MATHEMATICS 2nd YEAR

UNIT

07

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VECTORS

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Contents

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اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

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Scalar:- A quantity which has magnitude only is called scalar. e.g., time, volume, speed and work etc. The scalars are denoted by letters.

Vector:- A quantity which has both magnitude and direction is called vector. e.g., velocity, displacement, force and torque etc. A vector (say v) is denoted by $\vec{v}$ or $\underline{v}$ or by bold face letter **v**.

Geometrical interpretation of vector

Geometrically, a vector is represented by a line segment AB with A its initial point and B its terminal point. It is often found convenient to denote a vector by an arrow and is written either as $\overrightarrow{AB}$ or as a boldface symbol like **v** or in underlined form $\underline{v}$.

Magnitude of a vector:- Let v be a vector, then its magnitude is denoted by $|\vec{v}|$ or simply v. It is also called norm or length of vector. If the line $\overrightarrow{AB}$ represents a vector, then distance from pt. A to pt. B will be magnitude of $\overrightarrow{AB}$ and is denoted by $|\overrightarrow{AB}|$.

Unit vector:- A vector whose magnitude is one (unity) is called unit vector. Unit vector of $\underline{v}$ is written as $\hat{v}$ (read as v hat) and is defined as; $\hat{v} = \frac{\underline{v}}{|\underline{v}|}$

Null vector:- A vector whose magnitude is zero but no specific direction is called null or zero vector. It is denoted by $\vec{0}$.

Negative vectors:- Two vectors are said to be negative of each other if they have same magnitude but opposite direction. If $\overline{AB} = \underline{v}$ then $\overline{BA} = -\overline{AB} = -\underline{v}$ and $|\overline{BA}| = |-\overline{AB}|$ ($\because$ The magnitude of a vector is a non-negative number).

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Multiplication of vector by a Scalar:-

Let K be a scalar number ($K \in R$) and $\underline{v}$ be a vector, then $K\vec{v}$ is a vector which is K times to vector $\vec{v}$. (i) $\vec{v}$ and $K\vec{v}$ are in same direction if $K>0$. (ii) $\vec{v}$ and $K\vec{v}$ are in opposit direction if $K<0$.

Equal vectors:- Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be equal, if they have the same magnitude and same direction. i-e., $|\overrightarrow{AB}| = |\overrightarrow{CD}|$

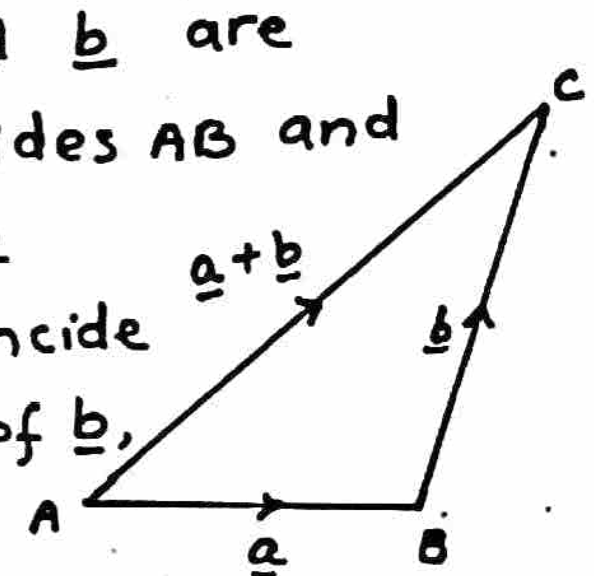
Parallel vectors:- Two vectors $\vec{u}$ and $\vec{v}$ are said to be parallel if $\vec{u} = K\vec{v}$ or $\vec{v} = K\vec{u}$. If $\vec{u} = K\vec{v}$ then $\vec{u}$ and $\vec{v}$ are in same direction if $K>0$, Also $\vec{u}$ and $\vec{v}$ are in opposite direction if $K<0$.

§ Addition of two vectors

Addition of two vectors is explained by following two laws:

(i) Triangle Law of addition:-

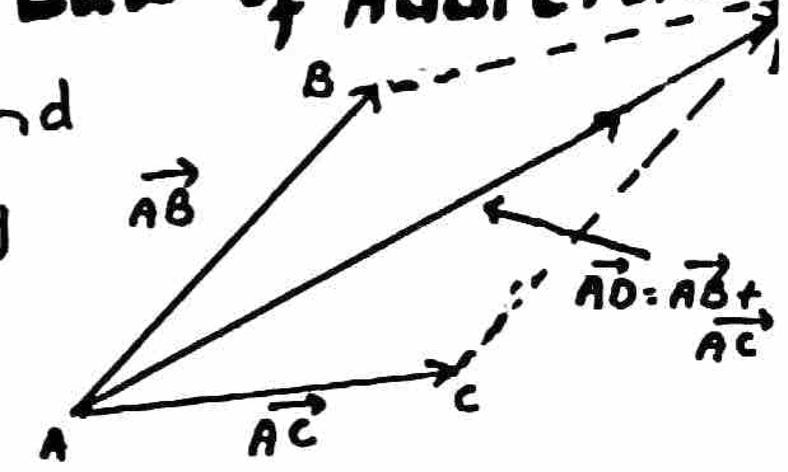
If two vectors $\underline{a}$ and $\underline{b}$ are represented by two sides AB and BC of $\triangle ABC$ s.that the terminal point of $\underline{a}$ coincide with the initial point of $\underline{b}$, then the third side AC of the triangle gives vector sum $\underline{a}+\underline{b}$.



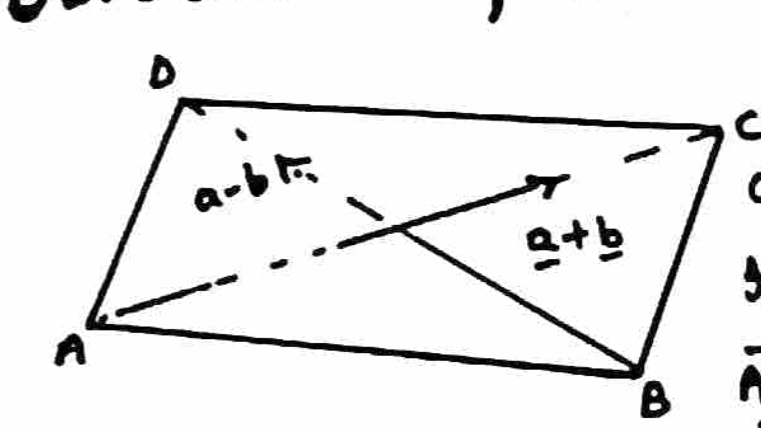
i-e., $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \Rightarrow \underline{a}+\underline{b} = \overrightarrow{AC}$

(ii) Parallelogram Law of Addition:-

If two vectors $\underline{u}$ and $\underline{v}$ are represented by two sides AB and AC of llgram as shown in fig, then diagonal AD give the sum of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ i-e., $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC} = \underline{u} + \underline{v}$



§ Subtraction of two vectors



Let ABCD be a llgram.

In fig

$\overrightarrow{AB} = \overrightarrow{DC} = \vec{a}$

$\overrightarrow{BC} = \overrightarrow{AD} = \vec{b}$

In $\triangle ABC$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\Rightarrow \overrightarrow{AC} = \underline{a} + \underline{b}$

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In ΔABD, $\vec{AB} + \vec{BD} = \vec{AD}$

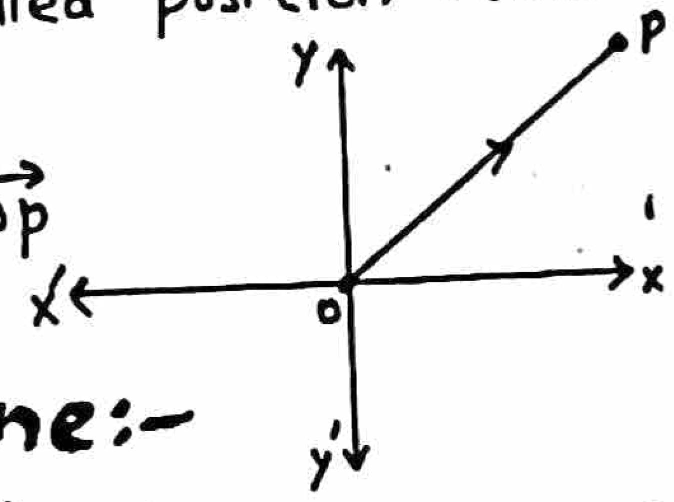
$\rightarrow \vec{BD} = \vec{AD} - \vec{AB}$

$\rightarrow \vec{BD} = \underline{b} - \underline{a}$ $\rightarrow \vec{DB} = -(\underline{b} - \underline{a})$

$\rightarrow \vec{DB} = \underline{a} - \underline{b}$

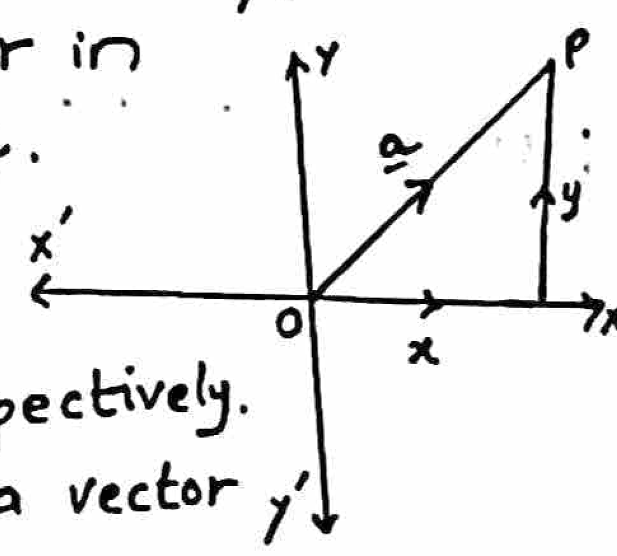
Position Vector:- The vector whose initial point is at origin O and terminal point is at P is called position vector of point P.

position vector of $P = \vec{OP}$



Vectors in plane:-

Let $\vec{OP} = \underline{a}$ be a vector in xy-plane. We resolve $\underline{a}$. x and y be the horizontal and vertical components of $\underline{a}$ respectively. Then $\underline{a} = [x, y]$ is a vector in xy-plane.



Addition:- Let $\vec{a} = [x_1, y_1]$, $\vec{b} = [x_2, y_2]$ be two vectors in plane, then

$\vec{a} + \vec{b} = [x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$

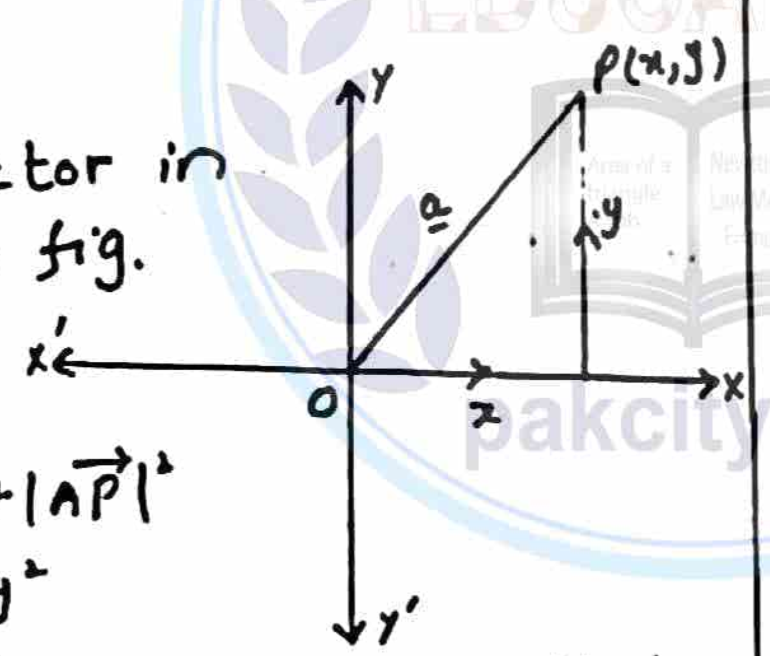
Subtraction:- Let $\vec{a} = [x_1, y_1]$, $\vec{b} = [x_2, y_2]$ be two vectors in plane, then

$\vec{a} - \vec{b} = [x_1, y_1] - [x_2, y_2] = [x_1 - x_2, y_1 - y_2]$

Note:- If K be any scalar and $\vec{a} = [x, y]$ then $K\vec{a} = K[x, y] = [Kx, Ky]$

Magnitude of a vector in plane:-

Let $\vec{a} = [x, y]$ be vector in plane as shown in fig. then by pythagora's theorem, $|\vec{OP}|^2 = |\vec{OA}|^2 + |\vec{AP}|^2$

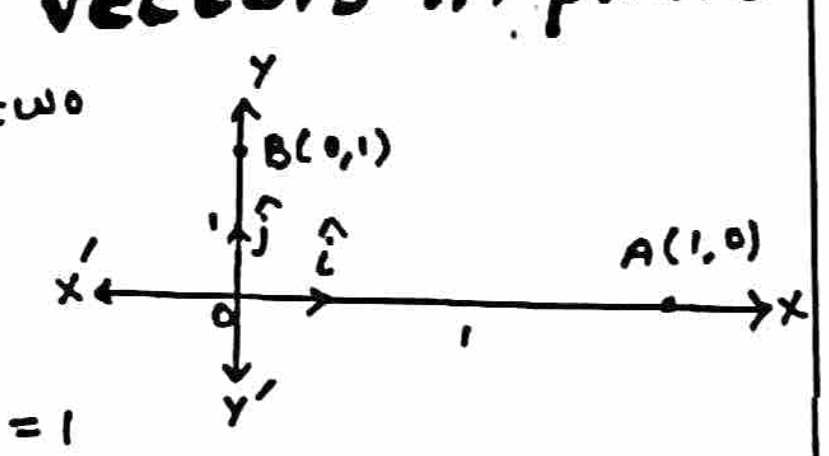


$\rightarrow |\vec{a}|^2 = x^2 + y^2$

$\rightarrow a = \sqrt{x^2 + y^2}$ which is magnitude of $\vec{a}$.

Another notation for representing vectors in plane

Let $\vec{OA}$ and $\vec{OB}$ be two unit vectors along x-axis and y-axis respectively. then $|\vec{OA}| = 1$ and $|\vec{OB}| = 1$



As $\vec{OA} = [1, 0]$, $\vec{OB} = [0, 1]$

then $\hat{i} = [1, 0]$, $\hat{j} = [0, 1]$

where $\hat{i}$ and $\hat{j}$ are called unit vectors along x-axis and y-axis respectively.

Now if $\vec{a} = [x, y]$ then $\vec{a}$ can be written as; $\vec{a} = [x, 0] + [0, y] = x[1, 0] + y[0, 1]$

$\rightarrow \vec{a} = x\hat{i} + y\hat{j}$

Example 1. For $\underline{v} = [1, -3]$ and $\underline{w} = [2, 5]$

(i) $\underline{v} + \underline{w} = [1, -3] + [2, 5] = [1+2, -3+5] = [3, 2]$

(ii) $4\underline{v} + 2\underline{w} = 4[1, -3] + 2[2, 5] = [4, -12] + [4, 10]$
$= [4+4, -12+10] = [8, -2]$

(iii) $\underline{v} - \underline{w} = [1, -3] - [2, 5] = [1-2, -3-5] = [-1, -8]$

(iv) $\underline{v} - \underline{v} = [1, -3] - [1, -3] = [1-1, -3+3] = [0, 0] = \vec{0}$

(v) $|\underline{v}| = \sqrt{(1)^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10}$

Example 2. Find the unit vector in the same direction as the vector $\underline{v} = [3, -4]$

Solution:- $\underline{v} = [3, -4] = 3\underline{i} - 4\underline{j}$

$|\underline{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$

Now $\hat{v} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{[3, -4]}{5} = \frac{1}{5}[3, -4]$

$\rightarrow \hat{v} = [\frac{3}{5}, -\frac{4}{5}]$ req. unit vector in the direction $\vec{v}$.

Example 3. Find a unit vector in the direction of the vector

(i) $\underline{v} = 2\underline{i} + 6\underline{j}$ (ii) $\underline{v} = [-2, 4]$

Solution:- $\underline{v} = 2\underline{i} + 6\underline{j} = [2, 6]$

$\rightarrow |\underline{v}| = \sqrt{(2)^2 + (6)^2} = \sqrt{4+36} = \sqrt{40} = \sqrt{4 \times 10}$

$|\underline{v}| = 2\sqrt{10}$

Now $\hat{v} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{[2, 6]}{2\sqrt{10}} = \dfrac{1}{2\sqrt{10}}[2, 6]$

$\hat{v} = [\frac{2}{2\sqrt{10}}, \frac{6}{2\sqrt{10}}] = [\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}]$

$\hat{v} = \frac{1}{\sqrt{10}}\hat{i} + \frac{3}{\sqrt{10}}\hat{j}$

(ii) $\underline{v} = [-2, 4]$

$\rightarrow |\underline{v}| = \sqrt{(-2)^2 + (4)^2} = \sqrt{4+16} = \sqrt{20}$

$|\underline{v}| = 2\sqrt{5}$

Now $\hat{v} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{[-2, 4]}{\sqrt{20}} = \frac{1}{\sqrt{20}}[-2, 4]$

$\rightarrow \hat{v} = [\frac{-2}{2\sqrt{5}}, \frac{4}{2\sqrt{5}}] = [\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}]$

$\hat{v} = \frac{-1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{j}$

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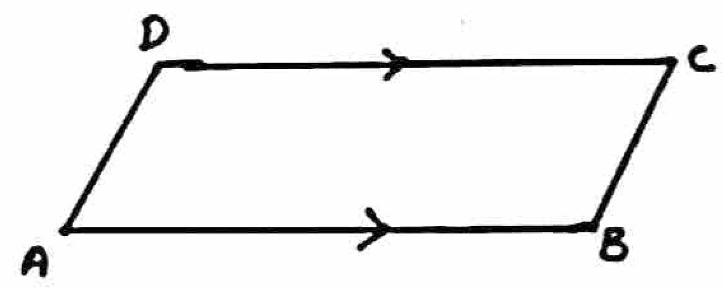
Example 4. If ABCD is a parallelogram s.that the points A, B and C are respectively $(-2,3)$, $(1,4)$, and $(0,-5)$. Find the coordinates of D.

Solution:- Let $D(x,y)$ be the req. point.



As ABCD is llgram.

$\therefore \overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$

Now $\overrightarrow{AB} = [1,4] - [-2,-3]$

$= [1+2, 4+3] = [3,7]$

Similarly, $\overrightarrow{DC} = [0,-5] - [x,y] = [0-x, -5-y]$

$\overrightarrow{DC} = [-x, -5-y]$

As $\overrightarrow{AB} = \overrightarrow{DC} \Rightarrow [3,7] = [-x, -5-y]$

$\rightarrow 3 = -x$, $7 = -5-y \rightarrow 7+5 = -y$

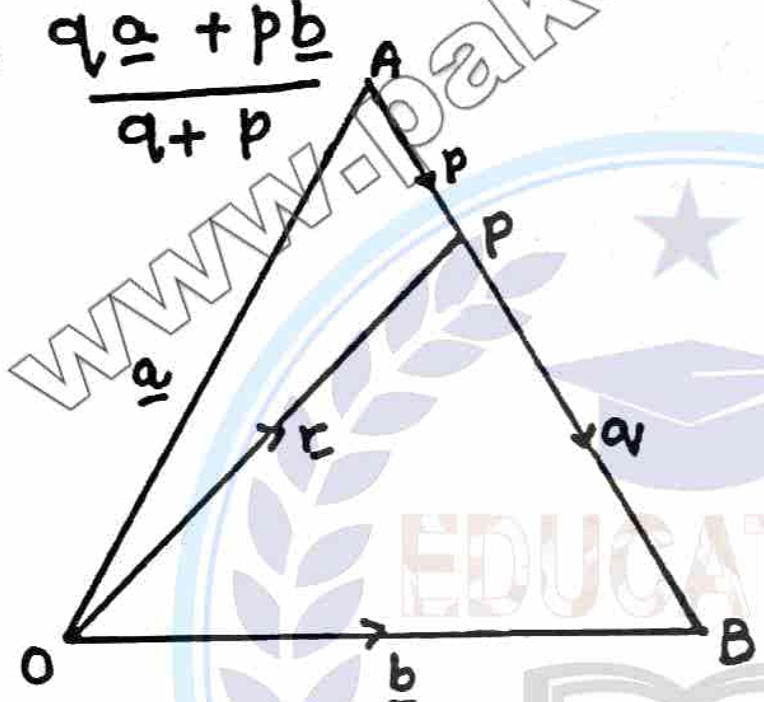
$\rightarrow x = -3$, $y = -12$

Hence coordinates of D are $(-3, -12)$.

The Ratio Formula

Statement:- Let A and B be two points whose position vectors (p.v) are $\underline{a}$ and $\underline{b}$ respectively. If a point P divides AB in the ratio $p:q$. then the position vector of P is given by $\underline{r} = \dfrac{q\underline{a} + p\underline{b}}{q+p}$

Proof:-



In fig,

$\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$

$\overrightarrow{OP} = \vec{r}$

Given that

$m\overrightarrow{AP} : m\overrightarrow{PB} = p : q \rightarrow \dfrac{m\overrightarrow{AP}}{m\overrightarrow{PB}} = \dfrac{p}{q}$

$q(m\overrightarrow{AP}) = p(m\overrightarrow{PB})$

$\rightarrow q(\overrightarrow{AP}) = p(\overrightarrow{PB})$

$\rightarrow q(\overrightarrow{OP} - \overrightarrow{OA}) = p(\overrightarrow{OB} - \overrightarrow{OP})$

$\rightarrow q(\vec{r} - \vec{a}) = p(\vec{b} - \vec{r})$

$\rightarrow q\vec{r} - q\vec{a} = p\vec{b} - p\vec{r} \rightarrow q\vec{r} + p\vec{r} = p\vec{b} + q\vec{a}$

$\rightarrow (q+p)\vec{r} = p\vec{b} + q\vec{a} \rightarrow \vec{r} = \dfrac{q\vec{a} + p\vec{b}}{q+p}$

Hence $\vec{r} = \dfrac{q\vec{a} + p\vec{b}}{q+p}$

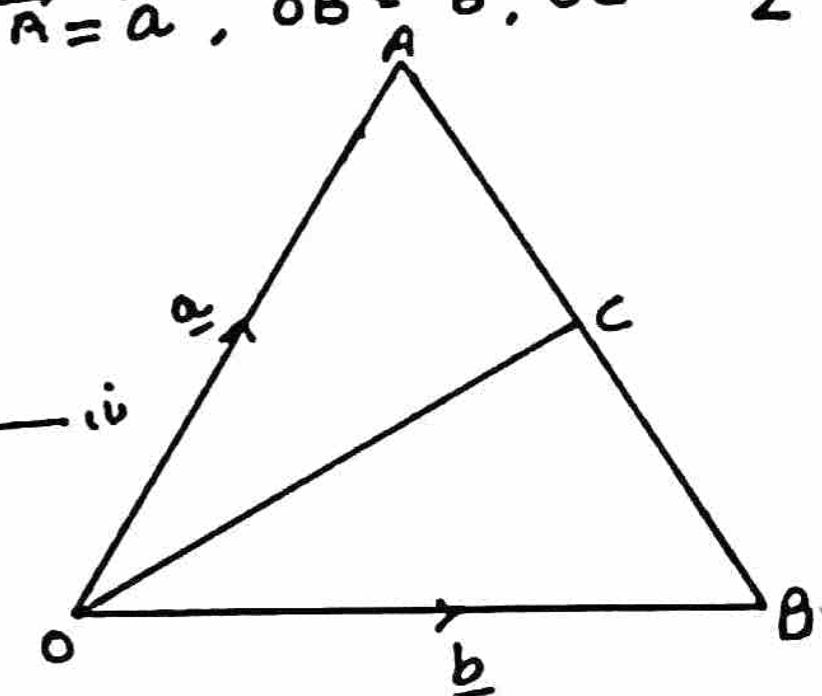
Corollary:- If P is the mid point of AB, then $p:q = 1:1$

so $\underline{r} = \dfrac{\vec{a} + \vec{b}}{2}$

Example 5. If $\underline{a}$ and $\underline{b}$ be the p.vs of A and B resp. w.r.t origin O, and C be pt. on $\overline{AB}$ s.that $\overrightarrow{OC} = \dfrac{a+b}{2}$, then show that C is the mid-point of AB.

Solution:-

Given that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \dfrac{\vec{a} + \vec{b}}{2}$



In fig.,

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$\rightarrow \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$ —— (i)

Similarly,

$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$

$\rightarrow \overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC}$ —— (ii)

$\because 2\overrightarrow{OC} = \vec{a} + \vec{b}$ (given)

$\rightarrow \overrightarrow{OC} + \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$

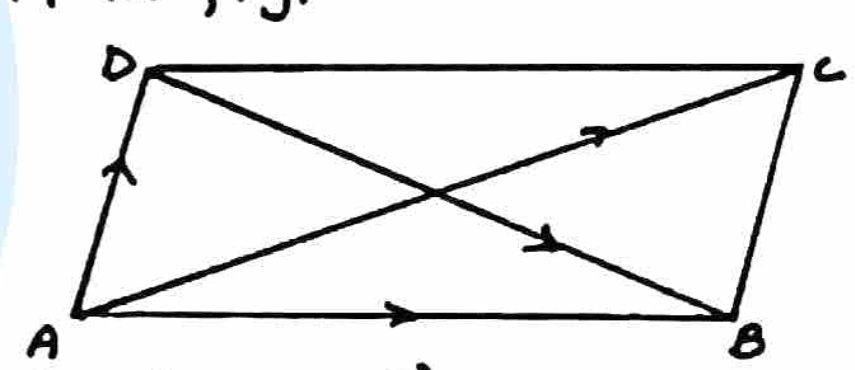
$\rightarrow \overrightarrow{OC} - \overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{OC}$

By (i) and (ii) $\overrightarrow{AC} = \overrightarrow{CB}$

$\rightarrow$ C is equidistant from A and B. Also A, B, C are collinear. so C is mid point of AB.

Example 6. Use vectors, to prove that the diagonals of a llgram bisect each other.

Solution:- Let ABCD be a llgram as shown in fig.



Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be position vectors of A, B, C and D resp of llgram

so $\overrightarrow{AB} = \overrightarrow{DC} \rightarrow \vec{b} - \vec{a} = \vec{c} - \vec{d}$

$\rightarrow \vec{b} + \vec{d} = \vec{a} + \vec{c}$ —— (i)

from fig $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are diagonals.

Now these diagonals will bisect each other if

p.v of mid point of $\overrightarrow{AC}$ = p.v of mid point of $\overrightarrow{BD}$

so p.v of mid pt. of $\overrightarrow{AC} = \dfrac{\vec{a} + \vec{c}}{2}$ —— (ii)

p.v of mid pt. of $\overrightarrow{BD} = \dfrac{\vec{b} + \vec{d}}{2}$

As $\vec{b} + \vec{d} = \vec{a} + \vec{c}$ so

p.v of mid pt. of $\overrightarrow{BD} = \dfrac{\vec{a} + \vec{c}}{2}$ —— (iii)

By (ii) and (iii) we conclude that diagonals bisect each other.

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Exercise 7.1

Q1. Write the vector $\overrightarrow{PQ}$ in the form $x\underline{i}+y\underline{j}$.

(i) $P=(2,3)$, $Q(6,-2)$

p.v of $P = \overrightarrow{OP} = 2\underline{i}+3\underline{j}$, p.v of $Q = \overrightarrow{OQ} = 6\underline{i}-2\underline{j}$

so $\overrightarrow{PQ} = \overrightarrow{OQ}-\overrightarrow{OP} = 6\underline{i}-2\underline{j}-(2\underline{i}+3\underline{j}) = 6\underline{i}-2\underline{j}-2\underline{i}-3\underline{j}$

$\rightarrow \overrightarrow{PQ} = 4\underline{i}-5\underline{j}$

(ii) $P=(0,5)$, $Q(-1,-6)$

p.v of $P = \overrightarrow{OP} = 0\underline{i}+5\underline{j}$, p.v of $Q = \overrightarrow{OQ} = -\underline{i}-6\underline{j}$

so $\overrightarrow{PQ} = \overrightarrow{OQ}-\overrightarrow{OP} = -\underline{i}-6\underline{j}-(0\underline{i}+5\underline{j}) = -\underline{i}-6\underline{j}-5\underline{j}$

$\rightarrow \overrightarrow{PQ} = -\underline{i}-11\underline{j}$

Q2. Find the magnitude of vector $\underline{u}$:

(i) $\underline{u} = 2\underline{i}-7\underline{j}$, (ii) $\underline{u} = \underline{i}+\underline{j}$, (iii) $\underline{u} = (3,-4)$

Solution:- (i) $|\underline{u}| = \sqrt{(2)^2+(-7)^2} = \sqrt{4+49} = \sqrt{53}$

(ii) $|\underline{u}| = \sqrt{(1)^2+(1)^2} = \sqrt{1+1} = \sqrt{2}$

(iii) $|\underline{u}| = \sqrt{(3)^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

Q3. If $\underline{u} = 2\underline{i}-7\underline{j}$, $\underline{v} = \underline{i}-6\underline{j}$ and $\underline{w} = -\underline{i}+\underline{j}$. Find the following vectors. (i) $\underline{u}+\underline{v}-\underline{w}$ (ii) $2\underline{u}-3\underline{v}+4\underline{w}$

(iii) $\frac{1}{2}\underline{u}+\frac{1}{2}\underline{v}+\frac{1}{2}\underline{w}$

Solution:- (i) $\underline{u}+\underline{v}-\underline{w}$

$= 2\underline{i}-7\underline{j}+\underline{i}-6\underline{j}-(-\underline{i}+\underline{j})$

$= 3\underline{i}-13\underline{j}+\underline{i}-\underline{j} = 4\underline{i}-14\underline{j}$

(ii) $2\underline{u}-3\underline{v}+4\underline{w} = 2(2\underline{i}-7\underline{j})-3(\underline{i}-6\underline{j})+4(-\underline{i}+\underline{j})$

$= 4\underline{i}-14\underline{j}-3\underline{i}+18\underline{j}-4\underline{i}+4\underline{j}$

$= -3\underline{i}+8\underline{j}$

(iii) $\frac{1}{2}\underline{u}+\frac{1}{2}\underline{v}+\frac{1}{2}\underline{w} = \frac{1}{2}(\underline{u}+\underline{v}+\underline{w})$

$= \frac{1}{2}(2\underline{i}-7\underline{j}+\underline{i}-6\underline{j}-\underline{i}+\underline{j}) = \frac{1}{2}(2\underline{i}-12\underline{j})$

$= \underline{i}-6\underline{j}$

Q4. Find the sum of vectors $\overline{AB}$ and $\overline{CD}$, given the four points $A(1,-1)$ $B(2,0)$, $C(-1,3)$ and $D(-2,2)$

Solution:- p.v of $A = \overrightarrow{OA} = \hat{i}-\hat{j}$

p.v of $B = \overrightarrow{OB} = 2\underline{i}+0\underline{j} = 2\underline{i}$

so $\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = 2\underline{i}-(\underline{i}-\underline{j}) = 2\underline{i}-\underline{i}+\underline{j} = \underline{i}+\underline{j}$

Now, p.v of $C = \overrightarrow{OC} = -\underline{i}+3\underline{j}$

p.v of $D = \overrightarrow{OD} = -2\underline{i}+2\underline{j}$

so $\overrightarrow{CD} = \overrightarrow{OD}-\overrightarrow{OC} = -2\underline{i}+2\underline{j}-(-\underline{i}+3\underline{j})$

$= -2\underline{i}+2\underline{j}+\underline{i}-3\underline{j} = -\underline{i}-\underline{j}$

Now, $\overrightarrow{AB}+\overrightarrow{CD} = \underline{i}+\underline{j}-\underline{i}-\underline{j} = 0\underline{i}+0\underline{j} = \overrightarrow{0}$

Q5. Find the vector from the point A to the origin where $\overrightarrow{AB} = 4\underline{i}-2\underline{j}$ and B is the point $(-2,5)$.

Solution:- $\overrightarrow{AO} = ?$ $O(0,0)$ (origin)

$\because \overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA}$, $\overrightarrow{OB} = [-2-0, 5-0]$

$\overrightarrow{OB} = [-2, 5] = -2\underline{i}+5\underline{j}$

$\rightarrow \overrightarrow{OA} = \overrightarrow{OB}-\overrightarrow{AB}$

$= -2\underline{i}+5\underline{j}-4\underline{i}+2\underline{j}$

$\overrightarrow{OA} = -6\underline{i}+7\underline{j} \rightarrow \overrightarrow{AO} = 6\underline{i}-7\underline{j}$

Q6. Find a unit vector in the direction of the vector given below: (i) $\underline{v} = 2\underline{i}-\underline{j}$ (ii) $\underline{v} = \frac{1}{2}\underline{i}+\frac{\sqrt{3}}{2}\underline{j}$

(iii) $\underline{v} = -\frac{\sqrt{3}}{2}\underline{i}-\frac{1}{2}\underline{j}$

Solution:- (i) $\underline{v} = 2\underline{i}-\underline{j}$, $|\underline{v}| = \sqrt{(2)^2+(-1)^2}$

$\rightarrow |\underline{v}| = \sqrt{4+1} = \sqrt{5}$

$\therefore \hat{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{2\underline{i}-\underline{j}}{\sqrt{5}} = \frac{1}{\sqrt{5}}(2\underline{i}-\underline{j})$

(ii) $\underline{v} = \frac{1}{2}\underline{i}+\frac{\sqrt{3}}{2}\underline{j}$, $|\underline{v}| = \sqrt{(\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$

$= \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$

$\therefore \hat{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{\frac{1}{2}\underline{i}+\frac{3}{2}\underline{j}}{1} = \frac{1}{2}\underline{i}+\frac{3}{2}\underline{j}$

(iii) $\underline{v} = -\frac{\sqrt{3}}{2}\underline{i}-\frac{1}{2}\underline{j}$, $|\underline{v}| = \sqrt{(-\frac{\sqrt{3}}{2})^2+(-\frac{1}{2})^2}$

$= \sqrt{\frac{3}{4}+\frac{1}{4}} = \sqrt{\frac{4}{4}} = 1$

$\therefore \hat{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{-\frac{\sqrt{3}}{2}\underline{i}-\frac{1}{2}\underline{j}}{1} = -\frac{\sqrt{3}}{2}\underline{i}-\frac{1}{2}\underline{j}$

Q7. If A, B and C are respectively the points $(2,-4)$, $(4,0)$ and $(1,6)$. Use vector method to find the coordinates of the point D if:

(i) ABCD is llgram (ii) ADBC is a llgram.

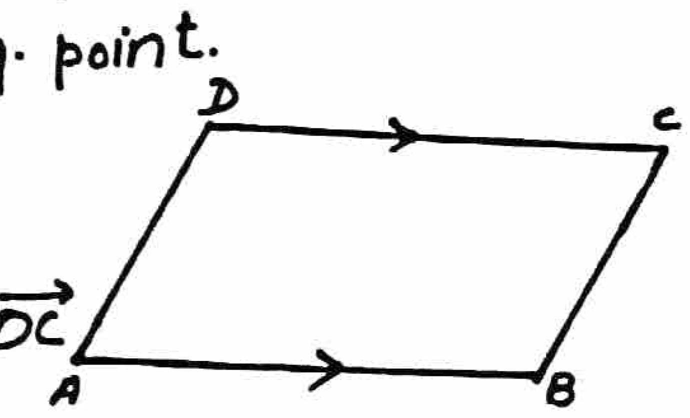
Solution:- (i) $A(2,-4)$, $B(4,0)$, $C(1,6)$

Let $D(x,y)$ be the req. point.

$\because$ ABCD is llgram.

So from fig.

$\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AB} \parallel \overrightarrow{DC}$



Now $\overrightarrow{AB} = [4,0]-[2,-4] = [4-2, 0+4] = [2,4]$

Also $\overrightarrow{DC} = [1,6]-[x,y] = [1-x, 6-y]$

$\because \overrightarrow{AB} = \overrightarrow{DC} \rightarrow [2,4] = [1-x, 6-y]$

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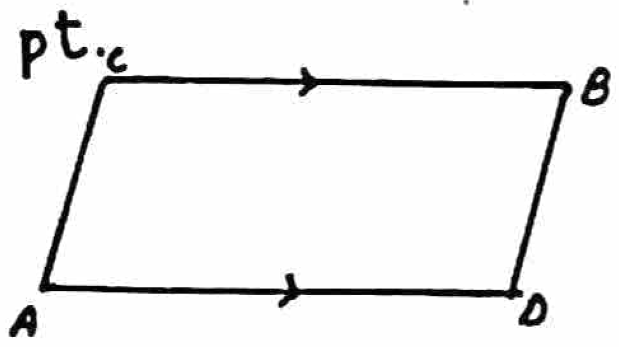
$\rightarrow 2 = 1 - x \;,\; 4 = 6 - y \rightarrow x = 1-2,\; y = 6-4$

$\rightarrow x = -1,\; y = 2$ so $D(x,y) = (-1,2)$

(ii) $\because A(2,-4), B(4,0)$ and $C(1,6)$

Let $D(x,y)$ be the req. pt.



As ADBC is ||gram.

So from fig $\overrightarrow{AD} = \overrightarrow{CB}$

and $\overrightarrow{AD} \parallel \overrightarrow{CB}$

Now $\overrightarrow{AD} = [x,y] - [2,-4] = [x-2, y+4]$

$\overrightarrow{CB} = [4,0] - [1,6] = [4-1, 0-6] = [3,-6]$

$\because \overrightarrow{AD} = \overrightarrow{CB} \rightarrow [x-2, y+4] = [3,-6]$

$\rightarrow x-2 = 3,\quad y+4 = -6 \rightarrow x = 5,\; y = -10$

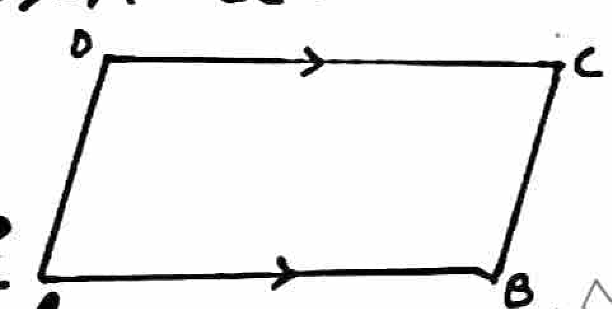
So $D(x,y) = D(5,-10)$

Q8. If B, C and D respectively $(4,1), (-2,3)$ and $(-8,0)$. Use vector method to find the coordinates of the point:

(i) A if ABCD is a ||gram (ii) E if AEBD is a ||gram

Solution:– (i) A if ABCD is a ||gram.

$\because B(4,1), C(-2,3), D(-8,0)$. Let $A(x,y)$ be req. point.



As ABCD is ||gram.

So $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AB} \parallel \overrightarrow{DC}$

Now $\overrightarrow{AB} = [4,1] - [x,y] = [4-x, 1-y]$

$\overrightarrow{DC} = [-2,3] - [-8,0] = [-2+8, 3-0] = [6,3]$

$\because \overrightarrow{AB} = \overrightarrow{DC} \rightarrow [4-x, 1-y] = [6,3]$

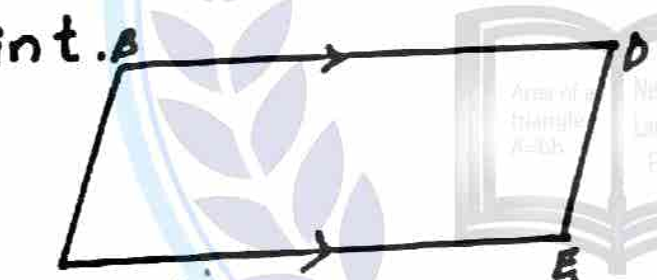
$\rightarrow 4-x = 6,\quad 1-y = 3 \rightarrow x = 4-6,\; y = 1-3$

$\rightarrow x = -2,\; y = -2$ So $A(-2,-2)$.

(ii) E if AEBD is ||gram.

$\because A(-2,-2), B(4,1), D(-8,0)$ — $A(-2,-2)$ from part (i)

Let $E(x,y)$ be the req. point.



As AEBD is ||gram. So $\overrightarrow{AE} = \overrightarrow{DB}$ and $\overrightarrow{AE} \parallel \overrightarrow{DB}$

Now $\overrightarrow{AE} = [x,y] - [-2,-2] = [x+2, y+2]$

$\overrightarrow{DB} = [4,1] - [-8,0] = [4+8, 1-0] = [12,1]$

As $\overrightarrow{AE} = \overrightarrow{DB} \rightarrow [x+2, y+2] = [12,1]$

$\rightarrow x+2 = 12,\; y+2 = 1 \rightarrow x = 12-2,\; y = 1-2$

$\rightarrow x = 10,\; y = -1$ so $E(x,y) = (10,-1)$

Q9. If O is the origin and $\overrightarrow{OP} = \overrightarrow{AB}$, find the point P when A and B are $(-3,7)$ and $(1,0)$ respectively.

Solution:– $A(-3,7),\; B(1,0)$

Let $P(x,y)$ be the req. point and $O(0,0)$ is the origin.

$\because \overrightarrow{OP} = \overrightarrow{AB}$

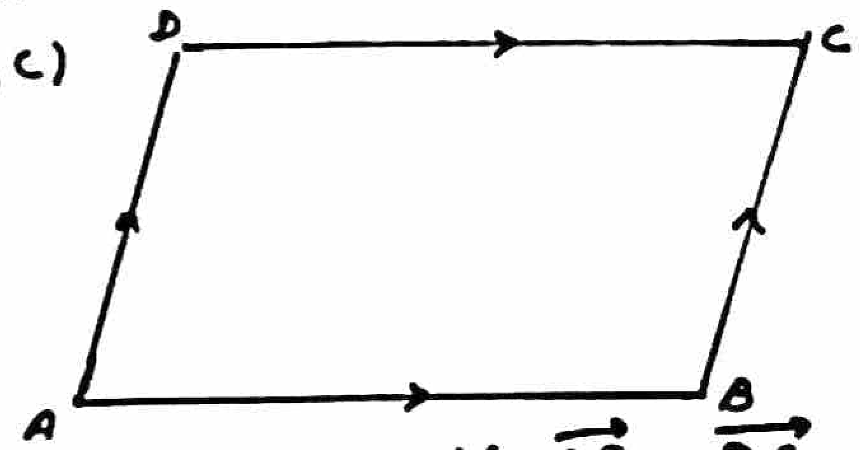
$[x-0, y-0] = [1+3, 0-7]$

$\rightarrow [x, y] = [4,-7] \rightarrow x = 4,\; y = -7$ so $P(x,y) = P(4,-7)$

Q10. Use vectors, to show that ABCD is a ||gram, when the points A, B, C and D are respectively $(0,0)$, $(a,0)$, (b,c) and $(b-a, c)$.

Solution:– Since $A(0,0), B(a,0), C(b,c)$

$D(b-a, c)$



ABCD will be ||gram if $\overrightarrow{AB} = \overrightarrow{DC}$

and $\overrightarrow{AD} = \overrightarrow{BC}$

Now $\overrightarrow{AB} = [a,0] - [0,0] = [a-0, 0-0]$

$= [a,0]$

$\overrightarrow{DC} = [b,c] - [b-a, c] = [b-b+a, c-c]$

$= [a,0]$

$\rightarrow \overrightarrow{AB} = \overrightarrow{DC}$

Also, $\overrightarrow{AD} = [b-a, c] - [0,0] = [b-a-0, c-0]$

$= [b-a, c]$

$\overrightarrow{BC} = [b,c] - [a,0] = [b-a, c-0]$

$= [b-a, c]$

$\rightarrow \overrightarrow{AD} = \overrightarrow{BC}$ Thus ABCD is a ||gram.

Q11. If $\overrightarrow{AB} = \overrightarrow{CD}$. Find the coordinates of the point A when points B, C, D are $(1,2), (-2,5), (4,11)$ respectively.

Solution:– $B(1,2), C(-2,5), D(4,11)$

Let $A(x,y)$ be required point. Now

$\overrightarrow{AB} = [1,2] - [x,y] = [1-x, 2-y]$

$\overrightarrow{CD} = [4,11] - [-2,5] = [4+2, 11-5] = [6,6]$

As $\overrightarrow{AB} = \overrightarrow{CD}$ (given)

$\rightarrow [1-x, 2-y] = [6,6] \rightarrow 1-x = 6,\; 2-y = 6$

$\rightarrow x = 1-6,\; y = 2-6 \rightarrow x = -5,\; y = -4$

So $A(x,y) = A(-5,-4)$.

Q12. Find the position vectors of the point of division of the line segments joining the following pair of points, in the given ratio:

(i) Point C with position vector $2\underline{i} - 3\underline{j}$ and point D with position vector $3\underline{i} + 2\underline{j}$ in the ratio 4:3

Solution:– Let $\vec{r}$ be req. vector then by ratio formula, $\vec{r} = \dfrac{p\underline{b} + q\underline{a}}{p+q}$ — (i)

Here $\underline{a} = 2\underline{i} - 3\underline{j}$, $\underline{b} = 3\underline{i} + 2\underline{j}$, $p:q = 4:3$

So by (i) $\rightarrow \vec{r} = \dfrac{4(3\underline{i}+2\underline{j}) + 3(2\underline{i}-3\underline{j})}{4+3}$

$\rightarrow \vec{r} = \dfrac{12\underline{i} + 8\underline{j} + 6\underline{i} - 9\underline{j}}{7} = \dfrac{18\underline{i} - \underline{j}}{7} = \dfrac{18}{7}\underline{i} - \dfrac{1}{7}\underline{j}$

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(ii) Point E with position vector $5\underline{j}$ and point F with position vector $4\underline{i}+\underline{j}$ in ratio 2 : 5.

Solution:- Let $\vec{r}$ be the req. vector then by ratio formula $\vec{r} = \dfrac{p\underline{b}+q\underline{a}}{p+q}$ — (i)

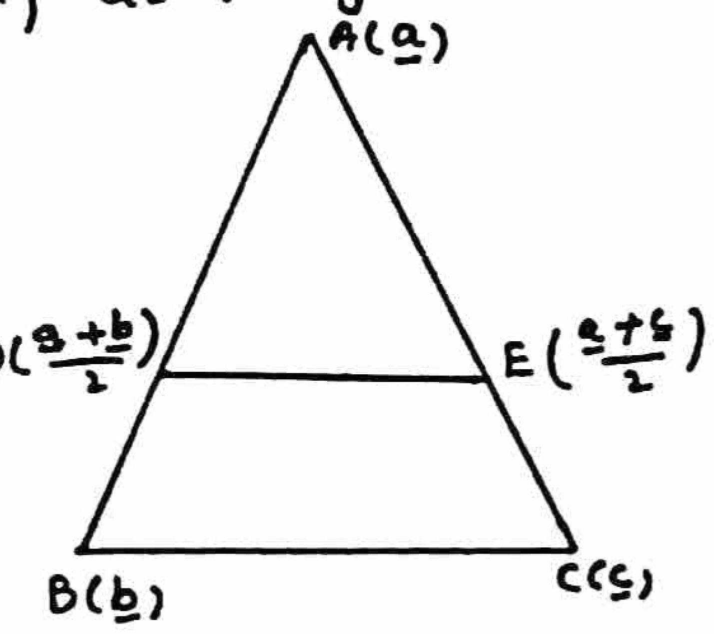
$\vec{a} = 5\underline{j}$, $\underline{b} = 4\underline{i}+\underline{j}$, $p:q = 2:5$

so by (i) → $\vec{r} = \dfrac{(2)(4\underline{i}+\underline{j}) + 5(5\underline{j})}{2+5} = \dfrac{8\underline{i}+2\underline{j}+25\underline{j}}{7}$

$\Rightarrow \vec{r} = \dfrac{8\underline{i}+27\underline{j}}{7} = \dfrac{8}{7}\underline{i} + \dfrac{27}{7}\underline{j}$

Q14. Prove that the line segment joining the mid points of two sides of a triangle is parallel to the third side and half as long.

Solution:-



Let $\underline{a}, \underline{b}, \underline{c}$ be the p.vectors of A, B, C respectively. then

p.v of D $= \dfrac{\underline{a}+\underline{b}}{2}$

p.v of E $= \dfrac{\underline{a}+\underline{c}}{2}$

$\because \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \underline{c} - \underline{b}$ ——— (i)

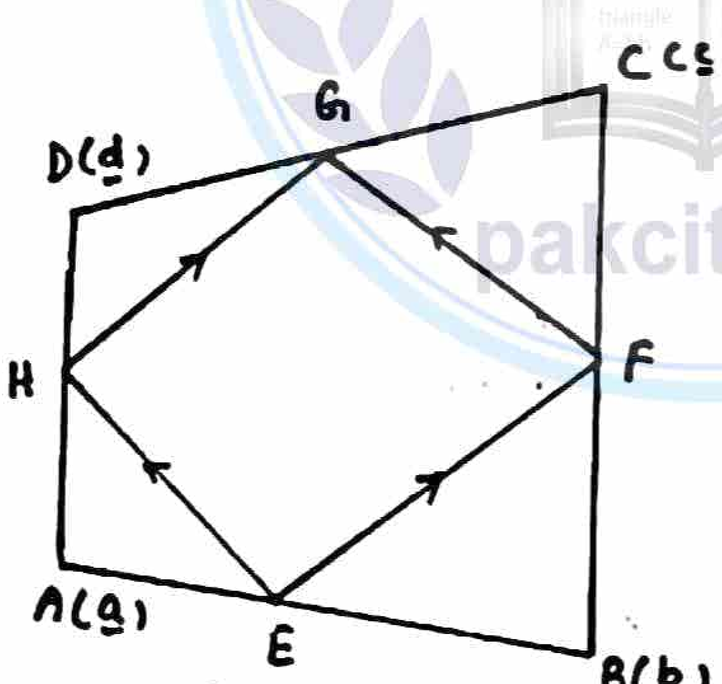
and $\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = \left(\dfrac{\underline{a}+\underline{c}}{2}\right) - \left(\dfrac{\underline{a}+\underline{b}}{2}\right)$

$= \dfrac{1}{2}(\underline{a}+\underline{c}-\underline{a}-\underline{b}) = \dfrac{1}{2}(\underline{c}-\underline{b})$

so $\overrightarrow{DE} = \dfrac{1}{2}\overrightarrow{BC}$ ——— (ii) Thus from (i) & (ii) $\overline{DE}$ & $\overline{BC}$ are || and DE is half to BC

Q15. Prove that the line segments joining the mid points of the sides of a quadrilateral taken in order form a parallelogram.

Solution:-



Suppose that $\underline{a}, \underline{b}, \underline{c}$ and $\underline{d}$ are position vectors of A, B, C and D respectively, then so

p.v of E $= \dfrac{\underline{a}+\underline{b}}{2}$, p.v of F $= \dfrac{\underline{b}+\underline{c}}{2}$

p.v of G $= \dfrac{\underline{c}+\underline{d}}{2}$, p.v of H $= \dfrac{\underline{a}+\underline{d}}{2}$

$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = \dfrac{\underline{b}+\underline{c}}{2} - \left(\dfrac{\underline{a}+\underline{b}}{2}\right) = \dfrac{\underline{b}+\underline{c}-\underline{a}-\underline{b}}{2} = \dfrac{\underline{c}-\underline{a}}{2}$

$\overrightarrow{FG} = \overrightarrow{OG} - \overrightarrow{OF} = \dfrac{\underline{c}+\underline{d}}{2} - \left(\dfrac{\underline{b}+\underline{c}}{2}\right) = \dfrac{\underline{c}+\underline{d}-\underline{b}-\underline{c}}{2} = \dfrac{\underline{d}-\underline{b}}{2}$

$\overrightarrow{HG} = \overrightarrow{OG} - \overrightarrow{OH} = \dfrac{\underline{c}+\underline{d}}{2} - \left(\dfrac{\underline{a}+\underline{d}}{2}\right) = \dfrac{\underline{c}+\underline{d}-\underline{a}-\underline{d}}{2} = \dfrac{\underline{c}-\underline{a}}{2}$

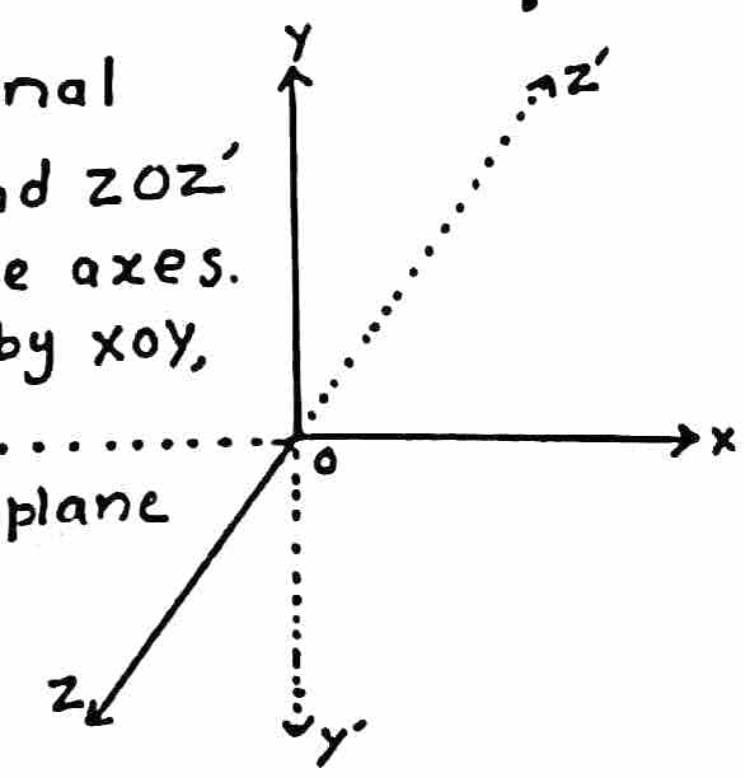
$\overrightarrow{EH} = \overrightarrow{OH} - \overrightarrow{OE} = \dfrac{\underline{a}+\underline{d}}{2} - \left(\dfrac{\underline{a}+\underline{b}}{2}\right) = \dfrac{\underline{a}+\underline{d}-\underline{a}-\underline{b}}{2} = \dfrac{\underline{d}-\underline{b}}{2}$

$\because \overrightarrow{EF} = \overrightarrow{HG} \Rightarrow \overrightarrow{EF}$ is || to $\overrightarrow{HG}$

and $\overrightarrow{FG} = \overrightarrow{EH} \Rightarrow \overrightarrow{FG}$ is || to $\overrightarrow{EH}$

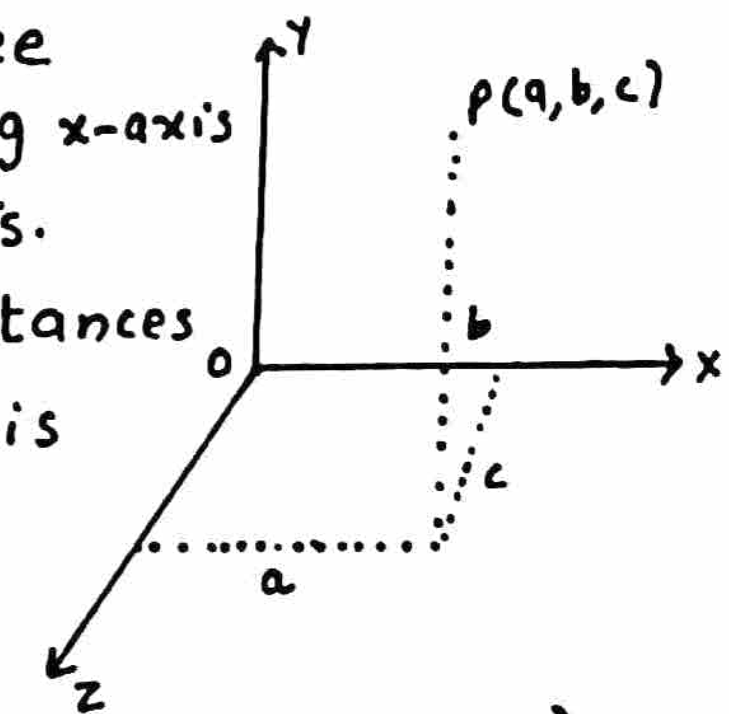
so EFGH is parallelogram.

Concept of vector in Space



In three dimensional space XOX', YOY' and ZOZ' are called coordinate axes. The planes made by XOY, YOZ and ZOX are called XY-plane, YZ-plane and ZX-plane resp. These planes are mutually orthogonal to each other.

Note:- If P is point in space then it will have three coordinates along x-axis, y-axis and z-axis.



If a, b, c are distances along x-axis, y-axis and z-axis resp. Then coordinates of point P are P(a, b, c) as shown in fig. Let $\vec{u}$ be vector in space whose position vector is $\overrightarrow{OP}$. then $\vec{u} = [x, y, z]$

Now **(i) Addition:-** Let $\underline{u} = [x_1, y_1, z_1]$, $\underline{v} = [x_2, y_2, z_2]$ then

$\underline{u}+\underline{v} = [x_1, y_1, z_1] + [x_2, y_2, z_2]$

$= [x_1+x_2 , y_1+y_2 , z_1+z_2]$

(ii) Difference:- Let $\underline{u} = [x_1, y_1, z_1]$, $\underline{v} = [x_2, y_2, z_2]$ then $\underline{u}-\underline{v} = [x_1, y_1, z_1] - [x_2, y_2, z_2]$

$\Rightarrow \underline{u}-\underline{v} = [x_1 - x_2, y_1 - y_2, z_1 - z_2]$

(iii) Scalar Multiplication:- If k be any scalar and $\underline{u} = [x, y, z]$ then

$k\underline{u} = k[x, y, z] = [kx, ky, kz]$

Magnitude of a vector (in Space):-

Let $\vec{u} = [x, y, z]$ be a vector in space then its length or norm or magnitude is denoted by $|\vec{u}|$ and defines as;

$$|\vec{u}| = \sqrt{x^2+y^2+z^2}$$

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Example 1:- For the vectors; $\underline{v} = [2,1,3]$ and $\underline{w} = [-1,4,0]$, we have the following.

(i) $\underline{v}+\underline{w} = [2,1,3]+[-1,4,0]$
$= [2-1, 1+4, 3+0] = [1,5,3]$

(ii) $\underline{v}-\underline{w} = [2,1,3]-[-1,4,0] = [2+1, 1-4, 3-0]$
$= [3,-3,3]$

(iii) $2\underline{w} = 2[-1,4,0] = [-2,8,0]$

(iv) $|\underline{v}-2\underline{w}| = |[2,1,3]-2[-1,4,0]| = |[2,1,3]-[-2,8,0]|$
$= |[2+2, 1-8, 3-0]| = |[4,-7,3]|$
$= \sqrt{(4)^2+(-7)^2+(3)^2} = \sqrt{16+49+9} = \sqrt{74}$

Properties of vectors:-

(i) Commutative property:-

For any two vectors $\underline{u}$ and $\underline{v}$
$\vec{u}+\vec{v} = \vec{v}+\vec{u}$

Proof:- Let $\vec{u} = [x_1,y_1,z_1]$, $\vec{v} = [x_2,y_2,z_2]$

$\Rightarrow \vec{u}+\vec{v} = [x_1,y_1,z_1]+[x_2,y_2,z_2]$
$= [x_1+x_2, y_1+y_2, z_1+z_2]$ (∵ In real numbers commutative law holds)
$= [x_2+x_1, y_2+y_1, z_2+z_1]$
$= [x_2,y_2,z_2]+[x_1,y_1,z_1]$
$= \vec{v}+\vec{w}$ Thus $\vec{u}+\vec{v} = \vec{v}+\vec{u}$

(ii) Associative property:-

For any three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$
$(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

Proof:- Let $\vec{u} = [x_1,y_1,z_1]$, $\vec{v} = [x_2,y_2,z_2]$
$\vec{w} = [x_3,y_3,z_3]$

$\Rightarrow (\vec{u}+\vec{v})+\vec{w} = ([x_1,y_1,z_1]+[x_2,y_2,z_2])+[x_3,y_3,z_3]$
$= [x_1+x_2, y_1+y_2, z_1+z_2]+[x_3,y_3,z_3]$
$= [(x_1+x_2)+x_3, (y_1+y_2)+y_3, (z_1+z_2)+z_3]$
$= [x_1+(x_2+x_3), y_1+(y_2+y_3), z_1+(z_2+z_3)]$ (∵ In real numbers associative law holds)
$= [x_1,y_1,z_1]+[x_2+x_3, y_2+y_3, z_2+z_3]$
$= [x_1,y_1,z_1]+([x_2,y_2,z_2]+[x_3,y_3,z_3])$
$= \vec{u}+(\vec{v}+\vec{w})$

(iii) Inverse property:-

For a vector $\vec{u}$, $\vec{u}+(-\vec{u}) = (-\vec{u})+\vec{u} = \vec{0}$

Proof:- Let $\vec{u} = [x,y,z]$ then
$-\vec{u} = [-x,-y,-z]$
then $\vec{u}+(-\vec{u}) = [x,y,z]+[-x,-y,-z]$
$= [x-x, y-y, z-z]$
$= [0,0,0] = \vec{0}$

similarly; $(-\vec{u})+\vec{u} = [-x,-y,-z]+[x,y,z]$
$= [-x+x, -y+y, -z+z] = [0,0,0] = \vec{0}$
Thus for $\vec{u}$ its additive inverse is $-\vec{u}$.

(iv) Distributive property:-

If k be any scalar and $\vec{u}, \vec{v}$ be two vectors then $k(\vec{u}+\vec{v}) = k\vec{u}+k\vec{v}$

Proof:- Let $\vec{u} = [x_1,y_1,z_1]$, $\vec{v} = [x_2,y_2,z_2]$

$k(\vec{u}+\vec{v}) = k([x_1,y_1,z_1]+[x_2,y_2,z_2])$
$= k[x_1+x_2, y_1+y_2, z_1+z_2]$
$= [k(x_1+x_2), k(y_1+y_2), k(z_1+z_2)]$
$= [kx_1+kx_2, ky_1+ky_2, kz_1+kz_2]$
$= [kx_1, ky_1, kz_1]+[kx_2, ky_2, kz_2]$
$= k[x_1,y_1,z_1]+k[x_2,y_2,z_2]$
$= k\vec{u}+k\vec{v}$

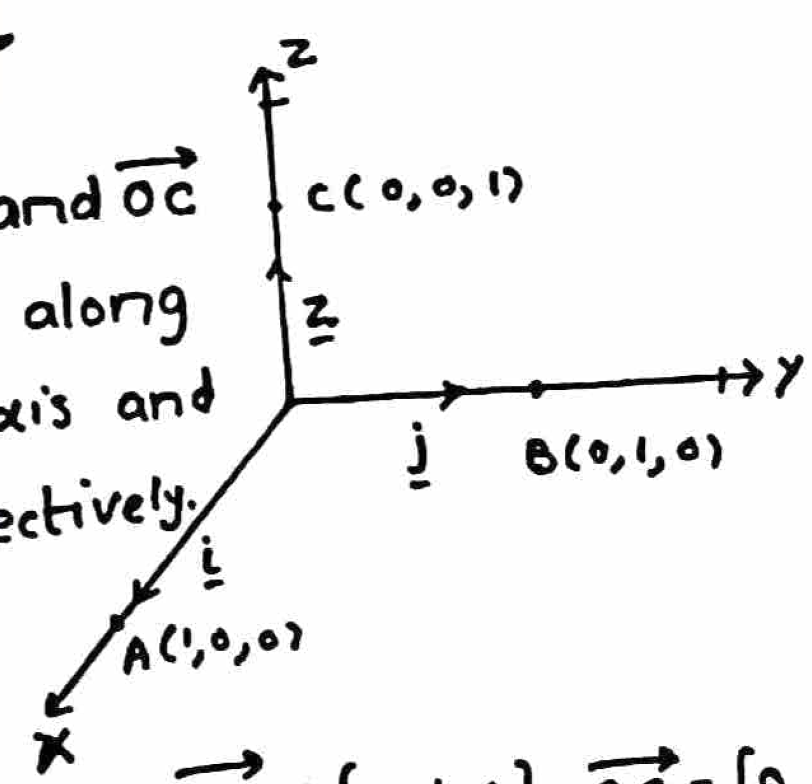
(v) Scalar Multiplication property

For a, b being scalars and $\vec{u}$ be a vector, then $a(b\vec{u}) = (ab)\vec{u}$

Proof:- Let $\vec{u} = [x,y,z]$
$\Rightarrow a(b\vec{u}) = a(b[x,y,z]) = a(bx,by,bz)$
$= [abx, aby, abz]$
$= ab[x,y,z] = (ab)\vec{u}$

Another notation for representing vectors in space:-

Let $\vec{OA}$, $\vec{OB}$ and $\vec{OC}$ be vectors along x-axis, y-axis and z-axis respectively.



As $\vec{OA} = [1,0,0]$, $\vec{OB} = [0,1,0]$, $\vec{OC} = [0,0,1]$
then $\underline{i} = [1,0,0]$, $\underline{j} = [0,1,0]$, $\underline{k} = [0,0,1]$
where $\underline{i}$, $\underline{j}$ and $\underline{k}$ are called unit vectors along x-axis, y-axis and z-axis. Now if $\vec{u} = [x,y,z]$ then $\vec{u}$ can be written as
$\vec{u} = [x,0,0]+[0,y,0]+[0,0,z]$
$= x[1,0,0]+y[0,1,0]+z[0,0,1]$
$\vec{u} = x\hat{i}+y\hat{j}+z\hat{k}$

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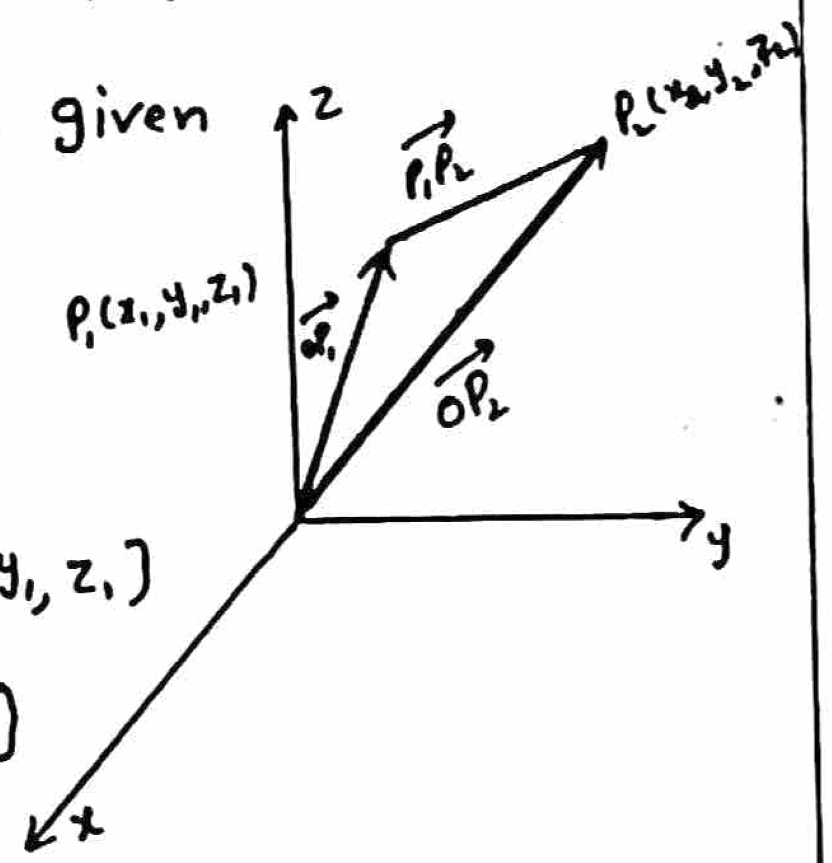
Distance between two points in space:-

If $\vec{OP_1}$ and $\vec{OP_2}$ are the position vectors of the points $P(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

The vector $\vec{P_1P_2}$ is given by



$\vec{P_1P_2} = \vec{OP_2} - \vec{OP_1}$

$= [x_2, y_2, z_2] - [x_1, y_1, z_1]$

$= [x_2 - x_1, y_2 - y_1, z_2 - z_1]$

Distance b/w P_1 and $P_2 = |\vec{P_1P_2}|$

$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

This is called distance formula b/w two points P_1 and P_2 in R.

Example 2. If $\underline{u} = 2\underline{i} + 3\underline{j} + \underline{k}$, $\underline{v} = 4\underline{i} + 6\underline{j} + 2\underline{k}$ and $\underline{w} = -6\underline{i} - 9\underline{j} - 3\underline{k}$, then (a) Find (i) $\underline{u} + 2\underline{v}$ (ii) $|\underline{u} - \underline{v} - \underline{w}|$
(b) Show that $\underline{u}, \underline{v}$ and $\underline{w}$ are parallel to each other.

Solution:- (a) (i). $\underline{u} + 2\underline{v} = 2\underline{i} + 3\underline{j} + \underline{k} + 2(4\underline{i} + 6\underline{j} + 2\underline{k})$

$= 2\underline{i} + 3\underline{j} + \underline{k} + 8\underline{i} + 12\underline{j} + 4\underline{k}$

$= 10\underline{i} + 15\underline{j} + 5\underline{k}$

(ii) $\underline{u} - \underline{v} - \underline{w} = (2\underline{i} + 3\underline{j} + \underline{k}) - (4\underline{i} + 6\underline{j} + 2\underline{k}) - (-6\underline{i} - 9\underline{j} - 3\underline{k})$

$= (2-4+6)\underline{i} + (3-6+9)\underline{j} + (1-2+3)\underline{k}$

$= 4\underline{i} + 6\underline{j} + 2\underline{k}$

$|\underline{u} - \underline{v} - \underline{w}| = \sqrt{(4)^2 + (6)^2 + (2)^2} = \sqrt{16+36+4}$

$= \sqrt{56}$

(b) $\underline{v} = 4\underline{i} + 6\underline{j} + 2\underline{k} = 2(2\underline{i} + 3\underline{j} + \underline{k})$

$\therefore \underline{v} = 2\underline{u}$

$\Rightarrow \underline{u}$ and $\underline{v}$ are ∥ vectors and have same direction.

Also, $\underline{w} = -6\underline{i} - 9\underline{j} - 3\underline{k} = -3(2\underline{i} + 3\underline{j} + \underline{k})$

$\underline{w} = -3\underline{u}$

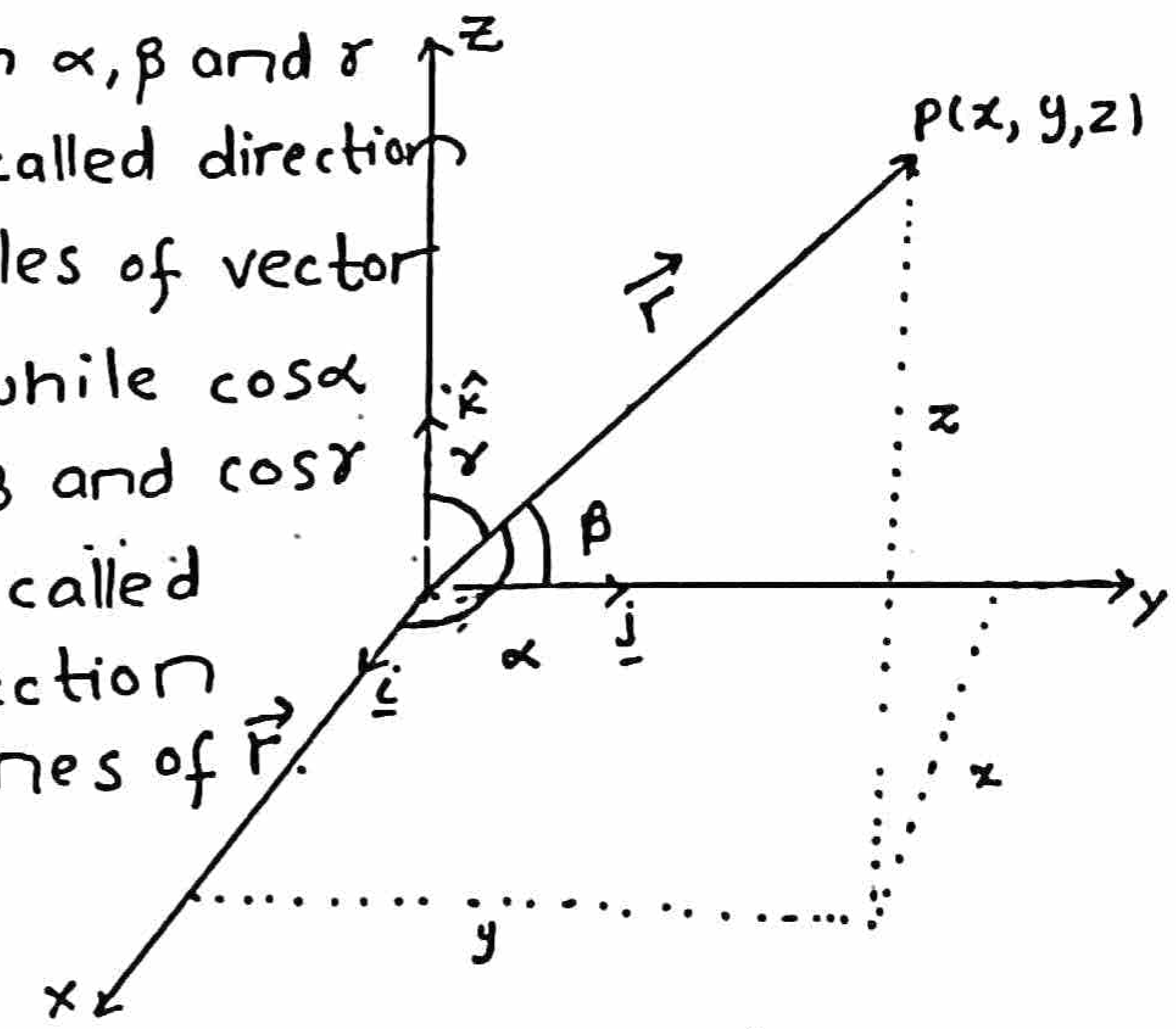
$\Rightarrow \underline{u}$ and $\underline{w}$ are ∥ vectors and have opposite direction.

Hence $\underline{u}, \underline{v}$ and $\underline{w}$ are ∥ to each other.

Direction angles and direction cosines of a vector:-

Let $\underline{r} = \vec{OP} = [x, y, z] = x\underline{i} + y\underline{j} + z\underline{k}$ be a vector s.that it makes angles α, β and γ along coordinate axes

then α, β and γ are called direction angles of vector $\vec{r}$. while $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called direction cosines of $\vec{r}$.



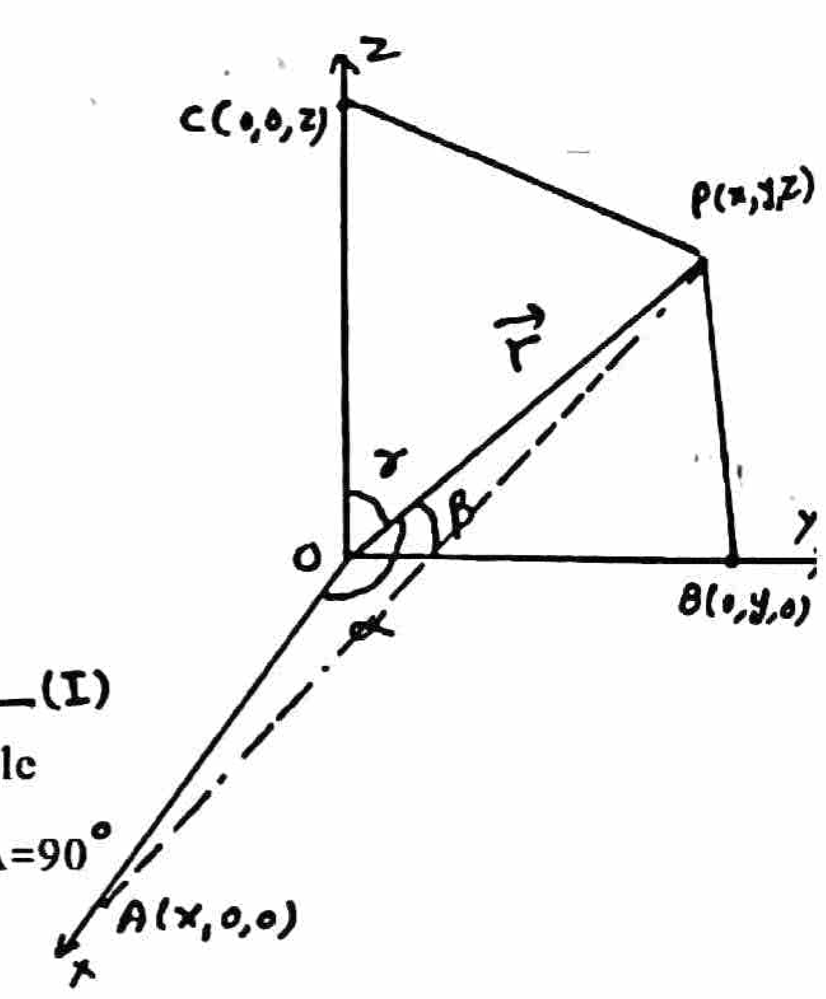
Important result:-

Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Proof:-

Let $\vec{r} = [x, y, z]$

$= x\underline{i} + y\underline{j} + z\underline{k}$



$\Rightarrow |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow r = \sqrt{x^2 + y^2 + z^2}$

$r^2 = x^2 + y^2 + z^2$ ——(I)

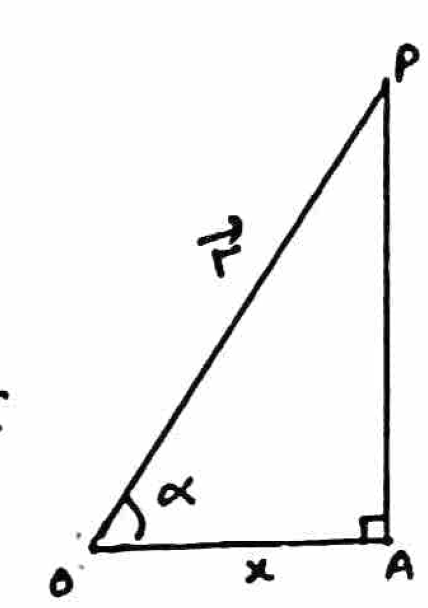
It can be visualized that the triangle OAP is right trianlge with $\angle A = 90°$

So In right △OAP,

$\cos\alpha = \frac{OA}{OP} = \frac{x}{r}$

similarly,

$\cos\beta = \frac{y}{r}$, $\cos\gamma = \frac{z}{r}$



So $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}$

$= \frac{x^2 + y^2 + z^2}{r^2} = \frac{r^2}{r^2}$

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\beta = 1$

Hence proved.

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Exercise 7.2

Q1. Let $A=(2,5)$, $B=(-1,1)$ and $C(2,-6)$. Find (i) $\overrightarrow{AB}$ (ii) $2\overrightarrow{AB}-\overrightarrow{CB}$ (iii) $2\overrightarrow{CB}-2\overrightarrow{CA}$

Solution:- (i) $\overrightarrow{AB}=(-1-2,1-5)=(-3,-4)$

or $\overrightarrow{AB}=-3\underline{i}-4\underline{j}$

(ii) $2\overrightarrow{AB}-\overrightarrow{CB}=2(-1-2,1-5)-(-1-2,1+6)$

$=2(-3,-4)-(-3,7)$

$=(-6,-8)-(-3,7)$

$=(-6+3,-8-7)=(-3,-15)$

or $2\overrightarrow{AB}-\overrightarrow{CB}=-3\underline{i}-15\underline{j}$

(iii) $2\overrightarrow{CB}-2\overrightarrow{CA}=2(-1-2,1+6)-2(2-2,5+6)$

$=2(-3,7)-2(0,11)$

$=(-6,14)-(0,22)$

$=(-6-0,14-22)=(-6,-8)$

$2\overrightarrow{CB}-2\overrightarrow{CA}=-6\underline{i}-8\underline{j}$

Q2. Let $\underline{u}=\underline{i}+2\underline{j}-\underline{k}$, $\underline{v}=3\underline{i}-2\underline{j}+2\underline{k}$, $\underline{w}=5\underline{i}-\underline{j}+3\underline{k}$. Find the indicated vector or number. (i) $\underline{u}+2\underline{v}+\underline{w}$ (ii) $\underline{v}-3\underline{w}$ (iii) $|3\underline{v}+\underline{w}|$

Solution:-

(i) $u+2\underline{v}+\underline{w}$

$=\underline{i}+2\underline{j}-\underline{k}+2(3\underline{i}-2\underline{j}+2\underline{k})+5\underline{i}-\underline{j}+3\underline{k}$

$=\underline{i}+2\underline{j}-\underline{k}+6\underline{i}-4\underline{j}+4\underline{k}+5\underline{i}-\underline{j}+3\underline{k}$

$=12\underline{i}-3\underline{j}+6\underline{k}$

(ii) $\underline{v}-3\underline{w}=3\underline{i}-2\underline{j}+2\underline{k}-3(5\underline{i}-\underline{j}+3\underline{k})$

$=3\underline{i}-2\underline{j}+2\underline{k}-15\underline{i}+3\underline{j}-9\underline{k}$

$=-12\underline{i}+\underline{j}-7\underline{k}$

(iii) $|3\underline{v}+\underline{w}|=|3(3\underline{i}-2\underline{j}+2\underline{k})+5\underline{i}-\underline{j}+3\underline{k}|$

$=|9\underline{i}-6\underline{j}+6\underline{k}+5\underline{i}-\underline{j}+3\underline{k}|$

$=|14\underline{i}-7\underline{j}+9\underline{k}|=\sqrt{(14)^2+(-7)^2+(9)^2}$

$=\sqrt{326}$

Q3. Find the magnitude of the vector $\underline{v}$ and write the direction cosines of $\underline{v}$. (i) $\underline{v}=2\underline{i}+3\underline{j}+4\underline{k}$ (ii) $\underline{v}=\underline{i}-\underline{j}-\underline{k}$ (iii) $\underline{v}=4\underline{i}-5\underline{j}$

Solution:- (i) $v=2\underline{i}+3\underline{j}+4\underline{k}$

$|\underline{v}|=\sqrt{(2)^2+(3)^2+(4)^2}=\sqrt{29}$

Direction cosines are

$\cos\alpha=\frac{2}{\sqrt{29}}$, $\cos\beta=\frac{3}{\sqrt{29}}$, $\cos\gamma=\frac{4}{\sqrt{29}}$

(ii) $\underline{v}=\underline{i}-\underline{j}-\underline{k}$

$|\underline{v}|=\sqrt{(1)^2+(-1)^2+(-1)^2}=\sqrt{3}$

Direction cosines are $\cos\alpha=\frac{1}{\sqrt{3}}$

$\cos\beta=\frac{-1}{\sqrt{3}}$, $\cos\gamma=\frac{-1}{\sqrt{3}}$

(iii) $\underline{v}=4\underline{i}-5\underline{j}$

$|\underline{v}|=\sqrt{(4)^2+(-5)^2}=\sqrt{16+25}=\sqrt{41}$

Direction cosines are $\cos\alpha=\frac{4}{\sqrt{41}}$, $\cos\beta=\frac{-5}{\sqrt{41}}$

Q4. Find α, so that $|\alpha\underline{i}+(\alpha+1)\underline{j}+2\underline{k}|=3$

Solution:- $|\alpha\underline{i}+(\alpha+1)\underline{j}+2\underline{k}|=3$

$\rightarrow \sqrt{\alpha^2+(\alpha+1)^2+(2)^2}=3$

squaring on both sides

$\alpha^2+(\alpha+1)^2+4=9$

$\rightarrow \alpha^2+\alpha^2+1+2\alpha+4-9=0 \rightarrow 2\alpha^2+2\alpha-4=0$

$\rightarrow \alpha^2+\alpha-2=0$ (÷ by 2)

$\alpha^2+2\alpha-\alpha-2=0$

$\rightarrow \alpha(\alpha+2)-1(\alpha+2)=0$

$\rightarrow (\alpha+2)(\alpha-1)=0$, $\alpha+2=0$, $\alpha-1=0$

$\rightarrow \alpha=-2$ or $\alpha=1$

Q5. Find a unit vector in the direction of $\underline{v}=\underline{i}+2\underline{j}-\underline{k}$

Solution:- $\underline{v}=\underline{i}+2\underline{j}-\underline{k}$

$|\underline{v}|=\sqrt{(1)^2+(2)^2+(-1)^2}=\sqrt{6}$

Now $\hat{v}=\frac{\underline{v}}{|\underline{v}|}=\frac{\underline{i}+2\underline{j}-\underline{k}}{\sqrt{6}}=\frac{1}{\sqrt{6}}\underline{i}+\frac{2}{\sqrt{6}}\underline{j}-\frac{1}{\sqrt{6}}\underline{k}$

which is req. unit vector in the direction of $\vec{v}$.

Q6. If $\underline{a}=3\underline{i}-\underline{j}-4\underline{k}$, $\underline{b}=-2\underline{i}-4\underline{j}-3\underline{k}$ and $\underline{c}=\underline{i}+2\underline{j}-\underline{k}$. Find a unit vector parallel to $3\underline{a}-2\underline{b}+4\underline{c}$

Solution:-

$3\underline{a}-2\underline{b}+4\underline{c}=3(3\underline{i}-\underline{j}-4\underline{k})-2(-2\underline{i}-4\underline{j}-3\underline{k})+4(\underline{i}+2\underline{j}-\underline{k})$

$=9\underline{i}-3\underline{j}-12\underline{k}+4\underline{i}+8\underline{j}+6\underline{k}+4\underline{i}+8\underline{j}-4\underline{k}$

$=17\underline{i}+13\underline{j}-10\underline{k}$

$|3\underline{a}-2\underline{b}+4\underline{c}|=\sqrt{(17)^2+(13)^2+(-10)^2}=\sqrt{289+169+100}$

$=\sqrt{558}$

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Unit vector $= \frac{3\underline{a}-2\underline{b}+4\underline{c}}{|3\underline{a}-2\underline{b}+4\underline{c}|} = \frac{17\underline{i}+13\underline{j}-10\underline{k}}{\sqrt{558}}$

$= \frac{17\underline{i}}{\sqrt{558}} + \frac{13}{\sqrt{558}}\underline{j} - \frac{10}{\sqrt{558}}\underline{k}$

Q7. Find a vector whose (i) magnitude is 4 and is parallel to $2\underline{i}-3\underline{j}+6\underline{k}$ (ii) magnitude is 2 and is parallel to $-\underline{i}+\underline{j}+\underline{k}$

Solution:- (i) Suppose $\vec{a} = 2\underline{i}-3\underline{j}+6\underline{k}$

$\because |\vec{a}| = \sqrt{(2)^2+(-3)^2+(6)^2} = \sqrt{4+9+36} = \sqrt{49} = 7$

so $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\underline{i}-3\underline{j}+6\underline{k}}{7} = \frac{2}{7}\underline{i}-\frac{3}{7}\underline{j}+\frac{6}{7}\underline{k}$

Let $\vec{b}$ be a vector having magnitude 4. i-e., $|\vec{b}| = 4$

$\because \vec{b}$ is ∥ to $\vec{a}$. so

$\hat{b} = \hat{a} = \frac{2}{7}\underline{i}-\frac{3}{7}\underline{j}+\frac{6}{7}\underline{k}$

$\Rightarrow \vec{b} = |\vec{b}|\hat{b} = 4(\frac{2}{7}\underline{i}-\frac{3}{7}\underline{j}+\frac{6}{7}\underline{k})$

$\Rightarrow \vec{b} = \frac{8}{7}\underline{i}-\frac{12}{7}\underline{j}+\frac{24}{7}\underline{k}$

(ii) Suppose $\vec{a} = -\underline{i}+\underline{j}+\underline{k}$ and

$|\vec{a}| = \sqrt{(-1)^2+(1)^2+(1)^2} = \sqrt{3}$ so

$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{-1}{\sqrt{3}}\underline{i}+\frac{1}{\sqrt{3}}\underline{j}+\frac{1}{\sqrt{3}}\underline{k}$

Let $\vec{b}$ be a vector having magnitude 2 i-e, $|\vec{b}| = 2$

$\because \vec{b}$ is ∥ to $\vec{a}$. So

$\hat{b} = \hat{a} = \frac{-1}{\sqrt{3}}\underline{i}+\frac{1}{\sqrt{3}}\underline{j}+\frac{1}{\sqrt{3}}\underline{k}$

$\Rightarrow \vec{b} = |\vec{b}|\hat{b} = 2(\frac{-1}{\sqrt{3}}+\frac{1}{\sqrt{3}}\underline{j}+\frac{1}{\sqrt{3}}\underline{k})$

$\Rightarrow \vec{b} = -\frac{2}{\sqrt{3}}\underline{i}+\frac{2}{\sqrt{3}}\underline{j}+\frac{2}{\sqrt{3}}\underline{k}$

Q8. If $\underline{u} = 2\underline{i}+3\underline{j}+4\underline{k}$, $\underline{v} = -\underline{i}+3\underline{j}-\underline{k}$ and $\underline{w} = \underline{i}+6\underline{j}+z\underline{k}$ represent the sides of a triangle. Find the value of z.

Solution:- Given that

$\underline{u} = 2\underline{i}+3\underline{j}+4\underline{k}$, $\underline{v} = -\underline{i}+3\underline{j}-\underline{k}$ and

$\underline{w} = \underline{i}+6\underline{j}+z\underline{k}$ Since $\underline{u}, \underline{v}$ and $\underline{w}$ are sides of triangle therefore

$\underline{u}+\underline{v} = \underline{w}$

$\Rightarrow 2\underline{i}+3\underline{j}+4\underline{k}-\underline{i}+3\underline{j}-\underline{k} = \underline{i}+6\underline{j}+z\underline{k}$

$\Rightarrow \underline{i}+6\underline{j}+3\underline{k} = \underline{i}+6\underline{j}+z\underline{k}$

Equating coefficient of $\underline{k}$ only, we have

$3 = z \Rightarrow z = 3$

Q9. The position vectors of the points A, B, C and D are $2\underline{i}-\underline{j}+\underline{k}$, $3\underline{i}+\underline{j}$, $2\underline{i}+4\underline{j}-2\underline{k}$ and $-\underline{i}-2\underline{j}+\underline{k}$. Show that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

Solution:- P.V of A is $\overrightarrow{OA} = 2\underline{i}-\underline{j}+\underline{k}$

P.V of B is $\overrightarrow{OB} = 3\underline{i}+\underline{j}$, P.V of C is $\overrightarrow{OC} = 2\underline{i}+4\underline{j}-2\underline{k}$

P.V of D is $\overrightarrow{OD} = -\underline{i}-2\underline{j}+\underline{k}$ Now,

$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = 3\underline{i}+\underline{j}-(2\underline{i}-\underline{j}+\underline{k})$

$= 3\underline{i}+\underline{j}-2\underline{i}+\underline{j}-\underline{k} = \underline{i}+2\underline{j}-\underline{k}$

$\overrightarrow{CD} = \overrightarrow{OD}-\overrightarrow{OC} = -\underline{i}-2\underline{j}+\underline{k}-(2\underline{i}+4\underline{j}-2\underline{k})$

$= -\underline{i}-2\underline{j}+\underline{k}-2\underline{i}-4\underline{j}+2\underline{k}$

$\overrightarrow{CD} = -3\underline{i}-6\underline{j}+3\underline{k} = -3(\underline{i}+2\underline{j}-\underline{k}) = -3\overrightarrow{AB}$

$\Rightarrow \overrightarrow{CD} = -3\overrightarrow{AB}$ Thus $\overrightarrow{AB}$ is ∥ to $\overrightarrow{CD}$.

Q10. We say that two vectors $\underline{v}$ and $\underline{w}$ in space are parallel if there is a scalar c such that $\underline{v} = c\underline{w}$. The vectors point in the same direction if $c>0$, and the vectors point in the opposite direction if $c<0$.

(a) Find two vectors of length 2 parallel to the vector $\underline{v} = 2\underline{i}-4\underline{j}+4\underline{k}$.

Solution:- $\underline{v} = 2\underline{i}-4\underline{j}+4\underline{k}$

$|\underline{v}| = \sqrt{(2)^2+(-4)^2+(4)^2} = \sqrt{4+16+16} = \sqrt{36} = 6$.

Now $\hat{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{2\underline{i}-4\underline{j}+4\underline{k}}{6} = \frac{1}{3}\underline{i}-\frac{2}{3}\underline{j}+\frac{2}{3}\underline{k}$

The two vectors of length 2 and ∥ to $\underline{v}$ are $2\hat{v}$ and $-2\hat{v}$. so

$2\hat{v} = 2(\frac{1}{3}\underline{i}-\frac{2}{3}\underline{j}+\frac{2}{3}\underline{k}) = \frac{2}{3}\underline{i}-\frac{4}{3}\underline{j}+\frac{4}{3}\underline{k}$

$-2\hat{v} = -2(\frac{1}{3}-\frac{2}{3}\underline{j}+\frac{2}{3}\underline{k}) = -\frac{2}{3}\underline{i}+\frac{4}{3}\underline{j}-\frac{4}{3}\underline{k}$

(b) Find the constant a so that the vectors $\underline{v} = \underline{i}-3\underline{j}+4\underline{k}$ and $\underline{w} = a\underline{i}+9\underline{j}-12\underline{k}$ are parallel.

Solution:- $\because \underline{v}$ and $\underline{w}$ are parallel so

$\underline{v} = c\underline{w} \Rightarrow \underline{i}-3\underline{j}+4\underline{k} = c(a\underline{i}+9\underline{j}-12\underline{k})$

$\Rightarrow \underline{i}-3\underline{j}+4\underline{k} = ac\underline{i}+9c\underline{j}-12c\underline{k}$

$\Rightarrow ac = 1$ ——(i), $-3 = 9c \Rightarrow c = -\frac{1}{3}$

So $a(-\frac{1}{3}) = 1 \Rightarrow a = -3$

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(c) Find a vector of length 5 in the direction opposite that of $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$

Solution:- $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$

$$|\underline{v}| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{1+4+9} = \sqrt{14}$$

$$\hat{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{\underline{i} - 2\underline{j} + 3\underline{k}}{\sqrt{14}} = \frac{1}{\sqrt{14}}\underline{i} - \frac{2}{\sqrt{14}}\underline{j} + \frac{3}{\sqrt{14}}\underline{k}$$

so required vector in opposit direction is $-5\hat{v}$.

$$= -5\left(\frac{1}{\sqrt{14}}\underline{i} - \frac{2}{\sqrt{14}}\underline{j} + \frac{3}{\sqrt{14}}\underline{k}\right) = \frac{-5}{\sqrt{14}}\underline{i} + \frac{10}{\sqrt{14}}\underline{j} - \frac{15}{\sqrt{14}}\underline{k}$$

(d) Find a and b so that the vectors $3\underline{i} - \underline{j} + 4\underline{k}$ and $a\underline{i} + b\underline{j} - 2\underline{k}$ are parallel.

Solution:- Let $\underline{u} = 3\underline{i} - \underline{j} + 4\underline{k}$ and $\underline{v} = a\underline{i} + b\underline{j} - 2\underline{k}$ $\because$ $\underline{u}$ and $\underline{v}$ are $\parallel$, so $\underline{u} = c\underline{v} \rightarrow 3\underline{i} - \underline{j} + 4\underline{k} = c(a\underline{i} + b\underline{j} - 2\underline{k})$

$\rightarrow 3\underline{i} - \underline{j} + 4\underline{k} = ac\underline{i} + bc\underline{j} - 2c\underline{k}$

$ac = 3$ —— (i) , $bc = -1$ —— (ii)

$-2c = 4 \rightarrow c = -2$ put in (i) and (ii)

so $a = \frac{-3}{2}$ and $b = \frac{1}{2}$

Q11. Find the direction cosines for the given vector:

(i) $\underline{v} = 3\underline{i} - \underline{j} + 2\underline{k}$ (ii) $6\underline{i} - 2\underline{j} + \underline{k}$

(iii) $\overrightarrow{PQ}$, where $P = (2,1,5)$ and $Q = (1,3,1)$

Solution:- (i) $\underline{v} = 3\underline{i} - \underline{j} + 2\underline{k}$

$\rightarrow |\underline{v}| = \sqrt{(3)^2 + (-1)^2 + (2)^2} = \sqrt{9+1+4} = \sqrt{14}$

Direction cosines are; $\cos\alpha = \frac{3}{\sqrt{14}}$, $\cos\beta = \frac{-1}{\sqrt{14}}$ and $\cos\gamma = \frac{2}{\sqrt{14}}$

(ii) Let $\underline{v} = 6\underline{i} - 2\underline{j} + \underline{k}$, $|\underline{v}| = \sqrt{(6)^2 + (-2)^2 + (1)^2}$

$|\underline{v}| = \sqrt{36+4+1} = \sqrt{41}$

Direction cosines are; $\cos\alpha = \frac{6}{\sqrt{41}}$, $\cos\beta = \frac{-2}{\sqrt{41}}$ and $\cos\gamma = \frac{1}{\sqrt{41}}$

(iii) $\overrightarrow{PQ} = (1-2, 3-1, 1-5) = (-1, 2, -4)$

$\overrightarrow{PQ} = -\underline{i} + 2\underline{j} - 4\underline{k}$

$|\overrightarrow{PQ}| = \sqrt{(-1)^2 + (2)^2 + (-4)^2} = \sqrt{1+4+16}$

$= \sqrt{21}$

Direction cosines are; $\cos\alpha = \frac{-1}{\sqrt{21}}$, $\cos\beta = \frac{2}{\sqrt{21}}$ and $\frac{-4}{\sqrt{21}}$

Q12. Which of the following triples can be the direction angles of a single vector:

(i) $45°, 45°, 60°$ (ii) $30°, 45°, 60°$ (iii) $45°, 60°, 60°$

Solution:- (i) Here $\alpha = 45°$, $\beta = 45°$, $\gamma = 60°$ then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$= \cos^2 45° + \cos^2 45° + \cos^2 60°$

$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{5}{4} \neq 1$

so given triples are not direction angles.

(ii) Here $\alpha = 30°$, $\beta = 45°$, $\gamma = 60°$ then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$= \cos^2 30° + \cos^2 45° + \cos^2 60°$

$= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{2} + \frac{1}{4}$

$= \frac{3+2+1}{4} = \frac{6}{4} = \frac{3}{2} \neq 1$

so given triples are not direction angles.

(iii) Here $\alpha = 45°$, $\beta = 60°$, $\gamma = 60°$ then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$= \cos^2 45° + \cos^2 60° + \cos^2 60°$

$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$

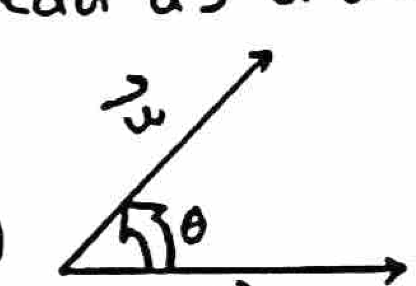
$= \frac{2+1+1}{4} = \frac{4}{4} = 1$

so given triples are direction angles.

The Scalar product of two vectors:- (Dot product)

Definition 1: The scalar product of two non-zero vectors $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u}\cdot\vec{v}$ (read as $\vec{u}$ dot $\vec{v}$) and defined as;

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$



where θ is angle from $\vec{u}$ to $\vec{v}$ and $0 \leq \theta \leq \pi$.

Definition 2:

(a) (In plane) If $\vec{u} = a_1\underline{i} + b_1\underline{j}$ and $\vec{v} = a_2\underline{i} + b_2\underline{j}$ are two non-zero vectors in plane then their scalar (dot) product is $\vec{u}\cdot\vec{v} = a_1a_2 + b_1b_2$

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(b) (In space) If $\vec{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$ and $\vec{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$ are two non-zero vectors in space then their scalar product is $\vec{u}\cdot\vec{v} = a_1a_2 + b_1b_2 + c_1c_2$

Note:- The dot product is also referred to the scalar product or the inner product.

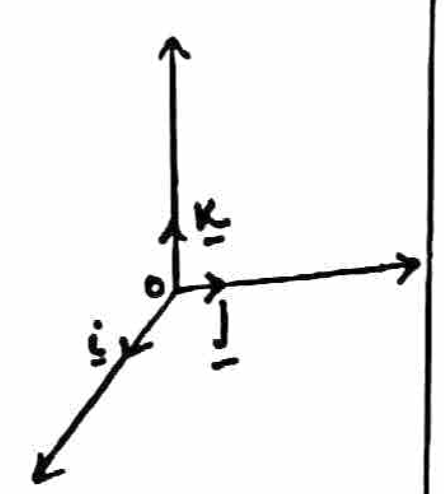
Deductions of the important results:-

For unit vectors $\underline{i}, \underline{j}, \underline{k}$ we have,

(a) $\underline{i}\cdot\underline{i} = |\underline{i}||\underline{i}|\cos 0^\circ = 1$
$\underline{j}\cdot\underline{j} = |\underline{j}||\underline{j}|\cos 0^\circ = 1$
$\underline{k}\cdot\underline{k} = |\underline{k}||\underline{k}|\cos 0^\circ = 1$

(b) $\underline{i}\cdot\underline{j} = |\underline{i}||\underline{j}|\cos 90^\circ = 0$
$\underline{j}\cdot\underline{k} = |\underline{j}||\underline{k}|\cos 90^\circ = 0$
$\underline{k}\cdot\underline{i} = |\underline{k}||\underline{i}|\cos 90^\circ = 0$

(c) prove that $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$



$\because \vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$
$= |\vec{v}||\vec{u}|\cos(-\theta)$
$= |\vec{v}||\vec{u}|\cos\theta$
$= \vec{v}\cdot\vec{u}$

$\Rightarrow \vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

Hence dot product of two vectors is commutative.

Perpendicular (orthogonal) vectors:-

Two non-zero vectors $\vec{u}$ and $\vec{v}$ are perpendicular if and only if $\vec{u}\cdot\vec{v} = 0$

($\because$ angle b/w $\vec{u}$ and $\vec{v}$ is $\frac{\pi}{2}$ so $\cos\frac{\pi}{2} = 0$)

Thus $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\frac{\pi}{2} \Rightarrow \vec{u}\cdot\vec{v} = 0$

Properties of Dot product:-

Let $\underline{u}, \underline{v}$ and $\underline{w}$ be vectors and let $c \in R$, then

(i) $\underline{u}\cdot\underline{v} = 0 \Rightarrow \underline{u} = 0$ or $\underline{v} = 0$

(ii) $\underline{u}\cdot\underline{v} = \underline{v}\cdot\underline{u}$ (Commutative property)

(iii) $\underline{u}\cdot(\underline{v}+\underline{w}) = \underline{u}\cdot\underline{v} + \underline{u}\cdot\underline{w}$ (dist. property)

(iv) $(c\underline{u})\cdot\underline{v} = c(\underline{u}\cdot\underline{v})$ (c is scalar)

(v) $\underline{u}\cdot\underline{u} = |\underline{u}|^2$

Analytical Expression of Dot Product $\underline{u}\cdot\underline{v}$:-

(Dot product of vectors in their components from)

Let $\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$ and
$\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$ be two non-zero vectors. Now

$\underline{u}\cdot\underline{v} = (a_1\underline{i} + b_1\underline{j} + c_1\underline{k})\cdot(a_2\underline{i} + b_2\underline{j} + c_2\underline{k})$

$= (a_1a_2)(\underline{i}\cdot\underline{i}) + a_1b_2(\underline{i}\cdot\underline{j}) + a_1c_2(\underline{i}\cdot\underline{k})$
$+ b_1a_2(\underline{j}\cdot\underline{i}) + b_1b_2(\underline{j}\cdot\underline{j}) + b_1c_2(\underline{j}\cdot\underline{k})$
$+ c_1a_2(\underline{k}\cdot\underline{i}) + c_1b_2(\underline{k}\cdot\underline{j}) + c_1c_2(\underline{k}\cdot\underline{k})$

$\Rightarrow \underline{u}\cdot\underline{v} = a_1a_2 + b_1b_2 + c_1c_2$

($\because \underline{i}\cdot\underline{i} = \underline{j}\cdot\underline{j} = \underline{k}\cdot\underline{k} = 1$ and $\underline{i}\cdot\underline{j} = \underline{j}\cdot\underline{k} = \underline{k}\cdot\underline{i} = 0$)

Example 1. (i) If $\underline{v} = [x_1, y_1]$ and $\underline{w} = [x_2, y_2]$ are two vectors in plane, then
$\underline{v}\cdot\underline{w} = x_1x_2 + y_1y_2$

Proof:- As $\underline{v} = [x_1, y_1] = x_1\underline{i} + y_1\underline{j}$
$\underline{w} = [x_2, y_2] = x_2\underline{i} + y_2\underline{j}$

so $\underline{v}\cdot\underline{w} = (x_1\underline{i} + y_1\underline{j})\cdot(x_2\underline{i} + y_2\underline{j})$
$= x_1x_2(\underline{i}\cdot\underline{i}) + x_1y_2(\underline{i}\cdot\underline{j}) + y_1x_2(\underline{j}\cdot\underline{i}) + y_1y_2(\underline{j}\cdot\underline{j})$

$\underline{v}\cdot\underline{w} = x_1x_2 + y_1y_2$ ($\because \underline{i}\cdot\underline{i} = \underline{j}\cdot\underline{j} = 1$, $\underline{i}\cdot\underline{j} = \underline{j}\cdot\underline{i} = 0$)

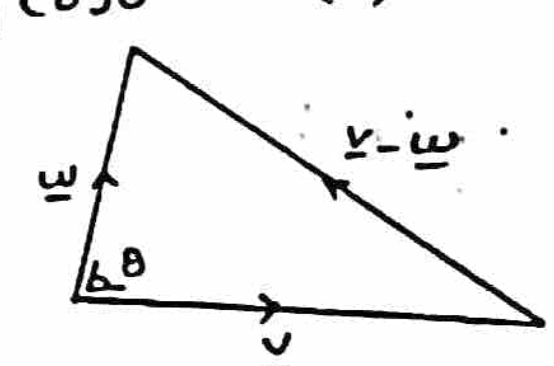
Hence proved.

(ii) If $\underline{v}$ and $\underline{w}$ are two non-zero vectors in the plane, then $\underline{v}\cdot\underline{w} = |\underline{v}||\underline{w}|\cos\theta$

Proof:- Let $\underline{v}$ and $\underline{w}$ be the sides of triangle then the third side, opposite to the angle θ, has length $|\underline{v}-\underline{w}|$

By Law of cosine,

$|\underline{v}-\underline{w}|^2 = |\underline{v}|^2 + |\underline{w}|^2 - 2|\underline{v}||\underline{w}|\cos\theta$ ——— (I)



$\because \underline{v} = [x_1, y_1]$, $\underline{w} = [x_2, y_2]$
then $\underline{v}-\underline{w} = [x_1 - x_2, y_1 - y_2]$

So eq (I) becomes as

$\Rightarrow |x_1 - x_2|^2 + |y_1 - y_2|^2 = |x_1 + y_1|^2 + |x_2 + y_2|^2 - 2|\underline{v}||\underline{w}|\cos\theta$

$\Rightarrow (x_1^2 + x_2^2 - 2x_1x_2) + (y_1^2 + y_2^2 - 2y_1y_2) = x_1^2 + y_1^2 + 2x_1y_1 + x_2^2 + y_2^2 + 2x_2y_2 - 2|\underline{v}||\underline{w}|\cos\theta$

$\Rightarrow -2x_1x_2 - 2y_1y_2 = -2|\underline{v}||\underline{w}|\cos\theta$

$\Rightarrow x_1x_2 + y_1y_2 = |\underline{v}||\underline{w}|\cos\theta$ ($\div$ by -2)

$\Rightarrow \underline{v}\cdot\underline{w} = |\underline{v}||\underline{w}|\cos\theta$ Hence proved

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Example 2:- If $\underline{u} = 3\underline{i} - \underline{j} - 2k$ and $\underline{v} = \underline{i} + 2\underline{j} - \underline{k}$

then $\underline{u}\cdot\underline{v} = (3)(1) + (-1)(2) + (-2)(-1)$

$= 3 - 2 + 2 = 3$

Example 3:- If $\underline{u} = 2\underline{i} - 4\underline{j} + 5\underline{k}$ and $\underline{v} = -4\underline{i} - 3\underline{j} - 4\underline{k}$, then

$\underline{u}\cdot\underline{v} = (2)(4) + (-4)(-3) + (5)(-4)$

$= 8 + 12 - 20 = 0$

$\rightarrow$ $\underline{u}$ and $\underline{v}$ are perpendicular.

Angle between two vectors

For two vectors $\underline{u}$ and $\underline{v}$,

(a) $\underline{u}\cdot\underline{v} = |\underline{u}||\underline{v}|\cos\theta$, $\because 0 \le \theta \le \pi$

$$\therefore \cos\theta = \frac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|}$$

(b) If $\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$ and $\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$ then

$\underline{u}\cdot\underline{v} = a_1a_2 + b_1b_2 + c_1c_2$

$|\underline{u}| = \sqrt{a_1^2 + b_1^2 + c_1^2}$, $|\underline{v}| = \sqrt{a_2^2 + b_2^2 + c_2^2}$

since $\cos\theta = \dfrac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|}$

$$\rightarrow \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Corollaries:- (i) If $\theta = 0$ or π, the vectors $\underline{u}$ and $\underline{v}$ are collinear.

(ii) If $\theta = \frac{\pi}{2}$, $\cos\theta = 0 \rightarrow \underline{u}\cdot\underline{v} = 0$ then vectors $\underline{u}$ and $\underline{v}$ are $\perp$ or othogonal.

Example 4:- Find the angle b/w the vectors $\underline{u} = 2\underline{i} - \underline{j} + \underline{k}$ and $\underline{v} = -\underline{i} + \underline{j}$

Solution:- $\underline{u}\cdot\underline{v} = (2\underline{i} - \underline{j} + \underline{k})\cdot(-\underline{i} + \underline{j})$

$= (2)(-1) + (-1)(1) + (1)(0) = -3$

$|\underline{u}| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$

$|\underline{v}| = \sqrt{(-1)^2 + (1)^2 + (0)^2} = \sqrt{1 + 1} = \sqrt{2}$

Now $\cos\theta = \dfrac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|} = \dfrac{-3}{\sqrt{6}\sqrt{2}} = -\dfrac{\sqrt{3}}{2}$

$\rightarrow \cos\theta = -\frac{\sqrt{3}}{2} \rightarrow \theta = \cos^{-1}(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$

Example 5:- Find a scalar α so that $2\underline{i} + \alpha\underline{j} + 5\underline{k}$ and $3\underline{i} + \underline{j} + \alpha\underline{k}$ are perpendicular.

Solution:- Let $\underline{u} = 2\underline{i} + \alpha\underline{j} + 5\underline{k}$ and $\underline{v} = 3\underline{i} + \underline{j} + \alpha\underline{k}$ $\because \underline{u}$ and $\underline{v}$ are $\perp$. so

$\underline{u}\cdot\underline{v} = 0 \rightarrow (2\underline{i} + \alpha\underline{j} + 5\underline{k})\cdot(3\underline{i} + \underline{j} + \alpha\underline{k}) = 0$

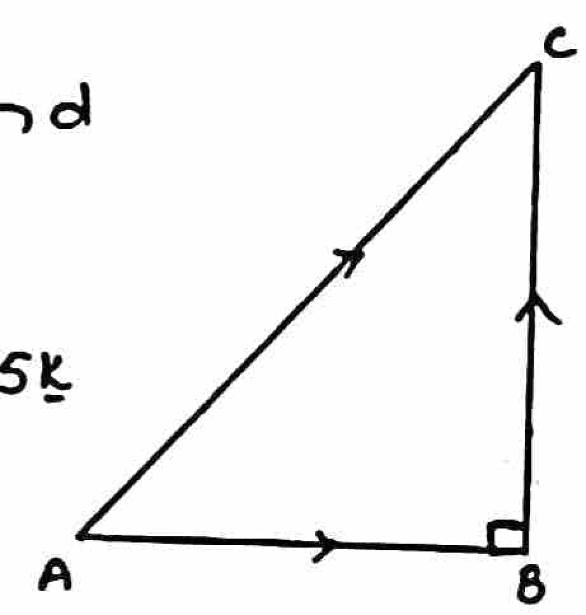
$\rightarrow (2)(3) + (\alpha)(1) + (5)(\alpha) = 0$

$\rightarrow 6 + \alpha + 5\alpha = 0 \rightarrow 6 + 6\alpha = 0$

$\rightarrow 6\alpha = -6 \rightarrow \alpha = -1$

Example 6:- Show that the vectors $2\underline{i} - \underline{j} + \underline{k}$, $\underline{i} - 3\underline{j} - 5\underline{k}$ and $3\underline{i} - 4\underline{j} - 4\underline{k}$ form the of a right triangle.

Solution:-



Let $\overrightarrow{AB} = 2\underline{i} - \underline{j} + \underline{k}$ and

$\overrightarrow{BC} = \underline{i} - 3\underline{j} - 5\underline{k}$ Now

$\overrightarrow{AB} + \overrightarrow{BC} = 2\underline{i} - \underline{j} + \underline{k} + \underline{i} - 3\underline{j} - 5\underline{k}$

$= 3\underline{i} - 4\underline{j} - 4\underline{k}$

$= \overrightarrow{AC}$ (third side)

$\rightarrow \overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{AC}$ form a ΔABC.

Now we show that ΔABC is right triangle.

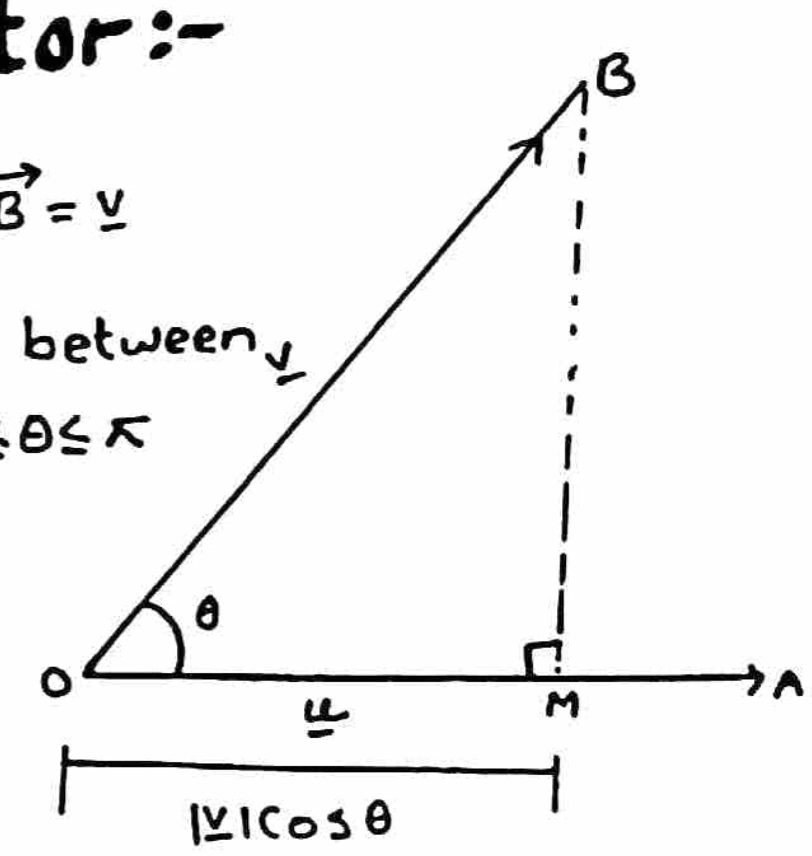
So $\overrightarrow{AB}\cdot\overrightarrow{BC} = (2\underline{i} - \underline{j} + \underline{k})\cdot(\underline{i} - 3\underline{j} - 5\underline{k})$

$= (2)(1) + (-1)(-3) + (1)(-5)$

$= 2 + 3 - 5 = 0$ $\therefore \overrightarrow{AB} \perp \overrightarrow{BC}$

Hence ΔABC is a right triangle.

Projection of one vector upon another vector:-



Let $\overrightarrow{OA} = \underline{u}$ and $\overrightarrow{OB} = \underline{v}$

Let θ be the angle between them, such that $0 \le \theta \le \pi$

Draw $\overline{BM} \perp \overline{OA}$.

Then $\overline{OM}$ is called the projection of $\underline{v}$ along $\underline{u}$. Now

In ΔOMB, $\cos\theta = \dfrac{|\overline{OM}|}{|\overline{OB}|} \rightarrow |\overline{OM}| = |\overline{OB}|\cos\theta$

$\rightarrow |\overline{OM}| = |\underline{v}|\cos\theta$ $\because \cos\theta = \dfrac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|}$

so $|\overline{OM}| = |\underline{v}|\cdot\dfrac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|}$

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$\therefore$ projection of $\mathbf{v}$ along $\mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|}$

Similarly,

Projection of $\mathbf{u}$ along $\mathbf{v} = \dfrac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}|}$

Example 7:- Show that the components of a vector are the projections of that vector along $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively.

Solution:- Let $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then

Projection of $\mathbf{v}$ along $\mathbf{i} = \dfrac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{i}|} = (a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot\mathbf{i} = a$

Projection of $\mathbf{v}$ along $\mathbf{j} = \dfrac{\mathbf{v}\cdot\mathbf{j}}{|\mathbf{j}|} = (a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot\mathbf{j} = b$

Projection of $\mathbf{v}$ along $\mathbf{k} = \dfrac{\mathbf{v}\cdot\mathbf{k}}{|\mathbf{k}|} = (a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot\mathbf{k} = c$

Hence proved.

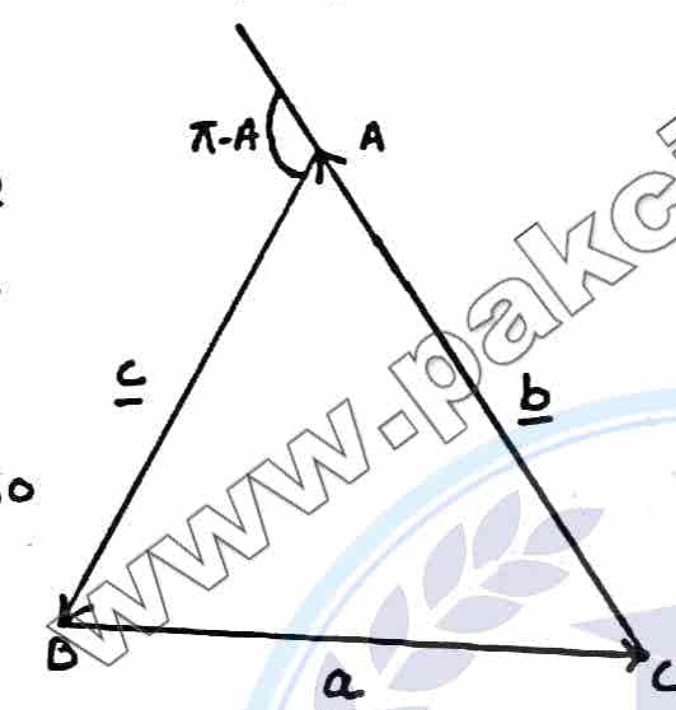
Example 8:- Prove that in any ΔABC

(i) $a^2 = b^2+c^2-2bc\cos A$ (Cosine Law)

(ii) $a = b\cos C + c\cos B$ (Projection Law)

Solution:- (i)

Let $\overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{CA} = \mathbf{b}$ and $\overrightarrow{AB} = \mathbf{c}$ be the sides of a ΔABC, as shown in fig., so



$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$

$\rightarrow \mathbf{a} = -(\mathbf{b}+\mathbf{c}) \rightarrow \mathbf{a}\cdot\mathbf{a} = -(\mathbf{b}+\mathbf{c})\cdot\mathbf{a}$

$\rightarrow \mathbf{a}\cdot\mathbf{a} = [-(\mathbf{b}+\mathbf{c})][-(\mathbf{b}+\mathbf{c})]$

$\rightarrow a^2 = (\mathbf{b}+\mathbf{c})(\mathbf{b}+\mathbf{c}) = \mathbf{b}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{b} + \mathbf{c}\cdot\mathbf{c}$

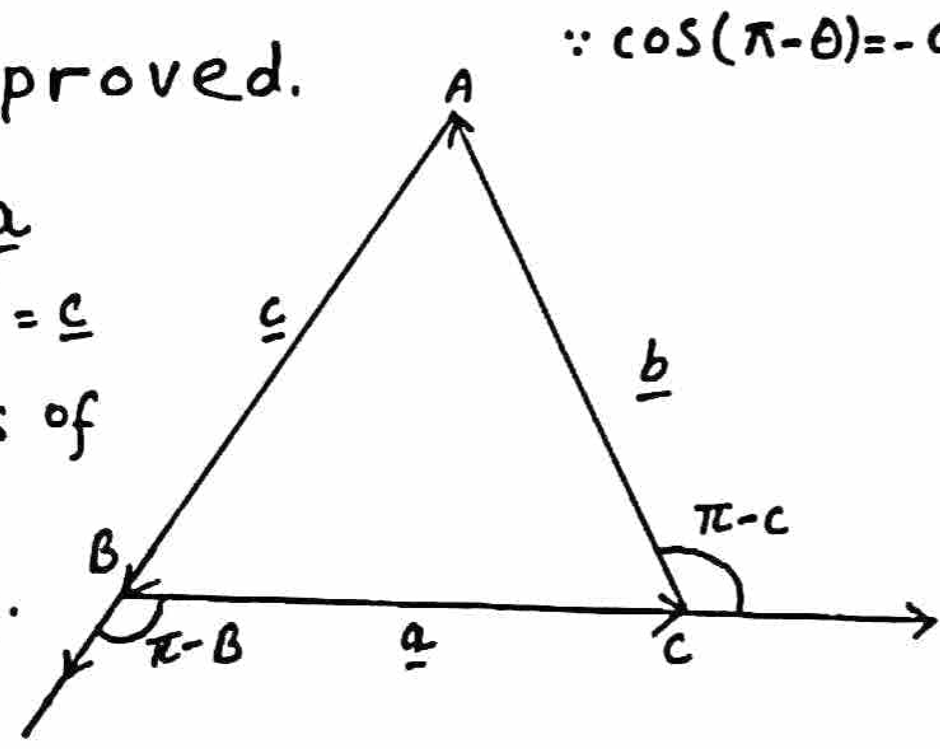
$\rightarrow a^2 = b^2 + c^2 + 2\mathbf{b}\cdot\mathbf{c}$ $\quad \because \mathbf{a}\cdot\mathbf{a} = a^2,\ \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{b}$

$\rightarrow a^2 = b^2 + c^2 + 2|\mathbf{b}||\mathbf{c}|\cos(\pi-A)$ $\quad$ ($\because$ angle from $\mathbf{b}$ to $\mathbf{c} = \pi - A$)

$a^2 = b^2 + c^2 - 2bc\cos A$ $\quad \because \cos(\pi-\theta) = -\cos\theta$

Hence proved.

(ii) Let $\overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{CA} = \mathbf{b}$, $\overrightarrow{AB} = \mathbf{c}$ be the sides of a ΔABC as shown in fig. so



$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$

$\rightarrow \mathbf{a} = -\mathbf{b} - \mathbf{c}$

$\rightarrow \mathbf{a}\cdot\mathbf{a} = -\mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{c}$

$\rightarrow \mathbf{a}\cdot\mathbf{a} = -\mathbf{a}\cdot\mathbf{b} - \mathbf{c}\cdot\mathbf{a}$ $\quad \because \mathbf{a}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a}$

$\rightarrow a^2 = -|\mathbf{a}||\mathbf{b}|\cos(\pi-C) - |\mathbf{c}||\mathbf{a}|\cos(\pi-B)$

$a^2 = ab\cos C + ca\cos B$ $\quad \because \cos(\pi-\theta) = -\cos\theta$

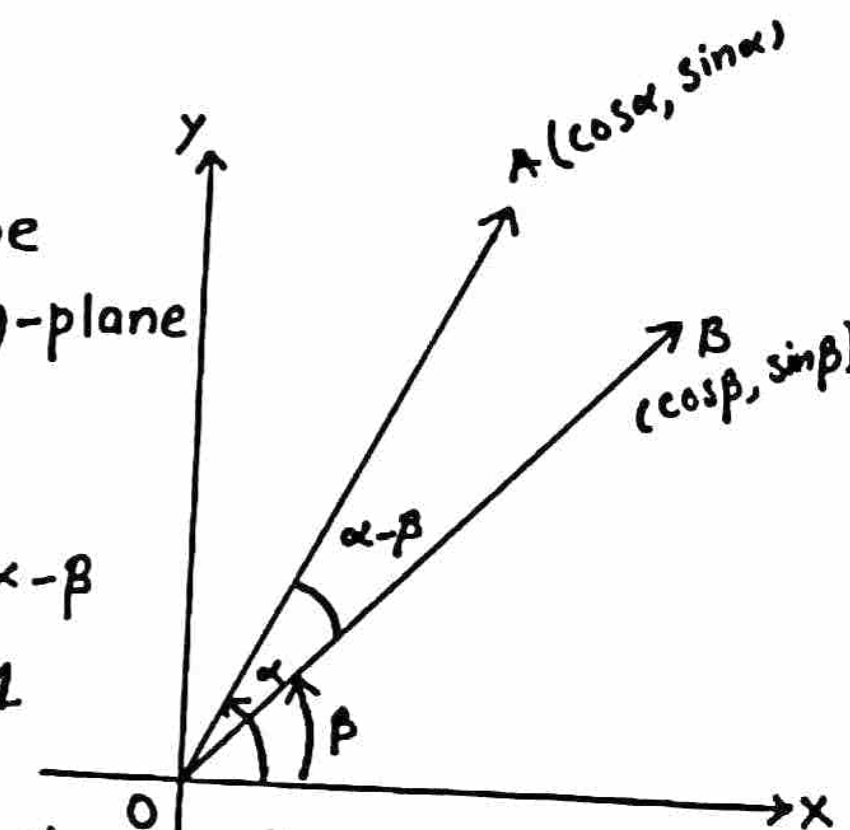
$\rightarrow a = b\cos C + c\cos B$ $\quad$ ($\because$ angle from $\mathbf{a}$ to $\mathbf{b}$ is $\pi - C$ and angle from $\mathbf{c}$ to $\mathbf{a}$ is $\pi - B$)

Hence proved.

Example 9:- Prove that

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Solution:-



Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be unit vectors in xy-plane such that

$\angle XOA = \alpha$

$\angle XOB = \beta$, $\angle BOA = \alpha - \beta$

also, $|\overrightarrow{OA}| = 1$, $|\overrightarrow{OB}| = 1$

and $\overrightarrow{OA} = \cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}$, $\overrightarrow{OB} = \cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j}$

$\therefore \overrightarrow{OA}\cdot\overrightarrow{OB} = (\cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j})\cdot(\cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j})$

$\rightarrow |\overrightarrow{OA}||\overrightarrow{OB}|\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\rightarrow \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Hence proved. $\quad \because |\overrightarrow{OA}| = |\overrightarrow{OB}| = 1$

Exercise 7.3

Q1. Find the cosine of angle θ between $\mathbf{u}$ and $\mathbf{v}$.

(i) $\mathbf{u} = 3\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\mathbf{v} = 2\mathbf{i}-\mathbf{j}+\mathbf{k}$, (iii) $\mathbf{u} = [-3,5]$, $\mathbf{v} = [6,-2]$

(ii) $\mathbf{u} = \mathbf{i}-3\mathbf{j}+4\mathbf{k}$, $\mathbf{v} = 4\mathbf{i}-\mathbf{j}+3\mathbf{k}$, (iv) $\mathbf{u} = [2,-3,1]$, $\mathbf{v} = [2,4,1]$

Solution:- (i) $\mathbf{u} = 3\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\mathbf{v} = 2\mathbf{i}-\mathbf{j}+\mathbf{k}$

$\mathbf{u}\cdot\mathbf{v} = (3\mathbf{i}+\mathbf{j}-\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = (3)(2)+(1)(-1)+(-1)(1)$
$= 6-1-1 = 4$

$|\mathbf{u}| = \sqrt{(3)^2+(1)^2+(-1)^2} = \sqrt{9+1+1} = \sqrt{11}$

$|\mathbf{v}| = \sqrt{(2)^2+(-1)^2+(1)^2} = \sqrt{4+1+1} = \sqrt{6}$

Now $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \dfrac{4}{\sqrt{11}\sqrt{6}} = \dfrac{4}{\sqrt{66}}$

(ii) $\mathbf{u} = \mathbf{i}-3\mathbf{j}+4\mathbf{k}$, $\mathbf{v} = 4\mathbf{i}-\mathbf{j}+3\mathbf{k}$

$\mathbf{u}\cdot\mathbf{v} = (\mathbf{i}-3\mathbf{j}+4\mathbf{k})\cdot(4\mathbf{i}-\mathbf{j}+3\mathbf{k}) = (1)(4)+(-3)(-1)+(4)(3)$
$= 4+3+12 = 19$

$|\mathbf{u}| = \sqrt{(1)^2+(-3)^2+(4)^2} = \sqrt{26}$, $|\mathbf{v}| = \sqrt{(4)^2+(-1)^2+(3)^2} = \sqrt{26}$

Now $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \dfrac{19}{\sqrt{26}\sqrt{26}} = \dfrac{19}{26}$

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(iii) $\underline{u} = [-3, 5] = -3\underline{i} + 5\underline{j}$, $\underline{v} = [6, -2] = 6\underline{i} - 2\underline{j}$

$\underline{u} \cdot \underline{v} = (-3\underline{i} + 5\underline{j}) \cdot (6\underline{i} - 2\underline{j}) = (-3)(6) + (5)(-2)$
$= -18 - 10 = -28$

$|\underline{u}| = \sqrt{(-3)^2 + (5)^2} = \sqrt{9+25} = \sqrt{34}$

$|\underline{v}| = \sqrt{(6)^2 + (-2)^2} = \sqrt{36+4} = \sqrt{40}$

Now $\cos\theta = \dfrac{\underline{u} \cdot \underline{v}}{|\underline{u}||\underline{v}|} = \dfrac{-28}{\sqrt{34}\sqrt{40}} = \dfrac{-28}{\sqrt{17 \times 2 \times 8 \times 5}}$

$\cos\theta = \dfrac{-28}{\sqrt{16 \times 85}} = \dfrac{-28}{4\sqrt{85}} = \dfrac{-7}{\sqrt{85}}$

(iv) $\underline{u} = [2, -3, 1] = 2\underline{i} - 3\underline{j} + \underline{k}$

$\underline{v} = [2, 4, 1] = 2\underline{i} + 4\underline{j} + \underline{k}$

$\underline{u} \cdot \underline{v} = (2\underline{i} - 3\underline{j} + \underline{k})(2\underline{i} + 4\underline{j} + \underline{k}) = (2)(2) + (-3)(4) + (1)(1)$
$= 4 - 12 + 1 = -7$

$|\underline{u}| = \sqrt{(2)^2 + (-3)^2 + (1)^2} = \sqrt{4+9+1} = \sqrt{14}$

$|\underline{v}| = \sqrt{(2)^2 + (4)^2 + (1)^2} = \sqrt{4+16+1} = \sqrt{21}$

Now $\cos\theta = \dfrac{\underline{u} \cdot \underline{v}}{|\underline{u}||\underline{v}|} = \dfrac{-7}{\sqrt{14}\sqrt{21}} = \dfrac{-7}{\sqrt{7 \times 2 \times 7 \times 3}}$

$\rightarrow \cos\theta = \dfrac{-7}{7\sqrt{6}} = \dfrac{-1}{\sqrt{6}}$

Q2. Calculate the projection of $\underline{a}$ along $\underline{b}$ and projection of $\underline{b}$ along $\underline{a}$ when: (i) $\underline{a} = \underline{i} - \underline{k}$, $\underline{b} = \underline{j} + \underline{k}$
(ii) $\underline{a} = 3\underline{i} + \underline{j} - \underline{k}$, $\underline{b} = -2\underline{i} - \underline{j} + \underline{k}$

Solution:- (i) $\underline{a} = \underline{i} - \underline{k}$, $\underline{b} = \underline{j} + \underline{k}$

$|\underline{a}| = \sqrt{(1)^2 + (0)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$

$|\underline{b}| = \sqrt{(0)^2 + (1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$

$\underline{a} \cdot \underline{b} = (\underline{i} - \underline{k}) \cdot (\underline{j} + \underline{k}) = (1)(0) + (0)(1) + (-1)(1)$
$= 0 + 0 - 1 = -1$

$\because$ projection of $\underline{a}$ along $\underline{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\underline{b}|} = \dfrac{-1}{\sqrt{2}}$

projection of $\underline{b}$ along $\underline{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\underline{a}|} = \dfrac{-1}{\sqrt{2}}$

(ii) $\underline{a} = 3\underline{i} + \underline{j} - \underline{k}$, $\underline{b} = -2\underline{i} - \underline{j} + \underline{k}$

$|\underline{a}| = \sqrt{(3)^2 + (1)^2 + (-1)^2} = \sqrt{9+1+1} = \sqrt{11}$

$|\underline{b}| = \sqrt{(-2)^2 + (-1)^2 + (1)^2} = \sqrt{4+1+1} = \sqrt{6}$

$\underline{a} \cdot \underline{b} = (3\underline{i} + \underline{j} - \underline{k}) \cdot (-2\underline{i} - \underline{j} + \underline{k})$
$= (3)(-2) + (1)(-1) + (-1)(1) = -6 - 1 - 1 = -8$

$\therefore$ projection of $\vec{a}$ along $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \dfrac{-8}{\sqrt{6}}$

Projection of $\vec{b}$ along $\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \dfrac{-8}{\sqrt{11}}$

Q3. Find a real number α so that the vectors $\underline{u}$ and $\underline{v}$ are perpendicular. (i) $\underline{u} = 2\alpha\underline{i} + \underline{j} - \underline{k}$, $\underline{v} = \underline{i} + \alpha\underline{j} + 4\underline{k}$
(ii) $\underline{u} = \alpha\underline{i} + 2\alpha\underline{j} - \underline{k}$, $\underline{v} = \underline{i} + \alpha\underline{j} + 3\underline{k}$

Solution:- (i) $\underline{u} = 2\alpha\underline{i} + \underline{j} - \underline{k}$, $\underline{v} = \underline{i} + \alpha\underline{j} + 4\underline{k}$

$\because \underline{u} \perp \underline{v}$. so $\underline{u} \cdot \underline{v} = 0$

$\rightarrow (2\alpha\underline{i} + \underline{j} - \underline{k}) \cdot (\underline{i} + \alpha\underline{j} + 4\underline{k}) = 0$

$\rightarrow (2\alpha)(1) + (1)(\alpha) + (-1)(4) = 0$

$\rightarrow 2\alpha + \alpha - 4 = 0 \rightarrow 3\alpha - 4 = 0$

$\rightarrow 3\alpha = 4 \rightarrow \alpha = 4/3$

(ii) $\underline{u} = \alpha\underline{i} + 2\alpha\underline{j} - \underline{k}$, $\underline{v} = \underline{i} + \alpha\underline{j} + 3\underline{k}$

$\because \underline{u} \perp \underline{v}$ so $\underline{u} \cdot \underline{v} = 0$

$\rightarrow (\alpha\underline{i} + 2\alpha\underline{j} - \underline{k}) \cdot (\underline{i} + \alpha\underline{j} + 3\underline{k}) = 0$

$\rightarrow (\alpha)(1) + (2\alpha)(\alpha) + (-1)(3) = 0$

$\rightarrow \alpha + 2\alpha^2 - 3 = 0 \Rightarrow 2\alpha^2 + \alpha - 3 = 0$

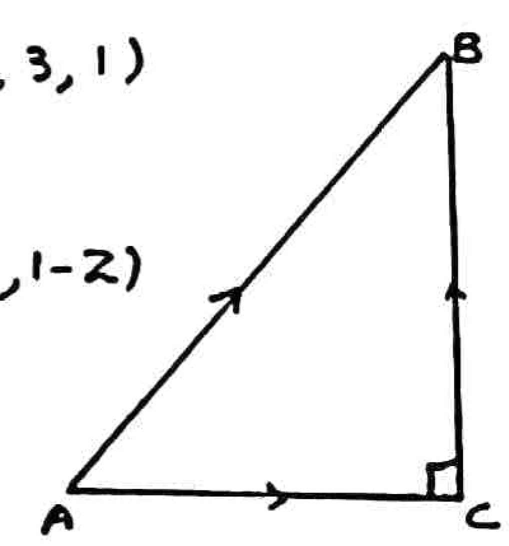
$\rightarrow 2\alpha^2 - 2\alpha + 3\alpha - 3 = 0 \Rightarrow 2\alpha(\alpha - 1) + 3(\alpha - 1) = 0$

$\rightarrow (2\alpha + 3)(\alpha - 1) = 0 \rightarrow 2\alpha + 3 = 0$ or $\alpha - 1 = 0$

$\rightarrow \alpha = -\frac{3}{2}$ or $\alpha = 1$

Q4. Find the number z so that the triangle with vertices $A(1,-1,0)$, $B(-2,2,1)$ and $C(0,2,z)$ is a right triangle with right angle at C.

Solution:- $\because A(1,-1,0), B(-2,2,1), C(0,2,z)$



Now $\vec{AB} = (-2-1, 2+1, 1-0) = (-3, 3, 1)$

$\vec{AB} = -3\underline{i} + 3\underline{j} + \underline{k}$

$\vec{CB} = (-2-0, 2-2, 1-z) = (-2, 0, 1-z)$

$= -2\underline{i} + 0\underline{j} + (1-z)\underline{k}$

$\vec{AC} = (0-1, 2+1, z-0) = (-1, 3, z)$

$\vec{AC} = -\underline{i} + 3\underline{j} + z\underline{k}$

$\because m\angle C = 90°$ so $\vec{AC} \cdot \vec{CB} = 0$

or $(-\underline{i} + 3\underline{j} + z\underline{k}) \cdot (-2\underline{i} + 0\underline{j} + (1-z)\underline{k}) = 0$

$\rightarrow (-1)(-2) + (3)(0) + (z)(1-z) = 0$

$\rightarrow 2 + z - z^2 = 0 \rightarrow z^2 - z - 2 = 0$

$\rightarrow z^2 - 2z + z - 2 = 0 \rightarrow z(z-2) + 1(z-2) = 0$

$\rightarrow (z-2)(z+1) = 0 \rightarrow z - 2 = 0$ or $z + 1 = 0$

$\rightarrow z = 2$ or $z = -1$

Q5. If $\underline{v}$ is a vector for which $\underline{v} \cdot \underline{i} = 0$, $\underline{v} \cdot \underline{j} = 0$, $\underline{v} \cdot \underline{k} = 0$, find $\underline{v}$.

Solution:- suppose $\underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$

$\because \underline{v} \cdot \underline{i} = 0 \rightarrow (v_1\underline{i} + v_2\underline{j} + v_3\underline{k}) \cdot \underline{i} = 0 \rightarrow v_1 = 0$

$\underline{v} \cdot \underline{j} = 0 \rightarrow (v_1\underline{i} + v_2\underline{j} + v_3\underline{k}) \cdot \underline{j} = 0 \rightarrow v_2 = 0$

$\underline{v} \cdot \underline{k} = 0 \rightarrow (v_1\underline{i} + v_2\underline{j} + v_3\underline{k}) \cdot \underline{k} = 0 \rightarrow v_3 = 0$

Hence $\underline{v} = 0\underline{i} + 0\underline{j} + 0\underline{k} \Rightarrow \underline{v} = \underline{0}$

It means $\underline{v}$ is a null vector.

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Q6. (i) Show that the vectors $3\underline{i}-2\underline{j}+\underline{k}$, $\underline{i}-3\underline{j}+5\underline{k}$ and $2\underline{i}+\underline{j}-4\underline{k}$ form a right triangle.

Solution:- Let $\underline{u} = 3\underline{i}-2\underline{j}+\underline{k}$

$\underline{v} = \underline{i}-3\underline{j}+5\underline{k}$, $\underline{w} = 2\underline{i}+\underline{j}-4\underline{k}$

$\underline{v}+\underline{w} = \underline{i}-3\underline{j}+5\underline{k}+2\underline{i}+\underline{j}-4\underline{k} = 3\underline{i}-2\underline{j}+\underline{k}$

$\underline{v}+\underline{w} = \underline{u}$ $\Rightarrow$ This shows that $\underline{u},\underline{v}$ and $\underline{w}$ are sides of triangle. Now

$\underline{u}\cdot\underline{w} = (3\underline{i}-2\underline{j}+\underline{k})\cdot(2\underline{i}+\underline{j}-4\underline{k})$

$= (3)(2)+(-2)(1)+(1)(-4) = 6-2-4 = 0$

Thus $\underline{u}$ and $\underline{w}$ are $\perp$ to each other. so $\underline{u},\underline{v}$ and $\underline{w}$ form a right triangle.

(ii) Show that the set of points $P=(1,3,2)$, $Q(4,1,4)$ and $R=(6,5,5)$ form a right triangle.

Solution:- $\because P(1,3,2),\ Q(4,1,4),\ R(6,5,5)$

Now $\overrightarrow{PQ} = (4-1, 1-3, 4-2) = (3,-2,2) = 3\underline{i}-2\underline{j}+2\underline{k}$

$\overrightarrow{QR} = (6-4, 5-1, 5-4) = (2,4,1) = 2\underline{i}+4\underline{j}+\underline{k}$

$\overrightarrow{PR} = (6-1, 5-3, 5-2) = (5,2,3) = 5\underline{i}+2\underline{j}+3\underline{k}$

As $\overrightarrow{PQ}+\overrightarrow{QR} = 3\underline{i}-2\underline{j}+2\underline{k}+2\underline{i}+4\underline{j}+\underline{k}$

$= 5\underline{i}+2\underline{j}+3\underline{k} = \overrightarrow{PR}$

Hence P, Q, R is a triangle. Now

$\overrightarrow{PQ}\cdot\overrightarrow{QR} = (3\underline{i}-2\underline{j}+2\underline{k})\cdot(2\underline{i}+4\underline{j}+\underline{k})$

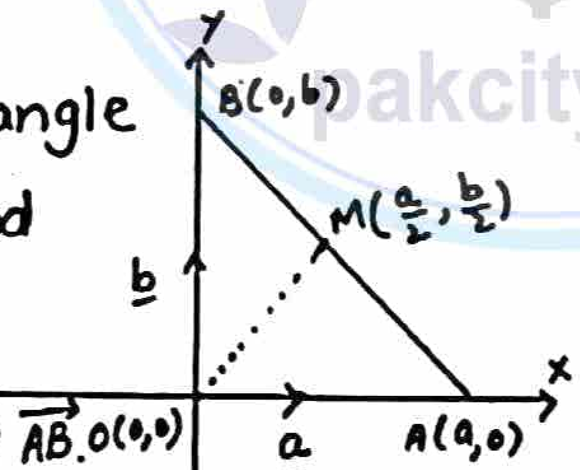
$= (3)(2)+(-2)(4)+(2)(1) = 6-8+2 = 0$

so $\overrightarrow{PQ}\perp\overrightarrow{QR}$. Thus P, Q, R is a right Δ.

Q7. Show that mid point of hypotenuse of right triangle is equidistant from its vertices.

Solution:-

Let AOB be a right triangle where O(0,0), A(a,0) and B(0,b). Let M be the mid point of hypotenuse $\overline{AB}$.



so co-ordinates of $M = \left(\frac{a+0}{2}, \frac{0+b}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$

Now $\overrightarrow{OM} = \left[\frac{a}{2}-0, \frac{b}{2}-0\right] = \left[\frac{a}{2}, \frac{b}{2}\right] = \frac{a}{2}\underline{i}+\frac{b}{2}\underline{j}$

$\Rightarrow |\overrightarrow{OM}| = \sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2} = \sqrt{\frac{a^2}{4}+\frac{b^2}{4}} \longrightarrow (I)$

$\overrightarrow{AM} = \left[\frac{a}{2}-a, \frac{b}{2}-0\right] = \left[-\frac{a}{2}, \frac{b}{2}\right] = -\frac{a}{2}\underline{i}+\frac{b}{2}\underline{j}$

$\Rightarrow |\overrightarrow{AM}| = \sqrt{\left(-\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2} = \sqrt{\frac{a^2}{4}+\frac{b^2}{4}} \longrightarrow (II)$

$\overrightarrow{BM} = \left[\frac{a}{2}-0, \frac{b}{2}-b\right] = \left[\frac{a}{2}, -\frac{b}{2}\right] = \frac{a}{2}\underline{i}-\frac{b}{2}\underline{j}$

$|\overrightarrow{BM}| = \sqrt{\left(\frac{a}{2}\right)^2+\left(-\frac{b}{2}\right)^2} = \sqrt{\frac{a^2}{4}+\frac{b^2}{4}}$ —— (III)

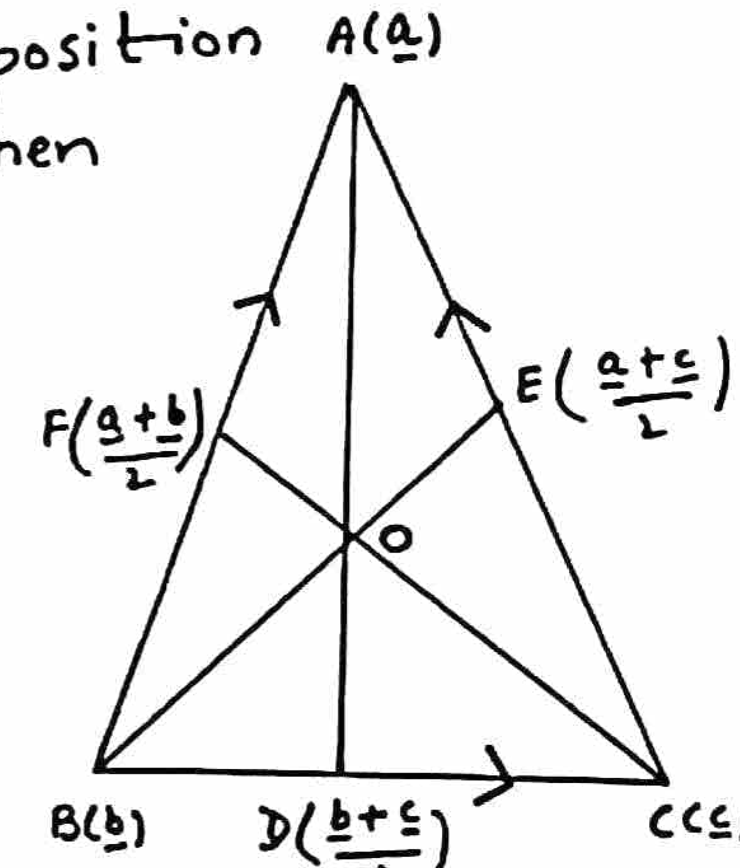
By (I), (II) and (III), $|\overrightarrow{OM}| = |\overrightarrow{AM}| = |\overrightarrow{BM}|$

Hence proved.

Q8. Prove that perpendicular bisectors of a triangle are concurrent.

Solution:-

Suppose $\underline{a},\underline{b},\underline{c}$ are position vectors of A, B, C then



P.v of $D = \overrightarrow{OD} = \frac{\underline{b}+\underline{c}}{2}$

P.v of $E = \overrightarrow{OE} = \frac{\underline{a}+\underline{c}}{2}$

P.v of $F = \overrightarrow{OF} = \frac{\underline{a}+\underline{b}}{2}$

Also,

$\overrightarrow{AB} = \underline{b}-\underline{a}$, $\overrightarrow{BC} = \underline{c}-\underline{a}$

$\overrightarrow{CA} = \underline{a}-\underline{c}$. Now

$\overrightarrow{OD}\perp\overrightarrow{BC}$ so $\overrightarrow{OD}\cdot\overrightarrow{BC} = 0$

$\rightarrow \left(\frac{\underline{b}+\underline{c}}{2}\right)\cdot(\underline{c}-\underline{b}) = 0 \rightarrow (\underline{c}+\underline{b})(\underline{c}-\underline{b}) = 0$

$\rightarrow c^2-b^2 = 0$ —— (I)

$\overrightarrow{OE}$ is $\perp$ to $\overrightarrow{CA}$ so $\overrightarrow{OE}\cdot\overrightarrow{CA} = 0$

$\rightarrow \left(\frac{\underline{a}+\underline{c}}{2}\right)(\underline{a}-\underline{c}) = 0 \rightarrow (\underline{a}+\underline{c})(\underline{a}-c) = 0$

$\rightarrow a^2-c^2 = 0$ —— (II)

Adding (I) and (II) $\rightarrow$

$\cancel{c^2} - b^2 = 0$
$a^2 - \cancel{c^2} = 0$
$a^2 - b^2 = 0$

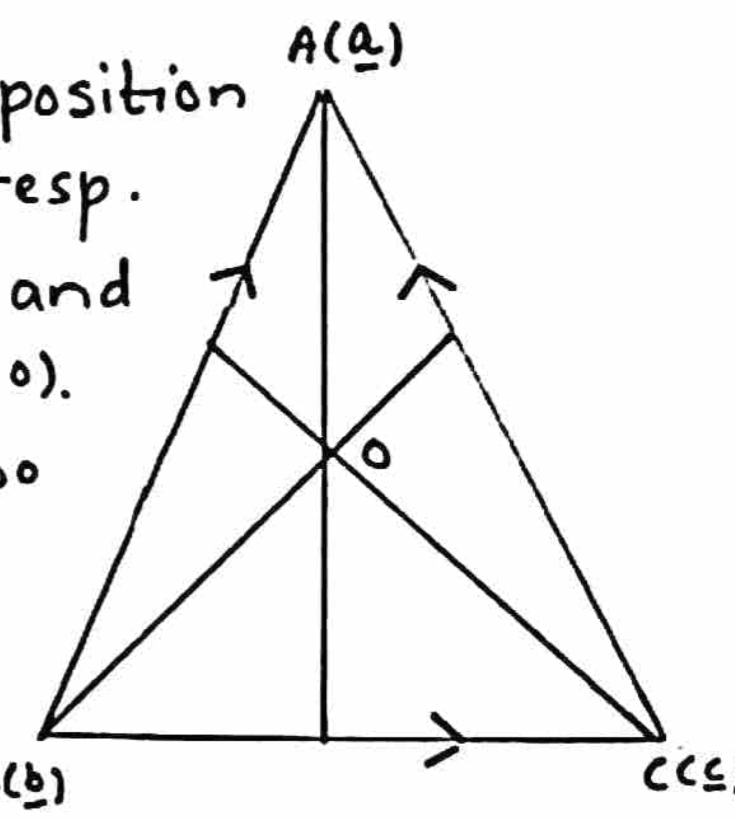
$\rightarrow (\vec{a}+\vec{b})(\underline{a}-\underline{b}) = 0 \rightarrow \left(\frac{\vec{a}+\vec{b}}{2}\right)(\underline{a}-\underline{b}) = 0$

$\rightarrow \overrightarrow{BA}$ is $\perp$ar to $\overrightarrow{OF}$. Hence proved

Q9. Prove that the altitudes of a triangle are concurrent.

Solution:-

Let $\underline{a},\underline{b},\underline{c}$ be the position vectors of A, B, C resp.



Let altitudes on $\overrightarrow{AB}$ and $\overrightarrow{BC}$ intersect at O(0,0).

$\because \overrightarrow{OB}$ is $\perp$ar to $\overrightarrow{CA}$ so

$\rightarrow \overrightarrow{OB}\cdot\overrightarrow{CA} = 0$

$\rightarrow \underline{b}\cdot(\vec{a}-\vec{c}) = 0$

$\rightarrow \underline{b}\cdot\underline{a} - \underline{b}\cdot\underline{c} = 0$ —— (I)

Also $\overrightarrow{OA}$ is $\perp$ar to $\overrightarrow{BC}$. so

$\rightarrow \overrightarrow{OA}\cdot\overrightarrow{BC} = 0 \Rightarrow \vec{a}\cdot(\vec{c}-\vec{b}) = 0$

$\rightarrow \underline{a}\cdot\underline{c} - \underline{a}\cdot\underline{b} = 0$ —— (II)

Adding (I) and (II). so

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$\underline{b}\cdot\underline{a} - \underline{b}\cdot\underline{c} + \underline{a}\cdot\underline{c} - \underline{a}\cdot\underline{b} = 0$

$\cancel{\underline{a}}\,\underline{b} - \underline{b}\cdot\underline{c} + \underline{a}\cdot\underline{c} - \cancel{\underline{a}}\,\underline{b} = 0$ $\quad \boxed{\underline{b}\cdot\underline{a} = \underline{a}\cdot\underline{b}}$

$\underline{c}\cdot(\underline{a} - \underline{b}) = 0$

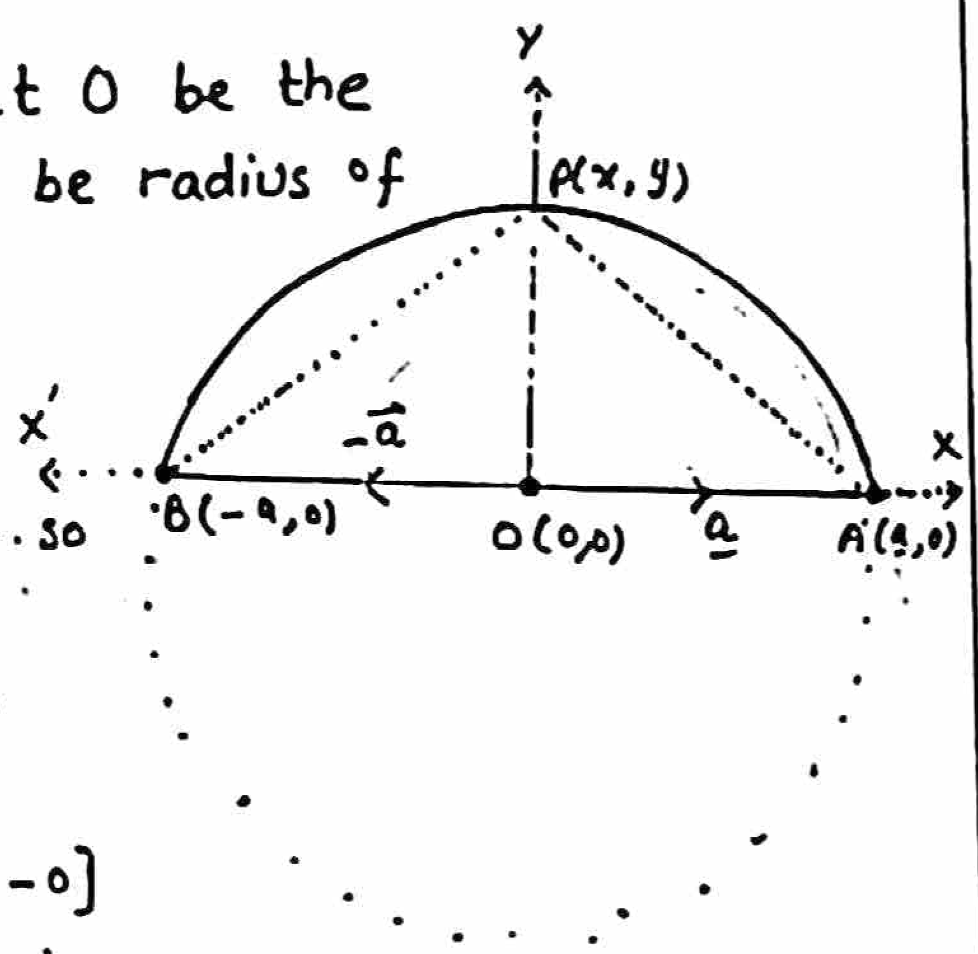
$\Rightarrow \overrightarrow{OC}\cdot\overrightarrow{BA} = 0 \quad \because \overrightarrow{BA} = \underline{a} - \underline{b}$

$\Rightarrow \overrightarrow{OC}$ is $\perp$ar to $\overrightarrow{BA}$ Hence proved.

Q10. Prove that angle in a semi-circle is a right angle.

Solution:- Let O be the centre and $\underline{a}$ be radius of semi-circle.

$\because x^2+y^2=a^2$ is the eq. of circle. so



$\overrightarrow{AP} = [x-a, y-0]$

$= (x-a)\underline{i} + y\underline{j}$

$\overrightarrow{BP} = [x+a, y-0]$

$\overrightarrow{BP} = (x+a)\underline{i} + y\underline{j}$

Now $\overrightarrow{AP}\cdot\overrightarrow{BP} = \left((x-a)\underline{i}+y\underline{j}\right)\cdot\left((x+a)\underline{i}+y\underline{j}\right)$

$= (x-a)(x+a) + y\cdot y = x^2 - a^2 + y^2$

$= x^2+y^2-a^2 = a^2 - a^2 \quad \because x^2+y^2=a^2$

$\overrightarrow{AP}\cdot\overrightarrow{BP} = 0 \Rightarrow \overrightarrow{AP}\perp\overrightarrow{BP}$ Hence angle in semi-circle is a right angle.

Q11. Prove that $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

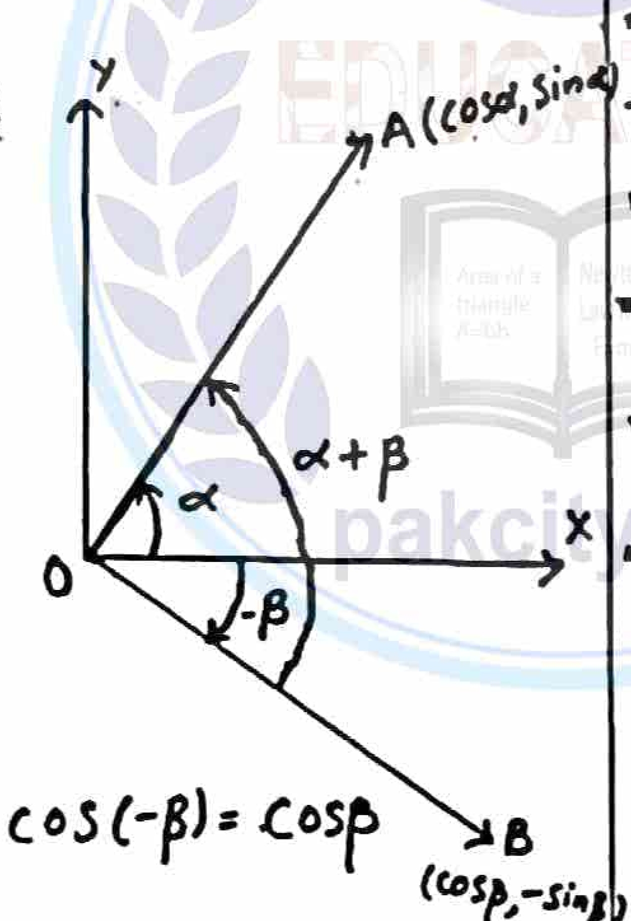
Solution:-

Suppose $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are unit vectors. and

$\angle XOA = \alpha$, $\angle XOB = -\beta$

$\angle BOA = \alpha+\beta$. Also

$|\overrightarrow{OA}| = |\overrightarrow{OB}| = 1$ As



$\overrightarrow{OA} = \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j}$

$\overrightarrow{OB} = \cos\beta\,\underline{i} + \sin(-\beta)\underline{j}$

$= \cos\beta\,\underline{i} - \sin\beta\,\underline{j} \quad \because \cos(-\beta) = \cos\beta$

Now

$\overrightarrow{OA}\cdot\overrightarrow{OB} = (\cos\alpha\,\underline{i} + \sin\alpha\,\underline{j})(\cos\beta\,\underline{i} - \sin\beta\,\underline{j})$

$|\overrightarrow{OA}||\overrightarrow{OB}|\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$(1)(1)\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\Rightarrow \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Thus proved.

Q12. Prove that in any triangle ABC

(i) $b = c\cos A + a\cos C$

(ii) $c = a\cos B + b\cos A$ (iii) $b^2 = c^2 + a^2 - 2ca\cos B$

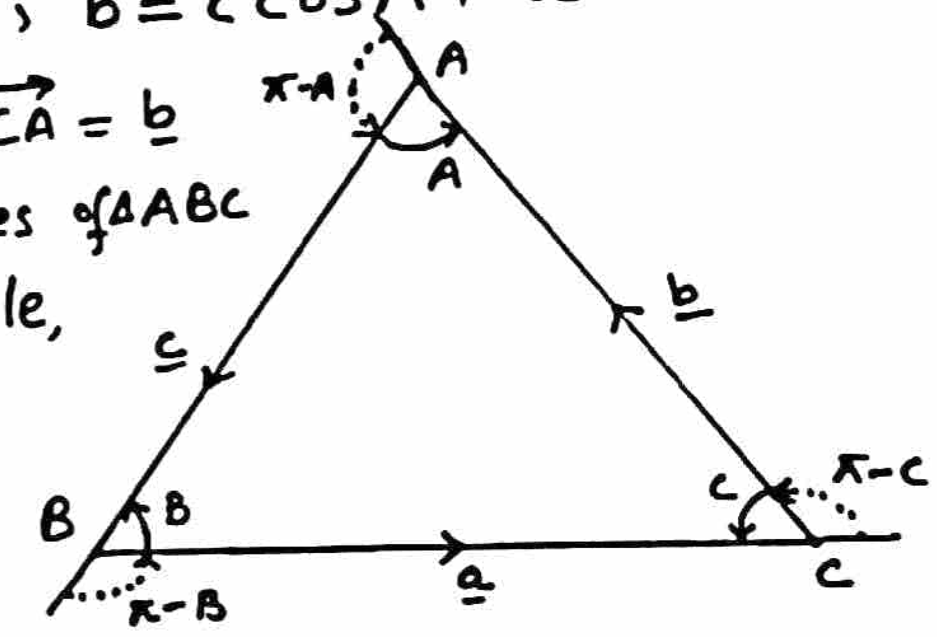
(iv) $c^2 = a^2 + b^2 - 2ab\cos C$

Solution:- (i) $b = c\cos A + a\cos C$

Let $\overrightarrow{BC} = \underline{a}$, $\overrightarrow{CA} = \underline{b}$

$\overrightarrow{AB} = \underline{c}$ be sides of ΔABC

In any triangle,



$\underline{a} + \underline{b} + \underline{c} = 0$

$\Rightarrow \underline{b} = -\underline{a} - \underline{c}$

Taking dot product with $\underline{b}$

$\Rightarrow \underline{b}\cdot\underline{b} = -\underline{b}\cdot\underline{a} - \underline{b}\cdot\underline{c}$

$\Rightarrow b^2 = -\underline{a}\cdot\underline{b} - \underline{b}\cdot\underline{c} \quad \because \underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a}$

$\Rightarrow b^2 = -|\underline{a}||\underline{b}|\cos(\pi - C) - |\underline{b}||\underline{c}|\cos(\pi - A)$

($\because$ angle from $\underline{a}$ to $\underline{b}$ is $\pi - C$ and angle from $\underline{b}$ to $\underline{c}$ is $\pi - A$.)

$\because \cos(\pi-\theta) = -\cos\theta$

$\Rightarrow b^2 = ab\cos C + bc\cos A$

$\Rightarrow b = a\cos C + c\cos A$ ($\div$ both sides by b)

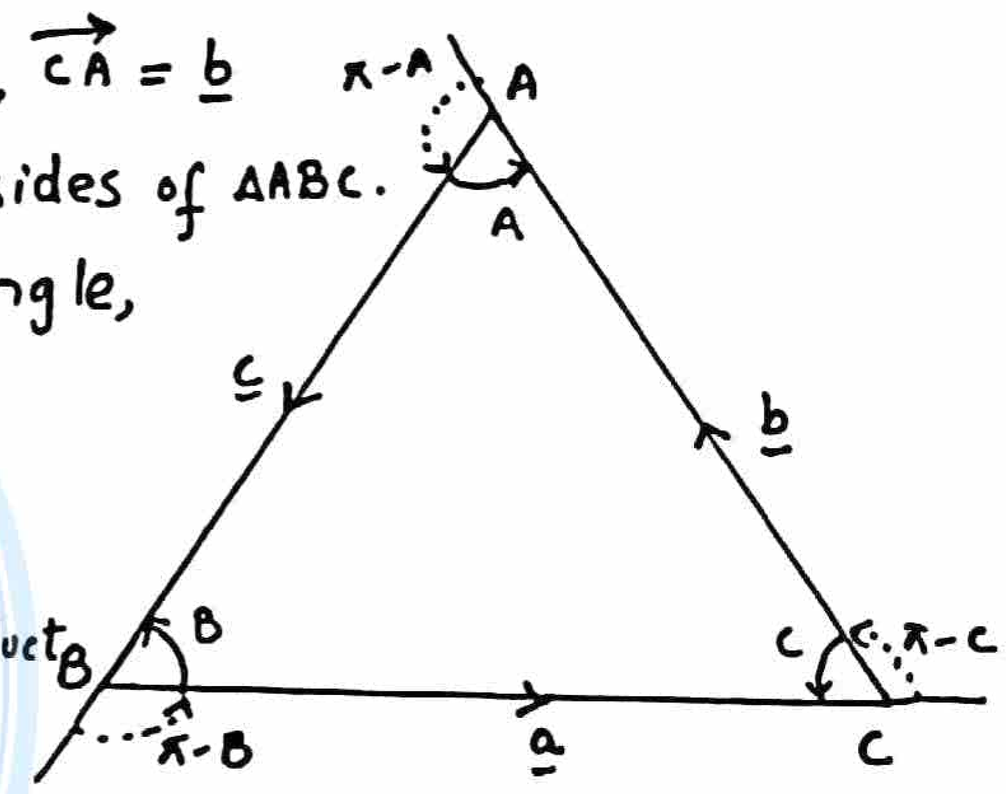
Hence proved.

(ii) $c = a\cos B + b\cos A$

Let $\overrightarrow{BC} = \underline{a}$, $\overrightarrow{CA} = \underline{b}$

$\overrightarrow{AB} = \underline{c}$ be sides of ΔABC.

In any triangle,



$\underline{a} + \underline{b} + \underline{c} = 0$

$\Rightarrow \underline{c} = -\underline{a} - \underline{b}$

Taking dot product with $\underline{c}$

$\Rightarrow \underline{c}\cdot\underline{c} = -\underline{c}\cdot\underline{a} - \underline{c}\cdot\underline{b}$

$\Rightarrow c^2 = -\underline{c}\cdot\underline{a} - \underline{b}\cdot\underline{c} \quad \because \underline{b}\cdot\underline{c} = \underline{c}\cdot\underline{b}$

$\Rightarrow c^2 = -|\underline{c}||\underline{a}|\cos(\pi - B) - |\underline{b}||\underline{c}|\cos(\pi - A)$

($\because$ angle from $\underline{c}$ to $\underline{a}$ is $\pi - B$ and angle from $\underline{b}$ to $\underline{c}$ is $\pi - A$.)

$\Rightarrow c^2 = ca\cos B + bc\cos A \quad \because \cos(\pi-\theta) = -\cos\theta$

$\Rightarrow c = a\cos B + b\cos A$ ($\div$ both sides by c)

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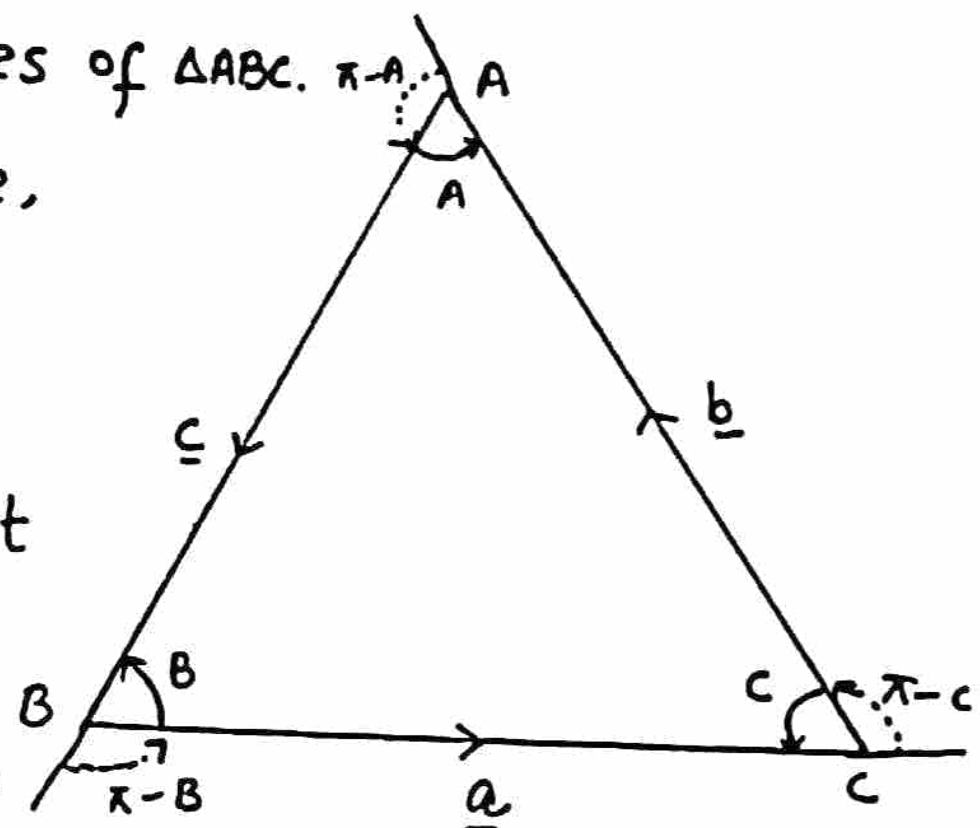
iii) $b^2 = c^2 + a^2 - 2ca\cos B$

Let $\overrightarrow{BC} = \underline{a}$, $\overrightarrow{CA} = \underline{b}$

$\overrightarrow{AB} = \underline{c}$ be sides of ΔABC.



In any triangle,

$\underline{a} + \underline{b} + \underline{c} = 0$

$\Rightarrow \underline{b} = -(\underline{a} + \underline{c})$

Take dot product with $\underline{b}$

$\Rightarrow \underline{b}\cdot\underline{b} = -(\underline{a}+\underline{c})\cdot\underline{b}$

$\Rightarrow b^2 = [-(\underline{a}+\underline{c})]\cdot[-(\underline{a}+\underline{c})]$

$\Rightarrow b^2 = (\underline{a}+\underline{c})(\underline{a}+\underline{c}) = \underline{a}\cdot\underline{a} + \underline{a}\cdot\underline{c} + \underline{c}\cdot\underline{a} + \underline{c}\cdot\underline{c}$

$= a^2 + \underline{c}\cdot\underline{a} + \underline{c}\cdot\underline{a} + c^2 \quad \because \underline{a}\cdot\underline{c} = \underline{c}\cdot\underline{a}$

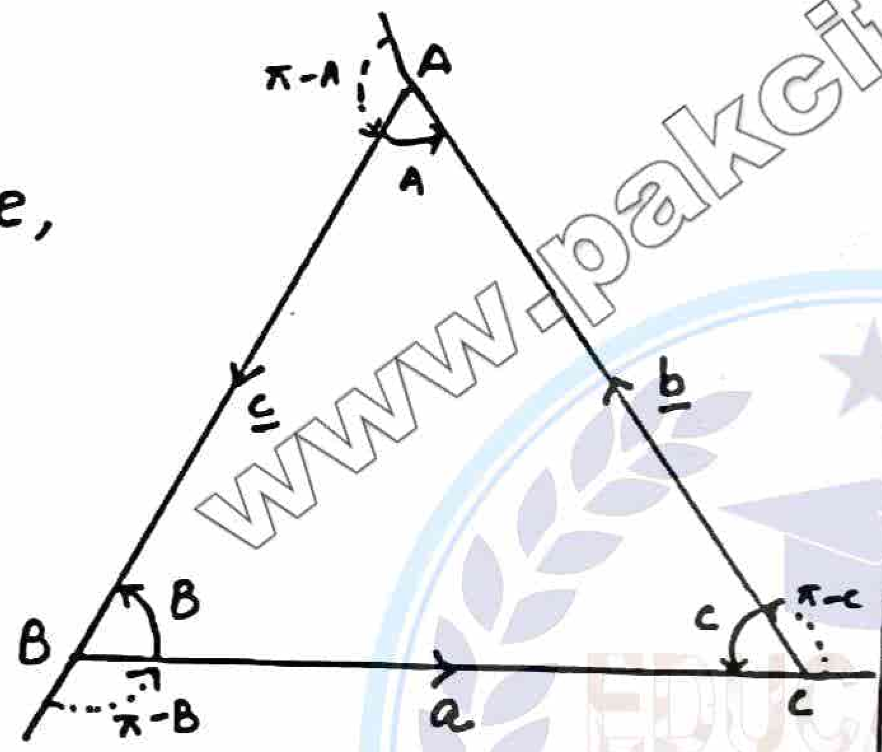
$\Rightarrow b^2 = a^2 + c^2 + 2|\underline{c}||\underline{a}|\cos(\pi - B)$

$\Rightarrow b^2 = a^2 + c^2 - 2ca\cos B$ ($\because$ Angle from $\underline{c}$ to $\underline{a}$ is $\pi - B$. $\because \cos(\pi-\theta) = -\cos\theta$)

Hence proved.

(iv) $c^2 = a^2 + b^2 - 2ab\cos C$.

Let $\overrightarrow{BC} = \underline{a}$, $\overrightarrow{CA} = \underline{b}$, $\overrightarrow{AB} = \underline{c}$ be the sides of ΔABC.



In any triangle,

$\underline{a} + \underline{b} + \underline{c} = 0$

$\Rightarrow \underline{c} = -(\underline{a} + \underline{b})$

Take dot product with $\underline{c}$

$\Rightarrow \underline{c}\cdot\underline{c} = -(\underline{a}+\underline{b})\cdot\underline{c}$

$\Rightarrow c^2 = [-(\underline{a}+\underline{b})][-(\underline{a}+\underline{b})]$

$= (\underline{a}+\underline{b})(\underline{a}+\underline{b}) = \underline{a}\cdot\underline{a} + \underline{a}\cdot\underline{b} + \underline{b}\cdot\underline{a} + \underline{b}\cdot\underline{b}$

$= a^2 + \underline{a}\cdot\underline{b} + \underline{a}\cdot\underline{b} + b^2 \quad \because \underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a}$

$c^2 = a^2 + b^2 + 2|\underline{a}||\underline{b}|\cos(\pi - C)$

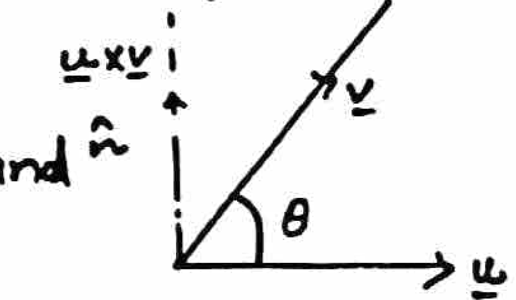
$\Rightarrow c^2 = a^2 + b^2 - 2ab\cos C$ ($\because$ Angle from $\underline{a}$ to $\underline{b}$ is $\pi - C$. $\cos(\pi-\theta) = -\cos\theta$)

Hence proved.

THE CROSS PRODUCT (VECTOR PRODUCT)

Let $\underline{u}$ and $\underline{v}$ be two non-zero vectors. The cross or vector product of $\underline{u}$ and $\underline{v}$, written as $\underline{u}\times\underline{v}$, is defined as $\underline{u}\times\underline{v} = (|\underline{u}||\underline{v}|\sin\theta)\hat{\underline{n}}$

where θ is the angle b/w the vectors, s.that $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector $\perp$ar to the plane of $\underline{u}$ and $\underline{v}$ with direction given by the right hand rule.



Right hand rule:-

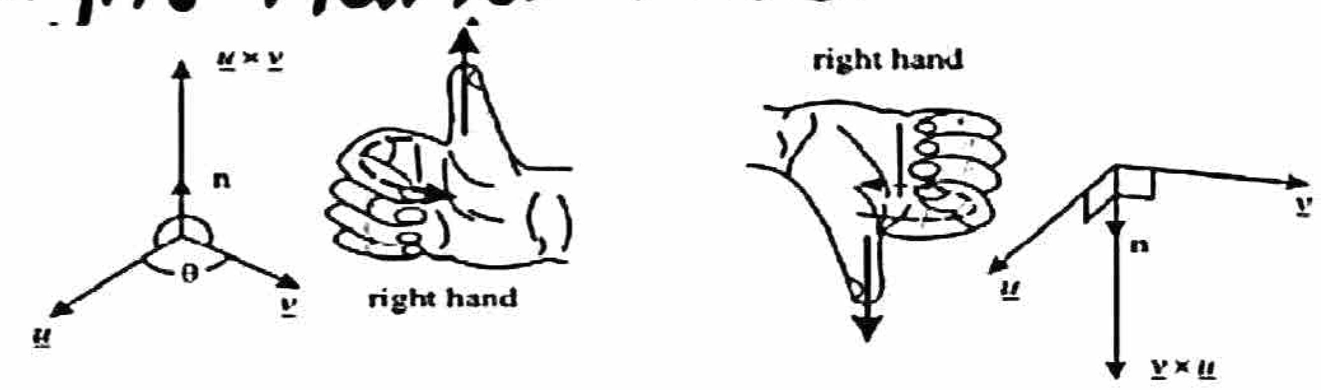


Figure (a) Figure (b)

If the fingers of right hand point along the vector $\underline{u}$ and then curl towards the vector $\underline{v}$, then the thumb will give the direction of $\hat{n}$ which is $\underline{u}\times\underline{v}$.

In fig (b) the right hand rule shows the direction of $\underline{v}\times\underline{u}$.

Derivation of useful results of cross products

(a) $\underline{i}\times\underline{i} = |\underline{i}||\underline{i}|\sin 0^\circ\,\hat{n} = 0$

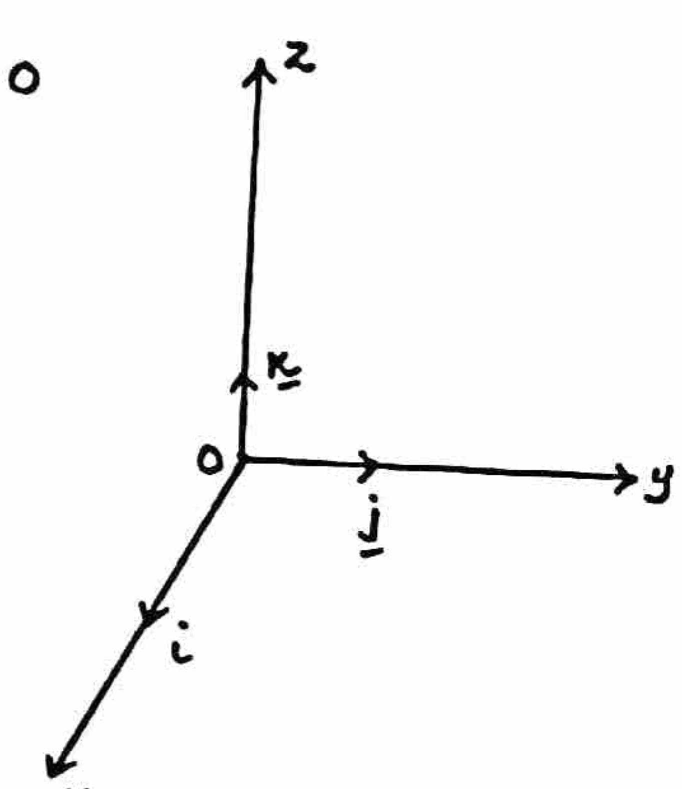
$\underline{j}\times\underline{j} = |\underline{j}||\underline{j}|\sin 0^\circ\,\hat{n} = 0$

$\underline{k}\times\underline{k} = |\underline{k}||\underline{k}|\sin 0^\circ\,\hat{n} = 0$

(b) $\underline{i}\times\underline{j} = |\underline{i}||\underline{j}|\sin 90^\circ\,\hat{\underline{k}} = \underline{k}$

$\underline{j}\times\underline{k} = |\underline{j}||\underline{k}|\sin 90^\circ\,\hat{\underline{i}} = \underline{i}$

$\underline{k}\times\underline{i} = |\underline{k}||\underline{i}|\sin 90^\circ\,\hat{\underline{j}} = \underline{j}$



(c) $\underline{u}\times\underline{v} = |\underline{u}||\underline{v}|\sin\theta\,\hat{n}$

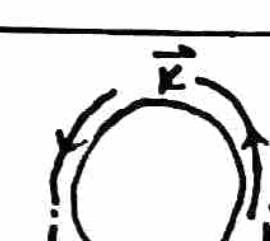
$= |\underline{v}||\underline{u}|\sin(-\theta)\hat{n}$

$= -|\underline{v}||\underline{u}|\sin\theta\,\hat{n}$

$\Rightarrow \underline{u}\times\underline{v} = -\underline{v}\times\underline{u}$

(d) $\underline{u}\times\underline{u} = |\underline{u}||\underline{u}|\sin 0^\circ\,\hat{\underline{n}} = 0$

Note:- The cross product of $\underline{i}, \underline{j}$ and $\underline{k}$ are written in cyclic pattern. It is helpful to remember.



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Properties of cross product

(i) $\underline{u}\times\underline{v}=0$ if $\underline{u}=0$ or $\underline{v}=0$ (ii) $\underline{u}\times\underline{v}=-\underline{v}\times\underline{u}$

(iii) $\underline{u}\times(\underline{v}\times\underline{w})=\underline{u}\times\underline{v}+\underline{u}\times\underline{w}$ (Distributive property)

(iv) $\underline{u}\times(k\underline{v})=(k\underline{u})\times\underline{v}=k(\underline{u}\times\underline{v})$ (v) $\underline{u}\times\underline{u}=0$

Analytical Expression of $\underline{u}\times\underline{v}$

(Determinant formula for $\underline{u}\times\underline{v}$)

Let $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$, $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$
then

$\underline{u}\times\underline{v}=(a_1\underline{i}+b_1\underline{j}+c_1\underline{k})\times(a_2\underline{i}+b_2\underline{j}+c_2\underline{k})$

$=a_1a_2(\underline{i}\times\underline{i})+a_1b_2(\underline{i}\times\underline{j})+a_1c_2(\underline{i}\times\underline{k})$
$+b_1a_2(\underline{j}\times\underline{i})+b_1b_2(\underline{j}\times\underline{j})+b_1c_2(\underline{j}\times\underline{k})$
$+c_1a_2(\underline{k}\times\underline{i})+c_1b_2(\underline{k}\times\underline{j})+c_1c_2(\underline{k}\times\underline{k})$

$=a_1b_2\underline{k}+a_1c_2\underline{j}-b_1a_2\underline{k}+b_1a_2\underline{i}$
$+c_1a_2\underline{j}-c_1b_2\underline{i}$

$\because \underline{i}\times\underline{i}=\underline{j}\times\underline{j}=\underline{k}\times\underline{k}=0$
$\underline{i}\times\underline{j}=\underline{k}$
$\underline{j}\times\underline{i}=-\underline{k}$

Re-arranging, we have

$\underline{u}\times\underline{v}=(b_1c_2-c_1b_2)\underline{i}-(a_1c_2-c_1a_2)\underline{j}+(a_1b_2-b_1a_2)\underline{k}$

The expansion of 3x3 determinant

$$\begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\a_1&b_1&c_1\\a_2&b_2&c_2\end{vmatrix}=(b_1c_2-c_1b_2)\underline{i}-(a_1c_2-c_1a_2)\underline{j}+(a_1b_2-b_1a_2)\underline{k}$$

Hence $\underline{u}\times\underline{v}=\begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\a_1&b_1&c_1\\a_2&b_2&c_2\end{vmatrix}$

Parallel vectors:-

If $\underline{u}$ and $\underline{v}$ are ∥ vectors ($\theta=0$, $\sin 0°=0$) then

$\underline{u}\times\underline{v}=|\underline{u}||\underline{v}|\sin\theta\,\hat{n}=0 \Rightarrow \underline{u}\times\underline{v}=0$

and if $\underline{u}\times\underline{v}=0$ then either $\sin\theta=0$ or $\underline{u}=0$ or $\underline{v}=0$

(i) If $\sin\theta=0 \Rightarrow \theta=180°$ or $0°$ which shows that vectors $\underline{u}$ and $\underline{v}$ are parallel.

(ii) If $\underline{u}=0$ or $\underline{v}=0$ then since zero vector has no specific direction, so zero vector is ∥ to every vector.

Note:- Zero vector is both ∥ and ⊥ar to every vector.

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Example 1. Find a vector ⊥ar to each of vectors $\underline{a}=2\underline{i}+\underline{j}+\underline{k}$ and $\underline{b}=4\underline{i}+2\underline{j}-\underline{k}$

Solution:- $\underline{a}=2\underline{i}+\underline{j}+\underline{k}$, $\underline{b}=4\underline{i}+2\underline{j}-\underline{k}$

$\because \underline{a}\times\underline{b}$ is a vector ⊥ar to both $\underline{a}$ and $\underline{b}$. so, $\underline{a}\times\underline{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&1&1\\4&2&-1\end{vmatrix}$

$=\hat{i}(-1-2)-\hat{j}(-2-4)+\hat{k}(4-4)=-3\hat{i}+6\hat{j}+0\hat{k}$

Example 2. If $\underline{a}=4\underline{i}+3\underline{j}+\underline{k}$ and $\underline{b}=2\underline{i}-\underline{j}+2\underline{k}$. Find a unit vector ⊥ar to both $\underline{a}$ and $\underline{b}$. Also find the sine of the angle b/w the vectors $\underline{a}$ and $\underline{b}$.

Solution:- $\underline{a}=4\underline{i}+3\underline{j}+\underline{k}$, $\underline{b}=2\underline{i}-\underline{j}+2\underline{k}$

$\because \underline{a}\times\underline{b}$ is a vector ⊥ar to both $\underline{a}$ and $\underline{b}$.

so $\underline{a}\times\underline{b}=\begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\4&3&1\\2&-1&2\end{vmatrix}=\underline{i}(6+1)-\underline{j}(8-2)+\underline{k}(-4-6)$

$\Rightarrow \underline{a}\times\underline{b}=7\underline{i}-6\underline{j}-10\underline{k}$

$|\underline{a}\times\underline{b}|=\sqrt{49+36+100}=\sqrt{185}$

Now unit vector ⊥ar to $\underline{a}$ and $\underline{b}$ $=\dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}=\dfrac{7\underline{i}-6\underline{j}-10\underline{k}}{\sqrt{185}}$

$=\dfrac{1}{\sqrt{185}}(7\underline{i}-6\underline{j}-10\underline{k})$

Also $|\underline{a}|=\sqrt{(4)^2+(3)^2+(1)^2}=\sqrt{16+9+1}=\sqrt{26}$

$|\underline{b}|=\sqrt{(2)^2+(-1)^2+(2)^2}=\sqrt{4+1+4}=3$

$\Rightarrow \sin\theta=\dfrac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|}=\dfrac{\sqrt{185}}{3\sqrt{26}}$

Example 3. Prove that $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

Solution:-

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be unit vectors in xy-plane s.that

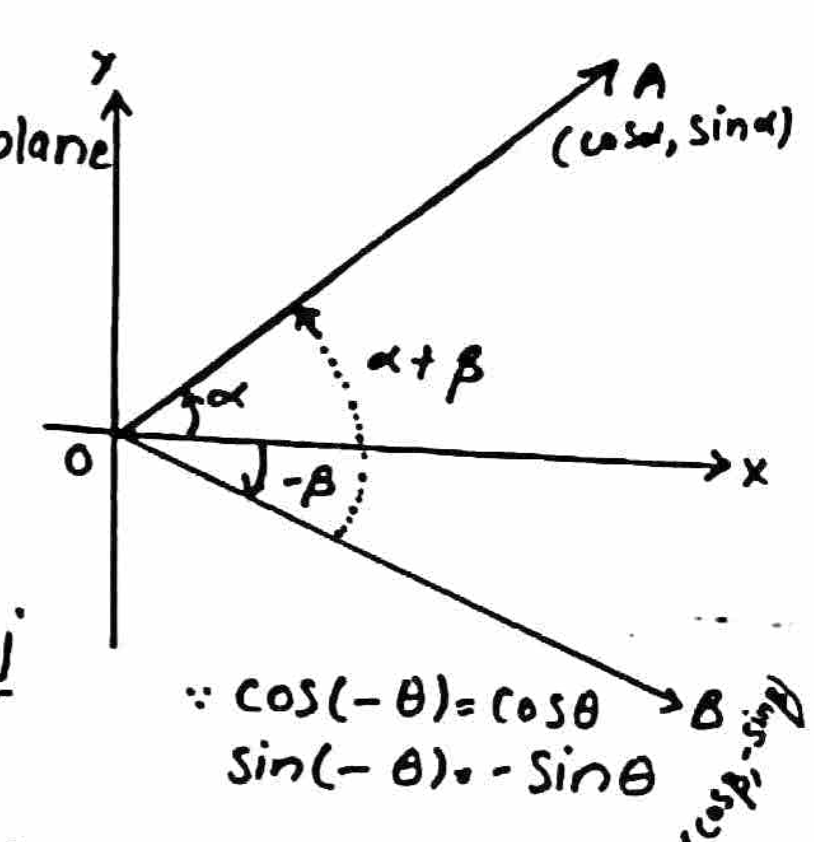
$\angle XOA=\alpha$, $\angle XOB=-\beta$

$\angle BOA=\alpha+\beta$

$|\overrightarrow{OA}|=|\overrightarrow{OB}|=1$ and

$\overrightarrow{OA}=\cos\alpha\,\underline{i}+\sin\alpha\,\underline{j}$

$\overrightarrow{OB}=\cos(-\beta)\underline{i}+\sin(-\beta)\underline{j}$
$=\cos\beta\,\underline{i}-\sin\beta\,\underline{j}$



$\because \cos(-\theta)=\cos\theta$
$\sin(-\theta)=-\sin\theta$

Now $\overrightarrow{OB}\times\overrightarrow{OA}=\begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\\cos\beta&-\sin\beta&0\\\cos\alpha&\sin\alpha&0\end{vmatrix}$

$\Rightarrow |\overrightarrow{OB}||\overrightarrow{OA}|\sin(\alpha+\beta)\underline{k}=(\sin\alpha\cos\beta+\cos\alpha\sin\beta)\underline{k}$

$\Rightarrow \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

Hence proved.

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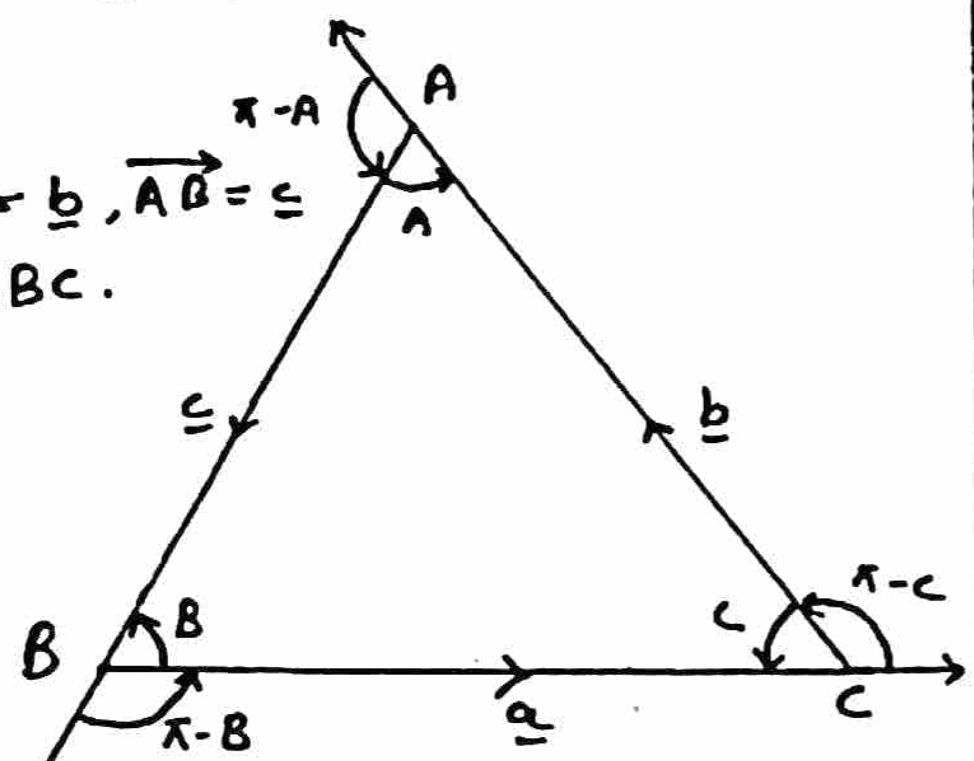
Example 4. In any ΔABC, prove that

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \text{ (Law of Sine)}$$

Solution:-



Let $\overrightarrow{BC}=\underline{a}$, $\overrightarrow{CA}=\underline{b}$, $\overrightarrow{AB}=\underline{c}$ be sides of ΔABC.

In any ΔABC,

$\underline{a}+\underline{b}+\underline{c}=0$

$\Rightarrow \underline{b}+\underline{c}=-\underline{a}$

Take cross product with $\underline{c}$

$\Rightarrow \underline{b}\times\underline{c}+\underline{c}\times\underline{c}=-\underline{a}\times\underline{c}$

$\Rightarrow \underline{b}\times\underline{c}=\underline{c}\times\underline{a}$ $\quad \because \underline{c}\times\underline{c}=0$, $-\underline{a}\times\underline{c}=\underline{c}\times\underline{a}$

$\Rightarrow |\underline{b}\times\underline{c}|=|\underline{c}\times\underline{a}|$

$\Rightarrow |\underline{b}||\underline{c}|\sin(\pi-A)=|\underline{c}||\underline{a}|\sin(\pi-B)$

($\because$ angle from $\underline{b}$ and $\underline{c}$ is $\pi-A$ and angle from $\underline{c}$ and $\underline{a}$ is $\pi-B$.)

$\Rightarrow b\cancel{c}\sin A=\cancel{c}a\sin B$ $\quad \because \sin(\pi-\theta)=\sin\theta$

$\Rightarrow b\sin A=a\sin B \Rightarrow \frac{a}{\sin A}=\frac{b}{\sin B}$ —— ①

Similarly, $\frac{a}{\sin A}=\frac{c}{\sin C}$ —— ②

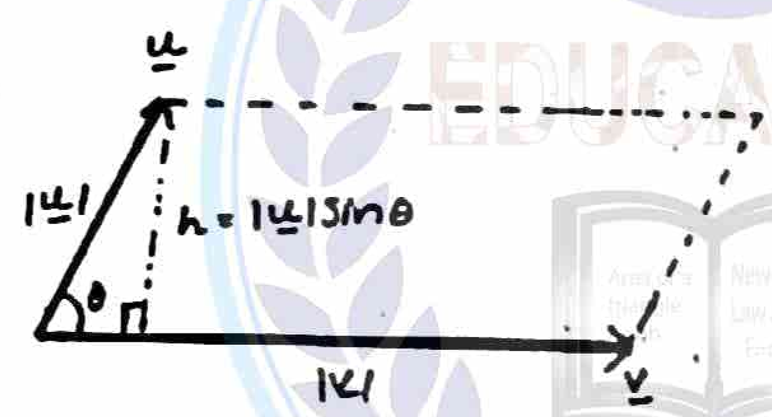
from ① and ②, we have

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

Hence proved.

Area of Parallelogram

If $\underline{u}$ and $\underline{v}$ are two non-zero vectors and θ be the angle b/w $\underline{u}$ and $\underline{v}$, then $|\underline{u}|$ and $|\underline{v}|$ represent the lengths of the adjacent sides of a llgram,



$\because$ Area of llgram = base × height = (base)(h)

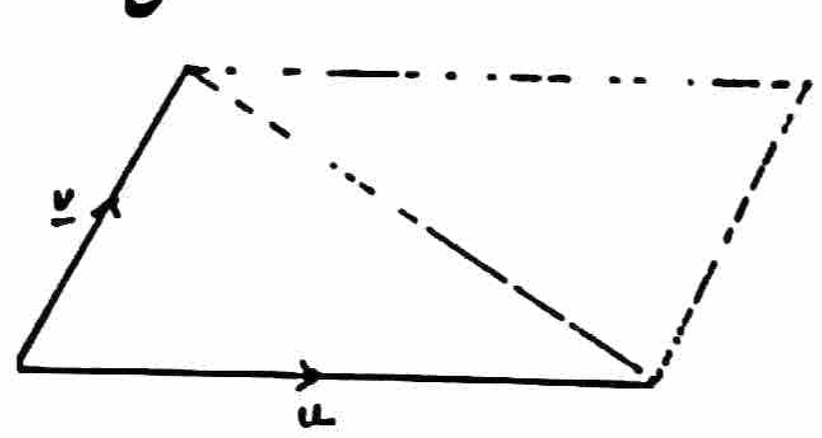
$= |\underline{u}||\underline{v}|\sin\theta$

Thus,

Area of llgram $= |\underline{u}\times\underline{v}|$

Area of Triangle

From fig, it is clear that



Area of triangle

$= \frac{1}{2}$(Area of llgram)

$= \frac{1}{2}|\underline{u}\times\underline{v}|$

$\Rightarrow$ Area of $\Delta = \frac{1}{2}|\underline{u}\times\underline{v}|$ where $\underline{u}$ and $\underline{v}$ are vectors along two adjacent sides of triangle.

Example 5. Find the area of triangle with vertices $A(1,-1,1)$, $B(2,1,-1)$ and $C(-1,1,2)$. Also find a unit vector ⊥ar to the plane ABC.

Solution:- $\overrightarrow{AB}=[2-1,1+1,-1-1]=[1,2,-2]$

$\Rightarrow \overrightarrow{AB}=\underline{i}+2\underline{j}-2\underline{k}$

$\overrightarrow{AC}=[-1-1,1+1,2-1]=[-2,2,1]=-2\underline{i}+2\underline{j}+\underline{k}$

Now, $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 1 & 2 & -2\\ -2 & 2 & 1\end{vmatrix}$

$=\underline{i}(2+4)-\underline{j}(1-4)\underline{j}+\underline{k}(2+4)=6\underline{i}+3\underline{j}+6\underline{k}$

$|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{(6)^2+(3)^2+(6)^2}=\sqrt{36+9+36}=\sqrt{81}$

$=9$

Area of $\Delta ABC=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=9/2$

Also, $\overrightarrow{AB}\times\overrightarrow{AC}$ is ⊥ar to plane ABC. so

unit vector $=\frac{\overrightarrow{AB}\times\overrightarrow{AC}}{|\overrightarrow{AB}\times\overrightarrow{AC}|}=\frac{1}{9}(6\underline{i}+3\underline{j}+6\underline{k})$

$=\frac{3}{9}(2\underline{i}+\underline{j}+2\underline{k})=\frac{1}{3}(2\underline{i}+\underline{j}+2\underline{k})$

Example 6. Find area of parallelogram whose vertices are $P(0,0,0)$, $Q(-1,2,4)$, $R(2,-1,4)$ and $S(1,1,8)$

Solution:- $\overrightarrow{PQ}=[-1-0,2-0,4-0]=[-1,2,4]$

$\overrightarrow{PQ}=-\underline{i}+2\underline{j}+4\underline{k}$

$\overrightarrow{PR}=[2-0,-1-0,4-0]=[2,-1,4]=2\underline{i}-\underline{j}+4\underline{k}$

Now $\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ -1 & 2 & 4\\ 2 & -1 & 4\end{vmatrix}$

$=\underline{i}(8+4)-\underline{j}(-4-8)+\underline{k}(1-4)$

$\overrightarrow{PQ}\times\overrightarrow{PR}=12\underline{i}+12\underline{j}-3\underline{k}$

Area of parallelogram $=|\overrightarrow{PQ}\times\overrightarrow{PR}|$

$=|12\underline{i}+12\underline{j}-3\underline{k}|=\sqrt{(12)^2+(12)^2+(-3)^2}$

$=\sqrt{144+144+9}=\sqrt{297}$

Example 7. If $\underline{u}=2\underline{i}-\underline{j}+\underline{k}$ and $\underline{v}=4\underline{i}+2\underline{j}-\underline{k}$ find by determinant formula

(i) $\underline{u}\times\underline{u}$ (ii) $\underline{u}\times\underline{v}$ (iii) $\underline{v}\times\underline{u}$

Solution:- (i) $\underline{u}\times\underline{u}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & -1 & 1\\ 2 & -1 & 1\end{vmatrix}=0$

($\because$ Two row are identical)

(ii) $\underline{u}\times\underline{v}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & -1 & 1\\ 4 & 2 & -1\end{vmatrix}=\underline{i}(1-2)-\underline{j}(-2-4)+\underline{k}(4+4)$

$=-\underline{i}+6\underline{j}+8\underline{k}$

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(iii) $v \times u = \begin{vmatrix} i & j & k \\ 4 & 2 & 1 \\ 2 & -1 & 1 \end{vmatrix} = (2-1)i - j(4+2) + k(-4-4)$
$= -i - 6j - 8k$

Exercise 7.4

Q1. Compute the cross product $a \times b$ and $b \times a$. check your answer by showing that a and b is perpendicular to $a \times b$ and $b \times a$.

(i) $a = 2i + j - k$, $b = i - j + k$

Solution:- $a \times b = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = i(1-1) - j(2+1) + k(-2-1)$
$= 0i - 3j - 3k$

$\rightarrow a \times b = -3j - 3k$ $\because a \times b = -b \times a$ or $b \times a = -a \times b$

so $b \times a = -(-3j - 3k) = 3j + 3k$

Checking:- $a \cdot (a \times b) = (2i + j - k) \cdot (0i - 3j - 3k)$
$= 2(0) + (1)(-3) + (-1)(-3) = 0 - 3 + 3 = 0$

$\rightarrow$ a is $\perp$ar to $a \times b$.

Now $b \cdot (a \times b) = (i - j + k) \cdot (-3j - 3k)$
$= (1)(0) + (-1)(-3) + (1)(-3) = 3 - 3 = 0$

$\rightarrow$ b is $\perp$ar to $a \times b$.

Now $a \cdot (b \times a) = (2i + j - k) \cdot (3j + 3k)$
$= (2)(0) + (1)(3) + (-1)(+3) = 3 - 3 = 0$

$\rightarrow$ a is $\perp$ar to $b \times a$.

Now $b \cdot (b \times a) = (i - j + k) \cdot (3j + 3k)$
$= (1)(0) + (-1)(3) + (1)(3) = 0 - 3 + 3 = 0$

$\rightarrow$ b is $\perp$ar to $b \times a$.

(ii) $a = i + j$, $b = i - j$

Solution:- $a \times b = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$

$a \times b = i(0-0) - j(0-0) + k(-1-1) = 0i - 0j - 2k$

$\rightarrow a \times b = -2k$ $\because a \times b = -b \times a$ or $b \times a = -a \times b$

$\rightarrow b \times a = -(-2k) = 2k$

Checking:- $a \cdot (a \times b) = (i + j) \cdot (-2k)$
$= (1)(0) + (1)(0) + 0(-2) = 0$

$\rightarrow$ a is $\perp$ar to $a \times b$.

$b \cdot (a \times b) = (i - j) \cdot (-2k) = (1)(0) + (-1)(0) + (0)(-2)$
$= 0$

$\rightarrow$ b is $\perp$ar to $a \times b$.

Now $a \cdot (b \times a) = (i + j) \cdot (2k) = (1)(0) + (1)(0) + (0)(2)$
$= 0$

$\rightarrow$ a is $\perp$ar to $b \times a$

$b \cdot (b \times a) = (i - j) \cdot (2k) = (1)(0) + (-1)(0) + (0)(2) = 0$

$\rightarrow$ b is $\perp$ar to $b \times a$.

(iii) $a = 3i - 2j + k$, $b = i + j$

Solution:- $a \times b = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$= i(0-1) - j(0-1) + k(3+2) = -i + j + 5k$

$\rightarrow a \times b = -i + j + 5k$ $\because a \times b = -b \times a$ or $b \times a = -a \times b$

$\rightarrow b \times a = -(-i + j + 5k) = i - j - 5k$

Checking:- $a \cdot (a \times b) = (3i - 2j + k) \cdot (i - j - 5k)$
$= (3)(1) + (-2)(-1) + ()(-5) = 3 + 2 - 5 = 0$

$\rightarrow$ a is $\perp$ar to $a \times b$.

$b \cdot (a \times b) = (i + j) \cdot (-i + j + 5k) = (1)(-1) + (1)(1) + 0(5)$
$= -1 + 1 + 0 = 0$

$\rightarrow$ b is $\perp$ar to $a \times b$.

Now $a \cdot (b \times a) = (3i - 2j + k) \cdot (i - j - 5k)$
$= (3)(1) + (-2)(-1) + (1)(-5)$
$= 3 + 2 - 5 = 0$

$\rightarrow$ a is $\perp$ar to $b \times a$.

$b \cdot (b \times a) = (i + j) \cdot (i - j - 5k)$
$= (1)(1) + (1)(-1) + (0)(-5) = 1 - 1 - 0 = 0$

$\rightarrow$ b is $\perp$ar to $b \times a$.

(iv) $a = -4i + j - 2k$, $b = 2i + j + k$

Solution:- $a \times b = \begin{vmatrix} i & j & k \\ -4 & 1 & -2 \\ 2 & 1 & 1 \end{vmatrix}$

$= i(1+2) - j(-4+4) + k(-4-2) = 3i - 0j - 6k$

$\rightarrow a \times b = 3i - 0j - 6k$ $\because a \times b = -b \times a$ or $b \times a = -a \times b$

so $b \times a = -(3i - 0j - 6k) = -3i + 0j + 6k$

Checking:- $a \cdot (a \times b) = (-4i + j - 2k) \cdot (-3i + 0j + 6k)$
$= (-4)(-3) + (1)(0) + (-2)(6) = 12 + 0 - 12 = 0$

$\rightarrow$ a is $\perp$ar to $a \times b$.

$b \cdot (a \times b) = (2i + j + k) \cdot (3i - 0j - 6k)$
$= (2)(3) + (1)(-0) + (1)(-6) = 6 - 0 - 6 = 0$

$\rightarrow$ b is $\perp$ar to $a \times b$.

Now $a \cdot (b \times a) = (-4i + j - 2k) \cdot (3i + 0j + 6k)$
$= (-4)(-3) + (1)(0) + (-2)(6) = 12 + 0 - 12 = 0$

$\rightarrow$ a is $\perp$ar to $b \times a$.

$b \cdot (b \times a) = (2i + j + k) \cdot (-3i + 0j + 6k)$
$= (2)(-3) + (1)(0) + (1)(6) = -6 + 0 + 6 = 0$

$\rightarrow$ b is $\perp$ar to $b \times a$.

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Q2. Find a unit vector perpendicular to the plane containing $\underline{a}$ and $\underline{b}$. Also find sine of the angle b/w them.

Solution:- (i) $\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k}$, $\underline{b} = 4\underline{i} + 3\underline{j} - \underline{k}$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = \underline{i}(6+9) - \underline{j}(-2+12) + \underline{k}(6+24) = 15\underline{i} - 10\underline{j} + 30\underline{k}$$

$$|\underline{a}\times\underline{b}| = \sqrt{(15)^2+(-10)^2+(30)^2} = \sqrt{225+100+900} = \sqrt{1225} = 35$$

Req. unit vector $= \dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|} = \dfrac{1}{35}(15\underline{i} - 10\underline{j} + 30\underline{k})$

Now $|\underline{a}| = \sqrt{(2)^2+(-6)^2+(-3)^2} = \sqrt{4+36+9} = \sqrt{49} = 7$

$|\underline{b}| = \sqrt{(4)^2+(3)^2+(-1)^2} = \sqrt{16+9+1} = \sqrt{26}$

$\because \sin\theta = \dfrac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|} = \dfrac{35}{7\sqrt{26}} = \dfrac{5}{\sqrt{26}}$

(ii) $\underline{a} = -\underline{i} - \underline{j} - \underline{k}$, $\underline{b} = 2\underline{i} - 3\underline{j} + 4\underline{k}$

Solution:- $\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix}$

$= \underline{i}(-4-3) - \underline{j}(-4+2) + \underline{k}(3+2)$

$= -7\underline{i} + 2\underline{j} + 5\underline{k}$, $|\underline{a}\times\underline{b}| = \sqrt{(-7)^2+(2)^2+(5)^2} = \sqrt{49+4+25} = \sqrt{78}$

Req. unit vector $= \dfrac{-7\underline{i}+2\underline{j}+5\underline{k}}{\sqrt{78}}$

Now $|\underline{a}| = \sqrt{(-1)^2+(-1)^2+(-1)^2} = \sqrt{3}$

$|\underline{b}| = \sqrt{(2)^2+(-3)^2+(4)^2} = \sqrt{4+9+16} = \sqrt{29}$

$\because \sin\theta = \dfrac{|\underline{a}\times\underline{b}|}{|\underline{a}||\underline{b}|} = \dfrac{\sqrt{78}}{\sqrt{3}\sqrt{29}} = \sqrt{\dfrac{26}{29}}$

(iii) $\underline{a} = 2\underline{i} - 2\underline{j} + 4\underline{k}$, $\underline{b} = -\underline{i} + \underline{j} - 2\underline{k}$

Solution:- $\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix}$

$= \underline{i}(4-4) - \underline{j}(-4+4) + \underline{k}(2-2) = 0\underline{i} - 0\underline{j} + 0\underline{k} = 0$

$\rightarrow \underline{a}\times\underline{b} = 0 =$ Null vector

$\because$ zero vector has no specific direction so any vector is required unit vector.

$\because \underline{a}\times\underline{b} = 0$ It means $\underline{a}$ and $\underline{b}$ are parallel, so $\theta = 0 \Rightarrow \sin\theta = \sin(0) = 0$

(iv) $\underline{a} = \underline{i} + \underline{j}$, $\underline{b} = \underline{i} - \underline{j}$

Solution:- $\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$

$\underline{a}\times\underline{b} = \underline{k}(-1-1) = -2\underline{k}$ (By expanding C_3)

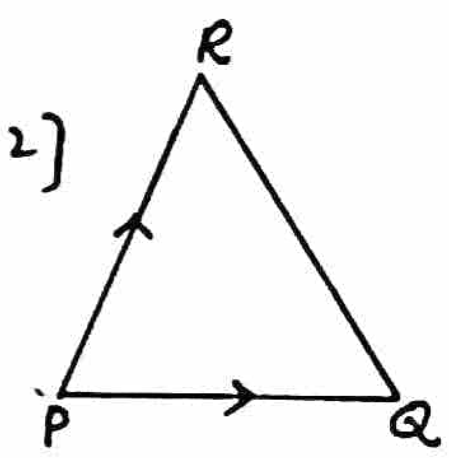
$|\underline{a}\times\underline{b}| = \sqrt{(-2)^2} = \sqrt{4} = 2$

Req. unit vector $= \dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|} = \dfrac{-2\underline{k}}{2} = -\underline{k}$

$\because \sin\theta = \dfrac{|\underline{a}\times\underline{b}|}{|\underline{a}|\times|\underline{b}|} = \dfrac{2}{\sqrt{2}\sqrt{2}} = \dfrac{2}{2} = 1$; $\because |\underline{a}| = \sqrt{(1)^2+(1)^2} = \sqrt{2}$, $|\underline{b}| = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$

Q3. Find the area of the triangle, determined by the point P, Q and R (i) P(0,0,0); Q(2,3,2); R(−1,1,4)

Solution:-



$\overrightarrow{PQ} = [2-0, 3-0, 2-0] = [2,3,2]$

$\overrightarrow{PQ} = 2\underline{i} + 3\underline{j} + 2\underline{k}$

$\overrightarrow{PR} = [-1-0, 1-0, 4-0] = [-1,1,4]$

$\overrightarrow{PR} = -\underline{i} + \underline{j} + 4\underline{k}$

Now $\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 2 \\ -1 & 1 & 4 \end{vmatrix}$

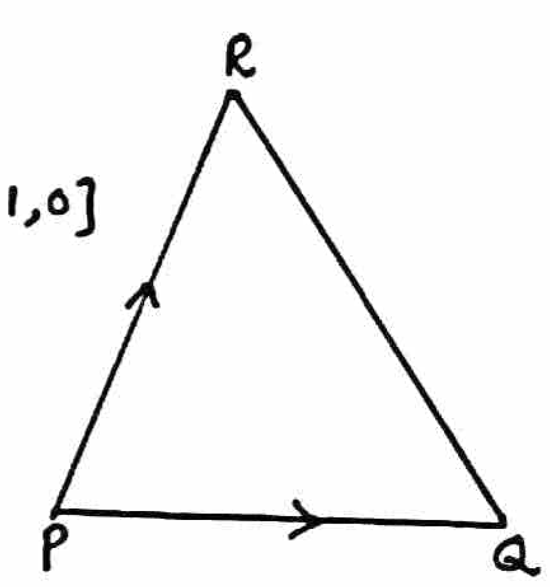
$= \underline{i}(12-2) - \underline{j}(8+2) + \underline{k}(2+3) = 10\underline{i} - 10\underline{j} + 5\underline{k}$

$|\overrightarrow{PQ}\times\overrightarrow{PR}| = \sqrt{(10)^2+(-10)^2+(5)^2} = \sqrt{100+100+25} = \sqrt{225} = 15.$

$\because$ Area of $\Delta PQR = \frac{1}{2}|\overrightarrow{PQ}\times\overrightarrow{PR}| = \frac{15}{2}$ sq. units

(ii) P(1,−1,−1), Q(2,0,−1), R(0,2,1)

Solution:-



$\overrightarrow{PQ} = [2-1, 0+1, -1+1] = [1,1,0]$

$\overrightarrow{PQ} = \underline{i} + \underline{j} + 0\underline{k}$

$\overrightarrow{PR} = [0-1, 2+1, 1+1] = [-1,3,2]$

$\overrightarrow{PR} = -\underline{i} + 3\underline{j} + 2\underline{k}$

$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ -1 & 3 & 2 \end{vmatrix} = \underline{i}(2-0) - \underline{j}(2-0) + \underline{k}(3+1) = 2\underline{i} - 2\underline{j} + 4\underline{k}$

$|\overrightarrow{PQ}\times\overrightarrow{PR}| = \sqrt{(2)^2+(-2)^2+(4)^2} = \sqrt{4+4+16} = \sqrt{24} = \sqrt{4\times6} = 2\sqrt{6}$

$\because$ Area of $\Delta PQR = \frac{1}{2}|\overrightarrow{PQ}\times\overrightarrow{PR}| = \frac{2\sqrt{6}}{2} = \sqrt{6}$ sq. units

Q4. Find the area of parallelogram, whose vertices are:

(i) A(0,0,0), B(1,2,3), C(2,−1,1), D(3,1,4)

Solution:- $\overrightarrow{AB} = [1-0, 2-0, 3-0] = [1,2,3]$

$\rightarrow \overrightarrow{AB} = \underline{i} + 2\underline{j} + 3\underline{k}$

$\overrightarrow{AC} = [2-0, -1-0, 1-0] = [2,-1,1] = 2\underline{i} - \underline{j} + \underline{k}$

$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} = \underline{i}(2+3) - \underline{j}(1-6) + \underline{k}(-1-4) = 5\underline{i} + 5\underline{j} - 5\underline{k}$

$|\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{(5)^2+(5)^2+(-5)^2} = \sqrt{25+25+25} = \sqrt{75} = \sqrt{25\times3} = 5\sqrt{3}$

$\rightarrow$ Area of $||gram_{ABCD} = 5\sqrt{3}$ sq. units

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(ii) $A(1,2,-1)$; $B(4,2,-3)$; $C(6,-5,2)$; $D(9,-5,0)$

Solution:- $\overrightarrow{AB} = [4-1,\ 2-2,\ -3+1] = [3,\ 0,\ -2]$

$\rightarrow \overrightarrow{AB} = 3i + 0j - 2k$

$\overrightarrow{AC} = [6-1,\ -5-2,\ 2+1] = [5,\ -7,\ 3] = 5i - 7j + 3k$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 3 & 0 & -2 \\ 5 & -7 & 3 \end{vmatrix} = i(0-14) - j(9+10) + k(-21-0) = -14i - 19j - 21k$$

$$|\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{(-14)^2+(-19)^2+(-21)^2} = \sqrt{196+361+441} = \sqrt{998}$$

$\rightarrow$ Area of llgram $_{ABCD}$ $= \sqrt{998}$ sq. units

(iii) $A(-1,1,1)$; $B(-1,2,2)$; $C(-3,4,-5)$; $D(-3,5,-4)$

Solution:- $\overrightarrow{AB} = [-1+1,\ 2-1,\ 2-1] = [0,\ 1,\ 1]$

$\rightarrow \overrightarrow{AB} = 0i + j + k$

$\overrightarrow{AC} = [-3+1,\ 4-1,\ -5-1] = [-2,\ 3,\ -6]$

$\rightarrow \overrightarrow{AC} = -2i + 3j - 6k$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{vmatrix} = i(-6-3) - j(0+2) + k(0+2) = -9i - 2j + 2k$$

$$|\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{(-9)^2+(-2)^2+(2)^2} = \sqrt{81+4+4} = \sqrt{89}$$

$\rightarrow$ Area of llgram $_{ABCD}$ $= \sqrt{89}$ sq. units

Q5. Which vectors, if any, are perpendicular or parallel

(i) $u = 5i - j + k$; $v = j - 5k$; $w = -15i + 3j - 3k$

Solution:- $u\cdot v = (5i - j + k)\cdot(j - 5k)$

$\rightarrow u\cdot v = (5)(0) + (-1)(1) + (1)(-5) = 0 - 1 - 5 = -6 \neq 0$

$\rightarrow$ u and v are not $\perp$r.

$v\cdot w = (j - 5k)\cdot(-15i + 3j - 3k)$

$= (0)(-15) + (1)(3) + (-5)(-3) = 0 + 3 + 15 = 18 \neq 0$

$\rightarrow$ v and w are not $\perp$r.

$u\cdot w = (5i - j + k)\cdot(-15i + 3j - 3k)$

$= (5)(-15) + (-1)(3) + (1)(-3)$

$= -75 - 3 - 3 = -81 \neq 0$

$\rightarrow$ u and w are not $\perp$r.

Now $w = -15i + 3j - 3k = -3(5i - j + k)$

$w = -3u$ $\rightarrow$ u and w are parallel.

(ii) $u = i + 2j - k$; $v = -i + j + k$; $w = -\frac{\pi}{2}i - \pi j + \frac{\pi}{2}k$

Solution:- $w = -\frac{\pi}{2}i - \pi j + \frac{\pi}{2}k = \frac{-\pi i - 2\pi j + \pi k}{2}$

$\rightarrow w = -\frac{\pi}{2}(i + 2j - k) = -\frac{\pi}{2}u$

$\rightarrow$ u and w are parallel.

Now $u\cdot v = (i + 2j - k)\cdot(-i + j + k)$

$= (1)(-1) + (2)(1) + (-1)(1) = -1 + 2 - 1 = 0$

$\rightarrow$ u and v are $\perp$r.

$v\cdot w = (-i + j + k)\cdot(-\frac{\pi}{2}i - \pi j + \frac{\pi}{2}k)$

$= (-1)(-\frac{\pi}{2}) + (1)(-\pi) + (1)(\frac{\pi}{2}) = \frac{\pi}{2} - \pi + \frac{\pi}{2}$

$= 2\frac{\pi}{2} - \pi = \pi - \pi = 0$

$\rightarrow$ v and w are $\perp$r.

For u and w no need to check because u and w are parallel.

Q6. Prove that: $a\times(b+c) + b\times(c+a) + c\times(a+b) = 0$

Solution:-

L.H.S $= a\times(b+c) + b\times(c+a) + c\times(a+b)$

$= a\times b + a\times c + b\times c + b\times a + c\times a + c\times b$

$= a\times b + a\times c + b\times c + (-a\times b) + (-a\times c) + (-b\times c)$

$= a\times b + a\times c + b\times c - a\times b - a\times c - b\times c$

$= 0$ $\quad$ ($\because b\times a = -a\times b$, $c\times a = -a\times c$, $c\times b = -b\times c$)

$=$ R.H.S

Hence proved.

Q7. If $a + b + c = 0$ then prove that $a\times b = b\times c = c\times a$

Proof:- $\because a + b + c = 0 \longrightarrow (1)$

Taking cross product with a of eq. (1)

$a\times(a + b + c) = a\times 0$

$\rightarrow a\times a + a\times b + a\times c = 0$

$\rightarrow 0 + a\times b + a\times c = 0$ $\quad \because a\times a = 0$

$\rightarrow a\times b = -a\times c$

$\rightarrow a\times b = c\times a$ —— (2) $\quad \because c\times a = -a\times c$

Now Taking cross product with b of eq(1)

$\rightarrow b\times(a + b + c) = b\times 0$

$\rightarrow b\times a + b\times b + b\times c = 0$

$\rightarrow b\times a + 0 + b\times c = 0$ $\quad \because b\times b = 0$

$\rightarrow b\times a = -b\times c$

$\rightarrow -a\times b = -b\times c$ $\quad \because -a\times b = b\times a$

$\rightarrow a\times b = b\times c$ —— (3)

By (2) and (3), we have

$a\times b = b\times c = c\times a$ Hence proved.

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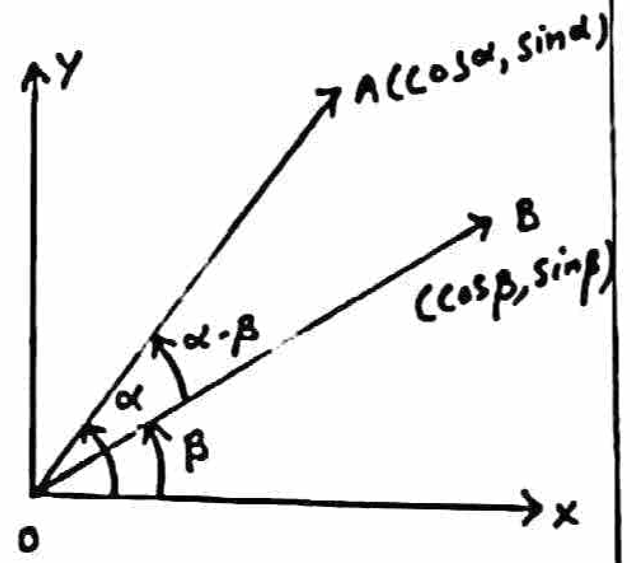
Q8. Prove that $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

Solution:-

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be unit vector in xy-plane s.that



$\angle XOA=\alpha, \angle XOB=\beta$

$\angle BOA=\alpha-\beta$ and

$|\overrightarrow{OA}|=|\overrightarrow{OB}|=1$ Also

$\overrightarrow{OA}=\cos\alpha\,\underline{i}+\sin\alpha\,\underline{j}$

$\overrightarrow{OB}=\cos\beta\,\underline{i}+\sin\beta\,\underline{j}$ Now

$$\overrightarrow{OB}\times\overrightarrow{OA}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \cos\beta & \sin\beta & 0\\ \cos\alpha & \sin\alpha & 0\end{vmatrix}$$ Expanding by c_3

$\rightarrow |\overrightarrow{OB}||\overrightarrow{OA}|\sin(\alpha-\beta)\,\underline{k}=\underline{k}\,(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$

$\rightarrow (1)(1)\sin(\alpha-\beta)\,\underline{k}=\underline{k}\,(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$

$\rightarrow \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

Hence proved.

Q9. If $\underline{a}\times\underline{b}=0$ and $\underline{a}\cdot\underline{b}=0$; what conclusion can be drawn about $\underline{a}$ and $\underline{b}$?

Solution:- $\because \underline{a}\times\underline{b}=0$

$\rightarrow |\underline{a}||\underline{b}|\sin\theta=0 \rightarrow \sin\theta=0$

$\rightarrow \theta=\sin^{-1}(0)=0$

$\rightarrow \theta=0,\pi$ so $\underline{a}$ and $\underline{b}$ are parallel.

and $\underline{a}\cdot\underline{b}=0$

$\rightarrow |\underline{a}||\underline{b}|\cos\theta=0 \rightarrow \cos\theta=0$

$\rightarrow \theta=\cos^{-1}(0)=90^\circ \rightarrow \theta=90^\circ$

so $\underline{a}$ and $\underline{b}$ are perpendicular.

At the same time parallel and $\perp$ar not possible so one vector should be zero or null.

Triple product of vectors

There are two types of triple product of vectors:

(a) **Scalar triple product:-**

$(\underline{u}\times\underline{v})\cdot\underline{w}$ or $\underline{u}\cdot(\underline{v}\times\underline{w})$

(b) **Vector triple Product:-**

$\underline{u}\times(\underline{v}\times\underline{w})$

In this section we shall study the scalar triple product only.

Definition:- Let $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$, $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$ and $\underline{w}=a_3\underline{i}+b_3\underline{j}+c_3\underline{k}$ be three vectors, the scalar triple product of vector $\underline{u},\underline{v}$ and $\underline{w}$ is defined by $\underline{u}\cdot(\underline{v}\times\underline{w})$ or $\underline{v}\cdot(\underline{w}\times\underline{u})$ or $\underline{w}\cdot(\underline{u}\times\underline{v})$.

The scalar triple product $\underline{u}\cdot(\underline{v}\times\underline{w})$ is written as; $\underline{u}\cdot(\underline{v}\times\underline{w})=[\underline{u}\ \underline{v}\ \underline{w}]$

Analytical Expression of $\underline{u}\cdot(\underline{v}\times\underline{w})$

Let $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$, $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$ and $\underline{w}=a_3\underline{i}+b_3\underline{j}+c_3\underline{k}$ so

$$\underline{v}\times\underline{w}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$$

$$=\underline{i}(b_2c_3-c_2b_3)-\underline{j}(a_2c_3-a_3c_2)+(a_2b_3-a_3b_2)\underline{k}$$

$$\therefore \underline{u}\cdot(\underline{v}\times\underline{w})=a_1(b_2c_3-c_2b_3)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)$$

$$\rightarrow \underline{u}\cdot(\underline{v}\times\underline{w})=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$$ which is called determinant formula for scalar triple product of $\underline{u},\underline{v}$ and $\underline{w}$.

Prove that

$\underline{u}\cdot(\underline{v}\times\underline{w})=\underline{v}\cdot(\underline{w}\times\underline{u})=\underline{w}\cdot(\underline{u}\times\underline{v})$

Proof:- We know that for

Let $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}$, $\underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}$

and $\underline{w}=a_3\underline{i}+b_3\underline{j}+c_3\underline{k}$

$$\underline{u}\cdot(\underline{v}\times\underline{w})=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$$

$$=-\begin{vmatrix}a_2 & b_2 & c_2\\ a_1 & b_1 & c_1\\ a_3 & b_3 & c_3\end{vmatrix}$$ Interchanging R_1 and R_2

$$=\begin{vmatrix}a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\\ a_1 & b_1 & c_1\end{vmatrix}$$ Interchanging R_2 and R_3

$$\therefore \underline{u}\cdot(\underline{v}\times\underline{w})=\underline{v}\cdot(\underline{w}\times\underline{u})$$

Now

$$\underline{v}\cdot(\underline{w}\times\underline{u})=\begin{vmatrix}a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\\ a_1 & b_1 & c_1\end{vmatrix}$$

$$=-\begin{vmatrix}a_3 & b_3 & c_3\\ a_2 & b_2 & c_2\\ a_1 & b_1 & c_1\end{vmatrix}$$ Interchanging R_1 and R_2

$$=\begin{vmatrix}a_3 & b_3 & c_3\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}$$ Interchanging R_2 and R_3

$$\therefore \underline{v}\cdot(\underline{w}\times\underline{u})=\underline{w}\cdot(\underline{u}\times\underline{v})$$

Hence $\underline{u}\cdot(\underline{v}\times\underline{w})=\underline{v}\cdot(\underline{w}\times\underline{u})=\underline{w}\cdot(\underline{u}\times\underline{v})$

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Note:- (i) Dot and cross may be interchanged without altering the value

i.e., $(\underline{u}\times\underline{v})\cdot\underline{w} = \underline{u}\cdot(\underline{v}\times\underline{w}) = [\underline{u}\ \underline{v}\ \underline{w}]$

$(\underline{v}\times\underline{w})\cdot\underline{u} = \underline{v}\cdot(\underline{w}\times\underline{u}) = [\underline{v}\ \underline{w}\ \underline{u}]$

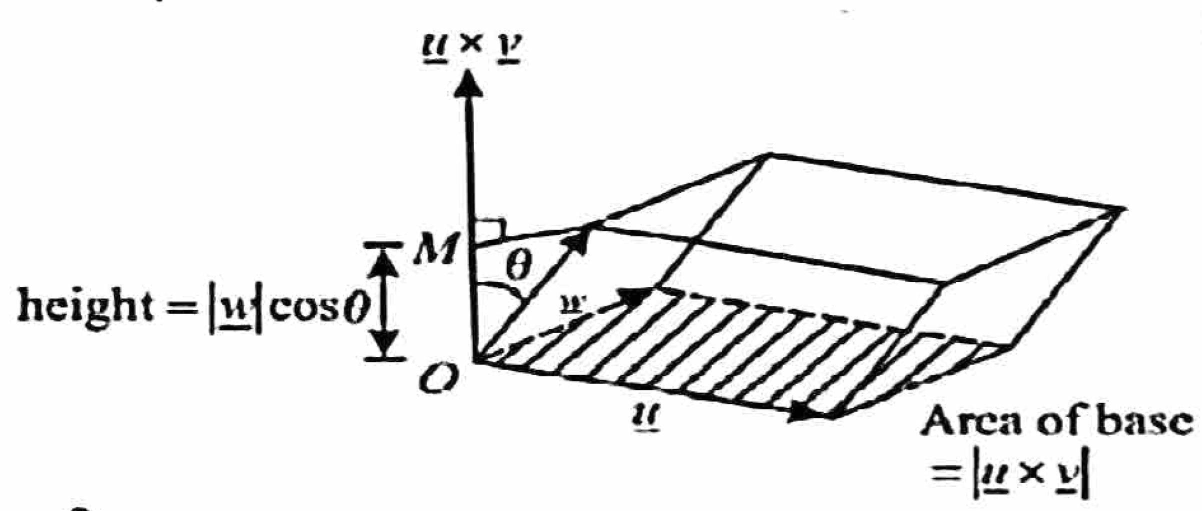
$(\underline{w}\times\underline{u})\cdot\underline{v} = \underline{w}\cdot(\underline{u}\times\underline{v}) = [\underline{w}\ \underline{u}\ \underline{v}]$

(ii) The value of the product changes if the order is non-cyclic.

(iii) $\underline{u}\cdot\underline{v}\cdot\underline{w}$ and $\underline{u}\times(\underline{v}\cdot\underline{w})$ are meaningless.

The volume of Parallelepiped

consider $\overrightarrow{OA} = \underline{u}$, $\overrightarrow{OB} = \underline{v}$ and $\overrightarrow{OC} = \underline{w}$ be the adjacent edges of parallelepiped OAFCDEGB.



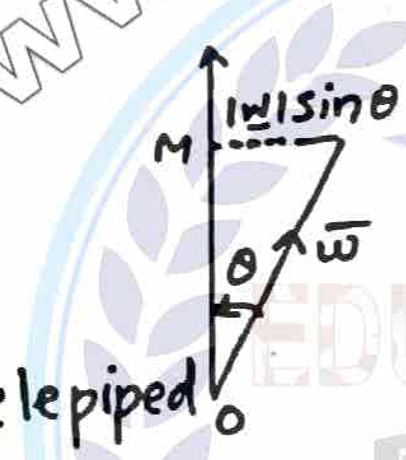
Let θ be the angle b/w $\underline{w}$ and $\underline{u}\times\underline{v}$

As $|\underline{u}\times\underline{v}|$ = area of llgram OAGB

= area of base of parallelepiped

Resolve $\underline{w}$ in components

In $\triangle COM$, $\cos\theta = \dfrac{|\overrightarrow{OM}|}{|\overrightarrow{OC}|}$



$\Rightarrow |\overrightarrow{OM}| = |\overrightarrow{OC}|\cos\theta$

$|\overrightarrow{OM}| = w\cos\theta$

As $|\overrightarrow{OM}|$ = height of parallelepiped

As volume of parallelepiped = (Area of base)(height)

$= |\underline{u}\times\underline{v}|\cdot|\underline{w}|\cos\theta$

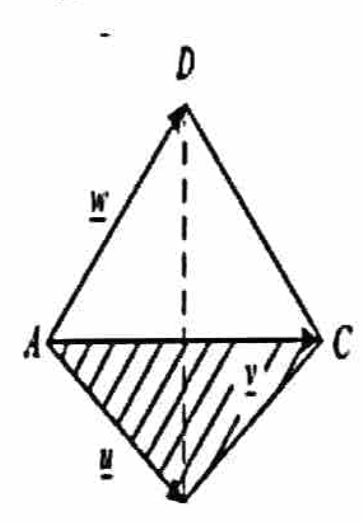
$\Rightarrow$ volume of parallelepiped $= \underline{w}\cdot(\underline{u}\times\underline{v})$

$\because \underline{w}\cdot(\underline{u}\times\underline{v}) = \underline{u}\cdot(\underline{v}\times\underline{w})$ $\left(\because \underline{w}\cdot(\underline{u}\times\underline{v}) = |\underline{w}||\underline{u}\times\underline{v}|\cos\theta\right)$

so volume of parallelepiped $= \underline{u}\cdot(\underline{v}\times\underline{w})$.

The volume of Tetrahedron

Volume of tetrahedron ABCD



$= \frac{1}{3}(\triangle ABC)$(heigh of D above the Place ABC)

$= \frac{1}{3}\cdot\frac{1}{2}|\underline{u}\times\underline{v}|(h)$

$= \frac{1}{6}$(Area of llgram with AB and AC as adjacent sides)(h)

$= \frac{1}{6}$(volume of parallelepiped with $\underline{u},\underline{v},\underline{w}$ edges)

Thus,

volume $= \frac{1}{6}(\underline{u}\times\underline{v})\cdot\underline{w} = \frac{1}{6}[\underline{u}\ \underline{v}\ \underline{w}]$

Properties of Scalar triple product:-

1. If $\underline{u},\underline{v}$ and $\underline{w}$ are coplaner, then the volume of parallelepiped so formed is zero i.e., the vectors $\underline{u},\underline{v},\underline{w}$ are coplaner $\Leftrightarrow (\underline{u}\times\underline{v})\cdot\underline{w} = 0$

2. If any two vectors of scalar triple product are equal, then its value is zero i.e., $[\underline{u}\ \underline{u}\ \underline{w}] = [\underline{u}\ \underline{v}\ \underline{v}] = 0$

Example 1. Find the volume of parallelepiped determined by $\underline{u} = \underline{i}+2\underline{j}-\underline{k}$, $\underline{v} = \underline{i}-2\underline{j}+3\underline{k}$, $\underline{w} = \underline{i}-7\underline{j}-4\underline{k}$

Solution:-

Volume of parallelepiped $= \underline{u}\cdot\underline{v}\times\underline{w}$

$= \begin{vmatrix} 1 & 2 & -1 \\ 1 & -2 & 3 \\ 1 & -7 & -4 \end{vmatrix}$

$= 1(8+21) - 2(-4-3) - 1(-7+2) = 29+14+5$

$\Rightarrow$ volume $= 48$

Example 2. Prove that four points $A(-3,5,-4)$, $B(-1,1,1)$, $C(-1,2,2)$ and $D(-3,4,-5)$ are coplaner.

Solution:- $\overrightarrow{AB} = [-1+3, 1-5, 1+4] = [2,-4,5]$

$\Rightarrow \overrightarrow{AB} = 2\underline{i} - 4\underline{j} + 5\underline{k}$

$\overrightarrow{AC} = [-1+3, 2-5, 2+4] = [2,-3,6] = 2\underline{i}-3\underline{j}+6\underline{k}$

$\overrightarrow{AD} = [-3+3, 4-5, -5+4] = [0,-1,-1] = 0\underline{i}-\underline{j}-\underline{k}$

Now,

$\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD}) = \begin{vmatrix} 2 & -4 & 5 \\ 2 & -3 & 6 \\ 0 & -1 & -1 \end{vmatrix}$

$= 2(3+6) + 4(-2-0) + 5(-2-0)$

$= 18 - 8 - 10 = 0$

Hence A, B, C and D are coplaner.

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Example 3. Find the volume of the tetrahedron whose vertices are $A(2,1,8)$, $B(3,2,9)$, $C(2,1,4)$ and $D(3,3,0)$

Solution:– $\overrightarrow{AB} = [3-2, 2-1, 9-8] = [1,1,1] = \underline{i}+\underline{j}+\underline{k}$

$\overrightarrow{AC} = [2-2, 1-1, 4-8] = [0,0,-4] = 0\underline{i}+0\underline{j}-4\underline{k}$

$\overrightarrow{AD} = [3-2, 3-1, 0-8] = [1,2,-8] = \underline{i}+2\underline{j}-8\underline{k}$

$\therefore$ volume of tetrahedron $= \frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]$

$$= \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & -8 \end{vmatrix}$$

$$= \frac{1}{6}\left\{-0+0-(-4)\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\right\} = \frac{1}{6}\left(4(2-1)\right) = \frac{4}{6}$$

$\rightarrow$ volume of tetrahedron $= \frac{2}{3}$

Example 4. Find the value of α, so that $\alpha\underline{i}+\underline{j}$, $\underline{i}+\underline{j}+3\underline{k}$ and $2\underline{i}+\underline{j}-2\underline{k}$ are coplaner.

Solution:– Let $\underline{u} = \alpha\underline{i}+\underline{j}$, $\underline{v} = \underline{i}+\underline{j}+3\underline{k}$ and $\underline{w} = 2\underline{i}+\underline{j}-2\underline{k}$ so

$$\underline{u}\cdot(\underline{v}\times\underline{w}) = 0 \rightarrow \begin{vmatrix} \alpha & 1 & 0 \\ 1 & 1 & 3 \\ 2 & 1 & -2 \end{vmatrix} = 0$$

$\rightarrow \alpha(-2-3) - 1(-2-6) = 0 \rightarrow -5\alpha + 8 = 0$

$\rightarrow -5\alpha = -8 \rightarrow \alpha = 8/5$

Example 5. Prove that the points whose position vectors are $A(-6\underline{i}+3\underline{j}+2\underline{k})$, $B(3\underline{i}-2\underline{j}+4\underline{k})$, $C(5\underline{i}+7\underline{j}+3\underline{k})$, $D(-13\underline{i}+17\underline{j}-\underline{k})$ are coplaner.

Solution:– $\overrightarrow{AB} = (3\underline{i}-2\underline{j}+4\underline{k}) - (-6\underline{i}+3\underline{j}+2\underline{k})$

$= 3\underline{i}-2\underline{j}+4\underline{k}+6\underline{i}-3\underline{j}-2\underline{k} = 9\underline{i}-5\underline{j}+2\underline{k}$

$\overrightarrow{AC} = (5\underline{i}+7\underline{j}+3\underline{k}) - (-6\underline{i}+3\underline{j}+2\underline{k})$

$= 5\underline{i}+7\underline{j}+3\underline{k}+6\underline{i}-3\underline{j}-2\underline{k}$

$\overrightarrow{AC} = 11\underline{i}+4\underline{j}+\underline{k}$

$\overrightarrow{AD} = (-13\underline{i}+17\underline{j}-\underline{k}) - (-6\underline{i}+3\underline{j}+2\underline{k})$

$= -13\underline{i}+17\underline{j}-\underline{k}+6\underline{i}-3\underline{j}-2\underline{k}$

$= -7\underline{i}+14\underline{j}-3\underline{k}$

Now

$$\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD}) = \begin{vmatrix} 9 & -5 & 2 \\ 11 & 4 & 1 \\ -7 & 14 & -3 \end{vmatrix}$$

$= 9(-12-14) + 5(-33+7) + 2(154+28)$

$= -234 - 130 + 364 = 0$

Thus A, B, C and D are coplaner.

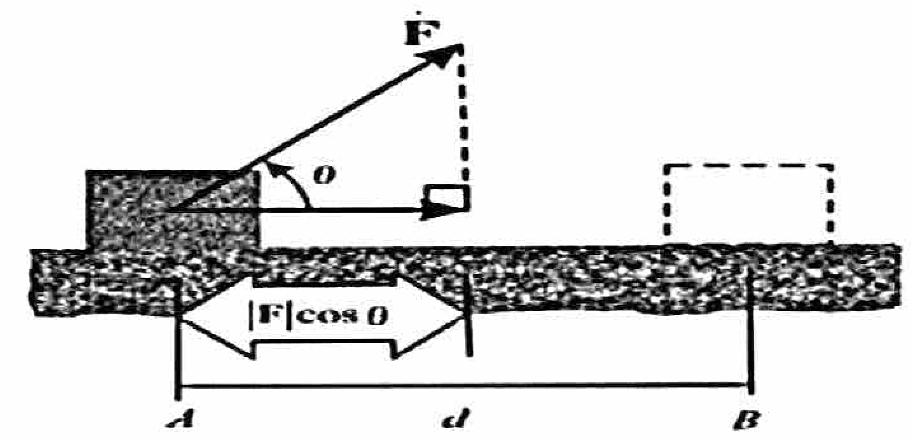
Application of vectors in Physics and Engineering

(a) Work done:–

If d is displacement (from A to B) when force $\overrightarrow{F}$ is applied on the particle then,

Work done = (Force).(Displacement)

$\rightarrow \underline{W} = \overrightarrow{F}\cdot\overrightarrow{AB} = \underline{F}\cdot\underline{d} \qquad \because \underline{d} = \overrightarrow{AB}$



Example 6. Find the work done by a constant force $\underline{F} = 2\underline{i}+4\underline{j}$, if its points of application to a body moves it from $A(1,1)$ to $B(4,6)$

Solution:– Force $= \underline{F} = 2\underline{i}+4\underline{j}$

Displacement $= \underline{d} = \overrightarrow{AB} = [4-1, 6-1] = (3,5)$

$\rightarrow \underline{d} = 3\underline{i}+5\underline{j}$ so

Work done $= \underline{F}\cdot\underline{d} = (2\underline{i}+4\underline{j})\cdot(3\underline{i}+5\underline{j})$

$= (2)(3)+(4)(5) = 6+20 = 26$ nt.m

Example 7. The constant forces $2\underline{i}+5\underline{j}+6\underline{k}$ and $-\underline{i}+2\underline{j}+\underline{k}$ act on a body which is displaced from $P(4,-3,2)$ to $Q(6,1,-3)$. Find the total work done.

Solution:– Let $\underline{F_1} = 2\underline{i}+5\underline{j}+6\underline{k}$

$\underline{F_2} = -\underline{i}+2\underline{j}+\underline{k}$, Total force $= \underline{F} = \underline{F_1}+\underline{F_2}$

$\rightarrow \underline{F} = 2\underline{i}+5\underline{j}+6\underline{k}-\underline{i}-2\underline{j}-\underline{k} = \underline{i}+3\underline{j}+5\underline{k}$

Displacement $= \underline{d} = \overrightarrow{PQ} = [6-4, 1+3, -3-2]$

$\underline{d} = [2,4,-5] = 2\underline{i}+4\underline{j}-\underline{k}$

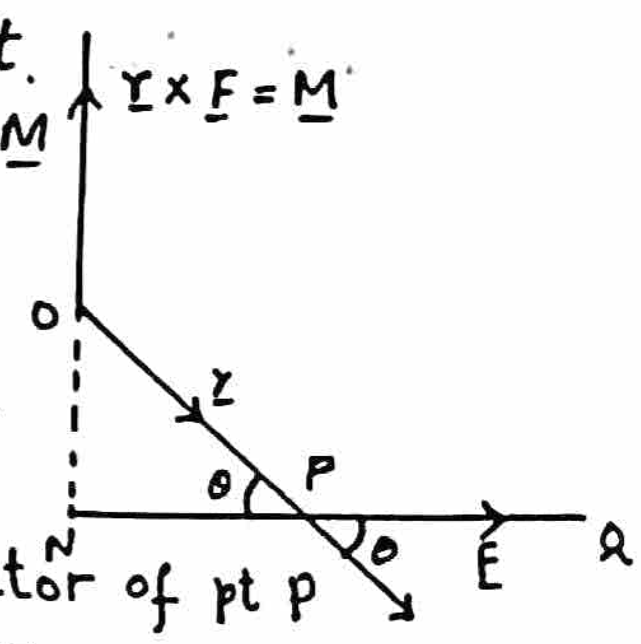
Now work done $= \underline{F}\cdot\underline{d} = (\underline{i}+3\underline{j}+5\underline{k})\cdot(2\underline{i}+4\underline{j}-\underline{k})$

$= (1)(2)+(3)(4)+(5)(-1)$

$= 2+12-5 = 9$ nt.m

(b) Moment of Force:–

Let $\underline{F}$ be the force at point P, the moment $\underline{M}$ of a force $\underline{F}$ about a point O is equal to vector product of $\underline{r}$ and $\underline{F}$.



Here $\underline{r}$ is position vector of pt P from O. i.e., $\underline{M} = \underline{r}\times\underline{F}$

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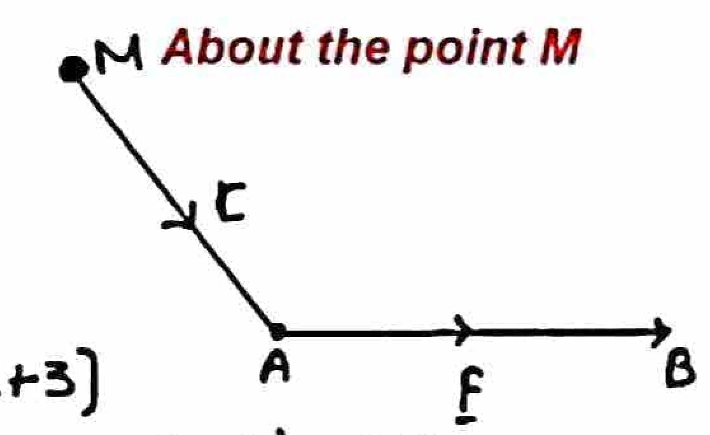
In fig., $\vec{F} = \vec{PQ}$, $\vec{r} = \vec{OP}$, θ is angle from $\underline{r}$ to $\underline{F}$, $\vec{M} = \vec{OP} \times \vec{PQ}$ so

$\underline{M} = \underline{r} \times \underline{F}$

Example 8. Find the moment about the point $M(-2,4,-6)$ of the force represented by $\vec{AB}$, where coordinates of points A and B are $(1,2,-3)$ and $(3,-4,2)$ resp.

Solution:-



$\underline{r} = \vec{MA}$
$= [1+2, 2-4, -3+6]$
$= [3,-2,3]$

$\underline{r} = 3\underline{i} - 2\underline{j} + 3\underline{k}$

$\underline{F} = \vec{AB} = [3-1, -4-2, 2+3]$

$\rightarrow \underline{F} = [2,-6,5] = 2\underline{i} - 6\underline{j} + 5\underline{k}$

so Moment $= \underline{r} \times \underline{F} = \vec{MA} \times \vec{AB}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 3 \\ 2 & -6 & 5 \end{vmatrix}$$

$= \underline{i}(-10+18) - \underline{j}(15-6) + \underline{k}(-18+14) = 8\underline{i} - 9\underline{j} - 14\underline{k}$

Exercise 7.5

Q1. Find the volume of the parallelepiped for which the given vectors are three edges. (i) $\underline{u} = 3\underline{i} + 2\underline{k}$; $\underline{v} = \underline{i} + 2\underline{j} + \underline{k}$; $\underline{w} = -\underline{j} + 4\underline{k}$

Solution:- Volume of parallelepiped

$$= \underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{vmatrix}$$

$= 3(8+1) - 1(0+2) = 27 - 2 = 25$

(ii) $\underline{u} = \underline{i} - 4\underline{j} - \underline{k}$; $\underline{v} = \underline{i} - \underline{j} - 2\underline{k}$; $\underline{w} = 2\underline{i} - 3\underline{j} + \underline{k}$

Solution:- Volume of parallelepiped

$$= \underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} 1 & -4 & -1 \\ 1 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$

$= 1(-1-6) - 1(-4-3) + 2(8+1) = -7 + 7 + 14 = 14$

(iii) $\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k}$; $\underline{v} = 2\underline{i} - \underline{j} - \underline{k}$; $\underline{w} = \underline{j} + \underline{k}$

Solution:- Volume of parallelepiped

$$= \underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} 1 & -2 & 3 \\ 2 & -1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$= 1(-1+1) - 2(-2-3) + 0 = 0 + 10 = 10$

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Q2. Verify that $\underline{a} \cdot \underline{b} \times \underline{c} = \underline{b} \cdot \underline{c} \times \underline{a} = \underline{c} \cdot \underline{a} \times \underline{b}$ if $\underline{a} = 3\underline{i} - \underline{j} + 5\underline{k}$, $\underline{b} = 4\underline{i} + 3\underline{j} - 2\underline{k}$ and $\underline{c} = 2\underline{i} + 5\underline{j} + \underline{k}$

Solution:- $\underline{a} \cdot \underline{b} \times \underline{c} = \begin{vmatrix} 3 & -1 & 5 \\ 4 & 3 & -2 \\ 2 & 5 & 1 \end{vmatrix}$

$= 3(3+10) - 4(-1-25) + 2(2-15)$

$= 3(13) - 4(-26) + 2(-13) = 39 + 104 - 26 = 117$

$\rightarrow \underline{a} \cdot \underline{b} \times \underline{c} = 117$

$\underline{b} \cdot \underline{c} \times \underline{a} = \begin{vmatrix} 4 & 3 & -2 \\ 2 & 5 & 1 \\ 3 & -1 & 5 \end{vmatrix}$

$= 4(25+1) - 2(15-2) + 3(3+10) = 104 - 26 + 39 = 117$

$\rightarrow \underline{b} \cdot \underline{c} \times \underline{a} = 117$

$\underline{c} \cdot \underline{a} \times \underline{b} = \begin{vmatrix} 2 & 5 & 1 \\ 3 & -1 & 5 \\ 4 & 3 & -2 \end{vmatrix}$

$= 2(2-15) - 3(-10-3) + 4(25+1)$

$= -26 + 39 + 104 = 117$

$\rightarrow \underline{c} \cdot \underline{a} \times \underline{b} = 117$

Thus $\underline{a} \cdot \underline{b} \times \underline{c} = \underline{b} \cdot \underline{c} \times \underline{a} = \underline{c} \cdot \underline{a} \times \underline{b}$

Q3. Prove that the vectors $\underline{i} - 2\underline{j} + 3\underline{k}$, $-2\underline{i} + 3\underline{j} - 4\underline{k}$ and $\underline{i} - 3\underline{j} + 5\underline{k}$ are coplaner.

Solution:- $\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k}$, $\underline{v} = -2\underline{i} + 3\underline{j} - 4\underline{k}$ and $\underline{w} = \underline{i} - 3\underline{j} + 5\underline{k}$ Now $\underline{u}, \underline{v}$ and $\underline{w}$ will be coplaner if $\underline{u} \cdot \underline{v} \times \underline{w} = 0$ so

$\underline{u} \cdot \underline{v} \times \underline{w} = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 1(15-12) + 2(-10+9) + 1(8-9)$

$= 3 - 2 - 1 = 3 - 3 = 0$

Thus $\underline{u}, \underline{v}$ and $\underline{w}$ are coplaner.

Q4. Find the constant α, such that the vectors are coplaner.

(i) $\underline{i} - \underline{j} + \underline{k}$; $\underline{i} - 2\underline{j} - \underline{k}$ and $3\underline{i} - \alpha\underline{j} + 5\underline{k}$

Solution:- Let $\underline{u} = \underline{i} - \underline{j} + \underline{k}$, $\underline{v} = \underline{i} - 2\underline{j} - \underline{k}$ and $\underline{w} = 3\underline{i} - \alpha\underline{j} + 5\underline{k}$ $\because \underline{u}, \underline{v}$ and $\underline{w}$ are coplaner. so $\underline{u} \cdot \underline{v} \times \underline{w} = 0$

$\rightarrow \begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & 3 \\ 3 & -\alpha & 5 \end{vmatrix} = 0$

$\rightarrow 1(-10-3\alpha) - 1(-5+\alpha) + 3(3+2) = 0$

$\rightarrow -10 - 3\alpha + 5 - \alpha + 15 = 0 \Rightarrow 10 - 4\alpha = 0$

$\rightarrow 10 = 4\alpha \rightarrow \alpha = \frac{10}{4} \rightarrow \alpha = \frac{5}{2}$

(ii) $\underline{i} - 2\alpha\underline{j} - \underline{k}$, $\underline{i} - \underline{j} + 2\underline{k}$ and $\alpha\underline{i} - \underline{j} + \underline{k}$

Solution:- Let $\underline{u} = \underline{i} - 2\alpha\underline{j} - \underline{k}$
$\underline{v} = \underline{i} - \underline{j} + 2\underline{k}$, $\underline{w} = \alpha\underline{i} - \underline{j} + \underline{k}$
$\because \underline{u}, \underline{v}$ and $\underline{w}$ are coplaner. so

$\underline{u} \cdot \underline{v} \times \underline{w} = 0$

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$$\rightarrow \begin{vmatrix} 1 & -2\alpha & -1 \\ 1 & -1 & 2 \\ \alpha & -1 & 1 \end{vmatrix} = 0$$

$\rightarrow 1(-1+2) - 1(-2\alpha - 1) + \alpha(-4\alpha - 1) = 0$

$\rightarrow 1 + 2\alpha + 1 - 4\alpha^2 - \alpha = 0 \rightarrow 4\alpha^2 + \alpha + 2 = 0$

$\rightarrow 4\alpha^2 - \alpha - 2 = 0$

using quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\rightarrow \alpha = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-2)}}{2(4)} = \frac{1 \pm \sqrt{1+32}}{8}$$

$$\rightarrow \alpha = \frac{1 \pm \sqrt{33}}{8}$$

Q5. (a) Find the value of (i) $2\underline{i} \times 2\underline{j} \cdot \underline{k}$ (ii) $3\underline{j} \cdot \underline{k} \times \underline{i}$ (iii) $[\underline{k}\ \underline{i}\ \underline{j}]$ (iv) $[\underline{i}\ \underline{i}\ \underline{k}]$

Solution:– (i) $2\underline{i} \times 2\underline{j} \cdot \underline{k} = 4(\underline{i} \times \underline{j}) \cdot \underline{k}$

$= 4\underline{k} \cdot \underline{k} = 4(1) = 4 \qquad \because \underline{i} \times \underline{j} = \underline{k},\ \underline{k} \cdot \underline{k} = 1$

(ii) $3\underline{j} \cdot \underline{k} \times \underline{i} = 3\underline{j} \cdot \underline{j} = 3(1) = 3 \qquad \because \underline{k} \times \underline{i} = \underline{j},\ \underline{j} \cdot \underline{j} = 1$

(iii) $[\underline{k}\ \underline{i}\ \underline{j}] = \underline{k} \cdot \underline{i} \times \underline{j} = \underline{k} \cdot \underline{k} = 1$

(iv) $[\underline{i}\ \underline{i}\ \underline{k}] = \underline{i} \cdot \underline{i} \times \underline{k} = \underline{i} \cdot (-\underline{j}) = -\underline{i} \cdot \underline{j} = 0$

$(\because \underline{i} \times \underline{k} = -\underline{j},\ \underline{i} \cdot \underline{j} = 0)$

(b) Prove that

$\underline{u} \cdot (\underline{v} \times \underline{w}) + \underline{v} \cdot (\underline{w} \times \underline{u}) + \underline{w} \cdot (\underline{u} \times \underline{v}) = 3\underline{u} \cdot (\underline{v} \times \underline{w})$

Solution:–

L.H.S $= \underline{u} \cdot (\underline{v} \times \underline{w}) + \underline{v} \cdot (\underline{w} \times \underline{u}) + \underline{w} \cdot (\underline{u} \times \underline{v})$

$= \underline{u} \cdot (\underline{v} \times \underline{w}) + \underline{u} \cdot (\underline{v} \times \underline{w}) + \underline{u} \cdot (\underline{v} \times \underline{w})$

$= 3\underline{u} \cdot (\underline{v} \times \underline{w}) \qquad \because \underline{v} \cdot (\underline{w} \times \underline{u}) = \underline{u} \cdot (\underline{v} \times \underline{w})$

$=$ R.H.S $\qquad$ Also $\underline{w} \cdot (\underline{u} \times \underline{v}) = \underline{u} \cdot (\underline{v} \times \underline{w})$

Hence proved.

Q6. Find volume of tetrahedron with the vertices (i) (0,1,2), (3,2,1), (1,2,1) and (5,5,6) (ii) (2,1,8), (3,2,9), (2,1,4) and (3,3,10)

Solution:– (i) Let A(0,1,2), B(3,2,1), C(1,2,1) and D(5,5,6) Now

$\overrightarrow{AB} = [3-0, 2-1, 1-2] = [3, 1, -1]$

$\overrightarrow{AC} = [1-0, 2-1, 1-2] = [1, 1, -1]$

$\overrightarrow{AD} = [5-0, 5-1, 6-2] = [5, 4, 4]$ so

volume of tetrahedron $= \frac{1}{6}(\overrightarrow{AB} \cdot \overrightarrow{AC} \times \overrightarrow{AD})$

$$= \frac{1}{6}\begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 5 & 4 & 4 \end{vmatrix}$$

$= \frac{1}{6}\{3(4+4) - 1(4+4) + 5(-1+1)\}$

$= \frac{1}{6}\{3(8) - 1(8) + 5(0)\} = \frac{1}{6}(24-8) = \frac{16}{6} = 8/3$

(ii) Let A(2,1,8), B(3,2,9), C(2,1,4) and D(3,3,10) Now

$\overrightarrow{AB} = [3-2, 2-1, 9-8] = [1, 1, 1]$

$\overrightarrow{AC} = [2-2, 1-1, 4-8] = [0, 0, -4]$

$\overrightarrow{AD} = [3-2, 3-1, 10-8] = [1, 2, 2]$

volume of tetrahedron $= \frac{1}{6}(\overrightarrow{AB} \cdot \overrightarrow{AC} \times \overrightarrow{AD})$

$$= \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & 2 \end{vmatrix} \quad \text{expanding by } R_2$$

$= \frac{1}{6}\{-0 + 0 - (-4)(2-1)\} = \frac{1}{6}(4(1)) = \frac{4}{6} = \frac{2}{3}$

Q7. Find the work done, if the point at which the constant force $\underline{F} = 4\underline{i} + 3\underline{j} + 5\underline{k}$ is applied to an object, moves from $P_1(3,1,-2)$ to $P_2(2,4,6)$

Solution:– Force $= \underline{F} = 4\underline{i} + 3\underline{j} + 5\underline{k}$

Displacement $= \underline{d} = \overrightarrow{P_1P_2} = [2-3, 4-1, 6+2] = [-1, 3, 8]$

$\rightarrow \underline{d} = -\underline{i} + 3\underline{j} + 8\underline{k}$

Work done $= \underline{F} \cdot \underline{d} = (4\underline{i} + 3\underline{j} + 5\underline{k}) \cdot (-\underline{i} + 3\underline{j} + 8\underline{k})$

$= (4)(-1) + (3)(3) + (5)(8) = -4 + 9 + 40 = 45$ nt.m

Q8. A particle, acted by constant forces $4\underline{i} + \underline{j} - 3\underline{k}$ and $3\underline{i} - \underline{j} - \underline{k}$ is displaced from A(1,2,3) to B(5,4,1). Find the work done.

Solution:– Let $\underline{F_1} = 4\underline{i} + \underline{j} - 3\underline{k}$

$\underline{F_2} = 3\underline{i} - \underline{j} - \underline{k}$ $\quad$ Total force $= \underline{F} = \underline{F_1} + \underline{F_2}$

$\rightarrow \underline{F} = 4\underline{i} + \underline{j} - 3\underline{k} + 3\underline{i} - \underline{j} - \underline{k} = 7\underline{i} - 0\underline{j} - 4\underline{k}$

Displacement $= \underline{d} = \overrightarrow{AB} = [5-1, 4-2, 1-3] = [4, 2, -2]$

$\rightarrow \underline{d} = 4\underline{i} + 2\underline{j} - 2\underline{k}$ so

work done $= \underline{F} \cdot \underline{d} = (7\underline{i} - 0\underline{j} - 4\underline{k}) \cdot (4\underline{i} + 2\underline{j} - 2\underline{k})$

$= 7(4) - 0(2) + (-4)(-2) = 28 + 8 = 36$ nt.m

Q9. A particle is displaced from the point A(5,-5,-7) to the point B(6,2,-2) under the action of constant forces defined by $10\underline{i} - \underline{j} + 11\underline{k}$, $4\underline{i} + 5\underline{j} + 9\underline{k}$ and $-2\underline{i} + \underline{j} - 9\underline{k}$. Show that the total work done by the forces is 102 units.

Solution:– Let $\underline{F_1} = 10\underline{i} - \underline{j} + 11\underline{k}$, $\underline{F_2} = 4\underline{i} + 5\underline{j} + 9\underline{k}$

$\underline{F_3} = -2\underline{i} + \underline{j} - 9\underline{k}$ so Total force $= \underline{F} = \underline{F_1} + \underline{F_2} + \underline{F_3}$

$\rightarrow \underline{F} = 12\underline{i} + 5\underline{j} + 11\underline{k}$ $\quad$ Displacement $= \underline{d} = \overrightarrow{AB}$

$\rightarrow \underline{d} = [6-5, 2+5, -2+7] = [1, 7, 5] = \underline{i} + 7\underline{j} + 5\underline{k}$

Work done $= \underline{W} = \underline{F} \cdot \underline{d} = (12\underline{i} + 5\underline{j} + 11\underline{k})(\underline{i} + 7\underline{j} + 5\underline{k})$

$= (12)(1) + (5)(7) + (9)(5) = 12 + 35 + 55 = 102$

Hence proved.

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Q10. A force of magnitude 6 units acting parallel to $2\underline{i}-2\underline{j}+\underline{k}$ displaces the point of application from $(1,2,3)$ to $(5,3,7)$. Find the work done.

Solution:– $\because |\underline{F}| = 6$ units

and $\underline{F} = |\underline{F}|\hat{\underline{F}} \Rightarrow \underline{F} = 6\hat{\underline{F}}$ —— (1)

$\because \underline{F}$ is parallel to $2\underline{i}-2\underline{j}+\underline{k}$ so

$$\hat{\underline{F}} = \frac{2\underline{i}-2\underline{j}+\underline{k}}{\sqrt{(2)^2+(-2)^2+(1)^2}} = \frac{2\underline{i}-2\underline{j}+\underline{k}}{\sqrt{9}} = \frac{2\underline{i}-2\underline{j}+\underline{k}}{3}$$

By (1) $\Rightarrow \underline{F} = 6\left(\frac{2\underline{i}-2\underline{j}+\underline{k}}{3}\right) = 4\underline{i}-4\underline{j}+2\underline{k}$

Let $A(1,2,3)$ and $B(5,3,7)$. so

Displacement $= \underline{d} = \overrightarrow{AB} = [5-1,\ 3-2,\ 7-3] = [4,1,4]$

$\Rightarrow \underline{d} = 4\underline{i}+\underline{j}+4\underline{k}$ Now

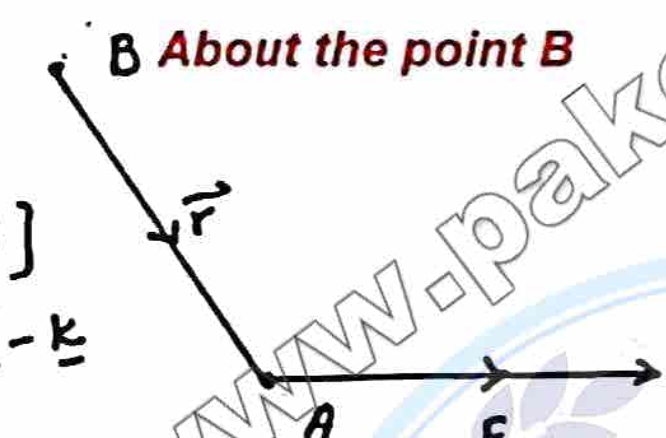
Work done $= \underline{F}\cdot\underline{d} = (4\underline{i}-4\underline{j}+2\underline{k})\cdot(4\underline{i}+\underline{j}+4\underline{k})$

$= (4)(4)+(-4)(1)+(2)(4) = 16-4+8$

Work $= 20$ nt.m

Q11. A force $\underline{F} = 3\underline{i}+2\underline{j}-4\underline{k}$ is applied at the point $(1,-1,2)$. Find the moment of the force $\underline{F}$ about the point $(2,-1,3)$

Solution:– Let $A(1,-1,2)$, $B(2,-1,3)$



$\underline{F} = 3\underline{i}+2\underline{j}-4\underline{k}$

$\underline{r} = \overrightarrow{BA} = [1-2, -1+1, 2-3]$

$\underline{r} = [-1,0,-1] = -\underline{i}-\underline{k}$

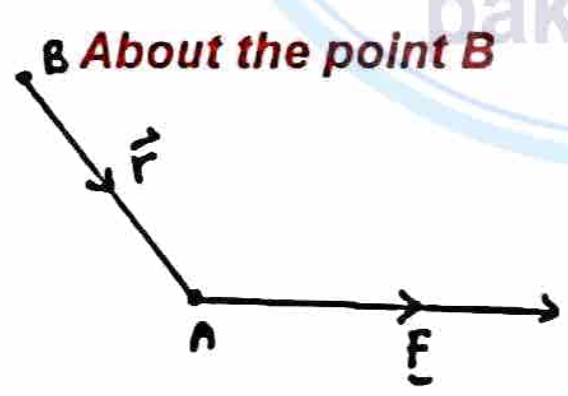
Now

Moment of force $= \underline{M} = \underline{r}\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix}$

$\underline{M} = \underline{i}(0+2)-\underline{j}(4+3)+\underline{k}(-2-0) = 2\underline{i}-7\underline{j}-2\underline{k}$

Q12. A force $\underline{F} = 4\underline{i}-3\underline{k}$, passes through the point $A(2,-2,5)$. Find the moment of $\underline{F}$ about the point $B(1,-3,1)$.

Solution:–



$\underline{F} = 4\underline{i}-3\underline{k}$

$\vec{r} = \overrightarrow{BA} = [2-1, -2+3, 5-1]$

$= [1,1,4]$

$\Rightarrow \vec{r} = \underline{i}+\underline{j}+4\underline{k}$

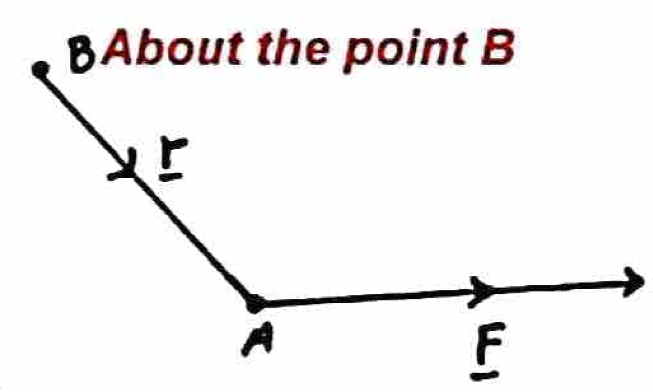
Moment of Force $= \underline{M} = \underline{r}\times\underline{F}$

$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 4 \\ 4 & 0 & -3 \end{vmatrix} = \underline{i}(-3-0)-\underline{j}(-3-16)+\underline{k}(0-4)$

$= -3\underline{i}+19\underline{j}-4\underline{k}$

Q13. Give a force $\underline{F} = 2\underline{i}+\underline{j}-3\underline{k}$ acting at a point $A(1,-2,1)$. Find the moment of $\underline{F}$ about the point $B(2,0,-2)$.

Solution:–



$\underline{F} = 2\underline{i}+\underline{j}-3\underline{k}$

$\underline{r} = \overrightarrow{BA}$

$= [1-2, -2-0, 1+2]$

$= [-1,-2,3]$

$\Rightarrow \underline{r} = -\underline{i}-2\underline{j}+3\underline{k}$

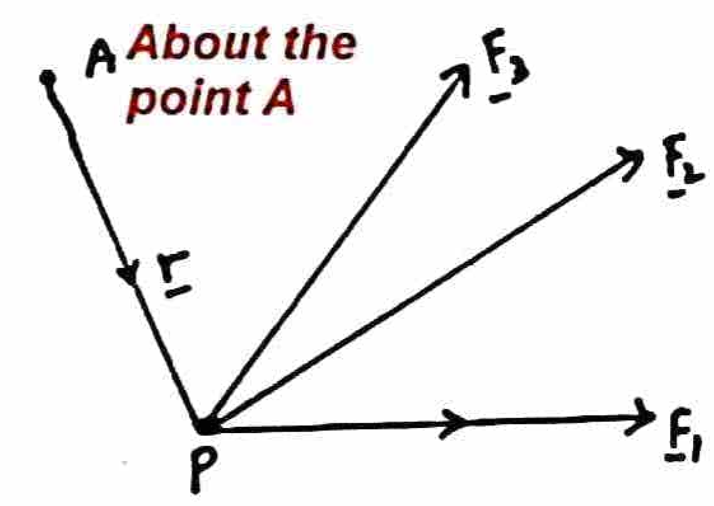
Moment of force $= \underline{M} = \underline{r}\times\underline{F}$

$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix} = \underline{i}(6-3)-\underline{j}(3-6)+\underline{k}(-1+4)$

$= 3\underline{i}+3\underline{j}+3\underline{k}$

Q14. Find the moment about $A(1,1,1)$ of each of concurrent forces $\underline{i}-2\underline{j}$, $3\underline{i}+2\underline{j}-\underline{k}$, $5\underline{j}+2\underline{k}$, where $P(2,0,1)$ is their point of concurrency.

Solution:–



Let

$\underline{F}_1 = \underline{i}-2\underline{j}$

$\underline{F}_2 = 3\underline{i}+2\underline{j}-\underline{k}$

$\underline{F}_3 = 5\underline{j}+2\underline{k}$

Total force $= \underline{F} = \underline{F}_1+\underline{F}_2+\underline{F}_3 = 4\underline{i}+5\underline{j}+\underline{k}$

$\vec{r} = \overrightarrow{AP} = [2-1, 0-1, 1-1] = [1,-1,0]$

$\underline{r} = \underline{i}-\underline{j}+0\underline{k}$

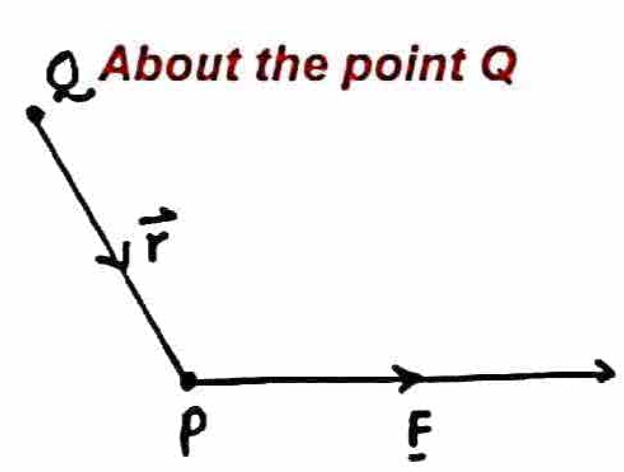
Moment of force $= \underline{M} = \underline{r}\times\underline{F}$

$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 4 & 5 & 1 \end{vmatrix} = \underline{i}(-1-0)-\underline{j}(1-0)+\underline{k}(5+4)$

$= -\underline{i}-\underline{j}+9\underline{k}$

Q15. A force $\underline{F} = 7\underline{i}+4\underline{j}-3\underline{k}$ is applied at $P(1,-2,3)$. Find its moment about the point $Q(2,1,1)$

Solution:–



$\underline{F} = 7\underline{i}+4\underline{j}-3\underline{k}$

$\underline{r} = \overrightarrow{QP}$

$= [1-2, -2-1, 3-1]$

$= [-1,-3,2]$

$\Rightarrow \underline{r} = -\underline{i}-3\underline{j}+2\underline{k}$

Moment of force $= \underline{M} = \underline{r}\times\underline{F}$

$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -3 & 2 \\ 7 & 4 & -3 \end{vmatrix} = \underline{i}(9-8)-\underline{j}(3-14)+\underline{k}(-4+21)$

$= \underline{i}+11\underline{j}+17\underline{k}$

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