive inverse of other.

Solution:-

According to the given condition

$\alpha = \alpha$ and $\beta = -\alpha$ then

$$\alpha+\beta = -\frac{b}{a} \rightarrow \alpha+(-\alpha) = -\frac{p}{1}$$

$$\alpha - \alpha = -p \rightarrow 0 = -p$$

$$\text{or } p = 0$$

$\alpha\beta = \frac{c}{a} \rightarrow \alpha(-\alpha) = \frac{q}{1}$

$\rightarrow -\alpha^2 = q$

so $p=0$ is req. condition

iv) multiplicative inverse of the other.

Solution:-

According to the given condition

$\alpha = \alpha$ and $\beta = \frac{1}{\alpha}$ so

$\alpha+\beta = -\frac{b}{a} \rightarrow \alpha + \frac{1}{\alpha} = -\frac{p}{1} = -p$

or $\alpha + \frac{1}{\alpha} = -p$

also $\alpha\beta = \frac{c}{a} \rightarrow \alpha(\frac{1}{\alpha}) = \frac{q}{1} = q$

$1 = q \rightarrow q = 1$

so $q = 1$ is req. condition.

Q4. If the roots of the equation $x^2 - px + q = 0$ differ by unity, Prove that $p^2 = 4q + 1$

Solution:- Let α, β be the roots of $x^2 - px + q = 0$ $\left(a=1,\ b=-p,\ c=q\right)$

so $\alpha + \beta = -\frac{b}{a} = -\frac{(-p)}{1} = p$

$\rightarrow \alpha + \beta = p$ and

$\alpha\beta = \frac{c}{a} \rightarrow \alpha\beta = \frac{q}{1} \rightarrow \alpha\beta = q$

According to the condition

$\alpha - \beta = 1$

$\rightarrow (\alpha-\beta)^2 = (1)^2$ squaring

$(\alpha+\beta)^2 - 4\alpha\beta = 1$ $\quad \because (a+b)^2 - (a-b)^2 = 4ab$

$\rightarrow p^2 - 4q = 1 \rightarrow p^2 = 1 + 4q$

which is required condition

Q5. Find the condition that $\frac{a}{x-a} + \frac{b}{x-b} = 5$ may have roots equal in magnitude but opposite in sign.

Solution:-

$\frac{a}{x-a} + \frac{b}{x-b} = 5$

'x' both sides by $(x-a)(x-b)$

$\rightarrow a(x-b) + b(x-a) = 5(x-a)(x-b)$

$\rightarrow ax - ab + bx - ab = 5(x^2 - ax - bx + ab)$

$\rightarrow ax + bx - 2ab = 5x^2 - 5ax - 5bx + 5ab$

$\rightarrow 5x^2 - 5bx - 5ax + 5ab - ax - bx + 2ab = 0$

$\rightarrow 5x^2 - 6ax - 6bx + 7ab = 0$

$\rightarrow 5x^2 - 6(a+b)x + 7ab = 0$

$A = 5,\ B = -6(a+b),\ C = ab$

According to given condition

$\alpha = \alpha$ and $\beta = -\alpha$ so

$\alpha + \beta = -\frac{b}{a} \rightarrow \alpha + (-\alpha) = -\frac{[-6(a+b)]}{5}$

$\rightarrow 0 = \frac{6(a+b)}{5} \rightarrow 6(a+b) = 0$

$\rightarrow a + b = 0$

also $\alpha\beta = \frac{c}{a} \rightarrow \alpha(-\alpha) = \frac{7ab}{5}$

$\rightarrow -\alpha^2 = \frac{7ab}{5}$

Hence $a + b = 0$ is req. condition

Q6. If the roots of $px^2 + qx + q = 0$ are α and β then prove that $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{p}{q}} = 0$

Solution:- $\because \alpha, \beta$ be the roots of $px^2 + qx + q = 0$ $\left(a=p,\ b=q,\ c=q\right)$

so $\alpha + \beta = -\frac{b}{a} \rightarrow \alpha + \beta = -\frac{q}{p} \longrightarrow (i)$

and $\alpha\beta = \frac{c}{a} \rightarrow \alpha\beta = \frac{q}{p}$

$\rightarrow \sqrt{\alpha\beta} = \sqrt{\frac{q}{p}} \rightarrow (ii)$

By $\frac{(i)}{(ii)} \rightarrow \frac{\alpha+\beta}{\sqrt{\alpha\beta}} = \frac{-q/p}{\sqrt{q/p}}$

$\rightarrow \frac{\alpha}{\sqrt{\alpha\beta}} + \frac{\beta}{\sqrt{\alpha\beta}} = \frac{-q/p}{\sqrt{q/p}}$

NOTE
$\because x = \sqrt{x}\sqrt{x} = (\sqrt{x})^2 = x$

$\rightarrow \frac{\sqrt{\alpha}\sqrt{\alpha}}{\sqrt{\alpha}\sqrt{\beta}} + \frac{\sqrt{\beta}\sqrt{\beta}}{\sqrt{\alpha}\sqrt{\beta}} = -\frac{\sqrt{q/p}\sqrt{q/p}}{\sqrt{q/p}}$

$\rightarrow \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} = -\sqrt{q/p}$

$\Rightarrow \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}} = 0$

Hence proved

Q7. If α, β are the roots of the equation $ax^2+bx+c=0$, form the equations whose roots are

i) α^2, β^2

Solution:- $\because \alpha, \beta$ be the roots of $ax^2+bx+c=0$

So $\alpha+\beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$

(For all parts)

Now

$S = \alpha^2+\beta^2$

$= (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(-\frac{b}{a}\right)^2 - 2\frac{c}{a} = \frac{b^2}{a^2} - \frac{2c}{a}$

$= \frac{b^2-2ac}{a^2}$

and $P = \alpha^2\beta^2 = (\alpha\beta)^2 = \left(\frac{c}{a}\right)^2 = \frac{c^2}{a^2}$

Req. equation is

$x^2 - Sx + P = 0$

$\Rightarrow x^2 - \left(\frac{b^2-2ac}{a^2}\right)x + \frac{c^2}{a^2} = 0$

$\Rightarrow a^2x^2 - (b^2-2ac)x + c^2 = 0$

ii) $\frac{1}{\alpha}, \frac{1}{\beta}$

$S = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta+\alpha}{\alpha\beta}$

$S = \frac{\alpha+\beta}{\alpha\beta} = \frac{-b/a}{c/a} = -\frac{b}{c}$

$P = \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) = \frac{1}{\alpha\beta} = \frac{1}{\frac{c}{a}} = \frac{a}{c}$

Req. equation is

$x^2 - Sx + P = 0$

$\Rightarrow x^2 - \left(-\frac{b}{c}\right)x + \frac{a}{c} = 0$

$\Rightarrow cx^2 + bx + a = 0$ ('x' by c)

iii) $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$

$S = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2+\alpha^2}{\alpha^2\beta^2}$

$= \frac{\alpha^2+\beta^2}{(\alpha\beta)^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$

$= \frac{\left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)}{\left(\frac{c}{a}\right)^2} = \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c^2}{a^2}}$

$= \frac{a^2}{c^2}\left[\frac{b^2-2ac}{a^2}\right] = \frac{b^2-2ac}{c^2}$

$P = \left(\frac{1}{\alpha^2}\right)\left(\frac{1}{\beta^2}\right) = \frac{1}{(\alpha\beta)^2} = \frac{1}{(c/a)^2} = \frac{1}{c^2/a^2}$

$P = \frac{a^2}{c^2}$

Req. equation is

$x^2 - Sx + P = 0$

$\Rightarrow x^2 - \left(\frac{b^2-2ac}{c^2}\right)x + \frac{a^2}{c^2} = 0$

$\Rightarrow c^2x^2 - (b^2-2ac)x + a^2 = 0$

iv) α^3, β^3

$S = \alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= \left(-\frac{b}{a}\right)^3 - 3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)$

$S = -\frac{b^3}{a^3} + \frac{3bc}{a^2} = \frac{-b^3+3abc}{a^3}$

$P = \alpha^3\beta^3 = (\alpha\beta)^3 = \left(\frac{c}{a}\right)^3 = \frac{c^3}{a^3}$

Req. equation is

$x^2 - Sx + P = 0$

$\Rightarrow x^2 - \left(\frac{3abc-b^3}{a^3}\right)x + \frac{c^3}{a^3} = 0$

$\Rightarrow a^3x^2 - (3abc-b^3)x + c^3 = 0$

v) $\frac{1}{\alpha^3}, \frac{1}{\beta^3}$

$S = \frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\beta^3+\alpha^3}{\alpha^3\beta^3}$

$= \frac{\alpha^3+\beta^3}{(\alpha\beta)^3} = \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}$

$$= \frac{\left(-\frac{b}{a}\right)^3 - 3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)}{\left(\frac{c}{a}\right)^3}$$

$$= \frac{a^3}{c^3}\left[-\frac{b^3}{a^3} + \frac{3bc}{a^2}\right] = \frac{a^3}{c^3}\left(\frac{-b^3+3abc}{a^3}\right)$$

$$S = \frac{3abc - b^3}{c^3}$$

$$P = \left(\frac{1}{\alpha^3}\right)\left(\frac{1}{\beta^3}\right) = \frac{1}{(\alpha\beta)^3} = \frac{1}{(c/a)^3}$$

$$P = \frac{a^3}{c^3}$$

Req. equation is

$$x^2 - Sx + P = 0$$

$$\Rightarrow x^2 - \left(\frac{3abc - b^3}{c^3}\right)x + \frac{a^3}{c^3} = 0$$

$$\Rightarrow c^3x^2 - (3abc - b^3)x + a^3 = 0$$

vi) $\alpha + \frac{1}{\alpha}$, $\beta + \frac{1}{\beta}$

$$S = \alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta}$$

$$= (\alpha+\beta) + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$$

$$= (\alpha+\beta) + \left(\frac{\beta+\alpha}{\alpha\beta}\right)$$

$$S = (\alpha+\beta) + \left(\frac{\alpha+\beta}{\alpha\beta}\right)$$

$$= \frac{-b}{a} + \frac{-b/a}{c/a} = -\frac{b}{a} - \frac{b}{c}$$

$$S = \frac{-bc-ba}{ac}$$

$$P = \left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right)$$

$$= \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}$$

$$= \alpha\beta + \frac{\alpha^2+\beta^2}{\alpha\beta} + \frac{1}{\alpha\beta}$$

$$= \alpha\beta + \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} + \frac{1}{\alpha\beta}$$

$$P = \alpha\beta + \frac{1}{\alpha\beta}\left[(\alpha+\beta)^2 - 2\alpha\beta + 1\right]$$

$$= \frac{c}{a} + \frac{1}{\frac{c}{a}}\left[\left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) + 1\right]$$

$$= \frac{c}{a} + \frac{a}{c}\left[\frac{b^2}{a^2} - \frac{2c}{a} + 1\right]$$

$$= \frac{c}{a} + \frac{a}{c}\left[\frac{b^2 - 2ac + a^2}{a^2}\right]$$

$$= \frac{c}{a} + \frac{b^2 - 2ac + a^2}{ac}$$

$$P = \frac{c^2 + b^2 - 2ac + a^2}{ac} = \frac{c^2 + a^2 - 2ac + b^2}{ac}$$

$$P = \frac{(a-c)^2 + b^2}{ac}$$

Req. equation is

$$x^2 - Sx + P = 0$$

$$\Rightarrow x^2 - \left(\frac{-bc-ba}{ac}\right)x + \frac{(a-c)^2 + b^2}{ac} = 0$$

$$x^2 + \left(\frac{bc+ab}{ac}\right)x + \frac{(a-c)^2 + b^2}{ac} = 0$$

$$\Rightarrow acx^2 + (ab+bc)x + (a-c)^2 + b^2 = 0$$

vii) $(\alpha-\beta)^2$, $(\alpha+\beta)^2$

$$S = (\alpha-\beta)^2 + (\alpha+\beta)^2$$

$$= \left[(\alpha+\beta)^2 - 4\alpha\beta\right] + (\alpha+\beta)^2$$

$$S = \left[\left(-\frac{b}{a}\right)^2 - 4\frac{c}{a}\right] + \left(-\frac{b}{a}\right)^2$$

$$= \left(\frac{b^2}{a^2} - \frac{4c}{a}\right) + \frac{b^2}{a^2}$$

$$= \frac{b^2 - 4ac}{a^2} + \frac{b^2}{a^2} = \frac{b^2 - 4ac + b^2}{a^2}$$

$$S = \frac{2b^2 - 4ac}{a^2} = \frac{2(b^2 - 2ac)}{a^2}$$

$$P = (\alpha-\beta)^2 \cdot (\alpha+\beta)^2$$

$$= \left[(\alpha+\beta)^2 - 4\alpha\beta\right](\alpha+\beta)^2$$

$$= \left(\left(-\frac{b}{a}\right)^2 - 4\frac{c}{a}\right)\left(-\frac{b}{a}\right)^2$$

$$= \left(\frac{b^2}{a^2} - \frac{4c}{a}\right)\left(\frac{b^2}{a^2}\right)$$

$$P = \frac{b^2 - 4ac}{a^2} \cdot \frac{b^2}{a^2} = \frac{b^2(b^2 - 4ac)}{a^4}$$

Req. equation is

$$x^2 - Sx + P = 0$$

$\rightarrow x^2 - 2\left(\frac{b^2-2ac}{a^2}\right)x + b^2\left(\frac{b^2-4ac}{a^4}\right) = 0$

$\rightarrow a^4x^2 - 2a^2(b^2-2ac)x + b^2(b^2-4ac) = 0$

viii) $-\frac{1}{\alpha^3}, -\frac{1}{\beta^3}$

$S = -\frac{1}{\alpha^3} + \left(-\frac{1}{\beta^3}\right) = \frac{-1}{\alpha^3} - \frac{1}{\beta^3}$

$= \frac{-\beta^3-\alpha^3}{\alpha^3\beta^3} = -\frac{(\alpha^3+\beta^3)}{(\alpha\beta)^3}$

$= -\frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}$

$= -\frac{(-b/a)^3 - 3\frac{c}{a}\left(-\frac{b}{a}\right)}{\left(\frac{c}{a}\right)^3}$

$= -\frac{a^3}{c^3}\left[-\frac{b^3}{a^3} + \frac{3bc}{a^2}\right]$

$S = -\frac{a^3}{c^3}\left(\frac{-b^3+3abc}{a^3}\right) = -\frac{a^3}{c^3}\left(\frac{3abc-b^3}{a^3}\right)$

$S = -\frac{(3abc-b^3)}{c^3}$

$P = \left(-\frac{1}{\alpha^3}\right)\left(-\frac{1}{\beta^3}\right) = \frac{1}{(\alpha\beta)^3}$

$P = \frac{1}{(c/a)^3} = \frac{a^3}{c^3}$

Req. equation is

$x^2 - Sx + P = 0$

$\rightarrow x^2 - \left[-\frac{(3abc-b^3)}{c^3}\right] + \frac{a^3}{c^3} = 0$

$\rightarrow c^3x^2 + (3abc-b^3) + a^3 = 0$

Q8. If α, β are the roots of $5x^2 - x - 2 = 0$, form the equation whose roots are $\frac{3}{\alpha}$ and $\frac{3}{\beta}$.

Solution:- $\because \alpha, \beta$ be the roots of

$5x^2 - x - 2 = 0 \quad (a=5, b=-1, c=-2)$

so $\alpha+\beta = \frac{-(-1)}{5} = \frac{1}{5}$

$\alpha\beta = \frac{-2}{5}$

given roots are $\frac{3}{\alpha}, \frac{3}{\beta}$

$S = \frac{3}{\alpha} + \frac{3}{\beta} = 3\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$

$= 3\left[\frac{\beta+\alpha}{\alpha\beta}\right] = 3\left(\frac{\alpha+\beta}{\alpha\beta}\right)$

$= 3\left(\frac{1/5}{-2/5}\right) = 3\left(\frac{1}{5} \times \frac{5}{-2}\right)$

$S = -\frac{3}{2}$

$P = \left(\frac{3}{\alpha}\right)\left(\frac{3}{\beta}\right) = \frac{9}{\alpha\beta} = \frac{9}{-2/5} = -\frac{45}{2}$

Req. equation is

$x^2 - Sx + P = 0$

$\rightarrow x^2 - \left(-\frac{3}{2}\right)x - \frac{45}{2} = 0$

$\rightarrow 2x^2 + 3x - 45 = 0$

Q9. If α and β are roots of $x^2 - 3x + 5 = 0$, form the equation whose roots are $\frac{1-\alpha}{1+\alpha}$ and $\frac{1-\beta}{1+\beta}$

Solution:- $\because \alpha, \beta$ are roots of

$x^2 - 3x + 5 = 0 \quad (a=1, b=-3, c=5)$

so $\alpha+\beta = -\frac{(-3)}{1} = 3$

$\alpha\beta = \frac{5}{1} = 5$

Roots of req. equation are

$\frac{1-\alpha}{1+\alpha}, \frac{1-\beta}{1+\beta}$

$S = \frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta}$

$= \frac{(1-\alpha)(1+\beta) + (1-\beta)(1+\alpha)}{(1+\alpha)(1+\beta)}$

$= \frac{1+\beta-\alpha-\alpha\beta+1+\alpha-\beta-\alpha\beta}{1+\beta+\alpha+\alpha\beta}$

$S = \frac{2-2\alpha\beta}{1+\alpha\beta+\alpha+\beta} = \frac{2-2(5)}{1+5+3}$

$S = -\frac{8}{9}$

$$P = \left(\frac{1-\alpha}{1+\alpha}\right)\left(\frac{1-\beta}{1+\beta}\right)$$

$$= \frac{(1-\alpha)(1-\beta)}{(1+\alpha)(1+\beta)} = \frac{1-\beta-\alpha+\alpha\beta}{1+\alpha+\beta+\alpha\beta}$$

$$P = \frac{1-(\alpha+\beta)+\alpha\beta}{1+\alpha+\beta+\alpha\beta} = \frac{1-3+5}{1+3+5}$$

$$P = \frac{3}{9} = \frac{1}{3}$$

Req. equation is

$$x^2 - Sx + P = 0$$

$$\Rightarrow x^2 - \left(-\frac{8}{9}\right)x + \frac{1}{3} = 0$$

$$\Rightarrow 9x^2 + 8x + 3 = 0$$

Nature of the roots of a Quadratic equation

We know that the roots of the quadratic equation $ax^2+bx+c=0$ are given by quadratic formula as $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

The expression b^2-4ac is called Discriminante (Disc).

* Nature of roots depends on the value of expression b^2-4ac (Disc)

Case I:- If $b^2-4ac > 0$ then roots are real and unequal

Case II:- If $b^2-4ac > 0$ but Perfect square then roots are real, rational and unequal.

Case III:- If $b^2-4ac > 0$ but not perfect square then roots are real, Irrational and unequal.

Case IV:- If $b^2-4ac < 0$ then roots are complex (imaginary) and unequal

Case V:- If $b^2-4ac = 0$ then roots are real and equal.

Example 1. Discuss the nature of the roots of the following equations:

i) $x^2+2x+3=0$

Solution:- $x^2+2x+3=0$

Here $a=1$, $b=2$, $c=3$

Disc $= b^2-4ac = (2)^2-4(1)(3)$

Disc $= 4-12 = -8 < 0$

Thus roots of given equation are complex/imaginary and distinct/unequal.

ii) $2x^2+5x-1=0$

Solution:- $2x^2+5x-1=0$

Here $a=2$, $b=5$, $c=-1$

Disc $= b^2-4ac = (5)^2-4(2)(-1)$

$= 25+8 = 33 > 0$ but not perfect square. Thus roots of given equation are real, irrational and unequal.

iii) $2x^2-7x+3=0$

Solution:- $2x^2-7x+3=0$

Here $a=2$, $b=-7$, $c=3$

Disc $= b^2-4ac = (-7)^2-4(2)(3)$

$= 49-24 = 25 = (5)^2 > 0$ but perfect square. Thus roots are real, rational and unequal.

iv) $9x^2-12x+4=0$

Solution:- $9x^2-12x+4=0$

Here $a=9$, $b=-12$, $c=4$

Disc $= b^2-4ac = (-12)^2-4(9)(4)$

Disc $= 144-144 = 0$

Thus roots are real and equal.

Example 2. For what value of m will the following equation have equal root? $(m+1)x^2+2(m+3)x+2m+3=0$

Solution:-

$(m+1)x^2+2(m+3)x+2m+3=0$

Here $a=m+1$, $b=2(m+3)$

$c=2m+3$

$$\text{Disc} = b^2 - 4ac$$
$$= [2(m+3)]^2 - 4(m+1)(2m+3)$$
$$= 4(m+3)^2 - 4(2m^2+3m+2m+3)$$
$$= 4(m^2+6m+9) - 4(2m^2+5m+3)$$
$$= 4m^2 + 24m + 36 - 8m^2 - 20m - 12$$
$$= -4m^2 + 4m + 24$$
$\Rightarrow$ $\text{Disc} = -4m^2 + 4m + 24$

Given that roots are equal so Disc = 0.

$\Rightarrow -4m^2 + 4m + 24 = 0$

$\Rightarrow m^2 - m - 6 = 0 \quad (\div \text{ by } -4)$

$\Rightarrow m^2 - 3m + 2m - 6 = 0$

$m(m-3) + 2(m-3) = 0$

$(m-3)(m+2) = 0$

$\Rightarrow m-3 = 0$ or $m+2 = 0$

$m = 3$ or $m = -2$

Thus for $m = 3$ or $m = -2$ roots will be equal.

Example 3. Show that the roots of the following equation are real.

$(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$

Also show that the roots will be equal only if $a = b = c$

Solution:-

$(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$

$x^2 - bx - ax + ab + x^2 - xc - bx + bc + x^2 - ax - cx + ac = 0$

$3x^2 - 2ax - 2bx - 2cx + ab + bc + ac = 0$

$3x^2 - 2(a+b+c)x + ab + bc + ca = 0$

$\text{Disc} = b^2 - 4ac$

$= [-2(a+b+c)]^2 - 4(3)(ab+bc+ca)$

$= 4(a+b+c)^2 - 12(ab+bc+ca)$

$= 4[(a+b+c)^2 - 3(ab+bc+ca)]$

$= 4[a^2+b^2+c^2+2ab+2bc+2ca - 3ab - 3bc - 3ca]$

$= 4[a^2+b^2+c^2 - ab - bc - ca]$

$= 2[2a^2+2b^2+2c^2 - 2ab - 2bc - 2ca]$

$= 2[a^2+a^2+b^2+b^2+c^2+c^2 - 2ab - 2bc - 2ca]$

$= 2[a^2+b^2-2ab + b^2+c^2-2bc + c^2+a^2-2ca]$

$= 2[(a-b)^2 + (b-c)^2 + (c-a)^2] > 0$

$\because$ disc > 0 Hence roots are real. Now roots will be equal if $b^2 - 4ac = 0$

$2((a-b)^2 + (b-c)^2 + (c-a)^2) = 0$

$\Rightarrow 2 \neq 0$, $(a-b)^2 + (b-c)^2 + (c-a)^2 = 0$

This is only possible if

$(a-b)^2 = 0$, $(b-c)^2 = 0$, $(c-a)^2 = 0$

$\Rightarrow a-b = 0$, $b-c = 0$, $c-a = 0$

$a = b$, $b = c$, $c = a$

$\Rightarrow a = b = c$

Exercise 4.7

Q1. Discuss the nature of the roots of the following equations:

i) $4x^2 + 6x + 1 = 0$

Solution:- $4x^2 + 6x + 1 = 0$

Here $a = 4$, $b = 6$, $c = 1$

$\text{Disc} = b^2 - 4ac = (6)^2 - 4(4)(1)$

$= 36 - 16 = 20 > 0$ but not pefect square

Thus roots are real, irrational and unequal.

ii) $x^2 - 5x + 6 = 0$

Solution:- $x^2 - 5x + 6 = 0$

Here $a = 1$, $b = -5$, $c = 6$

$\text{Disc} = b^2 - 4ac = (-5)^2 - 4(1)(6)$

$= 25 - 24 = 1 = (1)^2 > 0$ but perfect square. Thus roots are real, rational and unequal.

iii) $2x^2 - 5x + 1 = 0$

Solution:- $2x^2 - 5x + 1 = 0$

Here $a = 2$, $b = -5$, $c = 1$

$\text{Disc} = b^2 - 4ac = (-5)^2 - 4(2)(1)$

disc $= 25 - 8 = 17 > 0$ but not perfect square. Thus roots are real, irrational and unequal

iv) $25x^2 - 30x + 9 = 0$

Solution:- $25x^2 - 30x + 9 = 0$

Here $a = 25,\ b = -30,\ c = 9$

Disc $= b^2 - 4ac = (-30)^2 - 4(25)(9)$

Disc $= 900 - 900 = 0$

so roots are real and equal.

Q2. Show that the roots of the following equations will be real:

i) $x^2 - 2(m + \frac{1}{m})x + 3 = 0;\ m \neq 0$

Solution:-

$x^2 - 2(m + \frac{1}{m})x + 3 = 0$

Here $a = 1,\ b = -2(m + \frac{1}{m}),\ c = 3$

Disc $= b^2 - 4ac$

$= \left[-2(m + \frac{1}{m})\right]^2 - 4(1)(3)$

$= 4(m + \frac{1}{m})^2 - 12$

$= 4(m^2 + \frac{1}{m^2} + 2) - 12$

$= 4m^2 + \frac{4}{m^2} + 8 - 12$

$= 4m^2 + \frac{4}{m^2} - 4 = 4(m^2 + \frac{1}{m^2} - 1)$

$= 4(m^2 + \frac{1}{m^2} - 1 - 1 + 1)$

Disc $= 4(m^2 + \frac{1}{m^2} - 2 + 1)$

Disc $= 4\left[(m - \frac{1}{m})^2 + 1\right] > 0$

$\because$ disc > 0 Hence roots are real.

ii) $(b - c)x^2 + (c - a)x + (a - b) = 0,$
$a, b, c \in Q$

Solution:-

$(b - c)x^2 + (c - a)x + (a - b) = 0$

Here $A = b - C,\ B = c - a$
$C = a - b$

Disc $= B^2 - 4AC$

$= (c - a)^2 - 4(b - c)(a - b)$

$= c^2 + a^2 - 2ca - 4(ab - b^2 - ac + bc)$

$= c^2 + a^2 - 2ca - 4ab + 4b^2 + 4ac - 4bc$

$= a^2 + c^2 + 2ac - 4ab - 4bc + 4b^2$

disc $= (a + c - 2b)^2 > 0$

$\because$ disc > 0 Hence roots are real.

Q3. Show that the roots of the following equations will be rational:

i) $(p + q)x^2 - px - q = 0$

Solution:-

$(p + q)x^2 - px - q = 0$

Here $a = p + q,\ b = -p,\ c = -q$

Disc $= b^2 - 4ac$

$= (-p)^2 - 4(p + q)(-q)$

$= p^2 + 4pq + 4q^2$

Disc $= (p + 2q)^2 > 0$ and perfect square

Hence roots are rational.

ii) $px^2 - (p - q)x - q = 0$

Solution:-

$px^2 - (p - q)x - q = 0$

Here $a = p,\ b = -(p - q),\ c = -q$

Disc $= b^2 - 4ac$

$= \left[-(p - q)\right]^2 - 4(p)(-q)$

$= (p - q)^2 + 4pq$

$= p^2 + q^2 - 2pq + 4pq$

$= p^2 + q^2 + 2pq$

Disc $= (p + q)^2 > 0$ and perfect square

Hence roots are rational.

Q4. For what values of m will the roots of the following equations be equal?

i) $(m+1)x^2 + 2(m+3)x + m + 8 = 0$

Solution:-

$(m + 1)x^2 + 2(m + 3)x + m + 8 = 0$

Here $a = m + 1,\ b = 2(m + 3),\ c = m + 8$

Disc $= b^2 - 4ac$

$= \left[2(m + 3)\right]^2 - 4(m + 1)(m + 8)$

$= 4(m^2+6m+9) - 4(m^2+m+8m+8)$

$= 4m^2+24m+36-4m^2-4m-32m-32$

Disc $= -12m+4$

Given roots are equal

i.e., disc $= 0$

$\rightarrow -12m+4=0$

$\rightarrow 12m=4 \rightarrow m=\frac{1}{3}$

ii) $x^2 - 2(1+3m)x+7(3+2m)=0$

Solution:-

$x^2 - 2(1+3m)x + 7(3+2m)=0$

Here $a=1$, $b=-2(1+3m)$

$c=7(3+2m)$

Disc $= b^2-4ac$

$= [-2(1+3m)]^2 - 4(1)[7(3+2m)]$

$= 4(1+3m)^2 - 4(21+14m)$

$= 4(1+9m^2+6m) - 84 - 56m$

$= 4+36m^2+24m-84-56m$

Disc $= 36m^2-32m-80$

Given roots are equal

i.e., disc $=0$

$\rightarrow 36m^2-32m-80=0$

$\rightarrow 9m^2-8m-20=0$ $(\div \text{ by } 4)$

$9m^2-18m+10m-20=0$

$9m(m-2)+10(m-2)=0$

$(m-2)(9m+10)=0$

$\rightarrow m-2=0$, $9m+10=0$

$\rightarrow m=2$, $m=-\frac{10}{9}$

iii) $(1+m)x^2 - 2(1+3m)x+(1+8m)=0$

Solution:-

$(1+m)x^2-2(1+3m)x+(1+8m)=0$

Here $a=1+m$, $b=-2(1+3m)$

$c=1+8m$

Disc $= b^2-4ac$

$= [-2(1+3m)]^2 - 4(1+m)(1+8m)$

$= 4(1+3m)^2 - 4(1+m+8m+8m^2)$

$= 4(1+9m^2+6m) - 4(1+9m+8m^2)$

$= 4+36m^2+24m-4-36m-32m^2$

Disc $= 4m^2-12m$

Given roots are equal

i.e., disc $=0$

$\rightarrow 4m^2-12m=0$

$4m(m-3)=0$

$4m=0$, $m-3=0$

$m=0$, $m=3$

Q5. Show that the roots of $x^2+(mx+c)^2=a^2$ will be equal, if $c^2=a^2(1+m^2)$

Solution:-

$x^2+(mx+c)^2=a^2$

$\rightarrow x^2+m^2x^2+c^2+2mcx-a^2=0$

$(1+m^2)x^2+2mcx+c^2-a^2=0$

Here $A=1+m^2$, $B=2mc$

$C=c^2-a^2$

Given roots are equal so

$B^2-4AC=0$

$\rightarrow (2mc)^2-4(1+m^2)(c^2-a^2)=0$

$\rightarrow 4m^2c^2-4(c^2-a^2+m^2c^2-m^2a^2)=0$

$\rightarrow 4m^2c^2-4c^2+4a^2-4m^2c^2+4m^2a^2=0$

$\rightarrow -4c^2+4a^2+4m^2a^2=0$

$-c^2+a^2+m^2a^2=0$ $(\div \text{ by } 4)$

$\rightarrow c^2=a^2+m^2a^2$

$\rightarrow c^2=a^2(1+m^2)$

Hence proved

Q6. Show that the roots $(mx+c)^2=4ax$ will be equal, if $c=\frac{a}{m}$, $m\neq 0$

Solution:-

$(mx+c)^2=4ax$

$\rightarrow m^2x^2+c^2+2mcx-4ax=0$

$m^2x^2+2mcx-4ax+c^2=0$

$m^2x^2+(2mc-4a)x+c^2=0$

Here $A=m^2$, $B=2mc-4a$

$C=c^2$

Given roots are equal so

$B^2 - 4AC = 0$

$\rightarrow (2mc-4a)^2 - 4m^2c^2 = 0$

$\rightarrow 4m^2c^2 + 16a^2 - 16amc - 4m^2c^2 = 0$

$\rightarrow 16a^2 - 16amc = 0$

$\rightarrow a - mc = 0 \quad (\div 16a)$

$\rightarrow a = mc$

or $c = \frac{a}{m}$ Hence proved

Q7. Prove that $\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$ will have equal roots if $c^2 = a^2m^2 + b^2$; $a \neq 0, b \neq 0$

Solution:-

$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$

$\rightarrow b^2x^2 + a^2(mx+c)^2 = a^2b^2 \quad (\times \text{ by } a^2b^2)$

$\rightarrow b^2x^2 + a^2(m^2x^2 + c^2 + 2mcx) = a^2b^2$

$\rightarrow b^2x^2 + a^2m^2x^2 + a^2c^2 + 2a^2mcx - a^2b^2 = 0$

$(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2c^2 - a^2b^2 = 0$

Here $A = b^2 + a^2m^2$, $B = 2a^2mc$

$C = a^2c^2 - a^2b^2$

Given roots are equal so

$B^2 - 4AC = 0$

$\rightarrow (2a^2mc)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) = 0$

$\rightarrow 4a^4m^2c^2 - 4[a^2b^2c^2 - a^2b^4 + a^4m^2c^2 - a^4b^2m^2] = 0$

$\rightarrow 4a^4m^2c^2 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4m^2c^2 + 4a^4b^2m^2 = 0$

$\rightarrow -4a^2b^2c^2 + 4a^2b^4 + 4a^4b^2m^2 = 0$

$-c^2 + b^2 + a^2m^2 = 0 \quad (\div \text{ by } 4a^2b^2)$

$\rightarrow c^2 = b^2 + a^2m^2$

or $c^2 = a^2m^2 + b^2$

Hence proved.

Q8. Show that the roots of the equation $(a^2-bc)x^2 + 2(b^2-ca)x + c^2 - ab = 0$ will be equal, if either $a^3 + b^3 + c^3 = 3abc$ or $b = 0$

Solution:-

$(a^2-bc)x^2 + 2(b^2-ca)x + c^2 - ab = 0$

Here $A = a^2 - bc$, $B = 2(b^2 - ca)$

$C = c^2 - ab$

Given roots are equal so

$B^2 - 4AC = 0$

$\rightarrow [2(b^2-ca)]^2 - 4(a^2-bc)(c^2-ab) = 0$

$4(b^2-ca)^2 - 4(a^2-bc)(c^2-ab) = 0$

$(b^2-ca)^2 - (a^2-bc)(c^2-ab) = 0 \quad (\div \text{ by } 4)$

$b^4 + c^2a^2 - 2b^2ca - [a^2c^2 - a^3b - bc^3 + ab^2c] = 0$

$b^4 + c^2a^2 - 2ab^2c - a^2c^2 + a^3b + bc^3 - ab^2c = 0$

$\rightarrow a^3b + b^4 + bc^3 - 3ab^2c = 0$

$\rightarrow b(a^3 + b^3 + c^3 - 3abc) = 0$

$b = 0$ or $a^3 + b^3 + c^3 - 3abc = 0$

$b = 0$ or $a^3 + b^3 + c^3 = 3abc$

Hence proved

System of Two Equations Involving Two Variables

Simultaneous Equation:- A set of two or more than two equations in which the values of variables satisfy all the equation are called simultaneous equations or a system of equations.

Case I:- When One equation is linear and other is quadratic

If one of the equation is linear, we can find value of one variable in term of the other variable from linear equation. Put this value of one variable to the quadratic equation, we solve it.

See Example 1, Q 1,2,3,4,5,6,7

Example 1. Solve the system of equations: $x+y=7$ and $x^2-xy+y^2=13$

Solution:- $x+y=7 \longrightarrow (i)$

$x^2-xy+y^2=13 \longrightarrow (ii)$

By (i) $\rightarrow$ $y=7-x$ put in (ii)

(ii) $\rightarrow$ $x^2-x(7-x)+(7-x)^2=13$

$x^2-7x+x^2+49+x^2-14x=13$

$3x^2-21x+36=0$

$\rightarrow x^2-7x+12=0$ $(\div \text{ by } 3)$

$\rightarrow x^2-3x-4x+12=0$

$\rightarrow x(x-3)-4(x-3)=0$

$(x-3)(x-4)=0$

$x-3=0$, $x-4=0$

$x=3$, $x=4$

Put in (i) , Put in (i)

$3+y=7$, $4+y=7$

$\rightarrow y=7-3$, $y=7-4$

$y=4$, $y=3$

Hence $S.S=\{(3,4),(4,3)\}$

> **Note:-** Two quadratic equations in which xy term is missing and the coefficients of x^2 and y^2 are equal, give a linear equation by subtracting.
> See Example 2, Q 8, 9, 10

Example 2. Solve the following equations:

$x^2+y^2+4x=1$ and $x^2+(y-1)^2=10$

Solution:-

$x^2+y^2+4x=1 \longrightarrow (i)$

$x^2+(y-1)^2=10$

$\rightarrow x^2+y^2+1-2y=10$

$x^2+y^2-2y=10-1$

$x^2+y^2-2y=9 \longrightarrow (ii)$

By (i) − (ii) $\rightarrow$

$$\begin{array}{r} x^2+y^2+4x=1 \\ \underline{\overset{-}{x^2}+\overset{-}{y^2}\overset{+}{-2y}=\overset{-}{9}} \\ 4x+2y=-8 \end{array}$$

$\rightarrow 2y=-4x-8$

$y=-2x-4$

$y=-(2x+4) \longrightarrow$ (iii) Put in (i)

$x^2+(-(2x+4))^2+4x=1$

$x^2+(2x+4)^2+4x=1$

$x^2+4x^2+16+16x+4x=1$

$5x^2+20x+16-1=0$

$5x^2+20x+15=0$

$\rightarrow x^2+4x+3=0$ $(\div \text{ by } 5)$

$x^2+x+3x+3=0$

$x(x+1)+3(x+1)=0$

$(x+1)(x+3)=0$

$x+1=0$, $x+3=0$

$x=-1$, $x=-3$

Put in (iii) , Put in (iii)

$y=-(2(-1)+4)$, $y=-(2(-3)+4)$

$=-(-2+4)$, $y=-(-6+4)$

$y=-2$, $y=2$

Hence $S.S=\{(-1,-2),(-3,2)\}$

Exercise 4.8

Solve the following system of equations:

Q1. $2x-y=4$; $2x^2-4xy-y^2=6$

Solution:- $2x-y=4 \longrightarrow (i)$

$2x^2-4xy-y^2=6 \longrightarrow (ii)$

from (i) $-y=4-2x \rightarrow y=2x-4 \longrightarrow (iii)$

Put value of y in (ii)

$2x^2-4x(2x-4)-(2x-4)^2=6$

$2x^2-8x^2+16x-(4x^2+16-16x)=6$

$2x^2-8x^2+16x-4x^2-16+16x-6=0$

$-10x^2+32x-22=0$

$\rightarrow 10x^2-32x+22=0$

$\rightarrow 5x^2 - 16x + 11 = 0$ (÷ by 2)

$5x^2 - 5x - 11x + 11 = 0$

$5x(x-1) - 11(x-1) = 0$

$(x-1)(5x-11) = 0$

$x - 1 = 0$, $5x - 11 = 0$

$x = 1$, $x = \frac{11}{5}$

Put in (iii) , Put in (iii)

$y = 2(1) - 4$, $y = 2\left(\frac{11}{5}\right) - 4$

$y = -2$, $y = \frac{22-20}{5} = \frac{2}{5}$

$S.S = \left\{(1,-4), \left(\frac{11}{5}, \frac{2}{5}\right)\right\}$

Q2. $x+y=5$; $x^2+2y^2=17$

Solution:- $x + y = 5 \longrightarrow (i)$

$x^2 + 2y^2 = 17 \longrightarrow (ii)$

By (i) $\rightarrow y = 5 - x \longrightarrow (iii)$ put in (ii)

$x^2 + 2(5-x)^2 = 17$

$x^2 + 2(25 + x^2 - 10x) = 17$

$\rightarrow x^2 + 50 + 2x^2 - 20x - 17 = 0$

$3x^2 - 20x + 33 = 0$

$3x^2 - 9x - 11x + 33 = 0$

$\rightarrow 3x(x-3) - 11(x-3) = 0$

$(x-3)(3x-11) = 0$

$x - 3 = 0$, $3x - 11 = 0$

$x = 3$, $x = \frac{11}{3}$

Put in (iii) , Put in (iii)

$y = 5 - 3 = 2$, $y = 5 - \frac{11}{3}$

$y = \frac{15-11}{3} = \frac{4}{3}$

$S.S = \left\{(3,2), \left(\frac{11}{3}, \frac{4}{3}\right)\right\}$

Q3. $3x+2y=7$; $3x^2 = 25 + 2y^2$

Solution:- $3x + 2y = 7 \longrightarrow (i)$

$3x^2 - 2y^2 = 25 \longrightarrow (ii)$

From (i) $\rightarrow 2y = 7 - 3x \rightarrow y = \frac{7-3x}{2} \longrightarrow (iii)$

Put value of y in (ii)

$3x^2 - 2\left(\frac{7-3x}{2}\right)^2 = 25$

$\rightarrow 3x^2 - 2\left(\frac{49 + 9x^2 - 42x}{4}\right) - 25 = 0$

$3x^2 - \left(\frac{9x^2 - 42x + 49}{2}\right) - 25 = 0$

$\rightarrow 6x^2 - 9x^2 + 42x - 49 - 25 = 0$

$-3x^2 + 42x - 99 = 0$

$x^2 - 14x + 33 = 0$ (÷ by −3)

$x^2 - 3x - 11x + 33 = 0$

$x(x-3) - 11(x-3) = 0$

$(x-3)(x-11) = 0$

$x - 3 = 0$, $x - 11 = 0$

$x = 3$, $x = 11$

Put in (iii) , Put in (iii)

$y = \frac{7-3(3)}{2}$, $y = \frac{7-3(11)}{2}$

$y = \frac{7-9}{2} = \frac{-2}{2}$, $y = \frac{7-33}{2} = \frac{-26}{2}$

$y = -1$, $y = -13$

$S.S = \{(3,-1), (11,-13)\}$

Q4. $x+y=5$; $\frac{2}{x} + \frac{3}{y} = 2$, $x \neq 0, y \neq 0$

Solution:- $x + y = 5 \longrightarrow (i)$

$\frac{2}{x} + \frac{3}{y} = 2$

$\rightarrow 2y + 3x = 2xy \longrightarrow (ii)$ ('x' by xy)

By (i) $\rightarrow y = 5 - x \longrightarrow (iii)$ Put in (ii)

$2(5-x) + 3x = 2x(5-x)$

$10 - 2x + 3x = 10x - 2x^2$

$\rightarrow 10 + x - 10x + 2x^2 = 0$

$2x^2 - 9x + 10 = 0$

$\rightarrow 2x^2 - 4x - 5x + 10 = 0$

$2x(x-2) - 5(x-2) = 0$

$(x-2)(2x-5) = 0$

$x - 2 = 0$, $2x - 5 = 0$

$x = 2$, $x = \frac{5}{2}$

Put in (iii) , Put in (iii)

$y = 5 - 2$, $y = 5 - \frac{5}{2} = \frac{10-5}{2}$

$\rightarrow y = 3$, $y = \frac{5}{2}$

$S.S = \{(2,3), (\frac{5}{2}, \frac{5}{2})\}$

Q5. $x+y = a+b \; ; \; \frac{a}{x} + \frac{b}{y} = 2$

Solution:- $x+y = a+b \longrightarrow (i)$

$\frac{a}{x} + \frac{b}{y} = 2$

$\rightarrow ay + bx = 2xy \longrightarrow (ii)$ ('x' by xy)

By (i) $\rightarrow y = a+b-x \longrightarrow (iii)$ put in (ii)

$a(a+b-x) + bx = 2x(a+b-x)$

$\rightarrow a^2+ab-ax+bx = 2xa+2bx-2x^2$

$2x^2-2ax-2bx-ax+bx+a^2+ab=0$

$\rightarrow 2x^2-3ax-bx+a^2+ab=0$

$2x^2-(3a+b)x+a^2+ab=0$

using quadratic formula

$$x = \frac{-[-(3a+b)] \pm \sqrt{[-(3a+b)]^2-4(2)(a^2+ab)}}{2(2)}$$

$$= \frac{3a+b \pm \sqrt{(3a+b)^2-8(a^2+ab)}}{4}$$

$$x = \frac{3a+b \pm \sqrt{9a^2+b^2+6ab-8a^2-8ab}}{4}$$

$$x = \frac{3a+b \pm \sqrt{a^2+b^2-2ab}}{4}$$

$$x = \frac{3a+b \pm \sqrt{(a-b)^2}}{4}$$

$$x = \frac{3a+b \pm (a-b)}{4}$$

$x = \frac{3a+b+a-b}{4}$, $x = \frac{3a+b-(a-b)}{4}$

$= \frac{4a}{4}$, $x = \frac{3a+b-a+b}{4}$

$x = \frac{2a+2b}{4}$

$x = a$, $x = \frac{2(a+b)}{4}$

put in (iii)

$y = a+b-a$, $x = \frac{a+b}{2}$

$y = b$

put in (iii)

$y = a+b - \frac{a+b}{2}$

$y = \frac{2a+2b-a-b}{2}$

$y = \frac{a+b}{2}$

$S.S = \{(a,b), (\frac{a+b}{2}, \frac{a+b}{2})\}$

Q6. $3x+4y = 25 \; ; \; \frac{3}{x} + \frac{4}{y} = 2$

Solution:- $3x+4y = 25 \longrightarrow (i)$

$\frac{3}{x} + \frac{4}{y} = 2$

$\rightarrow 3y+4x = 2xy \longrightarrow (ii)$ ('x' by xy)

From (i) $\rightarrow 4y = 25-3x$

$\rightarrow y = \frac{25-3x}{4} \longrightarrow (iii)$ put in (ii)

$3\left(\frac{25-3x}{4}\right) + 4x = 2x\left(\frac{25-3x}{4}\right)$

$3(25-3x)+16x = 2x(25-3x)$ ('x' by 4)

$75-9x+16x = 50x-6x^2$

$\rightarrow 6x^2-50x+75-9x+16x=0$

$\rightarrow 6x^2-43x+75=0$

$\rightarrow 6x^2-18x-25x+75=0$

$6x(x-3)-25(x-3)=0$

$(x-3)(6x-25)=0$

$x-3=0$, $6x-25=0$

$x=3$, $x = \frac{25}{6}$

Put in (iii) , Put in (iii)

$y = \frac{25-3(3)}{4}$, $y = \frac{25-3(\frac{25}{6})}{4}$

$= \frac{25-9}{4} = \frac{16}{4}$, $y = \frac{25-\frac{25}{2}}{4}$

$y = 4$, $y = \frac{1}{4}(50-25)$

$y = \frac{1}{4}(25) = \frac{25}{4}$

$S.S = \{(3,4), (\frac{25}{6}, \frac{25}{4})\}$

Q7. $(x-3)^2+y^2 = 5 \; ; \; 2x = y+6$

Solution:- $(x-3)^2+y^2=5$

$\rightarrow x^2+9-6x+y^2=5$

$\rightarrow x^2+y^2-6x+9-5=0$

$x^2+y^2-6x+4=0 \longrightarrow (i)$

$2x = y+6$

$\rightarrow y = 2x-6 \longrightarrow (ii)$ put in (i)

$x^2+(2x-6)^2-6x+4=0$

$x^2 + 4x^2 - 24x + 36 - 6x + 4 = 0$

$\Rightarrow 5x^2 - 30x + 40 = 0$

$\Rightarrow x^2 - 6x + 8 = 0 \quad (\div \text{ by } 5)$

$\Rightarrow x^2 - 2x - 4x + 8 = 0$

$x(x-2) - 4(x-2) = 0$

$(x-2)(x-4) = 0$

$\Rightarrow x - 2 = 0 \quad , \quad x - 4 = 0$

$x = 2 \quad , \quad x = 4$

Put in (ii) , Put in (ii)

$y = 2(2) - 6 \quad , \quad y = 2(4) - 6$

$y = 4 - 6 \quad , \quad y = 8 - 6$

$y = -2 \quad , \quad y = 2$

$S.S = \{(2,-2), (4,2)\}$

Q8. $(x+3)^2 + (y-1)^2 = 5 ; \; x^2 + y^2 + 2x = 9$

Solution:- $(x+3)^2 + (y-1)^2 = 5$

$\Rightarrow x^2 + 9 + 6x + y^2 + 1 - 2y - 5 = 0$

$\Rightarrow x^2 + y^2 + 6x - 2y + 5 = 0 \longrightarrow (i)$

$x^2 + y^2 + 2x = 9$

$\Rightarrow x^2 + y^2 + 2x - 9 = 0 \longrightarrow (ii)$

By (ii) − (i) $\Rightarrow$

$$\begin{array}{r} x^2 + y^2 + 6x - 2y + 5 = 0 \\ \underline{\mp x^2 \mp y^2 \mp 2x \quad \pm 9 = 0} \\ 4x - 2y + 14 = 0 \end{array}$$

$\Rightarrow 2x - y + 7 = 0 \quad (\div \text{ by } 2)$

$y = 2x + 7 \longrightarrow (iii)$ put in (ii)

$x^2 + (2x+7)^2 + 2x - 9 = 0$

$\Rightarrow x^2 + 4x^2 + 49 + 28x + 2x - 9 = 0$

$5x^2 + 30x + 40 = 0$

$\Rightarrow x^2 + 6x + 8 = 0 \quad (\div \text{ by } 2)$

$x^2 + 2x + 4x + 8 = 0$

$x(x+2) + 4(x+2) = 0$

$(x+2)(x+4) = 0$

$x + 2 = 0 \quad , \quad x + 4 = 0$

$x = -2 \quad , \quad x = -4$

Put in (iii) , Put in (iii)

$y = 2(-2) + 7 \quad , \quad y = 2(-4) + 7$

$y = -4 + 7 \quad , \quad y = -8 + 7$

$y = 3 \quad , \quad y = -1$

$S.S = \{(-2,3), (-4,-1)\}$

Q9. $x^2 + (y+1)^2 = 18 ; \; (x+2)^2 + y^2 = 21$

Solution:- $x^2 + (y+1)^2 = 18$

$\Rightarrow x^2 + y^2 + 1 + 2y - 18 = 0$

$x^2 + y^2 + 2y - 17 = 0 \longrightarrow (i)$

$(x+2)^2 + y^2 = 21$

$\Rightarrow x^2 + 4 + 4x + y^2 = 21$

$x^2 + y^2 + 4x + 4 - 21 = 0$

$x^2 + y^2 + 4x - 17 = 0 \longrightarrow (ii)$

By (i) − (ii) $\Rightarrow$

$$\begin{array}{r} x^2 + y^2 + 2y - 17 = 0 \\ \underline{-x^2 \pm y^2 \pm 4x \mp 17 = 0} \\ 2y - 4x = 0 \end{array}$$

$\Rightarrow 2y = 4x \Rightarrow y = 2x \longrightarrow (iii)$ put in (ii)

$x^2 + (2x)^2 + 4x - 17 = 0$

$\Rightarrow x^2 + 4x^2 + 4x - 17 = 0$

$5x^2 + 4x - 17 = 0$

using quadratic formula

$x = \dfrac{-4 \pm \sqrt{(4)^2 - 4(5)(-17)}}{2(5)}$

$x = \dfrac{-4 \pm \sqrt{16 + 340}}{10}$

$x = \dfrac{-4 \pm \sqrt{356}}{10}$

$x = \dfrac{-4 \pm \sqrt{4 \times 89}}{10} = \dfrac{-4 \pm 2\sqrt{89}}{10}$

$x = \dfrac{2(-2 \pm \sqrt{89})}{10} = \dfrac{-2 \pm \sqrt{89}}{5}$

$x = \dfrac{-2 + \sqrt{89}}{5} \quad , \quad x = \dfrac{-2 - \sqrt{89}}{5}$

Put in (iii) , Put in (iii)

$y = 2\left(\dfrac{-2 + \sqrt{89}}{5}\right) \quad , \quad y = 2\left(\dfrac{-2 - \sqrt{89}}{5}\right)$

$y = \dfrac{-4 + 2\sqrt{89}}{5} \quad , \quad y = \dfrac{-4 - 2\sqrt{89}}{5}$

$S.S = \left\{\left(\frac{-2+\sqrt{89}}{5}, \frac{-4+2\sqrt{89}}{5}\right), \left(\frac{-2-\sqrt{89}}{5}, \frac{-4-\sqrt{89}}{5}\right)\right\}$

Q10.
$x^2+y^2+6x=1$
$x^2+y^2+2(x+y)=3$

Solution:- $x^2+y^2+6x=1 \longrightarrow (i)$

$x^2+y^2+2(x+y)=3$

$\Rightarrow x^2+y^2+2x+2y=3 \longrightarrow (ii)$

By (ii) − (i) $\Rightarrow$

$$\begin{array}{r} x^2+y^2+2x+2y=3 \\ \underline{\overset{-}{x^2} \overset{-}{+y^2} \overset{-}{+6x} \quad = \overset{+}{-1}} \\ -4x+2y=2 \end{array}$$

$\Rightarrow 2x-y=-1$

$\Rightarrow y=2x+1 \longrightarrow (iii)$ put in (i)

$x^2+(2x+1)^2+6x=1$

$x^2+4x^2+1+4x+6x=1$

$\Rightarrow 5x^2+10x+1-1=0$

$5x^2+10x=0$

$x^2+2x=0$ ($\div$ by 5)

$x(x+2)=0$

$x=0, \quad x=-2$

Put in (iii)	Put in (iii)
$y=2(0)+1$	$y=2(-2)+1$
$y=1$	$y=-4+1$
	$y=-3$

$S.S=\{(0,1), (-2,-3)\}$

Case II. When both equations are Quadratic

(a) When both equations contain x^2 and y^2 terms

See Example 1, Q 1, 2, 3

Example 1. Solve the equations
$x^2+y^2=25, \quad 2x^2+3y^2=66$

Solution:- $x^2+y^2=25$
$2x^2+3y^2=66$

Put $x^2=u$, $y^2=v$ then

$u+v=25 \longrightarrow (i)$
$2u+3v=66 \longrightarrow (ii)$

By (ii) − 2(i) $\Rightarrow$

$$\begin{array}{r} 2u+3v=66 \\ \underline{\overset{-}{2u} \overset{-}{+2v} = \overset{-}{50}} \\ v=16 \end{array}$$

put in (i)

$u+16=25$

$\Rightarrow u=25-16 \Rightarrow u=9$

Now $x^2=9, \quad y^2=16$

$\Rightarrow x=\pm 3, \quad y=\pm 4$

$S.S=\{(\pm 3, \pm 4)\}$

(b) When one of the equations is homogeneous in x and y

"Homogeneous equation":- The equation whose degree is 2 of every term is called homogeneous quadratic equation. e.g., $x^2-3xy+y^2=0$
$ax^2+2xy+by^2=0$

See Example 2, Q 4, 5, 6

Example 2. Solve the equations:
$x^2-3xy+2y^2=0; \quad 2x^2-3x+y^2=24$

Solution:-

$x^2-3xy+2y^2=0, \quad 2x^2-3x+y^2=24 \longrightarrow (i)$

$\Rightarrow x^2-2xy-xy+2y^2=0$

$\Rightarrow x(x-2y)-y(x-2y)=0$

$\Rightarrow (x-2y)(x-y)=0 \Rightarrow x-2y=0, \; x-y=0$

$x=2y$	$x=y$
Put in (i)	Put in (i)
$2(2y)^2-3(2y)+y^2-24=0$	$2y^2-3y+y^2-24=0$
$\Rightarrow 2(4y^2)-6y+y^2-24=0$	$3y^2-3y-24=0$
$8y^2-6y+y^2-24=0$	($\div$ by 3)
$9y^2-6y-24=0$ ($\div$ by 3)	

$3y^2 - 2y - 8 = 0$

$3y^2 - 6y + 4y - 8 = 0$

$3y(y-2) + 4(y-2) = 0$

$(y-2)(3y+4) = 0$

$y-2 = 0, 3y+4 = 0$

$y = 2, \quad y = \frac{-4}{3}$

If $y = 2$ then

$x = 2(2)$

$x = 4$

If $x = \frac{-4}{3}$ then

$x = 2(-\frac{4}{3})$

$x = -\frac{8}{3}$

$y^2 - y - 8 = 0$

using quadratic formula

$y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-8)}}{2(1)}$

$y = \frac{1 \pm \sqrt{1+32}}{2}$

$y = \frac{1 \pm \sqrt{33}}{2}$

$y = \frac{1+\sqrt{33}}{2}, y = \frac{1-\sqrt{33}}{2}$

If $y = \frac{1+\sqrt{33}}{2}$ then

$x = \frac{1+\sqrt{33}}{2}$

If $y = \frac{1-\sqrt{33}}{2}$ then

$x = \frac{1-\sqrt{33}}{2}$

$S.S = \{(4,2), (-\frac{4}{3}, -\frac{8}{3}), (\frac{1+\sqrt{33}}{2}, \frac{1+\sqrt{33}}{2}), (\frac{1-\sqrt{33}}{2}, \frac{1-\sqrt{33}}{2})\}$

(C) When both equations are non-homogeneous

see Example 3, Q 7, 8, 9, 10

Example 3. Solve the equations:

$x^2 - y^2 = 5 \quad ; \quad 4x^2 - 3xy = 18$

Solution:-

$x^2 - y^2 = 5 \longrightarrow (i)$

$4x^2 - 3xy = 18 \longrightarrow (ii)$

'x' (i) by 18 and (ii) by 5, we get

$18x^2 - 18y^2 = 90 \longrightarrow (iii)$

$20x^2 - 15xy = 90 \longrightarrow (iv)$

By (iv) − (iii) ⟹

$$\begin{array}{r} 20x^2 - 15xy = 90 \\ -18x^2 \mp 18y^2 = \underset{-}{90} \\ \hline 2x^2 - 15xy + 18y^2 = 0 \end{array}$$

$\Rightarrow 2x^2 - 15xy + 18y^2 = 0$ (Homogeneous equation)

$\Rightarrow 2x^2 - 12xy - 3xy + 18y^2 = 0$

$2x(x-6y) - 3y(x-6y) = 0$

$(x-6y)(2x-3y) = 0$

$x - 6y = 0$

$x = 6y$ put in (i)

$(6y)^2 - y^2 = 5$

$36y^2 - y^2 = 5$

$35y^2 = 5$

$y^2 = \frac{5}{35}$

$\Rightarrow y^2 = \frac{1}{7} \Rightarrow y = \pm\frac{1}{\sqrt{7}}$

$y = \frac{1}{\sqrt{7}}, \quad y = \frac{-1}{\sqrt{7}}$

If $y = \frac{1}{\sqrt{7}}$ then

$x = 6(\frac{1}{\sqrt{7}}) = \frac{6}{\sqrt{7}}$

If $y = -\frac{1}{\sqrt{7}}$ then

$x = 6(-\frac{1}{\sqrt{7}}) = -\frac{6}{\sqrt{7}}$

$2x - 3y = 0$

$x = \frac{3}{2}y$ Put in (i)

$(\frac{3y}{2})^2 - y^2 = 5$

$\frac{9y^2}{4} - y^2 = 5$

$\Rightarrow 9y^2 - 4y^2 = 20$

$5y^2 = 20$

$y^2 = 4 \Rightarrow y = \pm 2$

$y = 2$ and $y = -2$

If $y = 2$ then

$x = \frac{3}{2}(2) = 3$

If $y = -2$ then

$x = \frac{3}{2}(-2) = -3$

$S.S = \{(\frac{6}{\sqrt{7}}, \frac{1}{\sqrt{7}}), (-\frac{6}{\sqrt{7}}, -\frac{1}{\sqrt{7}}), (3,2), (-3,-2)\}$

Exercise 4.9

Solve the following systems of Equations:

Q1. $2x^2 = 6 + 3y^2 \; ; \; 3x^2 - 5y^2 = 7$

Solution:-

$2x^2 = 6 + 3y^2 \Rightarrow 2x^2 - 3y^2 = 6$

$3x^2 - 5y^2 = 7$

Put $x^2 = u, \; y^2 = v$ then

$2u - 3v = 6 \longrightarrow (i)$

$3u - 5v = 7 \longrightarrow (ii)$

'x' (i) by 3 and (ii) by 2

$6u - 9v = 18 \longrightarrow (iii)$

$6u - 10v = 14 \longrightarrow (iv)$

By (iv) − (iii) ⟹

$$\begin{array}{r} 6u - 10v = 14 \\ \underset{-}{6u} \underset{+}{-} 9v = \underset{-}{18} \\ \hline -v = -4 \end{array}$$

$\Rightarrow v = 4$ put in (i)

$2u - 3(4) = 6$

$2u - 12 = 6 \Rightarrow 2u = 6 + 12$

$2u = 18 \Rightarrow u = 9$

If $u=9$ then
$x^2=9$
$\rightarrow x=\pm 3$

If $v=4$ then
$y^2=4$
$\rightarrow y=\pm 2$

S.S $=\{(\pm 3, \pm 2)\}$

Q2. $8x^2=y^2$; $x^2+2y^2=19$

Solution:-

$8x^2=y^2 \rightarrow 8x^2-y^2=0$

$x^2+2y^2=19$

Put $x^2=u$ and $y^2=v$ then

$8u-v=0 \longrightarrow (i)$

$u+2v=19 \longrightarrow (ii)$

'x' (i) by 2 we get

$16u-2v=0 \longrightarrow (iii)$

By (ii)+(iii) $\rightarrow$
$$\begin{array}{r} u+2v=19 \\ 16u-2v=0 \\ \hline 17u=19 \end{array}$$

$\rightarrow u=\frac{19}{17}$ put in (i)

$8\left(\frac{19}{17}\right)-v=0 \rightarrow v=\frac{152}{17}$

If $u=\frac{19}{17}$ then
$x^2=\frac{19}{17}$
$\rightarrow x=\pm\sqrt{\frac{19}{17}}$

If $v=\frac{152}{17}$ then
$\rightarrow y^2=\frac{152}{17}$
$y=\pm\sqrt{\frac{152}{17}}$
$=\pm\sqrt{\frac{4\times 38}{17}}$
$=\pm 2\sqrt{\frac{38}{17}}$

S.S $=\left\{\pm\sqrt{\frac{19}{17}}, \pm 2\sqrt{\frac{38}{17}}\right\}$

Q3. $2x^2-8=5y^2$; $x^2-13=-2y^2$

Solution:-

$2x^2-8=5y^2 \rightarrow 2x^2-5y^2=8$

$x^2-13=-2y^2 \rightarrow x^2+2y^2=13$

Put $x^2=u$, $y^2=v$ then

$2u-5v=8 \longrightarrow (i)$

$u+2v=13 \longrightarrow (ii)$

'x' (ii) by 2 we get

$2u+4v=26 \longrightarrow (iii)$

By (iii) − (i) $\rightarrow$
$$\begin{array}{r} 2u+4v=26 \\ \underline{-}2u\,\underline{+}\,5v=\underline{-}8 \\ \hline 9v=18 \end{array}$$

$9v=18 \rightarrow v=2$ put in (ii)

$u+2(2)=13 \rightarrow u=13-4$, $u=9$

If $u=9$ then
$x^2=9$
$\rightarrow x=\pm 3$

If $v=2$ then
$y^2=2$
$\rightarrow y=\pm\sqrt{2}$

S.S $=\{(\pm 3, \pm\sqrt{2})\}$

Q4. $x^2-5xy+6y^2=0$; $x^2+y^2=45$

Solution:-

$x^2-5xy+6y^2=0$, $x^2+y^2=45 \longrightarrow (i)$

$\rightarrow x^2-3xy-2xy+6y^2=0$

$x(x-3y)-2y(x-3y)=0$

$(x-3y)(x-2y)=0$

$x-3y=0$, $x-2y=0$

$x=3y$, $x=2y$

If $x=3y$ then

(i) $\rightarrow (3y)^2+y^2=45$

$\rightarrow 9y^2+y^2=45$

$10y^2=45$

$y^2=\frac{45}{10}$

$y^2=\frac{9}{2}$

$y=\pm\frac{3}{\sqrt{2}}$

$y=\frac{3}{\sqrt{2}}$, $y=-\frac{3}{\sqrt{2}}$

If $y=\frac{3}{\sqrt{2}}$ then

$x=3\left(\frac{3}{\sqrt{2}}\right)$

$x=\frac{9}{\sqrt{2}}$

If $y=-\frac{3}{\sqrt{2}}$ then

$x=3\left(-\frac{3}{\sqrt{2}}\right)=\frac{-9}{\sqrt{2}}$

If $x=2y$ then

(i) $\rightarrow (2y)^2+y^2=45$

$\rightarrow 4y^2+y^2=45$

$5y^2=45$

$y^2=9$

$y=\pm 3$

$y=3$, $y=-3$

If $y=3$ then

$x=2(3)=6$

If $y=-3$ then

$x=2(-3)=-6$

S.S $=\left\{(6,3), (-6,-3), \left(\frac{9}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right), \left(\frac{-9}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right)\right\}$

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Q5. $12x^2 - 25xy + 12y^2 = 0$
$4x^2 + 7y^2 = 148$

Solution:-

$12x^2 - 25xy + 12y^2 = 0$, $4x^2 + 7y^2 = 148 \longrightarrow (i)$

$\rightarrow 12x^2 - 16xy - 9xy + 12y^2 = 0$

$4x(3x - 4y) - 3y(3x - 4y) = 0$

$(3x - 4y)(4x - 3y) = 0$

$3x - 4y = 0$, $4x - 3y = 0$

$3x = 4y$, $4x = 3y$

$x = \frac{4}{3}y$, $x = \frac{3y}{4}$

Put $x = \frac{4}{3}y$ in (i)

$4\left(\frac{4}{3}y\right)^2 + 7y^2 = 148$

$4\left(\frac{16}{9}y^2\right) + 7y^2 = 148$

$\frac{64}{9}y^2 + 7y^2 = 148$

$64y^2 + 63y^2 = 1332$

$127y^2 = 1332$

$y^2 = \frac{1332}{127}$

$y = \pm\sqrt{\frac{1332}{127}}$

$y = \pm\sqrt{\frac{4 \times 333}{127}}$

$y = \pm 2\sqrt{\frac{333}{127}}$

$y = 2\sqrt{\frac{333}{127}}$, and $y = -2\sqrt{\frac{333}{127}}$

If $y = 2\sqrt{\frac{333}{127}}$ then

$x = \frac{4}{3}\left(2\sqrt{\frac{333}{127}}\right)$

$x = \frac{8}{3}\sqrt{\frac{333}{127}}$

If $y = -2\sqrt{\frac{333}{127}}$ then

$x = \frac{4}{3}\left(-2\sqrt{\frac{333}{127}}\right)$

Put $x = \frac{3}{4}y$ in (i)

$4\left(\frac{3}{4}y\right)^2 + 7y^2 = 148$

$4\left(\frac{9}{16}y^2\right) + 7y^2 = 148$

$\frac{9}{4}y^2 + 7y^2 = 148$

$\rightarrow 9y^2 + 28y^2 = 592$

$37y^2 = 592$

$y^2 = \frac{592}{37}$

$y^2 = 16$

$\rightarrow y = \pm 4$

$y = 4$, $y = -4$

If $y = 4$ then

$x = \frac{3}{4}(4) = 3$

If $y = -4$ then

$x = \frac{3}{4}(-4) = -3$

$x = -\frac{8}{3}\sqrt{\frac{333}{127}}$

$S.S = \left\{\left(\frac{8}{3}\sqrt{\frac{333}{127}}, 2\sqrt{\frac{333}{127}}\right), \left(-\frac{8}{3}\sqrt{\frac{333}{127}}, -2\sqrt{\frac{333}{127}}\right), (3,4), (-3,-4)\right\}$

Q6. $12x^2 - 11xy + 2y^2 = 0$; $2x^2 + 7xy = 60$

Solution:-

$12x^2 - 11xy + 2y^2 = 0$, $2x^2 + 7xy = 60 \longrightarrow (i)$

$12x^2 - 8xy - 3xy + 2y^2 = 0$

$4x(3x - 2y) - y(3x - 2y) = 0$

$(3x - 2y)(4x - y) = 0$

$3x - 2y = 0$, $4x - y = 0$

$3x = 2y$, $4x = y$

$x = \frac{2}{3}y$, $x = \frac{1}{4}y$

Put $x = \frac{2}{3}y$ in (i)

$2\left(\frac{2}{3}y\right)^2 + 7\left(\frac{2}{3}y\right)y = 60$

$\frac{8}{9}y^2 + \frac{14}{3}y^2 = 60$

$\rightarrow 8y^2 + 42y^2 = 540$

$50y^2 = 540$

$y^2 = \frac{540}{50}$

$y^2 = \frac{54}{5}$

$y = \pm\sqrt{\frac{54}{5}} = \pm\sqrt{\frac{9 \times 6}{5}}$

$y = \pm 3\sqrt{\frac{6}{5}}$

$y = 3\sqrt{\frac{6}{5}}$, $y = -3\sqrt{\frac{6}{5}}$

If $y = 3\sqrt{\frac{6}{5}}$ then

$x = \frac{2}{3}\left(3\sqrt{\frac{6}{5}}\right)$

$x = 2\sqrt{\frac{6}{5}}$

If $y = -3\sqrt{\frac{6}{5}}$ then

$x = \frac{2}{3}\left(-3\sqrt{\frac{6}{5}}\right)$

$x = -2\sqrt{\frac{6}{5}}$

Put $x = \frac{1}{4}y$ in (i)

$2\left(\frac{1}{4}y\right)^2 + 7\left(\frac{1}{4}y\right)y = 60$

$\frac{2}{16}y^2 + \frac{7}{4}y^2 = 60$

$\frac{1}{8}y^2 + \frac{7}{4}y^2 = 60$

$y^2 + 14y^2 = 480$

$15y^2 = 480$

$y^2 = \frac{480}{15} = 32$

$y^2 = 32$

$y = \pm\sqrt{32}$

$= \pm\sqrt{16 \times 2}$

$y = \pm 4\sqrt{2}$

$y = 4\sqrt{2}$, $y = -4\sqrt{2}$

If $y = 4\sqrt{2}$ then

$x = \frac{1}{4}(4\sqrt{2}) = \sqrt{2}$

If $y = -4\sqrt{2}$ then

$x = \frac{1}{4}(-4\sqrt{2}) = -\sqrt{2}$

$S.S = \left\{\left(2\sqrt{\frac{6}{5}}, 3\sqrt{\frac{6}{5}}\right), \left(-2\sqrt{6/5}, -3\sqrt{6/5}\right), (\sqrt{2}, 4\sqrt{2}), (-\sqrt{2}, -4\sqrt{2})\right\}$

Q7. $x^2 - y^2 = 16$; $xy = 15$

Solution:-

$x^2 - y^2 = 16 \longrightarrow (i)$

$xy = 15 \longrightarrow (ii)$

'x' (i) by 15 and (ii) by 16, we get

$15x^2 - 15y^2 = 240 \longrightarrow (iii)$

$16xy = 240 \longrightarrow (iv)$

By (iii) − (iv) ➔ $15x^2 - 15y^2 = 240$

$\underline{-16xy \quad = -240}$

$15x^2 - 15y^2 - 16xy = 0$

➔ $15x^2 - 16xy - 15y^2 = 0$ (Homogeneous equation)

$15x^2 - 25xy + 9xy - 15y^2 = 0$

$5x(3x - 5y) + 3y(3x - 5y) = 0$

$(3x - 5y)(5x + 3y) = 0$

$3x - 5y = 0$

$3x = 5y$

$x = \frac{5}{3}y$

Put $x = \frac{5}{3}y$ in (ii)

$(\frac{5}{3}y)y = 15$

$\frac{5}{3}y^2 = 15$

➔ $y^2 = 15 \times \frac{3}{5}$

$y^2 = 9$

$y = \pm 3$

$y = 3, \quad y = -3$

If $y = 3$ then

$x = \frac{5}{3}(3)$

$x = 5$

If $y = -3$ then

$x = \frac{5}{3}(-3) = -5$

$5x + 3y = 0$

$5x = -3y$

$x = -\frac{3}{5}y$

Put $x = \frac{-3}{5}y$ in (ii)

$(-\frac{3}{5}y)y = 15$

$y^2 = 15 \times (\frac{5}{-3})$

$y^2 = -25$

➔ $y = \pm\sqrt{-25}$

$y = \pm 5i$

$y = 5i, \quad y = -5i$

If $y = 5i$ then

$x = -\frac{3}{5}(5i) = -3i$

If $y = -5i$ then

$x = -\frac{3}{5}(-5i) = 3i$

$S.S = \{(5,3), (-5,-3), (-3i, 5i), (3i, -5i)\}$

Q8. $x^2 + xy = 9$; $x^2 - y^2 = 2$

Solution:- $x^2 + xy = 9 \longrightarrow (i)$

$x^2 - y^2 = 2 \longrightarrow (ii)$

'x' (i) by 2 and (ii) by 9, we get

$2x^2 + 2xy = 18 \longrightarrow (iii)$

$9x^2 - 9y^2 = 18 \longrightarrow (iv)$

By (iv) − (iii) ➔ $9x^2 - 9y^2 = 18$

$\underline{-2x^2 \quad -2xy = -18}$

$7x^2 - 2xy - 9y^2 = 0$

➔ $7x^2 - 2xy - 9y^2 = 0$ (Homogeneous equation)

$7x^2 + 7xy - 9xy - 9y^2 = 0$

$7x(x + y) - 9y(x + y) = 0$

$(x + y)(7x - 9y) = 0$

$x + y = 0$

$x = -y$

Put in (ii)

$(-y)^2 - y^2 = 2$

$y^2 - y^2 = 2$

➔ $0 = 2$

Impossible

$7x - 9y = 0$

$7x = 9y$

$x = \frac{9}{7}y$

Put in (ii)

$(\frac{9}{7}y)^2 - y^2 = 2$

$\frac{81}{49}y^2 - y^2 = 2$

$81y^2 - 49y^2 = 98$

$32y^2 = 98$

$y^2 = \frac{98}{32} = \frac{49}{16}$

$y = \pm\frac{7}{4}$

$y = \frac{7}{4}, \quad y = -\frac{7}{4}$

If $y = \frac{7}{4}$ then

$x = \frac{9}{7}(\frac{7}{4}) = \frac{9}{4}$

If $x = -\frac{7}{4}$ then

$x = \frac{9}{7}(-\frac{7}{4}) = -\frac{9}{4}$

$S.S = \{(-\frac{9}{4}, -\frac{7}{4}), (\frac{9}{4}, \frac{7}{4})\}$

Q9. $y^2-7=2xy$; $2x^2+3=xy$

Solution:-

$y^2-7=2xy \Rightarrow y^2-2xy=7 \longrightarrow (i)$

$2x^2+3=xy \Rightarrow 2x^2-xy=-3 \longrightarrow (ii)$

'x' (i) by 3 and (ii) by 7, we get

$3y^2-6xy=21 \longrightarrow (iii)$

$14x^2-7xy=-21 \longrightarrow (iv)$

By (iii)+(iv) $\Rightarrow$ $3y^2-6xy=21$

$14x^2-7xy=-21$

$14x^2+3y^2-13xy=0$

$\Rightarrow 14x^2-13xy+3y^2=0$

$14x^2-7xy-6xy+3y^2=0$

$7x(2x-y)-3y(2x-y)=0$

$(2x-y)(7x-3y)=0$

$2x-y=0$, $7x-3y=0$

$x=\frac{y}{2}$; $x=\frac{3}{7}y$

Put in (i)

$y^2-2(\frac{1}{2}y)y=7$

$y^2-y^2=7$

$0=7$

Impossible

Put in (i)

$y^2-2(\frac{3}{7}y)y=7$

$y^2-\frac{6}{7}y^2=7$

$7y^2-6y^2=49$

$y^2=49$

$y=\pm7$

$y=7,\ y=-7$

If $y=7$ then $x=\frac{3}{7}(7)=3$

If $y=-7$ then $x=\frac{3}{7}(-7)=-3$

S.S = $\{(3,7),(-3,-7)\}$

Q10. $x^2+y^2=5$; $xy=2$

Solution:-

$x^2+y^2=5 \longrightarrow (i)$

$xy=2 \longrightarrow (ii)$

'x' (i) by 2 and (ii) by 5, we get

$2x^2+2y^2=10 \longrightarrow (iii)$

$5xy=10 \longrightarrow (iv)$

By (iii)-(iv) $\Rightarrow$ $2x^2+2y^2=10$

$5xy=10$

$2x^2-5xy+2y^2=0$

$\Rightarrow 2x^2-5xy+2y^2=0$

$2x^2-4xy-xy+2y^2=0$

$2x(x-2y)-y(x-2y)=0$

$(x-2y)(2x-y)=0$

$x-2y=0$; $2x-y=0$

$x=2y$; $x=\frac{1}{2}y$

Put in (ii)

$(2y)y=2$

$2y^2=2$

$y^2=1$

$y=\pm1$

$y=1,\ y=-1$

If $y=1$ then

$x=2(1)=2$

If $y=-1$ then

$x=2(-1)=-2$

Put in (ii)

$(\frac{1}{2}y)y=2$

$\frac{1}{2}y^2=2$

$y^2=4$

$y^2=4$

$y=\pm\sqrt{4}=\pm2$

$y=2,\ y=-2$

If $y=2$ then

$x=-\frac{1}{2}(2)=-1$

If $y=-2$ then

$y=-\frac{1}{2}(-2)=1$

S.S = $\{(2,1),(-2,-1),(1,2),(-1,-2)\}$

Problems on Quadratic Equations

Remember following steps to solve problems expressed symbolically, lead to quadratic equations in one or two variables.

1) Suppose the unknown quantities x or y etc.

2) Translate the problem into symbols

e.g., a) 5 is greater than 3 by 2 $= (5-3)$

b) x is greater than 3 by $x-3$

c) 5 is greater than y by $5-y$

d) x is greater than y by $x-y$

Example 1. Divide 12 into two parts such that the sum of their squares is greater than twic their product by 4.

Solution:- Let one part $= x$

other part $= 12-x$

Sum of the squares of the parts $= x^2 + (12-x)^2$

Twice the product of the parts $= 2x(12-x)$

By given condition

$x^2 + (12-x)^2 - 2x(12-x) = 4$

$\rightarrow x^2 + 144 + x^2 - 24x - 24x + 2x^2 = 4$

$\rightarrow 4x^2 - 48x + 140 = 0$

$\rightarrow x^2 - 12x + 35 = 0$ (÷ by 5)

$x^2 - 7x - 5x + 35 = 0$

$x(x-7) - 5(x-7) = 0$

$(x-7)(x-5) = 0$

$\rightarrow x-7 = 0, \quad x-5 = 0$

$x = 7, \quad x = 5$

If one part is 7 then other part is $12-7 = 5$

If one part is 5 then other part is $12-5 = 7$

Example 2. A man distributed Rs. 1000 equally among destitutes of his street. Had there been 5 more destitutes each one would have received Rs. 10 less. Find the number of destitutes.

Solution:-

Let number of destitutes $= x$

Total amount $= 1000$ Rs.

$\therefore$ Each destitute gets $= \frac{1000}{x}$ Rs.

For 5 more destitutes we have total destitutes $= x+5$

Now

Amount given to each destitute $= \frac{1000}{x+5}$

According to given condition

$\frac{1000}{x+5} = \frac{1000}{x} - 10$

$\rightarrow 1000x = 1000(x+5) - 10x(x+5)$

$1000x = 1000x + 5000 - 10x^2 - 50x$

$\rightarrow 0 = 5000 - 10x^2 - 50x$

$\rightarrow 10x^2 + 50x - 5000 = 0$

$x^2 + 5x - 500 = 0$ (÷ by 10)

$x^2 + 25x - 20x - 500 = 0$

$x(x+25) - 20(x+25) = 0$

$(x+25)(x-20) = 0$

$x+25 = 0, \quad x-20 = 0$

$x = -25, \quad x = 20$

(impossible)

Hence number of destitutes $= 20$

Example 3. The length of a room is 3 meters greater than its breadth. If the area of the room is 180 square meters, find length and the breadth of the room.

Solution:-

Let breadth of room = xm

then length of room = $x + 3$m

we know that

Area = length x width

$\Rightarrow$ Area = $(x+3) \times x$

$180 = x^2 + 3x$

$\Rightarrow x^2 + 3x - 180 = 0$

$x^2 + 15x - 12x - 180 = 0$

$x(x+15) - 12(x+15) = 0$

$(x+15)(x-12) = 0$

$x + 15 = 0$, $x - 12 = 0$

$x = -15$, $x = 12$

so breadth of room = 12m

length of room = 12 + 3 = 15m

Example 4. A number consists of two digits whose product is 8. If the digits are interchanged, the resulting number will exceed the original one by 18. Find the number.

Solution:-

Let unit digit = x

tens digit = y

then number = $x + 10y$

According to condition given

$xy = 8 \longrightarrow (i)$

$x + 10y + 18 = y + 10x$

$x + 10y + 18 - y - 10x = 0$

$\Rightarrow 9y - 9x + 18 = 0$

$y - x + 2 = 0$ ($\div$ by 9)

$\Rightarrow y = x - 2$ put in (i)

$x(x-2) = 8$

$\Rightarrow x^2 - 2x - 8 = 0$

$x^2 + 2x - 4x - 8 = 0$

$x(x+2) - 4(x+2) = 0$

$\Rightarrow (x+2)(x-4) = 0$

$x + 2 = 0$, $x - 4 = 0$

$x = -2$, $x = 4$

If $x = 4$ then , If $x = -2$ then

$y = 4 - 2 = 2$, $y = -2 - 2 = -4$

so number = $x + 10y$, so number = $x + 10y$

$= 4 + 10(2)$, $= -2 + 10(-4)$

$= 24$, $= -2 - 40$

$= -42$

Required number is 24 or -42.

Exercise 4.10

Q1. The product of one less than a certain positive number and two less than three time the number is 14 find the number?

Solution:-

Let x be +ive number then one less than +ive number = $x - 1$

two less than three times = $3x - 2$

According to given condition

$(x-1)(3x-2) = 14$

$\Rightarrow 3x^2 - 2x - 3x + 2 - 14 = 0$

$3x^2 - 5x - 12 = 0$

$\Rightarrow 3x^2 - 9x + 4x - 12 = 0$

$3x(x-3) + 4(x-3) = 0$

$(x-3)(3x+4) = 0$

$\Rightarrow x - 3 = 0$, $3x + 4 = 0$

$x = 3$, $x = -\frac{4}{3}$

So $x = 3$, $x = -\frac{4}{3}$ (impossible being -ive)

Q2. The sum of a positive number and its square is 380. Find a number.

Solution:-

Let a positive number $= x$

its square $= x^2$

According to given condition

$x + x^2 = 380$

$\rightarrow x + x^2 - 380 = 0 \rightarrow x^2 + x - 380 = 0$

$\rightarrow x^2 + 20x - 19x - 380 = 0$

$x(x+20) - 19(x+20) = 0$

$(x+20)(x-19) = 0$

$x + 20 = 0, \quad x - 19 = 0$

$x = -20, \quad x = 19$

$x = -20$ (impossible being –ive). so $x = 19$

Q3. Divide 40 into two parts such that the sum of their squares is greater than 2 times their product by 100.

Solution:-

Let one part $= x$, Another part $= 40 - x$

According to given condition

$\rightarrow x^2 + (40-x)^2 = 2x(40-x) + 100$

$\rightarrow x^2 + 1600 + x^2 - 80x = 80x - 2x^2 + 100$

$\rightarrow x^2 + 1600 + x^2 - 80x - 80x + 2x^2 - 100 = 0$

$\rightarrow 4x^2 - 160x + 1500 = 0$

$\rightarrow x^2 - 40x + 375 = 0 \quad (\div \text{ by } 4)$

$x^2 - 25x - 15x + 375 = 0$

$x(x-25) - 15(x-25) = 0$

$(x-25)(x-15) = 0$

$\rightarrow x - 25 = 0, \quad x - 15 = 0$

Thus $x = 25, \quad x = 15$

Q4. The sum of a positive number and its reciprocal is $\frac{26}{5}$. Find the number.

Solution:-

Let the number $= x$, its reciprocal is $= 1/x$

According to given condition

$x + \frac{1}{x} = \frac{26}{5}$

$\rightarrow 5\left(x + \frac{1}{x}\right) = 26$

$\rightarrow 5\left(\frac{x^2+1}{x}\right) = 26$

$\rightarrow 5(x^2+1) = 26x$

or $5x^2 + 5 - 26x = 0$

$\rightarrow 5x^2 - 26x + 5 = 0$

$5x^2 - 25x - x + 5 = 0$

$5x(x-5) - 1(x-5) = 0$

$(x-5)(5x-1) = 0$

$x - 5 = 0, \quad 5x - 1 = 0$

Thus $x = 5, \quad x = 1/5$

Q5. A number exceeds its square root by 56. Find the number.

Solution:-

Let the number $= x$, its square root is $= \sqrt{x}$

According to given condition

$x = \sqrt{x} + 56$

$\rightarrow x - 56 = \sqrt{x}$

$\rightarrow (x - 56)^2 = (\sqrt{x})^2$

$\rightarrow x^2 - 112x + 3136 = x$

$\rightarrow x^2 - 112x - x + 3136 = 0$

$x^2 - 113x + 3136 = 0$

$\rightarrow x^2 - 64x - 49x + 3136 = 0$

$x(x-64) - 49(x-64) = 0$

$(x-64)(x-49) = 0$

$x - 64 = 0, \quad x - 49 = 0$

$x = 64$, $x = 49$ does not satisfy given condition

so $x = 64$

Q 6. Find two consective number, whose product is 132.

Solution:-

Let x and $x+1$ be two consective numbers then

According to given condition

$x(x+1) = 132$

$\rightarrow x^2 + x - 132 = 0$

$\rightarrow x^2 + 12x - 11x - 132 = 0$

$\rightarrow x(x+12) - 11(x+12) = 0$

$(x+12)(x-11) = 0$

$x+12 = 0 , \quad x - 11 = 0$

$\rightarrow x = -12 , \quad x = 11$

Thus if $x = -12$ then $x+1 = -12+1 = -11$

if $x = 11$ then $x+1 = 11+1 = 12$

so required numbers are

$11, 12$ or $-11, -12$

Q7. The difference between the cubes of two consective even numbers is 296. Find them.

Solution:- Let x and $x+2$ be two consective even numbers then

according to given condition

$(x+2)^3 - x^3 = 296$

$\rightarrow x^3 + 8 + 6x^2 + 12x - x^3 - 296 = 0$

$(\because (a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2)$

$6x^2 + 12x - 288 = 0$

$\rightarrow x^2 + 2x - 48 = 0 \quad (\div \text{ by } 6)$

$x^2 + 8x - 6x - 48 = 0$

$x(x+8) - 6(x+8) = 0$

$(x+8)(x-6) = 0$

$\rightarrow x+8 = 0, \quad x - 6 = 0$

$x = -8 , \quad x = 6$

If $x = -8$ then $x+2 = -8+2 = -6$

If $x = 6$ then $x+2 = 6+2 = 8$

so required numbers are

$-8, -6$ or $6, 8$

Q8. A farmer bought some sheep for Rs. 9000. If he paid Rs. 100 less for each, he would have got 3 sheep more for the same money. How much sheep did he buy, when the rate in each case is uniform?

Solution:-

Let x be number of sheep.

Amount for x sheep $= 9000$

$\rightarrow$ Amount for 1 sheep $= \frac{9000}{x}$

According to condition

$\frac{9000}{x} - 100 = \frac{9000}{x+3}$

$\rightarrow x(x+3)\frac{9000}{x} - x(x+3)100 = \frac{9000}{x+3} \cdot x(x+3)$

$\rightarrow 9000(x+3) - (x^2+3x)100 = 9000x$

$\rightarrow 90(x+3) - (x^2+3x) = 90x \quad (\div \text{ by } 100)$

$\rightarrow 90x + 270 - x^2 - 3x - 90x = 0$

$\rightarrow -x^2 - 3x + 270 = 0$

$x^2 + 3x - 270 = 0$

$x^2 + 18x - 15x - 270 = 0$

$x(x+18) - 15(x+18) = 0$

$(x+18)(x-15) = 0$

$\rightarrow x+18 = 0, \quad x - 15 = 0$

$x = -18 , \quad x = 15$

Thus $x = 15$, $x = -18$ (not possible)

so $x = 15$ is number of sheep.

Q9. A man sold his stock of eggs for Rs. 240. If he had 2 dozen more, he would have got the same money by selling the whole for Rs. 0.50 per dozen cheaper. How many dozen eggs did he sell?

Solution:-

Let number of eggs $= x$ dozen

Rate of x dozen eggs $= 240$

$\rightarrow$ Rate of 1 dozen eggs $= \frac{240}{x}$

If 2 dozen are more than

rate of one dozen $= \frac{240}{x+2}$

According to condition

$$\frac{240}{x} - 0.5 = \frac{240}{x+2}$$

$$\rightarrow \frac{240 - 0.5x}{x} = \frac{240}{x+2}$$

$$(x+2)(240 - 0.5x) = 240x$$

$$240x - 0.5x^2 + 480 - x = 240x$$

$$\rightarrow 240x - 0.5x^2 + 480 - x - 240x = 0$$

$$\rightarrow -0.5x^2 - x + 480 = 0$$

$$\rightarrow x^2 + 2x - 960 = 0 \quad (\text{'x' by } -2)$$

$$x^2 + 32x - 30x - 960 = 0$$

$$\rightarrow x(x+32) - 30(x+32) = 0$$

$$(x+32)(x-30) = 0$$

$$\rightarrow x + 32 = 0, \quad x - 30 = 0$$

$$x = -32, \quad x = 30$$

$x = -32$ (Not possible)

so $x = 30$ dozen $=$ number of eggs

Q10. A cyclist travelled 48 km at a uniform speed. Had he travelled 2 km/hour slower, he would have taken 2 hours more to perform the journey. How long did he take to cover 48 km?

Solution:-

Let speed $= v$; time $= t$

According to condition

Distance $= s = vt = 48 \longrightarrow$ (i)

and $(v-2)(t+2) = 48 \longrightarrow$ (ii)

$\left(\because \begin{array}{l} \text{two km slow} = v-2 \\ \text{two hour more} = t+2 \end{array} \right)$

$$\rightarrow vt + 2v - 2t - 4 - 48 = 0$$

$$\rightarrow 48 + 2v - 2t - 4 - 48 = 0$$

$$2v - 2t - 4 = 0$$

$$\rightarrow v - t - 2 = 0 \quad (\div \text{ by } 2)$$

or $v = t + 2 \longrightarrow$ (iii) put in (i)

$$\text{(i)} \rightarrow (t+2)t = 48$$

$$t^2 + 2t - 48 = 0$$

$$t^2 + 8t - 6t - 48 = 0$$

$$\rightarrow t(t+8) - 6(t+8) = 0$$

$$\text{or } (t+8)(t-6) = 0$$

$$t + 8 = 0, \quad t - 6 = 0$$

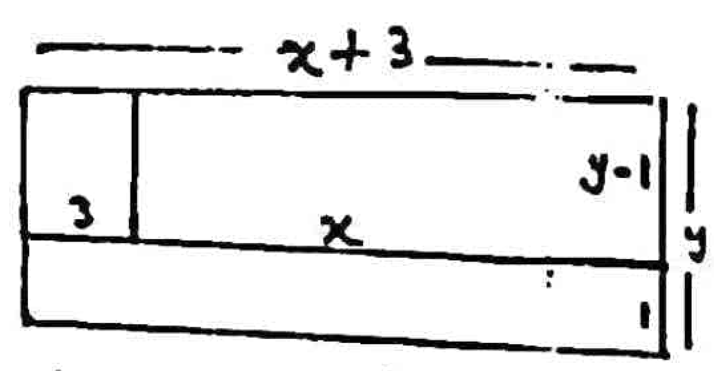
$\rightarrow t = -8$ (ignore), so $t = 6$

Q11. The area of a rectangular field is 297 square meters. Had it been 3 meters longer and one meter shorter, the area would have been 3 square meters more. Find its length and breadth.

Solution:- Let

length $= x$

breadth $= y$



According to the condition

$xy = 297 \longrightarrow$ (i) and

$$(x+3)(y-1) = 297 + 3$$

$$\rightarrow xy - x + 3y - 3 - 300 = 0$$

$\Rightarrow 297 - x + 3y - 3 - 300 = 0 \quad \because xy = 297$

or $-x + 3y - 6 = 0$

$x - 3y + 6 = 0$

or $x = 3y - 6$ put in (i)

$(3y-6)(y) = 297$

or $3y^2 - 6y - 297 = 0$

$\Rightarrow y^2 - 2y - 99 = 0 \quad (\div \text{ by } 3)$

$\Rightarrow y^2 - 11y + 9y - 99 = 0$

$\Rightarrow y(y-11) + 9(y-11) = 0$

$(y-11)(y+9) = 0$

$\Rightarrow y - 11 = 0, \quad y + 9 = 0$

$y = 11, \quad y = -9$ (Not possible)

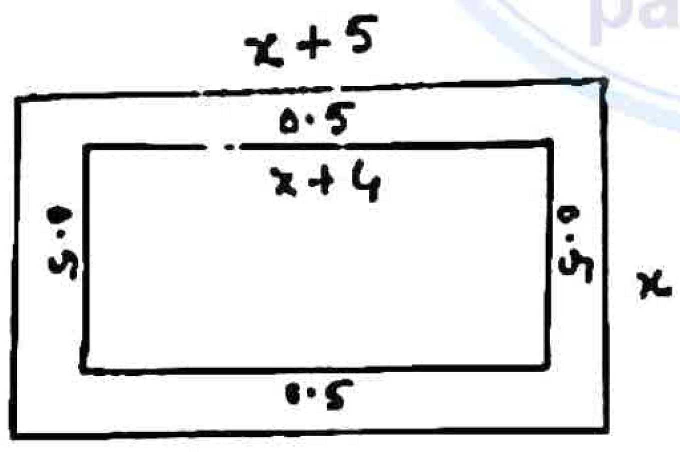
so when $y = 11$, $x = \frac{297}{11} = 27$

$\Rightarrow$ length $= 27$, breadth $= 11$

Q12. The length of a rectangular piece of paper exceeds its breadth by 5cm. If a strip 0.5 cm wide be cut all around the peice of paper, the area of the remaining part would be 500 square cms. Find its orignal dimensions.

Solution:-

Let
breadth $= x$,
length $= x+5$



New length $= x + 5 - 0.5 - 0.5 = x + 4$

and new breadth $= x - 0.5 - 0.5$
$= x - 1$

According to condition

$(x-1)(x+4) = 500$

$\Rightarrow x^2 + 4x - x - 4 - 500 = 0$

$x^2 + 3x - 504 = 0$

$\Rightarrow x^2 + 24x - 21x - 504 = 0$

$\Rightarrow x(x+24) - 21(x+24) = 0$

$(x+24)(x-21) = 0$

$\Rightarrow x + 24 = 0, \quad x - 21 = 0$

$x = -24$ Not possible, $\quad x = 21$

so when $x = 21$ then length $= x+5$
$= 21+5$
length $= 26$

Thus length $= 26$, breadth $= 21$

Q13. A number consists of two digits whose product is 18. If the digits are interchanged, the new number becomes 27 less than the orignal number. Find the number.

Solution:-

Let digits are x and y then

$xy = 18 \longrightarrow (i)$

Number $= 10x + y$

Reverse $= x + 10y$

According to condition

$x + 10y = 10x + y - 27$

$\Rightarrow 10x + y - x - 10y - 27 = 0$

$9x - 9y - 27 = 0$

$\Rightarrow x - y - 3 = 0 \quad (\div \text{ by } 9)$

$\Rightarrow y = x - 3$ put in (i)

$x(x-3) = 18$

$x^2 - 3x - 18 = 0$

$x^2 - 6x + 3x - 18 = 0$

$x(x-6) + 3(x-6) = 0$

$(x-6)(x+3) = 0$

$\Rightarrow x - 6 = 0, \quad x + 3 = 0$

$x = 6, \quad x = -3$ (ignore)

when $x = 6$ then $6y = 18 \Rightarrow y = 3$

so number $= 10x + y = 10(6) + 3$

number $= 63$

Q14. A number consists of two digits whose product is 14. If the digits are interchanged, the resulting number will exceed the orignal number 45. Find the number.

Solution:- Let digits are x and y

then Two digit number $= 10x + y$

Reversed $= x + 10y$

According to condition

$xy = 14 \longrightarrow (i)$ and

$(x + 10y) = (10x + y) + 45$

$\Rightarrow 10x + y - x - 10y + 45 = 0$

$9x - 9y + 45 = 0$

$\Rightarrow x - y + 5 = 0 \quad (\div \text{ by } 9)$

$\Rightarrow y = x + 5$ put in (i)

$x(x+5) = 14$

$\Rightarrow x^2 + 5x - 14 = 0$

$x^2 + 7x - 2x - 14 = 0$

$x(x+7) - 2(x+7) = 0$

$(x+7)(x-2) = 0$

$x + 7 = 0, \quad x - 2 = 0$

$x = -7$ (ignore), $\quad x = 2$

If $x = 2$ then

$2y = 14 \Rightarrow y = 7$ so

number $= 10x + y = 10(2) + 7$

$\Rightarrow$ number $= 20 + 7 = 27$

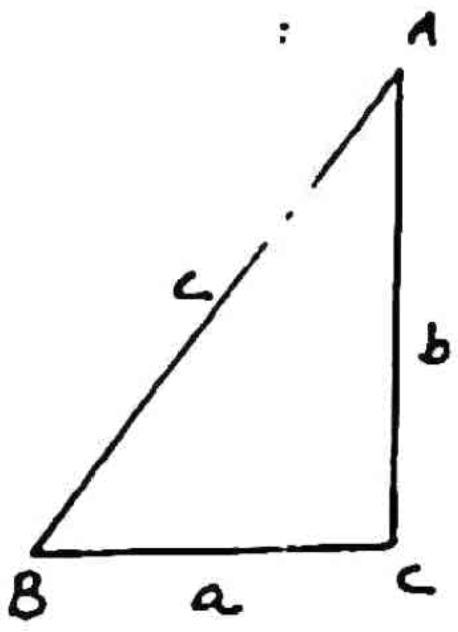
Q15. The area of a right triangle is 210 square meters. If its hypoteneuse is 37 meters long. Find the length of the base and the altitude.

Solution:-

Let base $= a$

and altitude $= b$

then



area $= \frac{1}{2}$(base)(altitude)

$\Rightarrow \frac{1}{2}ab = 210$

or $ab = 420 \longrightarrow (i)$

$2ab = 840 \longrightarrow (ii)$

By pythagoras theorem

$a^2 + b^2 = c^2$

$\Rightarrow a^2 + b^2 = (37)^2$

or $a^2 + b^2 = 1369 \longrightarrow (iii)$

By (iii) − (ii) $\Rightarrow$

$$\begin{array}{r} a^2 + b^2 = 1369 \\ \underline{-2ab = -840} \\ a^2 - 2ab + b^2 = 529 \end{array}$$

$\Rightarrow (a-b)^2 = (23)^2$

or $a - b = 23$

$\Rightarrow b = a - 23$ put in (i)

$(a - 23)a = 420$

$\Rightarrow a^2 - 23a - 420 = 0$

$a^2 - 35a + 12a - 420 = 0$

$a(a - 35) + 12(a - 35) = 0$

$(a - 35)(a + 12) = 0$

$a - 35 = 0, \quad a + 12 = 0$

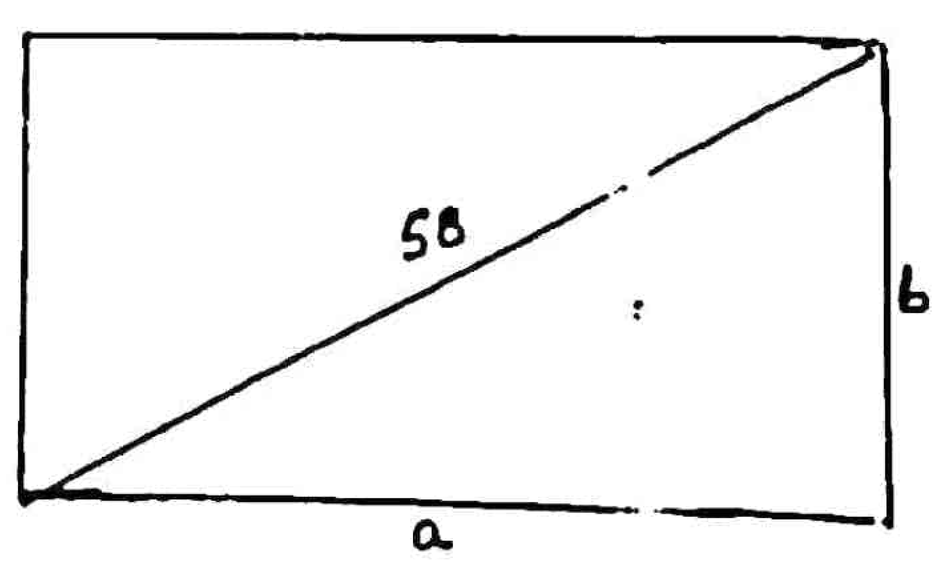
$a = 35, \quad a = -12$ (ignore)

when $a = 35$ then $b = \frac{420}{35} = 12$

so $a =$ base $= 35$, $b =$ Altitude $= 12$

Q16. The area of a rectangle is 1680 square meters. If its diagonal is 58 meters long, find the length and breadth of the rectangle.

Solution:-



Let base $= a$ and altitude $= b$

then

area $= (\text{length})(\text{breadth})$

$\Rightarrow 1680 = ab$

$\Rightarrow ab = 1680 \longrightarrow (i)$

$2ab = 3360 \longrightarrow (ii)$

By Pythagoras theorem,

$\Rightarrow a^2 + b^2 = c^2$

$\Rightarrow a^2 + b^2 = (58)^2$

$a^2 + b^2 = 3364 \longrightarrow (iii)$

By (iii) − (ii) $\Rightarrow$

$a^2 + b^2 = 3364$

$2ab = 3360$

$a^2 - 2ab + b^2 = 4$

or $(a-b)^2 = (\ )^2$

$\Rightarrow a - b = 2$

or $a = b + 2$ put in (i)

$(b+2)b = 1680$

$\Rightarrow b^2 + 2b - 1680 = 0$

$\Rightarrow b^2 + 42b - 40b - 1680 = 0$

$\Rightarrow b(b+42) - 40(b+42) = 0$

$(b+42)(b-40) = 0$

$b + 42 = 0$, $b - 40 = 0$

$b = -42$ (ignore) , $b = 40$

so when $b = 40$ then $a = 40 + 2$

or $a = 42$

so $a =$ length $= 42$, $b =$ breadth $= 40$

Q17. To do a piece of work, A takes 10 days more than B. Together they finish the work in 12 days. How long would B take to finish it alone?

Solution:-

Let B finishes in $= x$ days

Let A finishes in $= x + 10$

B's one days work $= \frac{1}{x}$

A's one days work $= \frac{1}{x+10}$

(A+B)'s one days work $= \frac{1}{x+10} + \frac{1}{x}$

According to condition

$\frac{1}{x+10} + \frac{1}{x} = \frac{1}{12}$

$\Rightarrow \frac{x + x + 10}{x(x+10)} = \frac{1}{12}$

$\Rightarrow \frac{2x+10}{x(x+10)} = \frac{1}{12}$

$\Rightarrow 12(2x+10) = x(x+10)$

$24x + 120 = x^2 + 10x$

$\Rightarrow x^2 + 10x - 24x - 120 = 0$

$x^2 - 14x - 120 = 0$

$x^2 - 20x + 6x - 120 = 0$

$x(x-20) + 6(x-20) = 0$

$(x-20)(x+6) = 0$

$x - 20 = 0$, $x + 6 = 0$

$x = 20$, $x = -6$ (ignore)

so $x = 20 =$ B finish work

so in 20 days B finish his work.

Q18. To complete a job, A and B take 4 days working together. A alone takes twice as long as B alone to finish the same job. How long would each one alone take to do the job?

Solution:-

Let B finishes in $= x$ days

Let A finishes in $= 2x$ days

B's one days work $= \frac{1}{x}$

A's one days work $= \frac{1}{2x}$

(A+B)'s one days work $= \frac{1}{x} + \frac{1}{2x}$

According to condition

$$\frac{1}{x}+\frac{1}{2x}=\frac{1}{4}$$

$$\rightarrow \frac{2+1}{2x}=\frac{1}{4} \rightarrow \frac{3}{2x}=\frac{1}{4}$$

$$\rightarrow 12=2x \rightarrow x=6$$

so $x=6$ = B finishes work

and $2x=2(6)=12$ = A finish work

Q19. An open box is to be made from a square piece of tin by cutting a piece 2dm square from each corner and then folding the sides of the remaining piece. If the capacity of the box is to be 128 dm, find the length of the side of the piece.

Solution:-

Let the sides are $(x-4), (x-4), 2$

Volume $= 2(x-4)(x-4)$

$$\rightarrow 128 = 2(x-4)^2$$

$$(x-4)^2 = 64 \rightarrow x-4=\pm 8$$

$$x = 4\pm 8 \rightarrow x=4+8=12$$

and $x=4-8=-4$ (ignore)

so $x=12$dm Hence sides are 2, 8, 8. ($\because x-4=12-4=8$)

Q20. A man invests Rs. 1,00,000 in two companies. His total profit is Rs. 3080, if he receives Rs. 1980 from one company and at the rate 1% more from the other, find the amount of each investment.

Solution:-

Suppose investment in I company $= x$

investment in II company $= 100000-x$

Profit from I at $y\% = 1980$

Profit from II at $(y+1)\% = 3080$

According to condition

$$\rightarrow x\left(\frac{y}{100}\right)=1980$$

$$\rightarrow xy = 198000 \longrightarrow (i)$$

and

$$[(y+1)\%][100000-x]=3080$$

$$\left(\frac{y+1}{100}\right)(100000-x)=3080$$

$$(y+1)(100000-x)=308000$$

$$100000y-xy+100000-x=308000$$

$$100000y-198000+100000-x-308000=0$$

$$100000y-x=406000 \longrightarrow (ii)$$

from (i) $\rightarrow x=\frac{198000}{y}$ put in (ii)

$$100000y-\frac{198000}{y}=406000$$

$$\rightarrow 100000y^2-198000=406000y$$

$$100000y^2-406000y-198000=0$$

($\div$ by 2000)

$$50y^2-203y-99=0$$

$$y=\frac{-(-203)\pm\sqrt{(-203)^2-4(50)(-99)}}{2(50)}$$

$$\rightarrow y=\frac{203\pm\sqrt{41209+19800}}{100}$$

$$y=\frac{203\pm\sqrt{61009}}{100}=\frac{203\pm 247}{100}$$

$y=\frac{203+247}{100}$, $y=\frac{203-247}{100}=\frac{-44}{100}$ (ignore)

$y=\frac{450}{100}=4.5$, so when $y=4.5$

then $x=\frac{198000}{4.5}=44000$

Thus Amount invested in I $= 44000$

in II $= 100000-44000=56000$