Assignment #02

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Subject Math

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Short Questions

(i)

Unit Matrix (LHR-2011)

A matrix in which the diagonal entries are 1 is called a Unit matrix, or identity.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } I_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \ldots \end{bmatrix}$$

(ii)

If $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$ find A_{12}, A_{22} (LHR-2011)

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 0 \\ -2 & 1 \end{vmatrix}$$

$$A_{12} = (-1)^3 (0+0)$$

$$\boxed{A_{12} = 0}$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix}$$

$$= (1)(1-6)$$

$$\boxed{A_{22} = -5}$$

(iii)

(LHR-2011)

If $A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$ find $(\bar{A})^t$

$$A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & i \end{bmatrix} \Rightarrow (\bar{A})^t = \begin{bmatrix} i & 1 \\ 1-i & -i \end{bmatrix}$$

$$(\bar{A})^t = \begin{bmatrix} i & 1 \\ 1-i & -i \end{bmatrix}$$

(iv)

If A and B are symmetric and AB = BA then Show that AB is symmetric. (LHR-2012)

Given A is symmetric $\Rightarrow A^t = A$ —(1)

B is symmetric $\Rightarrow B^t = A$ —(2)

Also $AB = BA$ —(3)

$$(AB)^t = B^t A^t$$

$(AB)^t = BA$ From equation (1) and (ii)

$(AB)^t = AB$ From equation (3)

Hence AB is symmetric.

(v)

If $A=\begin{bmatrix}3 & 2 & -1\\ 2 & 1 & 3\end{bmatrix}$ Find the value of $|AA^t|$ (LHR-2012)

$A=\begin{bmatrix}3 & 2 & -1\\ 2 & 1 & 3\end{bmatrix}$

$A^t=\begin{bmatrix}3 & 2\\ 2 & 1\\ -1 & 3\end{bmatrix}$

$AA^t=\begin{bmatrix}3 & 2 & -1\\ 2 & 1 & 3\end{bmatrix}\begin{bmatrix}3 & 2\\ 2 & 1\\ -1 & 3\end{bmatrix}$

$AA^t=\begin{bmatrix}9+4+1 & 6+2-3\\ 6+2-3 & 4+1+9\end{bmatrix}$

$|AA^t|=\begin{vmatrix}14 & 5\\ 5 & 14\end{vmatrix}$

$=(14)(14)-(5)(5) \Rightarrow 196-25$

$|AA^t|=171$

(vi)

If $A=\begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix}$ Show that $4A-3A=A$ (LHR-2012)

$A=\begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix}$

$4A=4\begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix} \Rightarrow \begin{bmatrix}8 & 12\\ 4 & 20\end{bmatrix}$

$3A=3\begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix} \Rightarrow \begin{bmatrix}6 & 9\\ 3 & 15\end{bmatrix}$

$4A-3A=\begin{bmatrix}8 & 12\\ 4 & 20\end{bmatrix}-\begin{bmatrix}6 & 9\\ 3 & 15\end{bmatrix}$

$=\begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix}$

$=$ R.H.S

(vii)

Evaluate $\begin{vmatrix}5 & -2 & -4\\ 3 & -1 & -3\\ -2 & 1 & 2\end{vmatrix}$ (LHR-2012)

$=5\begin{vmatrix}-1 & -3\\ 1 & 2\end{vmatrix}+2\begin{vmatrix}3 & -3\\ -2 & 2\end{vmatrix}-4\begin{vmatrix}3 & -1\\ -2 & 1\end{vmatrix}$

$=5(-2+3)+2(6-6)-4(3-2)$

$= 5 + 0 - 4$

$= 1$

(viii)

Show that AA^t is symmetric (LHR-2012)

$(AA^t)^t = (A^t)^t A^t$

$= AA^t \quad \because (A^t)^t = A$

Hence AA^t is symmetric

(ix)

If $A = \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}$ and $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ find the value of a and b. (LHR-2013+16+19)

$A \cdot A = \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix} \begin{bmatrix} 1 & -1 \\ a & b \end{bmatrix}$

$A^2 = \begin{bmatrix} 1-a & -1-b \\ a+ab & -a+b^2 \end{bmatrix}$

$\begin{bmatrix} 1-a & -1-b \\ a+ab & -a+b^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$1 - a = 0 \quad , \quad -1 - b = 0$

$\boxed{a = 1} \quad , \quad \boxed{b = -1}$

(x)

Solve the matrix equation for x; $3x - 2A = B$

If $A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$

(LHR-2013)

$3x - 2A = B$

$3x - 2\begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$

$3x = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 6 & -4 \\ -2 & 2 & 10 \end{bmatrix}$

$$3X = \begin{bmatrix} 6 & 3 & -3 \\ 3 & 6 & 9 \end{bmatrix}$$

$$X = \frac{\begin{bmatrix} 6 & 3 & -3 \\ 3 & 6 & 9 \end{bmatrix}}{3}$$

$$X = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

Q(xi)

If A is symmetric or skew symmetric show that A^2 is symmetric. (LHR-2013+14+19)

Given A is symmetric $\Rightarrow A^t = A$ —(i)

we have to prove A^2 is symmetric

$(A^2)^t = (A.A)^t \Rightarrow A^t.A^t$

$= A.A$ from Equation (1)

$= A^2$

Hence A^2 is symmetric.

Also

Given A is Skew Symmetric $\Rightarrow A^t = -A$ —(2)

we have to prove A^2 is Symmetric.

Now

$(A^2)^t = (A.A)^t$

$= A^t.A^t$

$= (-A).(-A) \because$ from (2)

$= A^2$

Hence A^2 is Symmetric

Q(xii)

State any properties of square matrix When $|A| = 0$ (LHR-2013)

* If a square matrix has two identical rows or two identical columns then $|A| = 0$
* If all the entries of a row (or a column) of a square matrix A are zero then $|A| = 0$

(xiii)

Define column matrix. (LHR-2013)

A matrix which has only one column is called column matrix.

Example:-

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

(xiv)

Find A^{-1} if $A = \begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix}$ (LHR-2013+16+21)

$$A = \begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix}$$

$$\text{Adj of } A = \begin{bmatrix} 1 & -3 \\ -1 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 5 & 3 \\ 1 & 1 \end{vmatrix}$$

$$|A| = (5)(1) - (3)(1)$$

$$|A| = 2$$

$$A^{-1} = \frac{\text{Adj of } A}{|A|} \Rightarrow \begin{bmatrix} 1/2 & -3/2 \\ -1/2 & 5/2 \end{bmatrix}$$

(xv)

If $A = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}$ show that $A^4 = I_2$ (LHR-2013)

$$A.A = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}$$

$$A^2 = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix}$$

$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \because i^2 = -1$

$A^2 . A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= I_2$

(xvi)

Define Inverse of a 2x2 Matrix. (LHR-2011)

let A be a non-singular square matrix of order **2.** If there exits a matrix B such that $\mathbf{AB = BA = I_2}$, then B is called the multiplicative inverse of A and is usually denoted by A^{-1} that is

$$\mathbf{B = A^{-1}}$$

Thus $\mathbf{AA^{-1} = A^{-1}A = I_2}$

(xvii)

Determine the matrix is singular or non-singular. (LHR-2013)

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} - 0 + 3\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$$

$1(4+2)+3(6-0)$

$|A| = 24$ Hence the matrix is non-singular.

(xviii) (LHR-2014)

If $A=\begin{bmatrix} a & -b \\ c & d \end{bmatrix}$ then find Adj of A

$$A=\begin{bmatrix} a & -b \\ c & d \end{bmatrix}$$

$$\text{Adj of } A=\begin{bmatrix} d & b \\ -c & a \end{bmatrix}$$

(xix)

Symmetric Matrix:- (LHR-2014) A square matrix $A=[a_{ij}]_{n\times n}$ is called symmetric if $A^t = A$

Skew symmetric Matrix:- A square matrix $A=[a_{ij}]_{n\times n}$ is called skew symmetric if $A^t = -A$

(xx)

If $A=\begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$ and $A^2=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then find the value of a and b. (LHR-2014+18)

$$A.A=\begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}\begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1+2a & 2+2b \\ a+ab & 2a+b^2 \end{bmatrix}$$

$$\begin{bmatrix} 1+2a & 2+2b \\ a+ab & 2a+b^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \because A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$1+2a = 0 \quad ; \quad 2+2b = 0$

$\boxed{a = -1/2} \quad ; \quad \boxed{b = -1}$

(LHR-2014+19+21+22) **(xxi)**

Without expansion show that $\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \alpha+\gamma & 1 \\ \gamma & \beta+\alpha & 1 \end{vmatrix} = 0$

$$\text{L.H.S} = \begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \alpha+\gamma & 1 \\ \gamma & \beta+\alpha & 1 \end{vmatrix} = 0$$

$$= \begin{vmatrix} \alpha+\beta+\gamma & \beta+\gamma & 1 \\ \alpha+\beta+\gamma & \alpha+\gamma & 1 \\ \alpha+\beta+\gamma & \beta+\alpha & 1 \end{vmatrix} \quad \text{by } C_1+C_2$$

$$= (\alpha+\beta+\gamma) \begin{vmatrix} 1 & \beta+\gamma & 1 \\ 1 & \alpha+\gamma & 1 \\ 1 & \beta+\alpha & 1 \end{vmatrix} \quad \text{Taking } (\alpha+\beta+\gamma) \text{ common from } C_1$$

$= \alpha+\beta+\gamma(0) \quad \because C_1 \text{ and } C_2 \text{ are identical}$

$= 0$

$= \text{R.H.S}$

(xxii)

Diagonal Matrix :- (LHR-2014+18) A matrix in which the some elements of the principle diagonal may be zero but not all is called a diagonal matrix.

Example: $[7]$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(XXIII)

(LHR-2014)

Show that if any two row of a determinant are identical then the value of determinant is zero.

If any two rows of a determinant are identical then the value of determinant is zero.

For example,

$$\begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} = 0$$

it can be proved by expanding the determinant.

L.H.S =

$$= a\begin{vmatrix} b & c \\ y & z \end{vmatrix} - b\begin{vmatrix} a & c \\ x & z \end{vmatrix} + c\begin{vmatrix} a & b \\ x & y \end{vmatrix}$$

$$= a(bz-cy) - b(az-cx) + c(ay-bx)$$

$$= \cancel{abz} - \cancel{acy} - \cancel{abz} + \cancel{bcx} + \cancel{acy} - \cancel{bcx}$$

$$= 0$$

$$= R.H.S$$

(XXIV)

Transpose of matrix (LHR-2015+17+18)

If A is a matrix of order $m \times n$ then an $n \times m$ matrix obtained by interchanging the rows and columns of A, is called the transpose of A.

(xxv)

Find the inverse of $A=\begin{bmatrix}2&1\\6&3\end{bmatrix}$ (LHR-2015)

$A=\begin{bmatrix}2&1\\6&3\end{bmatrix}$

Adj of $A=\begin{bmatrix}3&-1\\-6&2\end{bmatrix}$

$|A|=\begin{vmatrix}2&1\\6&3\end{vmatrix}$

$=(2)(3)-(6)(1)$

$=0$ A is Singular Matrix.

A^{-1} does not exit the value of $|A|$ is 0.

(xxvi)

(LHR-2015+22)

Without expansion show that $\begin{vmatrix}1&a^2&\frac{a}{bc}\\1&b^2&\frac{b}{ca}\\1&c^2&\frac{c}{ab}\end{vmatrix}=0$

$\begin{vmatrix}1&a^2&\frac{a}{bc}\\1&b^2&\frac{b}{ca}\\1&c^2&\frac{c}{ab}\end{vmatrix}=0$

L.H.S $=$

$=\frac{1}{abc}\begin{vmatrix}1&a^2&\frac{a}{bc}\times abc\\1&b^2&\frac{b}{ca}\times abc\\1&c^2&\frac{c}{ab}\times abc\end{vmatrix}$ → Multiplying Abc by C_3.

$=\frac{1}{abc}\begin{vmatrix}1&a^2&a^2\\1&b^2&b^2\\1&c^2&c^2\end{vmatrix}$

$=\frac{1}{abc}(0)$ $\because C_2$ and C_3 are identical

$=0$

$=$ R.H.S

(xxvii)

(LHR-2015+22)

Find x and y if $\begin{bmatrix} x+3 & 1 \\ -3 & 3x-4 \end{bmatrix} = \begin{bmatrix} y & 1 \\ -3 & 2x \end{bmatrix}$

$$\begin{bmatrix} x+3 & 1 \\ -3 & 3x-4 \end{bmatrix} = \begin{bmatrix} y & 1 \\ -3 & 2x \end{bmatrix}$$

$x+3 = y$; $3x-4 = 2x$

$x = y-3$ —(i) ; $3x-2x = 4$

$$\boxed{x = 4}$$

Put the value of x in(1)

$4 = y-3$

$4+3 = y$

$$\boxed{y = 7}$$

(xxviii)

Co-factor of an element (LHR-2015)

The cofactor of an element a_{ij} denoted by A_{ij} is defined as $A_{ij} = (-1)^{i+j} \times M_{ij}$ Where M_{ij} is the minor of the element a_{ij} of A or $|A|$.

(xxix)

(LHR-2015)

Without expansion show that $\begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix} = 0$

$$\text{L.H.S} = \begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix}$$

$$= 3x\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 5 & 3 \end{vmatrix}$$ Taking $3x$ common by C_3

$= 3x(0)$ $\because$ C_1 and C_3 are identical

$= 0$

$=$ R.H.S.

(xxx)

Find the inverse of $A = \begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$ (LHR-2015+18)

$$A = \begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$$

$$\text{Adj of } A = \begin{bmatrix} -i & -i \\ -i & 2i \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2i & i \\ i & -i \end{vmatrix}$$

$= (2i)(-i) - (i)(i)$

$= -2i^2 - i^2$

$= -2(-1) - (-1)$ $\because i^2 = -1$

$|A| = 3$

$$A^{-1} = \frac{\text{Adj of } A}{|A|} \Rightarrow \frac{\begin{bmatrix} -i & -i \\ -i & 2i \end{bmatrix}}{3}$$

$$A^{-1} = \begin{bmatrix} -i/3 & -i/3 \\ -i/3 & 2i/3 \end{bmatrix}$$

(xxxi)

(LHR-2016)

Find the the value of x if $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$

$$\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$$

$$3\begin{vmatrix}3 & 4\\ 1 & 0\end{vmatrix} - 1\begin{vmatrix}-1 & 4\\ x & 0\end{vmatrix} + x\begin{vmatrix}-1 & 3\\ x & 1\end{vmatrix} = -30$$

$$3(0-4) - 1(0-4x) + x(-1-3x) = -30$$

$$-12 + 4x - x + 3x^2 = -30$$

$$-3x^2 + 3x - 12 + 30 = 0$$

$$-3(x^2 - x - 6) = 0$$

$$x^2 - x - 6 = 0$$

$$x^2 + 2x - 3x - 6 = 0$$

$$x(x+2) - 3(x+2) = 0$$

$$(x+2)(x-3) = 0$$

$$x+2 = 0 \quad ; \quad x-3 = 0$$

$$\boxed{x = -2} \quad ; \quad \boxed{x = 3}$$

(LHR-2016)

Without expansion show that $\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix} = 0$

$$\text{L.H.S} = \begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix}$$

$$= \begin{vmatrix}4 & 2 & 3\\ 10 & 5 & 6\\ 16 & 8 & 9\end{vmatrix} \quad \text{By } C_1 + C_3$$

$$= 2\begin{vmatrix}2 & 2 & 3\\ 5 & 5 & 6\\ 8 & 8 & 9\end{vmatrix} \quad \text{Taking 2 common from } C_1$$

$$= 2(0) \quad \because C_1 \text{ and } C_2 \text{ are identical}$$

$$= 0$$

$$= \text{R.H.S.}$$

(xxxiii)

Square matrix: (LHR-2016) A matrix which has same number of rows and columns is called a square matrix. For example; $[0]$, $\begin{bmatrix} 2 & 5 \\ -1 & 6 \end{bmatrix}$

(xxxiv)

If $A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ 3 & 5 & 2 & -1 \end{bmatrix}$ then find $(A^t)^t$ (LHR-2016)

$$A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ 3 & 5 & 2 & -1 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$$

$$(A^t)^t = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ 3 & 5 & 2 & -1 \end{bmatrix}$$

(xxxv)

If $A = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}$ then find A^2 (LHR-2017)

$$A = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}$$

$$A.A = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix}$$

$$A^2 = \begin{bmatrix} i^2+0 & 0+0 \\ i-i & 0+i^2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \because i^2 = -1$$

(xxxvi)

If $B = \begin{bmatrix} 5 & -2 & 5 \\ 3 & -1 & 4 \\ -2 & 1 & -2 \end{bmatrix}$ then find B_{21}, B_{22} (LHR-2017+19)

$$B_{22} = (-1)^{2+2} \begin{vmatrix} 5 & 5 \\ -2 & -2 \end{vmatrix}$$

$$= (+1)(-10+10)$$

$$\boxed{B_{22} = 0}$$

$$B_{21} = (-1)^{2+1} \begin{vmatrix} -2 & 5 \\ 1 & -2 \end{vmatrix}$$

$$= (-1)(4-5)$$

$$\boxed{B_{21} = 1}$$

(xxxvii)

If $A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$ show that $A-(\bar{A})^t$ is skew-hermition. (LHR-2017)

$$A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & i \end{bmatrix}$$

$$(\bar{A})^t = \begin{bmatrix} -i & 1 \\ 1-i & +i \end{bmatrix}$$

$$A - (\bar{A})^t = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} - \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$$

$$A - (\bar{A})^t = \begin{bmatrix} i+i & 1+i-1 \\ 1-1+i & -i-i \end{bmatrix} \Rightarrow \begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$$

$$\left(\overline{A - (\bar{A})^t}\right) = \begin{bmatrix} -2i & -i \\ -i & 2i \end{bmatrix}$$

$$\left(\overline{A - (\bar{A})^t}\right)^t = \begin{bmatrix} -2i & -i \\ -i & 2i \end{bmatrix}$$

$$\left(\overline{A - (\bar{A})^t}\right)^t = -\begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$$

$$\left(\overline{A - (\bar{A})^t}\right)^t = -\left(A - (\bar{A})^t\right)$$

Hence $A - (\bar{A})^t$ is skew hermition.

(xxxviii)

(LHR-2018)

Without expansion show that $\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$

$$\text{L.H.S} = \begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 14 & 7 & 8 \\ 8 & 4 & 5 \\ 6 & 3 & 4 \end{vmatrix} \quad \text{By } C_1 + C_2$$

$$= 2\begin{vmatrix} 7 & 7 & 8 \\ 4 & 4 & 5 \\ 3 & 3 & 4 \end{vmatrix} \quad \text{Taking 2 common from } C_1$$

$$= 2(0) \quad \because C_1 \text{ and } C_2 \text{ are identical}$$

$$= 0$$

$$= \text{R.H.S}$$

(xxxix)

(LHR-2018)

Without expansion show that $\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix} = 0$

$$L.H.S = \begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} bc & ca & ab \\ \frac{1}{a}\times abc & \frac{1}{b}\times abc & \frac{1}{c}abc \\ a & b & c \end{vmatrix}$$ Multiplying R_2 by abc

$$= \frac{1}{abc}\begin{vmatrix} bc & ca & ab \\ bc & ca & ab \\ a & b & c \end{vmatrix}$$

$= \frac{1}{abc}(0)$ $\because$ R_1 and R_2 are identical

$= 0$

$= R.H.S$

(xxxx)

(LHR-2018)

Without expansion Show That $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

$$L.H.S = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix}$$

$$= \frac{1}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix}$$ Multiplying C_1 by x, C_2 by y, C_3 by z

$$= \frac{1}{\cancel{xyz}}\times \cancel{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$ Taking xyz common by R_3

$$= (-1)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix} \quad \text{By } R_1 \leftrightarrow R_3$$

$$= (-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} \quad \text{By } R_2 \leftrightarrow R_3$$

$= R.H.S.$

(xxxxi)

Find the matrix x if $x\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}$

Find the matrix X **(LHR-2019)**

$$X\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}$$

Let $XA = B$

$$X = BA^{-1} \quad \text{---(1)}$$

$$A = \begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj of } A}{|A|}$$

$$\text{Adj of } A = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 5 & 2 \\ -2 & 1 \end{vmatrix}$$

$$= (5)(1) - (2)(-2)$$

$$|A| = 9$$

$$A^{-1} = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} / 9$$

From Equation (1)

$$X = \frac{1}{9}\begin{bmatrix} -1 & 5 \\ 12 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$$

$$X = \frac{1}{9}\begin{bmatrix} -1+10 & +2+25 \\ 12+6 & -24+15 \end{bmatrix}$$

$$X = \frac{\begin{bmatrix} 9 & 27 \\ 18 & -9 \end{bmatrix}}{9}$$

$$X = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

(XXXXII)

If $A = \begin{bmatrix} 1 & 2 & -3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$ find A_{12} (LHR-2019)

$$A_{12} = (-1)^{1+2}\begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix}$$

$$= (-1)(-4-4)$$

$$= (-1)(-8)$$

$$\boxed{A_{12} = 8}$$

(XXXXIII)

(LHR-2021)

Find X if $2X-3A=B$ if $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$

$$2X - 3\begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

$$2X = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -3 & 6 \\ -6 & 12 & 15 \end{bmatrix}$$

$$2x = \begin{bmatrix} 3+3 & -1-3 & 0+6 \\ 4-6 & 2+12 & 1+15 \end{bmatrix}$$

$$X = \begin{bmatrix} 6 & -4 & 6 \\ -2 & 14 & 16 \end{bmatrix} / 2$$

$$X = \begin{bmatrix} 3 & -2 & 3 \\ -1 & 7 & 8 \end{bmatrix}$$

(xxxxiv)

If A and B are non-singular matrices then show that $(AB)^{-1} = B^{-1}A^{-1}$ (LHR-2022)

Consider $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= A.I.A^{-1} \quad \because BB^{-1} = I$

$= AA^{-1} \quad \because AI = A$

$= I \longrightarrow (1) \quad \because AA^{-1} = I$

Consider $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$

$= B^{-1}.I.B \quad \because A^{-1}A = I$

$= B^{-1}B \quad \because IB = I$

$= I \longrightarrow (2) \quad \because B^{-1}B = I$

From (1) and (2)

$(AB)^{-1} = B^{-1}A^{-1}$

(xxxxv)

Solve the system of linear equation.

$$3x - 5y = 1$$
$$-2x + y = -3$$

In matrix form

$$\begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Let

$$AX = B$$
$$X = A^{-1}B$$

$$A^{-1} = \frac{\text{Adj of } A}{|A|}$$

$$|A| = \begin{vmatrix} 3 & -5 \\ -2 & 1 \end{vmatrix}$$
$$= (3)(1) - (-2)(-5)$$
$$|A| = -7$$

$$\text{Adj of } A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}}{-7}$$

Now

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}}{-7}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-7}\begin{bmatrix} 1 - 15 \\ 2 - 9 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -14 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\boxed{x = 2}, \quad \boxed{y = 1}$$

Long Questions

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(LHR-2011)

(i)

Show that $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = (x+3)(x-1)^3$

$$L.H.S = \begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$

$$= \begin{vmatrix} x+3 & 1 & 1 & 1 \\ x+3 & x & 1 & 1 \\ x+3 & 1 & x & 1 \\ x+3 & 1 & 1 & x \end{vmatrix}$$ By $C_1 + (C_2 + C_3 + C_4)$

$$= (x+3) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$ Taking $(x+3)$ Common by C_1

$$= (x+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x-1 \end{vmatrix}$$ By $R_2 - R_1$, $R_3 - R_1$, $R_4 - R_1$

$= (x+3)(1)(x-1)(x-1)(x-1)$ ∵ By property of determinate of triangular matrix

$= (x+3)(x-1)^3$

$=$ R.H.S.

(ii)

(LHR-2011)

Find A_{12}, A_{22}, A_{32} and $|A|$ if $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ -2 & 1 \end{vmatrix}$$

$= (-1)^3(0+0)$

$\boxed{A_{12} = 0}$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix}$$

$= (1)(1-6)$

$\boxed{A_{12} = -5}$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix}$$

$= (-1)(0+0)$

$\boxed{A_{32} = 0}$

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{vmatrix}$$

$$|A| = 1\begin{vmatrix}-2 & 0\\ -2 & 1\end{vmatrix} - 2\begin{vmatrix}0 & 0\\ -2 & 1\end{vmatrix} + (-3)\begin{vmatrix}0 & -2\\ -2 & -2\end{vmatrix}$$

$$= 1(-2+0) - 2(0+0) - 3(0-4)$$

$$= -2 + 12$$

$$\boxed{|A| = 10}$$

(LHR-2012)

Show that $\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix} = a^3+b^3+c^3-3abc$

$$L.H.S. = \begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix}$$

$$= a\begin{vmatrix} a+c & b+c \\ a+b & c+a \end{vmatrix} - (b+c)\begin{vmatrix} b & b+c \\ c & c+a \end{vmatrix} + (a+b)\begin{vmatrix} b & c+a \\ c & a+b \end{vmatrix}$$

$$= a\left((a+c)^2 - (a+b)(b+c)\right) - (b+c)\left(b(c+a) - c(b+c)\right) + (a+b)\left(b(a+b) - c(c+a)\right)$$

$$= a\left((a^2+c^2+2ac) - (ab+ac+b^2+bc)\right) - (b+c)(bc+ab-bc-c^2) + (a+b)(ab+b^2-c^2-ac)$$

$$= a(a^2+c^2+2ac-ab-ac-b^2-bc) - (b+c)(ab-c^2) + (a+b)(ab+b^2-c^2-ac)$$

$$= a^3+ac^2+a^2c-a^2b-ab^2-abc - (ab^2-bc^2+abc-c^3) + (a^2b+ab^2-ac^2$$
$$-a^2c+ab^2+b^3-bc^2-abc$$

$$= a^3+\cancel{ac^2}+\cancel{a^2c}-\cancel{a^2b}-\cancel{ab^2}-abc-\cancel{ab^2}+\cancel{bc^2}-abc+c^3+\cancel{a^2b}+\cancel{ab^2}$$
$$-\cancel{ac^2}-\cancel{a^2c}+\cancel{ab^2}+b^3-\cancel{bc^2}-abc$$

$$= a^3+b^3+c^3-3abc$$

$$= R.H.S.$$

(iv)

(LHR-2013)

Solve the matrix equation for X; $3X-2A=B$

If $A=\begin{bmatrix}2 & 3 & -2\\ -1 & 1 & 5\end{bmatrix}$ and $B=\begin{bmatrix}2 & -3 & 1\\ 5 & 4 & -1\end{bmatrix}$

$$3X-2A=B$$

$$3X-2\begin{bmatrix}2 & 3 & -2\\ -1 & 1 & 5\end{bmatrix}=\begin{bmatrix}2 & -3 & 1\\ 5 & 4 & -1\end{bmatrix}$$

$$3X=\begin{bmatrix}2 & -3 & 1\\ 5 & 4 & -1\end{bmatrix}+\begin{bmatrix}4 & 6 & -4\\ -2 & 2 & 10\end{bmatrix}$$

$$3X=\begin{bmatrix}6 & 3 & -3\\ 3 & 6 & 9\end{bmatrix}$$

$$X=\frac{\begin{bmatrix}6 & 3 & -3\\ 3 & 6 & 9\end{bmatrix}}{3}$$

$$X=\begin{bmatrix}2 & 1 & -1\\ 1 & 2 & 3\end{bmatrix}$$

(v)

Solve the following system of linear equation by cramer's rule. (LHR-2014+16+21)

$$2x+2y+z=3$$

$$3x-2y-2z=1$$

$$5x+y-3z=2$$

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{vmatrix}$$

$$= 2\begin{vmatrix} -2 & -2 \\ 1 & -3 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ 5 & -3 \end{vmatrix} + 1\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}$$

$$= 2(6+2) - 2(-9+10) + 1(3+10)$$

$$= 2(8) - 2(1) + 1(13)$$

$$\boxed{|A| = 27}$$

$$x = \frac{\begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & -3 \end{vmatrix}}{|A|}$$

$$x = \frac{3\begin{vmatrix} -2 & -2 \\ 1 & -3 \end{vmatrix} - 2\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}}{27}$$

$$x = \frac{3(+6+2) - 2(-3+4) + 1(1+4)}{27}$$

$$x = \frac{3(8) - 2(1) + 1(5)}{27}$$

$$x = \frac{24 - 2 + 5}{27}$$

$$x = \frac{27}{27}$$

$$\boxed{x = 1}$$

Now

$$y = \frac{\begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}{|A|}$$

$$y = \frac{2\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} - 3\begin{vmatrix} 3 & -2 \\ 5 & -3 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix}}{27}$$

$$y = \frac{2(-3+4) - 3(-9+10) + (6-5)}{27}$$

$$y = \frac{2(1) - 3(1) + (1)}{27}$$

$$y = \frac{2-3+1}{27}$$

$$\boxed{y = 0}$$

$$z = \frac{\begin{vmatrix} 2 & 2 & 3 \\ 3 & -2 & 1 \\ 5 & 1 & 2 \end{vmatrix}}{|A|}$$

$$z = \frac{2\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} + 3\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}}{27}$$

$$z = \frac{2(-4-1) - 2(6-5) + 3(3+10)}{27}$$

$$z = \frac{2(5) - 2(1) + 3(13)}{27}$$

$$z = \frac{10 - 2 + 39}{27}$$

$$z = \frac{27}{27}$$

$$\boxed{z = 1}$$

(LHR-2022)

Solve by using matrices

$$x - 2y + z = -1$$
$$3x + y - 2z = 4$$
$$y - z = 1$$

In matrices form

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

Let

$$AX = B$$
$$X = A^{-1}B \quad \text{---(1)}$$

and . $A^{-1} = \dfrac{\text{Adj of } A}{|A|}$

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{vmatrix}$$

$$|A| = 1\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 1(-1+2) + 2(-3+0) + 1(3-0)$$

$$= 1 - 6 + 3$$

$$\boxed{|A| = -2}$$

Now

$$\text{Adj of } A = \begin{bmatrix} +\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} \\ +\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} & +\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \end{bmatrix}^t$$

$$\text{Adj of } A = \begin{bmatrix} +(-1+2) & -(-3+0) & +(3-0) \\ -(+2-1) & +(4-0) & -(1+0) \\ +(4-1) & -(-2-3) & +(1+6) \end{bmatrix}^t$$

$$\text{Adj of } A = \begin{bmatrix} 1 & 3 & 3 \\ -1 & -1 & -1 \\ 3 & 5 & 7 \end{bmatrix}^t$$

$$\text{Adj of } A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix}}{-2}$$

From equation (1)

$$X = \frac{\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}}{-2}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -1-4+3 \\ -3-4+5 \\ -3-4+7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\boxed{x = 1}$, $\boxed{y = 1}$, $\boxed{z = 0}$

(vii)

(LHR-2014+15)

Solve by using Crammer's rule

$$2x_1 - x_2 + x_3 = 5$$
$$4x_1 + 2x_2 + 3x_3 = 8$$
$$3x_1 - 4x_2 - x_3 = 3$$

Let

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}$$

$$|A| = 2\begin{vmatrix} 2 & 3 \\ -4 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 3 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 3 & -4 \end{vmatrix}$$

$$= 2(-2+12) + 1(-4-9) + 1(-16-6)$$

$$= 20 - 13 - 22$$

$$|A| = -15$$

Now

$$x_1 = \frac{\begin{vmatrix} 5 & -1 & 1 \\ 8 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}}{|A|}$$

$$x_1 = \frac{5\begin{vmatrix} 2 & 3 \\ -4 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 8 & 3 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 8 & 2 \\ 3 & -4 \end{vmatrix}}{-15}$$

$$x_1 = \frac{5(-2+12) + 1(-8-9) + (-32-6)}{-15}$$

$$x_1 = \frac{50-17-38}{-15}$$

$$x_1 = \frac{-5}{-15}$$

$$\boxed{x_1 = \frac{1}{3}}$$

NOW

$$x_2 = \frac{\begin{vmatrix} 2 & 5 & 1 \\ 4 & 8 & 3 \\ 3 & 3 & -1 \end{vmatrix}}{|A|}$$

$$= \frac{2\begin{vmatrix} 8 & 3 \\ 3 & -1 \end{vmatrix} - 5\begin{vmatrix} 4 & 3 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 4 & 8 \\ 3 & 3 \end{vmatrix}}{-15}$$

$$= \frac{2(-8-9) - 5(-4-9) + (12-24}{-15}$$

$$= \frac{-34 + 65 - 12}{-15}$$

$$\boxed{x_2 = -\frac{19}{15}}$$

Now

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & 5 \\ 4 & 2 & 8 \\ 3 & -4 & 3 \end{vmatrix}}{|A|}$$

$$= \frac{2\begin{vmatrix} 2 & 8 \\ -4 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 8 \\ 3 & 3 \end{vmatrix} + 5\begin{vmatrix} 4 & 2 \\ 3 & -4 \end{vmatrix}}{-15}$$

$$= \frac{2(6+32) + 1(12-24) + 5(-16-6)}{-15}$$

$$= \frac{76 - 12 - 110}{-15}$$

$$= \frac{-46}{-15}$$

$$\boxed{x_3 = \frac{46}{15}}$$

(LHR-2022)

Solve by using Crammer's rule

$$2x_1 - x_2 + x_3 = 8$$
$$x_1 + 2x_2 + 2x_3 = 6$$
$$x_1 - 2x_2 - x_3 = 1$$

Let

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}$$

$$|A| = 2\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix}$$

$$= 2(-2+4) + 1(-1-2) + 1(-2-2)$$

$$= 4 - 3 - 4$$

$$|A| = -3$$

Now

$$x_1 = \frac{\begin{vmatrix} 8 & -1 & 1 \\ 6 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}}{|A|}$$

$$= \frac{8\begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 6 & 2 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 6 & 2 \\ 1 & -2 \end{vmatrix}}{-3}$$

$$= \frac{8(-2+4) + 1(-6-2) + (-12-2)}{-3}$$

$$= \frac{16 - 8 - 14}{-3}$$

$$= \frac{-6}{-3}$$

$$\boxed{x_1 = 2}$$

Now

$$x_2 = \frac{\begin{vmatrix} 2 & 8 & 1 \\ 1 & 6 & +2 \\ 1 & 1 & -1 \end{vmatrix}}{|A|}$$

$$= \frac{2\begin{vmatrix} 6 & 2 \\ 1 & -1 \end{vmatrix} - 8\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 6 \\ 1 & 1 \end{vmatrix}}{-3}$$

$$= \frac{2(-6-2) - 8(-1-2) + 1(1-6)}{-3}$$

$$= \frac{-16+24-5}{-3}$$

$$= \frac{3}{-3}$$

$$\boxed{x_2 = -1}$$

Now

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & 8 \\ 1 & 2 & 6 \\ 1 & -2 & 1 \end{vmatrix}}{|A|}$$

$$= \frac{2\begin{vmatrix} 2 & 6 \\ -2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 6 \\ 1 & 1 \end{vmatrix} + 8\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix}}{-3}$$

$$= \frac{2(2+12) + 1(1-6) + 8(-2-2)}{-3}$$

$$= \frac{28-5-32}{-3}$$

$$= \frac{-9}{-3}$$

$$\boxed{x_3 = 3}$$

(iv)

(LHR-2015+16)

Solve by using matrices.

$$x_1 - 2x_2 + x_3 = -4$$
$$2x_1 - 3x_2 + 2x_3 = -6$$
$$2x_1 + 2x_2 + x_3 = 5$$

In matrix form

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix}$$

Let

$$AX = B$$
$$X = A^{-1}B \quad \text{—(1)}$$

and $|A| = \begin{vmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{vmatrix}$

$$|A| = 1\begin{vmatrix} -3 & 2 \\ 2 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & -3 \\ 2 & 2 \end{vmatrix}$$

$$= 1(-3-4) + 2(2-4) + 1(4+6)$$

$$= -7 - 4 + 10$$

$$|A| = -1$$

$$A^{-1} = \frac{\text{Adj of } A}{|A|}$$

$$\text{Adj of } A = \begin{bmatrix} \begin{vmatrix} -3 & 2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ -3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} \end{bmatrix}^t$$

$$\text{Adj of } A = \begin{bmatrix} (-3-4) & -(2-4) & (4+6) \\ -(-2-2) & (1-2) & -(2+4) \\ (-4+3) & -(2-2) & (-3+4) \end{bmatrix}^{t}$$

$$\text{Adj of } A = \begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix}}{-1}$$

$$A^{-1} = \begin{bmatrix} +7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix}$$

From equation (1)

$$X = A^{-1}B$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix} \begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -28+24+5 \\ 8-6+0 \\ 40-36-5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\boxed{x_1 = 1}$, $\boxed{x_2 = 2}$ $\boxed{x_3 = -1}$

Ex # 3.3

(LHR-2013)

Find the inverse of A if $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj of } A}{|A|}$$

$$|A| = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

$$= 1(-2+0) - 2(0-3) - 1(0+1)$$

$$= -2+6-1$$

$$\boxed{|A| = 3}$$

$$\text{Adj of } A = \begin{bmatrix} \begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} \end{bmatrix}^t$$

$$\text{Adj of } A = \begin{bmatrix} (-2-0) & -(0-3) & (0+1) \\ -(4+0) & (2+1) & -(0-2) \\ (6-1) & -(3+0) & (1-0) \end{bmatrix}^t$$

$$\text{Adj of } A = \begin{bmatrix} -2 & 3 & 1 \\ -4 & 3 & 2 \\ 5 & -3 & -1 \end{bmatrix}^t$$

$$\text{Adj of } A = \begin{bmatrix} -2 & -4 & 5 \\ 3 & 3 & -3 \\ 1 & 2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} -2 & -4 & 5 \\ 3 & 3 & -3 \\ 1 & 2 & -1 \end{bmatrix}}{3}$$

$$A^{-1} = \begin{bmatrix} -2/3 & -4/3 & 5/3 \\ 3/3 & 3/3 & -3/3 \\ 1/3 & 2/3 & -1/3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2/3 & -4/3 & 5/3 \\ 1 & 1 & -1 \\ 1/3 & 2/3 & -1/3 \end{bmatrix}$$

(xi)

(LHR-2017)

Verify that $(AB)^t = B^tA^t$ if $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$

$(AB)^t$

$$AB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1-3+0 & 1-2-2 \\ 0+9+0 & 0+6-1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -2 & -3 \\ 9 & 5 \end{bmatrix}$$

$$(AB)^t = \begin{bmatrix} -2 & 9 \\ -3 & 5 \end{bmatrix} \text{ ——— (1)}$$

B^tA^t

$$B^t = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$B^tA^t = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$B^tA^t = \begin{bmatrix} 1-3+0 & 0+9+0 \\ 1-2-2 & 0+6-1 \end{bmatrix}$$

$$B^tA^t = \begin{bmatrix} -2 & 9 \\ -3 & 5 \end{bmatrix} \text{ —— (ii)}$$

From (i) and (ii)

$$(AB)^t = B^tA^t$$

(xii)

(LHR-2012)

If $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix}$ then show that $A^{-1}A = I_3$

$$A^{-1} = \frac{\text{Adj of } A}{|A|}$$

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} -1 & 3 \\ -4 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & -1 \\ 2 & -4 \end{vmatrix}$$

$$= 2(-1+12) - 1(1-6) + 0$$

$$= 22+5$$

$$\boxed{|A| = 27}$$

$$\text{Adj of } A = \begin{bmatrix} +\begin{vmatrix} -1 & 3 \\ -4 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & -4 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ -4 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & -4 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \end{bmatrix}^t$$

$$\text{Adj of } A = \begin{bmatrix} (-1+12) & -(1-6) & (-4+2) \\ -(1+0) & (2-0) & -(-8-2) \\ (3+0) & -(6-0) & (-2-1) \end{bmatrix}^t$$

$$\text{Adj of } A = \begin{bmatrix} 11 & 5 & -2 \\ -1 & 2 & 10 \\ 3 & -6 & -3 \end{bmatrix}^t$$

$$\text{Adj of } A = \begin{bmatrix} 11 & -1 & 3 \\ 5 & 2 & -6 \\ -2 & 10 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} 11 & -1 & 3 \\ 5 & 2 & -6 \\ -2 & 10 & -3 \end{bmatrix}}{27}$$

$$A^{-1} = \begin{bmatrix} 11/27 & -1/27 & 1/9 \\ 5/27 & 2/27 & -2/9 \\ -2/27 & 10/27 & -1/9 \end{bmatrix}$$

Show that $A^{-1}A = I_3$

$$\text{L.H.S} = \begin{bmatrix} 11/27 & -1/27 & 1/9 \\ 5/27 & 2/27 & -2/9 \\ -2/27 & 10/27 & -1/9 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} \frac{22}{27} - \frac{1}{27} + \frac{2}{9} & \frac{11}{27} + \frac{1}{27} - \frac{4}{9} & 0 - \frac{3}{27} + \frac{1}{9} \\ \frac{10}{27} + \frac{2}{27} - \frac{4}{9} & \frac{5}{27} - \frac{2}{27} + \frac{8}{9} & 0 + \frac{6}{27} - \frac{2}{9} \\ \frac{-4}{27} + \frac{10}{27} - \frac{2}{9} & \frac{-2}{27} - \frac{10}{27} + \frac{4}{9} & 0 + \frac{30}{27} - \frac{1}{9} \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 27/27 & 0/27 & 0/27 \\ 0/27 & 27/27 & 0/27 \\ 0/27 & 0/27 & 27/27 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I_3$$

$$= R.H.S.$$

(xiii)

(LHR-2019)

Show that $\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = \ell^2(3a+\ell)$.

$$L.H.S = \begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix}$$

$$= a+b\begin{vmatrix} a+b & a \\ a & a+b \end{vmatrix} - a\begin{vmatrix} a & a \\ a & a+b \end{vmatrix} + a\begin{vmatrix} a & a+b \\ a & a \end{vmatrix}$$

$$= a+b\left[(a+\ell)^2 - a^2\right] - a\left[a(a+\ell) - a^2\right] + a\left[a^2 - a(a+b)\right]$$

$$= a+b\left[\cancel{a^2} + 2a\ell + \ell^2 - \cancel{a^2}\right] - a\left[\cancel{a^2} + a\ell - \cancel{a^2}\right] + a\left[\cancel{a^2} - \cancel{a^2} - ab\right]$$

$$= 2\cancel{a^2\ell} + a\ell^2 + 2a\ell^2 + \ell^3 - \cancel{a^2\ell} - \cancel{a^2\ell}$$

$$= 3a\ell^2 + \ell^3$$

$$= \ell^2(3a+\ell)$$

$$= R.H.S.$$

(LHR-2018)

If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ then find A^{-1} by using adjoint of the matrix.

$$A^{-1} = \frac{\text{Adj of } A}{|A|}$$

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= 1(2+1) - 0 + 2(0-2)$$

$$= 3 - 4$$

$$|A| = -1$$

$$\text{Adj of } A = \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{bmatrix}^t$$

$$\text{Adj of } A = \begin{bmatrix} (2+1) & -(0-1) & (0-2) \\ -(0+2) & (1-2) & -(-1+0) \\ (0-4) & -(1-0) & (2-0) \end{bmatrix}^t$$

$$\text{Adj of } A = \begin{bmatrix} 3 & 1 & -2 \\ -2 & -1 & 1 \\ -4 & -1 & 2 \end{bmatrix}^t$$

$$\text{Adj of } A = \begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} 3 & -2 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{bmatrix}}{-1}$$

$$A^{-1} = \begin{bmatrix} -3 & 2 & 4 \\ -1 & 1 & 1 \\ 2 & -1 & -2 \end{bmatrix}$$

Q (XV)

(LHR-2017) Show that $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

$$L.H.S = \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a & a^2 \\ a+b+c & b & b^2 \\ a+b+c & c & c^2 \end{vmatrix} \quad \text{By } C_1 + C_2$$

$$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \quad \text{Taking } a+b+c \text{ common by } C_1$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$ Taking transpose of the determinant.

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c^2 \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$$ By $C_1 - C_2,\ C_2 - C_3$

$$= (a+b+c)\left\{0-0+1\begin{vmatrix} a-b & b-c \\ a^2-b^2 & b^2-c^2 \end{vmatrix}\right\}$$

$$=(a+b+c)\left\{1\begin{vmatrix} a-b & b-c \\ (a-b)(a+b) & (b+c)(b-c) \end{vmatrix}\right\}$$

$$= (a+b+c)(a-b)(b-c)\begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix}$$ taking $(a-b)(b-c)$ common by C_1 and C_2

$$= (a+b+c)(a-b)(b-c)\left((b+c)-(a+b)\right)$$

$$= (a+b+c)(a-b)(b-c)(c-a)$$

$$= \text{R.H.S.}$$