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MATHEMATICS 1st YEAR

UNIT # 10

TRIGONOMETRIC IDENTITIES

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Sherazi Mathematics

اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Distance Formula:-

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points. If "d" denotes the distance between them, then

$$d = |PQ| = \sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$$

Example:- Find distance between the following points:

i) $A(3,8)$, $B(5,6)$

ii) $P(\cos x, \cos y)$, $Q(\sin x, \sin y)$

Solution:- (i) $A(3,8)$, $B(5,6)$

$$|AB| = \sqrt{(6-8)^2+(5-3)^2} = \sqrt{(-2)^2+(2)^2}$$

$$= \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

(ii) $P(\cos x, \cos y)$, $Q(\sin x, \sin y)$

$$|PQ| = \sqrt{(\sin y - \cos y)^2 + (\sin x - \cos x)^2}$$

$$= \sqrt{\sin^2 y + \cos^2 y - 2\sin y\cos y + \sin^2 x + \cos^2 x - 2\sin x\cos x}$$

$$\Rightarrow |PQ| = \sqrt{1 - 2\sin y\cos y + 1 - 2\sin x\cos x}$$

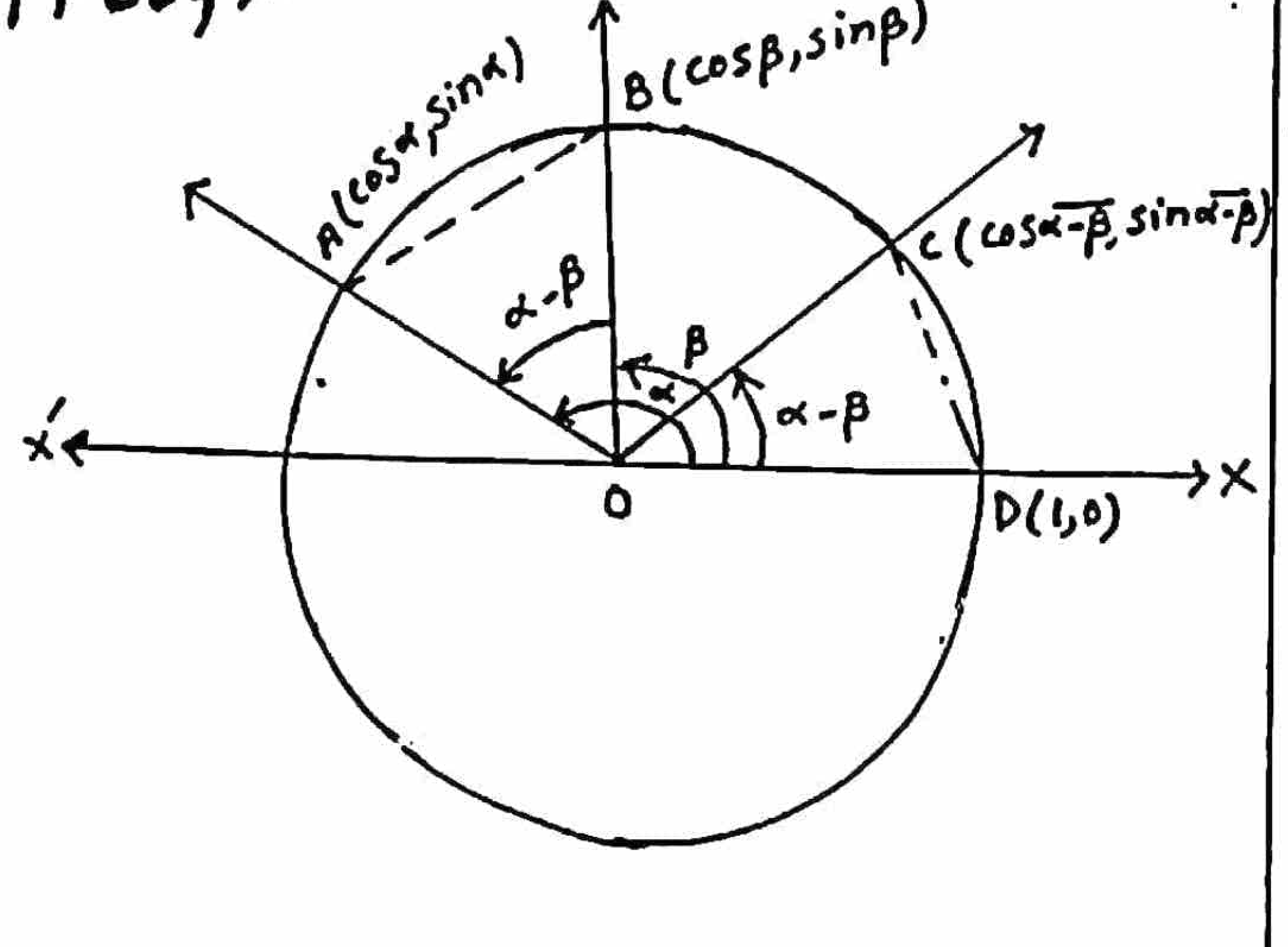
$$\Rightarrow |PQ| = \sqrt{2 - 2(\sin x\cos x + \sin y\cos y)}$$

Fundamental Law of trigonometry

Let α and β any two angles (real numbers) then $\boxed{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta}$

which is called the "Fundamental Law of trigonometry".

Proof:-



consider a unit circle at O.

where $\angle AOD = \alpha$, $\angle BOD = \beta$

$$\angle AOB = \angle COD = \alpha - \beta$$

Now $\triangle AOB$ and $\triangle COD$ are congruent

then $|AB| = |CD|$

$$\Rightarrow |AB|^2 = |CD|^2$$

using distance formula, we have

$$(\cos\alpha-\cos\beta)^2 + (\sin\alpha-\sin\beta)^2 = (\cos\overline{\alpha-\beta}-1)^2 + (\sin\overline{\alpha-\beta}-0)^2$$

$$\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta$$
$$= \cos^2\overline{\alpha-\beta} + 1 - 2\cos\overline{\alpha-\beta} + \sin^2\overline{\alpha-\beta}$$

$$\cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$
$$= \cos^2(\alpha-\beta) + \sin^2(\alpha-\beta) + 1 - 2\cos(\alpha-\beta)$$

$$1 + 1 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 1 + 1 - 2\cos(\alpha-\beta)$$

$$\Rightarrow 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 2 - 2\cos(\alpha-\beta)$$

subtract 2 from both sides

$$\Rightarrow -2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = -2\cos(\alpha-\beta)$$

$$\Rightarrow \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta) \quad \text{'}\div\text{' by } -2$$

or $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Hence proved.

Note:- we have proved this law for $\alpha > \beta > 0$, it is true for all values of α and β

Example 1. Find the value of $\cos\frac{\pi}{12}$.

Solution:- $\cos\frac{\pi}{12}$

$$\because \frac{\pi}{12} = \frac{180^\circ}{12} = 15^\circ = 45^\circ - 30^\circ$$
$$= \frac{\pi}{4} - \frac{\pi}{6}$$

so $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$$\Rightarrow \cos\frac{\pi}{12} = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$\cos\frac{\pi}{12} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\because \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Deductions from fundametal Law :-

1) we know that

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Put $\alpha = \frac{\pi}{2}$, we get

$\cos(\frac{\pi}{2}-\beta) = \cos\frac{\pi}{2}\cos\beta + \sin\frac{\pi}{2}\sin\beta$

$= (0)\cos\beta + (1)\sin\beta$

$\rightarrow \boxed{\cos(\frac{\pi}{2}-\beta) = \sin\beta}$ $\because \cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$

2) we know that

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Put $\beta = -\frac{\pi}{2}$, we get

$\cos(\alpha-(-\frac{\pi}{2})) = \cos\alpha\cos(-\frac{\pi}{2}) + \sin\alpha\sin(-\frac{\pi}{2})$

$\rightarrow \cos(\alpha+\frac{\pi}{2}) = \cos\alpha\cos\frac{\pi}{2} - \sin\alpha\sin\frac{\pi}{2}$

$= \cos\alpha(0) - \sin\alpha(1)$

$\cos(\alpha+\frac{\pi}{2}) = -\sin\alpha$ $\because \cos(-\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$, $\sin(-\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1$

$\rightarrow \boxed{\cos(\frac{\pi}{2}+\alpha) = -\sin\alpha}$

3) we know that

$\cos(\frac{\pi}{2}-\beta) = \sin\beta$

Put $\beta = \frac{\pi}{2}+\alpha$, we get

$\rightarrow \cos\left(\frac{\pi}{2}-(\frac{\pi}{2}+\alpha)\right) = \sin(\frac{\pi}{2}+\alpha)$

$\rightarrow \cos(\frac{\pi}{2}-\frac{\pi}{2}-\alpha) = \sin(\frac{\pi}{2}+\alpha)$

$\rightarrow \cos(-\alpha) = \sin(\frac{\pi}{2}+\alpha)$

$\rightarrow \cos\alpha = \sin(\frac{\pi}{2}+\alpha)$

$\rightarrow \boxed{\sin(\frac{\pi}{2}+\alpha) = \cos\alpha}$

4) we know that

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

replacing β by $-\beta$ we get

$\cos[\alpha-(-\beta)] = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$

$\rightarrow \boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$

$(\because \cos(-\beta) = \cos\beta,\ \sin(-\beta) = -\sin\beta)$

5) we know that

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Replacing α by $\frac{\pi}{2}+\alpha$, we get

$\cos\left((\frac{\pi}{2}+\alpha)+\beta\right) = \cos(\frac{\pi}{2}+\alpha)\cos\beta - \sin(\frac{\pi}{2}+\alpha)\sin\beta$

$\cos(\frac{\pi}{2}+(\alpha+\beta)) = -\sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\rightarrow -\sin(\alpha+\beta) = -(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$

$\rightarrow \boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$

$(\because \sin(\frac{\pi}{2}+\alpha) = -\cos\alpha,\ \cos(\frac{\pi}{2}+\alpha) = -\sin\alpha)$

6) we know that

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Replacing β by $-\beta$, we get

$\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$

$\boxed{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta}$

$\because \cos(-\beta) = \cos\beta,\ \sin(-\beta) = -\sin\beta$

7) we know that

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Let $\alpha = 2\pi$ and $\beta = \theta$

$\cos(2\pi-\theta) = \cos 2\pi\cos\theta + \sin 2\pi\sin\theta$

$\cos(2\pi-\theta) = (1)\cos\theta + (0)\sin\theta$

$\rightarrow \boxed{\cos(2\pi-\theta) = \cos\theta}$ $\because \cos 2\pi = 1$, $\sin 2\pi = 0$

8) we know that

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Let $\alpha = 2\pi$, $\beta = \theta$

$\sin(2\pi-\theta) = \sin 2\pi\cos\theta - \cos 2\pi\sin\theta$

$= (0)\cos\theta - (1)\sin\theta$

$\boxed{\sin(2\pi-\theta) = -\sin\theta}$

9) $\because \tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

Dividing up and down by $\cos\alpha\cos\beta$

$$= \frac{\dfrac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} + \dfrac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\dfrac{\cos\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Thus,

$$\boxed{\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}$$

10) $\because \tan(\alpha-\beta) = \dfrac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$

$$= \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

Dividing up and down by $\cos\alpha\cos\beta$

$$= \frac{\dfrac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} - \dfrac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\dfrac{\cos\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} + \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Thus

$$\boxed{\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}}$$

Trigonometric Ratios of Allied Angles:-

Allied angles

The angles associated with basic angles of measure θ to a right angle or multiple are called allied angles.

Examples: $90^\circ \pm \theta$, $180^\circ \pm \theta$, $270^\circ \pm \theta$, $360^\circ \pm \theta$ etc.

Remember some basic results of Allied angles

1) If θ is add to or subtracted from odd multiple of right angle, trigonometric ratios change into co-ratios and vice versa. i.e.,

$\sin \rightleftarrows \cos$, $\tan \rightleftarrows \cot$

$\sec \rightleftarrows \operatorname{cosec}$

$\sin \longrightarrow \cos$

$\sin(\frac{\pi}{2}-\theta) = \cos\theta$, $\sin(\frac{\pi}{2}+\theta) = \cos\theta$

$\sin(\frac{3\pi}{2}-\theta) = -\cos\theta$

$\sin(\frac{3\pi}{2}+\theta) = -\cos\theta$

$\cos \longrightarrow \sin$

$\cos(\frac{\pi}{2}-\theta) = \sin\theta$, $\cos(\frac{\pi}{2}+\theta) = -\sin\theta$

$\cos(\frac{3\pi}{2}-\theta) = -\sin\theta$, $\cos(\frac{3\pi}{2}+\theta) = \sin\theta$

$\tan \longrightarrow \cot$

$\tan(\frac{\pi}{2}-\theta) = \cot\theta$, $\tan(\frac{\pi}{2}+\theta) = -\cot\theta$

$\tan(\frac{3\pi}{2}-\theta) = \cot\theta$, $\tan(\frac{3\pi}{2}+\theta) = -\cot\theta$

2) If θ is add or subtracted from an even multiple of $\frac{\pi}{2}$, the trigonometric ratios shall remain the same.

3) So far as the sign of results is concerned, it is determined by the quadrant in which the terminal arm of the angle lies.

$\sin \longrightarrow \sin$

$\sin(\pi-\theta) = \sin\theta$, $\sin(\pi+\theta) = -\sin\theta$

$\sin(2\pi-\theta) = -\sin\theta$, $\sin(2\pi+\theta) = \sin\theta$

$\cos \longrightarrow \cos$

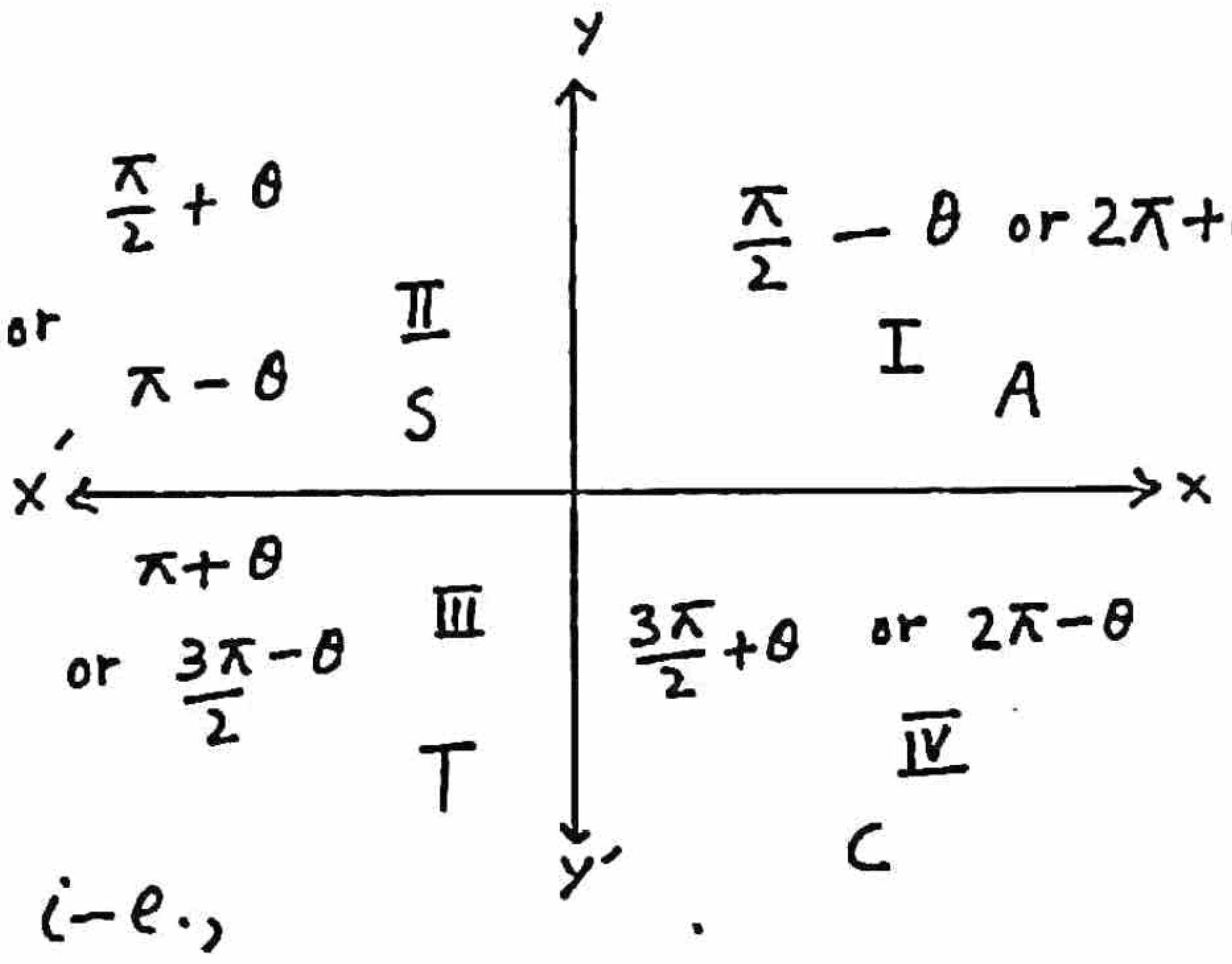
$\cos(\pi-\theta) = -\cos\theta$, $\cos(\pi+\theta) = -\cos\theta$

$\cos(2\pi-\theta) = \cos\theta$, $\cos(2\pi+\theta) = \cos\theta$

tan ⟶ tan

$\tan(\pi-\theta)=-\tan\theta$, $\tan(\pi+\theta)=\tan\theta$

$\tan(2\pi-\theta)=-\tan\theta$, $\tan(2\pi+\theta)=\tan\theta$



i.e.,

$\frac{\pi}{2}-\theta$, or $2\pi+\theta$ lies in Quad I

$\frac{\pi}{2}+\theta$ or $\pi-\theta$ lies in Quad II

$\frac{3\pi}{2}-\theta$ or $\pi+\theta$ lies in Quad III

$\frac{3\pi}{2}+\theta$ or $2\pi-\theta$ lies in Quad IV

Example 2. Without using the tables, write down the values of i) $\cos 315°$ ii) $\sin 540°$ iii) $\tan(-135°)$ iv) $\sec(-300°)$

Solution:- i) $\cos 315°$

$\cos 315° = \cos(270°+45°) = \cos(3\times90°+45°)$

$= +\sin 45°$

$\cos 315° = \frac{1}{\sqrt{2}}$ $\quad \because \cos(\frac{3\pi}{2}+\theta) = \sin\theta$

$\frac{3\pi}{2}+45°$ C

ii) $\sin 540°$

$= \sin(540°+0°) = \sin(6\frac{\pi}{2}+0°)$

$= \sin 0° = 0$ $\quad \because \sin(3\pi+\theta) = \sin\theta$

$3\pi+0°$

iii) $\tan(-135°)$

$= -\tan(135°)$ $\quad \because \tan(-\theta) = -\tan\theta$

$= -\tan(180°-45°) = -\tan(2\frac{\pi}{2}-45°)$

$= -(-\tan 45°)$ $\quad \because \tan(2\frac{\pi}{2}-\theta) = -\tan\theta$

$= -(-1) = 1$

$\pi-45°$

iv) $\sec(-300°)$

$= \sec 300°$ $\quad \because \sec(-\theta) = \sec\theta$

$= \sec(360°-60°) = \sec(4\times90°-60°)$

$= \sec(4\frac{\pi}{2}-60°)$ $\quad \because \sec(4\frac{\pi}{2}-\theta) = \sec\theta$

$= \sec 60° = \frac{1}{\cos 60°}$

$= \frac{1}{1/2} = 2$

$2\pi-\theta$ C

Example 3. Simplify

$$\frac{\sin(360°-\theta)\cos(180°-\theta)\tan(180°+\theta)}{\sin(90°+\theta)\cos(90°-\theta)\tan(360°+\theta)}$$

Solution:-

$\because \sin(360°-\theta) = -\sin\theta$

$\cos(180°-\theta) = -\cos\theta$

$\tan(180°+\theta) = \tan\theta$

$\sin(90°+\theta) = \cos\theta$

$\cos(90°-\theta) = \sin\theta$, $\tan(360°+\theta) = \tan\theta$

$90°+\theta$ $180°-\theta$ S	A $90°-\theta$ $360°+\theta$
$180°+\theta$ T	$360°-\theta$ C

Thus

$$\frac{\sin(360°-\theta)\cos(180°-\theta)\tan(180°+\theta)}{\sin(90°+\theta)\cos(90°-\theta)\tan(360°+\theta)}$$

$$= \frac{(-\sin\theta)(-\cos\theta)\tan\theta}{\cos\theta\sin\theta\tan\theta} = \frac{\sin\theta\cos\theta\tan\theta}{\cos\theta\sin\theta\tan\theta}$$

$= 1$

Exercise 10.1

Q1. Without using the tables, find the values of:

i) $\sin(-780°)$ ii) $\cot(-855°)$ iii) $\csc(2040°)$

iv) $\sec(-960°)$ v) $\tan(1110°)$ vi) $\sin(-300°)$

Solution:- i) $\sin(-780°)$

$= -\sin 780°$ $\quad \because \sin(-\theta) = -\sin\theta$

$= -\sin(720°+60°)$

$= -\sin(8\frac{\pi}{2}+60°)$

$= -\sin 60°$ $\quad \because \sin(\frac{8\pi}{2}+\theta) = \sin\theta$

$= -\frac{\sqrt{3}}{2}$

$\frac{8\pi}{2}+60°$

ii) $\cot(-855^\circ)$

$= -\cot 855^\circ$

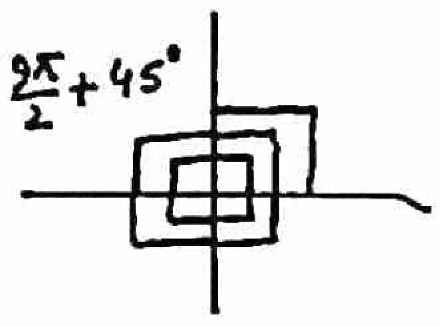
$= -\cot(810^\circ + 45^\circ)$

$= -\cot\left(9\frac{\pi}{2} + 45^\circ\right)$

$= -(-\tan 45^\circ)$

$= \tan 45^\circ = 1$

$\because \cot\left(\frac{9\pi}{2} + \theta\right) = -\tan\theta$



iii) $\operatorname{cosec} 2040^\circ$

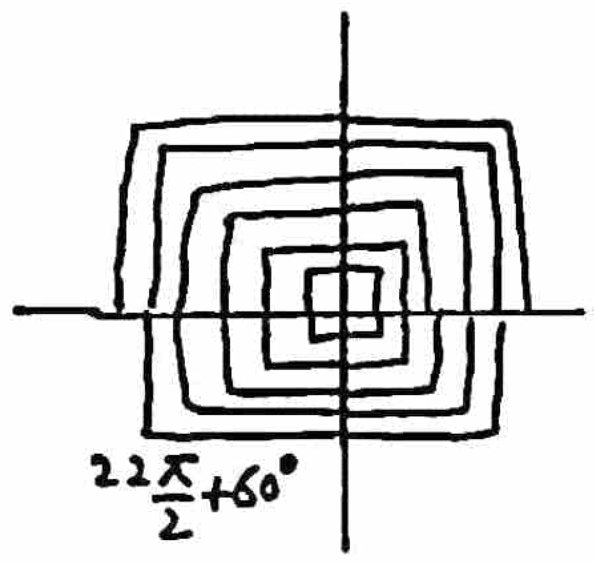
$= \operatorname{cosec}(1980^\circ + 60^\circ)$

$= \operatorname{cosec}\left(22\frac{\pi}{2} + 60^\circ\right)$

$= -\operatorname{cosec} 60^\circ$

$= -\frac{1}{\sin 60^\circ} = -\frac{1}{\sqrt{3}/2}$

$= -\frac{2}{\sqrt{3}}$ $\quad \left(\because \csc\left(\frac{22\pi}{2} + \theta\right) = -\csc\theta\right)$



iv) $\sec(-960^\circ)$ $\quad \because \sec(-\theta) = \sec\theta$

$= \sec 960^\circ$

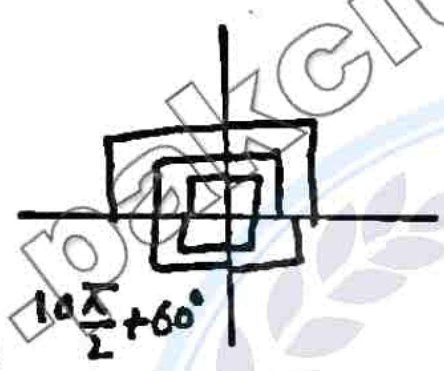
$= \sec(900^\circ + 60^\circ)$

$= \sec\left(10\frac{\pi}{2} + 60^\circ\right)$

$= -\sec 60^\circ = -\frac{1}{\cos 60^\circ}$

$= -\frac{1}{1/2} = -2$

$\because \sec\left(\frac{10\pi}{2} + \theta\right) = -\sec\theta$

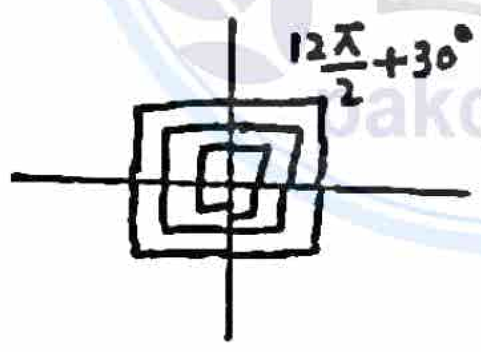


v) $\tan 1110^\circ$

$= \tan(1080^\circ + 30^\circ)$

$= \tan\left(12\frac{\pi}{2} + 30^\circ\right)$

$= \tan 30^\circ = \frac{1}{\sqrt{3}}$ $\quad \because \tan\left(\frac{12\pi}{2} + \theta\right) = \tan\theta$



vi) $\sin(-300^\circ)$ $\quad \because \sin(-\theta) = -\sin\theta$

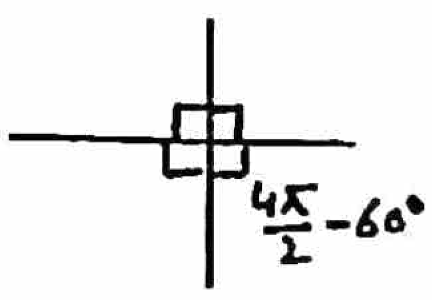
$= -\sin 300^\circ$

$= -\sin(360^\circ - 60^\circ)$

$= -\sin\left(4\frac{\pi}{2} - 60^\circ\right)$

$= -(-\sin 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$

$\because \sin\left(\frac{4\pi}{2} - \theta\right) = -\sin\theta$

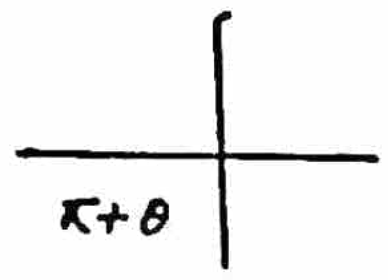


Q2. Express each of the following as a trigonometric function of an angle of positive degree measure of less then 45°.

Solution:– i) $\sin 196^\circ$

$= \sin(180^\circ + 16^\circ)$

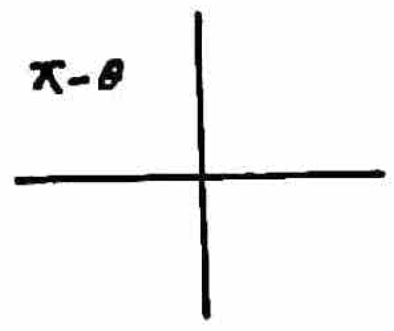
$= -\sin 16^\circ$ $\quad \because \sin(\pi + \theta) = -\sin\theta$



ii) $\cos 147^\circ$

$= \cos(180^\circ - 33^\circ)$

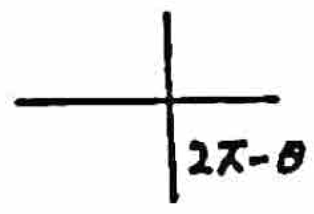
$= -\cos 33^\circ$ $\quad \because \cos(\pi - \theta) = -\cos\theta$



iii) $\sin 319^\circ$

$= \sin(360^\circ - 41^\circ)$

$= -\sin 41^\circ$ $\quad \because \sin(2\pi - \theta) = -\sin\theta$

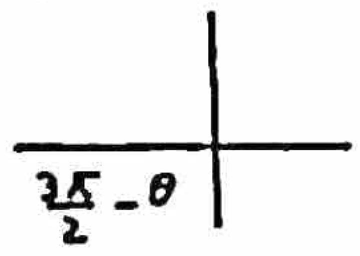


iv) $\cos 254^\circ$

$= \cos(270^\circ - 16^\circ)$

$= \cos\left(\frac{3\pi}{2} - 16^\circ\right)$ $\quad \because \cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$

$= -\sin 16^\circ$

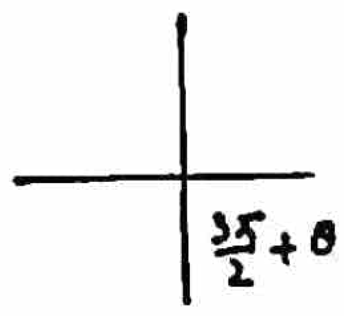


v) $\tan 294^\circ$

$= \tan(270^\circ + 24^\circ)$

$= \tan\left(\frac{3\pi}{2} + 24^\circ\right)$ $\quad \because \tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta$

$= -\cot 24^\circ$

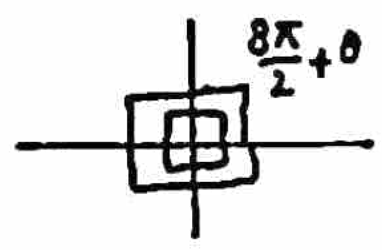


vi) $\cos 728^\circ$

$= \cos(720^\circ + 8^\circ)$

$= \cos\left(\frac{8\pi}{2} + 8^\circ\right)$ $\quad \because \cos\left(\frac{8\pi}{2} + \theta\right) = \cos\theta$

$= \cos 8^\circ$



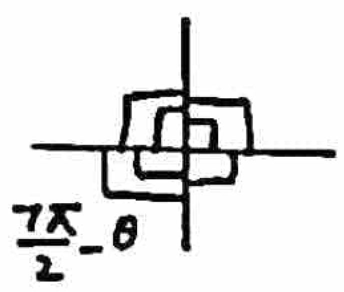
vii) $\sin(-625^\circ)$

$= -\sin 625^\circ$

$= -\sin(630^\circ - 5^\circ)$

$= -\sin\left(\frac{7\pi}{2} - 5^\circ\right)$ $\quad \because \sin\left(\frac{7\pi}{2} - \theta\right) = -\cos\theta$

$= -(-\cos 5^\circ) = \cos 5^\circ$



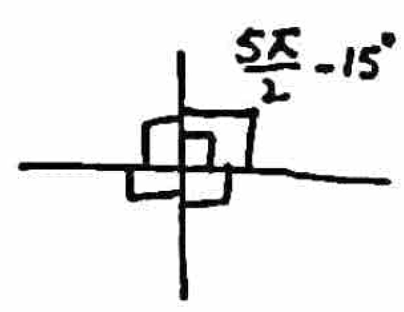
viii) $\cos(-435^\circ)$

$= \cos 435^\circ$

$= \cos(450^\circ - 15^\circ)$

$= \cos\left(\frac{5\pi}{2} - 15^\circ\right)$ $\quad \because \cos\left(\frac{5\pi}{2} - \theta\right) = \sin\theta$

$= \sin 15^\circ$

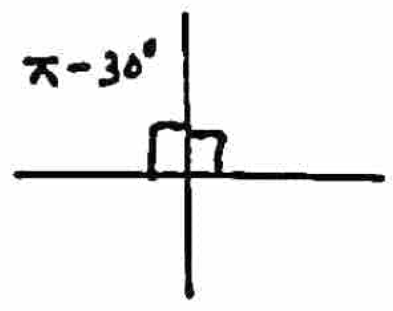


ix) $\sin 150^\circ$

$= \sin(180^\circ - 30^\circ)$

$= \sin(\pi - 30^\circ)$

$= \sin 30^\circ \quad \because \sin(\pi-\theta) = \sin\theta$



Q3. Prove the following

i) $\sin(180^\circ+\alpha)\sin(90^\circ-\alpha) = -\sin\alpha\cos\alpha$

Solution:-

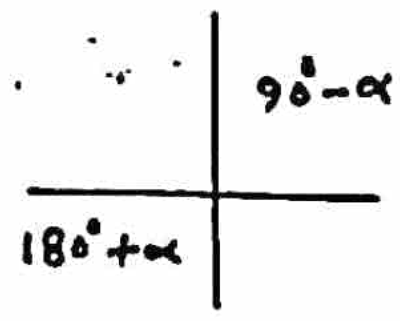
$L.H.S = \sin(180^\circ+\alpha)\sin(90^\circ-\alpha)$

$= \sin(\pi+\alpha)\sin(\frac{\pi}{2}-\alpha)$

$= -\sin\alpha\cos\alpha$

$= R.H.S$

Hence proved



$\because \sin(\pi+\alpha) = -\sin\alpha$
$\sin(\frac{\pi}{2}-\alpha) = \cos\alpha$

ii) $\sin 780^\circ \sin 480^\circ + \cos 120^\circ \sin 30^\circ = \frac{1}{2}$

Solution:-

$L.H.S = \sin 780^\circ \sin 480^\circ + \cos 120^\circ \sin 30^\circ$

$= \sin(720^\circ+60^\circ)\sin(450^\circ+30^\circ) + \cos(90^\circ+30^\circ)\sin 30^\circ$

$= \sin(\frac{8\pi}{2}+60^\circ)\sin(\frac{5\pi}{2}+30^\circ) + \cos(\frac{\pi}{2}+30^\circ)\sin 30^\circ$

$(\because \sin(\frac{8\pi}{2}+60^\circ) = \sin 60^\circ, \ \sin(\frac{5\pi}{2}+30^\circ) = \cos 30^\circ$
$\cos(\frac{\pi}{2}+30^\circ) = -\sin 30^\circ)$

$= \sin 60^\circ \cos 30^\circ - \sin 30^\circ \sin 30^\circ$

$= (\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2}) - (\frac{1}{2})(\frac{1}{2}) = \frac{3}{4} - \frac{1}{4}$

$= \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} = R.H.S$

Hence proved

iii) $\cos 306^\circ + \cos 234^\circ + \cos 162^\circ + \cos 18^\circ = 0$

Solution:-

$L.H.S = \cos 306^\circ + \cos 234^\circ + \cos 162^\circ + \cos 18^\circ$

$= \cos(360^\circ-54^\circ) + \cos(180^\circ+54^\circ) + \cos(180^\circ-18^\circ) + \cos 18^\circ$

$= \cos(2\pi-54^\circ) + \cos(\pi+54^\circ) + \cos(\pi-18^\circ) + \cos 18^\circ$

$(\because \cos(2\pi-\theta) = \cos\theta, \ \cos(\pi+\theta) = -\cos\theta$
$\cos(\pi-\theta) = -\cos\theta)$

$= \cancel{\cos 54^\circ} - \cancel{\cos 54^\circ} - \cancel{\cos 18^\circ} + \cancel{\cos 18^\circ}$

$= 0 = R.H.S$

Hence proved.

iv) $\cos 330^\circ \sin 600^\circ + \cos 120^\circ \sin 150^\circ = -1$

Solution:-

$L.H.S = \cos 330^\circ \sin 600^\circ + \cos 120^\circ \sin 150^\circ$

$= \cos(360^\circ-30^\circ)\sin(540^\circ+60^\circ) + \cos(90^\circ+30^\circ)\sin(180^\circ-30^\circ)$

$= \cos(2\pi-30^\circ)\sin(\frac{6\pi}{2}+60^\circ) + \cos(\frac{\pi}{2}+30^\circ)\sin(\pi-30^\circ)$

$= \cos 30^\circ(-\sin 60^\circ) - \sin 30^\circ \sin 30^\circ$

$= (\frac{\sqrt{3}}{2})(-\frac{\sqrt{3}}{2}) - (\frac{1}{2})(\frac{1}{2}) = \frac{-3}{4} - \frac{1}{4}$

$= \frac{-3-1}{4} = \frac{-4}{4} = -1 = R.H.S$

Hence proved

$(\because \cos(2\pi-\theta) = \cos\theta, \ \cos(\frac{\pi}{2}+\theta) = -\sin\theta$
$\sin(\frac{6\pi}{2}+\theta) = -\sin\theta, \ \sin(\pi-\theta) = \sin\theta)$

Q4. Prove that:

i) $\dfrac{\sin^2(\pi+\theta)\tan(\frac{3\pi}{2}+\theta)}{\cot^2(\frac{3\pi}{2}-\theta)\cos^2(\pi-\theta)\operatorname{cosec}(2\pi-\theta)} = \cos\theta$

Solution:-

$L.H.S = \dfrac{(\sin(\pi+\theta))^2\tan(\frac{3\pi}{2}+\theta)}{(\cot(\frac{3\pi}{2}-\theta))^2(\cos(\pi-\theta))^2\operatorname{cosec}(2\pi-\theta)}$

$\because \sin(\pi+\theta) = -\sin\theta, \ \cos(\pi-\theta) = -\cos\theta$
$\tan(\frac{3\pi}{2}+\theta) = -\cot\theta, \ \cot(\frac{3\pi}{2}-\theta) = \tan\theta$
$\operatorname{cosec}(2\pi-\theta) = -\operatorname{cosec}\theta \quad$ Thus

$L.H.S = \dfrac{(-\sin\theta)^2(-\cot\theta)}{(\tan\theta)^2(-\cos\theta)^2(-\operatorname{cosec}\theta)}$

$= \dfrac{\cancel{-}\sin^2\theta\cot\theta}{\cancel{-}\tan^2\theta\cos^2\theta\operatorname{cosec}\theta}$

$= \dfrac{\cancel{\sin^2}\theta\cot\theta}{\frac{\cancel{\sin^2}\theta}{\cancel{\cos^2}\theta}\cancel{\cos^2}\theta\operatorname{cosec}\theta} = \dfrac{\cos\theta}{\cancel{\sin\theta}}\cdot\cancel{\sin\theta}$

$= \cos\theta = R.H.S$

Hence proved.

ii) $\dfrac{\cos(90^\circ+\theta)\sec(-\theta)\tan(180^\circ-\theta)}{\sec(360^\circ-\theta)\sin(180^\circ+\theta)\cot(90^\circ-\theta)} = -1$

Solution:-

$\because \cos(90°+\theta) = -\sin\theta$, $\sec(-\theta) = \sec\theta$

$\tan(180°-\theta) = -\tan\theta$, $\sec(360°-\theta) = \sec\theta$

$\sin(180°+\theta) = -\sin\theta$, $\cot(90°-\theta) = \tan\theta$

so

$$L.H.S = \frac{-\cancel{\sin\theta}\,\cancel{\sec\theta}\,(-\cancel{\tan\theta})}{\cancel{\sec\theta}\,(-\cancel{\sin\theta})\,\tan\theta} = -1 = R.H.S$$

Hence proved

Q5. If α, β, γ are the angles of a triangle ABC, then prove that

i) $\sin(\alpha+\beta) = \sin\gamma$

Solution:- $\because \alpha, \beta, \gamma$ are angles of a triangle ABC, so

$$\alpha+\beta+\gamma = 180°$$

$$\Rightarrow \alpha+\beta = 180° - \gamma \longrightarrow (i)$$

$L.H.S = \sin(\alpha+\beta)$

$= \sin(180°-\gamma)$ $\quad \because \alpha+\beta = 180°-\gamma$ from (i)

$= \sin 180° \cos\gamma - \cos 180° \sin\gamma$

$= (0)\cos\gamma - (-1)\sin\gamma$

$= \sin\gamma = R.H.S$

Hence proved

ii) $\cos\left(\frac{\alpha+\beta}{2}\right) = -\sin\frac{\gamma}{2}$

$L.H.S = \cos\left(\frac{\alpha+\beta}{2}\right)$ $\quad \because \alpha+\beta = 180°-\gamma$ from (i)

$= \cos\left(\frac{180°-\gamma}{2}\right)$

$= \cos\left(90° - \frac{\gamma}{2}\right)$

$= \cos 90° \cos\frac{\gamma}{2} - \sin 90° \sin\frac{\gamma}{2}$

$= (0)\cos\frac{\gamma}{2} - (1)\sin\frac{\gamma}{2}$

$= -\sin\frac{\gamma}{2} = R.H.S$

Hence proved

iii) $\cos(\alpha+\beta) = -\cos\gamma$

Solution:-

$L.H.S = \cos(\alpha+\beta)$ $\quad \because \alpha+\beta = 180°-\gamma$ from (i)

$= \cos(180°-\gamma)$

$= \cos 180° \cos\gamma + \sin 180° \sin\gamma$

$= (-1)\cos\gamma + (0)\sin\gamma$

$= -\cos\gamma = R.H.S$

Hence proved

(iv) $\tan(\alpha+\beta) + \tan\gamma = 0$

Solution:-

$L.H.S = \tan(\alpha+\beta) + \tan\gamma$

$= \tan(180°-\gamma) + \tan\gamma$ $\quad \because \alpha+\beta = 180°-\gamma$ from (i)

$= \frac{\tan 180° - \tan\gamma}{1+\tan 180° \tan\gamma} + \tan\gamma$

$= \frac{0 - \tan\gamma}{1+(0)\tan\gamma} + \tan\gamma$

$= -\tan\gamma + \tan\gamma = 0 = R.H.S$

Hence proved.

Further Application of Basic Identities

Example 1. Prove that

$$\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha$$

Solution:-

$L.H.S = \sin(\alpha+\beta)\sin(\alpha-\beta)$

$= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$\because (a+b)(a-b) = a^2 - b^2$

$= \sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta$

$= \sin^2\alpha(1-\sin^2\beta) - (1-\sin^2\alpha)\sin^2\beta$

$= \sin^2\alpha - \cancel{\sin^2\alpha\sin^2\beta} - \sin^2\beta + \cancel{\sin^2\alpha\sin^2\beta}$

$= \sin^2\alpha - \sin^2\beta = R.H.S$

$= 1-\cos^2\alpha - (1-\cos^2\beta)$

$= 1-\cos^2\alpha - 1 + \cos^2\beta$

$= \cos^2\beta - \cos^2\alpha = R.H.S$

Hence proved

Example 2. Without using tables, find the values of all trigonometric of 75°.

Solution:- $\because 75^\circ = 45^\circ + 30^\circ$

$\Rightarrow \sin 75^\circ = \sin(45^\circ + 30^\circ)$

$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$\Rightarrow \sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$, $\csc 75^\circ = \frac{2\sqrt{2}}{\sqrt{3}+1}$

$\cos 75^\circ = \cos(45^\circ + 30^\circ)$

$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$

$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\Rightarrow \cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$, $\sec 75^\circ = \frac{2\sqrt{2}}{\sqrt{3}-1}$

$\because \tan 75^\circ = \frac{\sin 75^\circ}{\cos 75^\circ} = \frac{\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$

$\tan 75^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1}$, $\cot 75^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

Example 3. Prove that

$$\frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ} = \tan 56^\circ$$

Solution:-

$R.H.S = \tan 56^\circ$

$= \tan(45^\circ + 11^\circ)$

$= \frac{\tan 45^\circ + \tan 11^\circ}{1 - \tan 45^\circ \tan 11^\circ}$

$= \frac{1 + \tan 11^\circ}{1 - (1)\tan 11^\circ}$ $\quad \because \tan 45^\circ = 1$

$= \frac{1 + \frac{\sin 11^\circ}{\cos 11^\circ}}{1 - \frac{\sin 11^\circ}{\cos 11^\circ}} = \frac{\frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ}}{\frac{\cos 11^\circ - \sin 11^\circ}{\cos 11^\circ}}$

$= \frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ} = L.H.S$

Hence proved

Example 4. If $\cos\alpha = -\frac{24}{25}$, $\tan\beta = \frac{9}{40}$, the terminal side of the angle of measure α is in the II quadrant and that of β is in the III quadrant, find the values of:

i) $\sin(\alpha+\beta)$ ii) $\cos(\alpha+\beta)$

In which quadrant does the terminal side of the angle of measure $(\alpha+\beta)$ lie?

Solution:- $\because \cos\alpha = \frac{-24}{25}$ (α is in II quad)

$\because \sin\alpha = \pm\sqrt{1-\cos^2\alpha} = \pm\sqrt{1-\left(-\frac{24}{25}\right)^2}$

$= \pm\sqrt{1-\frac{576}{625}} = \pm\sqrt{\frac{625-576}{625}}$

$\sin\alpha = \pm\sqrt{\frac{49}{625}} = \pm\frac{7}{25}$

$\Rightarrow \sin\alpha = \frac{7}{25}$ ($\because \alpha$ is in II quad)

Also $\tan\beta = \frac{9}{40}$ (β is in III quad)

$\because \sec^2\beta = 1 + \tan^2\beta$

$= 1 + \left(\frac{9}{40}\right)^2 = 1 + \frac{81}{1600} = \frac{1600+81}{1600}$

$\sec^2\beta = \frac{1681}{1600} \Rightarrow \sec\beta = \pm\sqrt{\frac{1681}{1600}}$

$\Rightarrow \sec\beta = \pm\frac{41}{40} \Rightarrow \sec\beta = -\frac{41}{40}$ ($\because \beta$ is in III quad)

$\Rightarrow \cos\beta = -\frac{40}{41}$

$\because \tan\beta = \frac{\sin\beta}{\cos\beta} \Rightarrow \sin\beta = \tan\beta\cos\beta$

$\Rightarrow \sin\beta = \left(\frac{9}{40}\right)\left(-\frac{40}{41}\right) = -\frac{9}{41}$

$\Rightarrow \sin\beta = -\frac{9}{41}$ Now

i) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(\frac{7}{25}\right)\left(-\frac{40}{41}\right) + \left(-\frac{24}{25}\right)\left(-\frac{9}{41}\right)$

$= \frac{-280}{1025} + \frac{216}{1025} = \frac{-280+216}{1025}$

$\sin(\alpha+\beta) = \frac{-64}{1025} \longrightarrow (i)$

ii) $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$=\left(-\frac{24}{25}\right)\left(-\frac{40}{41}\right)-\left(\frac{7}{25}\right)\left(-\frac{9}{41}\right)$

$=\frac{960}{1025}+\frac{63}{1025}=\frac{960+63}{1025}$

$\cos(\alpha+\beta)=\frac{1023}{1025}\longrightarrow$ (ii)

Since $\sin(\alpha+\beta)$ is $-$ive (from (i) and (ii))
and $\cos(\alpha+\beta)$ is $+$ive

$\rightarrow \alpha+\beta$ lies in IV quadrant.

Example 5. If α,β,γ are the angles of $\triangle ABC$, prove that:

i) $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$

ii) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$

Solution:- i) $\because \alpha,\beta,\gamma$ are the angles of $\triangle ABC$, so

$\alpha+\beta+\gamma=180°$

$\rightarrow \alpha+\beta=180°-\gamma \longrightarrow$ (i)

$\therefore \tan(\alpha+\beta)=\tan(180°-\gamma)$

$\rightarrow \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=-\tan\gamma$ $\because \tan(\pi-\theta)=-\tan\theta$

$\rightarrow \tan\alpha+\tan\beta=-\tan\gamma+\tan\alpha\tan\beta\tan\gamma$

$\rightarrow \tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$

Hence proved

ii) from (i)

$\alpha+\beta=180°-\gamma$

$\rightarrow \frac{\alpha+\beta}{2}=\frac{180°-\gamma}{2}$

$\rightarrow \frac{\alpha}{2}+\frac{\beta}{2}=90°-\frac{\gamma}{2}$

$\therefore \tan\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)=\tan\left(90°-\frac{\gamma}{2}\right)$

$\frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}=\cot\frac{\gamma}{2}$ $\because \tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$

$\rightarrow \frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}=\frac{1}{\tan\frac{\gamma}{2}}$

$\rightarrow \tan\frac{\alpha}{2}\tan\frac{\gamma}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}=1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$

$\rightarrow \tan\frac{\alpha}{2}\tan\frac{\gamma}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=1$

or $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$

Hence proved

Example 6. Express $3\sin\theta+4\cos\theta$ in the form $r\sin(\theta+\phi)$, where the terminal side of the angle of measure ϕ is in the I quadrant

Solution:- $3\sin\theta+4\cos\theta$

Let $3=r\cos\phi$ —— (i)

$4=r\sin\phi$ —— (ii)

By $(i)^2+(ii)^2 \rightarrow (3)^2+(4)^2=r^2\cos^2\phi+r^2\sin^2\phi$

$\rightarrow 9+16=r^2(\cos^2\phi+\sin^2\phi)$

$\rightarrow 25=r^2(1) \rightarrow r^2=25 \rightarrow r=5$

By $\frac{(ii)}{(i)} \rightarrow \frac{r\sin\phi}{r\cos\phi}=\frac{4}{3}$

$\rightarrow \tan\phi=\frac{4}{3} \rightarrow \phi=\tan^{-1}\left(\frac{4}{3}\right)$

$\because 3\sin\theta+4\cos\theta=r\cos\phi\sin\theta+r\sin\phi\cos\theta$

$=r(\cos\phi\sin\theta+\sin\phi\cos\theta)$

$=r\sin(\theta+\phi)$

Here $r=5$, $\phi=\tan^{-1}\left(\frac{4}{3}\right)$

Exercise 10.2

Q1. Prove that:

i) $\sin(180°+\theta)=-\sin\theta$

Solution:- $L.H.S=\sin(180°+\theta)$

$=\sin180°\cos\theta+\cos180°\sin\theta$

$=(0)\cos\theta+(-1)\sin\theta=-\sin\theta=R.H.S$

Hence proved

ii) $\cos(180°+\theta)=-\cos\theta$

Solution:-

$L.H.S=\cos(180°+\theta)$

$=\cos180°\cos\theta-\sin180°\sin\theta$

$=(-1)\cos\theta-(0)\sin\theta=-\cos\theta=R.H.S$

Hence proved

$\because \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

iii) $\tan(270^\circ-\theta)=\cot\theta$

Solution:-

L.H.S $=\tan(270^\circ-\theta)$

$$=\frac{\sin(270^\circ-\theta)}{\cos(270^\circ-\theta)}$$

$$=\frac{\sin 270^\circ\cos\theta-\cos 270^\circ\sin\theta}{\cos 270^\circ\cos\theta+\cos 270^\circ\cos\theta}$$

$$=\frac{(-1)\cos\theta-(0)\sin\theta}{(0)\cos\theta+(-1)\sin\theta}=\frac{-\cos\theta}{-\sin\theta}$$

$=\cot\theta=$ R.H.S

Hence proved

iv) $\cos(\theta-180^\circ)=-\cos\theta$

Solution:-

L.H.S $=\cos(\theta-180^\circ)$

$=\cos\theta\cos180^\circ-\sin\theta\sin180^\circ$

$=\cos\theta(-1)-\sin\theta(0)$

$=-\cos\theta=$ R.H.S

Hence proved

v) $\cos(270^\circ+\theta)=\sin\theta$

Solution:-

L.H.S $=\cos(270^\circ+\theta)$

$=\cos270^\circ\cos\theta-\sin270^\circ\sin\theta$

$=(0)\cos\theta-(-1)\sin\theta$

$=\sin\theta=$ R.H.S

Hence proved

vi) $\sin(\theta+270^\circ)=-\cos\theta$

Solution:-

L.H.S $=\sin(\theta+270^\circ)$

$=\sin\theta\cos270^\circ+\cos\theta\sin270^\circ$

$=\sin\theta(0)+\cos\theta(-1)$

$=-\cos\theta=$ R.H.S

Hence proved

vii) $\tan(180^\circ+\theta)=\tan\theta$

Solution:-

L.H.S $=\tan(180^\circ+\theta)$

$$=\frac{\sin(180^\circ+\theta)}{\cos(180^\circ+\theta)}$$

$$=\frac{\sin180^\circ\cos\theta+\cos180^\circ\sin\theta}{\cos180^\circ\cos\theta-\sin180^\circ\sin\theta}$$

$$=\frac{(0)\cos\theta+(-1)\sin\theta}{(-1)\cos\theta-(0)\sin\theta}$$

$$=\frac{-\sin\theta}{-\cos\theta}=\frac{\sin\theta}{\cos\theta}=\tan\theta=\text{R.H.S}$$

Hence proved

viii) $\cos(360^\circ-\theta)=\cos\theta$

Solution:-

L.H.S $=\cos(360^\circ-\theta)$

$=\cos360^\circ\cos\theta+\sin360^\circ\sin\theta$

$=(1)\cos\theta+(0)\sin\theta$

$=\cos\theta=$ R.H.S

Hence proved

$\because \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$
$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

Q2. Find the values of the following:

i) $\sin15^\circ$ ii) $\cos15^\circ$ iii) $\tan15^\circ$
iv) $\sin105^\circ$ v) $\cos105^\circ$ vi) $\tan105^\circ$

(**Hint:** $15^\circ=(45^\circ-30^\circ)$ and $105^\circ=(60^\circ+45^\circ)$)

Solution:- i) $\sin15^\circ$

$\sin15^\circ=\sin(45^\circ-30^\circ)$

$=\sin45^\circ\cos30^\circ-\cos45^\circ\sin30^\circ$

$$=\left(\frac{1}{\sqrt2}\right)\left(\frac{\sqrt3}{2}\right)-\left(\frac{1}{\sqrt2}\right)\left(\frac12\right)$$

$$=\frac{\sqrt3}{2\sqrt2}-\frac{1}{2\sqrt2}=\frac{\sqrt3-1}{2\sqrt2}$$

ii) $\cos15^\circ$

$\cos15^\circ=\cos(45^\circ-30^\circ)$

$=\cos45^\circ\cos30^\circ+\sin45^\circ\sin30^\circ$

$$=\left(\frac{1}{\sqrt2}\right)\left(\frac{\sqrt3}{2}\right)+\left(\frac{1}{\sqrt2}\right)\left(\frac12\right)$$

$$=\frac{\sqrt3}{2\sqrt2}+\frac{1}{2\sqrt2}=\frac{\sqrt3+1}{2\sqrt2}$$

iii) $\tan15^\circ$

$\tan15^\circ=\tan(45^\circ-30^\circ)$

$$\left(\because \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\right)$$

$$= \frac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ}$$

$$= \frac{1-\frac{1}{\sqrt{3}}}{1+(1)\left(\frac{1}{\sqrt{3}}\right)} =$$

$$= \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

iv) $\sin 105^\circ$

$$\sin 105^\circ = \sin(60^\circ+45^\circ)$$

$$= \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

v) $\cos 105^\circ$

$$\cos 105^\circ = \cos(60^\circ+45^\circ)$$

$$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

vi) $\tan 105^\circ$

$$\tan 105^\circ = \tan(60^\circ+45^\circ)$$

$$\left(\because \tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\right)$$

$$= \frac{\tan 60^\circ + \tan 45^\circ}{1-\tan 60^\circ \tan 45^\circ}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}(1)} = \frac{\sqrt{3}+1}{1-\sqrt{3}}$$

Q3. Prove that:

i) $\sin(45^\circ+\alpha) = \frac{1}{\sqrt{2}}(\sin\alpha+\cos\alpha)$

Solution:-

$$L.H.S = \sin(45^\circ+\alpha)$$

$$= \sin 45^\circ \cos\alpha + \cos 45^\circ \sin\alpha$$

$$= \frac{1}{\sqrt{2}}\cos\alpha + \frac{1}{\sqrt{2}}\sin\alpha$$

$$= \frac{1}{\sqrt{2}}(\sin\alpha+\cos\alpha) = R.H.S$$

Hence proved

ii) $\cos(\alpha+45^\circ) = \frac{1}{\sqrt{2}}(\cos\alpha-\sin\alpha)$

Solution:-

$$L.H.S = \cos(\alpha+45^\circ)$$

$$= \cos\alpha\cos 45^\circ - \sin\alpha \sin 45^\circ$$

$$= \cos\alpha\left(\frac{1}{\sqrt{2}}\right) - \sin\alpha\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}(\cos\alpha-\sin\alpha) = R.H.S$$

Hence proved.

Q4. Prove that:

i) $\tan(45^\circ+A)\tan(45^\circ-A) = 1$

Solution:-

$$L.H.S = \tan(45^\circ+A)\tan(45^\circ-A)$$

$$\left(\because \tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\right.$$

$$\left. \text{and } \tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\right)$$

$$= \left(\frac{\tan 45^\circ+\tan A}{1-\tan 45^\circ\tan A}\right)\left(\frac{\tan 45^\circ-\tan A}{1+\tan 45^\circ\tan A}\right)$$

$$= \left(\frac{1+\tan A}{1-(1)\tan A}\right)\left(\frac{1-\tan A}{1+(1)\tan A}\right)$$

$$= \left(\frac{1+\cancel{\tan A}}{1-\cancel{\tan A}}\right)\left(\frac{1-\cancel{\tan A}}{1+\cancel{\tan A}}\right)$$

$$= 1 = R.H.S$$

Hence proved.

ii) $\tan\left(\frac{\pi}{4}-\theta\right) + \tan\left(\frac{3\pi}{4}+\theta\right) = 0$

Solution:-

$$L.H.S = \tan\left(\frac{\pi}{4}-\theta\right) + \tan\left(\frac{3\pi}{4}+\theta\right)$$

$$= \frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta} + \frac{\tan\frac{3\pi}{4}+\tan\theta}{1-\tan\frac{3\pi}{4}\tan\theta}$$

$$= \frac{1-\tan\theta}{1+(1)\tan\theta} + \frac{(-1)+\tan\theta}{1-(-1)\tan\theta}$$

$$= \left(\frac{1-\tan\theta}{1+\tan\theta}\right) - \left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= 0 = R.H.S$$

Hence proved.

iii) $\sin(\theta+\frac{\pi}{6})+\cos(\theta+\frac{\pi}{3}) = \cos\theta$

Solution:-

$$L.H.S = \sin(\theta+\frac{\pi}{6})+\cos(\theta+\frac{\pi}{3})$$

$$=\left(\sin\theta\cos\frac{\pi}{6}+\cos\theta\sin\frac{\pi}{6}\right)+\left(\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}\right)$$

$$=\left(\sin\theta(\frac{\sqrt{3}}{2})+\cos\theta(\frac{1}{2})\right)+\left(\cos\theta(\frac{1}{2})-\sin\theta(\frac{\sqrt{3}}{2})\right)$$

$$=\cancel{\frac{\sqrt{3}}{2}\sin\theta}+\frac{1}{2}\cos\theta+\frac{1}{2}\cos\theta-\cancel{\frac{\sqrt{3}}{2}\sin\theta}$$

$$= \cos\theta = R.H.S$$

Hence proved.

iv) $\dfrac{\sin\theta-\cos\theta\tan\frac{\theta}{2}}{\cos\theta+\sin\theta\tan\frac{\theta}{2}} = \tan\frac{\theta}{2}$

Solution:-

$$L.H.S = \frac{\sin\theta-\cos\theta\tan\frac{\theta}{2}}{\cos\theta+\sin\theta\tan\frac{\theta}{2}}$$

$$= \frac{\sin\theta-\cos\theta\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\cos\theta+\sin\theta\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}$$

$$= \frac{\frac{\sin\theta\cos\frac{\theta}{2}-\cos\theta\sin\frac{\theta}{2}}{\cancel{\cos\frac{\theta}{2}}}}{\frac{\cos\theta\cos\frac{\theta}{2}+\sin\theta\sin\frac{\theta}{2}}{\cancel{\cos\theta/2}}}$$

$$= \frac{\sin(\theta-\frac{\theta}{2})}{\cos(\theta-\frac{\theta}{2})} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$= \tan\frac{\theta}{2} = R.H.S$$

Hence proved.

v) $\dfrac{1-\tan\theta\tan\phi}{1+\tan\theta\tan\phi} = \dfrac{\cos(\theta+\phi)}{\cos(\theta-\phi)}$

Solution:-

$$L.H.S = \frac{1-\tan\theta\tan\phi}{1+\tan\theta\tan\phi}$$

$$= \frac{1-\frac{\sin\theta}{\cos\theta}\cdot\frac{\sin\phi}{\cos\phi}}{1+\frac{\sin\theta}{\cos\theta}\cdot\frac{\sin\phi}{\cos\phi}}$$

$$= \frac{\frac{\cos\theta\cos\phi-\sin\theta\sin\phi}{\cos\theta\cancel{\cos\phi}}}{\frac{\cos\theta\cos\phi+\sin\theta\sin\phi}{\cos\theta\cancel{\cos\phi}}}$$

$$= \frac{\cos(\theta+\phi)}{\cos(\theta-\phi)} = R.H.S$$

Hence proved.

Q5. Show that

$$\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha-\sin^2\beta = \cos^2\beta-\sin^2\alpha$$

Solution:-

$$L.H.S = \cos(\alpha+\beta)\cos(\alpha-\beta)$$

$$=(\cos\alpha\cos\beta-\sin\alpha\sin\beta)(\cos\alpha\cos\beta+\sin\alpha\sin\beta)$$

$$=\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta$$

$$=\cos^2\alpha(1-\sin^2\beta)-(1-\cos^2\alpha)\sin^2\beta$$

$$=\cos^2\alpha-\cancel{\cos^2\alpha\sin^2\beta}-\sin^2\beta-\cancel{\cos^2\alpha\sin^2\beta}$$

$$=\cos^2\alpha-\sin^2\beta = R.H.S$$

$$=1-\sin^2\alpha-(1-\cos^2\beta)$$

$$=1-\sin^2\alpha-1+\cos^2\beta$$

$$=\cos^2\beta-\sin^2\alpha = R.H.S$$

Hence proved

Q6. Show that:

$$\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)} = \tan\alpha$$

Solution:-

$$L.H.S = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}$$

$$= \frac{\sin\alpha\cos\beta+\cancel{\cos\alpha\sin\beta}+\sin\alpha\cos\beta-\cancel{\cos\alpha\sin\beta}}{\cos\alpha\cos\beta-\cancel{\sin\alpha\sin\beta}+\cos\alpha\cos\beta+\cancel{\sin\alpha\sin\beta}}$$

$$= \frac{\cancel{2}\sin\alpha\cancel{\cos\beta}}{\cancel{2}\cos\alpha\cancel{\cos\beta}} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha = R.H.S$$

Q7. Show that:

i) $\cot(\alpha+\beta) = \dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$

Solution:-

$$R.H.S = \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$

$$= \frac{\frac{1}{\tan\alpha\tan\beta} - 1}{\frac{1}{\tan\alpha} + \frac{1}{\tan\beta}} = \frac{\frac{1-\tan\alpha\tan\beta}{\cancel{\tan\alpha}\tan\beta}}{\frac{\tan\beta+\tan\alpha}{\cancel{\tan\alpha}\tan\beta}}$$

$$= \frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta} = \frac{1}{\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}$$

$$= \frac{1}{\tan(\alpha+\beta)} = \cot(\alpha+\beta) = L.H.S$$

Hence proved

ii) $\cot(\alpha-\beta) = \dfrac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$

Solution:-

$$R.H.S = \frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$$

$$= \frac{\frac{1}{\tan\alpha\tan\beta}+1}{\frac{1}{\tan\beta}-\frac{1}{\tan\alpha}}$$

$$= \frac{\frac{1+\tan\alpha\tan\beta}{\cancel{\tan\alpha}\tan\beta}}{\frac{\tan\alpha-\tan\beta}{\cancel{\tan\alpha}\tan\beta}} = \frac{1+\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$$

$$= \frac{1}{\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}} = \frac{1}{\tan(\alpha-\beta)}$$

$$= \cot(\alpha-\beta) = L.H.S$$

Hence proved

iii) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta} = \dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

Solution:-

$$L.H.S = \frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}$$

$$= \frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{\frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta}}$$

$$= \frac{\frac{\sin\alpha\cos\beta+\sin\beta\cos\alpha}{\cancel{\cos\alpha}\cos\beta}}{\frac{\sin\alpha\cos\beta-\sin\beta\cos\alpha}{\cancel{\cos\alpha}\cos\beta}}$$

$$= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$$

$$= \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = R.H.S$$

Hence proved

Q8. If $\sin\alpha = \frac{4}{5}$, $\cos\beta = \frac{40}{41}$, where $0<\alpha<\frac{\pi}{2}$ and $0<\beta<\frac{\pi}{2}$

show that $\sin(\alpha-\beta) = \frac{133}{205}$.

Solution:- $\sin\alpha = \frac{4}{5}$, $\cos\beta = \frac{40}{41}$

$\because 0<\alpha<\frac{\pi}{2}$ i.e., α lies in I^{st} quad

and $0<\beta<\frac{\pi}{2}$ i.e., β lies in I^{st} quad

$$\because \cos\alpha = \pm\sqrt{1-\sin^2\alpha} = \pm\sqrt{1-\left(\frac{4}{5}\right)^2}$$

$$= \pm\sqrt{1-\frac{16}{25}} = \pm\sqrt{\frac{25-16}{25}}$$

$$\cos\alpha = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

$$\rightarrow \cos\alpha = \frac{3}{5} \quad (\because \alpha \text{ lies in I quad})$$

Now $\sin\beta = \pm\sqrt{1-\cos^2\beta}$

$$= \pm\sqrt{1-\left(\frac{40}{41}\right)^2} = \pm\sqrt{1-\frac{1600}{1681}}$$

$$= \pm\sqrt{\frac{1681-1600}{1681}} = \pm\sqrt{\frac{81}{1681}}$$

$$\sin\beta = \pm\frac{9}{41} \rightarrow \sin\beta = \frac{9}{41} \quad (\because \beta \text{ lies in I quad})$$

$$L.H.S = \sin(\alpha-\beta)$$

$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \left(\frac{4}{5}\right)\left(\frac{40}{41}\right) - \left(\frac{3}{5}\right)\left(\frac{9}{41}\right)$$

$$= \frac{160}{205} - \frac{27}{205} = \frac{160-27}{205}$$

$$= \frac{133}{205}$$

$$\rightarrow \sin(\alpha-\beta) = \frac{133}{205} \quad \text{Hence proved}$$

Q9. If $\sin\alpha = \frac{4}{5}$ and $\sin\beta = \frac{12}{13}$ where $\frac{\pi}{2} < \alpha < \pi$ and $\frac{\pi}{2} < \beta < \pi$. Find

i) $\sin(\alpha+\beta)$ ii) $\cos(\alpha+\beta)$ iii) $\tan(\alpha+\beta)$

iv) $\sin(\alpha-\beta)$ v) $\cos(\alpha-\beta)$ vi) $\tan(\alpha-\beta)$

In which quadrants do the terminal sides of the angles of measures $(\alpha+\beta)$ and $(\alpha-\beta)$ lie?

Solution:- $\sin\alpha = \frac{4}{5}$, $\sin\beta = \frac{12}{13}$

$\because \frac{\pi}{2} < \alpha < \pi$ i-e., α lies in II quad

and $\frac{\pi}{2} < \beta < \pi$ i-e., β lies in II quad

$\because \cos\alpha = \pm\sqrt{1-\sin^2\alpha}$

$= \pm\sqrt{1-\left(\frac{4}{5}\right)^2}$

$= \pm\sqrt{1-\frac{16}{25}} = \pm\sqrt{\frac{25-16}{25}}$

$\cos\alpha = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$

$\Rightarrow \cos\alpha = -\frac{3}{5}$ ($\because \alpha$ lies in II quad)

Now $\cos\beta = \pm\sqrt{1-\sin^2\beta}$

$= \pm\sqrt{1-\left(\frac{12}{13}\right)^2} = \pm\sqrt{1-\frac{144}{169}}$

$= \pm\sqrt{\frac{169-144}{169}} = \pm\sqrt{\frac{25}{169}}$

$\cos\beta = \pm\frac{5}{13}$ $\Rightarrow \cos\beta = -\frac{5}{13}$ ($\because \beta$ is in II quad)

i) $\sin(\alpha+\beta)$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right)$

$= -\frac{20}{65} - \frac{36}{65} = \frac{-20-36}{65} = -\frac{56}{65}$

ii) $\cos(\alpha+\beta)$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{12}{13}\right)$

$= \frac{15}{65} - \frac{48}{65} = \frac{15-48}{65} = -\frac{33}{65}$

iii) $\tan(\alpha+\beta)$

$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{-56/65}{-33/65}$

$\tan(\alpha+\beta) = \frac{56}{33}$

iv) $\sin(\alpha-\beta)$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) - \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right)$

$= -\frac{20}{65} + \frac{36}{65} = \frac{-20+36}{65} = \frac{16}{65}$

v) $\cos(\alpha-\beta)$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{12}{13}\right)$

$= \frac{15}{65} + \frac{48}{65} = \frac{15+48}{65} = \frac{63}{65}$

vi) $\tan(\alpha-\beta)$

$\tan(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \frac{16/65}{63/65} = \frac{16}{63}$

$\because \sin(\alpha+\beta)$ is -ive and $\cos(\alpha+\beta)$ is -ive so $\alpha+\beta$ lies in III quadrant.

Also $\sin(\alpha-\beta)$ is +ive and $\cos(\alpha-\beta)$ is +ive so $\alpha-\beta$ lies in I quadrant.

Q10. Find $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, given that

i) $\tan\alpha = \frac{3}{4}$, $\cos\beta = \frac{5}{13}$ and neither the terminal side of the angle of measure α nor that of β is in the I quadrant.

Solution:- $\tan\alpha = \frac{3}{4}$ (α not in I quad)

so α lies in III quad.

$\cos\beta = \frac{5}{13}$ (β not in I quad, so β lies in IV quad)

$\because \sec^2\alpha = 1+\tan^2\alpha = 1+\left(\frac{3}{4}\right)^2$

$\sec^2\alpha = 1+\frac{9}{16} = \frac{16+9}{16} = \frac{25}{16}$

$\Rightarrow \sec\alpha = \pm\sqrt{\frac{25}{16}} = \pm\frac{5}{4}$

$\Rightarrow \sec\alpha = -\frac{5}{4}$ ($\because \alpha$ is in III quad)

or $\cos\alpha = -\frac{4}{5}$

Also $\because \sin\alpha = \pm\sqrt{1-\cos^2\alpha}$

$\sin\alpha = \pm\sqrt{1-\left(-\frac{4}{5}\right)^2}$

$\sin\alpha = \pm\sqrt{1-\frac{16}{25}} = \pm\sqrt{\frac{25-16}{25}}$

$\sin\alpha = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$

$\Rightarrow \sin\alpha = -\frac{3}{5}$ ($\because \alpha$ is in III quad)

$\because \sin\beta = \pm\sqrt{1-\cos^2\beta}$

$= \pm\sqrt{1-\left(\frac{5}{13}\right)^2} = \pm\sqrt{1-\frac{25}{169}}$

$= \pm\sqrt{\frac{169-25}{169}} = \pm\sqrt{\frac{144}{169}}$

$\sin\beta = \pm\frac{12}{13}$

$\Rightarrow \sin\beta = -\frac{12}{13}$ ($\because \beta$ is in IV quad)

Now

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right)$

$= -\frac{15}{65} + \frac{48}{65} = \frac{-15+48}{65} = \frac{33}{65}$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \left(-\frac{4}{5}\right)\left(\frac{5}{13}\right) - \left(-\frac{3}{5}\right)\left(-\frac{12}{13}\right)$

$= -\frac{20}{65} - \frac{36}{65} = \frac{-20-36}{65} = -\frac{56}{65}$

ii) $\tan\alpha = -\frac{15}{8}$ and $\sin\beta = -\frac{7}{25}$ and neither the terminal side of the angle of measure α nor that of β is in the IV quadrant.

Solution:– $\because \tan\alpha = -\frac{15}{8}$ (α is not in IV quad so α lies in II quad)

$\sin\beta = -\frac{7}{25}$ (β is not in IV quad so β lies in III quad).

$\because \sec^2\alpha = 1+\tan^2\alpha$

$= 1+\left(-\frac{15}{8}\right)^2 = 1+\frac{225}{64}$

$\Rightarrow \sec^2\alpha = \frac{64+225}{64} = \frac{289}{64}$

$\Rightarrow \sec\alpha = \pm\sqrt{\frac{289}{64}} = \pm\frac{17}{8}$

$\Rightarrow \sec\alpha = -\frac{17}{8}$ ($\because \alpha$ lies in II quad)

or $\cos\alpha = -\frac{8}{17}$

$\because \sin\alpha = \pm\sqrt{1-\cos^2\alpha}$

$\Rightarrow \sin\alpha = \pm\sqrt{1-\left(-\frac{8}{17}\right)^2} = \pm\sqrt{1-\frac{64}{289}}$

$= \pm\sqrt{\frac{289-64}{289}} = \pm\sqrt{\frac{225}{289}}$

$\sin\alpha = \pm\frac{15}{17}$

$\Rightarrow \sin\alpha = -\frac{15}{17}$ ($\because \alpha$ is in II quad)

$\because \cos\beta = \pm\sqrt{1-\sin^2\beta} = \pm\sqrt{1-\left(-\frac{7}{25}\right)^2}$

$= \pm\sqrt{1-\frac{64}{625}} = \pm\sqrt{\frac{625-64}{625}}$

$\cos\beta = \pm\sqrt{\frac{576}{625}} = \pm\frac{24}{25}$

$\Rightarrow \cos\beta = -\frac{24}{25}$ ($\because \beta$ is in III quad)

Now $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(\frac{15}{17}\right)\left(-\frac{24}{25}\right) + \left(-\frac{8}{17}\right)\left(-\frac{7}{25}\right)$

$= -\frac{360}{425} + \frac{56}{425} = \frac{-360+56}{425} = -\frac{304}{425}$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \left(-\frac{8}{17}\right)\left(-\frac{24}{25}\right) - \left(\frac{15}{17}\right)\left(-\frac{7}{25}\right)$

$= \frac{192}{425} + \frac{105}{425} = \frac{192+105}{425} = \frac{297}{425}$

Q11. Prove that: $\frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ} = \tan 37^\circ$

Solution:–

R.H.S $= \tan 37^\circ$

$= \tan(45^\circ - 8^\circ)$ $\left(\because \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\right)$

$= \frac{\tan 45^\circ - \tan 8^\circ}{1+\tan 45^\circ \tan 8^\circ}$

$= \frac{1-\tan 8^\circ}{1+(1)\tan 8^\circ} = \frac{1-\frac{\sin 8^\circ}{\cos 8^\circ}}{1+\frac{\sin 8^\circ}{\cos 8^\circ}}$

$= \frac{\frac{\cos 8^\circ - \sin 8^\circ}{\cancel{\cos 8^\circ}}}{\frac{\cos 8^\circ + \sin 8^\circ}{\cancel{\cos 8^\circ}}} = \frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ} = \text{L.H.S}$

Hence proved.

Q12. If α, β, γ are angles of a triangle ABC, show that

$$\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$$

Solution:- $\because \alpha, \beta, \gamma$ are angles of a triangle ABC, so

$$\alpha+\beta+\gamma = 180^\circ$$

$$\rightarrow \alpha+\beta = 180^\circ-\gamma$$

$$\rightarrow \frac{\alpha+\beta}{2} = \frac{180^\circ-\gamma}{2}$$

$$\rightarrow \frac{\alpha}{2}+\frac{\beta}{2} = 90^\circ-\frac{\gamma}{2}$$

$$\rightarrow \tan\left(\frac{\alpha}{2}+\frac{\beta}{2}\right) = \tan\left(90^\circ-\frac{\gamma}{2}\right)$$

$$\rightarrow \frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = \cot\frac{\gamma}{2} \qquad \because \tan(90^\circ-\theta) = \cot\theta$$

$$\rightarrow \frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = \frac{1}{\tan\frac{\gamma}{2}}$$

$$\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2} = 1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

$$\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = 1$$

Dividing both sides by $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

$$\frac{1}{\tan\frac{\beta}{2}}+\frac{1}{\tan\frac{\alpha}{2}}+\frac{1}{\tan\frac{\gamma}{2}} = \frac{1}{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}}$$

$$\rightarrow \cot\frac{\beta}{2}+\cot\frac{\alpha}{2}+\cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$$

$$\rightarrow \cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$$

Hence proved

Q13. If $\alpha+\beta+\gamma = 180^\circ$, show that

$$\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha = 1$$

Solution:-

$$\because \alpha+\beta+\gamma = 180^\circ$$

$$\rightarrow \alpha+\beta = 180^\circ-\gamma$$

$$\tan(\alpha+\beta) = \tan(180^\circ-\gamma)$$

$$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = -\tan\gamma \qquad \because \tan(\pi-\theta) = -\tan\theta$$

$$\tan\alpha+\tan\beta = -\tan\gamma(1-\tan\alpha\tan\beta)$$

$$\rightarrow \tan\alpha+\tan\beta = -\tan\gamma+\tan\alpha\tan\beta\tan\gamma$$

$$\rightarrow \tan\alpha+\tan\beta+\tan\gamma = \tan\alpha\tan\beta\tan\gamma$$

$$\frac{1}{\cot\alpha}+\frac{1}{\cot\beta}+\frac{1}{\cot\gamma} = \frac{1}{\cot\alpha\cot\beta\cot\gamma}$$

$$\frac{\cot\beta\cot\gamma+\cot\alpha\cot\gamma+\cot\alpha\cot\beta}{\cot\alpha\cancel{\cot\beta}\cot\gamma} = \frac{1}{\cot\alpha\cancel{\cot\beta}\cot\gamma}$$

$$\rightarrow \cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha = 1$$

Hence proved

Q14. Express the following in the form $r\sin(\theta+\phi)$ or $r\sin(\theta-\phi)$, where terminal sides of the angles of measures θ and ϕ are in the first quadrant:

i) $12\sin\theta+5\cos\theta$

Solution:- Let $12 = r\cos\phi \longrightarrow$ (i)

$5 = r\sin\phi \longrightarrow$ (ii)

By $(i)^2+(ii)^2 \rightarrow (12)^2+(5)^2 = r^2\cos^2\phi+r^2\sin^2\phi$

$$\rightarrow 144+25 = r^2(\cos^2\phi+\sin^2\phi) = r^2(1)$$

$$\rightarrow r^2 = 169 \quad \rightarrow r = 13$$

By $\frac{(ii)}{(i)} \rightarrow \frac{r\sin\phi}{r\cos\phi} = \frac{5}{12} \rightarrow \tan\phi = \frac{5}{12}$

or $\phi = \tan^{-1}\left(\frac{5}{12}\right)$

Now

$$12\sin\theta+5\cos\theta = r\cos\phi\sin\theta+r\sin\phi\cos\theta$$

$$= r(\sin\theta\cos\phi+\cos\theta\sin\phi) = r\sin(\theta+\phi)$$

where $r = 13$ and $\phi = \tan^{-1}\frac{5}{12}$

ii) $3\sin\theta-4\cos\theta$

Solution:- Let $3 = r\cos\phi \longrightarrow$ (i)

$-4 = r\sin\phi \longrightarrow$ (ii)

By $(i)^2+(ii)^2 \rightarrow (3)^2+(-4)^2 = r^2\cos^2\phi+r^2\sin^2\phi$

$$\rightarrow 9+16 = r^2(\cos^2\phi+\sin^2\phi) = r^2(1)$$

$$\rightarrow 25 = r^2 \rightarrow r = 5$$

By $\frac{(ii)}{(i)} \rightarrow \frac{r\sin\phi}{r\cos\phi} = \frac{-4}{3} \rightarrow \tan\phi = \frac{-4}{3}$

or $\phi = \tan^{-1}\left(-\frac{4}{3}\right)$

Now

$$3\sin\theta-4\cos\theta = r\cos\phi\sin\theta+r\sin\phi\cos\theta$$

$$= r(\sin\theta\cos\phi+\cos\theta\sin\phi)$$

$$= r\sin(\theta+\phi) \text{ where } r = 5$$

and $\phi = \tan^{-1}(-4/3)$

(iii) $\sin\theta - \cos\theta$

Solution:- Let $1 = r\cos\phi \longrightarrow (i)$
$-1 = r\sin\phi \longrightarrow (ii)$

By $(i)^2+(ii)^2 \rightarrow (1)^2+(-1)^2 = r^2\cos^2\phi + r^2\sin^2\phi$
$\rightarrow 1+1 = r^2(\cos^2\phi+\sin^2\phi) = r^2(1)$
$\rightarrow r^2 = 2 \rightarrow r = \sqrt{2}$

By $\frac{(ii)}{(i)} \rightarrow \frac{r\sin\phi}{r\cos\phi} = \frac{-1}{1} \rightarrow \tan\phi = -1$
$\rightarrow \phi = \tan^{-1}(-1)$

Now
$\sin\theta - \cos\theta = r\cos\phi\sin\theta + r\sin\phi\cdot\cos\theta$
$= r(\sin\theta\cos\phi + \cos\theta\sin\phi)$
$= r\sin(\theta+\phi)$ where $r = \sqrt{2}$
and $\phi = \tan^{-1}(-1)$

(iv) $5\sin\theta - 4\cos\theta$

Solution:- Let $5 = r\cos\phi \longrightarrow (i)$
$-4 = r\sin\phi \longrightarrow (ii)$

By $(i)^2+(ii)^2 \rightarrow (5)^2+(-4)^2 = r^2\cos^2\phi + r^2\sin^2\phi$
$\rightarrow 25+16 = r^2(\cos^2\phi+\sin^2\phi) = r^2(1)$
$\rightarrow r^2 = 41 \rightarrow r = \sqrt{41}$

By $\frac{(ii)}{(i)} \rightarrow -\frac{4}{5} = \frac{r\sin\phi}{r\cos\phi} \rightarrow \tan\phi = \frac{-4}{5}$
or $\phi = \tan^{-1}\left(\frac{-4}{5}\right)$

Now
$5\sin\theta - 4\cos\theta = r\cos\phi\sin\theta + r\sin\phi\cos\theta$
$= r(\sin\theta\cos\phi + \cos\theta\sin\phi)$
$= r\sin(\theta+\phi)$, where $r = \sqrt{41}$
and $\phi = \tan^{-1}(-\frac{4}{5})$

v) $\sin\theta + \cos\theta$

Solution:- Let $1 = r\cos\phi \longrightarrow (i)$
$1 = r\sin\phi \longrightarrow (ii)$

By $(i)^2+(ii)^2 \rightarrow (1)^2+(1)^2 = r^2\cos^2\phi + r^2\sin^2\phi$
$\rightarrow 1+1 = r^2(\cos^2\phi+\sin^2\phi) = r^2(1)$
$\rightarrow r^2 = 2 \rightarrow r = \sqrt{2}$

By $\frac{(ii)}{(i)} \rightarrow \frac{r\sin\phi}{r\cos\phi} = \frac{1}{1} \rightarrow \tan\phi = 1$
or $\phi = \tan^{-1}(1)$

Now
$\sin\theta + \cos\theta = r\cos\phi\sin\theta + r\sin\phi\cos\theta$
$= r(\sin\theta\cos\phi + \cos\theta\sin\phi)$
$= r\sin(\theta+\phi)$, where $r = \sqrt{2}$
and $\phi = \tan^{-1}(1)$

vi) $3\sin\theta - 5\cos\theta$

Solution:- Let $3 = r\cos\phi \longrightarrow (i)$
$-5 = r\sin\phi \longrightarrow (ii)$

By $(i)^2+(ii)^2 \rightarrow (3)^2+(-5)^2 = r^2\cos^2\phi + r^2\sin^2\phi$
$\rightarrow 9+25 = r^2(\cos^2\phi+\sin^2\phi) = r^2(1)$
$\rightarrow r^2 = 34 \rightarrow r = \sqrt{34}$

By $\frac{(ii)}{(i)} \rightarrow \frac{r\sin\phi}{r\cos\phi} = \frac{-5}{3} \rightarrow \tan\phi = -\frac{5}{3}$
or $\therefore \phi = \tan^{-1}(-5/3)$

Now,
$3\sin\theta - 5\cos\theta = r\cos\phi\sin\theta + r\sin\phi\cos\theta$
$= r(\sin\theta\cos\phi + \cos\theta\sin\phi)$
$= r\sin(\theta+\phi)$, where $r = \sqrt{34}$
and $\phi = \tan^{-1}(-5/3)$

Double angle Identities

i) $\sin 2\alpha = 2\sin\alpha\cos\alpha$

we know that
$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
Put $\beta = \alpha$ we get
$\sin(\alpha+\alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$
$\rightarrow \boxed{\sin 2\alpha = 2\sin\alpha\cos\alpha}$

ii) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

we know that
$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\cos\beta$
Put $\beta = \alpha$ we get
$\cos(\alpha+\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$
$\rightarrow \boxed{\cos 2\alpha = \cos^2\alpha - \sin^2\alpha}$
$\rightarrow \cos 2\alpha = \cos^2\alpha - (1-\cos^2\alpha)$
$= \cos^2\alpha - 1 + \cos^2\alpha$
$\rightarrow \boxed{\cos 2\alpha = 2\cos^2\alpha - 1}$
$\rightarrow \cos 2\alpha = 2(1-\sin^2\alpha) - 1$
$= 2 - 2\sin^2\alpha - 1$
$\rightarrow \boxed{\cos 2\alpha = 1 - 2\sin^2\alpha}$

iii) $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

we know that

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

put $\beta = \alpha$, we get

$$\rightarrow \tan(\alpha+\alpha) = \frac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}$$

$$\rightarrow \boxed{\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}}$$

Half angle Identities

i) $\cos\frac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$

we know that

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

similarly

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 \quad \text{(In form of half angle)}$$

$$\rightarrow 2\cos^2\frac{\alpha}{2} = 1+\cos\alpha$$

$$\rightarrow \cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$$

$$\rightarrow \boxed{\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}}$$

ii) $\sin\frac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$

we know that

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

similarly

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} \quad \text{(In form of half angle)}$$

$$\rightarrow 2\sin^2\frac{\alpha}{2} = 1-\cos\alpha$$

$$\rightarrow \sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$$

$$\rightarrow \boxed{\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}}$$

iii) $\tan\frac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$

we know that

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

Similarly

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \quad \text{(In form of half angle)}$$

$$\rightarrow \tan\frac{\alpha}{2} = \pm\frac{\sqrt{\frac{1-\cos\alpha}{2}}}{\sqrt{\frac{1+\cos\alpha}{2}}}$$

$$\rightarrow \boxed{\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}}$$

Triple angle Identities

i) $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

$$\begin{aligned}
L.H.S &= \sin 3\alpha \\
&= \sin(2\alpha+\alpha) \\
&= \sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha \\
&= 2\sin\alpha\cos\alpha\cos\alpha + (1-2\sin^2\alpha)\sin\alpha \\
&= 2\sin\alpha\cos^2\alpha + \sin\alpha - 2\sin^3\alpha \\
&= 2\sin\alpha(1-\sin^2\alpha) + \sin\alpha - 2\sin^3\alpha \\
&= 2\sin\alpha - 2\sin^3\alpha + \sin\alpha - 2\sin^3\alpha \\
&= 3\sin\alpha - 4\sin^3\alpha = R.H.S
\end{aligned}$$

Hence $\boxed{\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha}$

ii) $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

$$\begin{aligned}
L.H.S &= \cos 3\alpha \\
&= \cos(2\alpha+\alpha) \\
&= \cos 2\alpha\cos\alpha - \sin 2\alpha\sin\alpha \\
&= (2\cos^2\alpha - 1)\cos\alpha - 2\sin\alpha\cos\alpha\sin\alpha \\
&= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha \\
&= 2\cos^3\alpha - \cos\alpha - 2(1-\cos^2\alpha)\cos\alpha \\
&= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha \\
&= 4\cos^3\alpha - 3\cos\alpha = R.H.S
\end{aligned}$$

Hence $\boxed{\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha}$

iii) $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$

$$\begin{aligned}
L.H.S &= \tan 3\alpha \\
&= \tan(2\alpha+\alpha) \\
&= \frac{\tan 2\alpha + \tan\alpha}{1-\tan 2\alpha\tan\alpha}
\end{aligned}$$

$$= \frac{\frac{2\tan\alpha}{1-\tan^2\alpha} + \tan\alpha}{1 - \frac{2\tan\alpha}{1-\tan^2\alpha}.\tan\alpha}$$

$$= \frac{\frac{2\tan\alpha + \tan\alpha - \tan^3\alpha}{1-\cancel{\tan^2\alpha}}}{\frac{1-\tan^2\alpha - 2\tan^2\alpha}{1-\cancel{\tan^2\alpha}}}$$

$$= \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha} = R.H.S$$

Hence $\boxed{\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}}$

Example 1. Prove that

$$\frac{\sin A + \sin 2A}{1+\cos A + \cos 2A} = \tan A$$

Solution:- $L.H.S = \frac{\sin A + \sin 2A}{1+\cos A + \cos 2A}$

$$= \frac{\sin A + 2\sin A\cos A}{\cancel{1} + \cos A + 2\cos^2 A - \cancel{1}}$$

$$= \frac{\sin A\,(\cancel{1+2\cos A})}{\cos A\,(\cancel{1+2\cos A})} = \frac{\sin A}{\cos A}$$

$$= \tan A = R.H.S$$

Hence proved

Example 2. Show that

i) $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$ ii) $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

Solution:- i) $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

$$R.H.S = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$= \frac{2\tan\theta}{\sec^2\theta} = 2\frac{\sin\theta}{\cancel{\cos\theta}}.\cos^{\cancel{2}}\theta$$

$$= 2\sin\theta\cos\theta = \sin 2\theta = L.H.S$$

Hence proved

ii) $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

$$R.H.S = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$= \frac{1-\tan^2\theta}{\sec^2\theta}$$

$$= \left(1 - \frac{\sin^2\theta}{\cos^2\theta}\right)\cos^2\theta$$

$$= \left(\frac{\cos^2\theta - \sin^2\theta}{\cancel{\cos^2\theta}}\right)\cancel{\cos^2\theta}$$

$$= \cos^2\theta - \sin^2\theta$$

$$= \cos 2\theta = L.H.S$$

Hence proved

Example 3. Reduce $\cos^4\theta$ to an expression involving only function of multiples of θ, raised to the first power.

Solution:-

$$\cos^4\theta = (\cos^2\theta)^2 \longrightarrow (i)$$

$$\because \cos 2\theta = 2\cos^2\theta - 1$$

$$\Rightarrow 2\cos^2\theta = 1 + \cos 2\theta$$

$$\Rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

so (i) becomes as

$$\cos^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2$$

$$= \frac{1 + 2\cos\theta + \cos^2 2\theta}{4}$$

$$\Rightarrow \cos^4\theta = \frac{1}{4}\left[1 + 2\cos\theta + \cos^2 2\theta\right]$$

$$\because \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\Rightarrow \cos^2 2\theta = \frac{1+\cos 4\theta}{2} \quad \text{so}$$

$$\cos^4\theta = \frac{1}{4}\left[1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right]$$

$$= \frac{1}{4}\left[\frac{2 + 4\cos 2\theta + 1 + \cos 4\theta}{2}\right]$$

$$\Rightarrow \cos^4\theta = \frac{1}{8}\left[3 + 4\cos 2\theta + \cos 4\theta\right]$$

Exercise 10.3

Q1. Find the values of $\sin 2\alpha$, $\cos 2\alpha$, and $\tan 2\alpha$, when:

i) $\sin\alpha = \frac{12}{13}$ ii) $\cos\alpha = \frac{3}{5}$, where $0 < \alpha < \frac{\pi}{2}$

Solution:- i) $\sin\alpha = \frac{12}{13}$ (α is in I quad)

$\because \cos\alpha = \pm\sqrt{1-\sin^2\alpha}$

$= \pm\sqrt{1-\left(\frac{12}{13}\right)^2} = \pm\sqrt{1-\frac{144}{169}}$

$= \pm\sqrt{\frac{169-144}{169}} = \pm\sqrt{\frac{25}{169}}$

$\cos\alpha = \pm\frac{5}{13} \rightarrow \cos\alpha = \frac{5}{13}$ ($\because \alpha$ is in I quad)

$\because \sin 2\alpha = 2\sin\alpha\cos\alpha$

$\rightarrow \sin 2\alpha = 2\left(\frac{12}{13}\right)\left(\frac{5}{13}\right) = \frac{120}{169}$

$\because \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$= \left(\frac{5}{13}\right)^2 - \left(\frac{12}{13}\right)^2 = \frac{25}{169} - \frac{144}{169}$

$\rightarrow \cos 2\alpha = \frac{25-144}{169} = \frac{-119}{169}$

$\because \tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{120/169}{-119/169}$

$\rightarrow \tan 2\alpha = -\frac{120}{119}$

ii) $\cos\alpha = \frac{3}{5}$ (α is in I quad)

$\because \sin\alpha = \pm\sqrt{1-\cos^2\alpha}$

$= \pm\sqrt{1-\left(\frac{3}{5}\right)^2}$

$= \pm\sqrt{1-\frac{9}{25}} = \pm\sqrt{\frac{25-9}{25}}$

$\sin\alpha = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$

$\rightarrow \sin\alpha = \frac{4}{5}$ ($\because \alpha$ is in I quad)

$\because \sin 2\alpha = 2\sin\alpha\cos\alpha$

$\rightarrow \sin 2\alpha = 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}$

$\because \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$= \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25}$

$\rightarrow \cos 2\alpha = \frac{9-16}{25} = \frac{-7}{25}$

$\because \tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{24/25}{-7/25} = \frac{-24}{7}$

Prove the following identities:

Q2. $\cot\alpha - \tan\alpha = 2\cot 2\alpha$

Solution:- L.H.S $= \cot\alpha - \tan\alpha$

$= \frac{\cos\alpha}{\sin\alpha} - \frac{\sin\alpha}{\cos\alpha} = \frac{\cos^2\alpha - \sin^2\alpha}{\sin\alpha\cos\alpha}$

$= \frac{\cos 2\alpha}{\sin\alpha\cos\alpha} = \frac{2\cos 2\alpha}{2\sin\alpha\cos\alpha}$

$= \frac{2\cos 2\alpha}{\sin 2\alpha} = 2\cot 2\alpha =$ R.H.S

Hence proved

Q3. $\frac{\sin 2\alpha}{1+\cos 2\alpha} = \tan\alpha$

Solution:- L.H.S $= \frac{\sin 2\alpha}{1+\cos 2\alpha}$

$= \frac{2\sin\alpha\cos\alpha}{1+2\cos^2\alpha-1} = \frac{\cancel{2}\sin\alpha\cancel{\cos\alpha}}{\cancel{2}\cos^{\cancel{2}}\alpha}$

$= \frac{\sin\alpha}{\cos\alpha} = \tan\alpha =$ R.H.S

Hence proved

Q4. $\frac{1-\cos\alpha}{\sin\alpha} = \tan\frac{\alpha}{2}$

Solution:- L.H.S $= \frac{1-\cos\alpha}{\sin\alpha}$

$= \frac{\cancel{2}\sin^{\cancel{2}}\frac{\alpha}{2}}{\cancel{2}\cancel{\sin\frac{\alpha}{2}}\cos\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2} =$ R.H.S

Hence proved

Q5. $\frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} = \sec 2\alpha - \tan 2\alpha$

Solution:- L.H.S $= \frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha}$

$= \frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} \times \frac{\cos\alpha - \sin\alpha}{\cos\alpha - \sin\alpha}$

$= \frac{(\cos\alpha - \sin\alpha)^2}{\cos^2\alpha - \sin^2\alpha}$

$$= \frac{\cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha}{\cos 2\alpha}$$

$$= \frac{1 - \sin 2\alpha}{\cos 2\alpha} = \frac{1}{\cos 2\alpha} - \frac{\sin 2\alpha}{\cos 2\alpha}$$

$$= \sec 2\alpha - \tan 2\alpha = R.H.S$$

Hence proved

Q6. $\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} = \frac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}}$

Solution:- $L.H.S = \sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}$

$$= \sqrt{\frac{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}}$$

$$= \sqrt{\frac{(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2})^2}{(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2})^2}}$$

$$= \frac{\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}} = R.H.S$$

Hence proved

Q7. $\frac{\operatorname{cosec}\theta + 2\operatorname{cosec}2\theta}{\sec\theta} = \cot\frac{\theta}{2}$

Solution:- $L.H.S = \frac{\operatorname{cosec}\theta + 2\operatorname{cosec}2\theta}{\sec\theta}$

$$= \frac{1}{\sec\theta}\left[\frac{1}{\sin\theta} + \frac{2}{\sin 2\theta}\right]$$

$$= \cos\theta\left[\frac{1}{\sin\theta} + \frac{\cancel{2}}{\cancel{2}\sin\theta\cos\theta}\right]$$

$$= \cos\theta\left[\frac{1}{\sin\theta} + \frac{1}{\sin\theta\cos\theta}\right]$$

$$= \cancel{\cos\theta}\left[\frac{\cos\theta + 1}{\sin\theta\cancel{\cos\theta}}\right]$$

$$= \frac{1+\cos\theta}{\sin\theta} = \frac{\cancel{2}\cos^{\cancel{2}}\theta/2}{\cancel{2}\sin\theta/2\cancel{\cos\theta/2}}$$

$$= \frac{\cos\theta/2}{\sin\theta/2} = \because \cot\theta/2 = R.H.S$$

Hence proved

Q8. $1+\tan\alpha\tan 2\alpha = \sec 2\alpha$

Solution:-

$L.H.S = 1+\tan\alpha\tan 2\alpha$

$$= 1 + \frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin 2\alpha}{\cos 2\alpha}$$

$$= 1 + \frac{\sin\alpha}{\cancel{\cos\alpha}}\cdot\frac{2\sin\alpha\cancel{\cos\alpha}}{\cos^2\alpha - \sin^2\alpha}$$

$$= 1 + \frac{2\sin^2\alpha}{\cos^2\alpha - \sin^2\alpha}$$

$$= \frac{\cos^2\alpha - \sin^2\alpha + 2\sin^2\alpha}{\cos^2\alpha - \sin^2\alpha}$$

$$= \frac{\cos^2\alpha + \sin^2\alpha}{\cos 2\alpha} = \frac{1}{\cos 2\alpha}$$

$$= \sec 2\alpha = R.H.S$$

Hence proved

Q9. $\frac{2\sin\theta\sin 2\theta}{\cos\theta + \cos 3\theta} = \tan 2\theta\tan\theta$

Solution:- $L.H.S = \frac{2\sin\theta\sin 2\theta}{\cos\theta + \cos 3\theta}$

$$= \frac{2\sin\theta\sin 2\theta}{\cos\theta + 4\cos^3\theta - 3\cos\theta} \quad \left(\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta\right)$$

$$= \frac{2\sin\theta\sin 2\theta}{4\cos^3\theta - 2\cos\theta}$$

$$= \frac{\cancel{2}\sin\theta\sin 2\theta}{\cancel{2}\cos\theta(2\cos^2\theta - 1)} = \frac{\sin\theta\sin 2\theta}{\cos\theta\cdot\cos 2\theta}$$

$$= \tan\theta\tan 2\theta = R.H.S$$

Hence proved

Q10. $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$

Solution:-

$L.H.S = \frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta}$

$$= \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin(3\theta - \theta)}{\sin\theta\cos\theta} = \frac{\sin 2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\sin 2\theta}{2\sin\theta\cos\theta} = \frac{2\cancel{\sin 2\theta}}{\cancel{\sin 2\theta}} = 2 = R.H.S$$

Hence proved

Q11. $\frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta} = 4\cos 2\theta$

Solution:- $L.H.S = \frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta}$

$$= \frac{\cos 3\theta \sin\theta + \sin 3\theta \cos\theta}{\cos\theta \sin\theta}$$

$$= \frac{\sin(3\theta+\theta)}{\sin\theta\cos\theta} = \frac{\sin 4\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\sin 2\theta \cos 2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\cos 2\theta\,(2\cancel{\sin\theta\cos\theta})}{\cancel{\sin\theta\cos\theta}}$$

$$= 4\cos 2\theta = R.H.S$$

Hence proved

Q12. $\dfrac{\tan\frac{\theta}{2} + \cot\frac{\theta}{2}}{\cot\frac{\theta}{2} - \tan\frac{\theta}{2}} = \sec\theta$

Solution:-

$$L.H.S = \frac{\tan\frac{\theta}{2} + \cot\frac{\theta}{2}}{\cot\frac{\theta}{2} - \tan\frac{\theta}{2}}$$

$$= \frac{\dfrac{\sin\theta/2}{\cos\theta/2} + \dfrac{\cos\theta/2}{\sin\theta/2}}{\dfrac{\cos\theta/2}{\sin\theta/2} - \dfrac{\sin\theta/2}{\cos\theta/2}}$$

$$= \frac{\dfrac{\sin^2\theta/2 + \cos^2\theta/2}{\cancel{\cos\theta/2\,\sin\theta/2}}}{\dfrac{\cos^2\theta/2 - \sin^2\theta/2}{\cancel{\cos\theta/2\,\sin\theta/2}}}$$

$$= \frac{\sin^2\theta/2 + \cos^2\theta/2}{\cos^2\theta/2 - \sin^2\theta/2} = \frac{1}{\cos 2(\theta/2)}$$

$$= \frac{1}{\cos\theta} = \sec\theta = R.H.S$$

Hence proved

Q13. $\dfrac{\sin 3\theta}{\cos\theta} + \dfrac{\cos 3\theta}{\sin\theta} = 2\cot 2\theta$

Solution:-

$$L.H.S = \frac{\sin 3\theta}{\cos\theta} + \frac{\cos 3\theta}{\sin\theta}$$

$$= \frac{\sin 3\theta\sin\theta + \cos 3\theta\cos\theta}{\cos\theta\sin\theta}$$

$$= \frac{\cos 3\theta\cos\theta + \sin 3\theta\sin\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos(3\theta-\theta)}{\sin\theta\cos\theta} = \frac{\cos 2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\cos 2\theta}{2\sin\theta\cos\theta} = \frac{2\cos 2\theta}{\sin 2\theta}$$

$$= 2\cot 2\theta = R.H.S$$

Hence proved

Q14. Reduce $\sin^4\theta$ to an an expression involving only function of multiples of θ raised to the first power.

Solution:-

$$\sin^4\theta = (\sin^2\theta)^2 \longrightarrow (i)$$

$\because$ $\cos 2\theta = 1 - 2\sin^2\theta$

$\Rightarrow 2\sin^2\theta = 1 - \cos 2\theta$

$\Rightarrow \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

So (i) becomes as

$$\sin^4\theta = \left(\frac{1-\cos 2\theta}{2}\right)^2$$

$$= \frac{1 - 2\cos 2\theta + \cos^2 2\theta}{4}$$

$$= \frac{1}{4}\left[1 - 2\cos 2\theta + \cos^2 2\theta\right]$$

$\because$ $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$\Rightarrow \cos^2 2\theta = \dfrac{1+\cos 4\theta}{2}$ so

$$\sin^4\theta = \frac{1}{4}\left(1 - 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)$$

$$= \frac{1}{4}\left[\frac{2 - 4\cos 2\theta + 1 + \cos 4\theta}{2}\right]$$

$$\sin^4\theta = \frac{1}{8}\left[3 - 4\cos 2\theta + \cos 4\theta\right]$$

Q15. Find the values of $\sin\theta$ and $\cos\theta$ without using table or calculator, where θ

i) 18° ii) 36° iii) 54° iv) 72°

Hence prove that:

$$\cos 36^\circ \cos 72^\circ \cos 108^\circ \cos 144^\circ = \frac{1}{16}$$

Hint: Let $\theta = 18^\circ$

$\rightarrow 5\theta = 90^\circ$

$3\theta + 2\theta = 90^\circ$

$3\theta = 90^\circ - 2\theta$

$\sin 3\theta = \sin(90^\circ - 2\theta)$

Let $\theta = 36^\circ$

$5\theta = 180^\circ$

$3\theta + 2\theta = 180^\circ$

$3\theta = 180^\circ - 2\theta$

$\sin 3\theta = \sin(180^\circ - 2\theta)$ etc

Solution:- i) 18°

Let $\theta = 18^\circ$

$\rightarrow 5\theta = 90^\circ$

$\rightarrow 2\theta + 3\theta = 90^\circ$

$\rightarrow 2\theta = 90^\circ - 3\theta$

$\rightarrow \sin 2\theta = \sin(90^\circ - 3\theta)$

$\rightarrow 2\sin\theta\cos\theta = \cos 3\theta$

$2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$

$2\sin\theta\cancel{\cos\theta} = \cancel{\cos\theta}(4\cos^2\theta - 3)$

$2\sin\theta = 4(1-\sin^2\theta) - 3$

$= 4 - 4\sin^2\theta - 3$

$2\sin\theta = 1 - 4\sin^2\theta$

$\rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0$

Here $a = 4,\ b = 2,\ c = -1$

$$\sin\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\sin\theta = \frac{-2 \pm \sqrt{(2)^2 - 4(4)(-1)}}{2(4)}$$

$$\sin\theta = \frac{-2 \pm \sqrt{4+16}}{8}$$

$$\sin\theta = \frac{-2 \pm \sqrt{20}}{8} = \frac{-2 \pm 2\sqrt{5}}{8}$$

$$\sin\theta = \frac{2(-1 \pm \sqrt{5})}{8}$$

$$\rightarrow \sin\theta = \frac{-1 \pm \sqrt{5}}{4}$$

Put $\theta = 18^\circ$ so

$$\boxed{\sin\theta = \frac{-1+\sqrt{5}}{4}} \quad (\because 18^\circ \text{ lies in I quad})$$

$\because \cos^2\theta = 1 - \sin^2\theta$

$$\cos^2\theta = 1 - \left(\frac{-1+\sqrt{5}}{4}\right)^2 = 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$\cos^2\theta = 1 - \left(\frac{5+1-2\sqrt{5}}{16}\right) = \frac{16 - 6 + 2\sqrt{5}}{16}$$

$$\cos^2\theta = \frac{10+2\sqrt{5}}{16}$$

$$\cos\theta = \pm\sqrt{\frac{10+2\sqrt{5}}{16}}$$

$$\rightarrow \cos\theta = \pm\frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$\rightarrow \boxed{\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}} \quad (\because 18^\circ \text{ lies in I quad})$$

ii) 36°

Let $\theta = 36^\circ$

$\because \cos 2\theta = 2\cos^2\theta - 1$

Put $\theta = 18^\circ$ $\rightarrow \cos 2(18^\circ) = 2\cos^2 18^\circ - 1$

$\rightarrow \cos 36^\circ = 2(\cos 18^\circ)^2 - 1$

$$= 2\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2 - 1$$

$$\cos 36^\circ = 2\left(\frac{10+2\sqrt{5}}{16}\right) - 1$$

$$\cos 36^\circ = \frac{10+2\sqrt{5}}{8} - 1 = \frac{10+2\sqrt{5}-8}{8}$$

$$\rightarrow \cos 36^\circ = \frac{2+2\sqrt{5}}{8} = \frac{2(1+\sqrt{5})}{8}$$

$$\rightarrow \boxed{\cos 36^\circ = \frac{1+\sqrt{5}}{4}}$$

$\because \sin^2\theta = 1 - \cos^2\theta$

$\rightarrow \sin^2 36^\circ = 1 - \cos^2 36^\circ$

$$= 1 - (\cos 36^\circ)^2$$

$$= 1 - \left(\frac{1+\sqrt{5}}{4}\right)^2$$

$$= 1 - \frac{1+5+2\sqrt{5}}{16}$$

$$\sin^2 36^\circ = \frac{16 - 6 - 2\sqrt{5}}{16}$$

$$\sin^2 36^\circ = \frac{10-2\sqrt{5}}{16}$$

$$\rightarrow \boxed{\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}}$$

iii) 54°

Let $\theta = 54^\circ$

$$\because \cos 54^\circ = \cos(90^\circ - 54^\circ)$$

$$\cos 54^\circ = \sin 36^\circ = \frac{10-2\sqrt{5}}{4}$$

$$\rightarrow \boxed{\cos 54^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}}$$

$$\because \sin 54^\circ = \sin(90^\circ - 54^\circ)$$

$$= \therefore \cos 36^\circ = \frac{1+\sqrt{5}}{4}$$

$$\rightarrow \boxed{\sin 54^\circ = \frac{1+\sqrt{5}}{4}}$$

iv) 72°

Let $\theta = 72^\circ$

$$\because \sin 72^\circ = \sin(90^\circ - 18^\circ)$$

$$= \cos 18^\circ = \frac{\sqrt{10+2\sqrt{2}}}{4}$$

$$\rightarrow \boxed{\sin 72^\circ = \frac{10+2\sqrt{2}}{4}}$$

$$\cos 72^\circ = \cos(90^\circ - 18^\circ)$$

$$= \sin 18^\circ = \frac{-1+\sqrt{5}}{4}$$

$$\rightarrow \boxed{\cos 72^\circ = \frac{-1+\sqrt{5}}{4}}$$

Now

$$\text{L.H.S} = \cos 36^\circ \cos 72^\circ \cos 108^\circ \cos 144^\circ$$

$$= \cos 36^\circ \cos 72^\circ \cos(180^\circ - 72^\circ)\cos(180^\circ - 36^\circ)$$

$$= \cos 36^\circ \cos 72^\circ (-\cos 72^\circ)(-\cos 36^\circ)$$

$$= \cos^2 36^\circ \cos^2 72^\circ$$

$$= \left(\frac{1+\sqrt{5}}{4}\right)^2\left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= \left(\frac{1+5+2\sqrt{5}}{16}\right)\left(\frac{5+1-2\sqrt{5}}{16}\right)$$

$$= \left(\frac{6+2\sqrt{5}}{16}\right)\left(\frac{6-2\sqrt{5}}{16}\right)$$

$$= \frac{(6)^2-(2\sqrt{5})^2}{(16)^2}$$

$$= \frac{36-4(5)}{16\times 16} = \frac{36-20}{16\times 16}$$

$$= \frac{16}{16\times 16} = \frac{1}{16} = \text{R.H.S}$$

Hence proved.

Sum, Difference and Products of Sine and Cosines

We know that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \longrightarrow (i)$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \longrightarrow (ii)$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \longrightarrow (iii)$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \longrightarrow (iv)$$

By (i) + (ii) →

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$

By (i) − (ii) →

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$$

By (iii) + (iv) →

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$

By (iii) − (iv) →

$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$$

So we get four identities as:

$$\boxed{\begin{aligned} 2\sin\alpha\cos\beta &= \sin(\alpha+\beta) + \sin(\alpha-\beta) \\ 2\cos\alpha\sin\beta &= \sin(\alpha+\beta) - \sin(\alpha-\beta) \\ 2\cos\alpha\cos\beta &= \cos(\alpha+\beta) + \cos(\alpha-\beta) \\ -2\sin\alpha\sin\beta &= \cos(\alpha+\beta) - \cos(\alpha-\beta) \end{aligned}}$$

Now put $\alpha+\beta = P \longrightarrow (1)$

and $\alpha-\beta = Q \longrightarrow (2)$

By (1) + (2) $\rightarrow$ $2\alpha = P + Q$

$\rightarrow \alpha = \dfrac{P+Q}{2}$

By (1) − (2) $\rightarrow$ $2\beta = P - Q \rightarrow \beta = \dfrac{P-Q}{2}$

so

$$\boxed{\begin{aligned} \sin P + \sin Q &= 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2} \\ \sin P - \sin Q &= 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2} \\ \cos P + \cos Q &= 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2} \\ \cos P - \cos Q &= -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2} \end{aligned}}$$

Example 1. Express $2\sin 7\theta\cos 3\theta$ as a sum or difference.

Solution:- $2\sin 7\theta \cos 3\theta$

$= \sin(7\theta + 3\theta) + \sin(7\theta - 3\theta)$

$= \sin 10\theta + \sin 4\theta$

$(\because 2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta))$

Example 2. Prove without using tables/calculator, that

$\sin 19^\circ \cos 11^\circ + \sin 71^\circ \sin 11^\circ = \frac{1}{2}$

Solution:-

L.H.S $= \sin 19^\circ\cos 11^\circ + \sin 71^\circ \sin 11^\circ$

$= \frac{1}{2}[2\sin 19^\circ\cos 11^\circ + 2\sin 71^\circ\sin 11^\circ]$

$= \frac{1}{2}[2\sin 19^\circ \cos 11^\circ - (-2\sin 71^\circ \sin 11^\circ)]$

$= \frac{1}{2}[\sin(19^\circ+11^\circ) + \sin(19^\circ - 11^\circ) - (\cos(71^\circ+11^\circ) - \cos(71^\circ - 11^\circ))]$

$= \frac{1}{2}[\sin 30^\circ + \sin 8^\circ - \cos 82^\circ + \cos 60^\circ]$

$= \frac{1}{2}(\sin 30^\circ + \sin 8^\circ - \cos(90^\circ - 8^\circ) + \cos 60^\circ]$

$= \frac{1}{2}(\sin 30^\circ + \cancel{\sin 8^\circ} - \cancel{\sin 8^\circ} + \cos 60^\circ]$

$= \frac{1}{2}(\sin 30^\circ + \cos 60^\circ) = \frac{1}{2}(\frac{1}{2} + \frac{1}{2})$

$= \frac{1}{2}(1) = \frac{1}{2} =$ R.H.S

Hence proved

Example 3. Express $\sin 5x + \sin 7x$ as a product.

Solution:- $\sin 5x + \sin 7x$

$= 2\sin\left(\frac{5x+7x}{2}\right)\cos\left(\frac{5x-7x}{2}\right)$

$= 2\sin\left(\frac{12x}{2}\right)\cos\left(\frac{-2x}{2}\right)$

$= 2\sin 6x\cos(-x)$ $\quad \because \cos(-x) = \cos x$

$= 2\sin 6x\cos x$

$(\because \sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2})$

Example 4. Express $\cos A + \cos 3A + \cos 5A + \cos 7A$ as a product.

Solution:-

$\cos A + \cos 3A + \cos 5A + \cos 7A$

$= (\cos 3A + \cos A) + (\cos 7A + \cos 5A)$

$= 2\cos\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right) + 2\cos\left(\frac{7A+5A}{2}\right)\cos\left(\frac{7A-5A}{2}\right)$

$= 2\cos\left(\frac{4A}{2}\right)\cos\left(\frac{2A}{2}\right) + 2\cos\left(\frac{12A}{2}\right)\cos\left(\frac{2A}{2}\right)$

$= 2\cos 2A\cos A + 2\cos 6A\cos A$

$= 2\cos A(\cos 2A + \cos 6A)$

$= 2\cos A\left[2\cos\left(\frac{2A+6A}{2}\right)\cos\left(\frac{2A-6A}{2}\right)\right]$

$= 2\cos A\left[2\cos\left(\frac{8A}{2}\right)\cos\left(-\frac{4A}{2}\right)\right]$

$= 2\cos A[2\cos 4A\cos(-2A)]$

$= 2\cos A(2\cos 4A\cos 2A)$ $\quad \because \cos(-\theta) = \cos\theta$

$= 4\cos A\cos 4A\cos 2A$

$(\because \cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2})$

Example 5. Show that

$\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$

Solution:-

L.H.S $= \cos 20^\circ \cos 40^\circ \cos 80^\circ$

$= \frac{1}{2}(2\cos 20^\circ\cos 40^\circ)\cos 80^\circ$

$\because 2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

$= \frac{1}{2}[\cos(20^\circ+40^\circ) + \cos(20^\circ - 40^\circ)]\cos 80^\circ$

$= \frac{1}{2}[\cos 60^\circ + \cos(-20^\circ)]\cos 80^\circ$

$= \frac{1}{2}\left(\frac{1}{2} + \cos 20^\circ\right)\cos 80^\circ$

$= \frac{1}{4}\cos 80^\circ + \frac{1}{2}\cos 20^\circ \cos 80^\circ$

$= \frac{1}{4}\cos 80^\circ + \frac{1}{4}(2\cos 20^\circ \cos 80^\circ)$

$= \frac{1}{4}\cos 80^\circ + \frac{1}{4}[\cos(20^\circ+80^\circ) + \cos(20^\circ-80^\circ)]$

$= \frac{1}{4}\cos 80^\circ + \frac{1}{4}[\cos 100^\circ + \cos(-60^\circ)]$

$= \frac{1}{4}\cos 80^\circ + \frac{1}{4}(\cos 100^\circ + \cos 60^\circ)$

$= \frac{1}{4}(\cos 80^\circ + \cos 100^\circ + \frac{1}{2})$

$= \frac{1}{4}(\cos 80^\circ + \cos(180^\circ - 80^\circ) + \frac{1}{2})$

$= \frac{1}{4}(\cancel{\cos 80^\circ} - \cancel{\cos 80^\circ} + \frac{1}{2}) = \frac{1}{8} = R.H.S$

$\because \cos(180^\circ - \theta) = -\cos\theta$

Hence proved

Exercise 10.4

Q1. Express the following products as sums or differences:

i) $2\sin 3\theta \cos\theta$

Solution:- $2\sin 3\theta \cos\theta$

$\because 2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

$= \sin(3\theta+\theta) + \sin(3\theta-\theta)$

$= \sin 4\theta + \sin 2\theta$

ii) $2\cos 5\theta \sin 3\theta$

Solution:- $2\cos 5\theta \sin 3\theta$

$\because 2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

$= \sin(5\theta+3\theta) - \sin(5\theta-3\theta)$

$= \sin 8\theta - \sin 2\theta$

iii) $\sin 5\theta \cos 2\theta$

Solution:- $\sin 5\theta \cos 2\theta$

$= \frac{1}{2}[2\sin 5\theta \cos 2\theta]$

$= \frac{1}{2}(\sin(5\theta+2\theta) + \sin(5\theta-2\theta))$

$= \frac{1}{2}(\sin 7\theta + \sin 3\theta)$

iv) $2\sin 7\theta \sin 2\theta$

Solution:- $2\sin 7\theta \sin 2\theta$

$= -(-2\sin 7\theta \sin 2\theta)$

$(\because -2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta))$

$= -(\cos(7\theta+2\theta) - \cos(7\theta-2\theta))$

$= -(\cos 9\theta - \cos 5\theta)$

$= \cos 5\theta - \cos 9\theta$

v) $\cos(x+y)\sin(x-y)$

Solution:- $\cos(x+y)\sin(x-y)$

$= \frac{1}{2}(2\cos(x+y)\sin(x-y))$

$= \frac{1}{2}(\sin(x+y+x-y) - \sin(x+y-x+y))$

$= \frac{1}{2}(\sin 2x - \sin 2y)$

vi) $\cos(2x+30^\circ)\cos(2x-30^\circ)$

Solution:- $\cos(2x+30^\circ)\cos(2x-30^\circ)$

$= \frac{1}{2}[2\cos(2x+30^\circ)\cos(2x-30^\circ)]$

$= \frac{1}{2}[\cos(2x+30^\circ+2x-30^\circ) + \cos(2x+30^\circ-2x+30^\circ)]$

$= \frac{1}{2}[\cos 4x + \cos 60^\circ]$

vii) $\sin 12^\circ \sin 46^\circ$

Solution:- $\sin 12^\circ \sin 46^\circ$

$= -\frac{1}{2}(-2\sin 12^\circ \sin 46^\circ)$

$= -\frac{1}{2}(\cos(12^\circ+46^\circ) - \cos(12^\circ-46^\circ))$

$= -\frac{1}{2}(\cos 58^\circ - \cos(-34^\circ))$

$= -\frac{1}{2}(\cos 58^\circ + \cos 34^\circ)$

viii) $\sin(x+45^\circ)\sin(x-45^\circ)$

Solution:- $\sin(x+45^\circ)\sin(x-45^\circ)$

$= -\frac{1}{2}(-2\sin(x+45^\circ)\sin(x-45^\circ))$

$= -\frac{1}{2}(\cos(x+45^\circ+x-45^\circ) - \cos(x+45^\circ-x+45^\circ))$

$= -\frac{1}{2}(\cos 2x - \cos 90^\circ)$

$= \frac{1}{2}(\cos 90^\circ - \cos 2x)$

Q2. Express the following sums or differences as products:

i) $\sin 5\theta + \sin 3\theta$

Solution:- $\sin 5\theta + \sin 3\theta$

$\because \sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$

$= 2\sin\frac{5\theta+3\theta}{2}\cos\frac{5\theta-3\theta}{2}$

$= 2\sin\frac{8\theta}{2}\cos\frac{2\theta}{2} = 2\sin 4\theta\cos\theta$

ii) $\sin 8\theta - \sin 4\theta$

Solution:- $\sin 8\theta - \sin 4\theta$

$\because \sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$= 2\cos\frac{8\theta+4\theta}{2}\sin\frac{8\theta-4\theta}{2}$

$= 2\cos\frac{12\theta}{2}\sin\frac{4\theta}{2} = 2\cos 6\theta\sin 2\theta$

iii) $\cos 6\theta + \cos 3\theta$

Solution:- $\cos 6\theta + \cos 3\theta$

$\because \cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$

$= 2\cos\frac{6\theta+3\theta}{2}\cos\frac{6\theta-3\theta}{2}$

$= 2\cos\frac{9\theta}{2}\cos\frac{3\theta}{2}$

iv) $\cos 7\theta - \cos\theta$

Solution:- $\cos 7\theta - \cos\theta$

$\because \cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$= -2\sin\frac{7\theta+\theta}{2}\sin\frac{7\theta-\theta}{2}$

$= -2\sin\frac{8\theta}{2}\sin\frac{6\theta}{2}$

$= -2\sin 4\theta\sin 3\theta$

v) $\cos 12^\circ + \cos 48^\circ$

Solution:- $\cos 12^\circ + \cos 48^\circ$

$= 2\cos\frac{12^\circ+48^\circ}{2}\cos\frac{12^\circ-48^\circ}{2}$

$= 2\cos\frac{60^\circ}{2}\cos\left(-\frac{36^\circ}{2}\right)$

$= 2\cos 30^\circ\cos(-18^\circ)$

$= 2\cos 30^\circ\cos 18^\circ \qquad \because \cos(-\theta) = \cos\theta$

vi) $\sin(x+30^\circ)\sin(x-30^\circ)$

Solution:- $\sin(x+30^\circ)+\sin(x-30^\circ)$

$= 2\sin\left(\frac{x+30^\circ+x-30^\circ}{2}\right)\cos\left(\frac{x+30^\circ-x+30^\circ}{2}\right)$

$= 2\sin\left(\frac{2x}{2}\right)\cos\left(\frac{60^\circ}{2}\right)$

$= 2\sin x\cos 30^\circ$

Q3. Prove the following identities:

i) $\frac{\sin 3x - \sin x}{\cos x - \cos 3x} = \cot 2x$

Solution:-

L.H.S $= \frac{\sin 3x - \sin x}{\cos x - \cos 3x}$

$= \frac{\cancel{2}\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{-\cancel{2}\sin\left(\frac{x+3x}{2}\right)\sin\left(\frac{x-3x}{2}\right)}$

$= -\frac{\cos\left(\frac{4x}{2}\right)\sin\left(\frac{2x}{2}\right)}{\sin\left(\frac{4x}{2}\right)\sin\left(\frac{-2x}{2}\right)}$

$= -\frac{\cos 2x\sin x}{\sin 2x\sin(-x)} = \frac{\cancel{-}\cos 2x\cancel{\sin x}}{\cancel{-}\sin 2x\cancel{\sin x}} \qquad \because \sin(-\theta) = -\sin\theta$

$= \cot 2x = $ R.H.S

Hence proved

ii) $\frac{\sin 8x + \sin 2x}{\cos 8x + \cos 2x} = \tan 5x$

Solution:- L.H.S $= \frac{\sin 8x + \sin 2x}{\cos 8x + \cos 2x}$

$= \frac{\cancel{2}\sin\frac{8x+2x}{2}\cos\frac{8x-2x}{2}}{\cancel{2}\cos\frac{8x+2x}{2}\cos\frac{8x-2x}{2}}$

$= \frac{\sin\frac{10x}{2}\cancel{\cos\frac{6x}{2}}}{\cos\frac{10x}{2}\cancel{\cos\frac{6x}{2}}} = \frac{\sin 5x}{\cos 5x}$

$= \cot 5x = $ R.H.S

Hence proved

iii) $\frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \tan\frac{\alpha-\beta}{2}\cot\frac{\alpha+\beta}{2}$

Solution:-

L.H.S $= \frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta}$

$$= \frac{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}}$$

$$= \cot\frac{\alpha+\beta}{2}\tan\frac{\alpha-\beta}{2} = R.H.S$$

Hence proved

Q4. Prove that

i) $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$

Solution:-

$L.H.S = (\cos 20^\circ + \cos 100^\circ) + \cos 140^\circ$

$= 2\cos\left(\frac{20^\circ+100^\circ}{2}\right)\cos\left(\frac{20^\circ-100^\circ}{2}\right) + \cos 140^\circ$

$= 2\cos\left(\frac{120^\circ}{2}\right)\cos\left(-\frac{80^\circ}{2}\right) + \cos 140^\circ$

$= 2\cos 60^\circ \cos(-40^\circ) + \cos 140^\circ$

$= 2\left(\frac{1}{2}\right)\cos 40^\circ + \cos 140^\circ$

$= \cos 40^\circ + \cos 140^\circ$

$= 2\cos\left(\frac{40^\circ+140^\circ}{2}\right)\cos\left(\frac{40^\circ-140^\circ}{2}\right)$

$= 2\cos\left(\frac{180^\circ}{2}\right)\cos\left(\frac{-100^\circ}{2}\right)$

$= 2\cos 90^\circ \cos(-50^\circ)$

$= 2(0)\cos 50^\circ = 0 = R.H.S$

Hence proved

ii) $\sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right) = \frac{1}{2}\cos 2\theta$

Solution:-

$L.H.S = \sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right)$

$= -\frac{1}{2}\left[-2\sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right)\right]$

$= -\frac{1}{2}\left[\cos\left(\frac{\pi}{4}-\theta+\frac{\pi}{4}+\theta\right) - \cos\left(\frac{\pi}{4}-\theta-\frac{\pi}{4}-\theta\right)\right]$

$= -\frac{1}{2}\left[\cos 2\left(\frac{\pi}{4}\right) - \cos(-2\theta)\right]$

$= -\frac{1}{2}\left(\cos\frac{\pi}{2} - \cos 2\theta\right)$

$= -\frac{1}{2}(0 - \cos 2\theta) = \frac{1}{2}\cos 2\theta$

$= R.H.S$

Hence proved

iii) $\dfrac{\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta} = \tan 4\theta$

Solution:-

$L.H.S = \dfrac{\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta}$

$$= \frac{\left[2\sin\frac{\theta+3\theta}{2}\cos\frac{\theta-3\theta}{2}\right]+\left[2\sin\frac{5\theta+7\theta}{2}\cos\frac{5\theta-7\theta}{2}\right]}{\left[2\cos\frac{\theta+3\theta}{2}\cos\frac{\theta-3\theta}{2}\right]+\left[2\cos\frac{5\theta+7\theta}{2}\cos\frac{5\theta-7\theta}{2}\right]}$$

$$= \frac{\left[2\sin\frac{4\theta}{2}\cos\left(-\frac{2\theta}{2}\right)\right]+\left[2\sin\frac{12\theta}{2}\cos\left(-\frac{2\theta}{2}\right)\right]}{\left[2\cos\frac{4\theta}{2}\cos\left(-\frac{2\theta}{2}\right)\right]+\left[2\cos\frac{12\theta}{2}\cos\left(-\frac{2\theta}{2}\right)\right]}$$

$$= \frac{2\sin 2\theta\cos(-\theta) + 2\sin 6\theta\cos(-\theta)}{2\cos 2\theta\cos(-\theta) + 2\cos 6\theta\cos(-\theta)}$$

$$= \frac{2\sin 2\theta\cos\theta + 2\sin 6\theta\cos\theta}{2\cos 2\theta\cos\theta + 2\cos 6\theta\cos\theta}$$

$$= \frac{\cancel{2\cos\theta}(\sin 2\theta + \sin 6\theta)}{\cancel{2\cos\theta}(\cos 2\theta + \cos 6\theta)}$$

$$= \frac{\sin 2\theta + \sin 6\theta}{\cos 2\theta + \cos 6\theta}$$

$$= \frac{2\sin\frac{2\theta+6\theta}{2}\cos\frac{2\theta-6\theta}{2}}{2\cos\frac{2\theta+6\theta}{2}\cos\frac{2\theta-6\theta}{2}}$$

$$= \frac{\cancel{2}\sin\left(\frac{8\theta}{2}\right)\cancel{\cos\left(-\frac{4\theta}{2}\right)}}{\cancel{2}\cos\left(\frac{8\theta}{2}\right)\cancel{\cos\left(-\frac{4\theta}{2}\right)}}$$

$$= \frac{\sin 4\theta}{\cos 4\theta} = \tan 4\theta = R.H.S$$

Hence proved

Q5. Prove that

i) $\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ = \frac{1}{16}$

Solution:-

$L.H.S = \cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ$

$= \cos 20^\circ\cos 40^\circ\left(\frac{1}{2}\right)\cos 80^\circ$

$= \frac{1}{2}(\cos 20^\circ\cos 40^\circ)\cos 80^\circ$

$= \frac{1}{4}(2\cos 20^\circ\cos 40^\circ)\cos 80^\circ$

$= \frac{1}{4}\left[\cos(20^\circ+40^\circ)+\cos(20^\circ-40^\circ)\right]\cos 80^\circ$

$= \frac{1}{4}\left[\cos 60^\circ + \cos(-20^\circ)\right]\cos 80^\circ$

$= \frac{1}{4}\left[\frac{1}{2} + \cos 20^\circ\right]\cos 80^\circ$

$= \frac{1}{8}\cos 80^\circ + \frac{1}{4}\cos 20^\circ \cos 80^\circ$

$= \frac{1}{8}\cos 80^\circ + \frac{1}{8}(2\cos 20^\circ \cos 80^\circ)$

$= \frac{1}{8}\cos 80^\circ + \frac{1}{8}\left[\cos(20^\circ+80^\circ) + \cos(20^\circ-80^\circ)\right]$

$= \frac{1}{8}\cos 80^\circ + \frac{1}{8}\left[\cos 100^\circ + \cos(-60^\circ)\right]$

$= \frac{1}{8}\cos 80^\circ + \frac{1}{8}(\cos 100^\circ + \cos 60^\circ)$

$= \frac{1}{8}\cos 80^\circ + \frac{1}{8}\cos 100^\circ + \frac{1}{8}\left(\frac{1}{2}\right)$

$= \frac{1}{8}\cos 80^\circ + \frac{1}{8}\cos(180^\circ - 80^\circ) + \frac{1}{16}$

$= \frac{1}{8}\cancel{\cos 80^\circ} - \frac{1}{8}\cancel{\cos 80^\circ} + \frac{1}{16}$

$= \frac{1}{16} = R.H.S$ $\quad \because \cos(180^\circ - \theta) = -\cos\theta$

Hence proved

ii) $\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9} = \frac{3}{16}$

Solution:-

$L.H.S = \sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9}$

$= \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$

$= \sin 20^\circ \sin 40^\circ \left(\frac{\sqrt{3}}{2}\right)\sin 80^\circ$

$= \frac{\sqrt{3}}{2}(\sin 20^\circ \sin 40^\circ)\sin 80^\circ$

$= -\frac{\sqrt{3}}{4}(-2\sin 20^\circ \sin 40^\circ)\sin 80^\circ$

$= -\frac{\sqrt{3}}{4}\left[\cos(20^\circ+40^\circ) - \cos(20^\circ-40^\circ)\right]\sin 80^\circ$

$= -\frac{\sqrt{3}}{4}(\cos 60^\circ - \cos(-20^\circ))\sin 80^\circ$

$= -\frac{\sqrt{3}}{4}\left(\frac{1}{2} - \cos 20^\circ\right)\sin 80^\circ$

$= \left(-\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{4}\cos 20^\circ\right)\sin 80^\circ$

$= -\frac{\sqrt{3}}{8}\sin 80^\circ + \frac{\sqrt{3}}{4}\cos 20^\circ \sin 80^\circ$

$= -\frac{\sqrt{3}}{8}\sin 80^\circ + \frac{\sqrt{3}}{8}(2\cos 20^\circ \sin 80^\circ)$

$= -\frac{\sqrt{3}}{8}\sin 80^\circ + \frac{\sqrt{3}}{8}(\sin(20^\circ+80^\circ) - \sin(20^\circ-80^\circ))$

$= -\frac{\sqrt{3}}{8}\sin 80^\circ + \frac{\sqrt{3}}{8}(\sin 100^\circ - \sin(-60^\circ))$

$= -\frac{\sqrt{3}}{8}\sin 80^\circ + \frac{\sqrt{3}}{8}(\sin 100^\circ + \sin 60^\circ)$

$= -\frac{\sqrt{3}}{8}\sin 80^\circ + \frac{\sqrt{3}}{8}\sin 100^\circ + \frac{\sqrt{3}}{8}\sin 60^\circ$

$= -\frac{\sqrt{3}}{8}\sin 80^\circ + \frac{\sqrt{3}}{8}\sin(180^\circ - 80^\circ) + \frac{\sqrt{3}}{8}\left(\frac{\sqrt{3}}{2}\right)$

$= -\frac{\sqrt{3}}{8}\cancel{\sin 80^\circ} + \frac{\sqrt{3}}{8}\cancel{\sin 80^\circ} + \frac{3}{16}$

$= \frac{3}{16} = R.H.S$ $\quad \because \sin(\pi - \theta) = \sin\theta$

Hence proved

iii) $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{16}$

Solution:-

$L.H.S = \sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ$

$= \sin 10^\circ \left(\frac{1}{2}\right)\sin 50^\circ \sin 70^\circ$

$= \frac{1}{2}(\sin 10^\circ \sin 50^\circ)\sin 70^\circ$

$= -\frac{1}{4}(-2\sin 10^\circ \sin 50^\circ)\sin 70^\circ$

$= -\frac{1}{4}(\cos(10^\circ+50^\circ) - \cos(10^\circ-50^\circ))\sin 70^\circ$

$= -\frac{1}{4}(\cos 60^\circ - \cos(-40^\circ))\sin 70^\circ$

$= -\frac{1}{4}\left(\frac{1}{2} - \cos 40^\circ\right)\sin 70^\circ$

$= \left(-\frac{1}{8} + \frac{1}{4}\cos 40^\circ\right)\sin 70^\circ$

$= -\frac{1}{8}\sin 70^\circ + \frac{1}{4}\cos 40^\circ \sin 70^\circ$

$= -\frac{1}{8}\sin 70^\circ + \frac{1}{8}(2\cos 40^\circ \sin 70^\circ)$

$= -\frac{1}{8}\sin 70^\circ + \frac{1}{8}(\sin(40^\circ+70^\circ) - \sin(40^\circ-70^\circ))$

$= -\frac{1}{8}\sin 70^\circ + \frac{1}{8}(\sin 110^\circ - \sin(-30^\circ))$

$= -\frac{1}{8}\sin 70^\circ + \frac{1}{8}\sin 110^\circ + \frac{1}{8}\sin 30^\circ$

$= -\frac{1}{8}\sin 70^\circ + \frac{1}{8}\sin(180^\circ - 70^\circ) + \frac{1}{8}\sin 30^\circ$

$= -\frac{1}{8}\cancel{\sin 70^\circ} + \frac{1}{8}\cancel{\sin 70^\circ} + \frac{1}{8}\left(\frac{1}{2}\right)$

$= \frac{1}{16} = R.H.S$