MATHEMATICS 1st YEAR

UNIT # 11

TRIGONOMETRIC FUNCTIONS & THEIR GRAPHS

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Sherazi Mathematics

اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

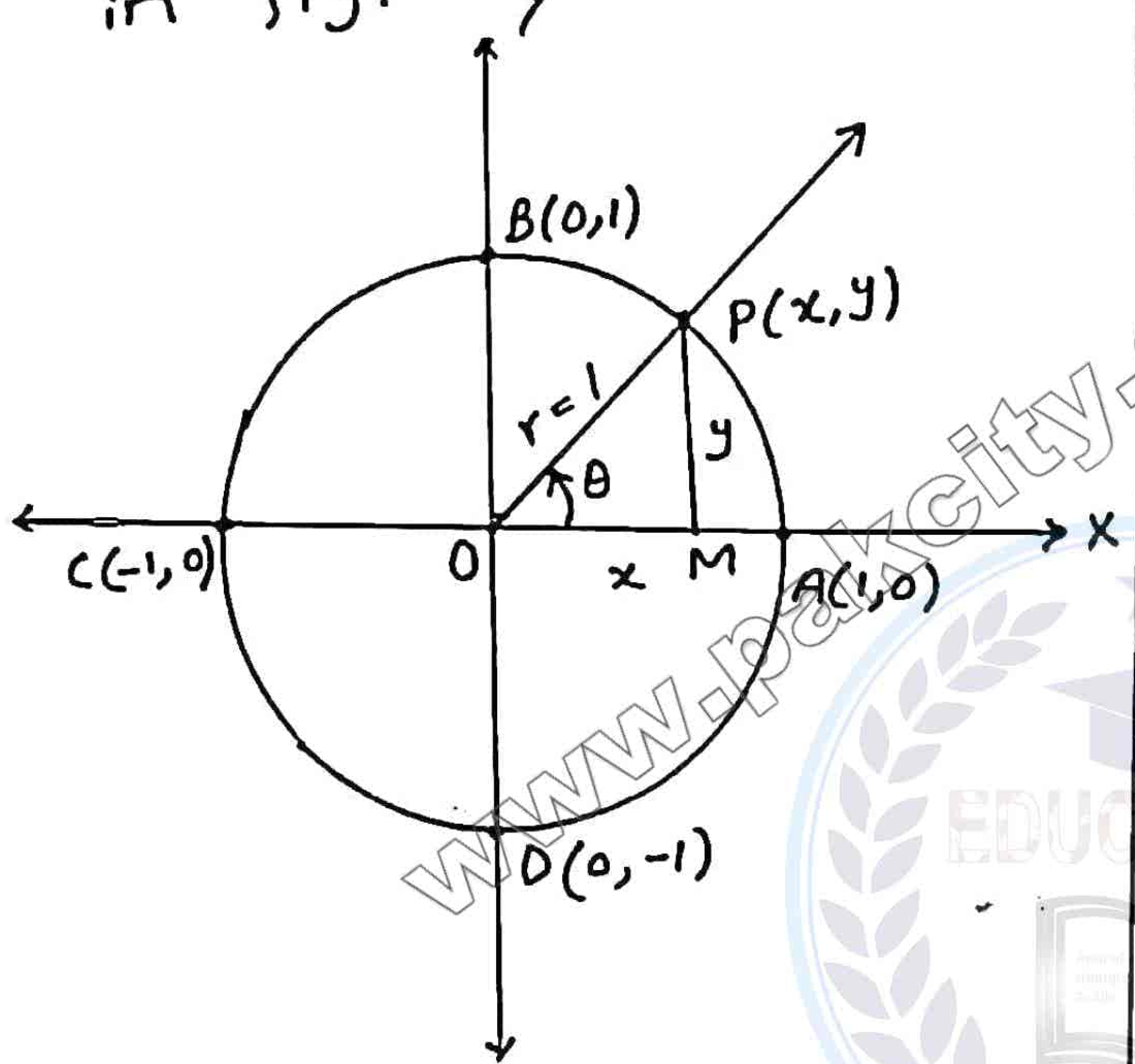
3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Domains and Ranges of Sine and Cosine Functions

Let $P(x,y)$ be any point on unit circle (radius=1) with centre at the origin O s.that $\angle XOP = \theta$ is standard position, shown in fig.



From fig

$\cos\theta = \frac{x}{r} \rightarrow \cos\theta = x \quad (\because r=1)$

and $\sin\theta = \frac{y}{r} \rightarrow \sin\theta = y \quad (\because r=1)$

Since $P(x,y)$ is a point on the unit circle

$\therefore -1 \le x \le 1 \quad \& \quad -1 \le y \le 1$

$\rightarrow -1 \le \cos\theta \le 1 \quad \& \quad -1 \le \sin\theta \le 1$

Thus $\sin\theta$ and $\cos\theta$ are functions of θ and their domain is $\mathbb{R}$ and range is $[-1, 1]$

Domains and Ranges of tangent and cotangent functions

from fig

i) $\tan\theta = \frac{y}{x}, \quad x \neq 0$

This shows that terminal side $\overrightarrow{OP}$ should not coincide with OY or OY′ (Y-axis) because on Y-axis $x = 0$

$\rightarrow \theta \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ (on Y-axis)

$\rightarrow \theta \neq (2n+1)\frac{\pi}{2}, n \in Z$

$\therefore$ Domain of $\tan\theta$

$= \mathbb{R} - \{x \mid x = (2n+1)\frac{\pi}{2}, n \in Z\}$

Range of $\tan\theta = \mathbb{R}$

ii) $\cot\theta = \frac{x}{y}, \quad y \neq 0$

This shows that terminal side $\overrightarrow{OP}$ should not coincide with OX or OX′ (X-axis) because on X-axis $y = 0$

$\rightarrow \theta \neq 0, \pm\pi, \pm 2\pi, \ldots$ (on X-axis)

$\rightarrow \theta \neq n\pi, n \in Z$

$\therefore$ Domain of $\cot\theta$

$= \mathbb{R} - \{x \mid x = n\pi, n \in Z\}$

Range of $\cot\theta = \mathbb{R}$

Domain and Range of Secant function

From fig

$$\sec\theta = \frac{r}{x}, \quad x \neq 0$$

$\Rightarrow \sec\theta = \frac{1}{x} \quad (\because r = 1)$

so terminal side $\overrightarrow{OP}$ should not coincide with OY or OY' (y-axis) because on y-axis $x = 0$.

$\Rightarrow \theta \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$\theta \neq (2n+1)\frac{\pi}{2}, \; n \in Z$

$\therefore$ Domain of $\sec\theta$

$= R - \{x \mid x = (2n+1)\frac{\pi}{2}, n \in Z\}$

As $\sec\theta$ attains all real values except those b/w -1 & 1 so

Range of $\sec\theta$

$= R - \{x \mid -1 < x < 1\}$

Domain and Range of cosecant function

from fig

$$\text{cosec}\,\theta = \frac{r}{y}, \quad y \neq 0$$

$\Rightarrow \text{cosec}\,\theta = \frac{1}{y} \quad (\because r = 1)$

so terminal side $\overrightarrow{OP}$ should not coincide with OX or OX' (x-axis) because on x-axis $y = 0$

$\Rightarrow \theta \neq 0, \pm\pi, \pm 2\pi, \ldots$

$\Rightarrow \theta \neq n\pi, \; n \in Z$

$\therefore$ Domain of $\text{cosec}\,\theta$

$= R - \{x \mid x = n\pi, n \in Z\}$

As $\text{cosec}\,\theta$ attains all real values except those b/w -1 & 1 so

Range of $\text{cosec}\,\theta$

$= R - \{x \mid -1 < x < 1\}$

Domain and Range of Trigonometric functions

Function	Domain	Range
$y = \sin x$	$-\infty < x < +\infty$	$-1 \le y \le 1$
$y = \cos x$	$-\infty < x < +\infty$	$-1 \le y \le 1$
$y = \tan x$	$-\infty < x < \infty,\ x \ne \frac{(2n+1)\pi}{2},\ n \in Z$	$-\infty < y < +\infty$
$y = \cot x$	$-\infty < x < \infty,\ x \ne n\pi,\ n \in Z$	$-\infty < y < +\infty$
$y = \sec x$	$-\infty < x < \infty,\ x \ne \frac{(2n+1)\pi}{2},\ n \in Z$	$y \ge 1$ or $y \le -1$
$y = \operatorname{cosec} x$	$-\infty < x < +\infty,\ x \ne n\pi,\ n \in Z$	$y \ge 1$ or $y \le -1$

Period of Trigonometric Functions

Period of a trigonometric function is the smallest +ve number which, when added to the orignal circular measure of the angle, gives the same value of the function.

Alternative definition "Periodic function"

A function $f(x)$ is said to be periodic if for a least positive number P, $f(x+P) = f(x)$ then P is called period of $f(x)$.

Theorem 1. Sine is a periodic function and its period is 2π.

Proof:- suppose P is period of sine function such that

$$\sin(\theta + P) = \sin\theta \quad , \forall \theta \in R \longrightarrow (i)$$

Put $\theta = 0$

$\rightarrow \sin(0 + P) = \sin 0$

$\rightarrow \sin P = 0 \quad \rightarrow P = \sin^{-1}(0)$

$\rightarrow P = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

If $P = \pi$ then

$(i) \rightarrow \sin(\theta + \pi) = \sin\theta$

False because $\sin(\pi + \theta) = -\sin\theta$

If $P = 2\pi$ then

$(i) \rightarrow \sin(\theta + 2\pi) = \sin\theta$

which is true because $\sin(2\pi + \theta) = \sin\theta$

Hence sine is a periodic function of period 2π.

Theorem 2. Cosine is a periodic function and its period is 2π.

Proof:- suppose P is period of cosine function such that

$$\cos(\theta + P) = \cos\theta \quad , \forall \theta \in R \longrightarrow (i)$$

Put $\theta = \frac{\pi}{2}$

$\rightarrow \cos(\frac{\pi}{2} + P) = \cos\frac{\pi}{2}$

$\rightarrow -\sin P = 0 \quad \because \cos(\frac{\pi}{2} + \theta) = -\sin\theta$

$\rightarrow \sin P = 0 \rightarrow P = \sin^{-1}(0)$

$\rightarrow P = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

If $p = \pi$ then

(i) $\rightarrow \cos(\theta+\pi) = \cos\theta$

false because $\cos(\pi+\theta) = -\cos\theta$

If $p = 2\pi$ then

(i) $\rightarrow \cos(\theta+2\pi) = \cos\theta$

which is true because

$\cos(2\pi+\theta) = \cos\theta$

Hence cosine is a periodic function of period 2π.

Theorem 3. Tangent is a periodic function and its period is π.

Proof:- suppose p is period of tangent function such that

$\tan(\theta+p) = \tan\theta \quad \forall\, \theta \in R \longrightarrow (i)$

Put $\theta = 0$

$\tan(0+p) = \tan 0$

$\rightarrow \tan p = 0 \rightarrow p = \tan^{-1}(0)$

$\rightarrow p = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

If $p = \pi$ then

(i) $\rightarrow \tan(\theta+\pi) = \tan\theta$

which is true because

$\tan(\pi+\theta) = \tan\theta$

Hence tangent is a periodic function of period π.

Theorem 4. cosecent is a periodic function and it period is 2π.

Proof:- suppose p is period of cosecent function such that

$\text{cosec}(\theta+p) = \text{cosec}\,\theta \quad \forall\, \theta \in R \longrightarrow (i)$

Put $\theta = \frac{\pi}{2}$

$\text{cosec}(\frac{\pi}{2}+p) = \text{cosec}\frac{\pi}{2}$

$\frac{1}{\sin(\frac{\pi}{2}+p)} = 1 \quad \because \text{cosec}\frac{\pi}{2} = \frac{1}{\sin\frac{\pi}{2}} = \frac{1}{1} = 1$

$\rightarrow 1 = \sin(\frac{\pi}{2}+p)$

$\rightarrow \cos p = 1$

$\rightarrow p = \cos^{-1}(1)$

$\rightarrow p = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

If $p = \pi$ then

(i) $\rightarrow \text{cosec}(\theta+\pi) = \text{cosec}\,\theta$

false because

$\text{cosec}(\pi+\theta) = \frac{1}{\sin(\pi+\theta)} = \frac{1}{-\sin\theta}$

$\rightarrow \text{cosec}(\pi+\theta) = -\text{cosec}\,\theta$

If $p = 2\pi$ then

(i) $\rightarrow \text{cosec}(\theta+2\pi) = \text{cosec}\,\theta$

which is true because

$\text{cosec}(2\pi+\theta) = \frac{1}{\sin(2\pi+\theta)} = \frac{1}{\sin\theta}$

$\rightarrow \text{cosec}(2\pi+\theta) = \text{cosec}\,\theta$

Hence cosecent is a periodic function of period 2π.

Theorem 5. Secent is a periodic function and its period is 2π.

Proof:- suppose p is period of secent function such that

$\sec(\theta+p) = \sec\theta \quad \forall\, \theta \in R \longrightarrow (i)$

Put $\theta = 0$

$\sec(0+p) = \sec 0$

$\rightarrow \sec p = 1 \quad \because \sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$

$\rightarrow p = \sec^{-1}(1)$, $p = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

If $p = \pi$ then

(i) $\rightarrow \sec(\theta+\pi) = \sec\theta$

false because $\sec(\pi+\theta) = \frac{1}{\cos(\pi+\theta)}$

$= \frac{1}{-\cos\theta} = -\sec\theta$

If $p = 2\pi$ then

(i) $\rightarrow \sec(\theta+2\pi) = \sec\theta$

which is true because

$$\sec(2\pi+\theta) = \frac{1}{\cos(2\pi+\theta)} = \frac{1}{\cos\theta}$$
$$= \sec\theta$$

Hence secent is a periodic function of period 2π.

Theorem 6. Cotangent is a periodic function and its period is π.

Proof:- suppose p is the period of cotangent function such that

$\cot(\theta+p) = \cot\theta \quad \forall\, \theta \in R \longrightarrow (i)$

Put $\theta = \frac{\pi}{2}$

$\cot(\frac{\pi}{2}+p) = \cot\frac{\pi}{2}$

$\rightarrow -\tan p = 0 \qquad \because \cot(\frac{\pi}{2}+\theta) = -\tan\theta$

$\rightarrow \tan p = 0$

$\rightarrow p = \tan^{-1}(0)$

$p = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots$

If $p = \pi$ then

$(i) \rightarrow \cot(\theta+\pi) = \cot\theta$

which is true because $\cot(\pi+\theta) = \cot\theta$

Hence cotangent function is periodic function of period π.

Now remember that

Function	Period	Take angle θ
sin	2π	0
cos	2π	$\frac{\pi}{2}$
tan	π	0
cosec	2π	$\frac{\pi}{2}$
sec	2π	0
cot	π	$\pi/2$

Example 1. Find the periods of:

i) $\sin 2x$ ii) $\tan\frac{x}{3}$

Solution:- i) $\sin 2x$

$\because$ period of $\sin x$ is 2π so

$\sin(2x+2\pi) = \sin 2x \qquad \because \sin(\theta+2\pi) = \sin\theta$

$\rightarrow \sin 2(x+\pi) = \sin 2x$

Thus period of $\sin 2x$ is π.

ii) $\tan\frac{x}{3}$

$\because$ Period of $\tan x$ is π. so

$\tan(\frac{x}{3}+\pi) = \tan\frac{x}{3} \qquad \because \tan(\theta+\pi) = \tan\theta$

$\rightarrow \tan\frac{1}{3}(x+3\pi) = \tan\frac{x}{3}$

Thus period of $\tan\frac{x}{3}$ is 3π.

Exercise 11.1

Find the periods of the following functions:

Q1. $\sin 3x$

Solution:- $\sin 3x$

$\because$ period of $\sin x$ is 2π. so

$\sin(3x+2\pi) = \sin 3x \qquad \because \sin(\theta+2\pi) = \sin\theta$

$\rightarrow \sin 3(x+\frac{2\pi}{3}) = \sin 3x$

$\rightarrow$ period of $\sin 3x$ is $\frac{2\pi}{3}$.

Q2. $\cos 2x$

Solution:- $\cos 2x$

$\because$ period of $\cos 2x$ is 2π so

$\cos(2x+2\pi) = \cos 2x \qquad \because \cos(\theta+2\pi) = \cos\theta$

$\rightarrow \cos 2(x+\pi) = \cos 2x$

$\rightarrow$ period of $\cos 2x$ is π.

Q3. $\tan 4x$

Solution:- $\tan 4x$

$\because$ period of $\tan x$ is π. so

$\tan(4x+\pi) = \tan 4x \qquad \because \tan(\theta+\pi) = \tan\theta$

$\rightarrow \tan 4(x+\frac{\pi}{4}) = \tan 4x$

$\rightarrow$ Period of $\tan 4x$ is $\pi/4$.

Q4. $\cot\frac{x}{2}$

Solution:- $\cot\frac{x}{2}$

$\because$ period of $\cot x$ is π. so

$\cot\left(\frac{x}{2}+\pi\right)=\cot\frac{x}{2}$ $\quad \because \cot(\theta+\pi) = \cot\theta$

$\rightarrow \cot\frac{1}{2}(x+2\pi)=\cot\frac{x}{2}$

$\rightarrow$ period of $\cot\frac{x}{2}$ is 2π.

Q5. $\sin\frac{x}{3}$

$\because$ period of $\sin x$ is 2π. so

$\sin\left(\frac{x}{3}+2\pi\right)=\sin\frac{x}{3}$ $\quad \because \sin(\theta+2\pi) = \sin\theta$

$\rightarrow \sin\frac{1}{3}(x+6\pi)=\sin\frac{x}{3}$

$\therefore$ period of $\sin\frac{x}{3}$ is 6π.

Q6. $\operatorname{cosec}\frac{x}{4}$

Solution:- $\operatorname{cosec}\frac{x}{4}$

$\because$ period of $\operatorname{cosec}x$ is 2π. So

$\operatorname{cosec}\left(\frac{x}{4}+2\pi\right)=\operatorname{cosec}\frac{x}{4}$

$\because \operatorname{cosec}(\theta+2\pi) = \operatorname{cosec}\theta$

$\rightarrow \operatorname{cosec}\frac{1}{4}(x+8\pi)=\operatorname{cosec}\frac{x}{4}$

$\therefore$ period of $\operatorname{cosec}\frac{x}{4}$ is 8π.

Q7. $\sin\frac{x}{5}$

Solution:- $\sin\frac{x}{5}$

$\because$ period of $\sin x$ is 2π. so

$\sin\left(\frac{x}{5}+2\pi\right)=\sin\frac{x}{5}$ $\quad \because \sin(\theta+2\pi) = \sin\theta$

$\rightarrow \sin\frac{1}{5}(x+10\pi)=\sin\frac{x}{5}$

$\therefore$ period of $\sin\frac{x}{5}$ is 10π.

Q8. $\cos\frac{x}{6}$

Solution:- $\cos\frac{x}{6}$

$\because$ period of $\cos x$ is 2π. so

$\rightarrow \cos\left(\frac{x}{6}+2\pi\right)=\cos\frac{x}{6}$ $\quad \because \cos(\theta+2\pi) = \cos\theta$

$\rightarrow \cos\frac{1}{6}(x+12\pi)=\cos\frac{x}{6}$

$\therefore$ period of $\cos\frac{x}{6}$ is 12π.

Q9. $\tan\frac{x}{7}$

Solution:- $\tan\frac{x}{7}$

$\because$ period of $\tan x$ is π. So

$\tan\left(\frac{x}{7}+\pi\right)=\tan\frac{x}{7}$ $\quad \because \tan(\theta+\pi) = \tan\theta$

$\rightarrow \tan\frac{1}{7}(x+7\pi)=\tan\frac{x}{7}$

$\therefore$ period of $\tan\frac{x}{7}$ is 7π.

Q10. $\cot 8x$

Solution:- $\cot 8x$

$\because$ period of $\cot x$ is π. so

$\cot(8x+\pi)=\cot 8x$ $\quad \because \cot(\theta+\pi) = \cot\theta$

$\rightarrow \cot 8\left(x+\frac{\pi}{8}\right)=\cot 8x$

$\therefore$ period of $\cot 8x$ is $\frac{\pi}{8}$.

Q11. $\sec 9x$

Solution:- $\sec 9x$

$\because$ period of $\sec x$ is 2π. so

$\sec(9x+2\pi)=\sec 9x$ $\quad \because \sec(\theta+2\pi) = \sec\theta$

$\rightarrow \sec 9\left(x+\frac{2\pi}{9}\right)=\sec 9x$

$\therefore$ period of $\sec 9x$ is $\frac{2\pi}{9}$.

Q12. $\operatorname{cosec}10x$

Solution:- $\operatorname{cosec}10x$

$\because$ period of $\operatorname{cosec}x$ is 2π. So

$\operatorname{cosec}(10x+2\pi)=\operatorname{cosec}10x$

$\rightarrow \operatorname{cosec}10\left(x+\frac{2\pi}{10}\right)=\operatorname{cosec}10x$ $\quad \because \operatorname{cosec}(\theta+2\pi) = \operatorname{cosec}\theta$

$\rightarrow \operatorname{cosec}10\left(x+\frac{\pi}{5}\right)=\operatorname{cosec}10x$

$\therefore$ period of $\operatorname{cosec}10x$ is $\frac{\pi}{5}$.

Q13. $3\sin x$

Solution:- $3\sin x$

$\because$ period of $\sin x$ is 2π. so

$3\sin(x+2\pi) = 3\sin x$ $\quad \because \sin(\theta+2\pi) = \sin\theta$

$\rightarrow$ period of $3\sin x$ is 2π.

Q14. $2\cos x$

Solution:- $2\cos x$

$\because$ period of $\cos x$ is 2π. so

$2\cos(x+2\pi) = 2\cos x$ $\quad \because \cos(\theta+2\pi) = \cos\theta$

$\rightarrow$ period of $2\cos x$ is 2π.

Q15. $3\cos\frac{x}{5}$

Solution:- $3\cos\frac{x}{5}$

$\because$ period of $\cos x$ is 2π. so

$3\cos\left(\frac{x}{5}+2\pi\right) = 3\cos\frac{x}{5}$ $\quad \because \cos(\theta+2\pi) = \cos\theta$

$\rightarrow 3\cos\frac{1}{5}(x+10\pi) = 3\cos\frac{x}{5}$

$\therefore$ period of $3\cos\frac{x}{5}$ is 10π.