MATHEMATICS 1st YEAR

UNIT # 02

SET,FUNCTIONS & GROUPS

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Contents

Sherazi Mathematics

اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Set:- "A well-defined collection of distinct objects is called a set."

* Sets are usually denoted by capital English alphabets such as A, B, C, ..., X, Y, Z
* The objects of a set are called its elements denoted by small letters such as a, b, c, ..., x, y, z.
e.g., $A = \{a, b, c\}$

There are three different ways of describing a set.

i) Descriptive Method:-

A method of describing a set in words is known as "Descriptive Method". e.g.,

A = set of all vowels of English alphabets

O = set of all odd numbers

ii) Tabular Method:-

A method of describing a set by writing the elements of the set within brackets is known as "Tabular method". e.g.,

$A = \{a, e, i, o, u\}$

$O = \{\pm1, \pm3, \pm5 \pm7, \ldots\}$

iii) Set-builder Method:-

A method of describing a set in which the elements of a set are denoted by an arbitrary variable (say x) stating common property or properties possessed by all elements of the set is known as set-builder Method. e.g.,

$A = \{x | x$ is vowels of english alphabets$\}$

$O = \{x | x$ is an odd number$\}$

The symbol used for membership of a set is $\in$ and $\notin$ means member not belong to a set.

Some Important Sets

N = The set of all natural numbers
$= \{1, 2, 3, \ldots\}$

W = The set of all whole numbers
$= \{0, 1, 2, \ldots\}$

Z = The set of all integers
$= \{0, \pm1, \pm2, \ldots\}$

Z' = The set of all negative integers
$= \{-1, -2, -3, \ldots\}$

O = The set of all odd integers
$= \{\pm1, \pm3, \pm5, \ldots\}$

E = The set of all even integers
$= \{0, \pm2, \pm4, \pm6, \ldots\}$

Q = The set of all rational numbers
$= \{x | x = \frac{p}{q}$ where $p, q \in Z$ and $q \neq 0\}$

Q' = The set of all irrational numbers
$= \{x | x \neq \frac{p}{q}$, where $p, q \in Z$ and $q \neq 0\}$

$\mathbb{R}$ = The set of all real numbers
$= Q \cup Q'$

KINDS OF SETS

Null Set:- A set which contains no element is called null or empty set. It is denoted by ϕ or $\{ \}$.

Order of a Set:-

The number of elements present in a set is called order of a set. Number or elements in a set A is denoted by: $n(A)$. e.g.,

If $A = \{1, 2, 3\}$, $n(A) = 3$

so order of set A = 3

Singleton Set:- A set having only one element is called singleton set. e.g.,

$A = \{4\}$, $B = \{x \mid x \in N \wedge 1 < x < 3\}$

Finite Set:- A set in which elements are countable or finite is called finite set e.g.,

$A = \{1, 2, 3, 4\}$, $B = \{x \mid x \in N \wedge 6 < x < 9\}$

Infinite Set:- A set in which elements are uncountable or infinite is called infinite set.

e.g., $A = \{x \mid x \in \mathbb{R} \wedge 0 < x < 1\}$

$B = \{1, 2, 3, \ldots\}$

Equal Sets:- Two sets A and B are equal i-e., $A = B$ if and only if they have the same elements that is, if and only if every element of each set is an element of the other set.

Mathematically, if $A = B$ iff $A \subseteq B$ and $B \subseteq A$ e.g.,

$A = \{1, 2, 3\}$, $B = \{2, 1, 3\}$

so $A = B$

Note:- Two sets A and B are said to be equal if they have

i) same order ii) same elements

One-to-One Correspondence:-

One-to-One (1-1) correspondence between two sets A and B is defined as: Each element of A can be paired with one and only one element of B and each element of B can be paired with one and only one element of A. e.g.,

Let

$A = \{a, b, c, d\}$

$B = \{1, 2, 3, 4\}$ so

$$\left\{\begin{array}{cccc} a & b & c & d \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 & 4 \end{array}\right\}$$

There is (1-1) correspondence between A and B.

* (1-1) correspondence cannot be established between a finite and infinite set.

Equivalent Sets:- Two sets A and B are said to be equivalent if (1-1) correspondence can be made between them. e.g.,

$A = \{2, 4, 6\}$, $B = \{a, b, c\}$ are equivalent sets. and

$A = \{1, 2, 3, \ldots\}$ and $B = \{2, 4, 6, \ldots 100\}$ are not equivalent sets.

Note:- * If A is equivalent to B then it is written as $A \sim B$ or $A \equiv B$

* Two equal sets are necessarily equivalent.

* Two equivalent set may or may not equal.

Subset:- Any set A is said to be subset of a set B if every element of the set A is also an element of set B. Mathematically;

$$A \subseteq B \text{ iff } x \in A \rightarrow x \in B$$

* If A is subset of B, then it is denoted by $A \subseteq B$ and read as "A is the subset of B."

* If A is subset of B, then $n(A) \leq n(B)$

* Every set is subset of itself and empty set is subset of every set.

* If $A \subseteq B$ then we may read it in two ways:

i) $A \subseteq B$ i-e., A is subset of B

ii) $B \supseteq A$ i-e., B is subset of A

Note:- Possible subsets of a set can be calculated by the formula:
2^n ($2^{\text{no. of elements present in the set}}$)

e.g., If $A = \{1, 2, 3\}$ then there will be 8 subsets of A are
$\{\ \}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}$
$\{2,3\}, \{1,2,3\}$ $(\because\ 2^3 = 8)$

Types of Subset

There are two types of subset
i) Proper subset
ii) Improper subset

1) Proper subset :- If A is subset of set B such that at least one element of B does not belong to A, then A is called proper subset of B denoted by $A \subset B$.

* All the subsets of any set (except the set itself) are its proper subsets.
* There is no of an empty and singleton set.

Example:- If $A = \{1, 2, 3\}$
subsets of A are
$\{\ \}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}$
$\{2,3\}, \{1,2,3\}$
Proper subsets of A are
$\{\ \}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}$
$\{2,3\}$

* If A is proper subset of a finit set B i.e., $A \subset B$ then $n(A) < n(B)$

Improper subset:-
If A is subset of set B such that $A = B$ then A is called improper subset of set B denoted by $A \subseteq B$. e.g., If
$A = \{1, 2, 3, 4, 5\}$ and $B = \{5, 3, 4, 1, 2\}$
then $A = B$

* We may say that two equal sets are also improper subsets of each other.
* Every set has one and only one improper subset.

Power Set:-
The collection of all possible subsets of any set A is called power set of A denoted by $P(A)$

Example 3. If $A = \{a, b\}$, then
$P(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

* Empty set is the subset of every set and every set is its own subset.

Example 4. If $B = \{1, 2, 3\}$ then
$P(B) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}$
$\{2,3\}, \{1,2,3\}\}$

Example 5. If $C = \{a, b, c, d\}$ then
$= \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}$
$\{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\},$
$\{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\}$

Example 6. If $D = \{a\}$ then
$P(D) = \{\phi, \{a\}\}$

Example 7. If $E = \{\ \}$ then
$P(E) = \{\phi\}$

Universal Set:- "The super set of all the sets under discussion is called universal set and is denoted by U."

* Universal set is also called Universe of Discourse.

Example 1. (Page # 31)

Check that $N = \{1, 2, 3 \ldots\}$ and $O = \{1, 3, 5, \ldots\}$ are equal or equivalent.

Solution:- Here

$$\left\{\begin{matrix} 1 & 2 & 3 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \\ 1 & 3 & 5 & \ldots \end{matrix}\right\}$$ (1-1) correspondence can be made between N and O. so $N \sim O$. But for equal sets they must have same elements so $N \neq O$.

Example 2. (Page # 32).

Let $A = \{a, b, c\}$, $B = \{c, a, b\}$ and $C = \{a, b, c, d\}$

clearly $A \subset C$, $B \subset C$

but $A = B$ and $B = A$

Remember, "When we do not want to distinguish between proper and improper subsets, we may use the symbol $\subseteq$ for the relationship."

It is easy to see that:

$N \subset Z \subset Q \subset \mathbb{R}$

* The empty set is a subset of every set.

Exercise 2.1

Q1. Write the following set in set builder notation:

Solution:-

i) $\{1, 2, 3, \ldots, 1000\}$

$\{x \mid x \in N \wedge x \leq 1000\}$

ii) $\{0, 1, 2, \ldots, 100\}$

$\{x \mid x \in W \wedge x \leq 100\}$

iii) $\{0, \pm 1, \pm 2, \ldots, \pm 1000\}$

$\{x \mid x \in Z \wedge -1000 \leq x \leq 1000\}$

iv) $\{0, -1, -2, \ldots, -500\}$

$\{x \mid x \in Z \wedge -500 \leq x \leq 0\}$

v) $\{100, 101, 102, \ldots, 400\}$

$\{x \mid x \in N \wedge 100 \leq x \leq 400\}$

vi) {Peshawar, Lahore, Karachi, Quetta}

$\{x \mid x$ is provincial capital of Pakistan$\}$

vii) $\{-100, -101, -102, \ldots, -500\}$

$\{x \mid x \in Z \wedge -500 \leq x \leq -100\}$

viii) {January, June, July}

$\{x \mid x$ is month of solar year, starting with J$\}$

ix) The set of all odd natural numbers

$\{x \mid x \in N$ and x is odd$\}$

x) The set of all rational numbers

$\{x \mid x \in Q\}$

xi) The set of all real numbers 1 and 2

$\{x \mid x \in R \wedge 1 < x < 2\}$

xii) The set of all integers between -100 and 1000

$\{x \mid x \in Z \wedge -100 < x < 1000\}$

Q2. Write each of the following sets in descriptive and tabular forms:-

Solution:-

i) $\{x | x \in N \wedge x \leq 10\}$

Tabular form:- $\{1, 2, 3, 4, \ldots, 10\}$

Des. form:- Set of first ten natural numbers

ii) $\{x | x \in N \wedge 4 < x < 12\}$

Tabular form:- $\{5, 6, 7, \ldots, 11\}$

Des. form:- Set of natural numbers between 4 and 12.

iii) $\{x | x \in Z \wedge -5 < x < 5\}$

Tabular form:- $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

Des. form:- Set of integers between -5 and 5.

iv) $\{x | x \in E \wedge 2 < x \leq 4\}$

Tabular form:- $\{4\}$

Des. form:- set of even numbers between 2 and 5

v) $\{x | x \in P \wedge x < 12\}$

Tabular form:- $\{2, 3, 5, 7, 11\}$

Des. form:- set of Prime numbers less than 12

vi) $\{x | x \in O \wedge 3 < x < 12\}$

Tabular form:- $\{5, 7, 9, 11\}$

Des. form:- set of odd integers between 3 and 12.

vii) $\{x | x \in E \wedge 4 \leq x \leq 10\}$

Tabular form:- $\{4, 6, 8, 10\}$

Des. form:- Set of even numbers from 4 to 10

viii) $\{x | x \in E \wedge 4 < x < 6\}$

Tabular form:- $\{\ \}$

Des. form:- Set of even numbers between 4 and 6

ix) $\{x | x \in O \wedge 5 \leq x \leq 7\}$

Tabular form:- $\{5, 7\}$

Des. form:- Set of odd integers from 5 to 7

x) $\{x | x \in O \wedge 5 \leq x < 7\}$

Tabular form:- $\{5\}$

Des. form:- set of odd integers between 4 and 6

xi) $\{x | x \in N \wedge x + 4 = 0\}$

Tabular form:- $\{\ \}$ $\because x = -4 \notin N$

Des. form:- Set of natural numbers satisfying $x + 4 = 0$

xii) $\{x | x \in Q \wedge x^2 = 2\}$

Tabular form:- $\{\ \}$ $\because x = \pm\sqrt{2} \notin Q$ but $x = \pm\sqrt{2} \in Q'$

Des. form:- Set of rational nos. satisfying equation $x^2 = 2$

xiii) $\{x | x \in \mathbb{R} \wedge x = x\}$

Tabular form:- $\mathbb{R}$

Des. form:- set of real nos. satisfying equation $x = x$

xiv) $\{x | x \in Q \wedge x = -x\}$

Tabular form:- $\{0\}$ $\because x = -x$
$\rightarrow x + x = 0$
$\rightarrow 2x = 0 \rightarrow x = 0$ $\because 2 \neq 0$

Descriptive form:- set of rational numbers satisfying equation $x = -x$

xv) $\{x | x \in \mathbb{R} \wedge x \neq 2\}$

Tabular form:- $\mathbb{R} - \{2\}$

Des. form:- set of real nos. except 2

xvi) $\{x | x \in \mathbb{R} \wedge x \notin Q\}$

Tabular form:- Q'

Des. form:- Set of all real numbers which are not rational.

Q3. Which of the following sets are finite and which of these are infinite?

Solution:-

i) The set of students of your class

Finite

ii) The set of all schools in Pakistan

Finite

iii) The set of natural numbers between 3 and 10.

Finite

iv) The set of rational numbers between 3 and 10

Infinite

v) The set of real numbers between 0 and 10.

Infinite

vi) The set of rationals between 0 and 1

Infinite

vii) The set of whole numbers between 0 and 1

Finite

viii) The set of all leaves of trees in Pakistan

Infinite

ix) $P(N)$ Infinite

x) $P\{a,b,c\}$ Finite

xi) $\{1,2,3,4,\ldots\}$ Infinite

xii) $\{1,2,3,\ldots\ldots,100000000\}$ Finite

xiii) $\{x \mid x \in \mathbb{R} \wedge x \neq x\}$ Finite

xiv) $\{x \mid x \in \mathbb{R} \wedge x^2 = -16\}$ Finite

xv) $\{x \mid x \in Q \wedge x^2 = 5\}$ Finite

xvi) $\{x \mid x \in Q \wedge 0 \leq x \leq 1\}$ Infinite

Q4. Write two proper subsets of each of the following sets:-

i) $\{a,b,c\}$

Solution:-

$\{a\}, \{b\}$

ii) $\{0,1\}$

Solution:-

$\{0\}, \{1\}$

iii) N

Solution:-

$N = \{1,2,3,\ldots\}$

$\{1\}, \{2\}$

iv) Z

Solution:-

$Z = \{0, \pm 1, \pm 2, \ldots\}$

$\{1\}, \{2\}$

v) $\mathbb{R}$

Solution:-

$\mathbb{R}$ = Set of real nos.

$\{1\}, \{2\}$

vi) Q

Solution:-

Q = set of rational nos.

$\{1\}, \{2\}$

vii) W

Solution:-

W = set of whole numbers

$\{1\}, \{2\}$

viii) $\{x \mid x \in Q \wedge 0 \leq x \leq 2\}$

Solution:-

$\{1\}, \{2\}$

Q5. Is there any set which has no proper subset? If so name the set.

Solution:- Yes, ϕ is set which has no proper subset.

Q6. What is the difference between $\{a,b\}$ and $\{\{a,b\}\}$?

Solution:-

$\{a,b\}$ is a set with two elements a and b

and $\{\{a,b\}\}$ is a set with one element $\{a,b\}$.

Q7. Which of the following sentences are true and which of them are false?

Solution:-

			Reason
i)	$\{1,2\} = \{2,1\}$	True	(Because order of elements in sets does not matter.)
ii)	$\phi \subseteq \{\{2,1\}\}$	True	(Because empty set is subset of every set.)
iii)	$\{a\} \subseteq \{\{a\}\}$	False	(Because an element can not be the subset of any set.)
iv)	$\{a\} \in \{\{a\}\}$	True	(Because {a} is an element of the set {{a}}.)
v)	$a \in \{\{a\}\}$	False	(Because "a" is not an element of the set {{a}}.)
vi)	$\phi \in \{\{a\}\}$	False	(Because a set can not be the element of the other set.)

Q8. What is the number of elements of the Power set of each of the following sets?

Solution:-

i) $\{\ \}$

Power set of $\{\ \}$ has elements $= 2^0 = 1$

ii) $\{0,1\}$

Power set of $\{0,1\}$ has elements $= 2^2 = 4$

iii) $\{1,2,3,4,5,6,7\}$

Power set of $\{1,2,3,4,5,6,7\}$ has elements

$2^7 = 128$

iv) $\{0,1,2,3,4,5,6,7\}$

Power set of $\{0,1,2,3,4,5,6,7\}$ has elements $= 2^8 = 256$

v) $\{a,\{b,c\}\}$

Power set of $\{a,\{b,c\}\}$ has elements $= 2^2 = 4$

vi) $\{\{a,b\},\{b,c\},\{d,e\}\}$

Power set of $\{\{a,b\},\{b,c\},\{d,e\}\}$ has element $= 2^3 = 8$

Q9. Write down the power set of each of the following sets:-

i) $\{9,11\}$

Solution:- $\{9,11\}$

Power set is $\{\phi, \{9\}, \{11\}, \{9,11\}\}$

ii) $\{+,-,\times,\div\}$

Solution:- $\{+,-,\times,\div\}$

Power set is

$\{\phi, \{+\}, \{-\}, \{\times\}, \{\div\}, \{+,-\}, \{+,\times\}, \{+,\div\}, \{-,\times\}, \{-,\div\}, \{\times,\div\}, \{+,-,\times\}, \{+,-,\div\}, \{+,\times,\div\}, \{-,\times,\div\}, \{+,-,\times,\div\}\}$

iii) $\{\phi\}$

Solution:- $\{\phi\}$

Power set is $\{\phi, \{\phi\}\}$

iv) $\{a,\{b,c\}\}$

Solution:- $\{a,\{b,c\}\}$

Power set is $\{\phi, \{a\}, \{\{b,c\}\}, \{a,\{b,c\}\}\}$

Q10. Which pairs of sets are equivalent? Which of them are also equal?

i) $\{a,b,c\}$, $\{1,2,3\}$

Equivalent

ii) The set of the first 10 whole numbers, $\{0,1,2,3,\ldots,9\}$

Equal

iii) Set of angles of a quadrilateral ABCD,
set of sides of the same quadrilateral **Equivalent**

iv) Set of the sides of a hexagon ABCDEF,
set of angles of the same hexagon;

Equivalent

v) $\{1,2,3,4,\ldots\}$, $\{2,4,6,8,\ldots\}$

Equivalent

vii) $\{5,10,15,\ldots,55555\}$,
$\{5,10,15,20,\ldots\}$

Neither equivalent nor Equal sets

Operations on Sets

Union of two sets:- Union of two sets A and B denoted by $A \cup B$ is the set of all elements, which belongs to A or B. Symbolically;

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Example:- If $A = \{1,2,3\}$ and $B = \{2,3,4,5\}$ then

$$A \cup B = \{1,2,3,4,5\}$$

Intersection of two sets:-

Intersection of two sets A and B, denoted by $A \cap B$ is the set of all elements, which belong to A and B. Symbolically.,

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Example:- If $A = \{1,2,3\}$ and $B = \{2,3,4,5\}$ then $A \cap B = \{2,3\}$

Disjoint Sets:- If intersection of two sets A and B is empty set then sets A and B are called disjoint sets.

Example:- $O \cap E = \phi$ where O is set of odd integers and E is set of even integers.

Overlapping Sets:- If the intersection of two sets A and B is non-empty set but neither is subset of the other, the sets are called overlapping sets.

Example:- Let $A = \{1,2,3,4\}$
$B = \{3,4,5,6\}$; $A \cap B = \{3,4\}$
= overlaping set

Complement of a set:-

If U is universal set, then U/A or U−A is called complement of A, denoted A' or A^c. Thus $A^c = U - A$

Symbolically $A' = \{x \mid x \in U \wedge x \notin A\}$

Example:- If $U = N$ then
$E' = O$ and $O' = E$

Example 1. If U = set of alphabets of English language
C = set of consonants, W = set of vowels
then $C' = W$ and $W' = C$

Difference of two sets:-

The difference $A - B$ or A/B of two sets A and B is the set of elements which belong to A but not belonging to B.

Symbolically;

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Example 2. If $A = \{1,2,3,4,5\}$ and $B = \{4,5,6,7,8,9,10\}$ then

$A - B = \{1,2,3\}$ and $B - A = \{6,7,8,9,10\}$

Notice that $A - B \neq B - A$

Venn Diagrams

(Named by "JOHN VENN" The English Logician and Mathematician (1834 - 1883 A.D.)

"It is the picture representation of given sets in the form of rectangle and circles".

- In Venn Diagram, rectangular region represents universal set U and circular region represent given sets.

Venn Diagrams of given sets

When A and B are Disjoint Sets ($A \cap B = \phi$)

Results:-

$n(A \cup B) = n(A) + n(B)$

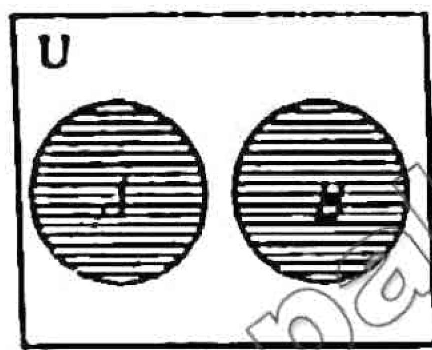
$n(A \cap B) = 0$

$n(A - B) = n(A)$

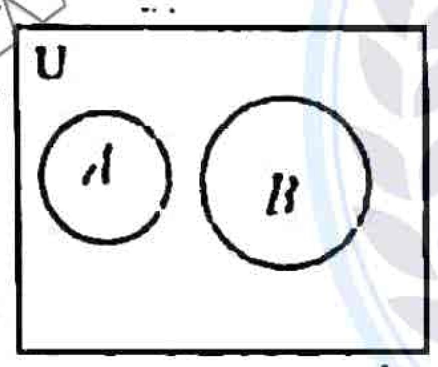
$n(B - A) = n(B)$

$n(B) \leq n(A \cup B)$

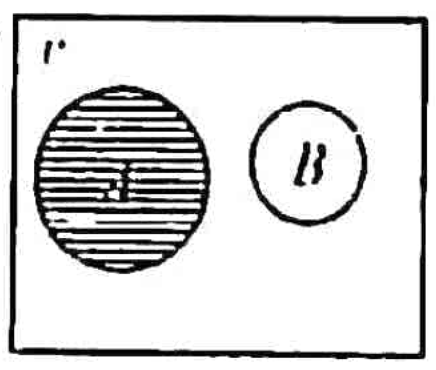
$n(A) \leq n(A \cup B)$



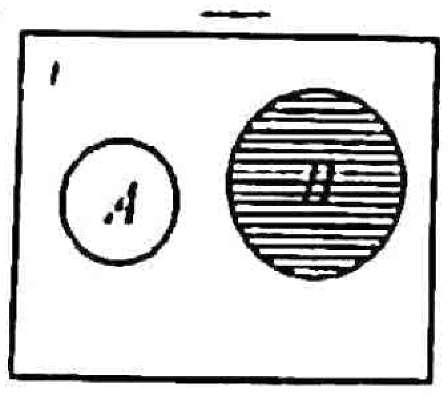
$A \cup B$



$A \cap B$

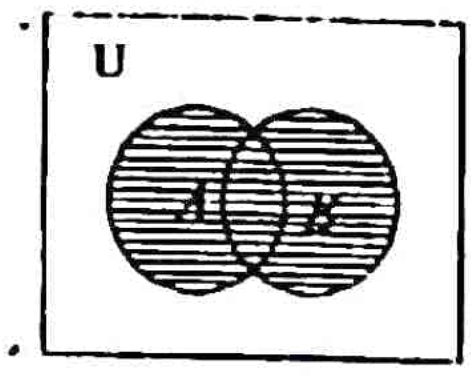


$A - B$

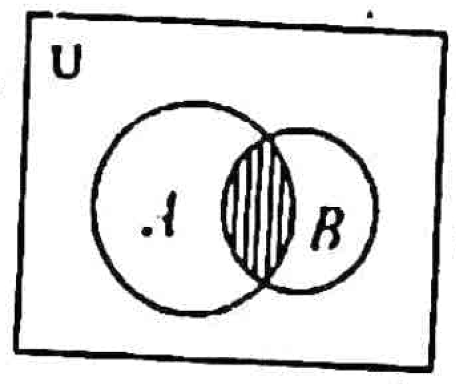


$B - A$

When A and B are Overlapping ($A \cap B \neq \phi$)



$A \cup B$



$A \cap B$

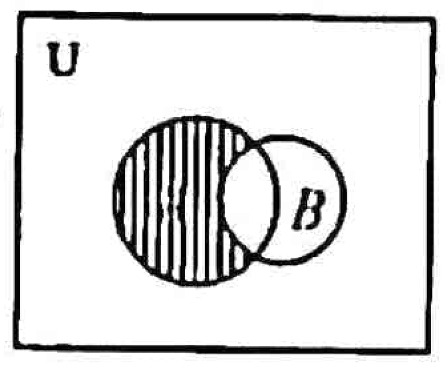
Results:-

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A \cap B) \leq n(A)$

$n(A \cap B) \leq n(B)$

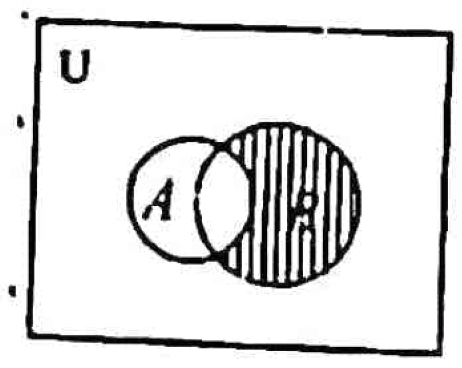
$n(A - B) = n(A) - n(B)$



$A - B$

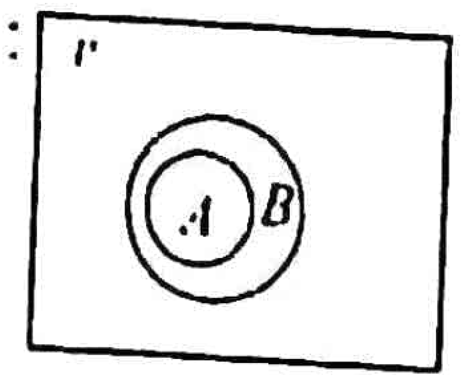
$n(B - A) = n(B) - n(A \cap B)$

$n(A \cap B) = n(A) + n(B) - n(A \cup B)$



$B - A$

When A is subset of B ($A \subseteq B$)



$A - B$

Results:-

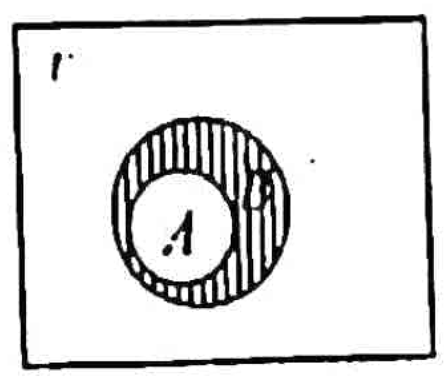
$n(A \cup B) = n(B)$

$n(A \cap B) = n(A)$

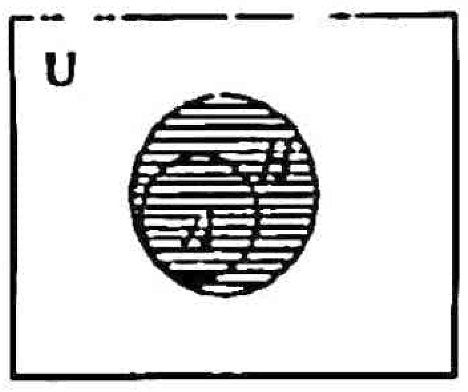
$n(A) \leq n(B)$

$n(A - B) = 0$

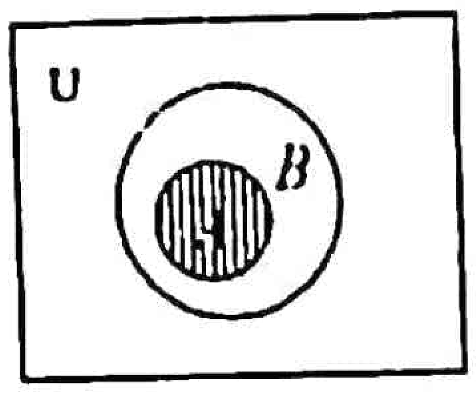
$n(B - A) = n(B) - n(A)$



$B - A$

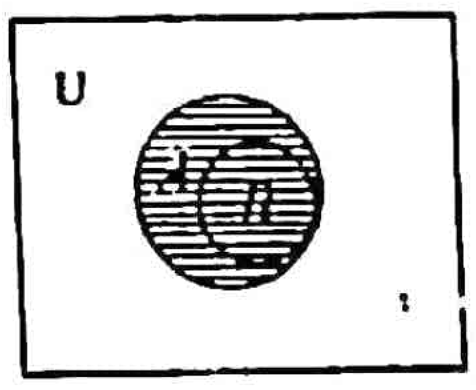


$A \cup B$



$A \cap B$

When B is subset of A
$(B \subseteq A)$



$A \cup B$

Results:-

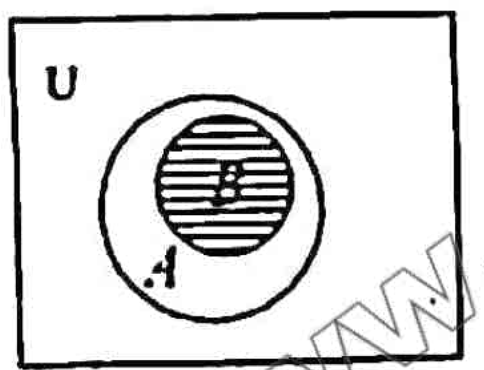
$n(B \cup A) = n(A)$

$n(A \cap B) = n(B)$

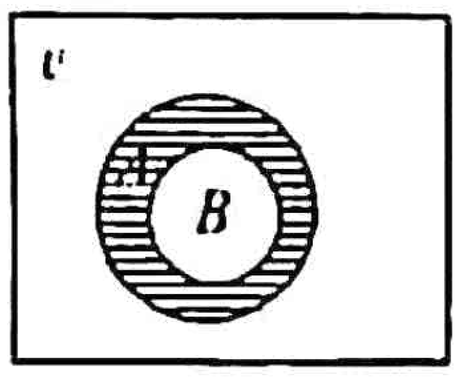
$n(B) \leq n(A)$

$n(B - A) = 0$

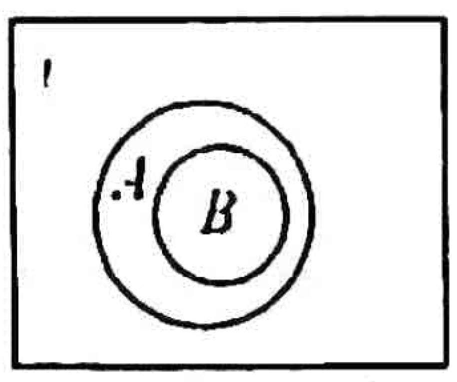
$n(A - B) = n(A) - n(B)$



$A \cap B$



$A - B$



$B - A$

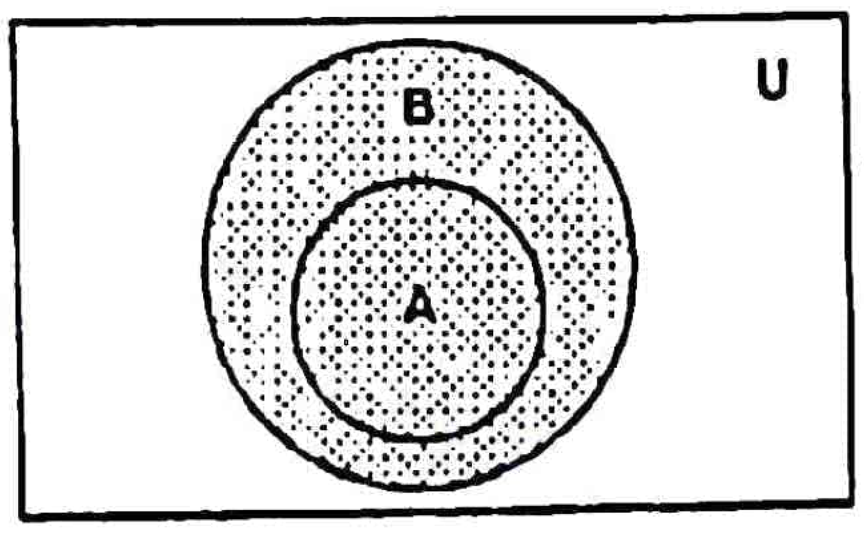
Note:- Shaded area gives required region or required result.

Exercise 2.2

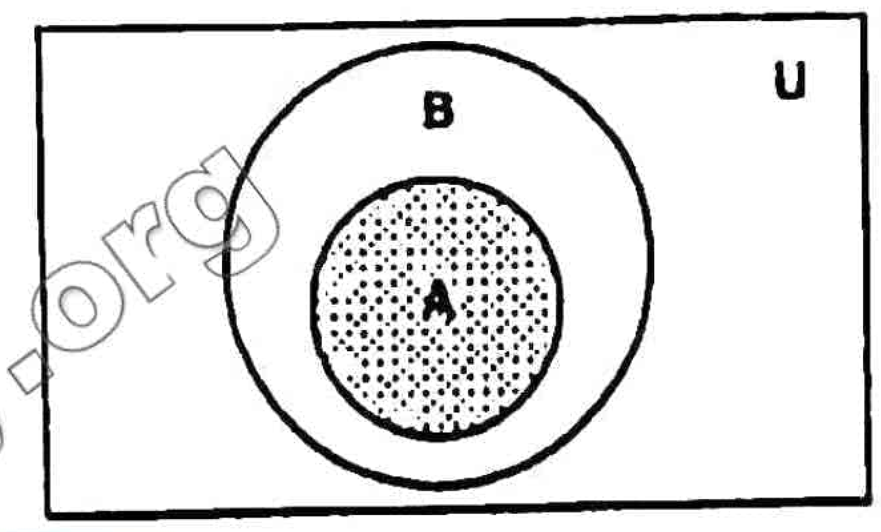
Q1. Exhibit $A \cup B$ and $A \cap B$ by Venn diagrams in the following cases:-

i) $A \subseteq B$

Solution:- $A \subseteq B$



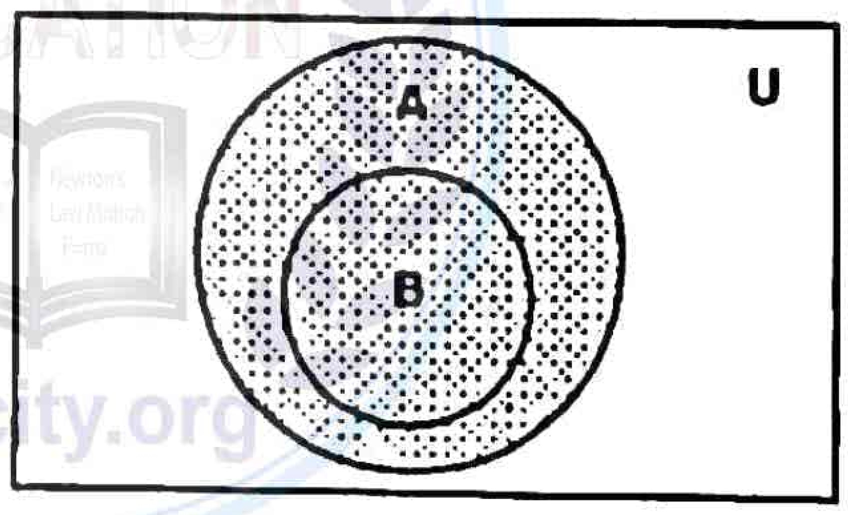
$A \cup B$



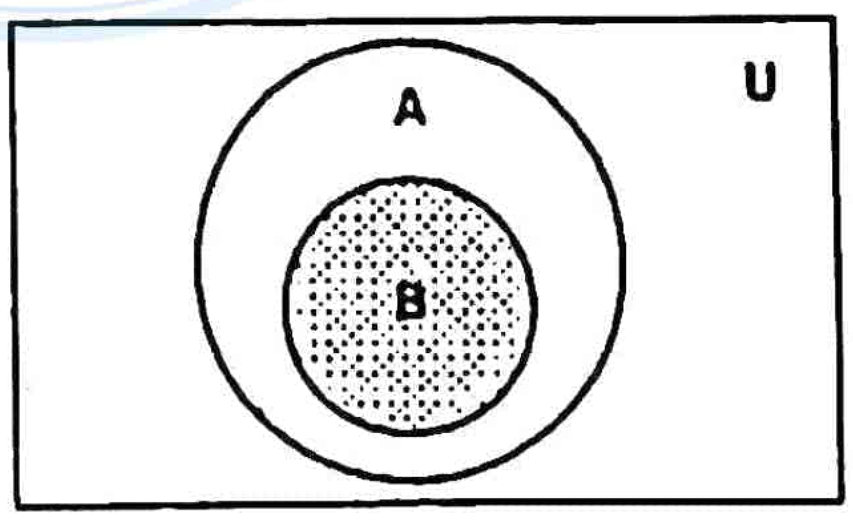
$A \cap B$

ii) $B \subseteq A$

Solution:- $B \subseteq A$



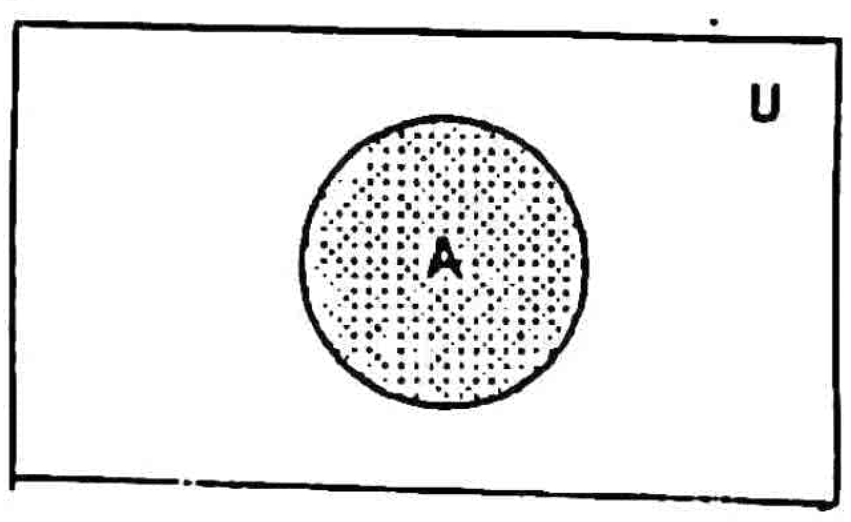
$A \cup B$



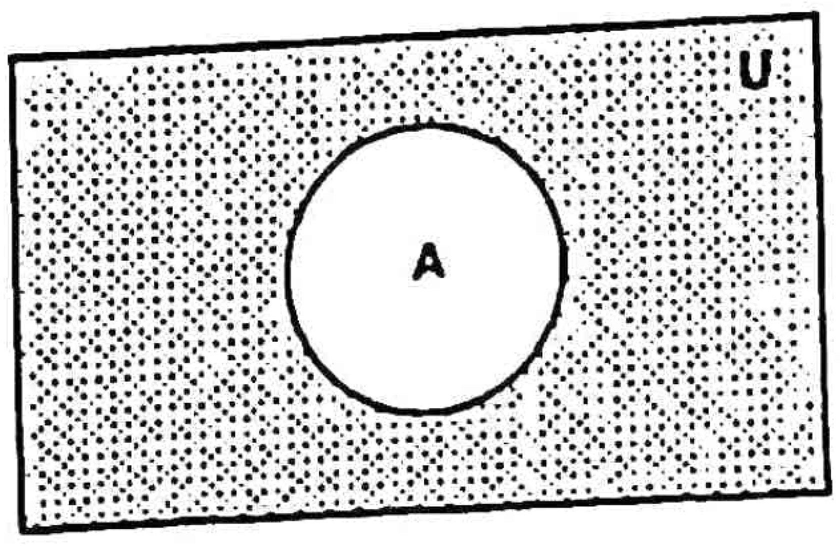
$A \cap B$

iii) $A \cup A'$

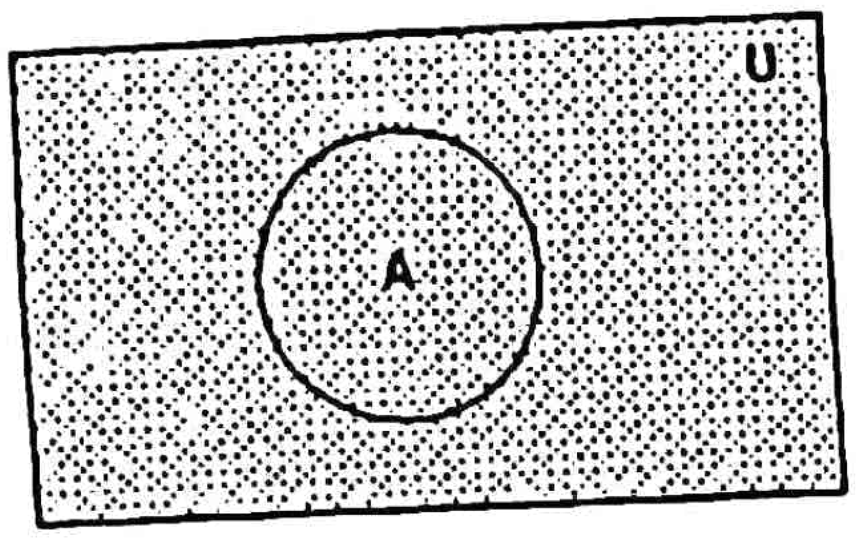
Solution:- $A \cup A'$



A



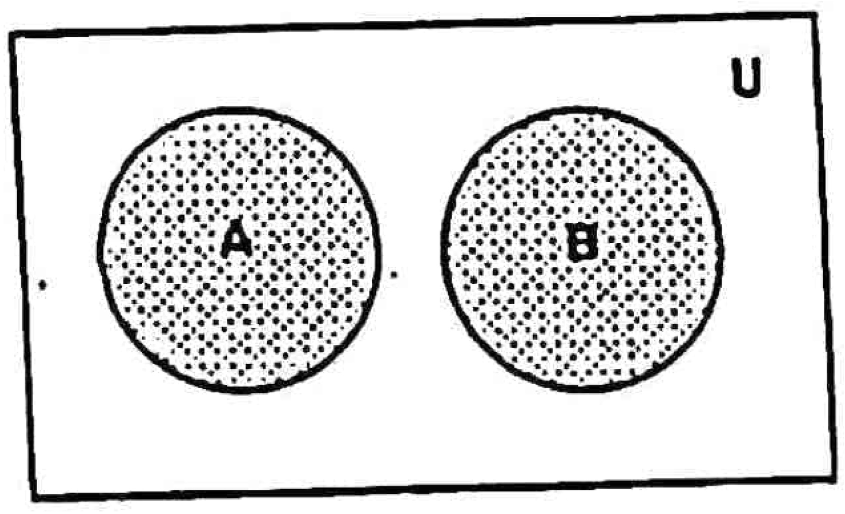
A'



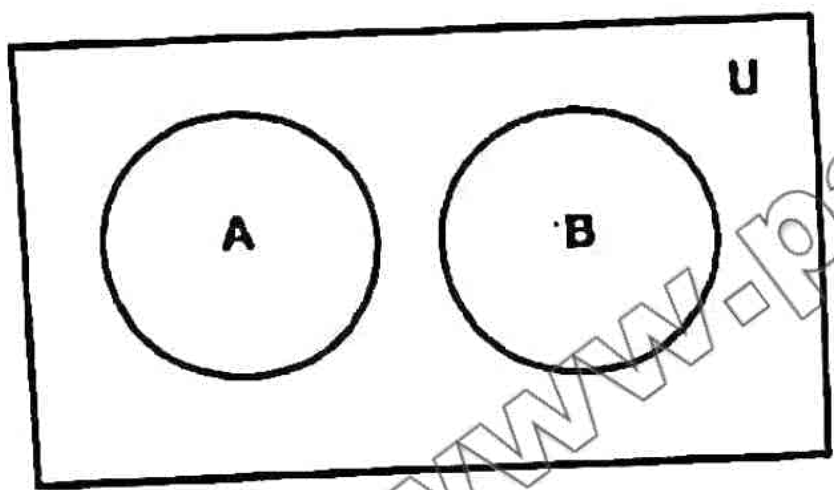
$A \cup A'$

iv) A and B are disjoint sets

Solution:-



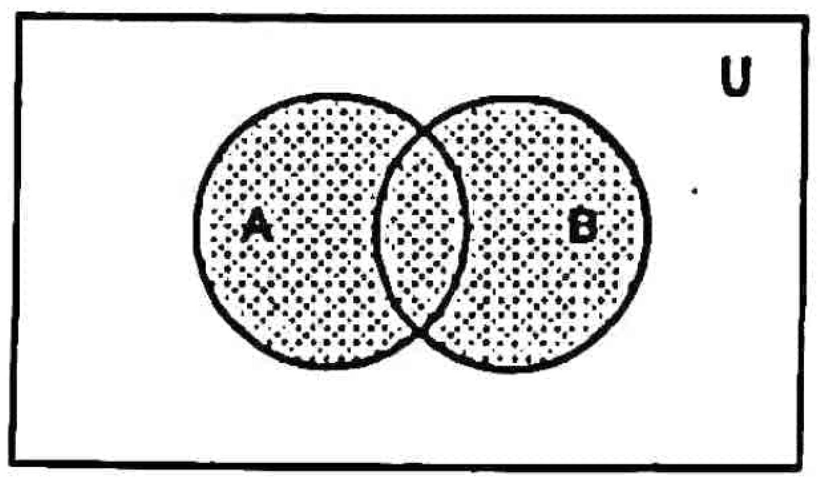
$A \cup B$



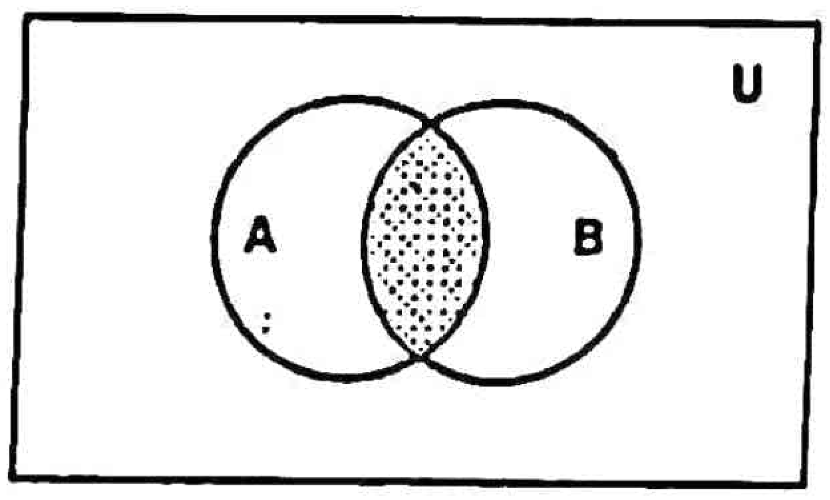
$A \cap B$

v) A and B are overlapping sets

Solution:-



$A \cup B$

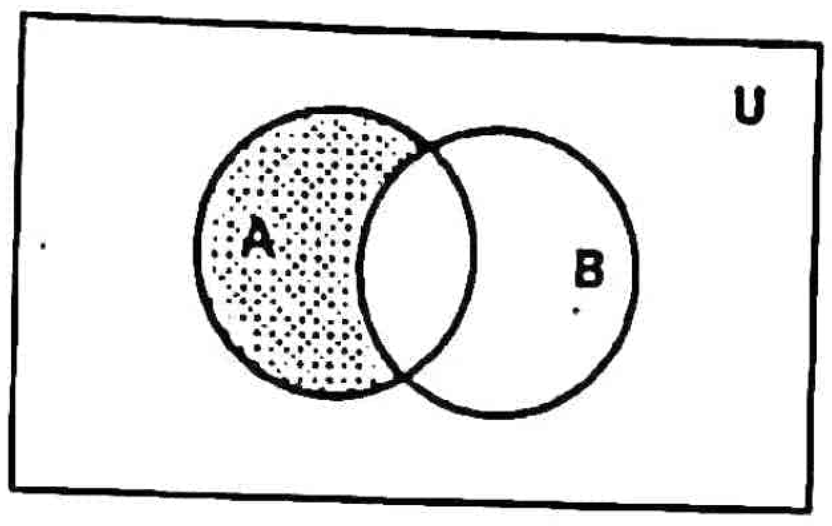


$A \cap B$

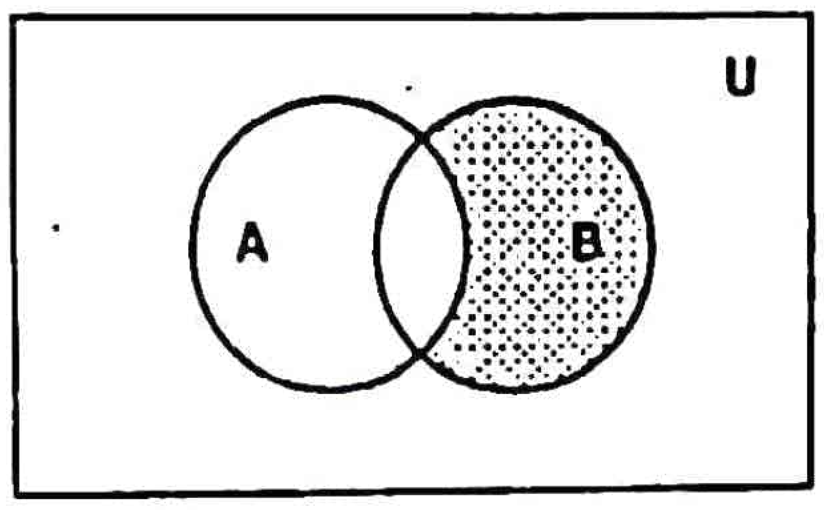
Q2. Show A-B and B-A by Venn Diagrams when:-

i) A and B are overlapping sets

Solution:-



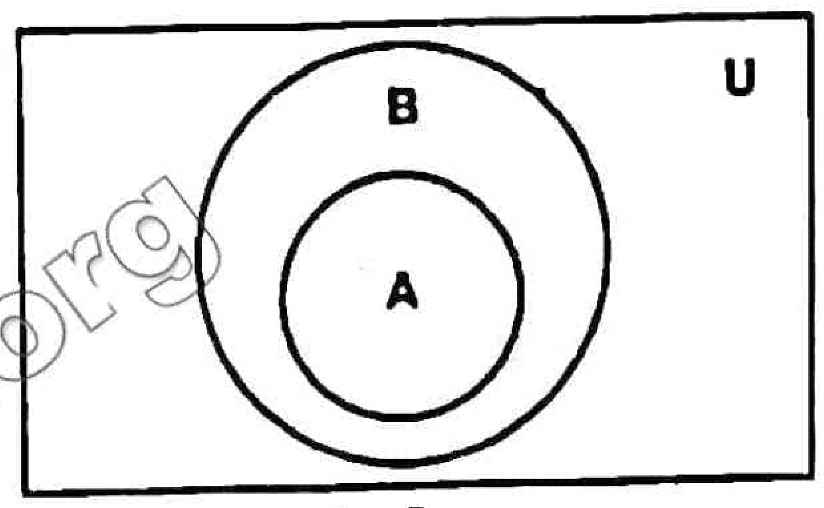
$A-B$



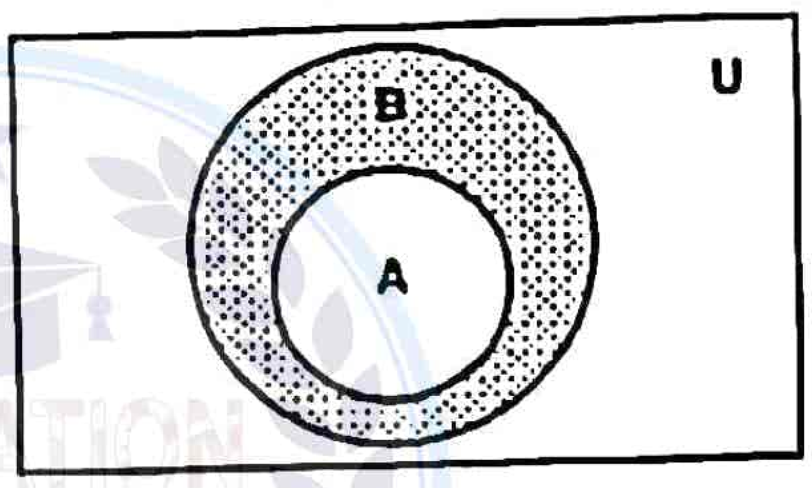
$B-A$

ii) $A \subseteq B$

Solution :-



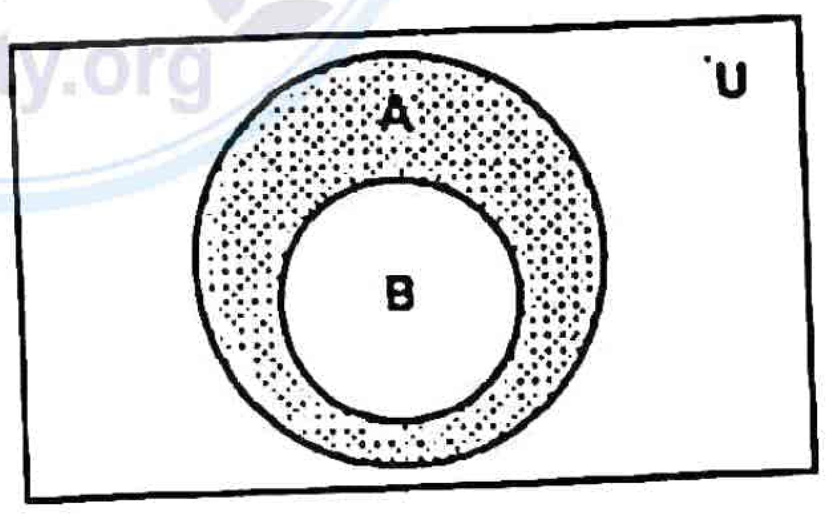
$A-B$



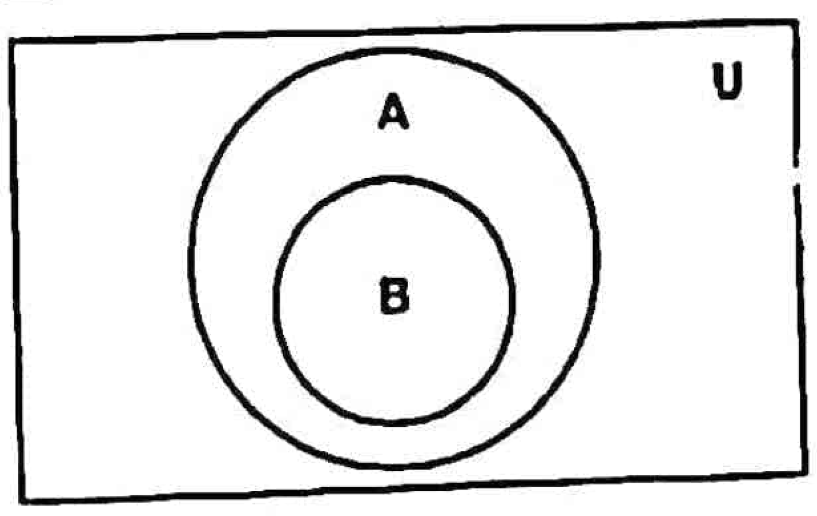
$B-A$

iii) $B \subseteq A$

Solution:-



$A-B$



$B-A$

Q3. Under what conditions on A and B are the following statements true?

Solution:-

i) $A \cup B = A$

If $B \subseteq A$

ii) $A \cup B = B$

If $A \subseteq B$

iii) $A - B = A$ If $A \cap B = \phi$

iv) $A \cap B = B$ If $B \subseteq A$

v) $n(A \cup B) = n(A) + n(B)$

If A and B are disjoint sets

vi) $n(A \cap B) = n(A)$

If $A \subseteq B$

vii) $A - B = A$

If A and B are disjoint sets or $A \cap B = \phi$

viii) $n(A \cap B) = 0$

If $A \cap B = \phi$

ix) $A \cup B = U$

If $B = A'$ or $B' = A$

x) $A \cup B = B \cup A$

It is always true

xi) $n(A \cap B) = n(B)$

If $B \subseteq A$

xii) $U - A = \phi$

If $U = A$

Q4. Let $U = \{1,2,3,4,5,6,7,8,9,10\}$

$A = \{2,4,6,8,10\}$, $B = \{1,2,3,4,5\}$

and $C = \{1,3,5,7,9\}$

Solution:-

i) A^c

$A^c = U - A$

$= \{1,2,3,4,5,6,7,8,9,10\} - \{2,4,6,8,10\}$

$= \{1,3,5,7,9\} = C$

ii) B^c

$B^c = U - B$

$= \{1,2,3,4,5,6,7,8,9,10\} - \{1,2,3,4,5\}$

$= \{6,7,8,9,10\}$

iii) $A \cup B$

$A \cup B = \{2,4,6,8,10\} \cup \{1,2,3,4,5\}$

$\Rightarrow A \cup B = \{1,2,3,4,5,6,8,10\}$

iv) $A - B$

$A - B = \{2,4,6,8,10\} - \{1,2,3,4,5\}$

$\Rightarrow A - B = \{6,8,10\}$

v) $A \cap C$

$A \cap C = \{2,4,6,8,10\} \cap \{1,2,3,4,5\} = \phi$

vi) $A^c \cup C^c$

$A^c = U - A = \{1,2,3,4,5,6,7,8,9,10\} - \{2,4,6,8,10\}$

$A^c = \{1,3,5,7,9\}$

$C^c = U - C = \{1,2,3,4,5,6,7,8,9,10\} - \{1,3,5,7,9\}$

$C^c = \{2,4,6,8,10\}$

$A^c \cup C^c = \{1,3,5,7,9\} \cup \{2,4,6,8,10\}$

$= \{1,2,3,4,5,6,7,8,9,10\}$

vii) $A^c \cup C$

$A^c = U - A = \{1,2,3,4,5,6,7,8,9,10\} - \{2,4,6,8,10\}$

$= \{1,3,5,7,9\}$

$A^c \cup C = \{1,3,5,7,9\} \cup \{1,3,5,7,9\}$

$= \{1,3,5,7,9\}$

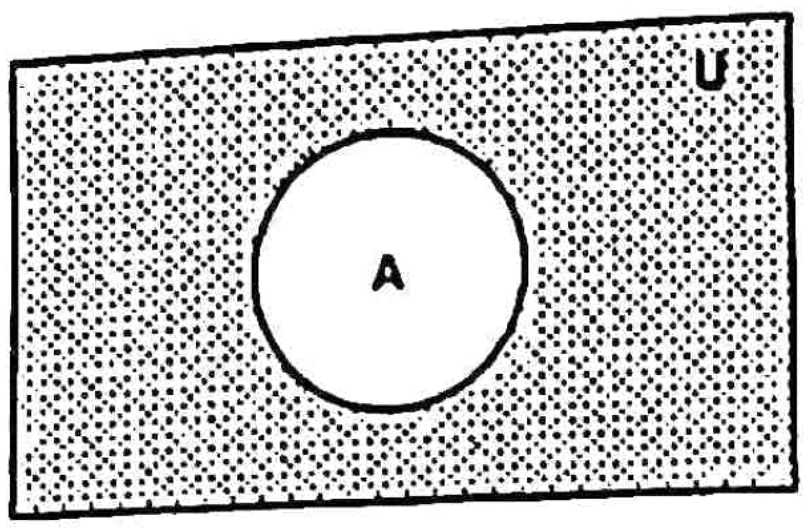
$U^c = U - U = \{1,2,3,\ldots,10\} - \{1,2,3,\ldots,10\}$

$U^c = \phi$

Q5. Using the venn Diagrams, if necessary, find the singleton set equa to the following.

i) A^c

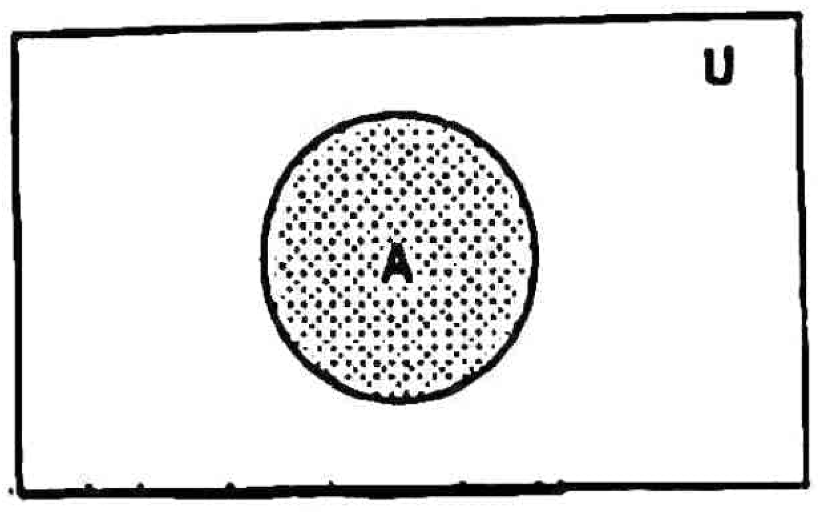
Solution:- $\because A^c = U - A$



shaded area shows A^c

ii) $A \cap U$

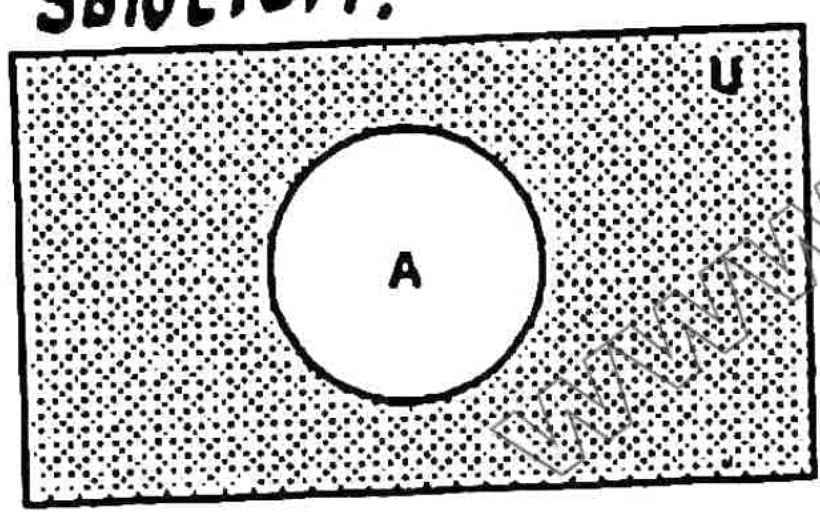
Solution:- $\because A \cap U = A$



shaded area shows $A \cap U$

iii) $A \cup U$

Solution:- $\because A \cup U = U$

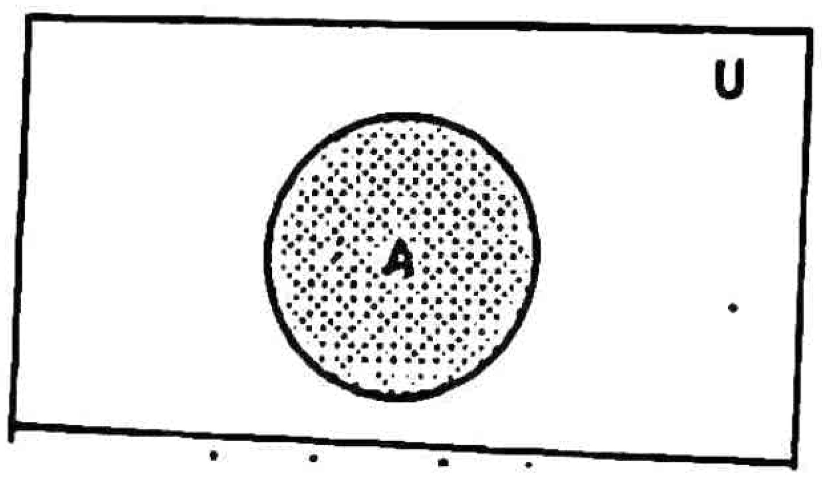


shaded area shows $A \cup U$

iv) $A \cup \phi$

Solution:-

$\because A \cup \phi = A$



shaded area shows $A \cup \phi$

v) $\phi \cap \phi$

Solution:-

$\because \phi \cap \phi = \{ \ \}$

Venn diagram is not necessary

Q6. Use the venn diagrams to verify

i) $A - B = A \cap B'$

ii) $(A-B)^c \cap B = B$

Solution :- i) $A - B = A \cap B'$

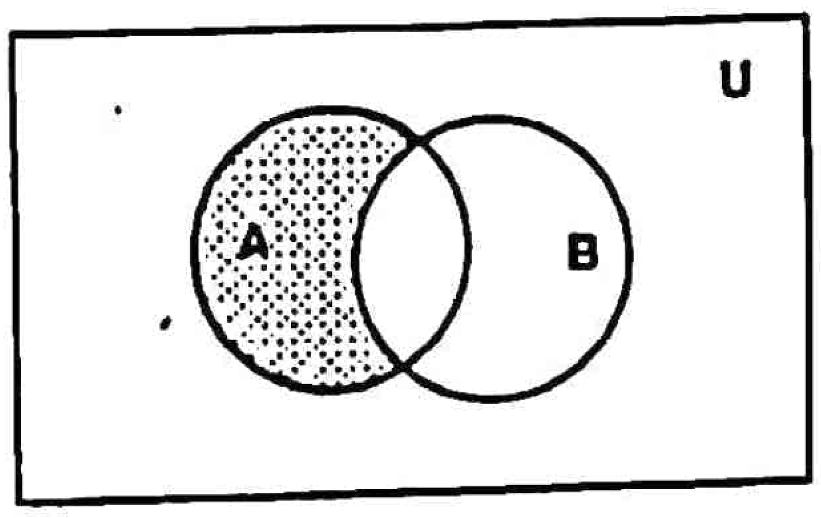


fig (i)

$A - B$

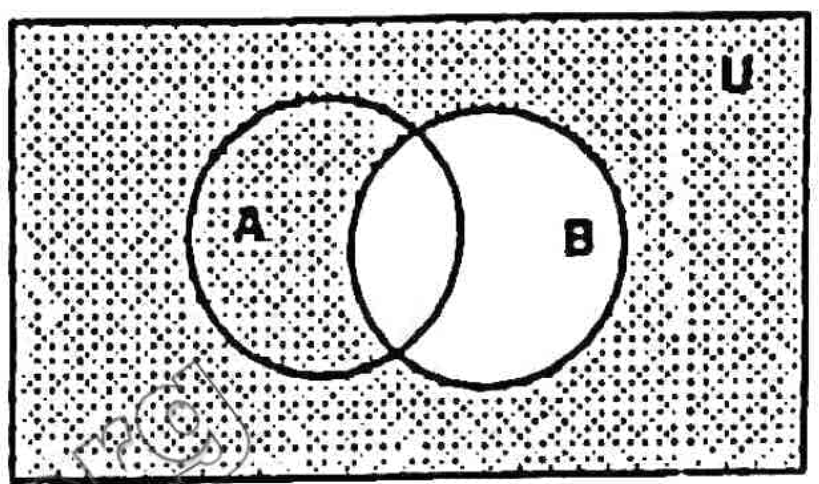


fig (ii)

$B' = U - B$

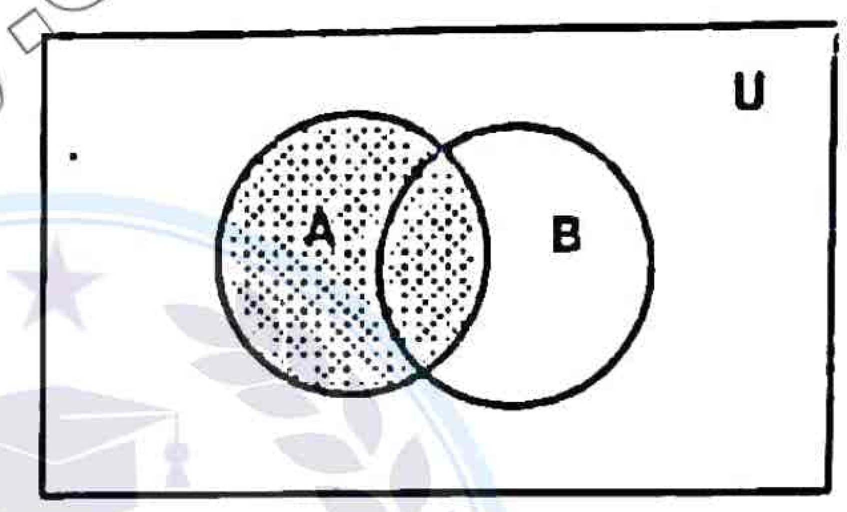
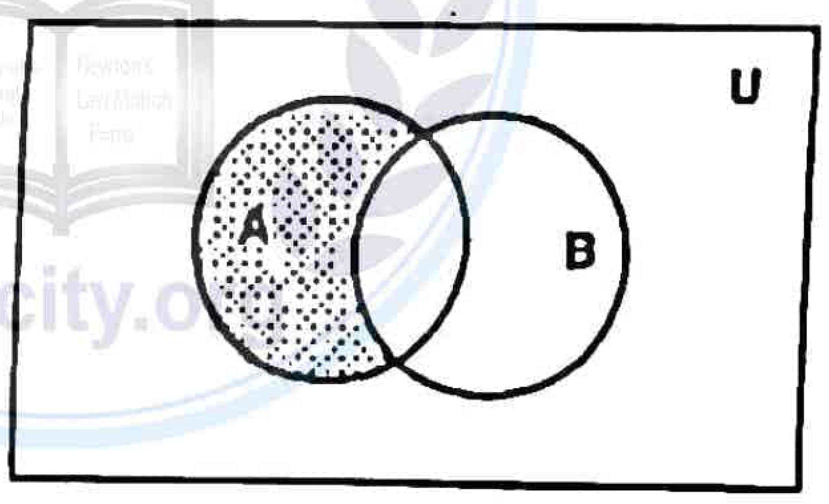


fig (iii)

A

Now from fig (ii) and fig (iii) :



$A \cap B'$

fig (iv)

Hence from fig (i) and f (iv) we conclude that $A - B = A \cap B'$

ii) $(A - B)^c \cap B = B$

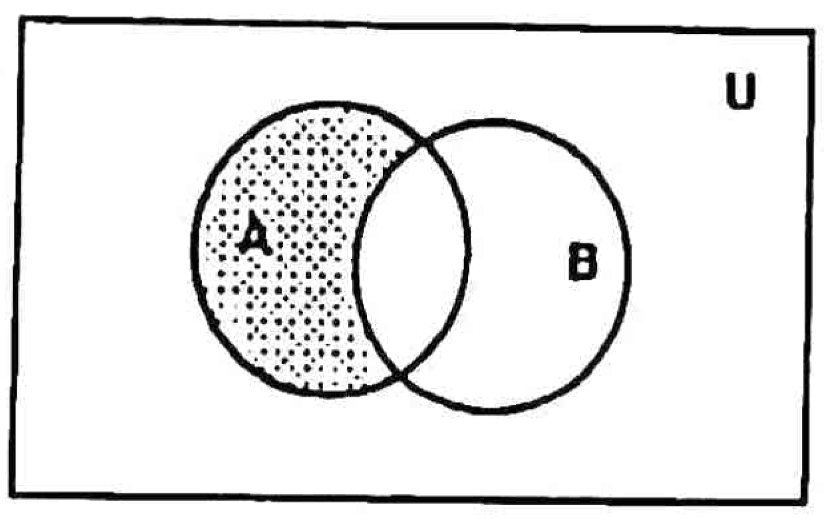


fig (i)

$A - B$

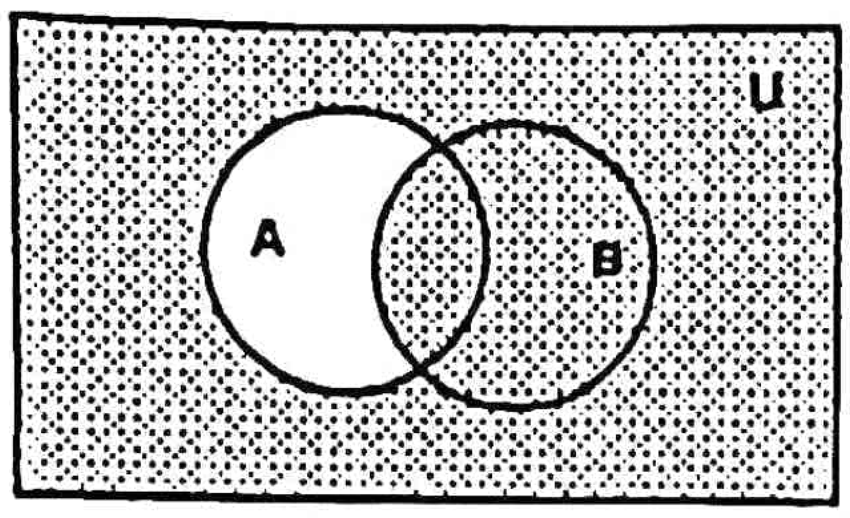


fig (ii)

$(A-B)^c = U-(A-B)$

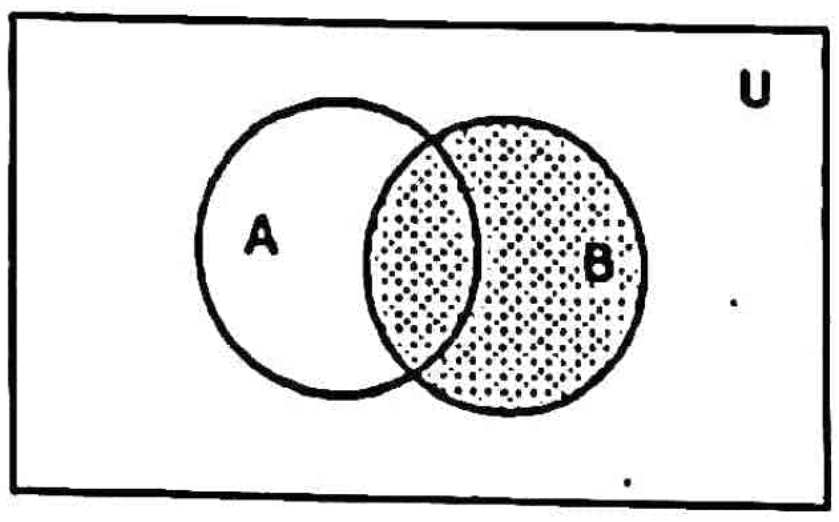


fig (iii)

B

from fig (ii) and fig (iii)

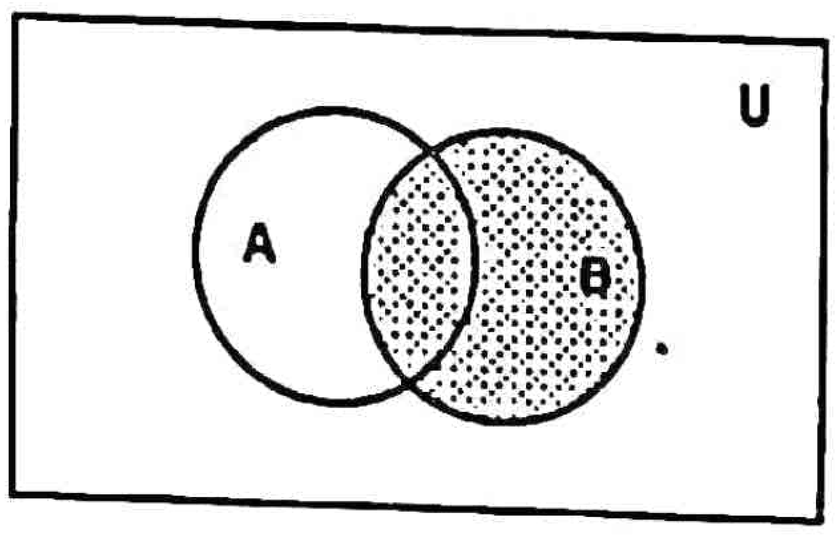


fig (iv)

$(A-B)^c \cap B$

from fig (iii) and fig (iv) we conclude that

$(A-B)^c \cap B = B$

Properties of Union and Intersection

i) Commutative property of Union $A \cup B = B \cup A$

Proof:-

Let $x \in A \cup B$

$\to x \in A$ or $x \in B$

$\to x \in B$ or $x \in A$

$\to x \in B \cup A$ As $x \in A \cup B \to x \in B \cup A$

so $A \cup B \subseteq B \cup A$ ..... (i)

Conversely,

Let $y \in B \cup A$

$\to y \in B$ or $y \in A$

$\to y \in A$ or $y \in B \to y \in A \cup B$

As $y \in B \cup A \to y \in A \cup B$

so $B \cup A \subseteq A \cup B$ ..... (ii)

From (i) and (ii) we conclude that $A \cup B = B \cup A$

ii) Associative Property of Union

$A \cup (B \cup C) = (A \cup B) \cup C$

Proof:-

Let $x \in A \cup (B \cup C)$

$\to x \in A$ or $x \in (B \cup C)$

$\to x \in A$ or ($x \in B$ or $x \in C$)

$\to$ ($x \in A$ or $x \in B$) or $x \in C$

$\to x \in (A \cup B)$ or $x \in C$

$\to x \in (A \cup B) \cup C$

As $x \in A \cup (B \cup C) \to x \in (A \cup B) \cup C$

so $A \cup (B \cup C) \subseteq (A \cup B) \cup C$ ..... (i)

Conversely,

Let $y \in (A \cup B) \cup C$

$\to y \in (A \cup B)$ or $y \in C$

($y \in A$ or $y \in B$) or $y \in C$

$\to y \in A$ or ($y \in B$ or $y \in C$)

$\to y \in A$ or $y \in (B \cup C)$

$\to y \in A \cup (B \cup C)$

As $y \in (A \cup B) \cup C \to y \in A \cup (B \cup C)$

so $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ ..... (ii)

from (i) and (ii) we conclude that $A \cup (B \cup C) = (A \cup B) \cup C$

iii) Commutative property of Intersection $A \cap B = B \cap C$

Proof:-

Let $x \in A \cap B$

$\to x \in A$ and $x \in B$

$\to x \in B$ and $x \in A$

$\to x \in B \cap A$

As $x \in A \cap B \to x \in B \cap A$

so $A \cap B \subseteq B \cap A$ ..... (i)

Conversely,

Let $y \in B \cap A$

Let $y \in B \cap A$

$\rightarrow y \in B$ and $y \in A$

$\rightarrow y \in A$ and $y \in B$

$\rightarrow y \in A \cap B$

As $y \in B \cap A \rightarrow y \in A \cap B$

so $B \cap A \subseteq A \cap B$ ..... (ii)

from (i) and (ii) we conclude that $A \cap B = B \cap A$

iv) Associative property of Intersection

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Proof:-

Let $x \in A \cap (B \cap C)$

$\rightarrow x \in A$ and $x \in (B \cap C)$

$\rightarrow x \in A$ and ($x \in B$ and $x \in C$)

$\rightarrow$ ($x \in A$ and $x \in B$) and $x \in C$

$\rightarrow x \in (A \cap B)$ and $x \in C$

$\rightarrow x \in (A \cap B) \cap C$

As $x \in A \cap (B \cap C) \rightarrow x \in (A \cap B) \cap C$

so $A \cap (B \cap C) \subseteq (A \cap B) \cap C$ ..... (i)

Conversely,

Let $y \in (A \cap B) \cap C \rightarrow y \in (A \cap B)$ and $y \in C$

$\rightarrow$ ($y \in A$ and $y \in B$) and $y \in C$

$\rightarrow y \in A$ and ($y \in B$ and $y \in C$)

$\rightarrow y \in A$ and $y \in (B \cap C)$

$\rightarrow y \in A \cap (B \cap C)$

As $y \in (A \cap B) \cap C \rightarrow y \in A \cap (B \cap C)$

so $A \cap (B \cap C) \subseteq (A \cap B) \cap C$ ..... (ii)

from (i) and (ii) we conclude that $A \cap (B \cap C) = (A \cap B) \cap C$

v) Distributivity of Union over Intersection

$$A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$$

Proof:-

Let $x \in A \cup (B \cap C)$

$\rightarrow x \in A$ or $x \in (B \cap C)$

$\rightarrow x \in A$ or ($x \in B$ and $x \in C$)

$\rightarrow$ ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$)

$\rightarrow$ ($x \in A \cup B$) and ($x \in A \cup C$)

$\rightarrow x \in (A \cup B) \cap (A \cup C)$

As $x \in A \cup (B \cap C) \rightarrow x \in (A \cup B) \cap (A \cup C)$

so $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ ..... (i)

Conversely,

Let $y \in (A \cup B) \cap (A \cap C)$

$\rightarrow y \in (A \cup B)$ and $y \in (A \cap C)$

$\rightarrow$ ($y \in A$ or $y \in B$) and ($y \in A$ or $y \in C$)

$\rightarrow y \in A$ or ($y \in B$ and $y \in C$)

$\rightarrow y \in A$ or $y \in (B \cap C)$

$\rightarrow y \in A \cup (B \cap C)$

As $y \in (A \cup B) \cap (A \cap C) \subseteq A \cup (B \cap C)$ ..... (ii)

from (i) and (ii) we conclude that $A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$

vi) Distributivity of Intersection over Union

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof:-

Let $x \in A \cap (B \cup C)$

$\rightarrow x \in A$ and $x \in (B \cup C)$

$\rightarrow x \in A$ and ($x \in B$ or $x \in C$)

$\rightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

$\rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

$\rightarrow x \in (A \cap B) \cup (A \cap C)$

As $x \in A \cap (B \cup C) \rightarrow x \in (A \cap B) \cup (A \cap C)$

so $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ ..... (i)

Conversely,

Let $y \in (A \cap C) \cup (B \cap C)$

$\rightarrow (y \in A \text{ and } y \in C) \text{ or } (y \in B \text{ and } y \in C)$

$\rightarrow (y \in A \text{ or } y \in B) \text{ and } y \in C$

$\rightarrow y \in (A \cup B) \text{ and } y \in C$

$\rightarrow y \in (A \cup B) \cap C$

As $y \in (A \cap C) \cup (B \cap C) \rightarrow y \in (A \cup B) \cap C$

so $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$ ..... (ii)

from (i) and (ii) we conclude

that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

De Morgan's Laws

vii) $(A \cup B)' = A' \cap B'$

viii) $(A \cap B)' = A' \cup B'$

Proof:- vii) $(A \cup B)' = A' \cap B'$

Let $x \in (A \cup B)'$

$\rightarrow x \notin (A \cup B)$

$\rightarrow x \notin A$ and $x \notin B$

$\rightarrow x \in A'$ and $x \in B'$

$\rightarrow x \in (A' \cap B')$

As $x \in (A \cup B)' \rightarrow x \in (A' \cap B')$

so $(A \cup B)' \subseteq (A' \cap B')$ ..... (i)

Conversely

Let $y \in A' \cap B'$

$\rightarrow y \in A'$ and $y \in B'$

$\rightarrow y \notin A$ and $y \notin B$

$\rightarrow y \notin (A \cup B)$

$\rightarrow y \in (A \cup B)'$

As $y \in A' \cap B' \rightarrow y \in (A \cup B)'$

so $A' \cap B' \subseteq (A \cup B)'$ ...... (ii)

from (i) and (ii) we conclude

that $(A \cup B)' = A' \cap B'$

viii) $(A \cap B)' = A' \cup B'$

Let $x \in (A \cap B)'$

$\rightarrow x \notin (A \cap B)$

$x \notin A$ and $x \notin B$

$\rightarrow x \in A'$ and $x \in B'$

$\rightarrow x \in (A' \cup B')$

As $x \in (A \cap B)' \rightarrow x \in (A' \cup B')$

so $(A \cap B)' \subseteq A' \cup B'$ ...... (i)

Conversely,

Let $y \in A' \cup B'$

$\rightarrow y \in A'$ or $y \in B'$

$\rightarrow y \notin A$ or $y \notin B$

$y \notin (A \cap B)$

$y \in (A \cap B)'$

As $y \in (A' \cup B') \rightarrow y \in (A \cap B)'$

$\rightarrow A' \cup B' \subseteq (A \cap B)'$ ..... (ii)

from (i) and (ii) we conclude

that $(A \cap B)' = A' \cup B'$

Example 1. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$ and $C = \{3, 4, 5, 6, 7, 8\}$

then verify that **i) $A \cup B = B \cup A$**

$A \cup B = \{1, 2, 3\} \cup \{2, 3, 4, 5\}$

$= \{1, 2, 3, 4, 5\}$

$B \cup A = \{2, 3, 4, 5\} \cup \{1, 2, 3\}$

$= \{1, 2, 3, 4, 5\}$

Hence $A \cup B = B \cup A$

ii) verify that $A \cup (B \cup C) = (A \cup B) \cup C$

$B \cup C = \{2, 3, 4, 5\} \cup \{3, 4, 5, 6, 7, 8\}$

$= \{2, 3, 4, 5, 6, 7, 8\}$

$A \cup (B \cup C) = \{1, 2, 3\} \cup \{2, 3, 4, 5, 6, 7, 8\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8\} \longrightarrow$ (i)

$A \cup B = \{1, 2, 3\} \cup \{2, 3, 4, 5\}$

$= \{1, 2, 3, 4, 5\}$

$(A \cup B) \cup C = \{1, 2, 3, 4, 5\} \cup \{3, 4, 5, 6, 7, 8\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8\} \longrightarrow$ (ii)

$\therefore A\cup(B\cup C) = A\cup(B\cup C)$ from (i) and (ii)

iii) $A\cap B = B\cap A$

$A\cap B = \{1,2,3\}\cap\{2,3,4,5\} = \{2,3\}$

$B\cap A = \{2,3,4,5\}\cap\{1,2,3\} = \{2,3\}$

$\rightarrow A\cap B = B\cap A$

iv) $A\cap(B\cap C) = (A\cap B)\cap C$

$B\cap C = \{2,3,4,5\}\cap\{3,4,5,6,7,8\}$
$= \{3,4,5\}$

$A\cap(B\cap C) = \{1,2,3\}\cap\{3,4,5\}$
$= \{3\} \longrightarrow (i)$

$A\cap B = \{1,2,3\}\cap\{2,3,4,5\}$
$= \{2,3\}$

$(A\cap B)\cap C = \{2,3\}\cap\{3,4,5,6,7,8\}$
$= \{3\} \longrightarrow (ii)$

$\rightarrow A\cap(B\cap C) = (A\cap B)\cap C$ by (i) and (ii)

v) $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

$B\cap C = \{2,3,4,5\}\cap\{3,4,5,6,7,8\}$
$= \{3,4,5\}$

$A\cup(B\cap C) = \{1,2,3\}\cup\{3,4,5\}$
$= \{1,2,3,4,5\} \longrightarrow (i)$

$A\cup B = \{1,2,3\}\cup\{2,3,4,5\} = \{1,2,3,4,5\}$

$A\cup C = \{1,2,3\}\cup\{3,4,5,6,7,8\}$
$= \{1,2,3,4,5,6,7,8\}$

$(A\cup B)\cap(A\cup C) = \{1,2,3,4,5\}\cup\{1,2,3,4,5,6,7,8\}$
$= \{1,2,3,4,5\} \longrightarrow (ii)$

$\rightarrow A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$ By (i) and (ii)

vi) $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

$B\cup C = \{2,3,4,5\}\cup\{3,4,5,6,7,8\}$
$= \{2,3,4,5,6,7,8\}$

$A\cap(B\cup C) = \{1,2,3\}\cap\{2,3,4,5,6,7,8\}$
$= \{2,3\} \longrightarrow (i)$

$A\cap B = \{1,2,3\}\cap\{2,3,4,5\} = \{3\}$

$A\cap C = \{1,2,3\}\cap\{3,4,5,6,7,8\} = \{2,3\}$

$(A\cap B)\cup(A\cap C) = \{3\}\cup\{2,3\}$
$= \{2,3\} \longrightarrow (ii)$

$\rightarrow A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$ By (i) and (ii)

vii) $(A\cup B)' = A'\cap B'$

Let $U = \{1,2,3,4,\dots,10\}$

$A\cup B = \{1,2,3\}\cup\{2,3,4,5\}$
$= \{1,2,3,4,5\}$

$(A\cup B)' = U-(A\cup B)$
$= \{1,2,3,4,\dots,10\}-\{1,2,3,4,5\}$
$= \{6,7,8,9,10\} \longrightarrow (i)$

$A' = U-A = \{1,2,3,\dots,10\}-\{1,2,3\}$
$= \{4,5,6,\dots,10\}$

$B' = U-B = \{1,2,3,\dots,10\}-\{2,3,4,5\}$
$= \{1,6,7,8,9,10\}$

$A'\cap B' = \{4,5,6,\dots,10\}\cap\{1,6,7,8,9,10\}$
$= \{6,7,8,9,10\} \longrightarrow (ii)$

$\rightarrow (A\cup B)' = A'\cap B'$ from (i) and (ii)

viii) $(A\cap B)' = A'\cup B'$

Let $U = \{1,2,3,4,\dots,10\}$

$A\cap B = \{1,2,3\}\cap\{2,3,4,5\}$
$= \{2,3\}$

$(A\cap B)' = U-(A\cap B)$
$= \{1,2,3,\dots,10\}-\{2,3\}$
$= \{1,4,5,6,7,8,9,10\} \longrightarrow (i)$

$A' = U-A = \{1,2,3,\dots,10\}-\{1,2,3\}$
$= \{4,5,6,\dots,10\}$

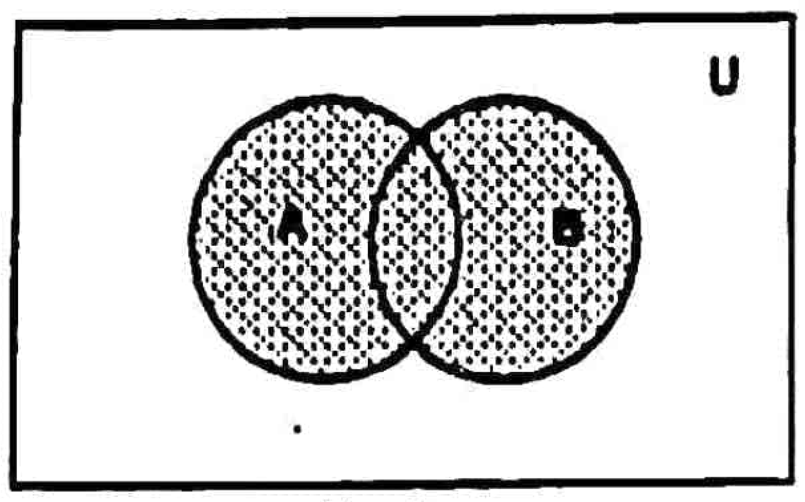
$B' = U-B = \{1,2,3,\dots,10\}-\{2,3,4,5\}$
$= \{1,6,7,8,9,10\}$

$A'\cup B' = \{4,5,6,\dots,10\}\cup\{1,6,7,8,\dots,10\}$
$= \{1,4,5,6,7,8,9,10\} \longrightarrow (ii)$

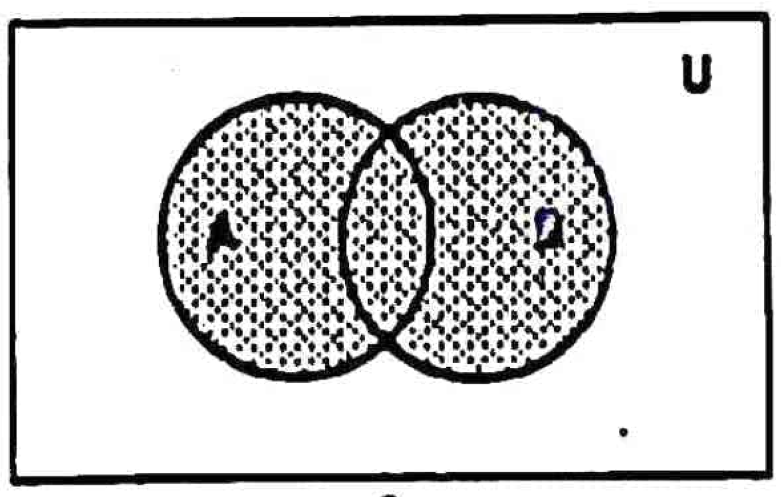
$\rightarrow (A\cap B)' = A'\cup B'$ from (i) and (ii)

Verification of the properties with the help of Venn Diagram

i) $A \cup B = B \cup A$

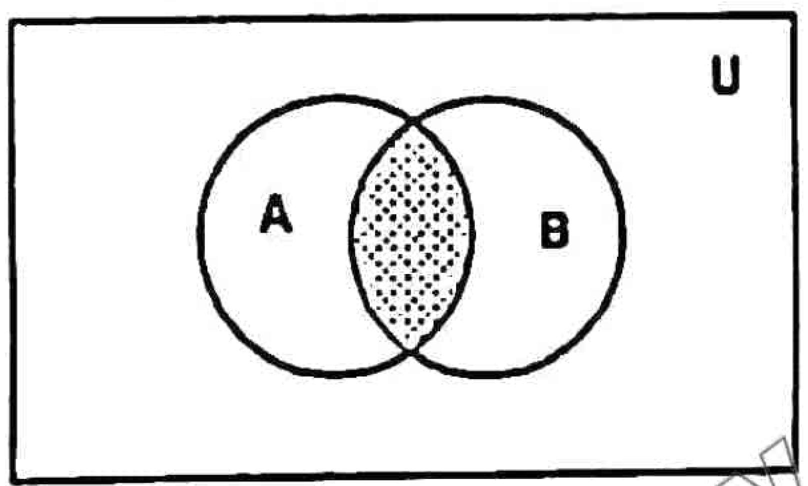


$A \cup B$

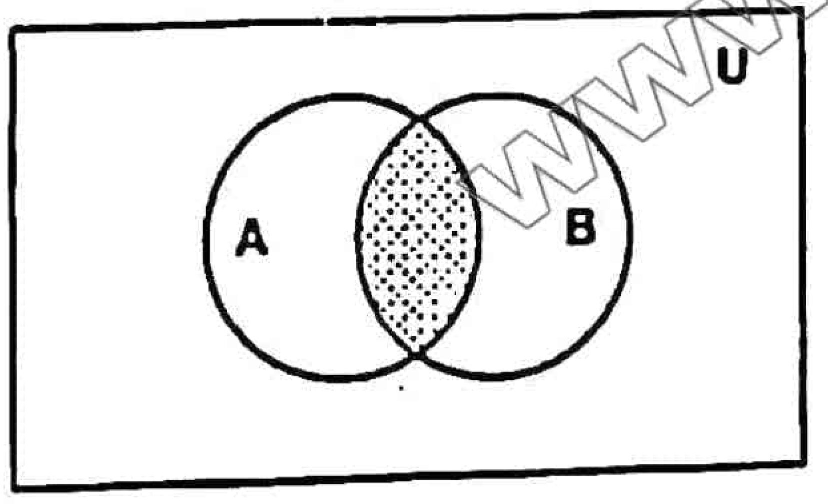


$B \cup A$

ii) $A \cap B = B \cap A$



$A \cap B$



$B \cap A$

iii) $A \cup (B \cup C) = (A \cup B) \cup C$

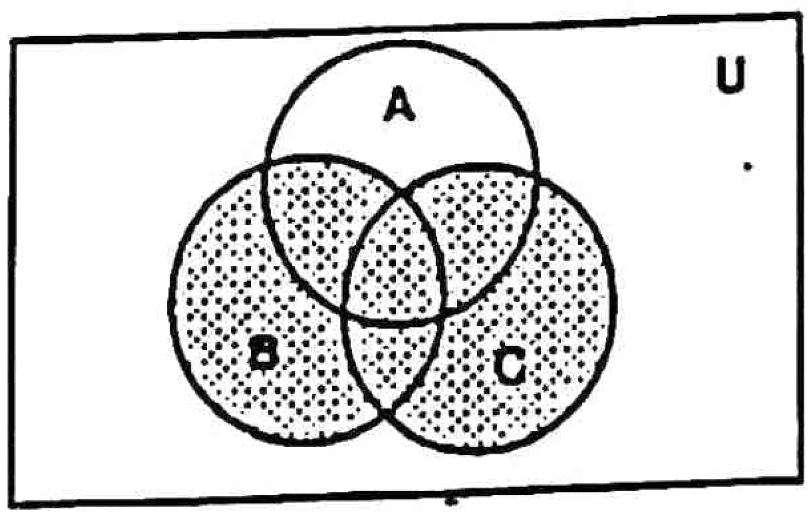


fig (i)
$B \cup C$

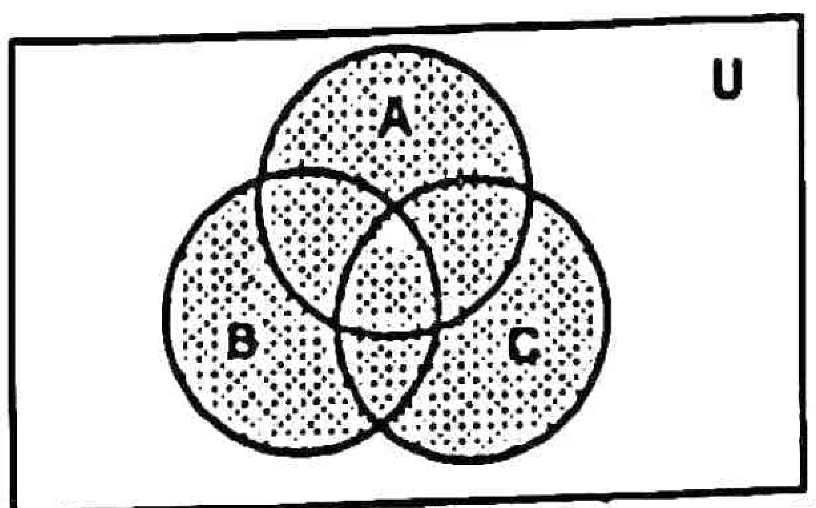


fig (ii)
$A \cup (B \cup C)$

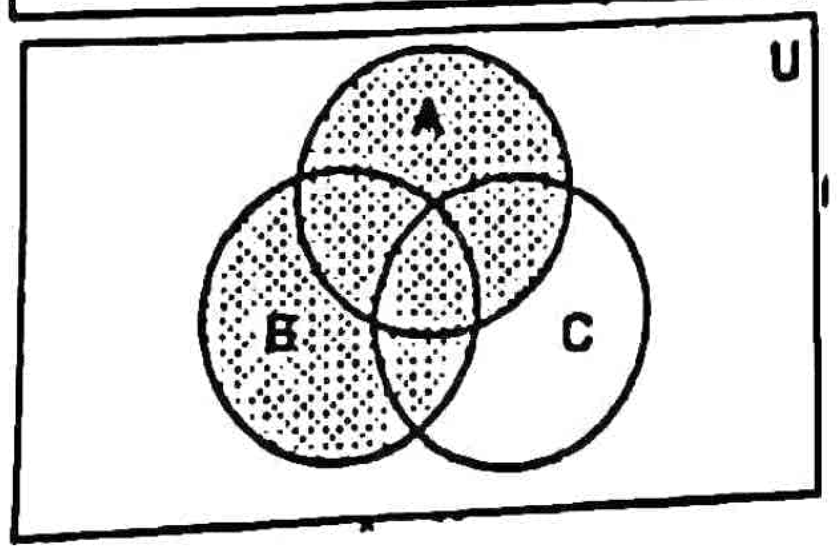


fig (iii)
$A \cup B$

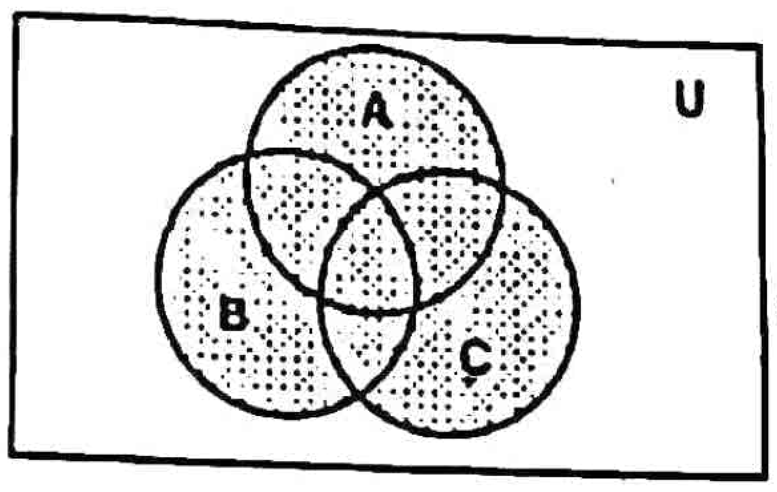


fig (iv)
$(A \cup B) \cup C$

from fig (ii) and fig (iv) we conclude that
$A \cup (B \cup C) = (A \cup B) \cup C$.

IV) $A \cap (B \cap C) = (A \cap B) \cap C$

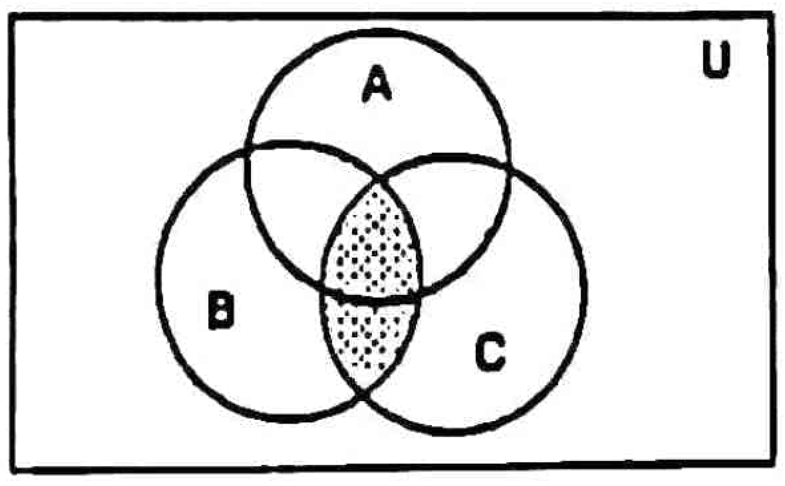


fig (i)
$B \cap C$

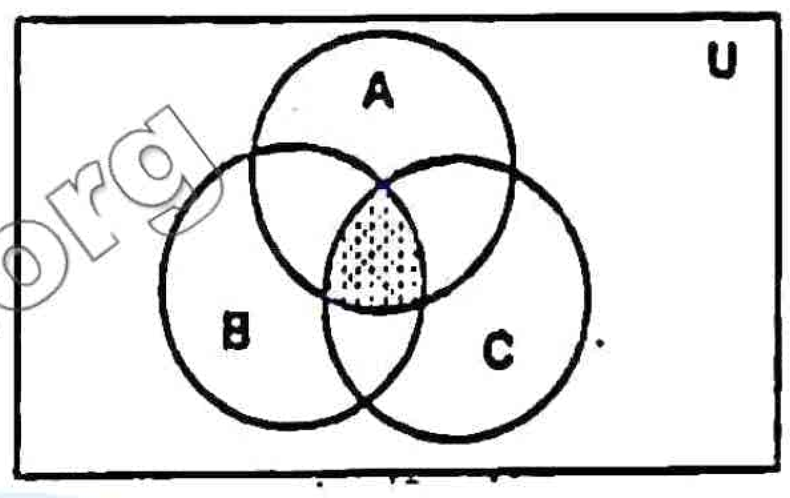


fig (ii)
$A \cap (B \cap C)$

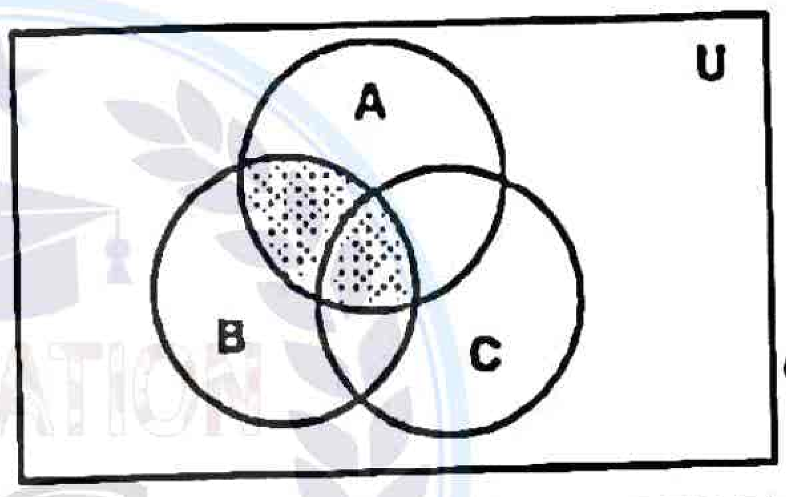


fig (iii)
$A \cap B$

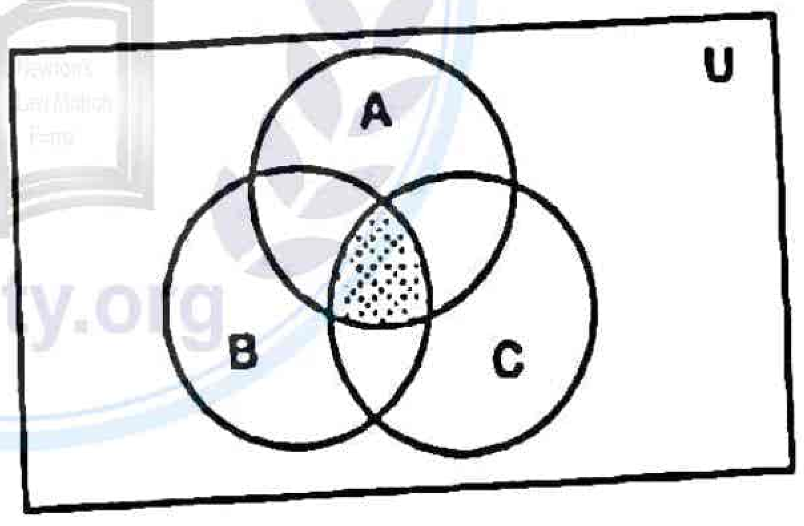


fig (iv)
$(A \cap B) \cap C$

from fig (ii) and fig (iv) it is verified that $A \cap (B \cap C) = (A \cap B) \cap C$

v) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

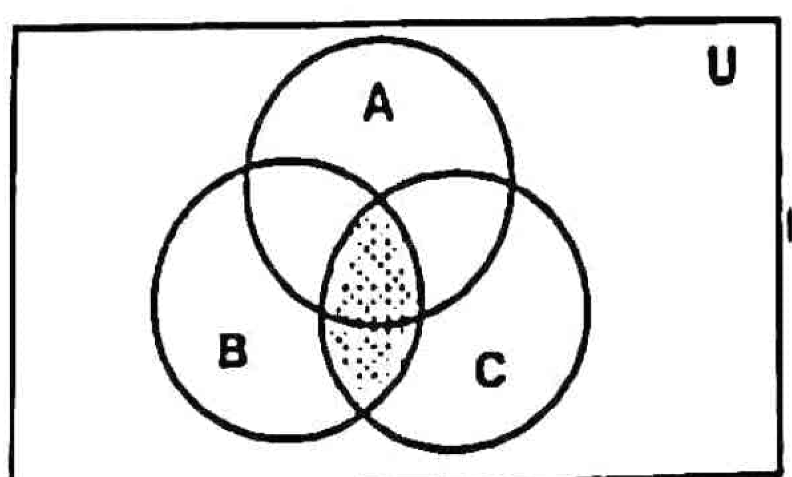


fig (i)
$B \cap C$

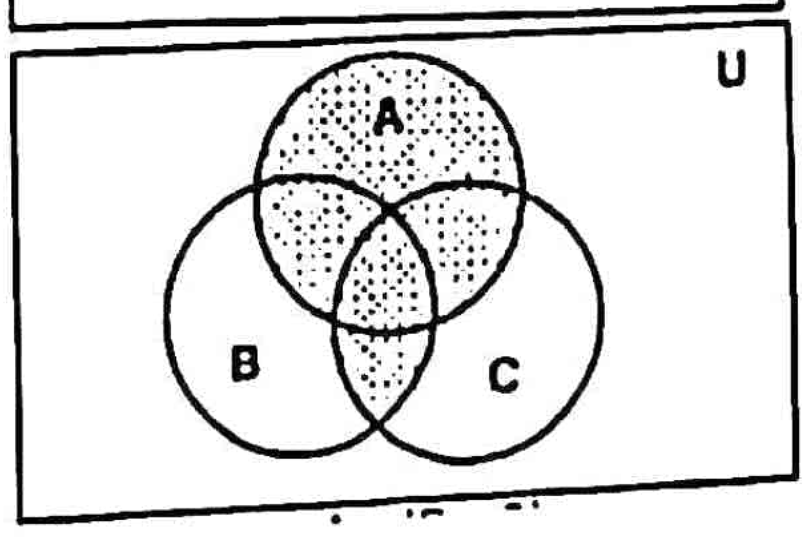


fig (ii)
$A \cup (B \cap C)$

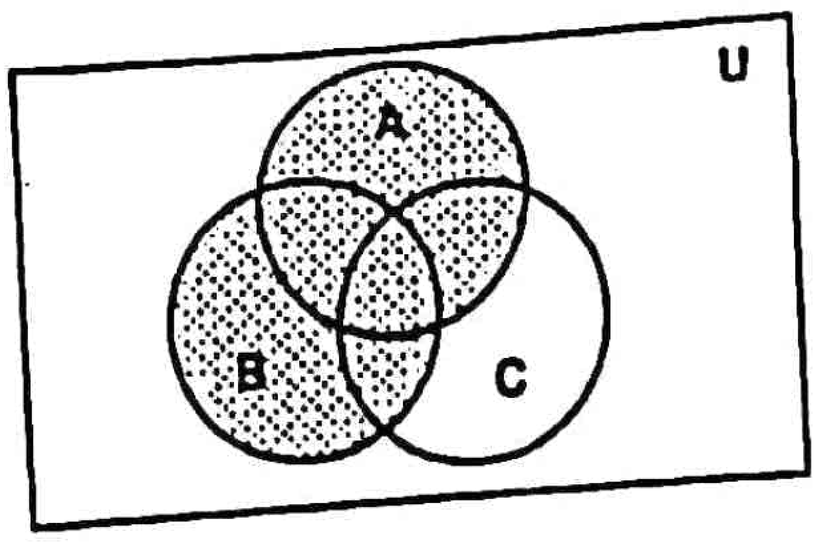


fig (iii)
$A \cup B$

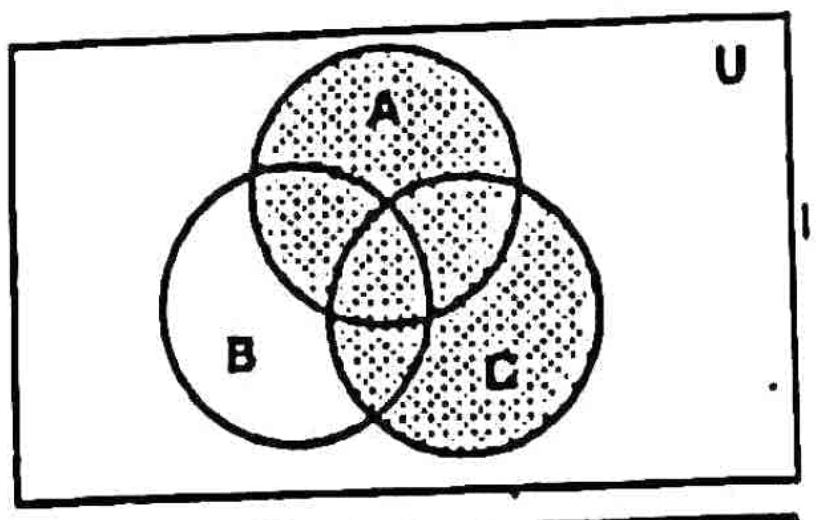


fig (iv)
$A \cup C$

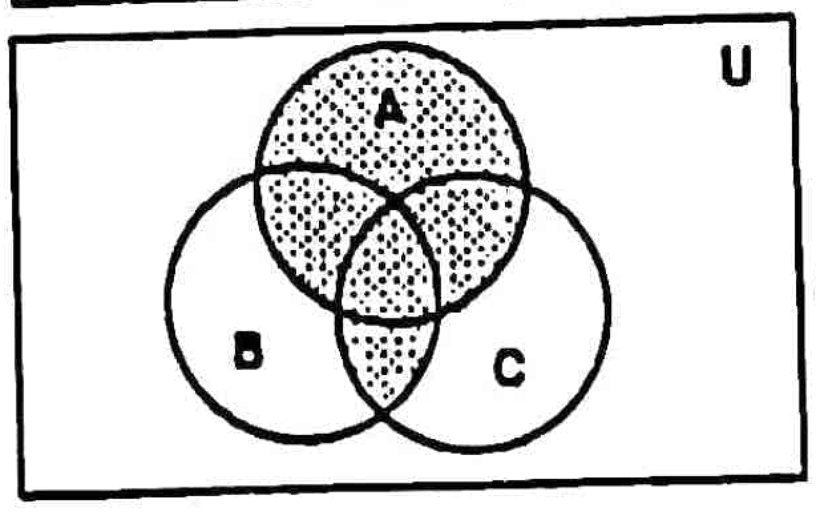


fig (v)
$(A \cup B) \cap (A \cup C)$

from fig (ii) and fig (v) it is verified that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

vi) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

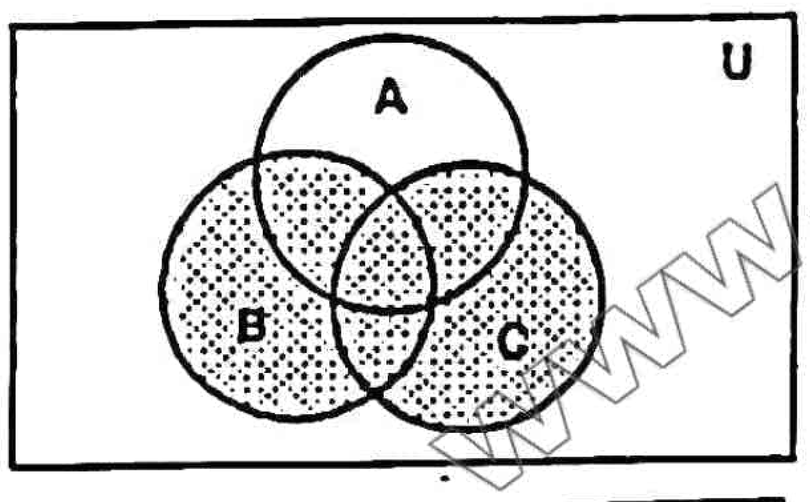


fig (i)
$B \cup C$

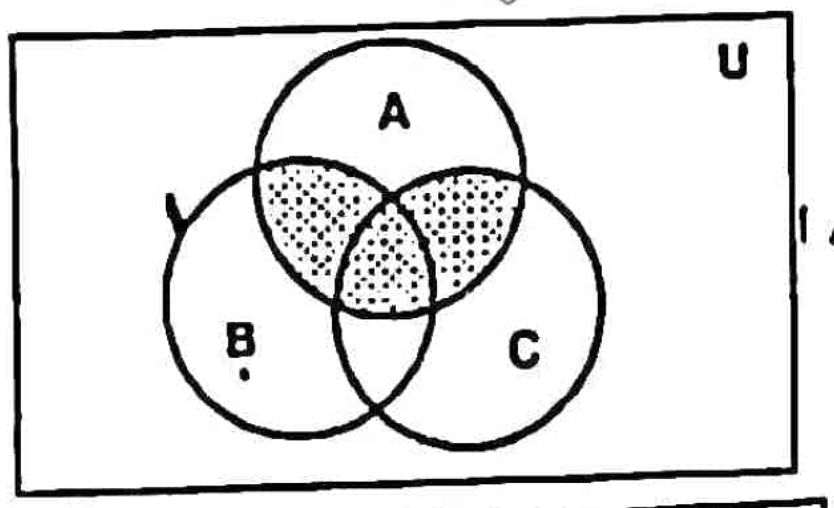


fig (ii)
$A \cap (B \cup C)$

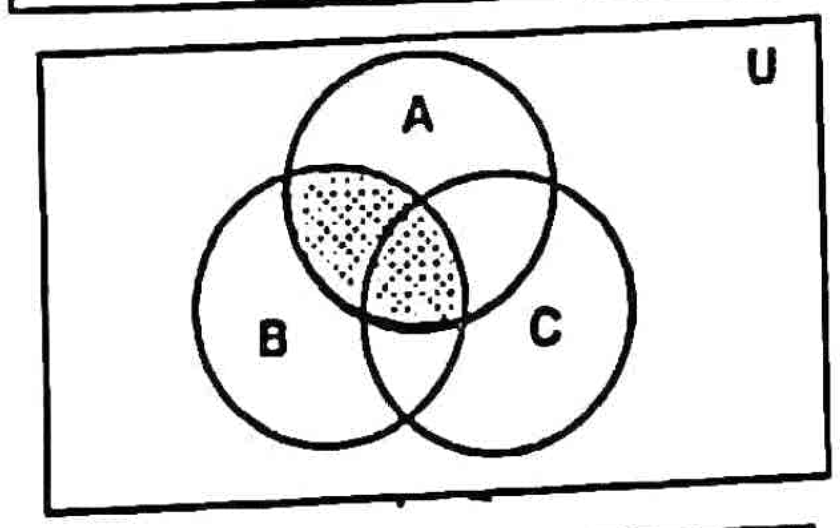


fig (iii)
$A \cap B$

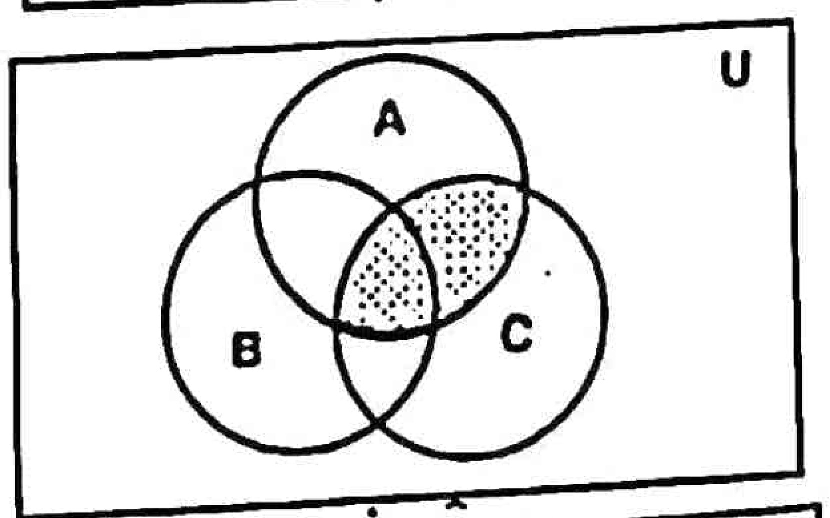


fig (iv)
$A \cap C$

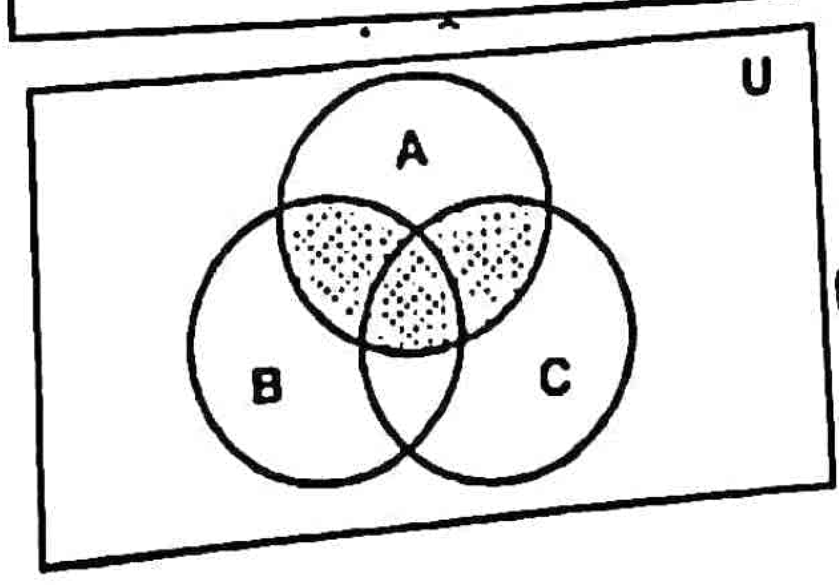


fig (v)
$(A \cap B) \cup (A \cap C)$

from fig (ii) and fig (v) it is verified that
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

vii) $(A \cup B)' = A' \cap B'$ (DeMorgan's Law)

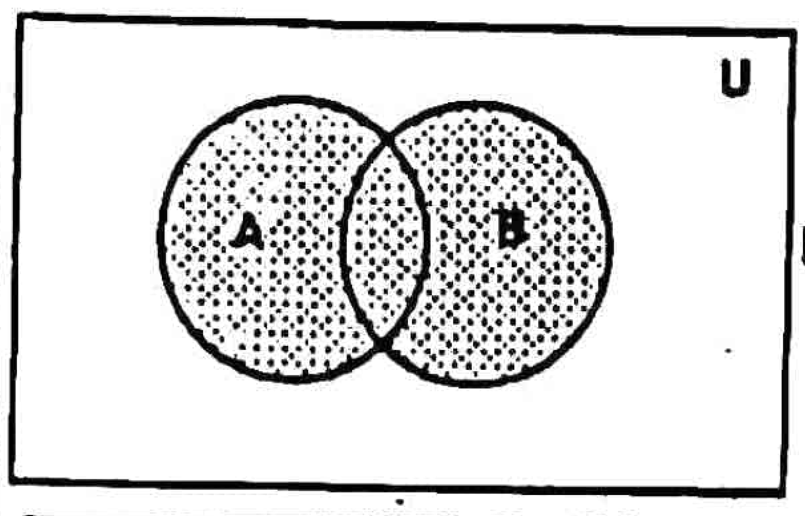


fig (i)
$A \cup B$

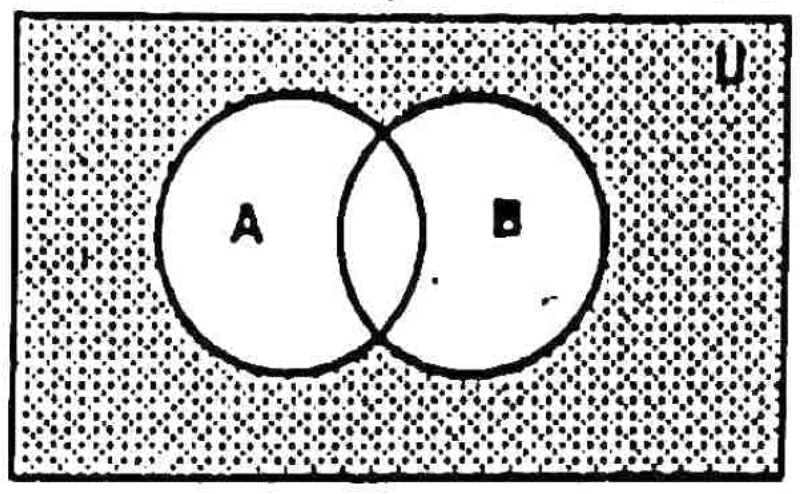


fig (ii)
$(A \cup B)'$

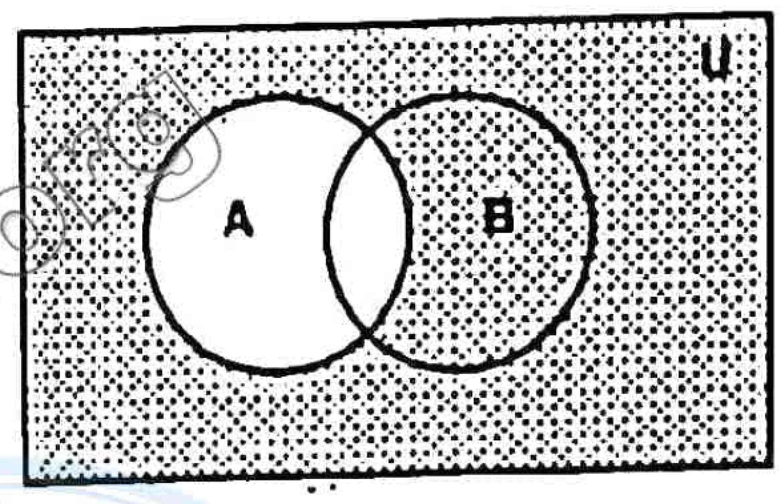


fig (iii)
A'

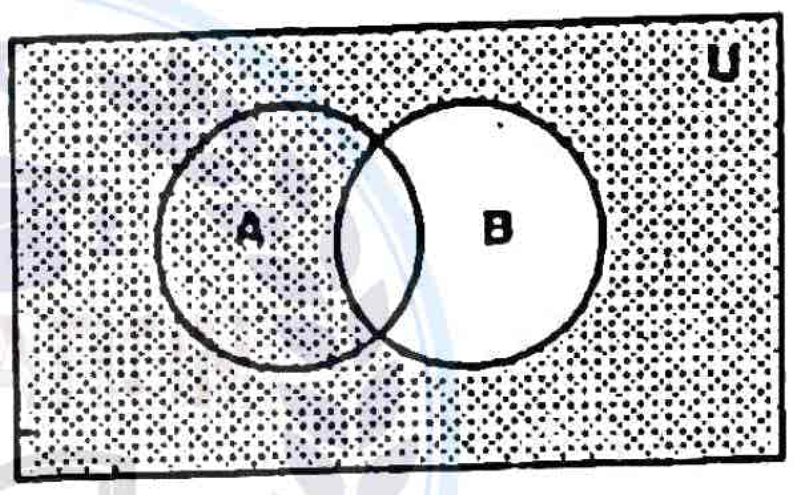


fig (iv)
B'

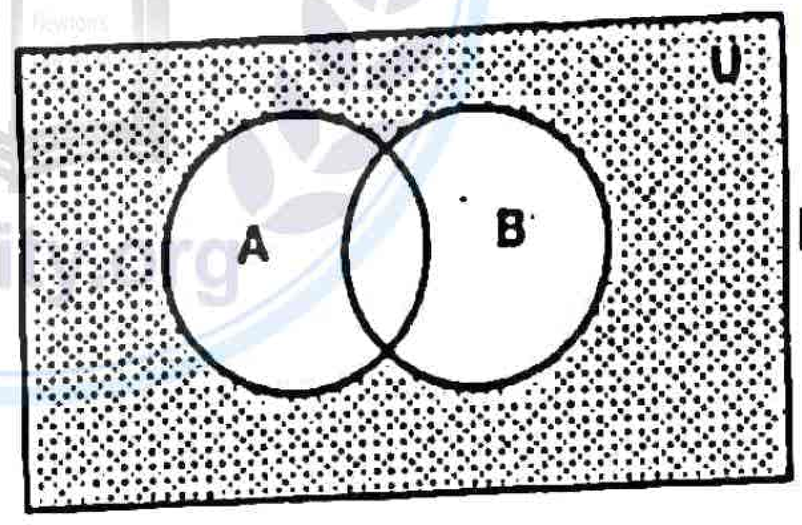


fig (v)
$A' \cap B'$

from fig (ii) and fig (v) it is verified that $(A \cup B)' = A' \cap B'$

viii) $(A \cap B)' = A' \cup B'$ (DeMorgan's Law)

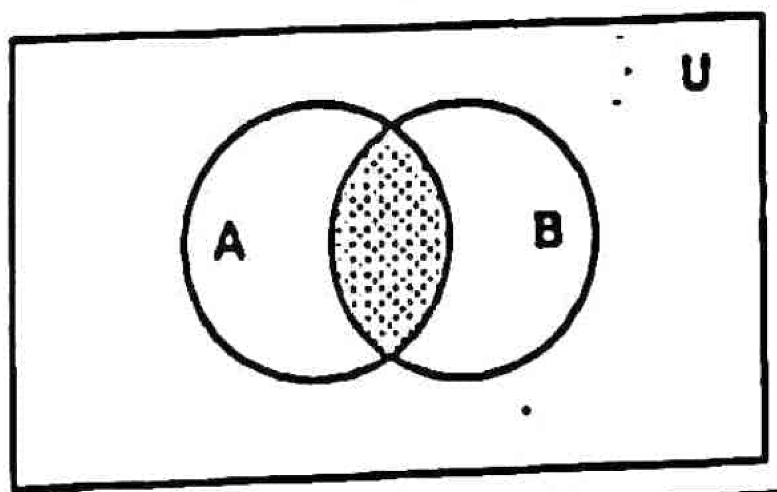


fig (i)
$A \cap B$

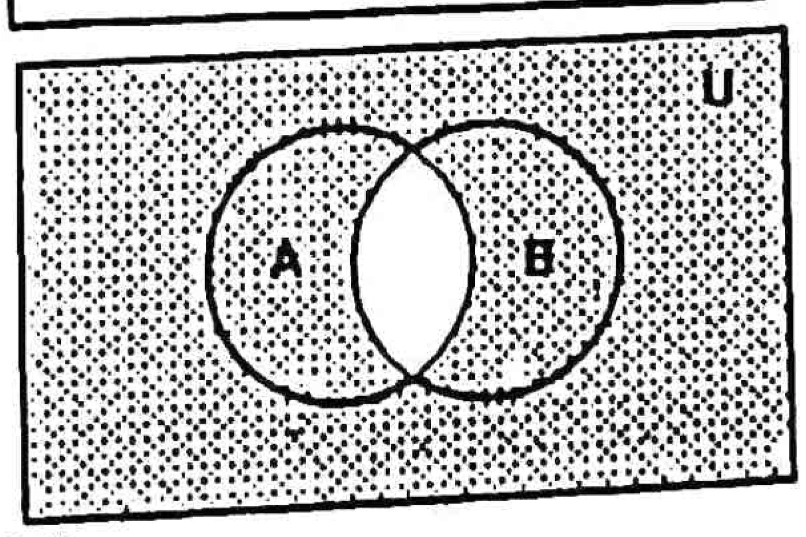


fig (ii)
$(A \cap B)'$

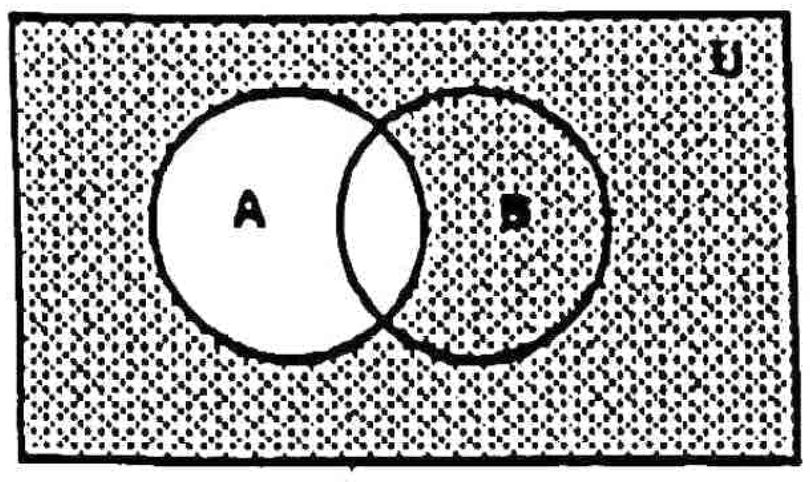


fig (iii)
A'

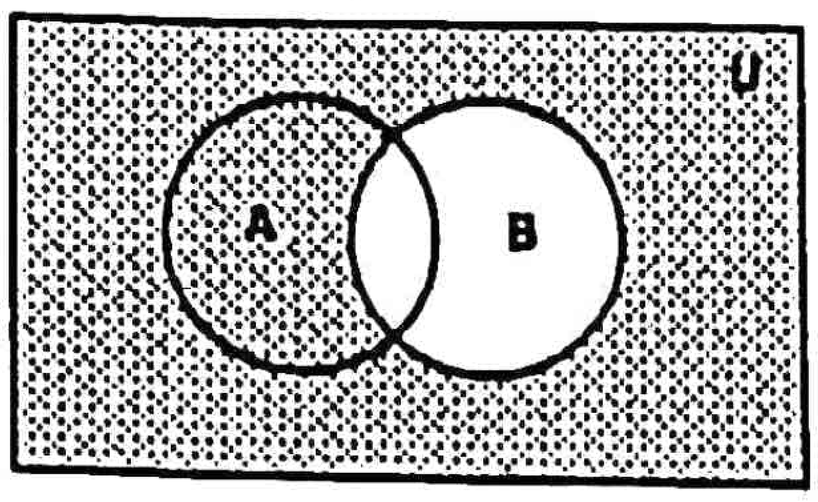


fig (iv)
B'

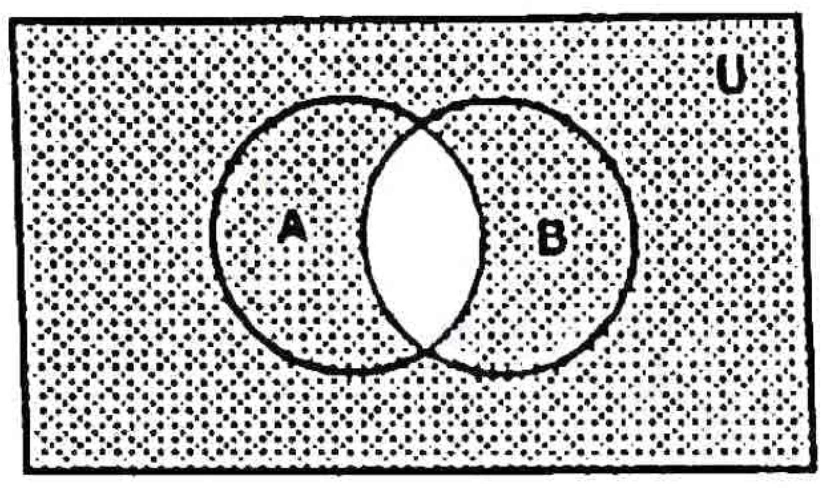


fig (v)
$A' \cup B'$

from fig (ii) and fig (v) it is verified that $(A\cap B)' = A'\cup B'$

Exercise 2.3

Q1. Verify the commutative properties of union and intersection for following pairs of sets:-

i) $A = \{1,2,3,4,5\}$, $B = \{4,6,8,10\}$

Solution:- a) $A\cup B = B\cup A$

$A\cup B = \{1,2,3,4,5\}\cup\{4,6,8,10\}$
$= \{1,2,3,4,5,6,8,10\}$

$B\cup A = \{4,6,8,10\}\cup\{1,2,3,4,5\}$
$= \{1,2,3,4,5,6,8,10\}$

$\therefore A\cup B = B\cup A$

b) $A\cap B = B\cap A$

$A\cap B = \{1,2,3,4,5\}\cap\{4,6,8,10\} = \{4\}$

$B\cap A = \{4,6,8,10\}\cap\{1,2,3,4,5\} = \{4\}$

$\therefore A\cap B = B\cap A$

ii) N, Z

Solution:-
N = set of natural numbers
Z = set of integers

a) $N\cup Z = Z\cup N$

Since $N\cup Z = Z$ & $Z\cup N = Z$ $(N\subset Z)$
so $N\cup Z = Z\cup N$

b) $N\cap Z = Z\cap N$

Since $N\cap Z = N$ & $Z\cap N = N$ $(N\subset Z)$
So $N\cap Z = Z\cap N$

iii) $A = \{x \mid x\in\mathbb{R} \wedge x\geq 0\}$, $B = \mathbb{R}$

Solution:- A(Set of non-negative real numbers)
B(Set of all real numbers)

a) $A\cup B = B\cup A$

$A\cup B = \{x\mid x\in\mathbb{R}\wedge x\geq 0\}\cup\mathbb{R} = \mathbb{R}$ $(A\subset B)$

$B\cup A = \mathbb{R}\cup\{x\mid x\in\mathbb{R}\wedge x\geq 0\} = \mathbb{R}$

so $A\cup B = B\cup A$

b) $A\cap B = B\cap A$

$A\cap B = \{x\mid x\in\mathbb{R}\wedge x\geq 0\}\cap\mathbb{R}$
$= \{x\mid x\in\mathbb{R}\wedge x\geq 0\}$

$B\cap A = \mathbb{R}\cap\{x\mid x\in\mathbb{R}\wedge x\geq 0\}$ $(A\subset B)$
$= \{x\mid x\in\mathbb{R}\wedge x\geq 0\}$

so $A\cap B = B\cap A$

Q2. Verify the properties for the sets A, B and C given below:-

Solution:-

a) $A = \{1,2,3,4\}$, $B = \{3,4,5,6,7,8\}$
$C = \{5,6,7,9,10\}$

1) Associativity of union
$A\cup(B\cup C) = (A\cup B)\cup C$

L.H.S $= A\cup(B\cup C)$
$= \{1,2,3,4\}\cup[\{3,4,5,6,7,8\}\cup\{5,6,7,9,10\}]$
$= \{1,2,3,4\}\cup\{3,4,5,6,7,8,9,10\}$
$= \{1,2,3,4,5,6,7,8,9,10\} \longrightarrow (i)$

R.H.S $= (A\cup B)\cup C$
$= [\{1,2,3,4\}\cup\{3,4,5,6,7,8\}]\cup\{5,6,7,9,10\}$
$= \{1,2,3,4,5,6,7,8\}\cup\{5,6,7,9,10\}$
$= \{1,2,3,4,5,6,7,8,9,10\} \longrightarrow (ii)$

By (i) and (ii)
$A\cup(B\cup C) = (A\cup B)\cup C$

ii) Associativity of intersection
$A\cap(B\cap C) = (A\cap B)\cap C$

L.H.S $= A\cap(B\cap C)$
$= \{1,2,3,4\}\cap[\{3,4,5,6,7,8\}\cap\{5,6,7,9,10\}]$
$= \{1,2,3,4\}\cap\{5,6,7\} = \{\ \} \longrightarrow (i)$

$R.H.S = (A\cap B)\cap C$

$= [\{1,2,3,4\}\cap\{3,4,5,6,7,8\}]\cap\{5,6,7,9,10\}$

$= \{3,4\}\cap\{5,6,7,9,10\} = \{\ \} \rightarrow (ii)$

By (i) and (ii)

$A\cap(B\cap C) = (A\cap B)\cap C$

iii) Distributivity of Union over intersection

$A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

$L.H.S = A\cup(B\cap C)$

$= \{1,2,3,4\}\cup[\{3,4,5,6,7,8\}\cap\{5,6,7,9,10\}]$

$= \{1,2,3,4\}\cup\{5,6,7\}$

$= \{1,2,3,4,5,6,7\} \longrightarrow (i)$

$R.H.S = (A\cup B)\cap(A\cup C)$

$= [\{1,2,3,4\}\cup\{3,4,5,6,7,8\}]\cap[\{1,2,3,4\}\cup\{5,6,7,9,10\}]$

$= \{1,2,3,4,5,6,7,8\}\cap\{1,2,3,4,\ldots,7,9,10\}$

$= \{1,2,3,4,5,6,7\} \longrightarrow (ii)$

By (i) and (ii)

$A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

iv) Distributivity of intersection over union

$A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

$L.H.S = A\cap(B\cup C)$

$= \{1,2,3,4\}\cap[\{3,4,5,6,7,8\}\cup\{5,6,7,9,10\}]$

$= \{1,2,3,4\}\cap\{3,4,5,6,7,8,9,10\}$

$= \{3,4\} \longrightarrow (i)$

$R.H.S = (A\cap B)\cup(A\cap C)$

$= [\{1,2,3,4\}\cap\{3,4,5,6,7,8\}]\cup[\{1,2,3,4\}\cap\{5,6,7,9,10\}]$

$= \{3,4\}\cup\{\ \} = \{3,4\} \longrightarrow (ii)$

By (i) and (ii) $L.H.S = R.H.S$

b) $A = \phi$, $B = \{0\}$, $C = \{0,1,2\}$

i) Associativity of Union

$A\cup(B\cup C) = (A\cup B)\cup C$

$L.H.S = A\cup(B\cup C)$

$= \phi\cup[\{0\}\cup\{0,1,2\}]$

$= \phi\cup\{0,1,2\} = \{0,1,2\} \longrightarrow (i)$

$R.H.S = (A\cup B)\cup C$

$= [\phi\cup\{0\}]\cup\{0,1,2\} = \{0\}\cup\{0,1,2\}$

$= \{0,1,2\} \longrightarrow (ii)$

from (i) and (ii) $A\cup(B\cup C) = (A\cup B)\cup C$

ii) Associativity of Intersection

$A\cap(B\cap C) = (A\cap B)\cap C$

$L.H.S = A\cap(B\cap C)$

$= \phi\cap[\{0\}\cap\{0,1,2\}]$

$= \phi\cap\{0\} = \phi \longrightarrow (i)$

$R.H.S = A\cap(B\cap C)$

$= \phi\cap[\{0\}\cap\{0,1,2\}]$

$= \phi\cap\{0\} = \phi \longrightarrow (ii)$

By (i) and (ii) $A\cap(B\cap C) = (A\cap B)\cap C$

iii) Distributivity of Union over intersection

$A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

$L.H.S = A\cup(B\cap C)$

$= \phi\cup[\{0\}\cap\{0,1,2\}] = \phi\cup\{0\}$

$= \{0\} \longrightarrow (i)$

$R.H.S = (A\cup B)\cap(A\cup C)$

$= [(\phi\cup\{0\})]\cap[(\phi\cup\{0,1,2\})]$

$= \{0\}\cap\{0,1,2\} = \{0\} \longrightarrow (ii)$

By (i) and (ii) $L.H.S = R.H.S$

iv) Distributivity of $\cap$ over $\cup$

$A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

$L.H.S = A\cap(B\cup C)$

$= \phi\cap[\{0\}\cup\{0,1,2\}]$

$= \phi\cap\{0,1,2\} = \phi \longrightarrow (i)$

$R.H.S = [(\phi\cap\{0\})]\cup[(\phi\cap\{0,1,2\})]$

$= \phi\cup\phi = \phi \longrightarrow (ii)$

By (i) and (ii)

$L.H.S = R.H.S$

c) N, Z, Q

$N = \{1,2,3,4,\ldots\}$, $Z = \{0,\pm1,\pm2,\pm3,\ldots\}$

Q = set of rational nos.

i) Associativity of Union

$N\cup(Z\cup Q) = (N\cup Z)\cup Q$

$N\cup Q = Z\cup Q \quad (\because N\subset Z\subset Q)$

$\rightarrow Q = Q$

L.H.S = R.H.S proved

ii) Associativity of Intersection

$N\cap(Z\cap Q) = (N\cap Z)\cap Q$

$N\cap Z = N\cap Q \quad (\because N\subset Z\subset Q)$

$N = N$

L.H.S = R.H.S proved

iii) Distributivity of $\cup$ over $\cap$

$N\cup(Z\cap Q) = (N\cup Z)\cap(N\cup Q)$

$N\cup Z = Z\cap Q \quad (\because N\subset Z\subset Q)$

$Z = Z$ proved

L.H.S = R.H.S

iv) Distributivity of $\cap$ over $\cup$

$N\cap(Z\cup Q) = (N\cap Z)\cup(N\cap Q)$

$N\cap Q = N\cup N \quad (\because N\subset Z\subset Q)$

$N = N$

L.H.S = R.H.S proved

Q3. Verify De Morgan's Laws for the following sets:

$U = \{1,2,3,\ldots,20\}$, $A = \{2,4,6,\ldots,20\}$ and $B = \{1,3,5,\ldots,19\}$

i) $(A\cap B)' = A'\cup B'$

L.H.S = $(A\cap B)'$ where

$A\cap B = \{2,4,6,\ldots,20\}\cap\{1,3,5,\ldots,20\}$

$A\cap B = \{\ \} = \phi$

$(A\cap B)' = U - A\cap B$

$(A\cap B)' = U - \phi = U \longrightarrow (i)$

R.H.S = $A'\cup B'$ where

$A' = U - A = \{1,2,3,\ldots,20\} - \{2,4,6,\ldots,20\}$

$A' = \{1,3,5,\ldots,19\}$

$B' = U - B = \{1,2,3,\ldots,20\} - \{1,3,5,\ldots,19\}$

$B' = \{2,4,6,\ldots,20\}$

$A'\cup B' = \{1,3,5,\ldots,19\}\cup\{2,4,6,\ldots,20\}$

$= \{1,2,3,4,\ldots,19,20\}$

$A'\cup B' = U \longrightarrow (ii)$

By (i) and (ii) $(A\cap B)' = A'\cup B'$

ii) $(A\cup B)' = A'\cap B'$

L.H.S = $(A\cup B)'$ where

$A\cup B = \{2,4,6,\ldots,20\}\cup\{1,3,5,\ldots,19\}$

$= \{1,2,3,4,5,\ldots,19,20\} = U$

$(A\cup B)' = U - A\cup B = U - U = \phi \longrightarrow (i)$

R.H.S = $A'\cap B'$

$A' = U - A = \{1,2,3,4,\ldots,20\} - \{2,4,6,\ldots,20\}$

$A' = \{1,3,5,\ldots,19\}$

$B' = U - B = \{1,2,3,\ldots,20\} - \{1,3,5,\ldots,19\}$

$B' = \{2,4,6,\ldots,20\}$

$A'\cap B' = \{1,2,3,\ldots,19\}\cap\{2,4,6,\ldots,20\}$

$= \phi \longrightarrow (ii)$

By (i) and (ii) $\rightarrow$ L.H.S = R.H.S

Q4. Let U = The set of the English alphabet

$A = \{x \mid x \text{ is a vowel}\}$,

$B = \{y \mid y \text{ is a consonant}\}$

Verify De Morgan's Laws for these sets.

i) $(A\cup B)' = A'\cap B'$

L.H.S = $(A\cup B)'$ where

$A\cup B = \{x \mid x \text{ is vowel}\}\cup\{y \mid y \text{ is a consonant}\}$

$A\cup B$ = Set of English alphabet = U

$(A\cup B)' = U - A\cup B = U - U = \phi \longrightarrow (i)$

$A' = U - A = U - \{x \mid x \text{ is vowel}\}$

$A' = \{y \mid y \text{ is a consonent}\}$

$B' = U - B = U - \{y \mid y \text{ is consonent}\}$

$B' = \{x/x \text{ is vowel}\}$

$A' \cap B' = \{y|y \text{ is consonent}\} \cap \{x/x \text{ is vowel}\}$

$= \{ \ \ \} = \phi \rightarrow (ii)$

from (i) and (ii)

$(A \cup B)' = A' \cap B'$

ii) $(A \cap B)' = A' \cup B'$

L.H.S $= (A \cap B)'$ where

$A \cap B = \{x/x \text{ is vowel}\} \cap \{y|y \text{ is consonent}\}$

$A \cap B = \{ \ \ \}$

$(A \cap B)' = U - A \cap B = U - \{ \ \ \} = U \rightarrow (i)$

$A' \cup B' = \{y|y \text{ is consonent}\} \cup \{x/x \text{ is vowel}\}$

$A' \cup B' =$ set of English alphabet $= U \rightarrow (ii)$

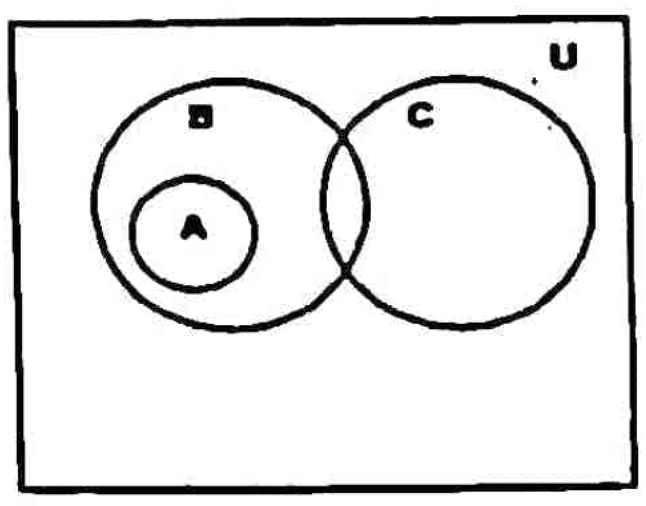
By (i) and (ii) $(A \cap B)' = A' \cup B'$

Q5. With the help of venn diagrams, verify the two distributive properties in the following cases w.r.t Union and Intersection.

i) $A \subseteq B$, $A \cap C = \phi$ and B and C are overlapping.

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

From given informations we have venn diagram as



$A \subseteq B$, $A \cap C = \phi$
B and C are overlapping.

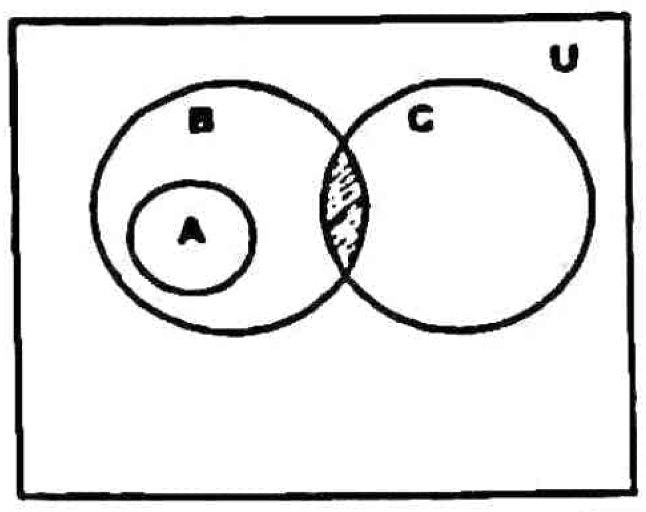


fig (i)
$B \cap C$

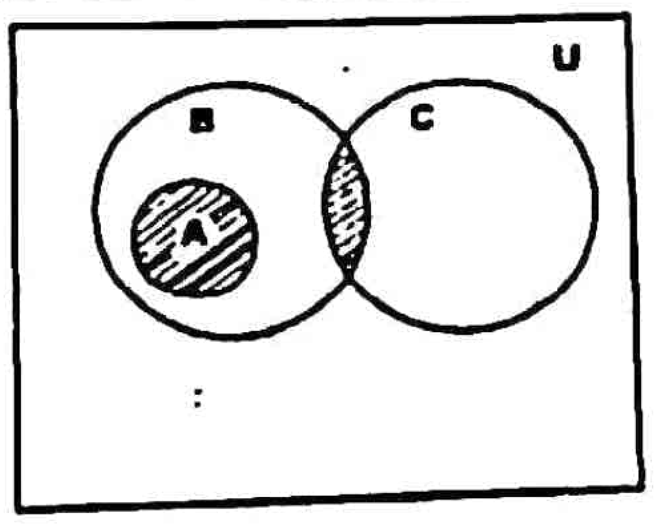


fig (ii)
$A \cup (B \cap C)$

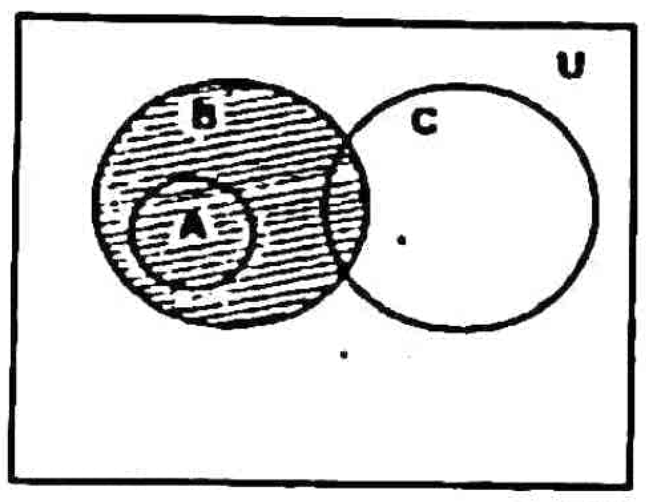


fig (iii)
$A \cup B$

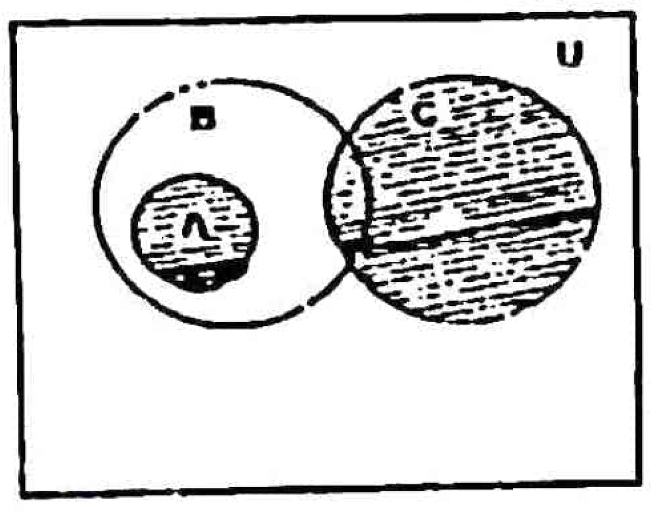


fig (iv)
$A \cup C$

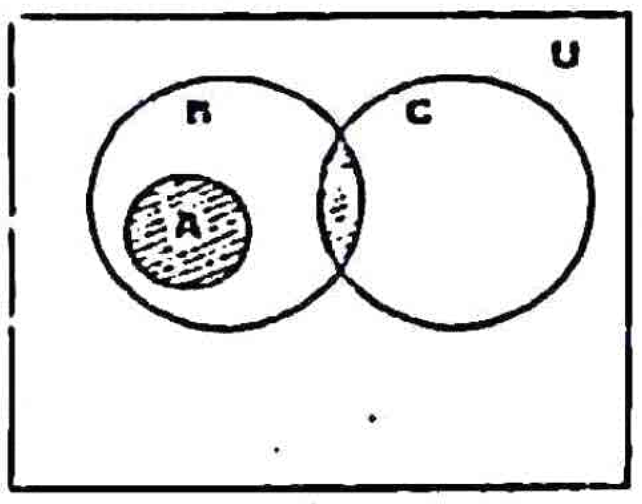


fig (v)
$(A \cup B) \cap (A \cup C)$

from fig (ii) and fig (v) it is varified that
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

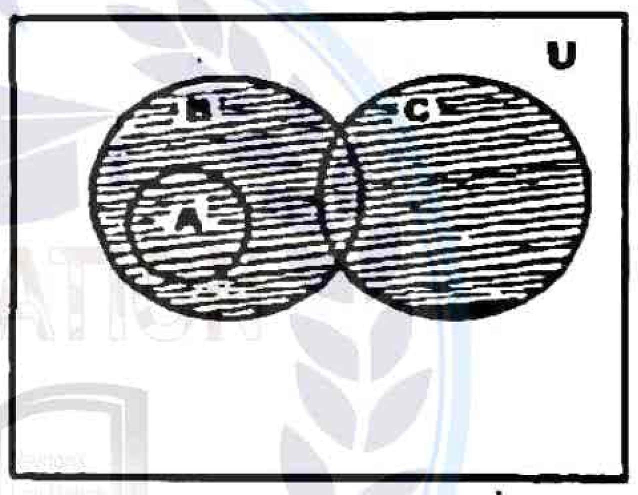


fig (i)
$B \cup C$

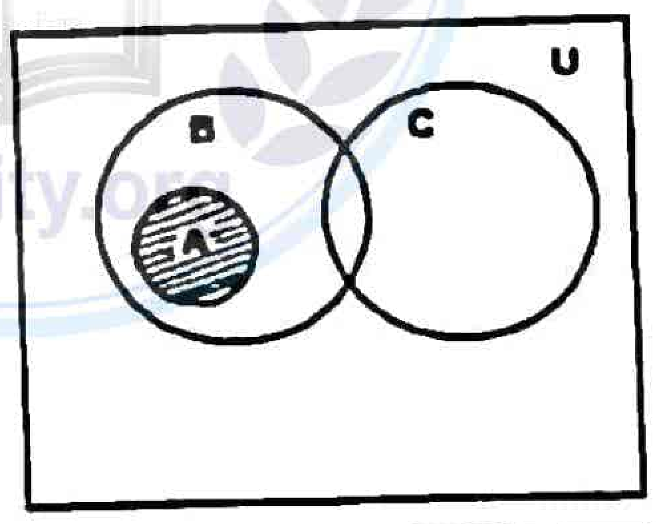


fig (ii)
$A \cap (B \cup C)$

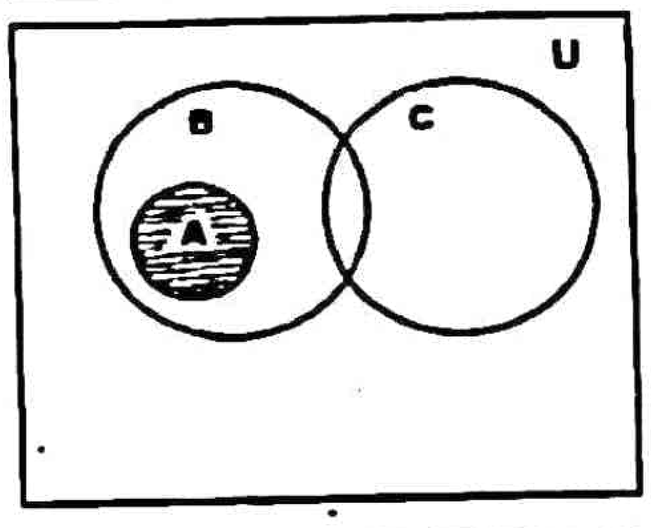


fig (iii)
$A \cap B$

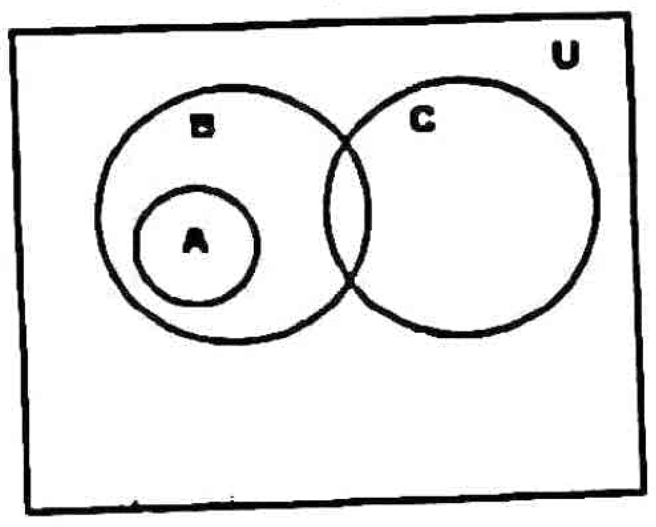
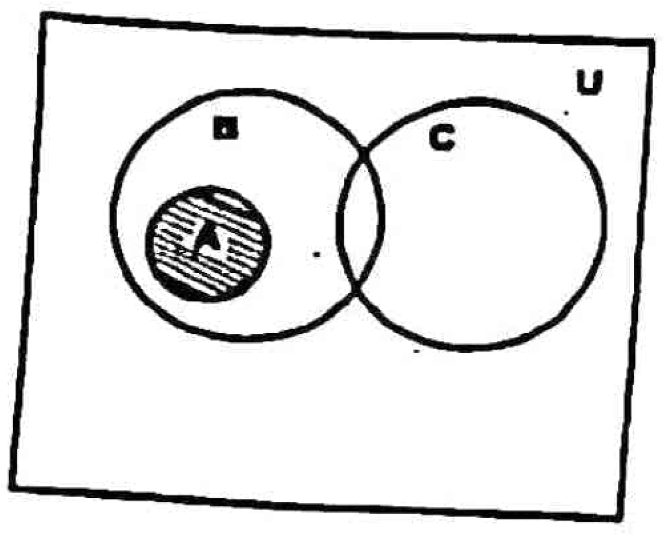


fig (iv)
$A \cap C$ $\because A \cap C = \phi$



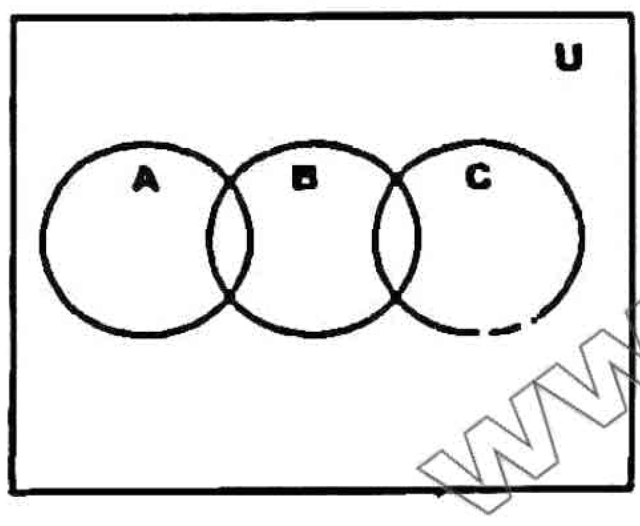
fig(v)
$(A\cap B)\cup(A\cap C)$

from fig (ii) and fig (v) it is varified that
$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

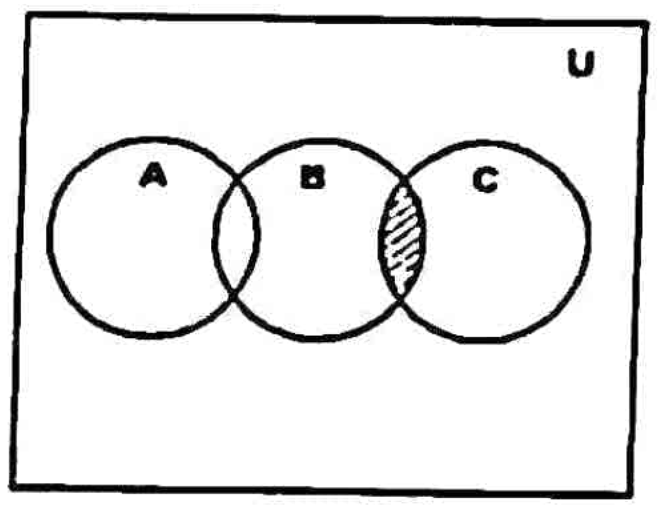
ii) A and B are overlapping, B and C are overlapping, but A and C are disjoint.

a) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

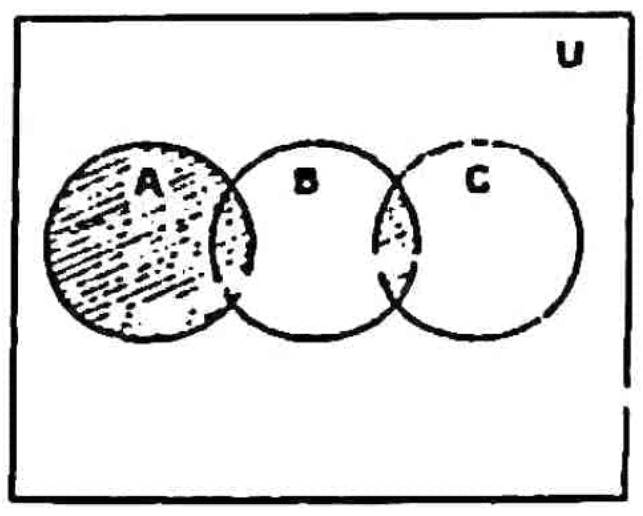
According to given informations we have venn diagram as



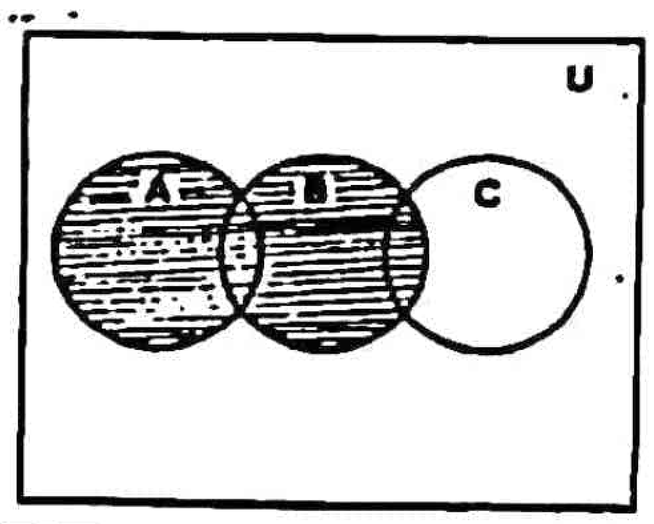
$A\cap B\neq\phi$
$B\cap C=\phi$
$A\cap C=\phi$



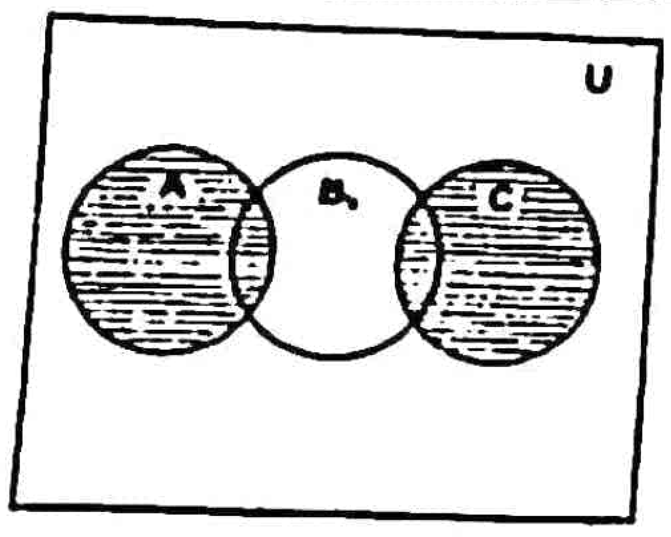
fig(i)
$B\cap C$



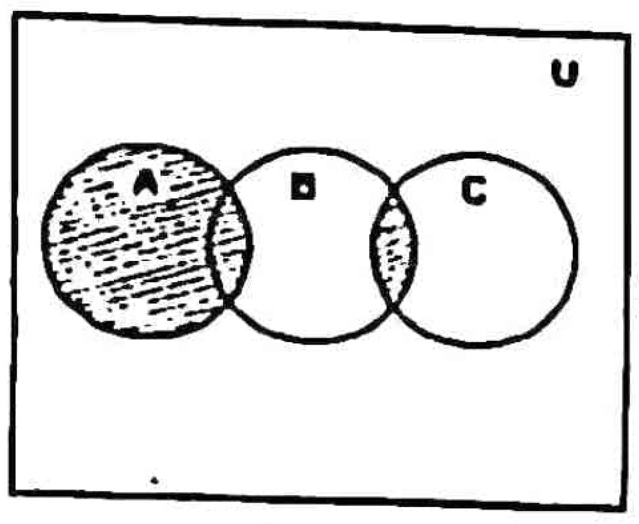
fig(ii)
$A\cup(B\cap C)$



fig(iii)
$A\cup B$



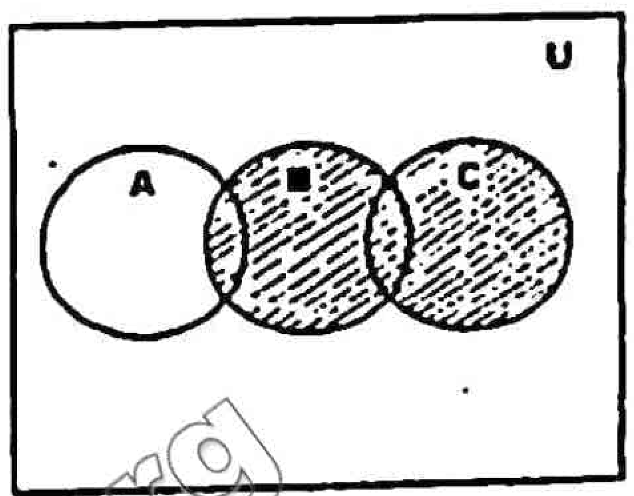
fig(iv)
$A\cup C$



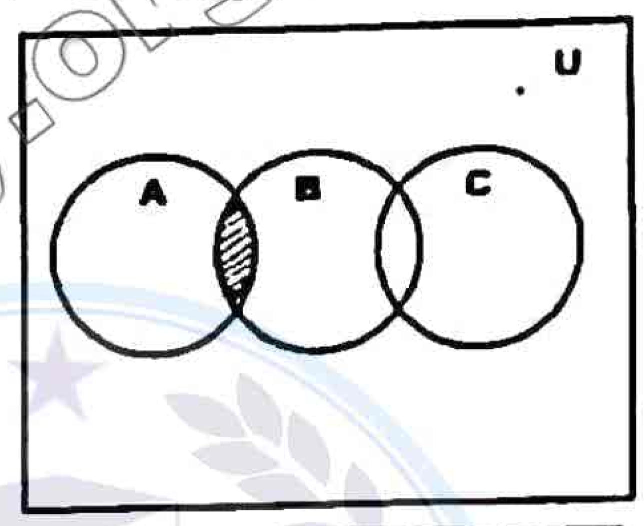
fig(v)
$(A\cup B)\cap(A\cup C)$

from fig (ii) and fig (v) it is varified that
$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

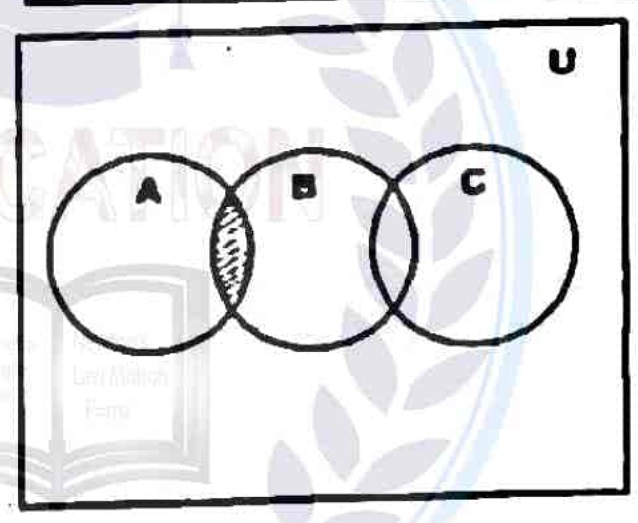
b) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$



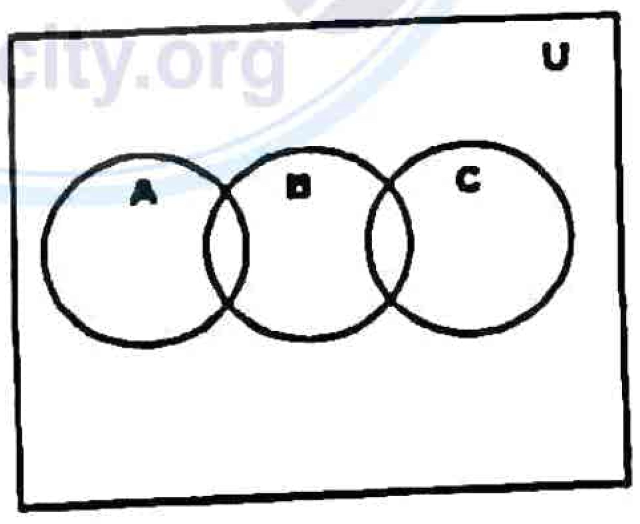
fig(i)
$B\cup C$



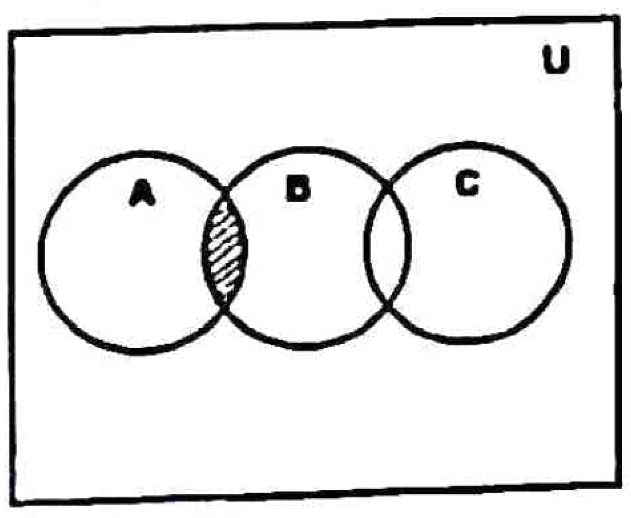
fig(ii)
$A\cap(B\cup C)$



fig(iii)
$A\cap B$



fig(iv)
$A\cap C$ $\quad\because A\cap C=\phi$



fig(v)
$(A\cap B)\cup(A\cap C)$

From fig (ii) and fig(v) it is varified that
$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

Q6. Taking any set, say $A = \{1,2,3,4,5\}$ verify the following

i) $A \cup \phi = A$

Solution:-

$L.H.S = A \cup \phi$

$= \{1,2,3,4,5\} \cup \phi$

$= \{1,2,3,4,5\} = A = R.H.S$

ii) $A \cup A = A$

Solution:-

$L.H.S = A \cup A$

$= \{1,2,3,4,5\} \cup \{1,2,3,4,5\}$

$= \{1,2,3,4,5\} = A = R.H.S$

iii) $A \cap A = A$

Solution:-

$L.H.S = A \cap A$

$= \{1,2,3,4,5\} \cap \{1,2,3,4,5\}$

$= \{1,2,3,4,5\} = R.H.S$

Q7. If $U = \{1,2,3,4,5,\ldots,20\}$ and $A = \{1,3,5,\ldots,19\}$ verify the following:

i) $A \cup A' = U$

Solution:-

$L.H.S = A \cup A'$ where

$A' = U - A = \{1,2,3,\ldots,20\} - \{1,3,5,\ldots,19\}$

$A' = \{2,4,6,\ldots,20\}$ so

$A \cup A' = \{1,3,5,\ldots,19\} \cup \{2,4,6,\ldots,20\}$

$= \{1,2,3,4,\ldots,19,20\} = U = R.H.S$

ii) $A \cap U = A$

Solution:-

$L.H.S = A \cap U$

$= \{1,3,5,\ldots,19\} \cap \{1,2,3,\ldots,20\}$

$= \{1,3,5,\ldots,19\}$

$= A = R.H.S$

iii) $A \cap A' = \phi$

Solution:-

$L.H.S = A \cap A'$

$= \{1,3,5,\ldots,19\} \cap \{2,4,6,\ldots,20\}$

$= \phi = R.H.S$

Q8. From suitable properties of union and intersection deduce the following results:

i) $A \cap (A \cup B) = A \cup (A \cap B)$

Solution:-

$L.H.S = A \cap (A \cup B)$

$= (A \cap A) \cup (A \cap B)$ (using distributive law)

$= A \cup (A \cap B)$ $\quad \because A \cap A = A$

$= R.H.S$

ii) $A \cup (A \cap B) = A \cap (A \cup B)$

Solution:-

$L.H.S = A \cup (A \cap B)$

$= (A \cup A) \cap (A \cup B)$ (Distributive law)

$= A \cap (A \cup B)$

$= R.H.S$

Q9. Using Venn diagrams, verify the following results.

i) $A \cap B' = A$ iff $A \cap B = \phi$

suppose $A \cap B' = A$

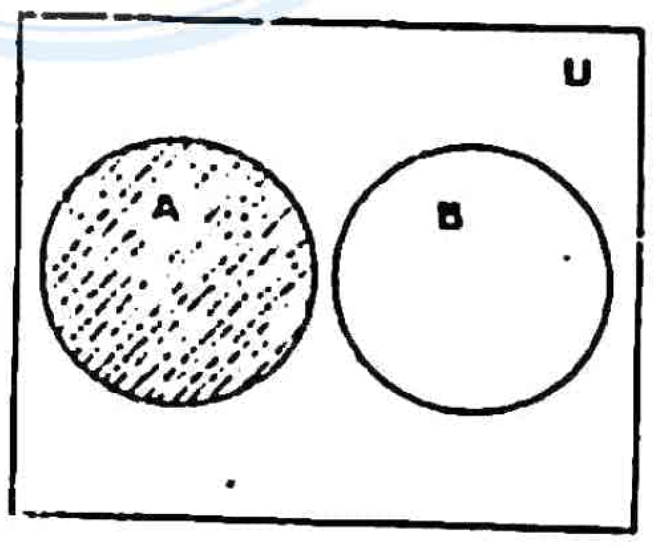


fig (i)

$A \cap B' = A$

We are to prove that $A \cap B = \phi$

from fig (i) $A \cap B' = A$

showing A and B are disjoint

so $A \cap B = \phi$

Conversely,

suppose $A \cap B = \phi$

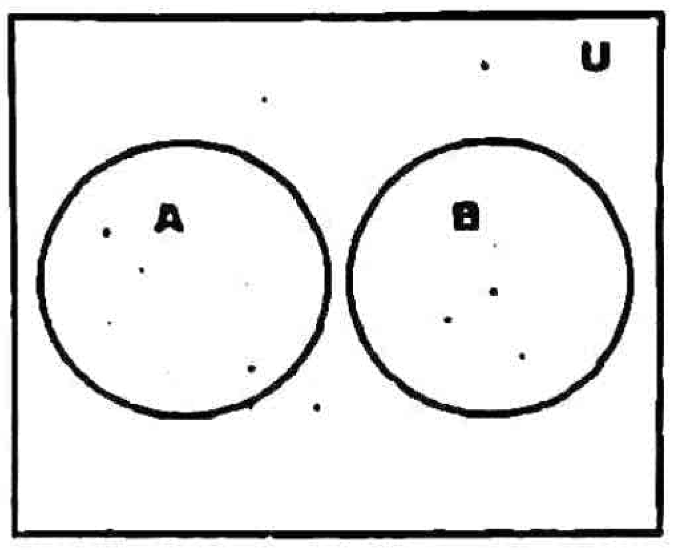


fig (ii)

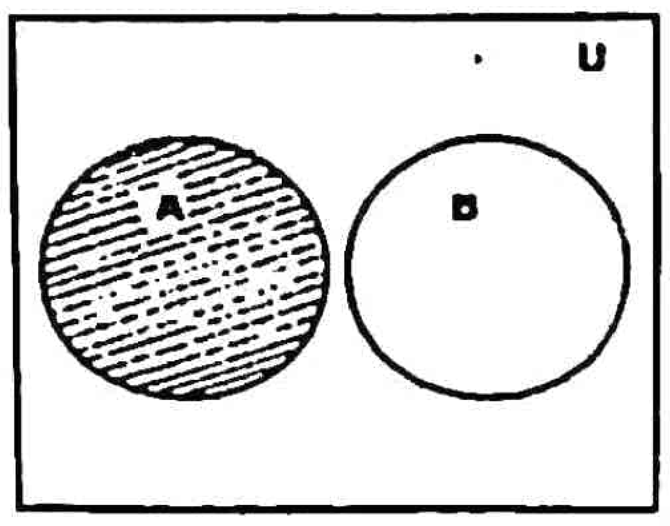
$A \cap B = \phi$

As $A \cap B = \phi$

$\rightarrow$ A and B are disjoint

$\rightarrow$ A will be subset of B'

so $A \cap B' = A$ as shown in fig (iii)



$A \cap B' = A$

ii) $(A - B) \cup B = A \cup B$

consider A and B are overlapping sets, then

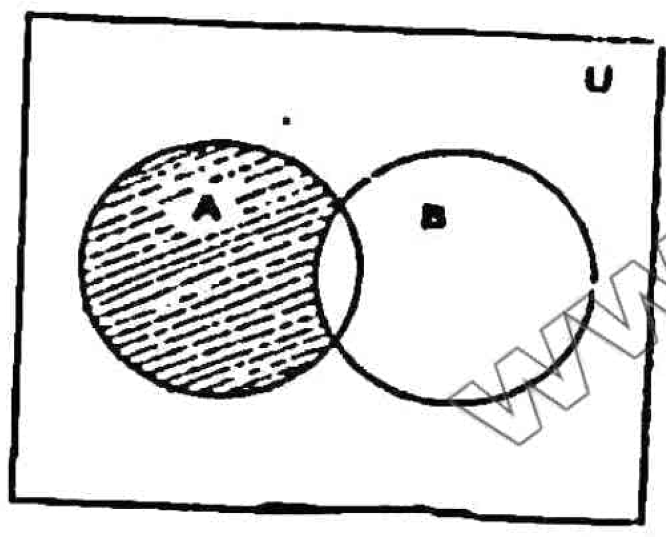


fig (i)

$A - B$

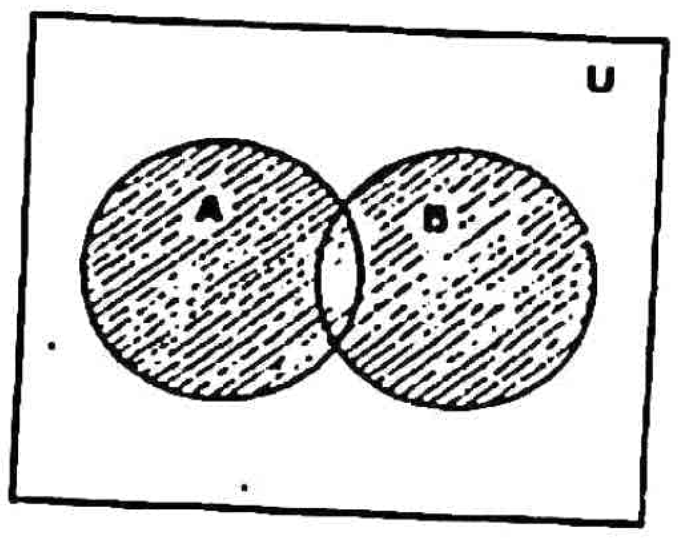


fig (ii)

$(A-B) \cup B$

from fig (ii) it is clear that

$(A - B) \cup B = A \cup B$

iii) $(A - B) \cap B = \phi$

Consider A and B are overlapping sets, then

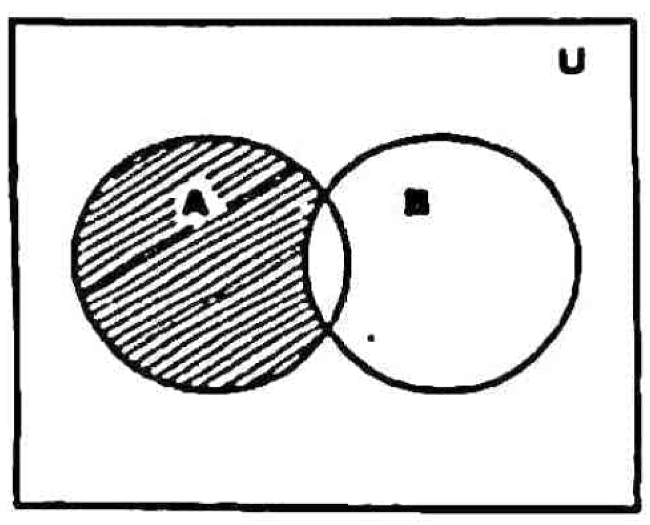


fig (i)

$A - B$

from fig (i) A−B and B have nothing common so nothing will be shaded to show $(A-B) \cap B$ as shown in fig (ii)

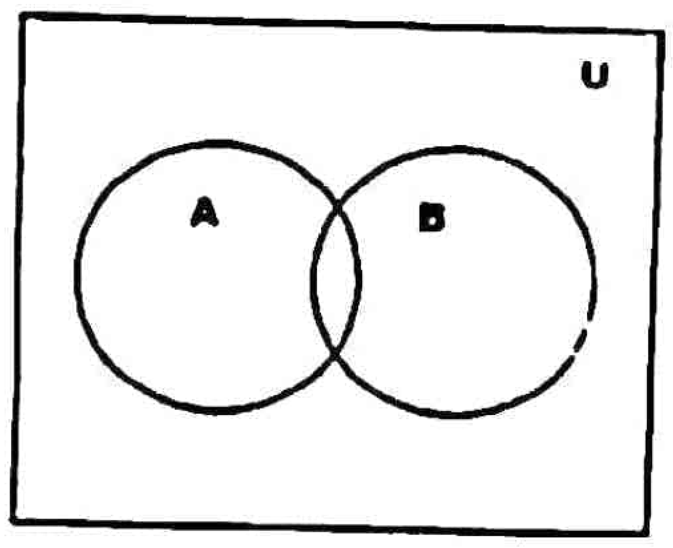


fig (ii)

So $(A-B) \cap B = \phi$

iv) $A \cup B = A \cup (A' \cap B)$

consider A and B are overlapping sets, then

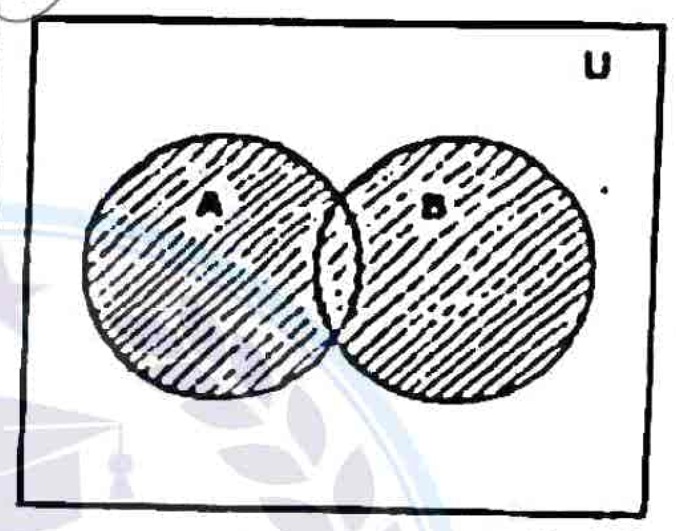


fig (i)

$A \cup B$

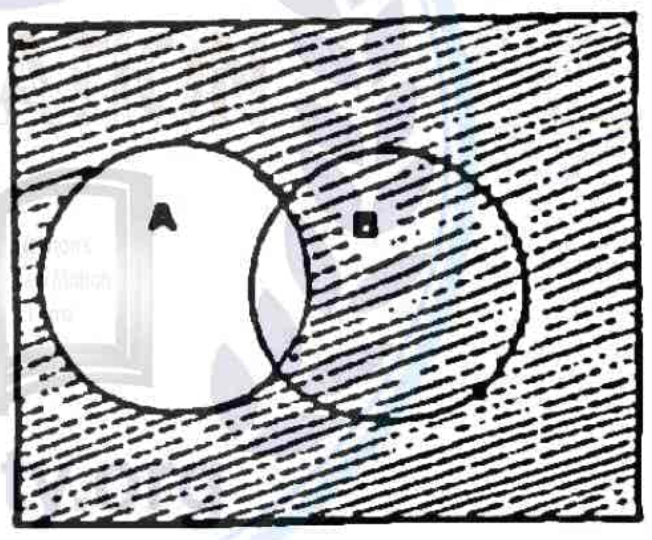


fig (ii)

$A' = U - A$

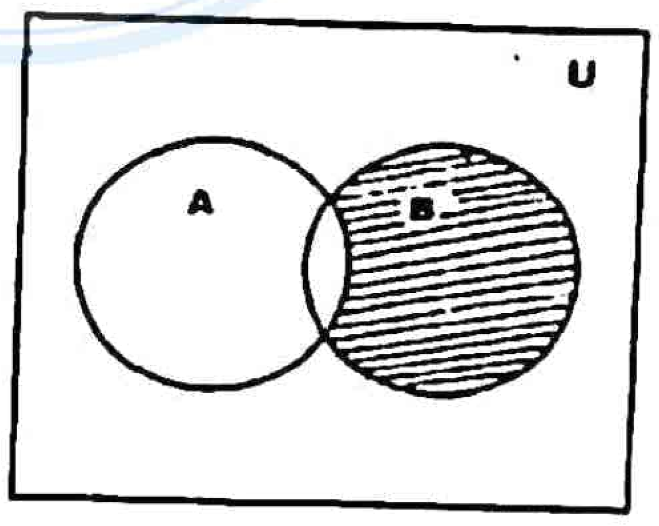


fig (iii)

$A' \cap B$

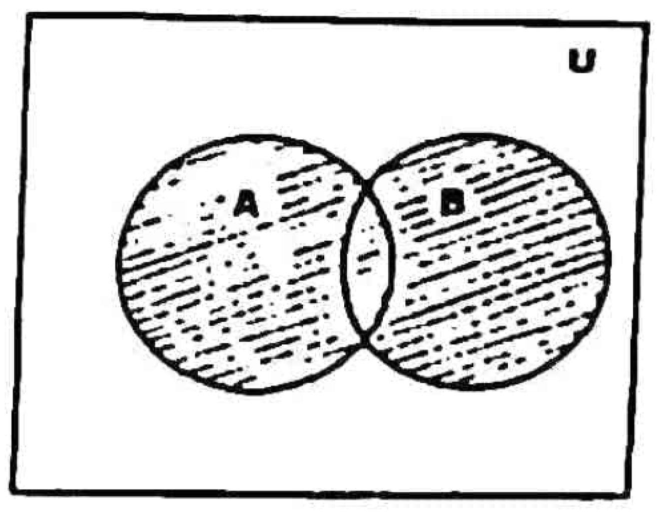


fig (iv)

$A \cup (A' \cap B)$

from fig (i) and fig (iv) it is verified that

$A \cup B = A \cup (A' \cap B)$

Induction:- A result on the basis of limited observations is called induction.

Deduction:- A result or (conclusion) on the basis of well known facts is called deduction.

Proposition:- Any statement which is either true or false but not both is called proposition. Propositions are denoted by $p, q, r, \ldots$

Aristotelian Logic :- According to Aristotle there could be only two possibilities - a proposition could be either true or false and there could not be any third possibility. e.g., the statement $a = b$ can be either true or false.

Non Aristotelian Logic :- Logic in which there is scope for a third or fourth possibility is called non-Aristotelian.

Negation:- Negation of a proposition means to reject that proposition. If p is proposition variable then negation of p is denoted by $\sim p$.

Note:-
If p is true, $\sim p$ is false.
If p is false, $\sim p$ is true.
If $\sim p$ is true, p is falls.
If $\sim p$ is false, p is true

The truth table is given as

p	$\sim p$
T	F
F	T

Conjunction:- Let p and q be two propositions then their conjunction is denoted by $p \wedge q$ and read as "p and q"

- A conjunction is true only if both p and q are true.
- A conjunction is false if at least one of p and q is false.

The truth table is given as

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.

i) Lahore is capital of Punjab and Quetta is capital of Balochistan.

ii) $4 < 5 \wedge 8 < 10$

iii) $4 < 5 \wedge 8 > 10$

iv) $2+2=3 \wedge 6+6=10$ so

(i) and (ii) are true whereas (iii) and (iv) are false.

Disjunction:- Let p and q be two propositions then their disjunction is denoted by $p \vee q$ and read as "p or q".

- A disjunction is false only if both p and q are false.
- A disjunction is true if at least one of p and q is true.

The truth table is given as

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 2.

i) 10 is positive integer or 0 is a rational number. Find truth value of this disjunction.

Solution:-
$\because$ first component is true, so
$\therefore$ disjunction is true

ii) A triangle can have two right angles or Lahore is the capital of sindh.

Solution:-
Both the components being false, the composite proposition is false.

Implication or conditional

Let p and q be two propositions, then p implication q is denoted by $p \rightarrow q$ and read as "p implies q" (or if p then q)

Here p is called hypothesis or antecedent while q is called consequent or conclusion.

* A conditional statement is false only when hypothesis is true, otherwise conditional statement is always true.

The truth table is given below.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional:-

The proposition $p \rightarrow q \wedge q \rightarrow p$ is shortly written as $p \leftrightarrow q$ read as p if and only if (iff) q is called biconditional or equivalent statement.

* A biconditional statement is true if both p and q are true.
* A bicondinal statement is true if both p and q are fallse.
* A biconditional statement is false when any one of p, q is falls.

we draw up its truth table as:

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Conditionals related with a given conditional:

Converse:- Let $p \rightarrow q$ be given conditional then converse of $p \rightarrow q$ is $q \rightarrow p$.

Inverse:- Let $p \rightarrow q$ be given conditional then inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$

Contrapositive:- Let $p \rightarrow q$ be given conditional then contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

The truth table of above conditionals is given as:

				Given conditional	Converse	Inverse	Contrapositive
p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	F	T	T	T	T

Note:- (i) From the table it is clear that converse and inverse are equivalent to each other.

(ii) From the table it is clear that any conditional and its contrapositive are equivalent to each other.

Example 3. Prove that in any universe the empty set ϕ is a subset of any set A.

First Proof:— Let U be universal set consider the conditional:

$\forall x \in U,\ x \in \phi \rightarrow x \in A$ → (i)

the antecedent of this conditional is false because no $x \in U$, is a member of ϕ.

Hence the conditional is true.

Second Proof:— (By contrapositive)

The contrapositive of conditional (i) is

$\forall x \in U,\ x \notin A \rightarrow x \notin \phi$

The consequent of this conditional is true. Therefore the conditional is true.

∴ Hence proved

Example 4. Construct the truth table of $[(p \rightarrow q \wedge p \rightarrow q)]$

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q \wedge p \rightarrow q)]$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Tautologies:—

A statement which is necessarily true for all the cases is called a tautology.

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

From table we observe that $p \vee \sim p$ is true for all cases. so $p \vee \sim p$ is a tautology.

Contradiction:—

A statement which is necessarily false for all the cases is called contradiction.

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

From table we observe that $p \wedge \sim p$ is a contradiction.

Contingency:— A statement which is neither tautology nor contradiction is called contingency. e.g., $(p \rightarrow q) \wedge (p \vee q)$ is contingency.

Quantifiers:— The word or symbols which convey the idea of quantity or number are called quantifiers.

There are two quantifiers

i) Universal quantifiers

The symbol $\forall$ (for all) is called universal quantifiers.

ii) Existential quantifiers

The symbol $\exists$ (there exist) is called existential quantifiers.

Exercise 2.4

Q1. Write the converse, inverse and contrapositive of the following conditionals.

i) $\sim p \rightarrow q$ ii) $q \rightarrow p$ iii) $\sim p \rightarrow \sim q$ iv) $\sim q \rightarrow \sim p$

Part	Conditional	Converse	Inverse	Contra positive
i.	$\sim p \rightarrow q$	$q \rightarrow \sim p$	$p \rightarrow \sim q$	$\sim q \rightarrow p$
ii.	$q \rightarrow p$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$\sim p \rightarrow \sim q$
iii.	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$	$p \rightarrow q$	$q \rightarrow p$
iv.	$\sim q \rightarrow \sim p$	$\sim p \rightarrow \sim q$	$q \rightarrow p$	$p \rightarrow q$

Q2. Construct truth tables for the following statements:-

i) $(p \rightarrow \sim p) \vee (p \rightarrow q)$

p	q	$\sim p$	$p \rightarrow \sim p$	$p \rightarrow q$	$(p \rightarrow \sim p) \vee (p \rightarrow q)$
T	T	F	F	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	T	T	T

ii) $(p \wedge \sim p) \rightarrow p$

p	q	$\sim p$	$p \wedge \sim p$	$(p \wedge \sim q) \rightarrow q$
T	T	F	F	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

iii) $\sim (p \rightarrow q) \rightarrow p$

p	q	$p \rightarrow q$	$\sim (p \rightarrow q)$	$\sim q$	$p \wedge \sim q$	$\sim (p \rightarrow q) \leftrightarrow (P \wedge \sim q)$
T	T	T	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	F	T	F	T	F	T

Q3. Show that each of the following statements is a tautology:-

i) $(p \wedge q) \rightarrow p$

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

ii) $p \rightarrow (p \vee q)$

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

iii) $\sim(p \rightarrow q) \rightarrow p$

p	q	$p \rightarrow q$	$\sim(p \rightarrow q)$	$\sim(p \rightarrow q) \rightarrow p$
T	T	T	F	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

iv) $\sim q \wedge (p \rightarrow q) \rightarrow \sim p$

p	q	$\sim q$	$p \rightarrow q$	$\sim q \wedge (p \rightarrow q)$	$\sim p$	$\sim q \wedge (p \rightarrow q) \rightarrow \sim p$
T	T	F	T	F	F	T
T	F	T	F	F	F	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T

Q4. Determine whether each of the following is a tautology, a contingency or an absurdity:–

i) $p \wedge \sim p$

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

$\because$ all values of $p \wedge \sim p$ are false so $p \wedge \sim p$ is an absurdity (contradiction)

ii) $p \rightarrow (q \rightarrow p)$

p	q	$q \rightarrow p$	$p \rightarrow (q \rightarrow p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

$\because$ all values of $p \rightarrow (q \rightarrow p)$ are true so $p \rightarrow (q \rightarrow p)$ is a tautology.

iii) $q \vee (\sim q \vee p)$

p	q	$\sim q$	$\sim q \vee p$	$q \vee (\sim q \vee p)$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	T

$\because$ all values of $q \vee (\sim q \vee p)$ are true so $q \vee (\sim p \vee p)$ is a tautology.

Q5. Prove that $p\vee(\sim p\wedge\sim q)\vee(p\wedge q)=p\vee(\sim p\wedge\sim q)$

p	q	$\sim p$	$\sim q$	$\sim p\wedge\sim q$	$p\wedge q$	$p\vee(\sim p\wedge\sim q)$ R.H.S	$p\vee(\sim p\wedge\sim q)\vee(p\wedge q)$ L.H.S
T	T	F	F	F	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	F	F	F	F
F	F	T	T	T	F	T	T

$\because$ last two columns are same.

$\therefore$ $p\vee(\sim p\wedge\sim q)\vee(p\wedge q)=p\vee(\sim p\wedge\sim q)$ proved.

Truth Sets, A link between Set Theory and Logic

Example 1. Give logical proofs of the following theorems:- (A, B and C are any sets)

i) $(A\cup B)'=A'\cap B'$

Solution:- Its logical form is $\sim(p\vee q)=\sim p\wedge\sim q$

$\because$ '2' variables, so rows of table $=2^2=4$

p	q	$\sim p$	$\sim q$	$p\vee q$	$\sim(p\vee q)$ L.H.S	$\sim p\wedge\sim q$ R.H.S
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

$\because$ last two columns are same.

$\rightarrow \sim(p\vee q)=\sim p\wedge\sim q \rightarrow (A\cup B)'=A'\cap B'$

ii) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

Its logical form is $p\wedge(q\vee r)=(p\wedge q)\vee(p\wedge r)$

$\because$ '3' variables, so rows of table $=2^3=8$

p	q	r	$q\vee r$	$p\wedge(q\vee r)$	$p\wedge q$	$p\wedge r$	$(p\wedge q)\vee(p\wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F
1	2	3	4	5	6	7	8

$\because$ from column ⑤ and ⑧ it is varified that

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

$$\Rightarrow A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad \text{proved.}$$

Exercise 2.5

Convert the following theorems to logical form and prove them by constructing truth tables:-

Q1. $(A \cap B)' = A' \cup B'$

The logical form is $\sim(p \wedge q) = \sim p \vee \sim q$

$\because$ '2' variables, so rows $= 2^2 = 4$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

$\because$ last two columns are same.

$$\Rightarrow \sim(p \wedge q) = \sim p \vee \sim q \Rightarrow (A \cap B)' = A' \cup B'$$

Q2. $(A \cup B) \cup C = A \cup (B \cup C)$

The logical form is $(p \vee q) \vee r = p \vee (q \vee r)$

$\because$ '3' variables, so rows $= 2^3 = 8$

p	q	r	$p \vee q$	$q \vee r$	$(p \vee q) \vee r$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

$\because$ last two columns are same

$$\Rightarrow (p \vee q) \vee r = p \vee (q \vee r)$$

$$\Rightarrow (A \cup B) \cup C = A \cup (B \cup C)$$

Q3. $(A \cap B) \cap C = A \cap (B \cap C)$

The logical form is $(p \wedge q) \wedge r = p \wedge (q \wedge r)$

$\because$ '3' variables, so rows $= 2^3 = 8$

p	q	r	$p\wedge q$	$q\wedge r$	$(p\wedge q)\wedge r$	$p\wedge(q\wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

$\because$ last two columns are same.

$\Rightarrow (p\wedge q)\wedge r = p\wedge(q\wedge r)$

$\Rightarrow (A\cap B)\cap C = A\cap(B\cap C)$ proved

Q4. $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

The logical form is $p\vee(q\wedge r) = (p\vee q)\wedge(p\vee r)$

$\because$ '3' variables, so rows $= 2^3 = 8$

p	q	r	$p\vee q$	$q\wedge r$	$p\vee r$	$p\vee(q\wedge r)$	$(p\vee q)\wedge(p\vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	T	F	T	T	T
T	F	F	T	F	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

$\because$ last two columns are same.

$\Rightarrow p\vee(q\wedge r) = (p\vee q)\wedge(p\vee r)$

$\Rightarrow A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$ proved.

Relation/Binary relation:-

Let A and B be two non-empty sets and $A\times B$ be their Cartesian product, then relation or binary relation from A to B is subset of $A\times B$.

Domain:- The set of the first elements of the ordered pairs forming a relation is called its domain.

Range:- The set of the second elements of the ordered pairs forming a relation is called its range.

Note:- In general

$$A\times B \neq B\times A$$

Example 1. Let c_1, c_2, c_3 be three children and m_1, m_2 be two men such that father of both c_1, c_2 is m_1 and father of c_3 is m_2. Find the relation $\{(\text{child}, \text{father})\}$

Solution:-

Let C = set of childrens
$= \{c_1, c_2, c_3\}$

F = set of fathers $= \{m_1, m_2\}$

Now

$C \times F = \{(c_1, m_1), (c_1, m_2), (c_2, m_1)$
$(c_2, m_2), (c_3, m_1), (c_3, m_2)\}$

r = set of ordered pairs
$= (\text{child}, \text{father})$

$r = \{(c_1, m_1), (c_2, m_1), (c_3, m_2)\}$

$\text{Dom } r = \{c_1, c_2, c_3\}$

$\text{Ran } r = \{m_1, m_2\}$

Example 2. Let $A = \{1, 2, 3\}$, Determine the relation r such that $x r y$ iff $x < y$

Solution:-

$A = \{1, 2, 3\}$

$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2)$
$(2,3), (3,1), (3,2), (3,3)\}$

By given condition

$r = \{(x, y) \mid x, y \in A \wedge x < y\}$

$r = \{(1,2), (1,3), (2,3)\}$

$\text{Dom } r = \{1, 2\}$, $\text{Ran } r = \{2, 3\}$

Example 3. Let $A = \mathbb{R}$, the set of real numbers, Determine the relation r such that $x r y$ iff $y = x + 1$

Solution:-

$A = \mathbb{R}$

$A \times A = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in A\}$

Now

$r = \{(x, y) \mid y = x + 1\}$

There are unlimited relations satisfying $y = x + 1$ so clearly

$\text{Dom } r = \mathbb{R}$, $\text{Ran } r = \mathbb{R}$

Function:-

Let A and B be two non-empty sets, then f is called function from A to B written as

$f : A \longrightarrow B$ and

defined as;

i) $\text{Dom } f = A$

ii) No two ordered pairs of f have first elements equal. e.g.,

i) Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ and $f = \{(1, a), (2, b), (3, c)\}$

Here $\text{Dom } f = \{1, 2, 3\} = A$ Also No two ordered pairs of f have first elements equal.

Hence f is function

ii) Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$ and $h = \{(1, a), (2, b)\}$ Here

$\text{Dom } h = \{1, 2\} \neq A$ so condition (i) of function is not satisfied.

Hence h is not function.

Into function:- If $f: A \longrightarrow B$ be a function such that Range of f is proper subset of B (i-e., Rang of $f \neq B$) then f is said to be function from A into B.

Onto (Surjective) function:- If $f: A \longrightarrow B$ be a function such that Range of $f = B$ then f is called onto function.

(1-1) and into (Injective) function:- If $f: A \longrightarrow B$ be into function further more there is no repetition in the second element any two ordered pairs (i-e., each element of A have distinct image in B) then f is said to be (1-1) and into function.

(1-1) and Onto (bijective) function:- If $f: A \longrightarrow B$ be onto function furthur more there is no repetition in the second element of any two ordered pairs of f. then f is said to be an one-one and onto function.

Set Builder Notation for a function:-

The function $f = \{(x,y) \mid y = mx+c\}$ is called a linear function. If we draw a linear function then its graph will be a straight line.

The function $f = \{(x,y) \mid y = ax^2+bx+c\}$ is called quadratic function. If we draw a quadratic function its graph will be a parabola.

Example 4. Give rough sketch of the functions

i) $\{(x,y) \mid 3x+y=2\}$

Solution:-

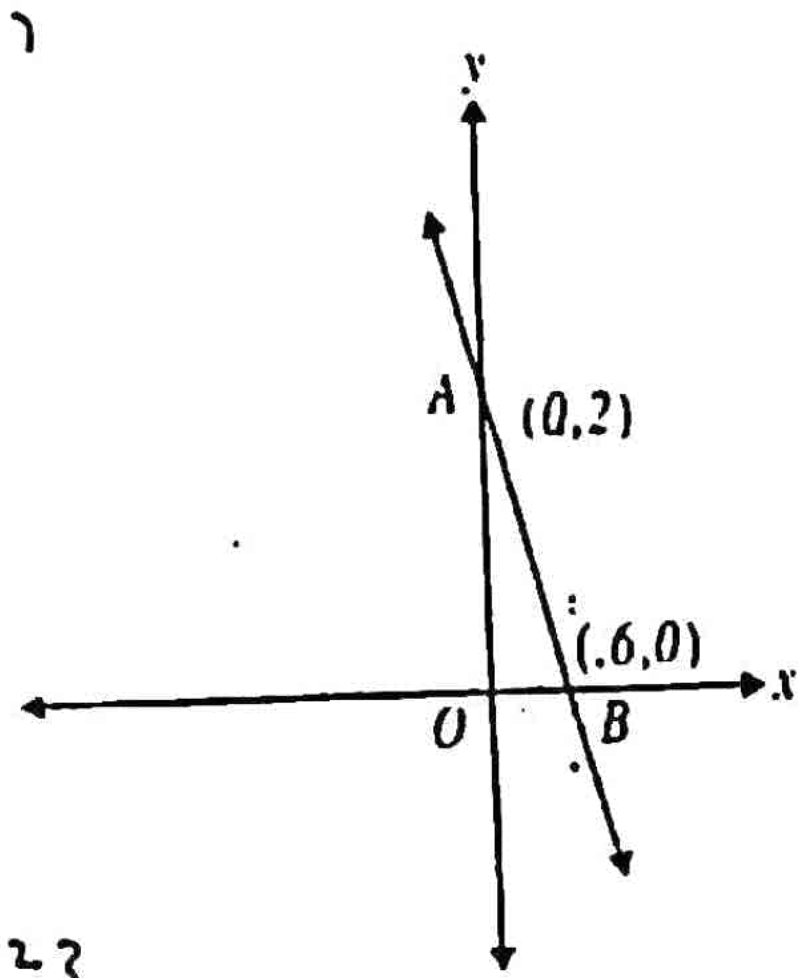
$\because 3x+y=2 \rightarrow y=-3x+2$

Put $x=0$, $y=2$ so pt $(0,2)$

Put $y=0$, $-3x-2=0 \rightarrow x=\frac{2}{3}=0.6$

So pt $(\frac{2}{3}, 0)$

Now graph of given linear equation is shown in fig.



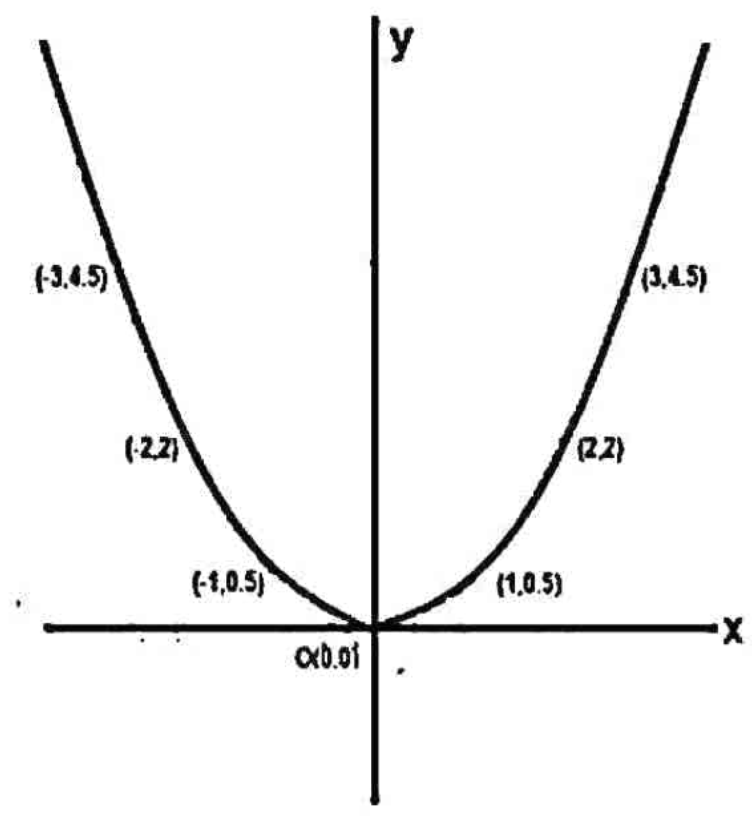
ii) $\{(x,y) \mid y = \frac{1}{2}x^2\}$

Solution:-

$\because y = \frac{1}{2}x^2$

x	0	±1	±2	±3
y	0	0.5	2	4.5

Now graph of given quadratic equation is shown in fig.,



Inverse of a function:-

If $f: A \rightarrow B$ be a bijective function then its inverse is denoted by f^{-1} and defined as $f^{-1}: B \rightarrow A$. In this case Domain of f^{-1} = Range of f

Range of f^{-1} = Domain of f

e.g., :

Let $f = \{(x,y) \mid y = mx + c\}$

$\rightarrow f^{-1} = \{(y,x) \mid x = my + c\}$

and

Let $f = \{(x,y) \mid x, y \in Z^{+} \wedge y = x^2\}$

$\rightarrow f^{-1} = \{(y,x) \mid x, y \in Z^{+} \wedge x = y^2\}$

> In other words, f^{-1} can be obtained by interchanging components of ordered pairs of f

+ Inverse of a function is not necessarily a function.

Example 5. Not available (missing) in text book

Example 6. Find the inverse of i) $\{(1,1), (2,4), (3,9), (4,16), \ldots x \in Z^{+}\}$

Tell which of these are functions

Solution:-

Let $r = \{(1,1), (2,4), (3,9), (4,16), \ldots x \in Z^{+}\}$

r is a function because no repitition in domain.

$r^{-1} = \{(1,1), (4,2), (9,3), (16,4), \ldots x \in Z^{+}\}$

Similarly, relation r^{-1} is function because no repitition in domain.

ii) $\{(x,y) \mid y = 2x+3,\ x \in \mathbb{R}\}$

Solution:-

Let $r = \{(x,y) \mid y = 2x+3,\ x \in \mathbb{R}\}$

r is a function because for each value of $x \in \mathbb{R}$ there is a unique value of y

(i.e., no repitition in domain.)

$r^{-1} = \{(x,y) \mid x = 2y+3,\ x \in \mathbb{R}\}$

$= \{(x,y) \mid y = \frac{x-3}{2},\ x \in \mathbb{R}\}$

r^{-1} is a function because for each value of $x \in \mathbb{R}$ there is a unique value of y

(i.e., no repitition in domain.)

iii) $\{(x,y) \mid x^2+y^2 = a^2, |x| \le a, |y| \le a\}$

Solution:- Let

$r = \{(x,y) \mid x^2+y^2 = a^2, |x| \le a, |y| \le a\}$

$= \{(x,y) \mid y = \pm\sqrt{a^2-x^2}, |x| \le a, |y| \le a\}$

r is not a function because for each value of x there exist two values of y.

(i.e., repitition in domain.)

$r^{-1} = \{(x,y) \mid x^2+y^2 = a^2, |x| \le a, |y| \le a\}$

$= \{(x,y) \mid y = \pm\sqrt{a^2-x^2}, |x| \le a, |y| \le a\}$

r^{-1} is not a function because for each value of x there exist two values of y.

(i.e., repitition in domain.)

Exercise 2.6

Q1. For $A = \{1, 2, 3, 4\}$, find the following relations in A. state the domain and range of each relation. Also draw the graph of each.

i) $\{(x,y) \mid y = x\}$

Solution:-

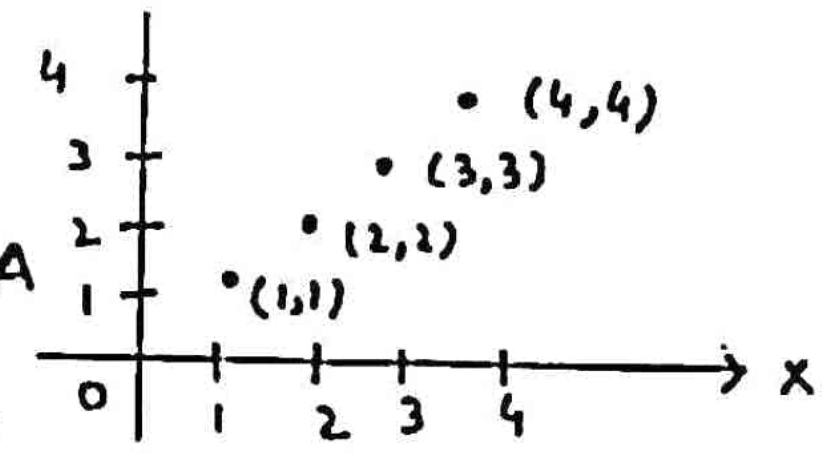
Given that

$A = \{1, 2, 3, 4\}$

$A \times A = \{(1,1), (1,2), (1,3), (1,4)$
$(2,1), (2,2), (2,3), (2,4), (3,1)$
$(3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$

According to the given condition

$R = \{(1,1), (2,2), (3,3), (4,4)\}$



Dom $R = \{1, 2, 3, 4\} = A$

Ran $R = \{1, 2, 3, 4\} = A$

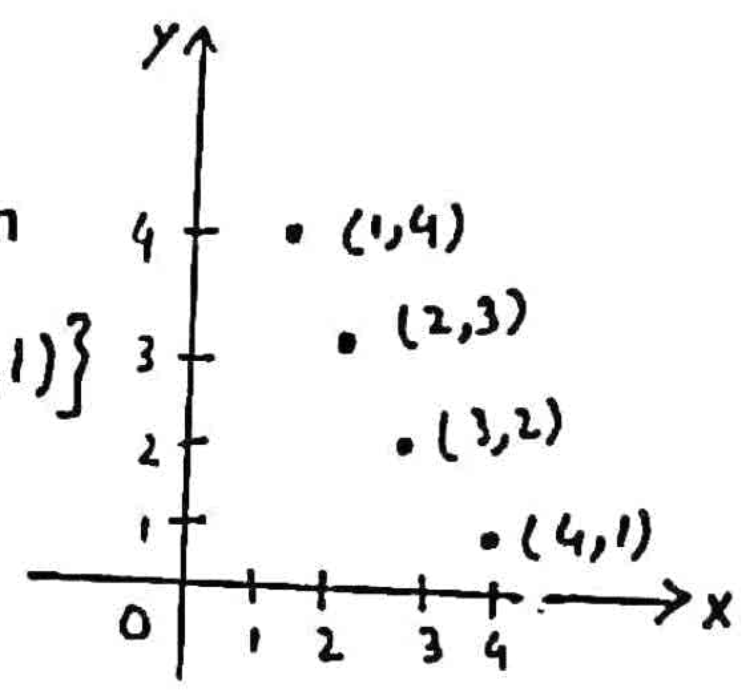
ii) $\{(x,y) \mid y + x = 5\}$

According to the given condition

$R = \{(1,4), (2,3), (3,2), (4,1)\}$

Dom $R = \{1, 2, 3, 4\}$

Ran $R = \{1, 2, 3, 4\}$



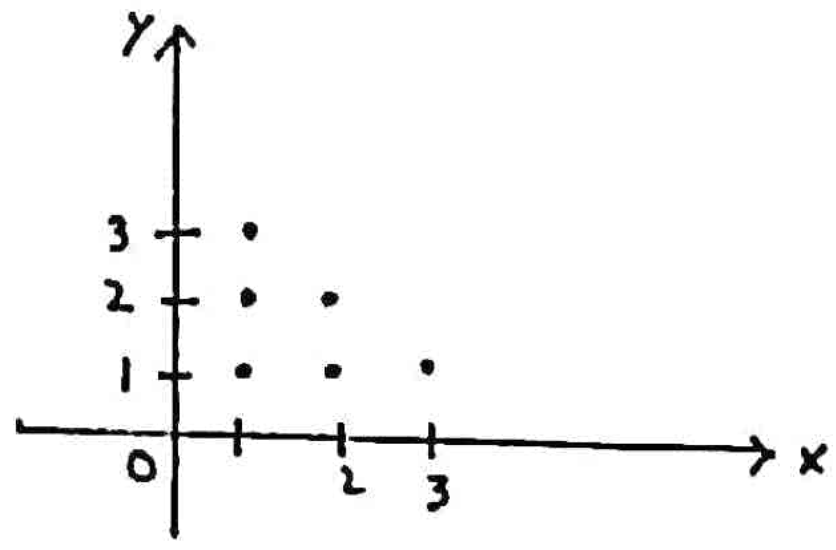
iii) $\{(x,y) \mid x+y<5\}$

According to given condition

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}$

Dom $R = \{1,2,3\}$

Ran $R = \{1,2,3\}$



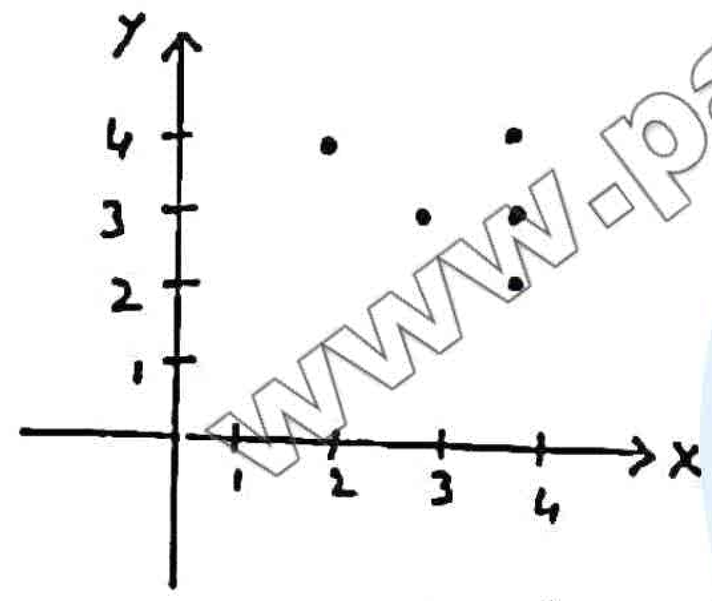
iv) $\{(x,y) \mid x+y>5\}$

According to given condition

$R = \{(2,4), (3,3), (4,2), (4,3), (4,4)\}$

Dom $R = \{2,3,4\}$

Ran $R = \{2,3,4\}$



Q2. Repeate Q-1 when $A = \mathbb{R}$, the set of real numbers, which of the real lines are functions.

Solution:–

Given $A = \mathbb{R}$ = set of real nos.

i) $R = \{(x,y) \mid y = x\}$

Dom $R = \mathbb{R}$

No two ordered pairs of R have first element equal. so R is a function

ii) $R = \{(x,y) \mid x+y=5\}$

Dom $R = \mathbb{R}$

No two ordered pairs of R have first element equal. so R is a function.

iii) $R = \{(x,y) \mid x+y<5\}$

Dom $R = \mathbb{R}$

There are so many ordered pairs of R (i.e., $(1,2), (1,3), (3,1), (2,2), (3,0)$ .....) having first element same. so R is not function.

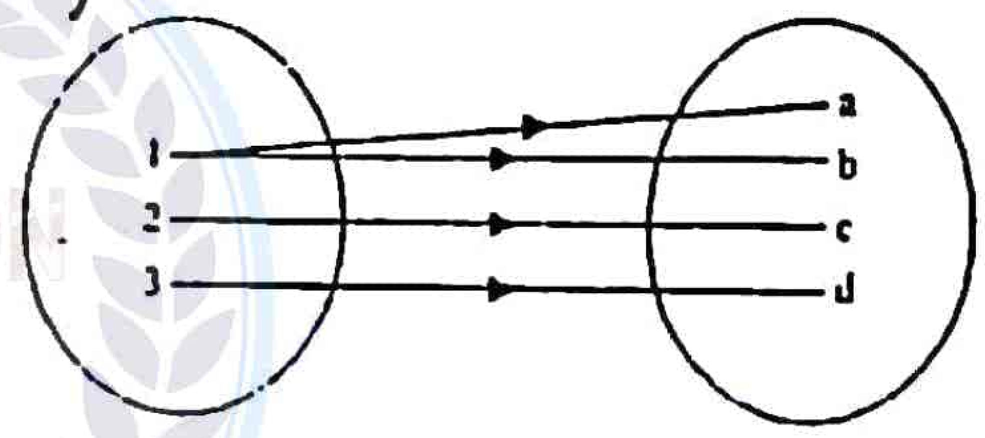
iv) $R = \{(x,y) \mid x+y>5\}$

Dom $R = \mathbb{R}$

There are so many ordered pairs of R (i.e., $(1,5), (1,6), (3,3), (4,4)$ .....) having first elements equal. So R is not a function.

Q3. Which of the following diagrams represent functions and of which type?

i)



Solution:–

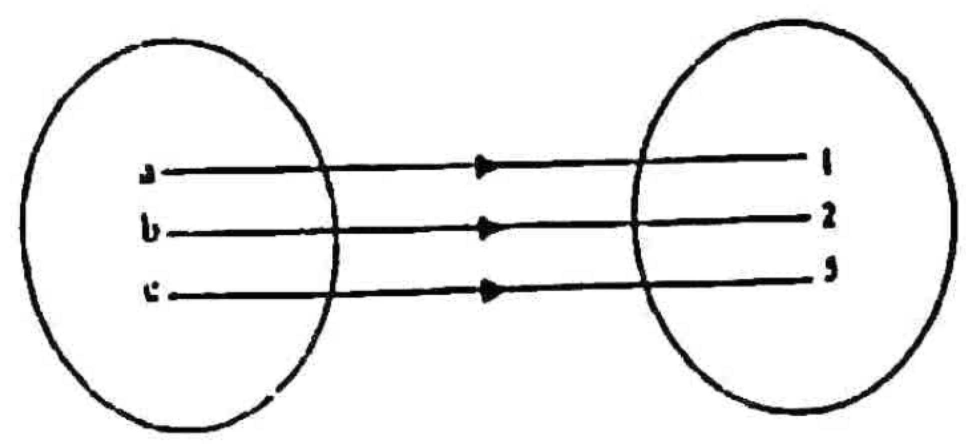
Here $f = \{(1,a), (1,b), (2,c), (3,d)\}$

Dom $f = \{1,2,3\} = A$

Rang $f = \{a,b,c,d\} = B$

Ordered pairs $(1,a)$ and $(1,b)$ have first element equal. so f is not a function.

ii)



Solution:–

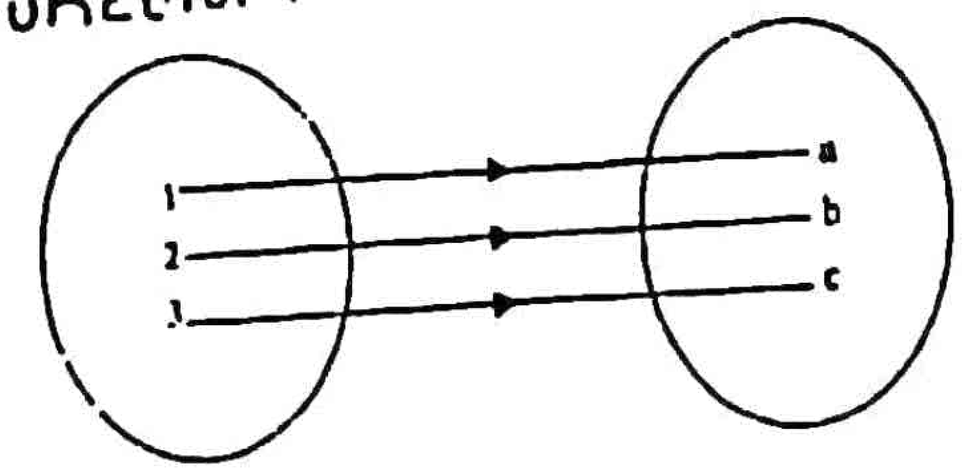
Here $f = \{(a,1), (b,3), (c,5)\}$

Dom $f = \{a,b,c\} = A$

Ran $f = \{1,3,5\} = B$

f is one-one and also onto function. so f is bijective function.

iii)



Solution:-

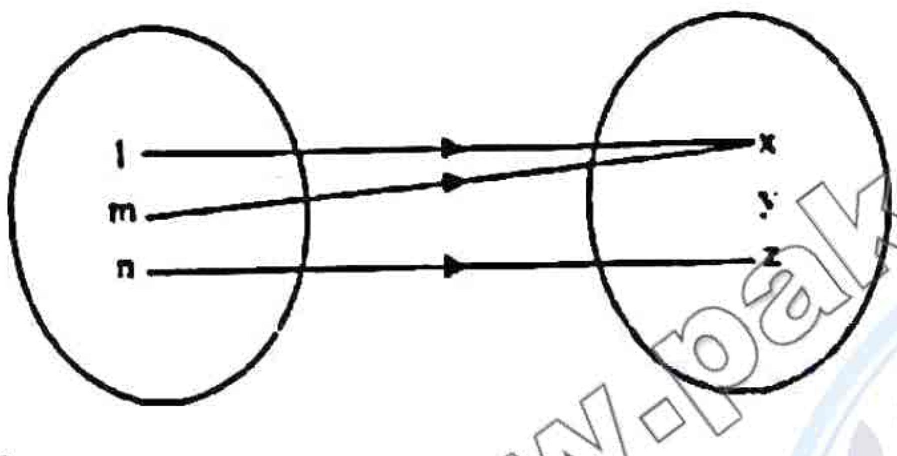
Here $f = \{(1,a), (2,b), (3,c)\}$

Dom $f = \{1,2,3\} = A$

Ran $f = \{a,b,c\} = B$

f is one-one as well as onto function. so f is bijective function.

iv)



Solution:-

Here $f = \{(l,x), (m,y), (n,z)\}$

Dom $f = \{l, m, n\} = A$

Ran $f = \{x, z\} \subset B$

So f is into function

Q4. Find the inverse of the following relations. Tell whether each relation and its inverse is a function or not:-

i) $\{(2,1), (3,2), (4,3), (5,4), (6,5)\}$

Solution:-

Let $r = \{(2,1), (3,2), (4,3), (5,4), (6,5)\}$

r is a function because no repitition in domain.

$r^{-1} = \{(2,1), (2,3), (3,4), (4,5), (5,6)\}$

r^{-1} is a function because no repitition in domain.

ii) $\{(1,3), (2,5), (3,7), (4,9), (5,11)\}$

Solution:-

Let $r = \{(1,3), (2,5), (3,7), (4,9), (5,11)\}$

r is a function because no repitition in domain.

$r^{-1} = \{(2,1), (3,2), (4,3), (5,4), (6,5)\}$

r^{-1} is a function because no repitition in domain.

iii) $\{(x,y) \mid y = 2x+3,\ x \in \mathbb{R}\}$

Solution:-

Let $r = \{(x,y) \mid y = 2x+3,\ x \in \mathbb{R}\}$

r is a function because for each value of $x \in \mathbb{R}$ there is a unique value of y

(i.e., no repitition in domain.)

$r^{-1} = \{(x,y) \mid x = 2y+3,\ x \in \mathbb{R}\}$

$= \{(x,y) \mid y = \frac{x-3}{2},\ x \in \mathbb{R}\}$

r^{-1} is a function because for each value of $x \in \mathbb{R}$ there is a unique value of y

(i.e., no repitition in domain.)

iv) $\{(x,y) \mid y^2 = 4ax,\ x \geq 0\}$

Solution:- Let $r = \{(x,y) \mid y^2 = 4ax,\ x \geq 0\}$

$= \{(x,y) \mid y = \pm 2\sqrt{ax}, x \geq 0\}$

r is not a function because for each value of x there are two values of y. (i.e., repitition in domain)

$r^{-1} = \{(x,y) \mid x^2 = 4ay,\ x \geq 0\}$

$r^{-1} = \{(x,y) \mid y = \frac{x^2}{4a},\ x \geq 0\}$

r^{-1} is a function because for each value of x there is a unique value of y

(i.e., no repitition in domain.)

v) $\{(x,y) \mid x^2+y^2 = 9,\ |x| \leq 3,\ |y| \leq 3\}$

Solution:-

Let $r = \{(x,y) \mid x^2+y^2 = 9,\ |x| \leq 3,\ |y| \leq 3\}$

$r = \{(x,y) \mid y = \pm\sqrt{9-x^2},\ |x| \leq 3,\ |y| \leq 3\}$

r is not a function because for each value of x there exist two values of y.

(i.e., repitition in domain.)

Let $r^{-1} = \{(x,y) \mid x^2+y^2 = 9,\ |x| \leq 3,\ |y| \leq 3\}$

$r^{-1} = \{(x,y) \mid y = \pm\sqrt{9-x^2},\ |x| \leq 3,\ |y| \leq 3\}$

r^{-1} is not a function because for each value of x there exist two values of y.

(i.e., repitition in domain.)

Binary Operations

Let G be a non-empty set then binary operation on G (is a function) denoted by $*$ (read as star), and defined as $* : G \times G \longrightarrow G$

i.e., for all $a, b \in G$ $a * b \in G$

Remember, i) $(G, *)$ will be a non-empty set.

ii) If $*$ is a binary operation on G, then G is said to be closed under binary operation $*$

Example 1. Ordinary addition, multiplication are operations on N i.e., N is closed w.r.t ordinary addition and multiplication

$\because \forall a, b \in N, \ a+b \in N \wedge a \cdot b \in N$

Example 2. Ordinary addition and multiplication are operations on E, the set of even natural numbers. It is worth nothing that addition is not an operation on O, the set of odd natural numbers.

Example 3. Let

E = set of all even natural numbers

O = set of all odd natural numbers

As

$E + E = E$

(sum of two even numbers is also an even number)

$E + O = O$

(sum of an even number and odd number is an odd no.)

$O + O = E$

(sum of two odd numbers is an even number)

shortly we have table

so $\{E, O\}$ is closed under '+'

$\oplus$	E	O
E	E	O
O	O	E

Example 4. The set $G = \{1, -1, i, -i\}$ where $i = \sqrt{-1}$ is closed w.r.t '×' but not w.r.t '+'

Solution:-

$\forall a, b \in G$

$a + b \notin G$

$\because 1, -1 \in G \rightarrow 1 + (-1) = 0 \notin G$

also $1, i \in G \rightarrow 1 + i \notin G$

so G is not closed w.r.t '+'

Now we construct '×' table as

$\otimes$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

From the table it is verified that $a, b \in G \rightarrow a \cdot b \in G$

so G is closed w.r.t '×'.

Example 5. Let $G = \{1, \omega, \omega^2\}$ then show that G is closed w.r.t '×' but not w.r.t '+'

Solution:-

Here $\forall a, b \in G \rightarrow a + a \notin G$

$\because 1, \omega \in G \rightarrow 1 + \omega = -\omega^2 \notin G$

also $\omega, \omega^2 \in G \rightarrow \omega + \omega^2 = -1 \notin G$

$(\because 1 + \omega + \omega^2 = 0 \rightarrow \omega + \omega^2 = -1, \ 1 + \omega = -\omega^2)$

so G is not closed w.r.t '+'

Now we construct '×' table as

$\otimes$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

It is clear from table it is clear that $\forall a, b \in G \rightarrow a \cdot b \in G$

so G is closed w.r.t '×'

Residue classes Modulo n under '$*$'

consider '$*$' be a binary operation on a non-empty set S.

Let $a, b \in S$ Now we find $a * b$

i) If $a \dot{*} b < n$ then we take $a+b$ as ordinary sum $a \dot{*} b$ of a and b.

ii) If $a \dot{*} b \geqslant n$ then we take

$$a \dot{*} b = r$$

where r is the remainder obtained after dividing $a+b$ by n. then binary operation '$\dot{*}$' is called addition modulo 'n'.

Example 6. Give the table for addition of elements of the set of residue classes modulo 5.

Solution:-

⊕	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Here $n = 5$

As $1+4 = 5$

But for modulo 5

$1+4 \equiv 0$

$\because 5 \overline{\smash{)}\,5}$, quotient 1, remainder 0 so $1+4 \equiv 0$

$\because 2+4 = 6$ But for modulo 5

$2+4 \equiv 1$ $\because 5 \overline{\smash{)}\,6}$, quotient 1, remainder 1

As $3+4 = 7$ But for modulo 5

$3+4 \equiv 2$ $5 \overline{\smash{)}\,7}$, quotient 1, remainder 2

$4+4 = 8$ But for modulo 5

$4+4 \equiv 3$ $5 \overline{\smash{)}\,8}$, quotient 1, remainder 3

* In set theory, the symbol $\equiv$ is called congruence symbol and used to say " is identical with".

Example 7. Give the table for addition of elements of the set of residue classes modulo 4.

Solution:-

⊕	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Example 8. Give the table for multiplication of elements of the set of residue classes modulo 4.

Solution:-

⊗	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Example 9. Give the table for multiplication of elements of the set of residue classes modulo 8.

Solution:-

⊗	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

Properties of Binary Operations

Commutativity:-

If $\forall\, a, b \in S$ $\quad a \dot{*} b = b \dot{*} a$

then $\dot{*}$ is called commutative in S

Associativity:-

If $\forall\, a, b, c \in S$

$$a \dot{*} (b \dot{*} c) = (a \dot{*} b) \dot{*} c$$

then $\dot{*}$ is called associative in S

Existence of an identity element:-

If $\forall\, a \in S$ there exists $e \in S$ such that $a \dot{*} e = e \dot{*} a = a$

then e is called identity w.r.t $\dot{*}$

Existence of inverse of each element:-

If $\forall\, a \in S$ $\exists\, a' \in S$

such that

$a * a' = a' * a = e$

then a' is called inverse of a w.r.t $*$

Theorem:- In a set S having a binary operation $*$, a left identity and a right identity are the same.

Proof:- Let $e', e'' \in S$

Suppose e' be left identity and $e'' \in S$. By def

$e' * e'' = e'' \longrightarrow (i)$

Also suppose e'' be right identity and $e' \in S$. By def

$e' * e'' = e' \longrightarrow (ii)$

By (i) and (ii) $e'' = e'$

Hence left and right identities are same.

Theorem:- In a set S having an associative binary operation, left inverse of an element is equal to its right inverse.

Proof:- Let $a, a', a'' \in S$

suppose a' be left inverse of a and a'' be right inverse of a. Now

$a' * (a * a'') = a' * e$

($\because a''$ is right inverse of a)

$\Rightarrow a' * (a * a'') = a' \longrightarrow (i)$

Also

$(a' * a) * a'' = e * a''$

($\because a'$ is left inverse of a)

$\Rightarrow (a' * a) * a'' = a'' \longrightarrow (ii)$

Given

$a' * (a * a'') = (a' * a) * a''$

By (i) and (ii)

$\Rightarrow a' = a''$

Hence left and right inverses are equal.

Example 10. Let $A = \{1, 2, 3, \ldots, 20\}$, the set of first 20 natural nos.

Ordinary addition is not a binary operation on A

$\because 10 + 25 = 25 \notin A$ not closed w.r.t '+'

Example 11. Addition and multiplication are commutative and associative operations on the sets $N, Z, Q, \mathbb{R}$

e.g., $4 \times 5 = 5 \times 4$

$2 + (3 + 5) = (2 + 3) + 5$ etc

Exercise 2.7

Q1. Complete the table, indicating by a tick mark those properties which are satisfied by the specified set of numbers.

Property ↓ / Set of numbers →		Natural	Whole	Integers	Rational	Reals
Closure	⊕	✓	✓	✓	✓	✓
	⊗	✓	✓	✓	✓	✓
Associative	⊕	✓	✓	✓	✓	✓
	⊗	✓	✓	✓	✓	✓
Identity	⊕		✓	✓	✓	✓
	⊗	✓	✓	✓	✓	✓
Inverse	⊕			✓	✓	✓
	⊗					
Commutative	⊕	✓	✓	✓	✓	✓
	⊗	✓	✓	✓	✓	✓

Q2. What are the field axioms? In what respect does the field of real numbers differ from that of complex numbers?

Solution:- A non-empty set F is called field if

i. It is abelian group under '+'

ii. Non zero elements of F from abelian group under 'x'

iii. Distributive laws held i-e.,

$a\cdot(b+c) = a\cdot b + a\cdot c$

& $(a+b)\cdot c = a\cdot c + b\cdot c$

Also set of real no's is subfield of set of complex numbers.

Q3. Show that the adjoining table is that of multiplication of the elements of the set of residue classes modulo 5.

※	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Solution:-

The zero's in C_2 and R_2 are obtained by multiplication of 1,2,3,4 with '0'.

→ It is a 'x' table.

every element is less than 5. So the table is a 'x' table of the set of elements residue classes modulo 5.

Q4. Prepare a table of the elements of the set of residue classes modulo 4.

Solution:-

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Clearly $\{0,1,2,3\}$ is the set of residues classes modulo 4.

Q5. Which of the following binary operations shown in tables (a) and (b) is commutative?

$*$	a	b	c	d
a	a	c	b	d
b	b	c	b	a
c	c	d	b	c
d	a	a	b	b

(a)

$*$	a	b	c	d
a	a	c	b	d
b	c	d	b	a
c	b	b	a	c
d	d	a	c	d

(b)

Solution:-

In table (a)

$a * c = b \longrightarrow (i)$

and $c * a = c \longrightarrow (ii)$

By (i) and (ii) $a * c \neq c * a$

so Binary operation is not commutative.

In table (b)

$a * b = c \longrightarrow (i)$ and $b * a = c \longrightarrow (ii)$

By (i) and (ii)

$a * b = b * a$

so binary operation is commutative.

Q6. Supply the missing elements of the third row of the given table so that the operation $*$ may be associative.

$*$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	.	.	.	.
d	d	c	c	d

Solution:-

we want to find $c*a$, $c*b$, $c*c$, $c*d$ from table

$\because c = d * b$

$c * a = (d * b) * a$,

$= d * (b * a)$ ($\because$ Associative)

$c * a = d * b = c \rightarrow c * a = c$

Also $c = d * b$

$\rightarrow c * c = (d * b) * c$

$= d * (b * c)$

$= d * c = c$

$\rightarrow c * c = c$

Again $c = d * b$

$\rightarrow c * b = (d * b) * b$

$= d * (b * b)$

$= d * a = d$

$\rightarrow c * b = d$

Also $c = d * b$

$\rightarrow c * d = (d * b) * d$

$= d * (b * d)$

$= d * d = b$

$\rightarrow c * d = b$

so third row will be completed as

c	c	d	c	d

Q7. What operation is represented by the adjoining tables? Name the identity element of the relevant set, if it exists. Is the operation associative? Find the inverses of 0, 1, 2, 3 if they exist?

$*$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Solution:-

i) The operation used the set of residue class mod 4 is '+'

ii) The identity element is zero.
$\because 0+0=0,\ 0+1=1,\ 0+2=2,\ 0+3=3$

iii) The operation is associative

e.g., $(1+2)+3 = 1+(2+3)$

$3+3 = 1+1 \rightarrow 2=2$

similarly it can be verified for any other choice of elements.

iv) $\because 1+3 = 3+1 = 0$ 1 and 3 are inverse of each other.

$2+2=0$ also

$0+0=0$

Groups

"Groupoid":- A non-empty set which is closed under given binary operation $\ast$ is called groupoid. It is denoted as $(S, \ast)$

Example 1. The $\{E, O\}$ is closed under addition

$\because E+E=E \quad ; \quad O+E=O$

$E+O=O \quad ; \quad O+O=E$

$\therefore \{E ; O\}$ is groupoid

Example 2. The set of Natural numbers is not closed under operation of subtraction

For $4, 5 \in N$, $4-5=-1 \notin N$

Thus $(N, -)$ is not groupoid under '$-$'

Semi-group:- A non-empty set is called semi group if

i. it is closed under binary operation given.

ii. The binary operation is associative.

Example 4. The set of natural nos 'N' under binary operation '+' is semi-group operation '+' is semi-group

i. i.e., B.O '+' is defined in N

ii. for any three elements

$a, b, c \in N$

$(a+b)+c = a+(b+c)$

i.e., Associative law holds.

$\rightarrow$ Both conditions for semi-group are satisfied.

Monoid:- A non empty set is called Monoid.

i. It is closed w.r.t given binary operation. $\ast$

ii. Binary operation $\ast$ is associative

iii. The set has identity element w.r.t Binary operation $\ast$

Example:- If $Z' = \{0, 1, 2, 3, \ldots\}$

i. Z' is closed w.r.t '+'

ii. Binary operation '+' is associative

iii. '0' is identity element w.r.t Binary operation '+'.

$\therefore$ Given set is Monoid.

Group:- A non-empty set G is called a group w.r.t Binary operation $\ast$

i. It is closed under Binary operation $\ast$ if

i.e, $\forall a, b \in G;\ a \ast b \in G$

ii. Binary Operation is associative

$\forall a, b, c \in G;\ (a \ast b) \ast c = a \ast (b \ast c)$

iii. G has identity element w.r.t Binary operation $\ast$

i.e, $\forall\ a \in G \ \exists\ e \in G$ s.t

$a \ast e = e \ast a = a$ then

e is identity element.

iv. Every element of G

has an inverse in G w.r.t Binary operation i.e.,

$a \circledast a' = a' \circledast a = e$

where $a' \in G$ is called inverse of $a \in G$ w.r.t Binary operation $\circledast$.

Abelian group:- A group G under Binary operation $\circledast$ is called abelian group if Binary operation is commutative i.e., $\forall a, b \in G$;

$a \circledast b = b \circledast a$

Finite group:- A group G having finite number of elements is called finite group.

Infinite group:- A group G having infinite number of elements is called infinite group.

Example:- If $S = \{0, 1, 2\}$ The operation '+' performed in table show that S is abelian under '+'

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Solution:-

i) Clearly S is closed under '+'

ii) The operation is associative

i.e., $0+(1+2) = 0+0 = 0$

$(0+1)+2 = 1+2 = 0$

iii) Identity element '0' exists

iv) Inverses of all elements, exist, e.g.,

$0+0=0$, $1+2=0$, $2+1=0$

$\rightarrow 0^{-1}=0$, $1^{-1}=2$, $2^{-1}=1$

Also clearly $1+2=0=2+1$

Hence the result.

Example:- Consider the set $S=\{1,-1,i,-i\}$ set up multiplication table and show that the set is an abelian group under 'x'.

$\otimes$	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

Solution:-

i) Clearly S is closed w.r.t 'x'

ii) 'x' is associative

iii) '1' is identity element

iv) Inverse of each element exists.

i.e., $1^{-1} = -1$, $i^{-1} = -i$

$-1^{-1} = 1$, $-i^{-1} = i$

v) 'x' is commutative in case of C. Hence the result.

Example:- Let G be the set of all 2x2 non-singular real matrices, then under the real multiplication of matrices, G is a non-abelian group.

Solution:-

Let G be the set of all 2x2 non singular matrices over R. So let AB is also matrices of same order $\in G$.

i) 'x' is closed in G

ii) Matrix multiplication is associative

$(A\cdot B).C = A(B\cdot C)$ $\forall A,B,C \in G$

iii) The unit matrix $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \in G$

I is identity element in G

iv) The inverse of each element exists in G

$\because$ $\forall$ $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

$$A^{-1} = \frac{adj A}{|A|} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad-bc} \in G$$

All four conditions are satisfied. Hence G is group under '$\times$'.

$\because AB \neq BA$

$\Rightarrow G$ is not abelian group under '$\times$'.

Left Cancellation Law

If a, b, c are elements of group G then

$ab = ac \Rightarrow b = c$

Proof:-

$ab = ac$

$\Rightarrow a^{-1}(ab) = a^{-1}(ac)$

$\Rightarrow (a^{-1}a)b = (a^{-1}a)c$ ($\because$ Associative Law)

$\Rightarrow eb = ec$ ($\because a^{-1}a = e$)

$\Rightarrow b = c$ proved

Right Cancellation Law

If a, b, c are elements of group G then

$ba = ca \Rightarrow b = c$

Proof:-

$ba = ca$

$\Rightarrow (ba)a^{-1} = (ca)a^{-1}$

$\Rightarrow b(aa^{-1}) = c(aa^{-1})$ ($\because$ Assoc. Law)

$\Rightarrow be = ce$ ($\because aa^{-1} = e$)

$\Rightarrow b = c$ proved.

Reversal Law of inverses

If a, b are elements of a group G, then show that

$(ab)^{-1} = b^{-1}a^{-1}$

Proof:-

$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1}$ (Assoc. Law)

$= aea^{-1}$

$= aa^{-1}$

$= e$

Also, $(b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}a)b$

$= b^{-1}eb$

$= b^{-1}b$

$= e$

$\Rightarrow ab$ and $b^{-1}a^{-1}$ are inverse of each other.

$\Rightarrow$ Inverse of ab is $b^{-1}a^{-1}$.

i.e., $(ab)^{-1} = b^{-1}a^{-1}$

Solution of linear equations

a, b being elements of a group G, solve the following equations:

i) $ax = b$ ii) $xa = b$

Solution:- i) $ax = b$

$\Rightarrow a^{-1}(ax) = a^{-1}b$

$\Rightarrow (a^{-1}a)x = a^{-1}b$ (Assoc. Law)

$ex = a^{-1}b$ ($\because a^{-1}a = e$)

$\Rightarrow x = a^{-1}b$

ii) $xa = b$

$\Rightarrow (xa)a^{-1} = ba^{-1}$

$\Rightarrow x(aa^{-1}) = ba^{-1}$ (Assoc. Law)

$\Rightarrow xe = ba^{-1}$ ($\because aa^{-1} = e$)

$\Rightarrow x = ba^{-1}$

Theorem:- If $(G, *)$ is a group with e its identity, then e is unique.

Proof:- we suppose contrary that e and e' be two different identities in G. Now $\forall\ a \in G$

$a * e = a \rightarrow (i)$

$a * e' = a \rightarrow (ii)$

so by (i) and (ii)

$a * e = a * e'$

By left cancellation law

$e = e'$ (a contradiction)

Hence identity of a group is always true.

Theorem:- If $(G, *)$ is a group and $a \in G$, there is a unique inverse.

Proof:- We suppose contrary that a' and a'' be two different inverses of $a \in G$ where G being group. then

$a * a' = e \longrightarrow (i)$

$a * a'' = e \longrightarrow (ii)$

By (i) and (ii)

$a * a' = a * a''$

By Left cancellation law

$a' = a''$ (a contradiction)

Hence inverse of each element of a group G is unique.

Exercise 2.8

Q1. Operation $\oplus$ performed on the two-member set $G = \{0,1\}$ is shown in the adjoining table. Answer the following questions:-

$\oplus$	0	1
0	0	1
1	1	0

Solution:-

i) Name the identity element if it exists?

Here 0 is identity in G.

ii) What is the inverse of 1?

$\because 1+1 = 0$ (i.e., identity element)

so inverse of 1 is 1

iii) Is the set G, under the given operation a group? Abelian or nonAbelian?

* G is closed under '+'
* G is associative w.r.t '+'
* $0 \in G$ is identity w.r.t '+'
* Inverse of each element exists in G i.e., $0+0=0$ so $0^{-1}=0$, $1+1=0$ so $1^{-1}=1$
* Commutative law holds in G i.e., $1+0 = 0+1$

$\rightarrow$ so G is abelian group under '+'

Q2. The operation $\oplus$ as performed on the set $\{0,1,2,3\}$ is shown in the adjoining table, show that the set is an Abelian group?

Solution:-

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

i) S is closed under '+' (It is clear from table)

ii) It is clear the set is associative w.r.t '+'

iii) '0' is identity element

iv) Each element has inverse

$\because 1+3 = 3+1 = 0$ and

$0+0 = 0$ and $2+2 = 0$

v) $\forall\ 1, 0 \in G = \{0,1,2,3\}$, $1+0 = 0+1 = 1$

$\rightarrow$ G is abelian

Q3. For each of the following sets, determine whether or not the set forms a group with respect to the indicated operation.

i) The set of rational no's. $\times$

Solution:-

Q = set of rational no's.
is not a group w.r.t '$\times$'
$\because$ inverse of 0 w.r.t '$\times$' does not exist.

ii) The set of rational no's. $+$

Solution:-

$(Q, +)$ is a group

iii) The set of +ive rational numbers $\times$

Solution:-

It is a group w.r.t '$\times$'

iv) The set of integers $+$

Solution:-

It is a group w.r.t '+'

v) The set of integers $\times$

Solution:-

It is not a group.
$\because$ '$\times$' inverse of '0' does not exist.

Q4. Show that the adjoining table represents the sums of the elements of the set $\{E, O\}$. What is the identity element of the set? Show that the set is an abelian group.

$\oplus$	E	O
E	E	O
O	O	E

Solution:-

i) $E + E = E$ (even)
$E + O = O$ (odd)
$O + O = E$ (Even)

Here E is identity element.

ii) * Table shows the set satisfies the closure law w.r.t '+'. $\because$ all elements of table $\in \{E, O\}$

* The set is associative under '+'

$(O+E)+O = O+(E+O)$
$O+O = O+O$
$E = E$

* E is identity $\in \{E, O\}$

* Each element has inverse
$(O+E = E+O$ and $E+E = O$ and $O+O = O)$

* commutative law holds
$(O+E = E+O)$

So set $\{E, O\}$ is abelian group.

Q5. Show that the set $\{1, \omega, \omega^2\}$, when $\omega^3 = 1$, is an abelian group w.r.t ordinary multiplication.

Solution:- Let $S = \{1, \omega, \omega^2\}$

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

i) Clearly from table S is closed under '$\times$'

ii) $1, \omega, \omega^2 \in S$

$(1 \cdot \omega) \cdot \omega^2 = 1 \cdot (\omega \cdot \omega^2)$
$\omega \cdot \omega^2 = 1 \cdot \omega^3$
$\omega^3 = \omega^3 \rightarrow 1 = 1$

$\therefore$ Assoc. Law holds under '$\times$'

iii) 1 is identity element under '$\times$'

iv) Inverse of each element exists.

As $1\times 1 = 1 \rightarrow 1^{-1} = 1$

$\omega \times \omega^2 = 1 \rightarrow \omega^{-1} = \omega^2$

$\omega^2 \times \omega = 1 \rightarrow (\omega^2)^{-1} = \omega$

v) Commutative law holds under '$\times$' (clear from table)

$\rightarrow$ S is an abelian group under '$\times$'

Q6. If G is a group under the operation $*$ and $a, b \in G$, find the solutions of the equations $a * x = b$; $x * a = b$

Solution:-

For $a * x = b$

$\rightarrow a^{-1}(a * x) = a^{-1} * b$

$\rightarrow (a^{-1}a) * x = a^{-1} * b$ (Assoc. law)

$\rightarrow e * x = a^{-1} * b \quad \because a^{-1}a = e$

$x = a^{-1} * b$

For $x * a = b$

$\rightarrow (x * a)a^{-1} = b * a^{-1}$

$\rightarrow x * (aa^{-1}) = b * a^{-1}$ (Assoc. law)

$\rightarrow x * e = b * a^{-1} \quad (\because aa^{-1} = e)$

$\rightarrow x = b * a^{-1}$

Q7. Show that the set consisting of the elements of the form $a + \sqrt{3}b$ (a, b being rational), is an abelian group w.r.t addition.

Solution:-

Let $G = \{a + b\sqrt{3} \mid a, b \in Q\}$

i) G is closed w.r.t '+'

Let $x = a + b\sqrt{3}$ $\quad a, b, c, d \in Q$

$y = c + d\sqrt{3}$

So $x, y \in G$

Now

$x + y = (a + \sqrt{3}b) + (c + \sqrt{3}d)$

$= (a + c) + \sqrt{3}(b + d) \in G$

$\because (a + c), (b + d) \in Q$

ii) Associative law w.r.t '+' holds in G

Let $x = a + \sqrt{3}b$, $y = c + \sqrt{3}d$

$z = e + \sqrt{3}f \quad a, b, c, d, e, f \in Q$

then $x, y, z \in G$

$x + (y + z) = (a + \sqrt{3}b) + [(c + \sqrt{3}d) + (e + \sqrt{3}f)]$

$= (a + \sqrt{3}b) + [(c + e) + \sqrt{3}(d + f)]$

$= (a + c + e) + \sqrt{3}(b + d + f)$

$= [(a + c) + \sqrt{3}(b + d)] + (e + \sqrt{3}f)$

$= (x + y) + z$

iii) $0 + \sqrt{3}\,0 \in G$ is identity w.r.t '+'

iv) For $(a + \sqrt{3}b) \in G$, $\exists -a - \sqrt{3}b \in G$

$\because (a + \sqrt{3}b) + (-a - \sqrt{3}b) = 0 + \sqrt{3}0$

i.e., inverse of each element of G exists.

v) Commutative law w.r.t '+' holds in G.

i.e., $x + y = y + x \quad \forall x, y \in G$

As $x = a + \sqrt{3}b$, $y = c + \sqrt{3}d$

$x + y = (a + \sqrt{3}b) + (c + \sqrt{3}d)$

$= (a + c) + \sqrt{3}(b + d)$

$= (c + a) + \sqrt{3}(d + b)$

$= y + x$

$\rightarrow$ G is abelian group.

Q8. Determine whether, $(P(S), *)$, where $*$ stands for intersection is a semi-group a monoid or neither. If it is monoid, specify its identity.

Solution:- Given that

$P(S) =$ Power set of set S.

Here $*$ means $\cap$

i) $P(S)$ is closed w.r.t $*$

Let $S_1, S_2 \in P(S)$

then $S_1 * S_2 \in P(S)$

i.e., $S_1 \cap S_2 \in P(S)$

ii) $P(S)$ holds associative law w.r.t $*$.

Let $S_1, S_2, S_3 \in P(S)$

then $S_1 * (S_2 * S_3) = (S_1 * S_2) * S_3$

$\rightarrow S_1 \cap (S_2 \cap S_3) = (S_1 \cap S_2) \cap S_3$

iii) $P(S)$ has no identity element.

so $P(S)$ is semi-group under $*$ But $P(S)$ is not monoid under $*$.

Q9. Complete the following table to obtain a semi-group under $*$

$*$	a	b	c
a	c	a	b
b	a	b	c
c	–	–	a

Solution:-

From table

$a * a = c \longrightarrow (i)$

Now

$c * a = (a * a) * a$ By (i)

$= a * (a * a)$ ($\because$ Assoc. Law)

$= a * c$

$= b$ (from table)

$\rightarrow c * a = b$

Also

$c * b = (a * a) * b$ By (i)

$= a * (a * b)$

$= a * a$ (from table)

$= c$ By (i)

$\rightarrow c * b = c$

so third row becomes as

c	b	c	a

Q10. Prove that all 2x2 non-singular matrices over the real field form a non-abelian group under multiplication.

Solution:-

See Example at Page # 46