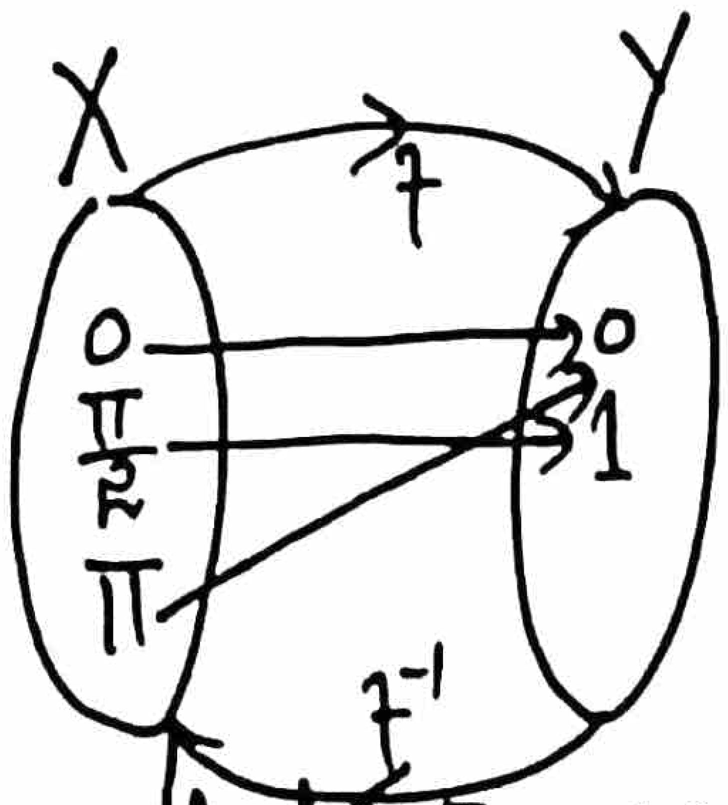
apter#13
sc Part-1

INVERSE TRIGONOMETRIC FUNCTIONS

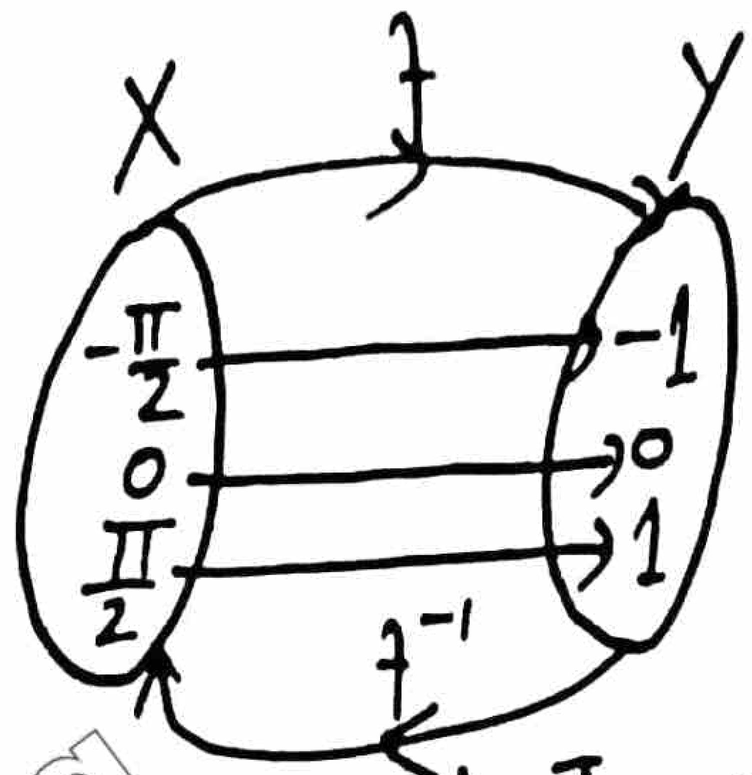
We learned that Only a one-to-one function will have an Inverse. If a function is not one-to-one, it may be possible to restrict its Domain to make it one-to-one so that its Inverse can be found.

For Example:-

$y = \sin x$



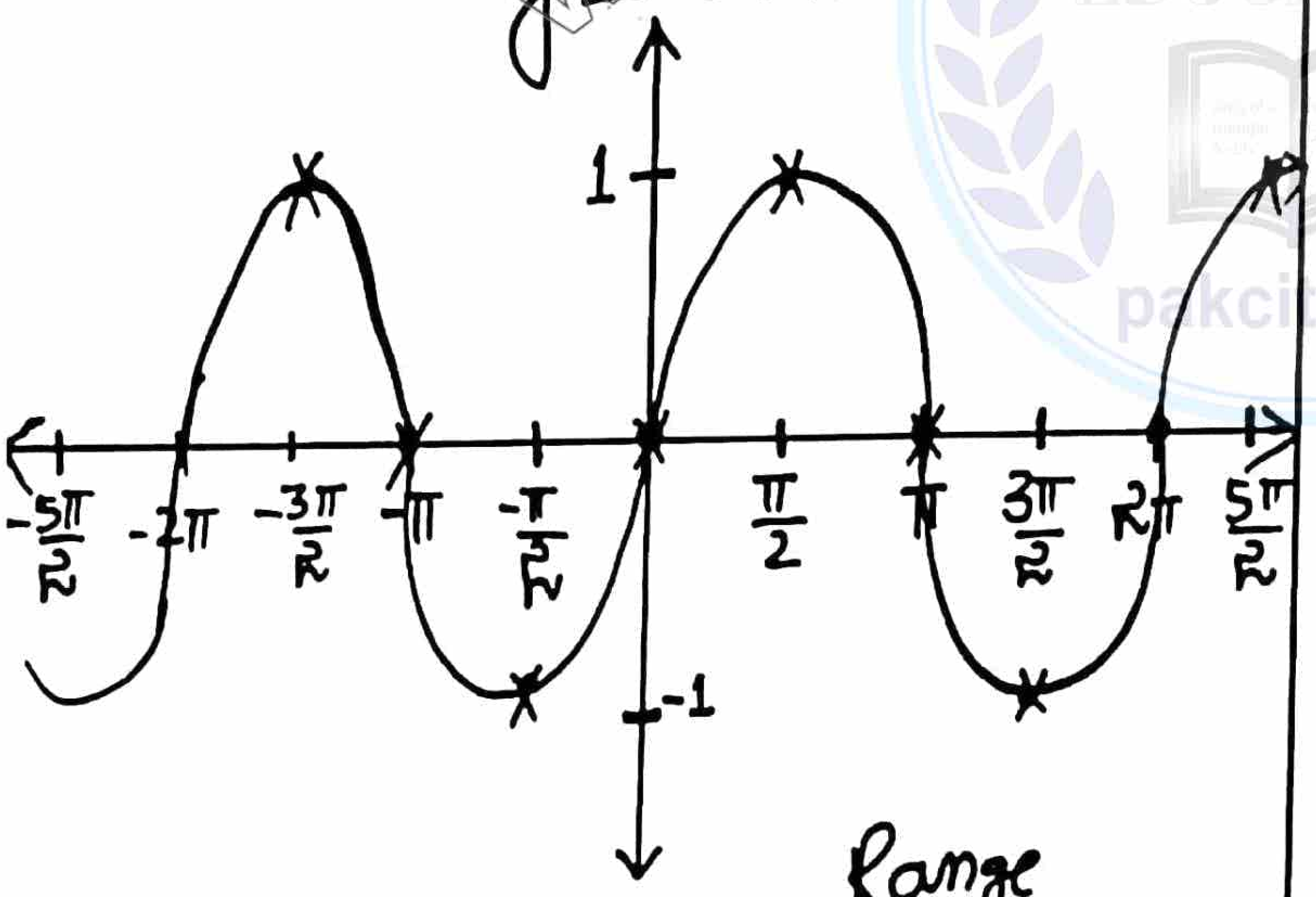
We see that Inverse of function is not a function So it is not Invertible function.



We see that Inverse of a function is also a function So it is Inveritble.

THE Inverse Sine Function

$y = \sin x$



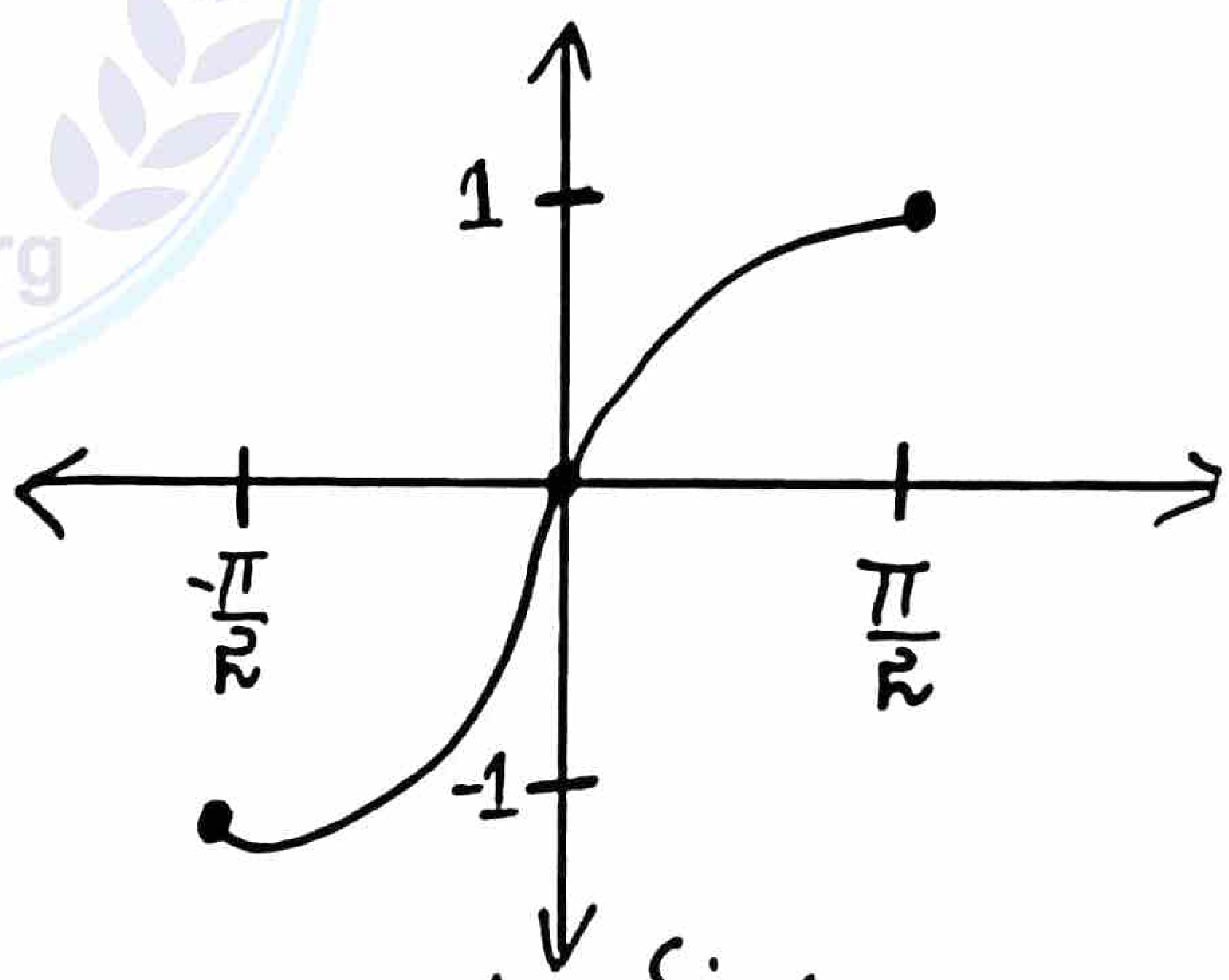
Domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$ Range (Principal Domain)

$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \longrightarrow [-1, 1]$

$\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \longrightarrow [-1, 1]$

The Portion which is Nearest to Origion is called Principal Domain.

Principal Sine Function

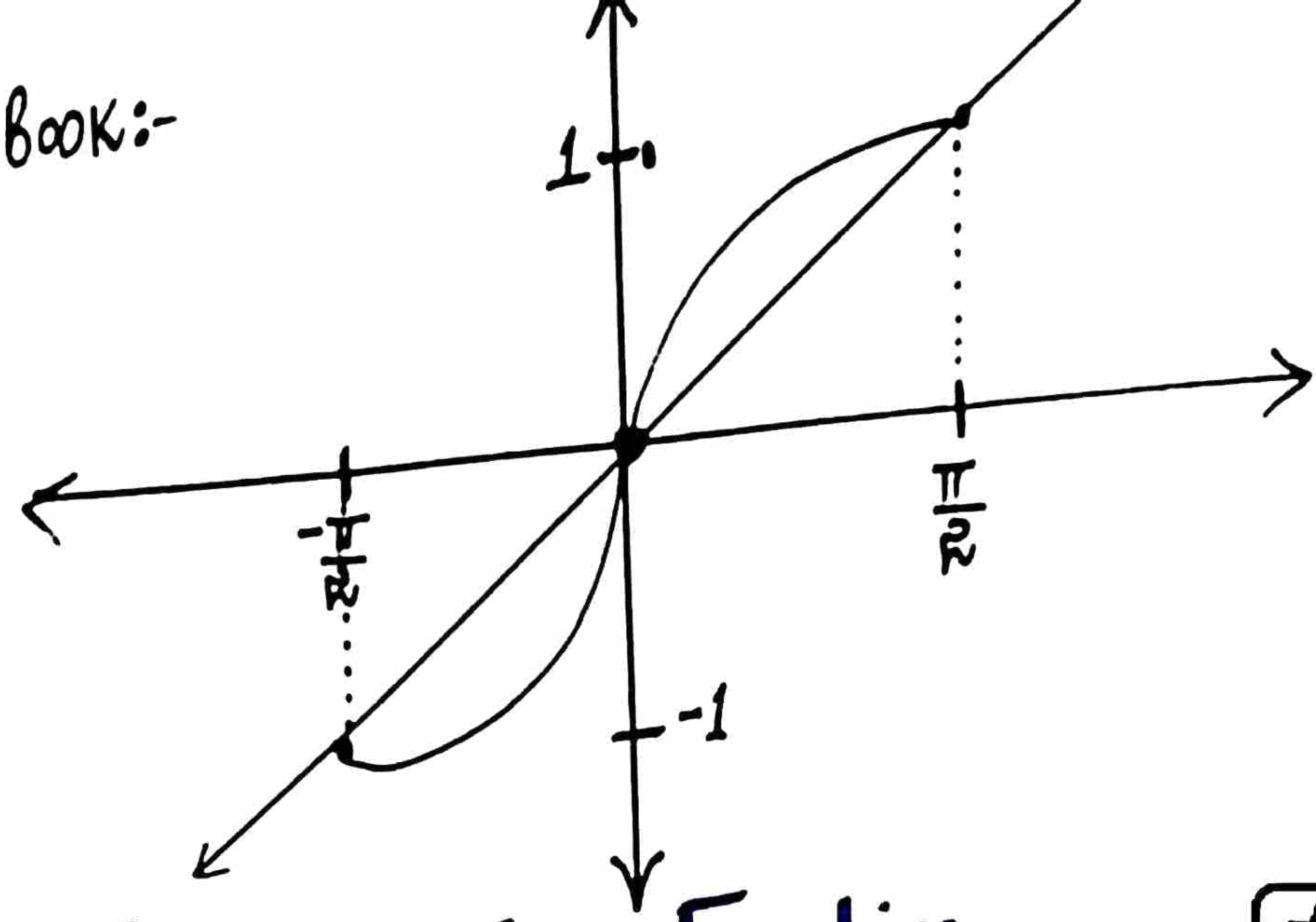


$y = \sin x$

Domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Range $[-1, 1]$

Graph On Book:-

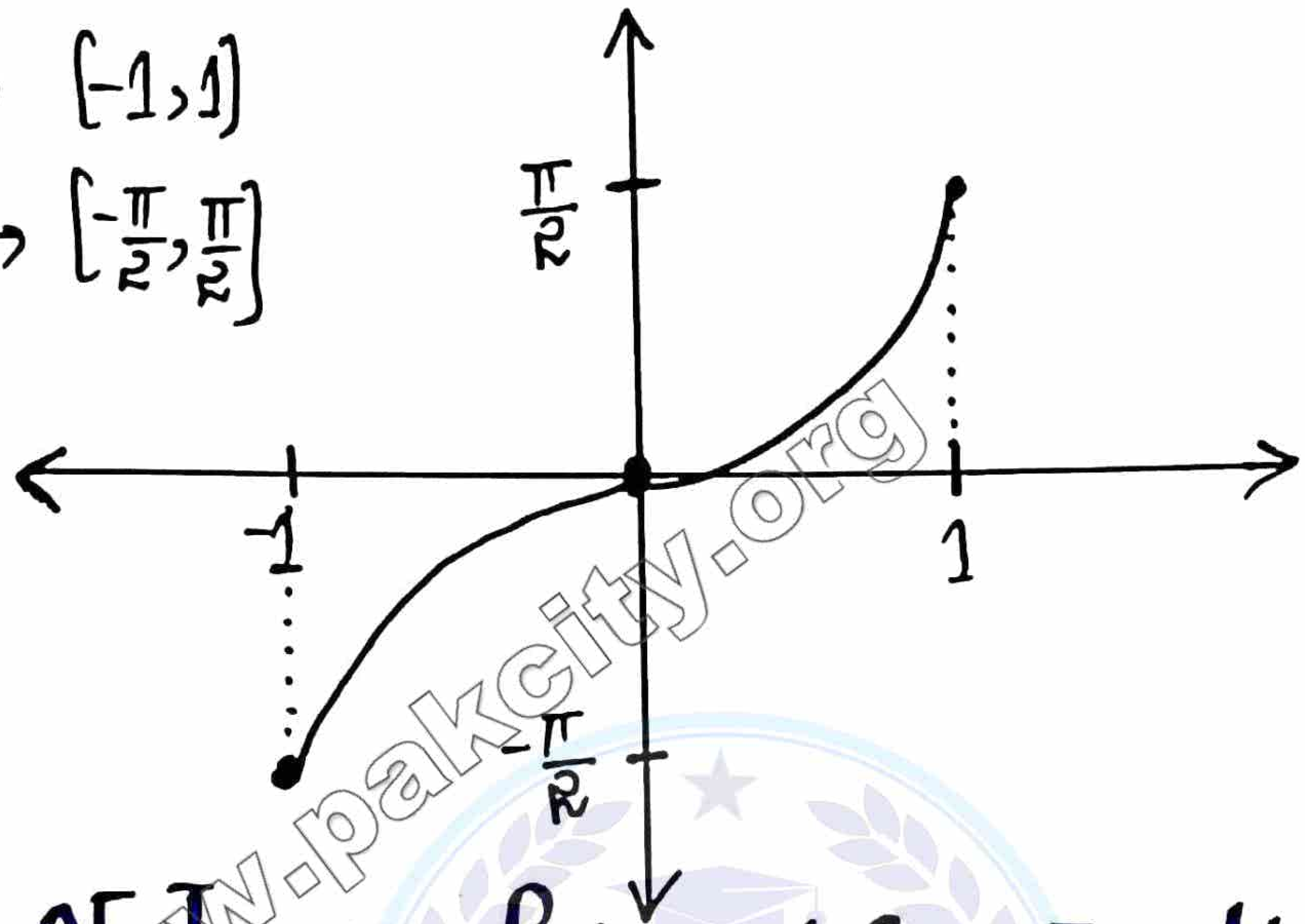


Inverse Principal Sine Function:-

$$y = \sin^{-1} x$$

Domain → $[-1, 1]$

Range → $[-\frac{\pi}{2}, \frac{\pi}{2}]$



Properties Of Inverse Principal Sine Function

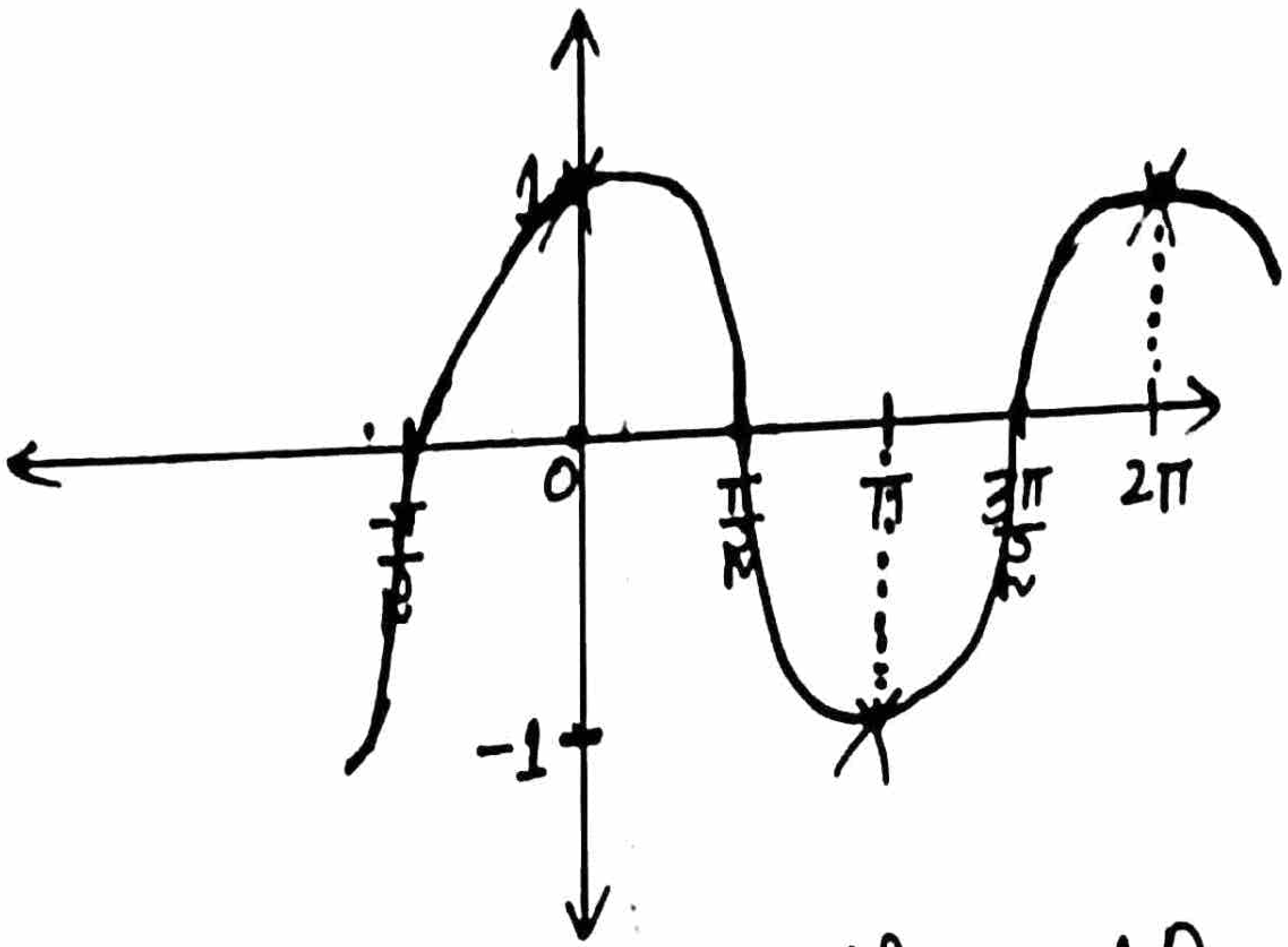
- Inverse of Principal Sine is bijective.
- Bounded Function. (Restricted Domain)
- Increasing Function.
- Symmetric About Origion. (Reflection of Points Showing in the opposite Quadrant)
- $\sin^{-1}(-x) = -\sin^{-1}(x)$

Note:- * If Y-axis reflect than it is a even function.

* If Opposite Quadrants Reflects then odd function.

The Inverse Cosine Function:-

$y = \cos x \qquad \mathbb{R} \rightarrow [-1, 1]$



$[0, \pi] \longrightarrow [-1, 1] \rightarrow$ (Principal Domain)

$[\pi, 2\pi] \longrightarrow [-1, 1]$

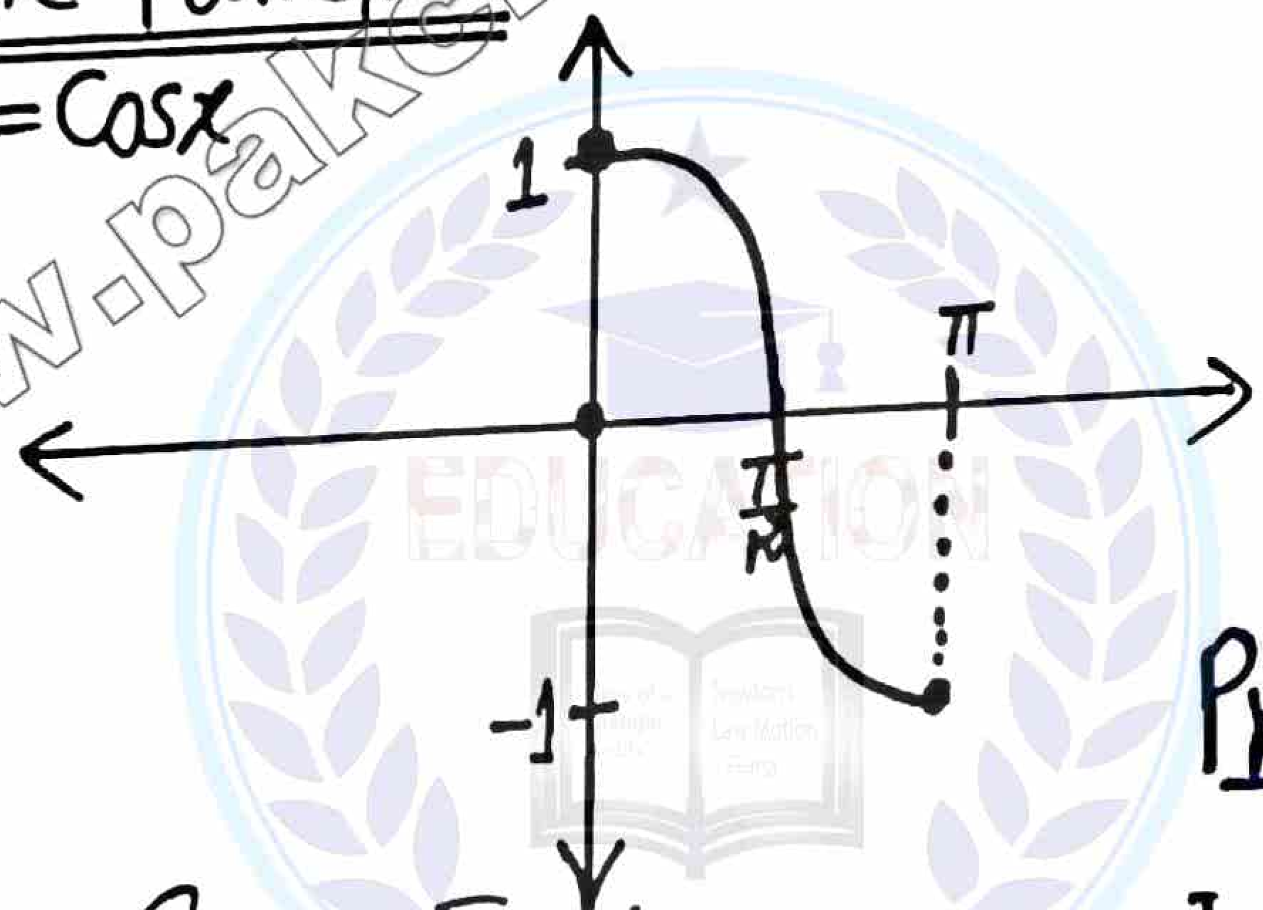
$[2\pi, 3\pi] \longrightarrow [-1, 1]$

Principal Cosine Function:-

$y = \cos x$

Domain: $[0, \pi]$

Range: $[-1, 1]$

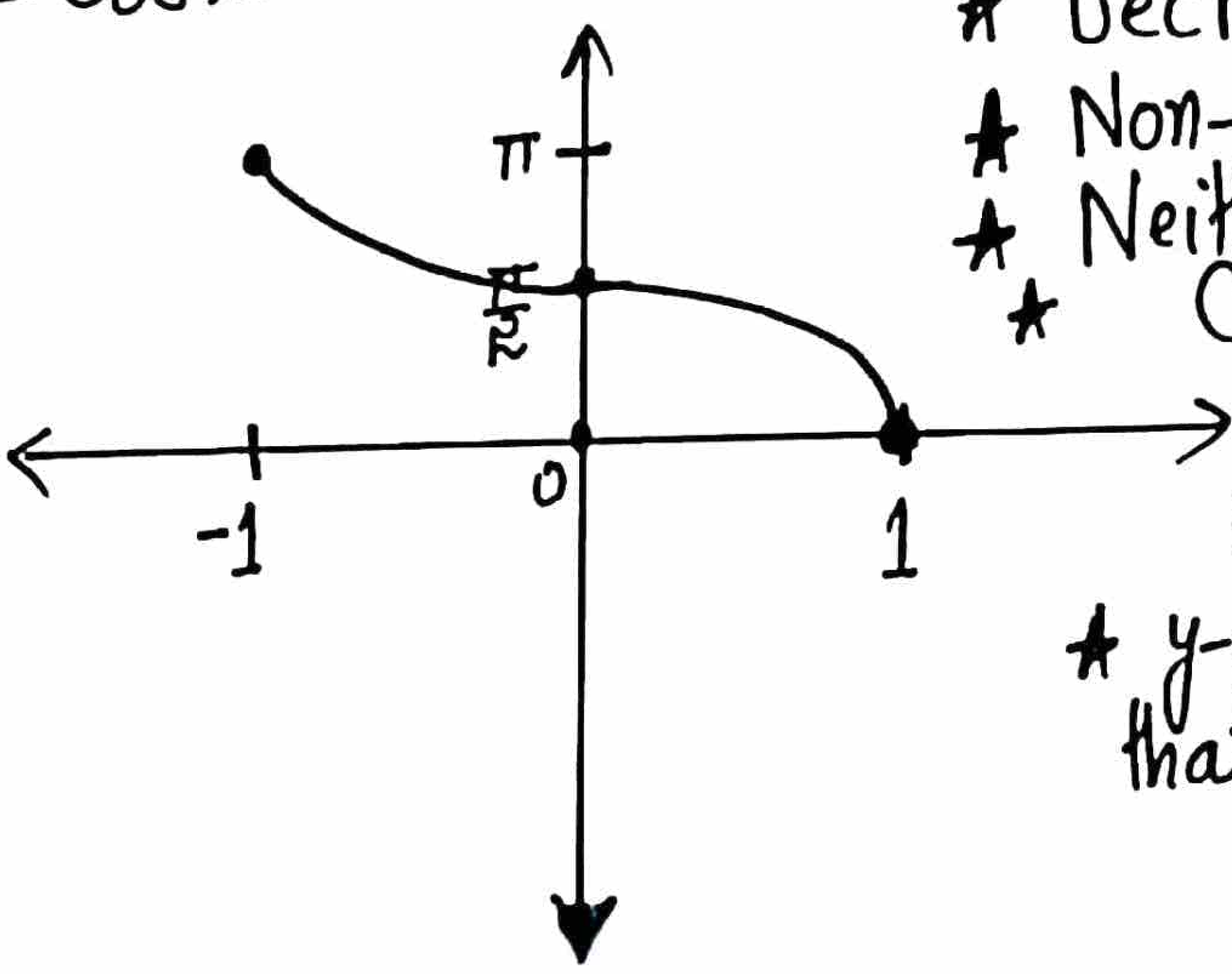


Inverse Principal Cosine Function:-

$y = \cos^{-1} x$

Domain: $[-1, 1]$

Range: $[0, \pi]$



Properties: (Inverse Principle Cosine Function)

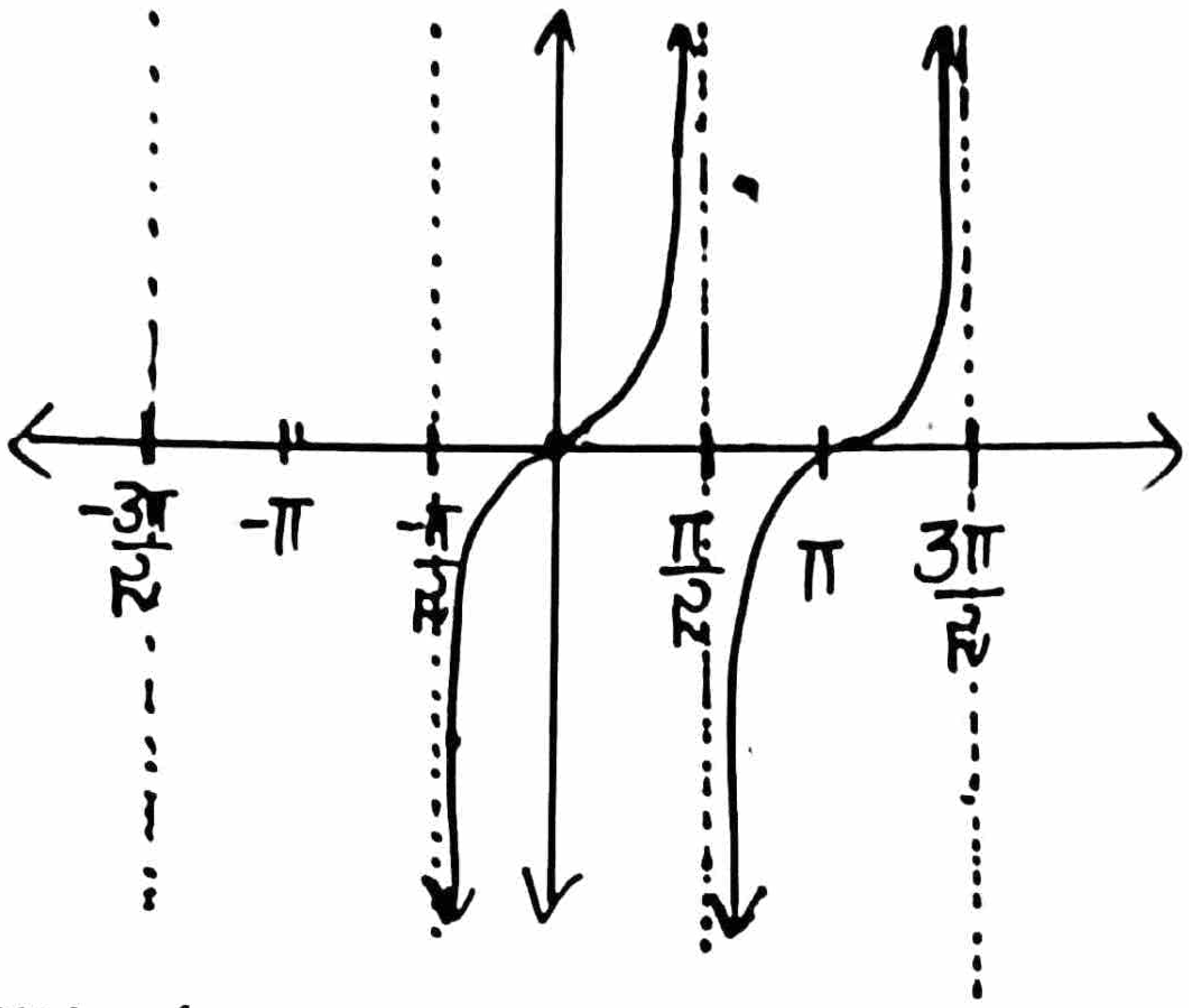
- Inverse Principal Cosine is also bijective.
- Decreasing function.
- Non-Periodic function.
- Neither even nor odd.
- $\cos^{-1}(-x) \neq \cos^{-1} x$
  $\neq -\cos^{-1} x$

$\cos^{-1}(-x) = \pi - \cos^{-1}(+x)$

- y-axis does not Reflect that is why it is not even

★ The Inverse Tangent Function:-

$y = \tan x \qquad \mathbb{R} - \{(2n+1)\frac{\pi}{2}\} \to \mathbb{R}$



Now $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$ (Principal Domain)

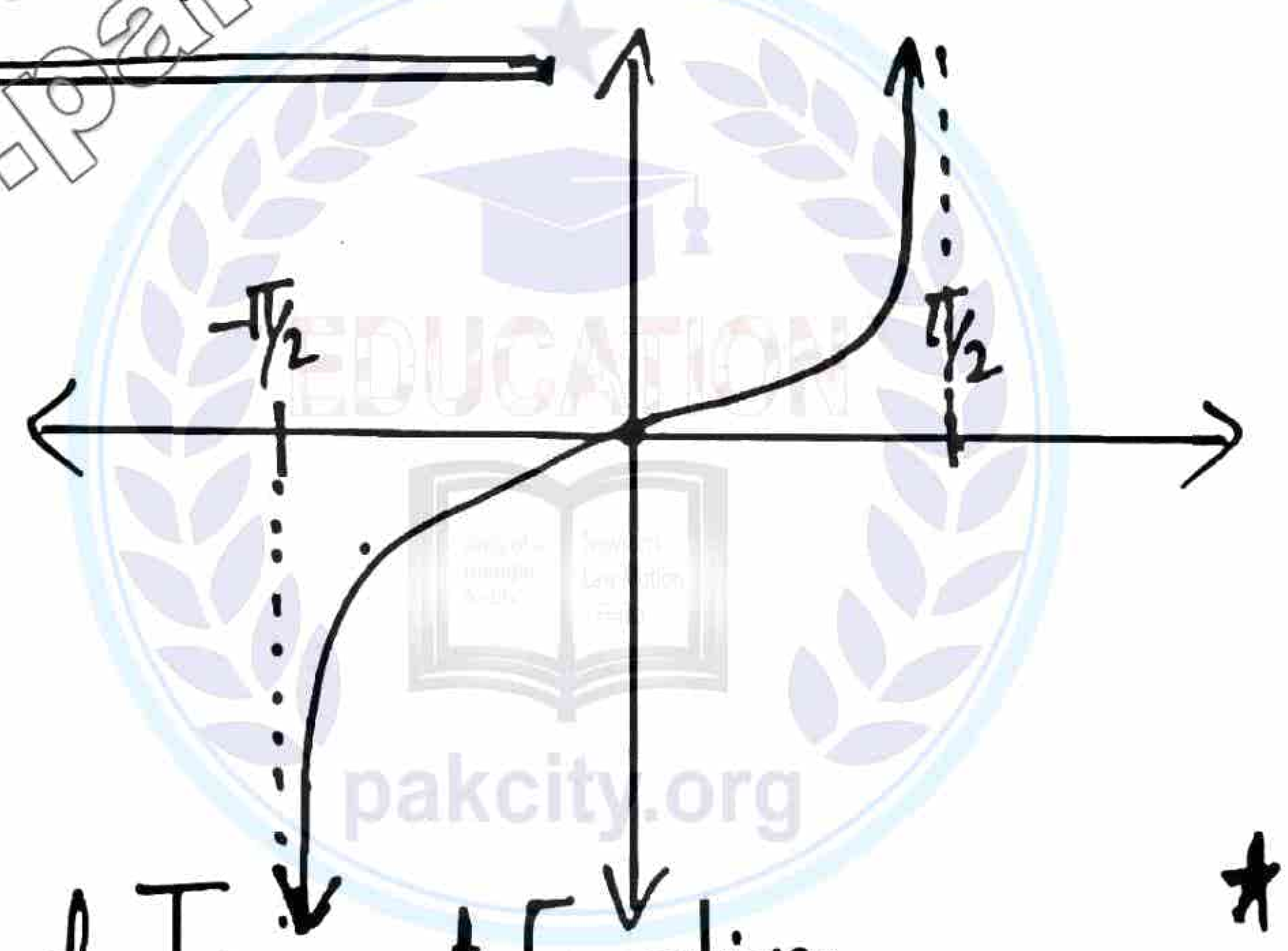
$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \longrightarrow \mathbb{R}$

$\left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \longrightarrow \mathbb{R}$

★ Principal Tangent Function:-

Domain: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Range: $\mathbb{R}$

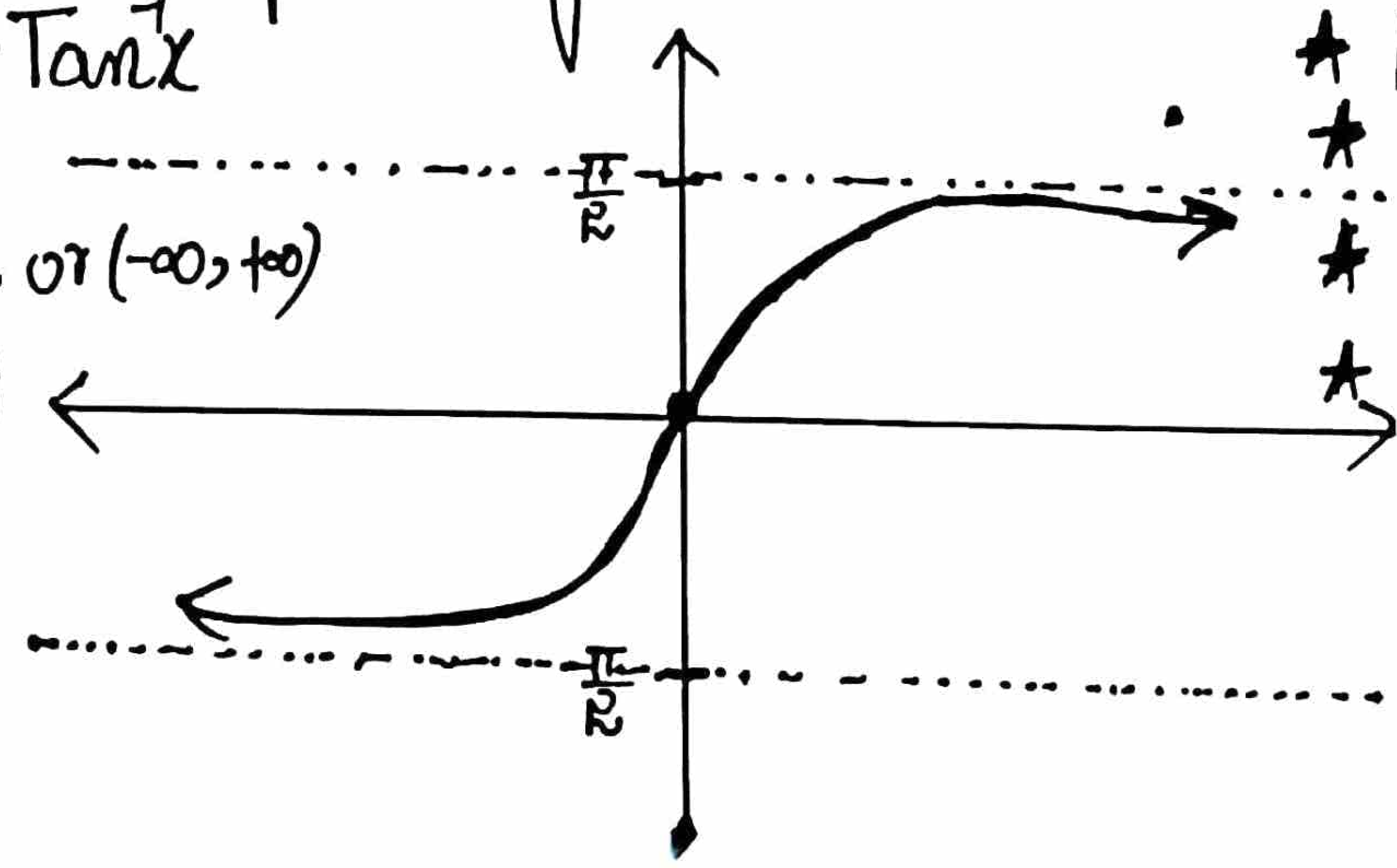


★ Inverse Principal Tangent Function:-

$y = \tan^{-1} x$

Domain:- ($\mathbb{R}$ or $(-\infty, +\infty)$)

Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

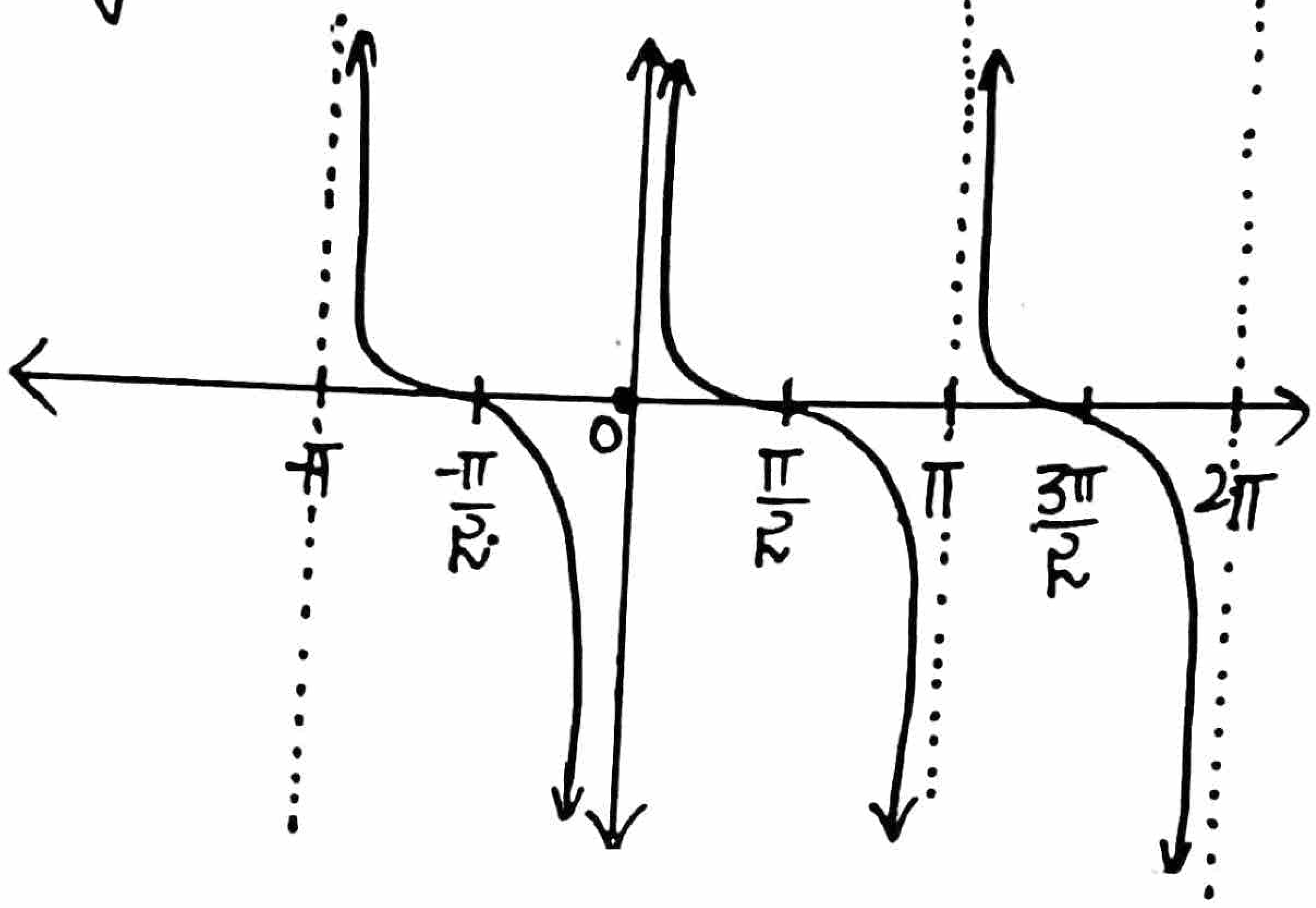


Properties: (Inverse Principal Tangent Function)

- ★ Inverse Tangent is bijective.
- ★ Non- Periodic Function
- ★ Increasing Function
- ★ Symmetric About origin.
- ★ $\tan^{-1}(-x) = -\tan^{-1}x$

★ The Inverse Cotagent Function:-

$y = \cot x \qquad \mathbb{R} - \{n\pi\} \longrightarrow \mathbb{R}$



$(0,\pi) \longrightarrow \mathbb{R}$ (Principal Domain)

$(\pi,2\pi) \longrightarrow \mathbb{R}$

$(2\pi,3\pi) \longrightarrow \mathbb{R}.$

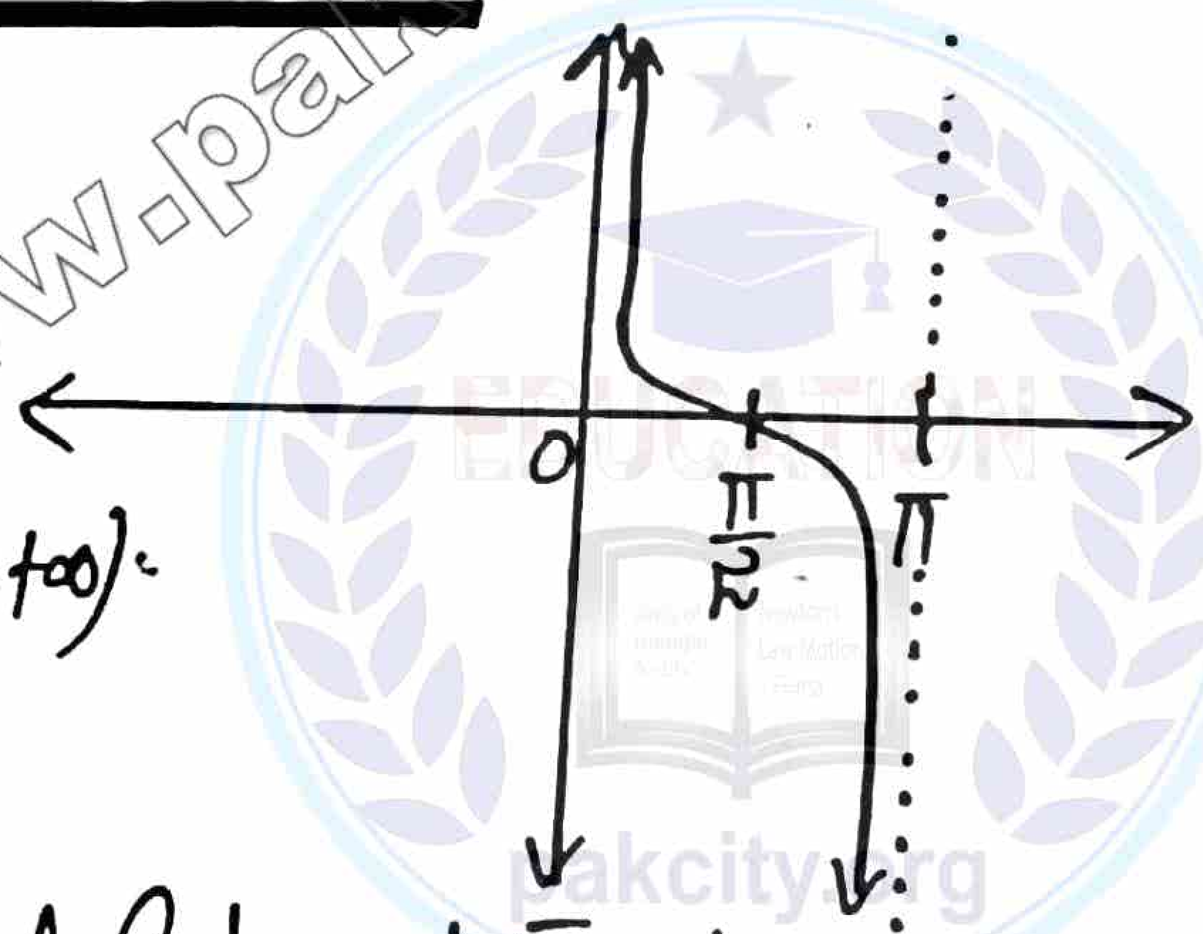
$(-\pi,0) \longrightarrow \mathbb{R}$

★ Principal Cotagent Function:-

$y = \cot x$

Domain: $(0,\pi)$

Range: $\mathbb{R}$ or $(-\infty,+\infty)$.

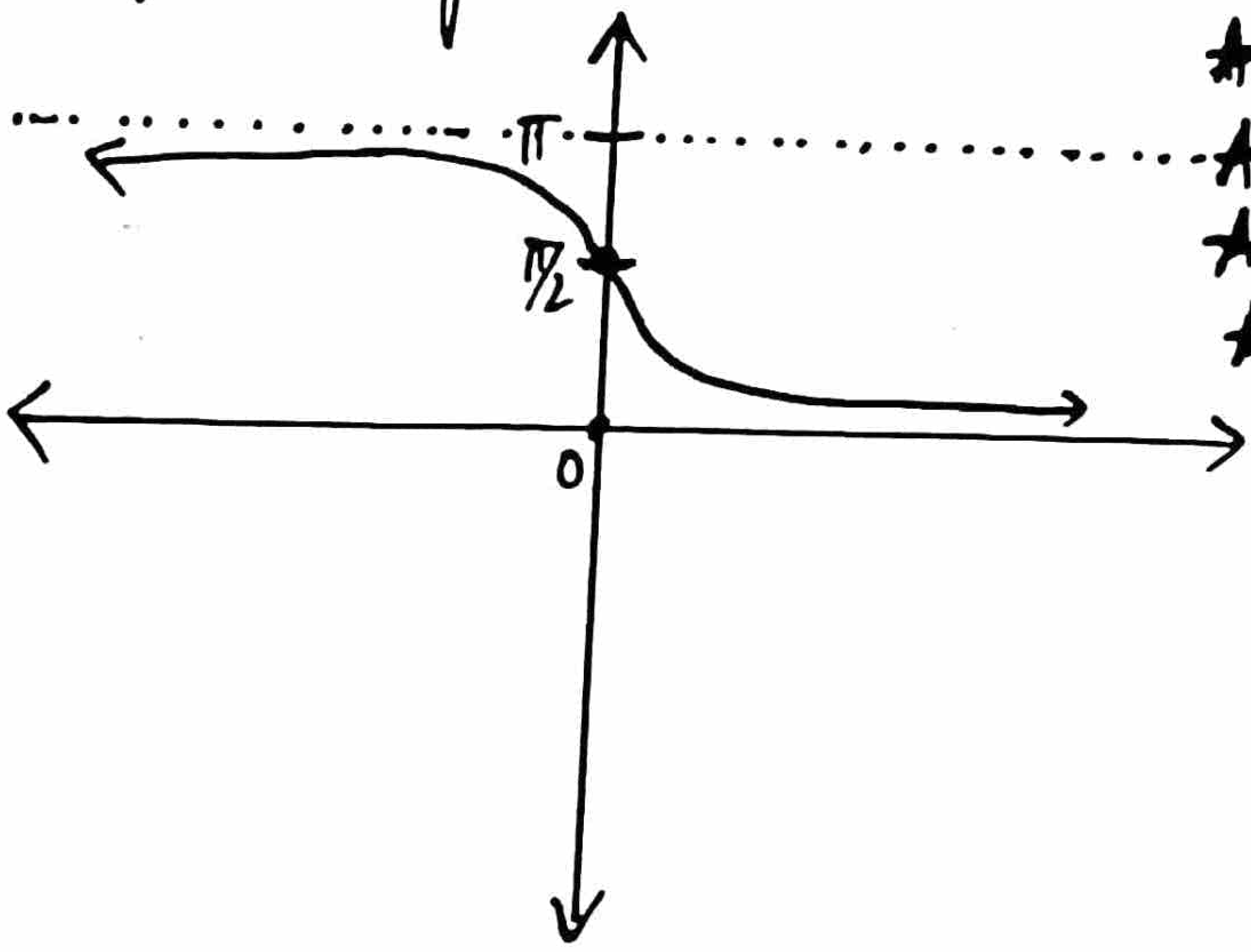


★ Inverse Principal Cotagent Function:-

$y = \cot^{-1} x$

Domain $= \mathbb{R}$

Range $= (0,\pi)$

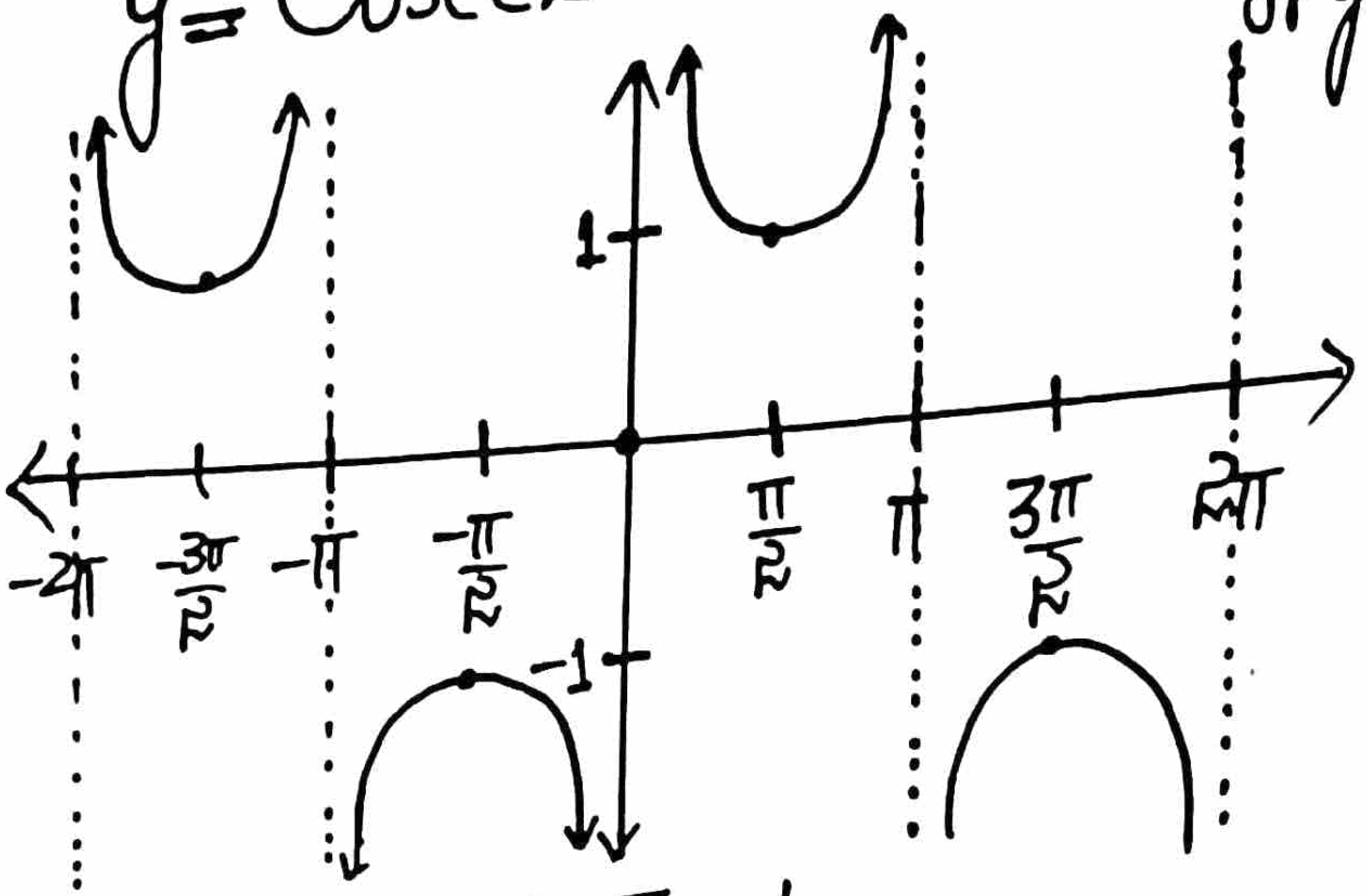


Properties Of Inverse Principle Cotagent Function:-

- ★ Non-Periodic function.
- ★ Bijective function.
- ★ Decreasing function.
- ★ Neither even nor odd.
- ★ $\cot^{-1}(-x) \neq \cot^{-1}(x)$
  $\neq -\cot^{-1}(x)$
  $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

★ The Inverse Cosecant Function:-

$y = \operatorname{Cosec} x$ $\quad \mathbb{R}-\{n\pi\} \rightarrow y \geq 1$ or $y \leq -1$

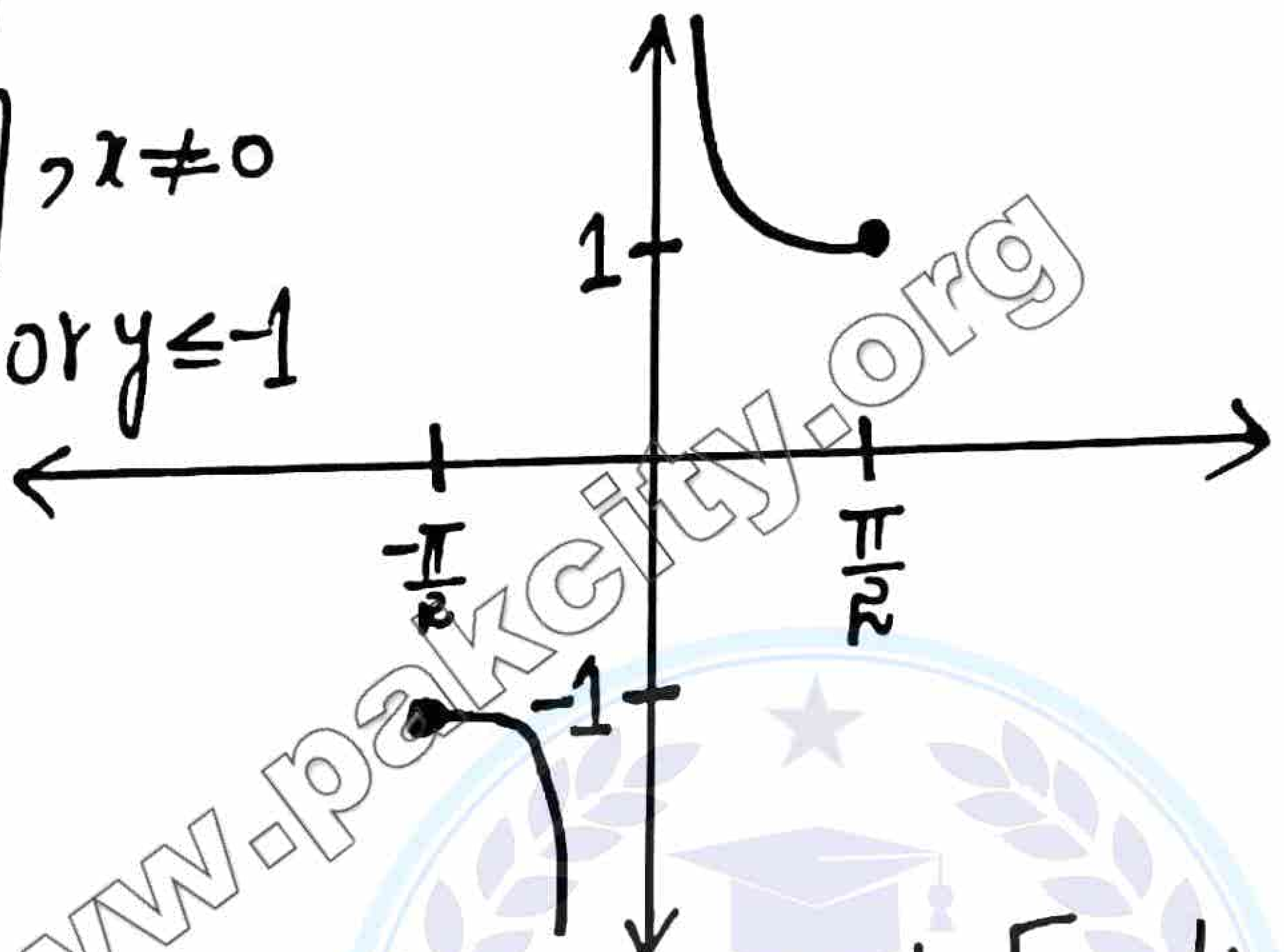


★ The Principal Cosecant Function:-

$y = \operatorname{Cosec} x$

Domain: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right], x \neq 0$

Range: $y \geq 1$ or $y \leq -1$

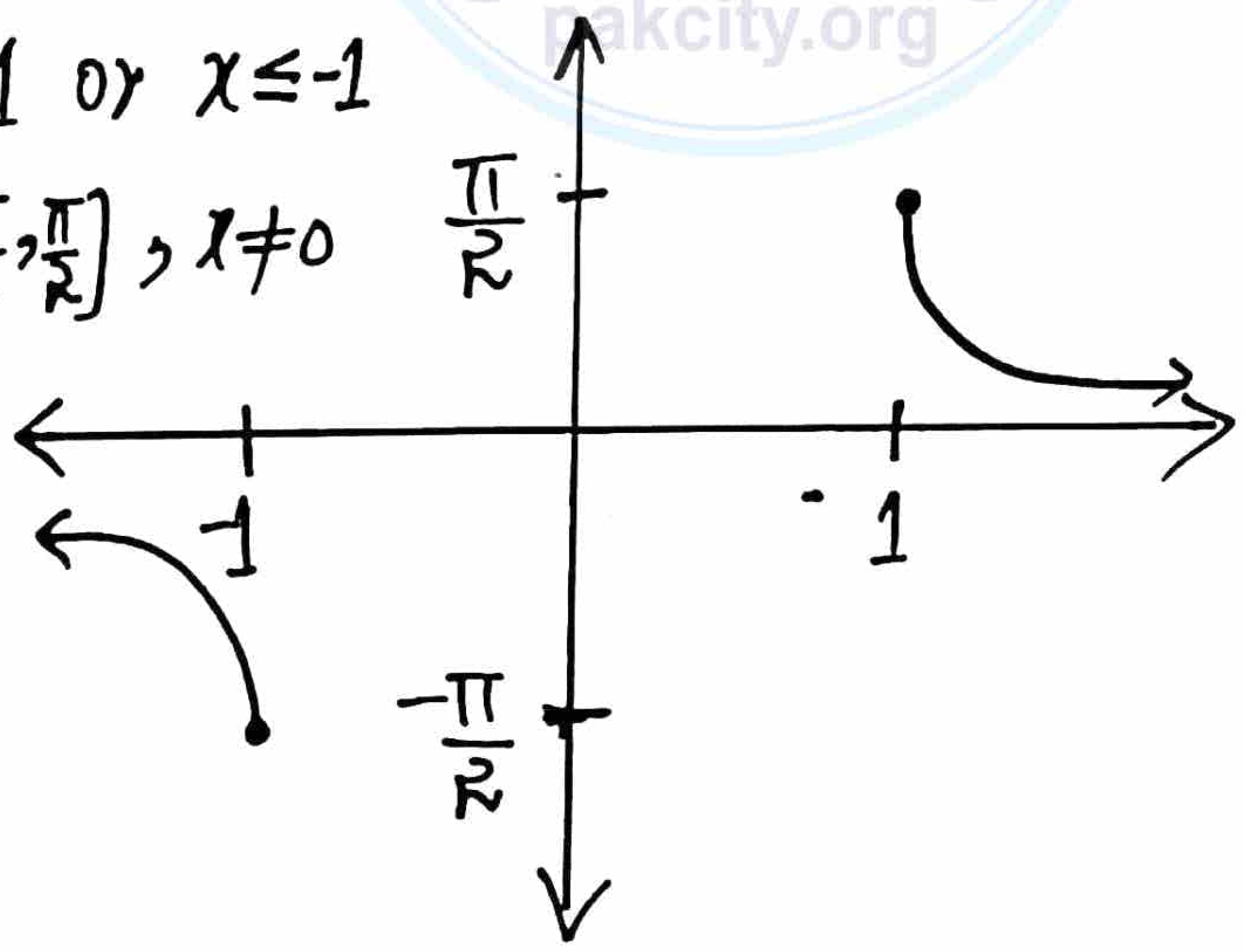


★ The Inverse Principal Cosecant Function:

$y = \operatorname{Cosec}^{-1} x$

Domain: $x \geq 1$ or $x \leq -1$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right], x \neq 0$

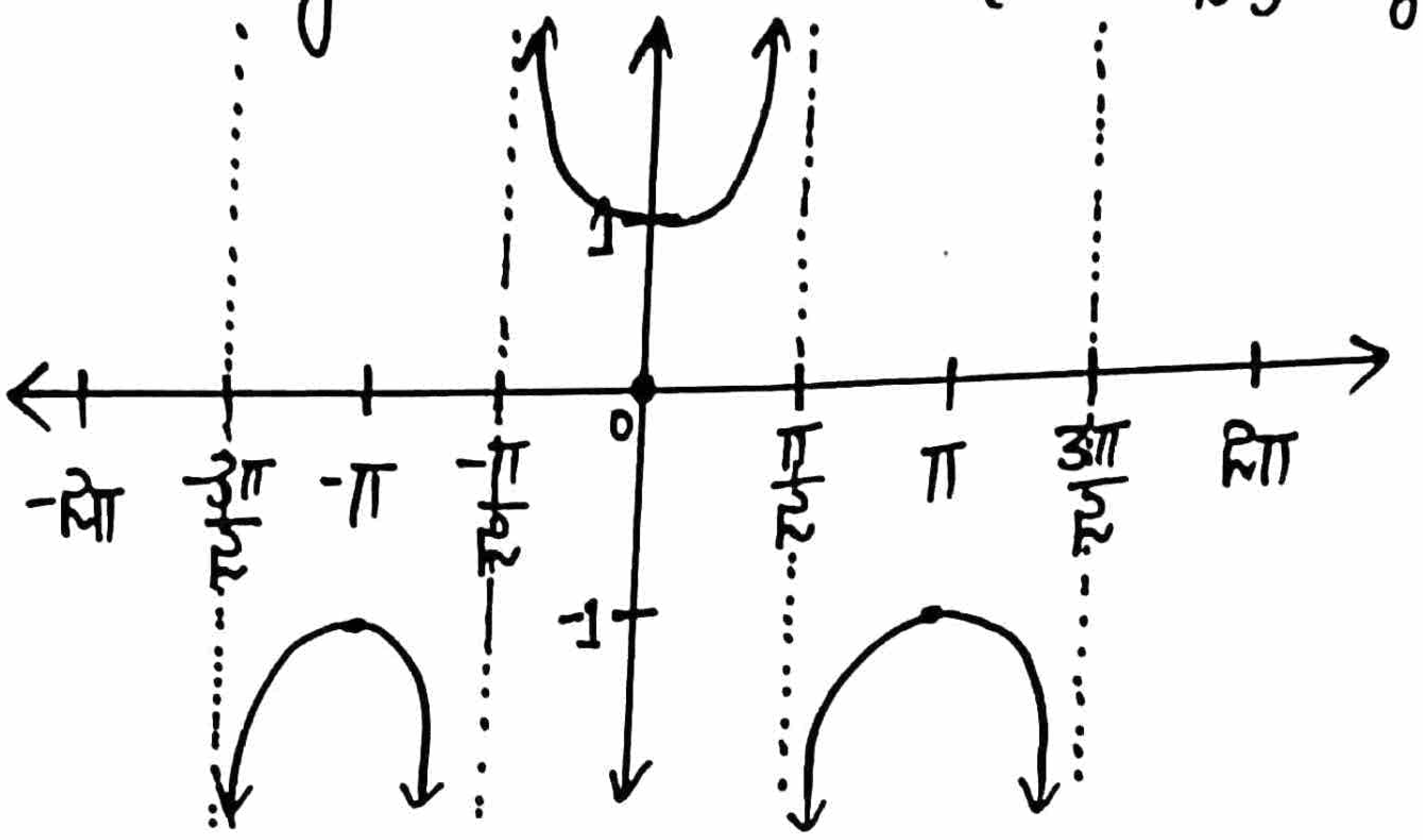


Properties of Inverse Principal Cosecant Function:-

- ★ Non-Periodic
- ★ Bijective.
- ★ Decreasing function
- ★ Odd function
- ★ $\operatorname{Cosec}^{-1}(-x) = -\operatorname{Cosec}^{-1}(x)$

★ The Inverse Secant Function:-

$y = \sec x \quad \mathbb{R} - \left\{ (2n+1)\frac{\pi}{2} \right\} \rightarrow y \geq 1$ or $y \leq -1$

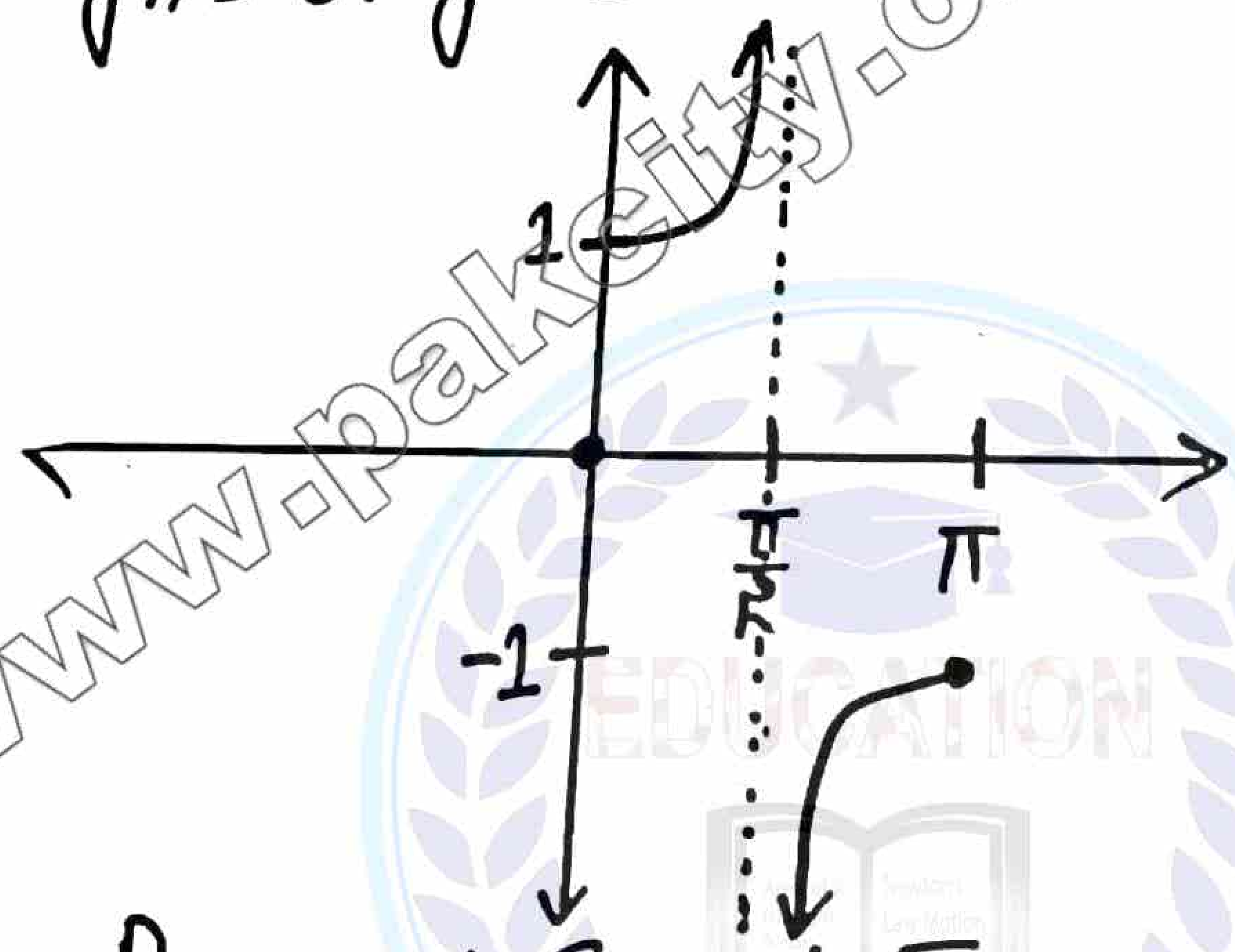


★ The Principal Secant Function

$y = \sec x$

Domain: $[0, \pi]$, $x \neq \frac{\pi}{2}$

Range: $y \geq 1$ or $y \leq -1$

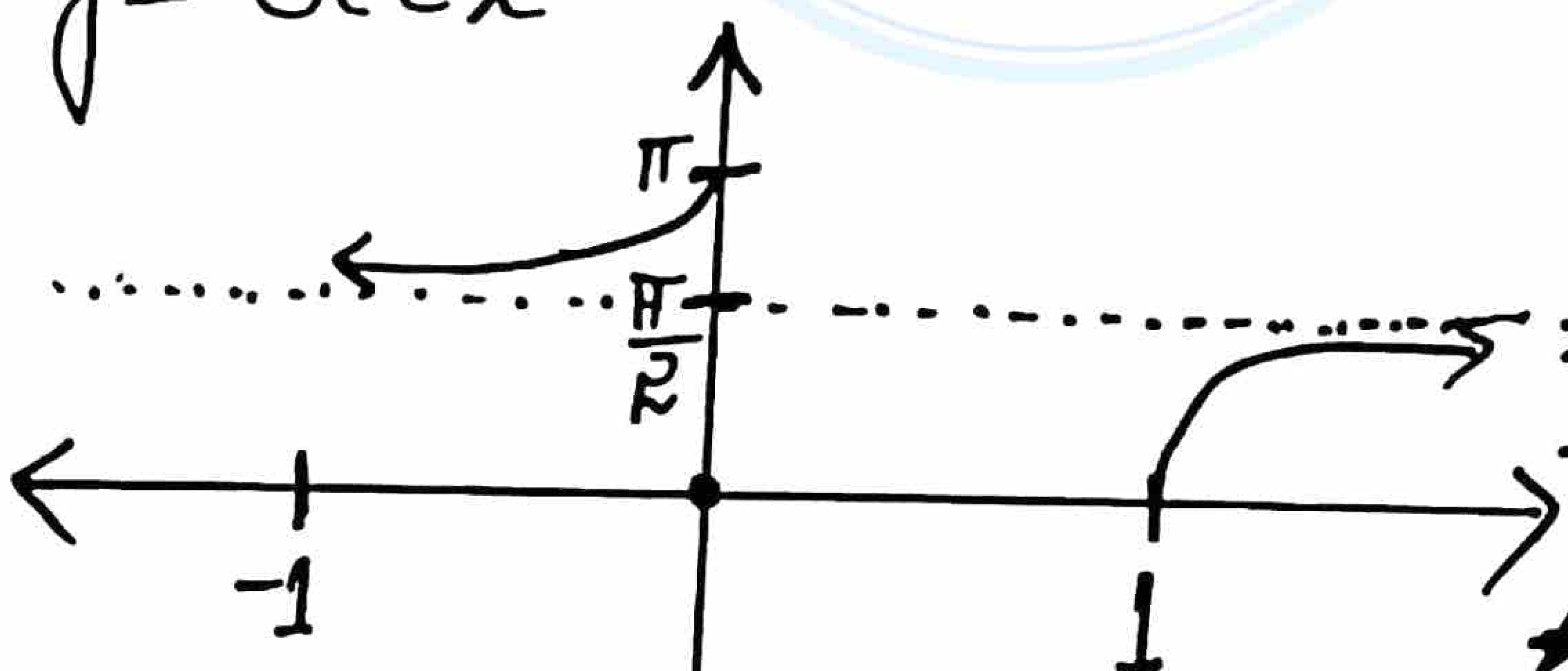


★ The Inverse Principal Secant Function:-

$y = \sec^{-1} x$

Domain:-

Range:-



Domain $= x \geq 1$ or $x \leq -1$

Range $= [0, \pi]$, $y \neq \frac{\pi}{2}$

Properties of Inverse Principal Secant Function

Properties:-

- ★ Non-Periodic.
- ★ Bijective.
- ★ Piecewise Increasing.
- ★ Neither even nor odd
- ★ $\sec^{-1}(-x) \neq \sec^{-1}(x)$
  $\neq -\sec^{-1}(x)$

$\sec^{-1}(-x) = \pi - \sec^{-1}(x)$

Domains And Ranges of Trigonometric Functions:-

Function	Domain	Range
$y = \sin x$	$-\infty < x < +\infty$	$-1 \le y \le 1$
$y = \cos x$	$-\infty < x < +\infty$	$-1 \le y \le 1$
$y = \tan x$	$-\infty < x < +\infty,\ x \ne \frac{(2n+1)\pi}{2},\ n \in Z$	$-\infty < y < +\infty$
$y = \cot x$	$-\infty < x < +\infty,\ x \ne n\pi,\ n \in Z$	$-\infty < y < +\infty$
$y = \sec x$	$-\infty < x < +\infty,\ x \ne \frac{(2n+1)\pi}{2},\ n \in Z$	$y \ge 1$ or $y \le -1$
$y = \operatorname{cosec} x$	$-\infty < x < +\infty,\ x \ne n\pi,\ n \in Z$	$y \ge 1$ or $y \le -1$

Lecturer Mathematics
Asad Ali Khalid
BS(HONS) Math (GCUF)
M.Phill Math (NTUF)
B.ed (AIOU) (2015-2016)

Chapter NO. 13:-

Inverse Trigonometric Functions

*Exercise 13.1

Q.NO.1:-

Evaluate without using tables/calculators:

(i) $\sin^{-1}(1)$

let

$$\sin^{-1}(1) = y$$

$$\sin y = 1$$

$$y = \frac{\pi}{2}$$

(ii) $\sin^{-1}(-1)$

let

$$\sin^{-1}(-1) = y$$

$$\sin y = -1$$

$$y = -\frac{\pi}{2}$$

(iii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

let

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$$

$$\cos y = \frac{\sqrt{3}}{2}$$

$$y = \frac{\pi}{6}$$

(iv) $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

let

$$\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = y$$

$$\tan y = \frac{-1}{\sqrt{3}}$$

$$y = \frac{-\pi}{6}$$

(v) $\cos^{-1}\left(\frac{1}{2}\right)$

Let

$$\cos^{-1}\left(\frac{1}{2}\right) = y$$

$$\cos y = \frac{1}{2}$$

$$y = \frac{\pi}{3}$$

(vi) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

let

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = y$$

$$\tan y = \frac{1}{\sqrt{3}}$$

$$y = \frac{\pi}{6}$$

(vii) $\cos^{-1}(-1)$

let

$$\cos^{-1}(-1) = y$$

$$\cos y = -1$$

$$y = -\frac{\pi}{4}$$

(viii) $\operatorname{cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right)$

let

$$\operatorname{cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right) = y$$

$$\operatorname{cosec} y = \frac{-2}{\sqrt{3}}$$

$$y = -\frac{\pi}{3}$$

(x) $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

Let

$$\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right) = y$$

$$\sin y = \frac{-1}{\sqrt{2}}$$

$$y = \frac{-\pi}{4}$$

Q.NO.26:-

Without using table / calculator show that:

(i) $\tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13}$

L.H.S:-

Let $\tan^{-1}\frac{5}{12} = y$

$$\tan y = \frac{5}{12}$$

As;

$$1 + \tan^2 y = \sec^2 y$$

$$1 + \left(\frac{5}{12}\right)^2 = \sec^2 y$$

$$1 + \frac{25}{144} = \sec^2 y$$

$$\frac{144+25}{144} = \sec^2 y$$

$$\sqrt{\frac{169}{144}} = \sqrt{\sec^2 y}$$

$$\frac{13}{12} = \sec y$$

$$\cos y = \frac{12}{13}$$

$$\frac{\sin y}{\cos y} = \tan y$$

$$\sin y = \tan y \cdot \cos y$$

$$\sin y = \frac{5}{\cancel{12}} \times \frac{\cancel{12}}{13}$$

$$\sin y = \frac{5}{13}$$

$$y = \sin^{-1}\frac{5}{13}$$

$$\tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13}$$

ii) $$2\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25}$$

L.H.S:-

let $$2\cos^{-1}\frac{4}{5} = y$$

$$\cos y = \frac{4}{5}$$

$$\sin^2 y = 1 - \cos^2 y$$

$$= 1 - \left(\frac{4}{5}\right)^2$$

$$= 1 - \frac{16}{25}$$

$$= \frac{25-16}{25}$$

$$\sqrt{\sin^2 y} = \sqrt{\frac{9}{25}}$$

$$\sin y = \frac{3}{5}$$

$$\sin 2y = 2\sin y \cos y$$

$$\sin 2y = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)$$

$$\sin 2y = \frac{24}{25}$$

$$2y = \sin^{-1}\frac{24}{25}$$

$$2\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25}$$

(iii) $\cos^{-1}\frac{4}{5} = \cot^{-1}\frac{4}{3}$

L.H.S:-

let $\cos^{-1}\frac{4}{5} = y$

$$\cos y = \frac{4}{5}$$

$$\sin y = \sqrt{1 - \cos^2 y}$$

$$= \sqrt{1 - \left(\frac{4}{5}\right)^2}$$

$$= \sqrt{1 - \frac{16}{25}}$$

$$= \sqrt{\frac{25 - 16}{25}}$$

$$\sin y = \frac{9}{5}$$

$$\frac{\cos y}{\sin y} = \cot y$$

$$\frac{4}{\cancel{8}} \times \frac{\cancel{8}}{3} = \cot y$$

$$\cot y = \frac{4}{3}$$

$$y = \cot^{-1}\frac{4}{3}$$

$$\cos^{-1}\frac{4}{5} = \cot^{-1}\frac{4}{3}$$

Q.NO. 3:-

Find the value of each expression:

(i) $\cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)$

let

$$\sin^{-1}\frac{1}{\sqrt{2}} = y$$

$$\sin y = \frac{1}{\sqrt{2}}$$

$$y = \frac{\pi}{4}$$

Now;

$$= \cos(y)$$

$$= \cos\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}$$

(ii) $\sec\left(\cos^{-1}\frac{1}{2}\right)$

let

$$\cos^{-1}\frac{1}{2} = y$$

$$\cos y = \frac{1}{2}$$

$$y = \frac{\pi}{3}$$

Now;

$$= \sec(y)$$

$$= \sec\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{\cos\left(\frac{\pi}{3}\right)}$$

$$= \frac{1}{1/2}$$

$$= 2$$

(iii) $\tan\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$

let

$$\cos^{-1}\frac{\sqrt{3}}{2} = y$$

$$\cos y = \frac{\sqrt{3}}{2}$$

$$y = \frac{\pi}{3}$$

NOW; $= \tan(y)$

$$= \tan\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{\sqrt{3}}$$

iv) $\operatorname{cosec}(\tan^{-1}(-1))$

let

$$\tan^{-1}(-1) = y$$

$$\tan y = -1$$

$$y = -\frac{\pi}{4}$$

NOW;

$$= \operatorname{cosec}(y)$$

$$= \operatorname{cosec}\left(-\frac{\pi}{4}\right)$$

$$= -\frac{1}{\sin\left(\frac{\pi}{4}\right)}$$

$$= -\frac{1}{1/\sqrt{2}}$$

$$= -\sqrt{2}$$

(v) $\sec\left(\sin^{-1} \frac{-1}{2}\right)$

let

$$\sin^{-1}\left(\frac{-1}{2}\right) = y$$

$$\sin y = \frac{-1}{2}$$

$$y = \frac{-\pi}{6}$$

Now;

$$= \sec(y)$$

$$= \sec\left(\frac{-\pi}{6}\right)$$

$$= -\frac{1}{\cos\left(\frac{\pi}{6}\right)}$$

$$= -\frac{1}{\sqrt{3}/2}$$

$$= \frac{-2}{\sqrt{3}}$$

(vi) $\tan(\tan^{-1}(-1))$

Let

$$\tan^{-1}(-1) = y$$

$$\tan y = -1$$

$$y = -\frac{\pi}{4}$$

Now;

$$= \tan(y)$$

$$= \tan\left(-\frac{\pi}{4}\right)$$

$$= -1$$

(vii) $\sin\left(\sin^{-1}\frac{1}{2}\right)$

Let

$$\sin^{-1}\left(\frac{1}{2}\right) = y$$

$$\sin y = \frac{1}{2}$$

$$y = \frac{\pi}{6}$$

Now;

$$= \sin(y)$$

$$= \sin\left(\frac{\pi}{6}\right)$$

$= \frac{1}{2}$

(viii) $\tan\left(\sin^{-1}\left(\frac{-1}{2}\right)\right)$

let

$\sin^{-1}\left(\frac{-1}{2}\right) = y$

$\sin y = \frac{-1}{2}$

$y = \frac{-\pi}{6}$

Now;

$= \tan(y)$

$= \tan\left(\frac{-\pi}{6}\right)$

$= \frac{-1}{\sqrt{3}}$

(ix) $\sin(\tan^{-1}(-1))$

let

$\tan^{-1}(-1) = y$

$\tan y = 1$

$y = \frac{\pi}{6}$

NOW;

$= \sin(y)$

$= \sin\left(\frac{\pi}{6}\right)$

$= \frac{1}{2}$

Important Formulas

1- $\sin^{-1}A + \sin^{-1}B = \sin^{-1}\left(A\sqrt{1-B^2} + B\sqrt{1-A^2}\right)$

2- $\sin^{-1}A - \sin^{-1}B = \sin^{-1}\left(A\sqrt{1-B^2} - B\sqrt{1-A^2}\right)$

3- $\cos^{-1}A + \cos^{-1}B = \cos^{-1}\left(AB - \sqrt{(1-A^2)(1-B^2)}\right)$

4- $\cos^{-1}A - \cos^{-1}B = \cos^{-1}\left(AB + \sqrt{(1-A^2)(1-B^2)}\right)$

5- $\tan^{-1}A + \tan^{-1}B = \tan^{-1}\frac{A+B}{1-AB}$

6- $\tan^{-1}A - \tan^{-1}B = \tan^{-1}\frac{A-B}{1+AB}$

Exercise 13.2

Q.NO. 13:-

Show that $\cos(\sin^{-1}x) = \sqrt{1-x^2}$

L.H.S:-

$\cos(\sin^{-1}x)$

Let

$$y = \sin^{-1}x$$

$$\sin y = x$$

$$= \cos y$$

$$\therefore \quad \cos y = \sqrt{1-\sin^2 y}$$

$$= \sqrt{1-x^2} \quad = R.H.S$$

Q.NO.14:-

Show that $\sin(2\cos^{-1}x) = 2x\sqrt{1-x^2}$

L.H.S

$$= \sin(2\cos^{-1}x)$$

Let

$$y = \cos^{-1}x$$

$$\cos y = x$$

$$= \sin 2y$$

$$\therefore \quad \sin 2y = 2\sin y \cos y$$

$$= 2\sqrt{1-\cos^2} \cos y$$

$$= 2\sqrt{1-x^2}\,(x)$$

$$= 2x\sqrt{1-x^2}$$

$$= R.H.S$$

Q.NO.19:-

Show that $\tan^{-1}(\sin^{-1}x) = \dfrac{x}{\sqrt{1-x^2}}$

L.H.S:-

$$\sin^{-1}x = y$$

$$\sin y = x$$

$$= \tan y$$

$$= \frac{\sin y}{\cos y}$$

$$= \frac{\sin y}{\sqrt{1-\sin^2}}$$

$$\because \sin^2 y + \cos^2 y = 1$$

$$\sqrt{\cos^2 y} = \sqrt{1-\sin^2 y}$$

$$\cos y = \sqrt{1-\sin^2 y}$$

$$= \frac{x}{\sqrt{1-x^2}}$$

$$= R.H.S$$

Q.NO.20:-

Given that $x = \sin^{-1}\frac{1}{2}$, find the values of the following trigonometric functions:

$\sin x, \cos x, \tan x, \cot x, \sec x, \operatorname{cosec} x$

$$\sin x = \frac{1}{2} = \frac{P}{H}$$

By pythagoras theorem

$$H^2 = B^2 + P^2$$

$$(2)^2 = B^2 + (1)^2$$

$$4 - 1 = B^2$$

$$\sqrt{3} = \sqrt{B^2}$$

$$B = \sqrt{3}$$

$$\operatorname{cosec} x = 2$$

$$\cos x = \frac{B}{H} = \frac{\sqrt{3}}{2}$$

$$\sec x = \frac{2}{\sqrt{3}}$$

$$\tan x = \frac{P}{B} = \frac{1}{\sqrt{3}}$$

$$\cot x = \sqrt{3}$$

Q.NO. 15:-

Show that $\cos(2\sin^{-1}x) = 1 - 2x^2$

L.H.S:-

$$= \cos(2\sin^{-1}x)$$

let

$$\sin^{-1}x = y$$

$$\sin y = x$$

$$= \cos 2y$$

$$\therefore \cos 2y = 1 - 2\sin^2 y$$

As; $\sin y = x$

$$= 1 - 2x^2$$

$$= R.H.S$$

Q.NO.16:-

Show that $\tan^{-1}(-x) = -\tan^{-1}x$

L.H.S:-

$\tan^{-1}(-x) + \tan^{-1}(x) = 0$

$\because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\left[\frac{A+B}{1-AB}\right]$

$= \tan^{-1}\left[\frac{\cancel{-x}+\cancel{x}}{1-(-x)(x)}\right]$

$= \tan^{-1}\left[\frac{0}{1+x^2}\right]$

$= \tan^{-1}(0)$

$= 0$

$=$ R.H.S

Q.NO.17:-

Show that $\sin^{-1}(-x) = -\sin^{-1}x$

$\sin^{-1}(-x) + \sin^{-1}(x) = 0$

L.H.S:-

$= \sin^{-1}(-x) + \sin^{-1}(x)$

$\because \sin^{-1}A + \sin^{-1}B = \sin^{-1}\left[A\sqrt{1-B^2} + B\sqrt{1-A^2}\right]$

$= \sin^{-1}\left[-x\sqrt{1-x^2} + x\sqrt{1-x^2}\right]$

$= \sin^{-1}(-x+x)$

$= 0$

$=$ R.H.S

Q.NO.18:-

Show that $\cos^{-1}(-x) = \pi - \cos^{-1}x$

$\cos^{-1}(-x) + \cos^{-1}(x) = \pi$

L.H.S:-

$= \cos^{-1}(-x) + \cos^{-1}(x)$

$\because \cos^{-1}A + \cos^{-1}B = \cos^{-1}\left[AB - \sqrt{(1-A^2)(1-B^2)}\right]$

$= \cos^{-1}\left(-x^2 - \sqrt{(1-x^2)(1-x^2)}\right]$

$= \cos^{-1}\left[-x^2 - \sqrt{(1-x^2)^2}\right]$

$= \cos^{-1}\left[-x^2 - (1-x^2)\right]$

$= \cos^{-1}\left[\cancel{-x^2} - 1 + \cancel{x^2}\right]$

$= \cos^{-1}(-1)$

$= \pi$

$=$ R.H.S

Q.NO.1:-

$\sin^{-1}\frac{5}{13} + \sin^{-1}\frac{7}{25} = \cos^{-1}\frac{253}{325}$

L.H.S:-

$= \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{7}{25}$

$\because \sin^{-1}A + \sin^{-1}B = \sin^{-1}\left[A\sqrt{1-B^2} + B\sqrt{1-A^2}\right]$

$= \sin^{-1}\left[\frac{5}{13}\sqrt{1-\left(\frac{7}{25}\right)^2} + \frac{7}{25}\sqrt{1-\left(\frac{5}{13}\right)^2}\right]$

$= \sin^{-1}\left[\frac{5}{13}\sqrt{1-\frac{49}{625}} + \frac{7}{25}\sqrt{1-\frac{25}{169}}\right]$

$$= \sin^{-1}\left[\frac{5}{13}\sqrt{\frac{625-49}{625}} + \frac{7}{25}\sqrt{\frac{169-25}{169}}\right]$$

$$= \sin^{-1}\left[\frac{5}{13}\left(\frac{24}{25}\right) + \frac{7}{25}\left(\frac{12}{13}\right)\right]$$

$$= \sin^{-1}\left(\frac{120}{325} + \frac{84}{325}\right)$$

$$= \sin^{-1}\left(\frac{120+84}{325}\right)$$

$$= \sin^{-1}\left(\frac{204}{325}\right)$$

let

$$\sin^{-1}\left(\frac{204}{325}\right) = y$$

$$\sin y = \frac{204}{325} = \frac{P}{H}$$

By pythagoras theorem

$$H^2 = B^2 + P^2$$

$$(325)^2 = B^2 + (204)^2$$

$$105625 - 41616 = B^2$$

$$\sqrt{64009} = \sqrt{B^2}$$

$$253 = B$$

$$\cos y = \frac{B}{H} = \frac{253}{325}$$

$$y = \cos^{-1}\frac{253}{325}$$

Q.NO.6:-

$$\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \sin^{-1}\frac{77}{85}$$

L.H.S:-

$$= \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17}$$

$$= \sin^{-1}\left[\frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2} + \frac{8}{17}\sqrt{1-\left(\frac{3}{5}\right)^2}\right]$$

$$= \sin^{-1}\left[\frac{3}{5}\sqrt{1-\frac{64}{289}} + \frac{8}{17}\sqrt{1-\frac{9}{25}}\right]$$

$$= \sin^{-1}\left[\frac{3}{5}\sqrt{\frac{289-64}{289}} + \frac{8}{17}\sqrt{\frac{25-9}{25}}\right]$$

$$= \sin^{-1}\left[\frac{3}{5}\sqrt{\frac{225}{289}} + \frac{8}{17}\sqrt{\frac{16}{25}}\right]$$

$$= \sin^{-1}\left[\frac{3}{5}\left(\frac{15}{17}\right) + \frac{8}{17}\left(\frac{4}{5}\right)\right]$$

$$= \sin^{-1}\left(\frac{45}{85} + \frac{32}{85}\right)$$

$$= \sin^{-1}\left(\frac{45+32}{85}\right)$$

$$= \sin^{-1}\frac{77}{85} = R.H.S$$

Q.NO.2:-

$$\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{9}{19}$$

L.H.S:-

$$= \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5}$$

$$\because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\left[\frac{A+B}{1-AB}\right]$$

$$= \tan^{-1}\left[\frac{\frac{1}{4}+\frac{1}{5}}{1-\left(\frac{1}{4}\right)\left(\frac{1}{5}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\frac{5+4}{20}}{1-\frac{1}{20}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{9}{20}}{\frac{20-1}{20}}\right]$$

$$= \tan^{-1}\frac{9}{19}$$

$$= R.H.S$$

Q.NO.11:-

$$\tan^{-1}\frac{1}{11} + \tan^{-1}\frac{5}{6} = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}$$

$$\because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\left[\frac{A+B}{1-AB}\right]$$

$$\tan^{-1}\left[\frac{\frac{1}{11}+\frac{5}{6}}{1-\left(\frac{1}{11}\right)\left(\frac{5}{6}\right)}\right] = \tan^{-1}\left[\frac{\frac{1}{3}+\frac{1}{2}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)}\right]$$

$$\tan^{-1}\left[\frac{\frac{6+55}{66}}{\frac{66-5}{66}}\right] = \tan^{-1}\left[\frac{\frac{2+3}{8}}{\frac{6-1}{6}}\right]$$

$$\tan^{-1}\frac{\cancel{61}}{\cancel{61}} = \tan^{-1}\frac{\cancel{8}}{\cancel{8}}$$

$$\tan^{-1}(1) = \tan^{-1}(1)$$

$$\frac{\pi}{4} = \frac{\pi}{4}$$

Q.NO.9:-

$$\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19} = \frac{\pi}{4}$$

L.H.S:-

$$= \tan^{-1}\left[\frac{\left(\frac{3}{4}+\frac{3}{5}\right)}{1-\left(\frac{3}{4}\right)\left(\frac{3}{5}\right)}\right] - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\left[\frac{\frac{15+12}{\cancel{20}}}{\frac{20-9}{\cancel{20}}}\right] - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\frac{27}{11} - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\left[\frac{\frac{27}{11}-\frac{8}{9}}{1+\left(\frac{27}{11}\right)\left(\frac{8}{9}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\frac{513-88}{\cancel{209}}}{\frac{209+216}{\cancel{209}}}\right]$$

$$= \tan^{-1}\frac{\cancel{425}}{\cancel{425}}$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

$$= R.H.S$$

Q.NO.10:-

$$\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$$

L.H.S:-

$$= \sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{25}$$

$$= \sin^{-1}\left[\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2} + \frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right] + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left[\frac{4}{5}\sqrt{1-\frac{25}{169}} + \frac{5}{13}\sqrt{1-\frac{16}{25}}\right] + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left[\frac{4}{5}\sqrt{\frac{169-25}{169}} + \frac{5}{13}\sqrt{\frac{25-16}{25}}\right] + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left[\frac{4}{5}\sqrt{\frac{144}{169}} + \frac{5}{13}\sqrt{\frac{9}{25}}\right] + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left[\frac{4}{5}\left(\frac{12}{13}\right) + \frac{5}{13}\left(\frac{3}{5}\right)\right] + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left[\frac{48}{65} + \frac{15}{65}\right] + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left(\frac{48+15}{65}\right) + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\frac{63}{65} + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left[\frac{63}{65}\sqrt{1-\left(\frac{16}{65}\right)^2} + \frac{16}{65}\sqrt{1-\left(\frac{63}{65}\right)^2}\right]$$

$$= \sin^{-1}\left[\frac{63}{65}\sqrt{1-\frac{256}{4225}} + \frac{16}{65}\sqrt{1-\frac{3969}{4225}}\right]$$

$$= \sin^{-1}\left[\frac{63}{65}\sqrt{\frac{4225-256}{4225}} + \frac{16}{65}\sqrt{\frac{4225-3969}{4225}}\right]$$

$$= \sin^{-1}\left[\frac{63}{65}\left(\frac{63}{65}\right) + \frac{16}{65}\left(\frac{16}{65}\right)\right]$$

$$= \sin^{-1}\left[\frac{3969}{4225} + \frac{256}{4225}\right]$$

$$= \sin^{-1}\left[\frac{3969+256}{4225}\right]$$

$$= \sin^{-1}\left[\frac{4225}{4225}\right]$$

$$= \sin^{-1}(1)$$

$$= \frac{\pi}{2}$$

$$= R.H.S$$

Q.NO.3:-

$$2\tan^{-1}\frac{2}{3} = \sin^{-1}\frac{12}{13}$$

L.H.S:-

let $y = \tan^{-1}\frac{2}{3}$

$$\tan y = \frac{2}{3}$$

By pythagoras theorem

$$H^2 = B^2 + P^2$$

$$H^2 = (3)^2 + (2)^2$$

$$H^2 = 9 + 4$$

$$\sqrt{H^2} = \sqrt{13}$$

$$H = \sqrt{13}$$

$$\sin y = \frac{P}{H} = \frac{2}{\sqrt{13}}$$

$$\cos y = \frac{B}{H} = \frac{3}{\sqrt{13}}$$

We know that;

$$\therefore \sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin 2y = 2\sin y \cos y$$

$$\sin 2y = 2\left(\frac{2}{\sqrt{13}}\right)\left(\frac{3}{\sqrt{13}}\right)$$

$$\sin 2y = \frac{12}{(\sqrt{13})^2}$$

$$\sin 2y = \frac{12}{13}$$

$$2y = \sin^{-1}\frac{12}{13}$$

$$2\tan^{-1}\frac{2}{3} = \sin^{-1}\frac{12}{13}$$

Q.NO.4:-

$$\tan^{-1}\frac{120}{119} = 2\cos^{-1}\frac{12}{13}$$

R.H.S:-

let $y = \cos^{-1}\frac{12}{13}$

$$\cos y = \frac{12}{13} = \frac{B}{H}$$

By pythagoras theorem

$$H^2 = B^2 + P^2$$

$$(13)^2 = (12)^2 + P^2$$

$$169 = 144 + P^2$$

$$169 - 144 = P^2$$

$$\sqrt{25} = \sqrt{P^2}$$

$$5 = P$$

$$\tan y = \frac{P}{B} = \frac{5}{12}$$

We know that

$$\therefore \tan 2y = \frac{2\tan y}{1-\tan^2 y}$$

$$\tan 2y = \left[\frac{2\left(\frac{5}{12}\right)}{1-\left(\frac{5}{12}\right)^2}\right]$$

$$\tan 2y = \left[\frac{\left(\frac{10}{12}\right)}{1-\frac{25}{144}}\right]$$

$$\tan 2y = \left[\frac{\frac{10}{12}}{\frac{144-25}{144}}\right]$$

$$\tan 2y = \frac{10}{\cancel{12}_1} \times \frac{\cancel{144}^{12}}{119}$$

$$\tan 2y = \frac{120}{119}$$

$$2y = \tan^{-1}\frac{120}{119}$$

$$2\cos^{-1}\frac{12}{13} = \tan^{-1}\frac{120}{119}$$

Q.NO. 7β-

$$\sin^{-1}\frac{77}{85} - \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{15}{17}$$

L.H.Sβ-

$$\because \sin^{-1}A - \sin^{-1}B = \sin^{-1}\left[A\sqrt{1-B^2} - B\sqrt{1-A^2}\right]$$

$$\sin^{-1}\frac{77}{85} - \sin^{-1}\frac{3}{5} = \sin^{-1}\left[\frac{77}{85}\sqrt{1-\left(\frac{3}{5}\right)^2} - \frac{3}{5}\sqrt{1-\left(\frac{77}{85}\right)^2}\right]$$

$$= \sin^{-1}\left[\frac{77}{85}\sqrt{1-\frac{9}{25}} - \frac{3}{5}\sqrt{1-\frac{5929}{7225}}\right]$$

$$= \sin^{-1}\left[\frac{77}{85}\sqrt{\frac{25-9}{25}} \cdot \frac{3}{5}\sqrt{\frac{7225-5929}{7225}}\right]$$

$$= \sin^{-1}\left[\frac{77}{85}\sqrt{\frac{16}{25}} - \frac{3}{5}\sqrt{\frac{1296}{7225}}\right]$$

$$= \sin^{-1}\left[\frac{77}{85}\left(\frac{4}{5}\right) - \frac{3}{5}\left(\frac{36}{85}\right)\right]$$

$$= \sin^{-1}\left[\frac{308}{425} - \frac{108}{425}\right]$$

$$= \sin^{-1}\left[\frac{308-108}{425}\right]$$

$$= \sin^{-1}\left(\frac{\cancel{200}^{\,8}}{\cancel{425}_{\,17}}\right)$$

$$= \sin^{-1}\frac{8}{17} = \frac{P}{H}$$

By pythagoras theorem

$$H^2 = B^2 + P^2$$

$$(17)^2 = B^2 + (8)^2$$

$$289 - 64 = B^2$$

$$\sqrt{225} = \sqrt{B^2}$$

$$15 = B$$

$$\cos y = \frac{B}{H} = \frac{15}{17}$$

$$y = \cos^{-1}\frac{15}{17}$$

So proved

Q.NO.12:-

$$2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$$

L.H.S:-

$$= 2\tan^{-1}\frac{1}{3}$$

$$2\tan^{-1}\frac{1}{3} = \tan^{-1}\left[\frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}\right]$$

$$= \tan^{-1}\left[\frac{\frac{2}{3}}{1-\frac{1}{9}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{2}{3}}{\frac{9-1}{9}}\right]$$

$$= \tan^{-1}\left[\frac{\cancel{2}^{1}}{\cancel{3}_{1}} \times \frac{\cancel{9}^{3}}{\cancel{8}_{4}}\right]$$

$$= \tan^{-1}\left[\frac{3}{4}\right]$$

$$\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left[\frac{\frac{3}{4}+\frac{1}{7}}{1-\left(\frac{3}{4}\right)\left(\frac{1}{7}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\frac{21+4}{28}}{1-\frac{3}{28}}\right]$$

$$= \tan^{-1}\left[\frac{25/28}{\frac{28-3}{28}}\right]$$

$$= \tan^{-1}\left[\frac{25/28}{25/28}\right]$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

$$= R.H.S$$