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MATHEMATICS 2nd YEAR

UNIT # 06

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CONIC SECTIONS
(6.1-6.6)

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Muhammad Salman Sherazi
M.Phil (Math)

Contents

Sherazi Mathematics

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اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Cone:- A solid figure generated by a straight line passing through a fixed point and revolving about a fixed line is called cone.

Axis of Cone:- The fixed line passing through vertex and perpendicular to the centre of base is called axis of cone.

Apex:- The fixed point of cone is called vertex or apex of cone.

Napes of Cone:- The cone has two parts called napes. The part above the vertex is upper nape. The part below the vertex is lower nape.

Types of Cone

Cones are named according to shape of their base. If the base of cone is circle, then it is called circular cone. If the base of cone is ellipse, then it is called elliptic cone.

" A cone having its axis perpendicular circular base is called Right Circular Cone.

Conic Section:-

Some standard conic sections are: Circle, Ellipse Parabola and Hyperbola.

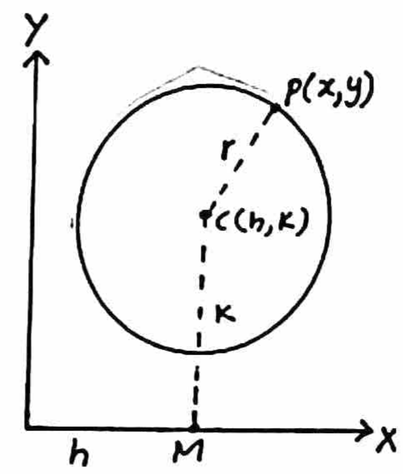
We first study the properties of a circle. Other conics will be taken up later.

Circle:- A set of all points in a plane which are equidistant from a fixed point is called circle. The fixed point is called centre and fixed distance is called radius of the circle.

Point circle:- If the plane passes through vertex of cone, the intersection is a single point or point circle or if $r=0$

Equation of Circle

Let $P(x,y)$ be any point on the circle having centre (h,k)



from figure,

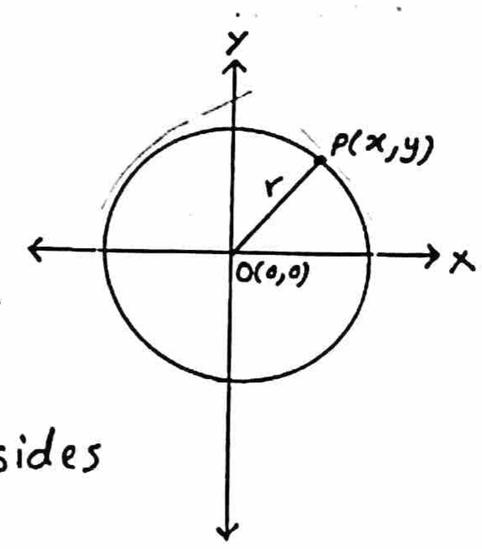
radius $= |CP| = r$

using distance formula

$r^2 = (x-h)^2 + (y-k)^2$

Hence required circle is; $(x-h)^2 + (y-k)^2 = r^2$

Note:- If $P(x,y)$ be any point on the circle having centre at origin $O(0,0)$ and radius r



$\because |OP| = r$

using distance formula,

$r = \sqrt{(x-0)^2 + (y-0)^2}$

$r = \sqrt{x^2+y^2}$

squaring both sides

$r^2 = x^2 + y^2$

$\rightarrow x^2 + y^2 = r^2$

$\because$ A circle having radius zero is called point circle. so

$x^2 + y^2 = r^2$ put $r=0$

$\rightarrow x^2 + y^2 = 0$ which is equation of point circle.

Parametric Equations of circle

Let $P(x,y)$ be any point on the circle having centre at origin and radius r.

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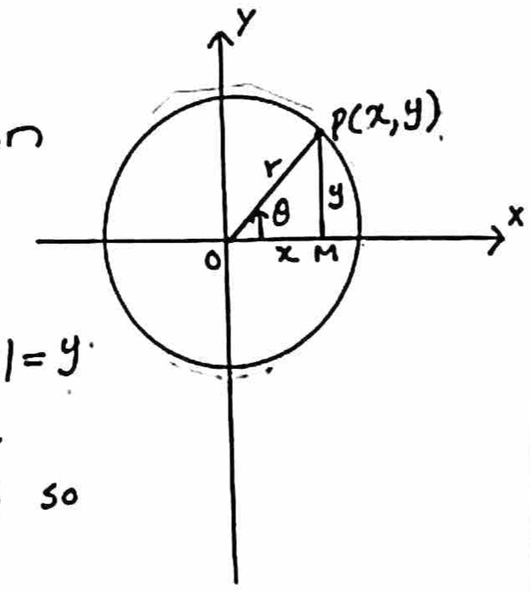
Draw ⊥ar PM from point P on x-axis



In ΔPOM

$|OM| = x$; $|PM| = y$

$|OP| = r$ and

$m\angle MOP = \theta$ so

$\cos\theta = \frac{|OM|}{|OP|}$

$\rightarrow \cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$

$\sin\theta = \frac{|PM|}{|OP|} \rightarrow \sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$

Thus $x = r\cos\theta$ and $y = r\sin\theta$ are called parametric eqs of circle where θ is parameter.

Example 1. Write an equation of the circle with centre $(-3, 5)$ and radius 7.

Solution:- Here

centre $= (h, k) = (-3, 5)$

$\rightarrow h = -3, \; k = 5$

radius $= r = 7$

$\because (x-h)^2 + (y-k)^2 = r^2$

$\rightarrow (x-(-3))^2 + (y-5)^2 = (7)^2$

$\rightarrow (x+3)^2 + (y-5)^2 = 49$

$\rightarrow x^2 + 6x + 9 + y^2 + 25 - 10y = 49$

$x^2 + y^2 + 6x - 10y + 9 + 25 - 49 = 0$

$\rightarrow x^2 + y^2 + 6x - 10y - 15 = 0$

which is req. eq. of circle.

General Form of an Equation of a Circle

Theorem:- The equation

$$x^2 + y^2 + 2gx + 2fy + c = 0 \longrightarrow (I)$$

represents a circle g, f and c being constants.

By (I)

$x^2 + y^2 + 2gx + 2fy = -c$

adding g^2 and f^2 on both sides

$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$

$\rightarrow (x+g)^2 + (y+f)^2 = g^2 + f^2 - c$

$\rightarrow [x-(-g)]^2 + [y-(-f)]^2 = (\sqrt{g^2+f^2-c})^2$

It is of the form

$(x-h)^2 + (y-k)^2 = r^2$

where centre $= (-g, -f)$

radius $= \sqrt{g^2+f^2-c}$

* The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ is called "General equation of circle".
* Every equation of 2nd degree in x and y represents an eq. of circle if; i) coefficients of x^2 and y^2 is 1. ii) It does not involve product xy.

Example 2. Show that the equation $5x^2 + 5y^2 + 24x + 36y + 10 = 0$ represents a circle. Also find its centre and radius.

Solution:-

$\because 5x^2 + 5y^2 + 24x + 36y + 10 = 0$

$(\div$ by $5)$

$\rightarrow x^2 + y^2 + \frac{24}{5}x + \frac{36}{5}y + 2 = 0$

$\rightarrow x^2 + y^2 + 2\left(\frac{12}{5}\right)x + 2\left(\frac{18}{5}\right)y + 2 = 0$

It is of the form

$x^2 + y^2 + 2gx + 2fy + c = 0$

where $g = \frac{12}{5}$, $f = \frac{18}{5}$, $c = 2$

$\because$ centre $= (-g, -f) = \left(-\frac{12}{5}, \frac{-18}{5}\right)$

radius $= \sqrt{g^2+f^2-c}$

$= \sqrt{\left(-\frac{12}{5}\right)^2 + \left(-\frac{18}{5}\right)^2 - 2}$

$= \sqrt{\frac{144}{25} + \frac{324}{25} - 2}$

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$$= \sqrt{\frac{144+324-50}{25}} = \sqrt{\frac{418}{25}}$$

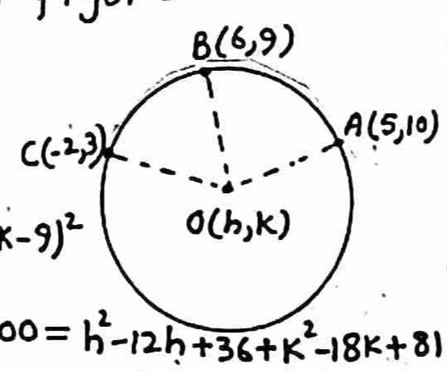
$$= \frac{\sqrt{418}}{5}$$

A Circle Passing Three Non-Collinear points

Example 3. Find an equation of the circle which passes through the points $A(5,10)$, $B(6,9)$ and $C(-2,3)$

Solution:-

It is clear from figure that



$|AO|^2 = |BO|^2$.

$(h-5)^2+(k-10)^2 = (h-6)^2+(k-9)^2$

$\rightarrow h^2-10h+25+k^2-20k+100 = h^2-12h+36+k^2-18k+81$

$-10h-20k+125 = -12h-18k+117$

$2h-2k+8=0$

or $h-k+4=0 \longrightarrow (i)$

Also $|AO|^2=|CO|^2$

$(h-5)^2+(k-10)^2 = (h-(-2))^2+(k-3)^2$

$h^2-10h+25+k^2-20k+100 = h^2+4h+4+k^2-6k+9$

$-10h-20k+125 = 4h-6k+13$

$\rightarrow -10h-4h-20k+6k+125-13=0$

or $-14h-14k+112=0$

or $h+k-8=0 \longrightarrow (ii)$ $(\div \text{ by } -14)$

By (i) + (ii) $\rightarrow$

$$\begin{array}{r} h-k+4=0 \\ h+k-8=0 \\ \hline 2h-4=0 \rightarrow 2h=4 \end{array}$$

$\rightarrow h=2$ put in (ii), we get

$2+k-8=0 \rightarrow k-6=0$

or $k=6$

$\because r = |AO| = \sqrt{(h-5)^2+(k-10)^2}$

$\rightarrow r=\sqrt{(2-5)^2+(6-10)^2}$ $\left(\because \begin{array}{l} h=2 \\ k=6 \end{array}\right)$

$r=\sqrt{(-3)^2+(-4)^2}$

$r=\sqrt{9+16}=\sqrt{25}=5$

Now eq. of circle is

$(x-h)^2+(y-k)^2=r^2$

$\rightarrow (x-2)^2+(y-6)^2=(5)^2$

$\rightarrow x^2+4-4x+y^2+36-12y=25$

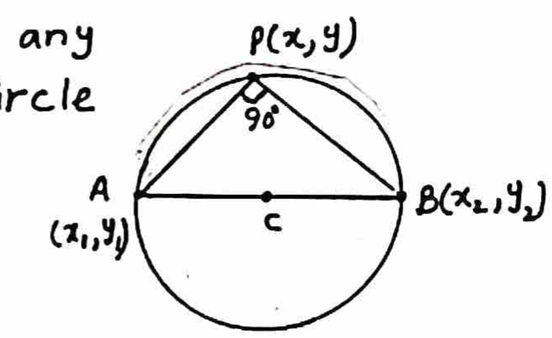
$\rightarrow x^2+y^2-4x-12y+15=0$

A circle passing through two points and having its centre on a given line

Example 4. Find an equation of circle having the join of $A(x_1,y_1)$ and $B(x_2,y_2)$ as a diameter.

Solution:-

Let $P(x,y)$ be any point on the circle having $A(x_1,y_1)$ and $B(x_2,y_2)$ as ends of



In fig $m\angle APB = 90°$

($\because$ angle inscribed in a semi-circle is $90°$)

so $\overline{AP} \perp \overline{BP}$

$\rightarrow$ (slope of $\overline{AP}$)(slope of $\overline{BP}$) $=-1$

$$\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right)=-1$$

$\rightarrow (y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$

$\rightarrow (y-y_1)(y-y_2)+(x-x_1)(x-x_2)=0$

or $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

which is required eq. of circle.

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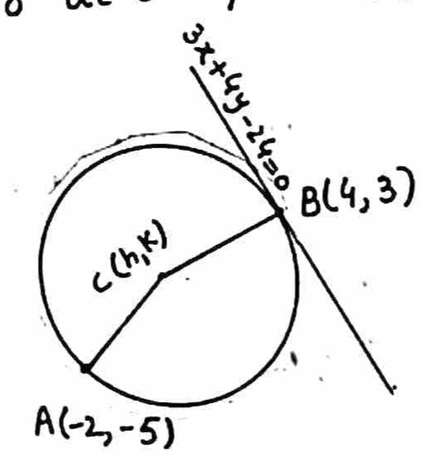
A circle passing through two points and equation of tangent at one of these points is known

Example 5. Find an equation of the circle passing through the point $(-2,-5)$ and touching the line $3x+4y-24=0$ at the point $(4,3)$

Solution:-

clearly from fig



$|AC|^2 = |BC|^2$

$\rightarrow (h-(-2))^2+(K-(-5))^2 = (h-4)^2+(K-3)^2$

$\rightarrow (h+2)^2+(K+5)^2 = (h-4)^2+(K-3)^2$

$\rightarrow h^2+4+4h+K^2+25+10K = h^2+16-8h+K^2+9-6K$

$\rightarrow 4h+10K+29 = -8h-6K+25$

$\rightarrow 12h+16K+4 = 0$

$\rightarrow 3h+4K+1 = 0 \longrightarrow (i)$

given line is $3x+4y-24=0$

slope of given line is $= \frac{-3}{4}$

slope of radial segment $BC = \frac{K-3}{h-4}$

$\because$ tangent line is $\perp$ar to radial segment so

$$\left(\text{slope of tangent line}\right)\left(\text{slope of radial segment } BC\right) = -1$$

$\rightarrow \left(\frac{-3}{4}\right)\left(\frac{K-3}{h-4}\right) = -1$

$\rightarrow -3(K-3) = -4(h-4)$

$\rightarrow -3K+9 = -4h+16$

$\rightarrow 4h-3K+9-16 = 0$

$\rightarrow 4h-3K-7 = 0 \longrightarrow (ii)$

By $3(i)+4(ii) \rightarrow$

$$\begin{array}{r} 9h+12K+3=0 \\ 16h-12K-28=0 \\ \hline 25h-25=0 \end{array}$$

$25h = 50 \rightarrow h = 1$ put in (i)

$3(1)+4K+1=0 \rightarrow 4K+4=0$

$K=-1$

As $r = |BC| = \sqrt{(h-4)^2+(K-3)^2}$

$r = \sqrt{(1-4)^2+(-1-3)^2}$ $\quad \because h=1, K=-1$

$\rightarrow r = \sqrt{(-3)^2+(-4)^2} = \sqrt{9+16}$

$\rightarrow r = 5$

Now eq. of circle is

$(x-h)^2+(y-K)^2 = r^2$

$\rightarrow (x-1)^2+(y-(-1))^2 = (5)^2$

$\rightarrow x^2+1-2x+y^2+1+2y = 25$

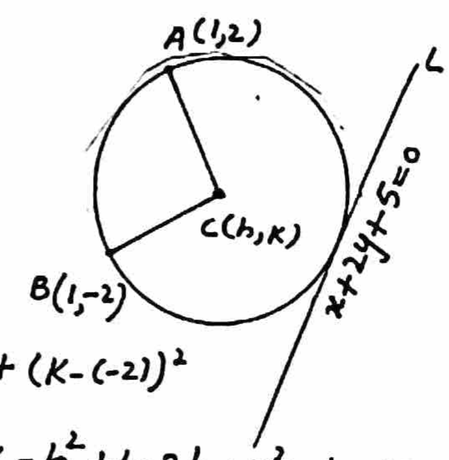
$\rightarrow x^2+y^2-2x+2y-23=0$

A circle passing through two points and touching a given line

Example 6. Find an equation of the circle passing through the points $A(1,2)$ and $B(1,-2)$ and touching the line $x+2y+5=0$

Solution:-

clearly from fig,



$|AC|^2 = |BC|^2$

$\rightarrow (h-1)^2+(K-2)^2 = (h-1)^2+(K-(-2))^2$

$\rightarrow h^2+1-2h+K^2+4-4K = h^2+1-2h+K^2+4+4K$

$\rightarrow -4K = 4K \rightarrow 4K+4K=0$

$\rightarrow 8K=0$ or $K=0$

Now $|AC| = |CL|$ where L is tangent line

$\sqrt{(h-1)^2+(K-2)^2} = \frac{|1(h)+2(K)+5|}{\sqrt{(1)^2+(2)^2}}$

$\rightarrow \sqrt{h^2+1-2h+(0-2)^2} = \frac{|h+2(0)+5|}{\sqrt{5}}$ $\quad \because K=0$

$\rightarrow \sqrt{h^2-2h+1+4} = \frac{|h+5|}{\sqrt{5}}$

$\rightarrow \sqrt{5}\sqrt{h^2-2h+5} = |h+5|$

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squaring both sides, we get

$5(h^2-2h+5) = (h+5)^2$

$\rightarrow 5h^2-10h+25 = h^2+25+10h$

$\rightarrow 4h^2-20h=0 \rightarrow 4h(h-5)=0$

$4h=0, \quad h-5=0$

$\rightarrow h=0, \quad h=5$

when $h=0$, $K=0$ then

$r=|AC| = \sqrt{(h-1)^2+(K-2)^2}$

$r = \sqrt{(0-1)^2+(0-2)^2} = \sqrt{1+4}$

$r=\sqrt{5}$ Now eq. of circle is $(x-0)^2+(y-0)^2=(\sqrt{5})^2$

$\rightarrow x^2+y^2=5$

when $h=5$, $K=0$ then

$r=|AC| = \sqrt{(h-1)^2+(K-2)^2}$

$r = \sqrt{(5-1)^2+(0-2)^2} = \sqrt{16+4}$

$r=\sqrt{20}$ Now eq. of circle is $(x-5)^2+(y-0)^2=(\sqrt{20})^2$

$\rightarrow x^2+25-10x+y^2=20$

$\rightarrow x^2+y^2-10x+5=0$

Thus $x^2+y^2-5=0$ and $x^2+y^2-10x+5=0$ are eqs of circles.

Exercise 6.1

Q1. In each of the following, find an equation of circle with

(a) centre at $(5,-2)$ and radius 4

Solution:- Here

centre $=(h,K)=(5,-2)$

$\rightarrow h=5, \quad K=-2$

radius $=r=4$

$\because$ Equation of circle is

$(x-h)^2+(y-K)^2=r^2$

$\rightarrow (x-5)^2+(y-(-2))^2=(4)^2$

$\rightarrow x^2+25-10x+y^2+4+2y=16$

$\rightarrow x^2+y^2-10x+2y+13=0$

(b) centre at $(\sqrt{2},-3\sqrt{3})$ and radius $2\sqrt{2}$

Solution:- Here

Centre $=(h,K)=(\sqrt{2},-3\sqrt{3})$

$\rightarrow h=\sqrt{2}, \quad K=-3\sqrt{3}$

radius $=r=2\sqrt{2}$

$\because$ Equation of circle is

$(x-h)^2+(y-K)^2=r^2$

$\rightarrow (x-\sqrt{2})^2+(y-(-3\sqrt{3}))^2=(2\sqrt{2})^2$

$\rightarrow (x-\sqrt{2})^2+(y+3\sqrt{3})^2=4(2)$

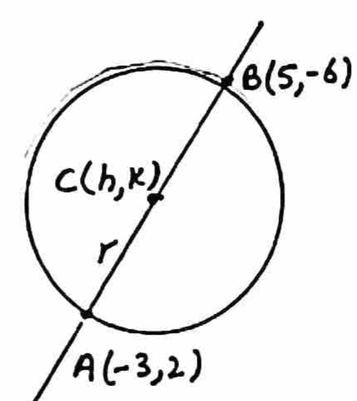
$\rightarrow x^2-2\sqrt{2}x+(\sqrt{2})^2+y^2+2(3\sqrt{3})y+(3\sqrt{3})^2=8$

$x^2-2\sqrt{2}x+2+y^2+6\sqrt{3}y+9(3)-8=0$

$x^2+y^2-2\sqrt{2}+6\sqrt{3}y+21=0$

(c) ends of a diameter at $(-3,2)$ and $(5,-6)$

Solution:-



centre is (h,K) so

$(h,K)=\left(\frac{-3+5}{2}, \frac{-6+2}{2}\right)$

$(h,K)=(1,-2)$

$\rightarrow h=1, \quad K=-2$

$r=|AC| = \sqrt{(h+3)^2+(K-2)^2}$

$r = \sqrt{(1+3)^2+(-2-2)^2} = \sqrt{16+16}$

$r=\sqrt{32}$

$\because$ Eq. of circle is

$(x-h)^2+(y-K)^2=r^2$

$\rightarrow (x-1)^2+(y+2)^2=(\sqrt{32})^2$

$\rightarrow x^2+1-2x+y^2+4+4y=32$

$\rightarrow x^2+y^2-2x+4y+5-32=0$

$x^2+y^2-2x+4y-27=0$

Q2. Find the centre and radius of the given with the given equation

(a) $x^2+y^2+12x-10y=0$

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Solution:-

$x^2+y^2+12x-10y=0$

compare with

$x^2+y^2+2gx+2fy+c=0$

$2g=12 \rightarrow g=6$, $2f=-10$

$\rightarrow f=-5$, $c=0$

centre is $(-g,-f)=(-6,5)$

Radius $= r = \sqrt{g^2+f^2-c}$

$r=\sqrt{(6)^2+(-5)^2-0}=\sqrt{36+25}$

$r=\sqrt{61}$

(b) $5x^2+5y^2+14x+12y-10=0$

Solution:-

$5x^2+5y^2+14x+12y-10=0$

$\rightarrow x^2+y^2+\frac{14}{5}x+\frac{12}{5}y-2=0$ $(\div$ by $5)$

compare with

$x^2+y^2+2gx+2fy+c=0$

$2g=\frac{14}{5} \rightarrow g=\frac{7}{5}$, $2f=\frac{12}{5}$

$\rightarrow f=\frac{6}{5}$

and $c=-2$

centre is $(-g,-f)=(-\frac{7}{5},-\frac{6}{5})$

radius $= r = \sqrt{g^2+f^2-c}$

$\rightarrow r=\sqrt{(\frac{7}{5})^2+(\frac{6}{5})^2-(-2)}$

$r=\sqrt{\frac{49}{25}+\frac{36}{25}+2}$

$r=\sqrt{\frac{49+36+50}{25}}=\sqrt{\frac{135}{25}}$

$r=\sqrt{\frac{27}{5}}$

(c) $x^2+y^2-6x+4y+13=0$

Solution:-

$x^2+y^2-6x+4y+13=0$

compare with

$x^2+y^2+2gx+2fy+c=0$

$2g=-6 \rightarrow g=-3$

$2f=4 \rightarrow f=2$, $c=13$

Centre is $(-g,-f)=(3,-2)$

Radius $= r = \sqrt{g^2+f^2-c}$

$r=\sqrt{(-3)^2+(2)^2-13}$

$r=\sqrt{9+4-13}=\sqrt{13-13}=0$

$\rightarrow r=0$

(d) $4x^2+4y^2-8x+12y-25=0$

Solution:-

$4x^2+4y^2-8x+12y-25=0$

$\rightarrow x^2+y^2-2x+3y-\frac{25}{4}=0$ $(\div$ by $4)$

compare with

$x^2+y^2+2gx+2fy+c=0$

$\rightarrow 2g=-2 \rightarrow g=-1$

$2f=3 \rightarrow f=\frac{3}{2}$, $c=\frac{-25}{4}$

Centre is $(-g,-f)=(1,-\frac{3}{2})$

Radius $= r = \sqrt{g^2+f^2-c}$

$r=\sqrt{(1)^2+(-\frac{3}{2})^2-(\frac{-25}{4})}$

$r=\sqrt{1+\frac{9}{4}+\frac{25}{4}}$

$r=\sqrt{\frac{4+9+25}{4}}=\sqrt{\frac{38}{4}}$

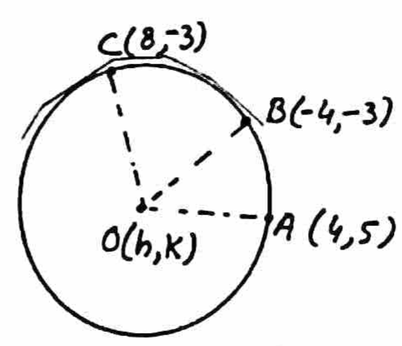
$r=\sqrt{\frac{19}{2}}$

Q3. Write an equation of circle that passes through the given points:

(a) $A(4,5)$, $B(-4,-3)$, $C(8,-3)$

Solution:-

It is clear from figure that,



$|AO|^2=|BO|^2$

$\rightarrow (h-4)^2+(k-5)^2=(h+4)^2+(k+3)^2$

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$h^2-8h+16+K^2-10K+25=h^2+8h+16+K^2+6K+9$

$-8h-10K+41=8h+6K+25$

$\rightarrow -8h-8h-10K-6K+41-25=0$

$\rightarrow -16h-16K+16=0$

$\rightarrow -h-K+1=0 \longrightarrow (i)$

Also $|AO|^2=|CO|^2$

$\rightarrow (h-4)^2+(K-5)^2=(h-8)^2+(K-(-3))^2$

$h^2+16-8h+K^2+25-10K=h^2+64-16h+K^2+9+6K$

$-8h-10K+41=-16h+6K+73$

$\rightarrow 8h-16K-32=0$

$\rightarrow h-2K-4=0 \longrightarrow (ii)$

By (i) + (ii) $\rightarrow$

$$\begin{array}{r} -h-K+1=0 \\ h-2K-4=0 \\ \hline -3K-3=0 \end{array}$$

$\rightarrow -3K=3 \rightarrow K=-1$ put in (i)

$-h-(-1)+1=0 \rightarrow -h+1+1=0$

$-h+2=0 \rightarrow -h=-2 \rightarrow h=2$

$\because r=|AO|=\sqrt{(h-4)^2+(K-5)^2}$

$=\sqrt{(2-4)^2+(-1-5)^2}=\sqrt{(-2)^2+(-6)^2}$

$r=\sqrt{4+36}=\sqrt{40}$ $(\because h=2, K=-1)$

Now eq. of circle is

$(x-h)^2+(y-K)^2=r^2$

$\rightarrow (x-2)^2+(y-(-1))^2=(\sqrt{40})^2$

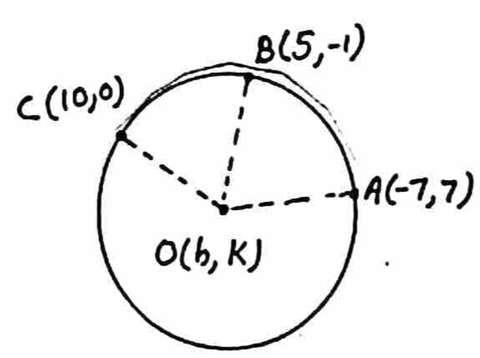
$\rightarrow x^2+4-4x+y^2+1+2y-40=0$

$\rightarrow x^2+y^2-4x+2y-35=0$

(b) $A(-7,7)$, $B(5,-1)$, $C(10,0)$

Solution:-

clearly from fig,



$|AO|^2=|BO|^2$

$\rightarrow (h+7)^2+(K-7)^2=(h-5)^2+(K+1)^2$

$h^2+14h+49+K^2-14K+49=h^2-10h+25+K^2+2K+1$

$\rightarrow 14h-14K+98=-10h+2K+26$

$\rightarrow 14h+10h-14K+98-26=0$

$24h-16K+72=0$

$\rightarrow 3h-2K+9=0 \longrightarrow (i)$

Also $|AO|^2=|CO|^2$

$(h+7)^2+(K-7)^2=(h-10)^2+(K-0)^2$

$h^2+49+14h+K^2-14K+49=h^2-20h+100+K^2$

$14h-14K+98+20h-100=0$

$\rightarrow 34h-14K-2=0$

$\rightarrow 17h-7K-1=0 \longrightarrow (ii)$ (÷ by 2)

$\times 17(i)-3(ii) \rightarrow 51h-34K+153=0$

$$\begin{array}{r} 51h-21K-3=0 \\ \hline -13K+156=0 \end{array}$$

$\rightarrow -13K=-156 \rightarrow K=12$ put in (i)

$3h-2(12)+9=0$

$\rightarrow 3h-24+9=0 \rightarrow 3h-15=0$

$h=5$ Ans.

$r=|AO|=\sqrt{(h+7)^2+(K-7)^2}$

$=\sqrt{(5+7)^2+(12-7)^2}$

$=\sqrt{(12)^2+(5)^2}=\sqrt{144+25}$

$r=\sqrt{169}=13$

Now eq. of circle is

$(x-h)^2+(y-K)^2=r^2$

$\rightarrow (x-5)^2+(y-12)^2=(13)^2$

$\rightarrow x^2+25-10x+y^2+144-24y=169$

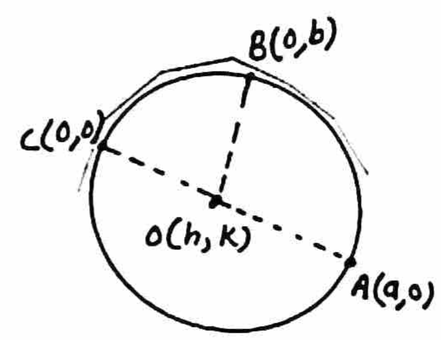
$\rightarrow x^2+y^2-10x-24y+25+144-169=0$

or $x^2+y^2-10x-24y=0$

(c) $A(a,0)$, $B(0,b)$, $C(0,0)$

Solution:-

Clearly from fig



$|AO|^2=|BO|^2$

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$(h-a)^2 + (K-0)^2 = (h-0)^2 + (K-b)^2$

$\rightarrow h^2+a^2-2ah+K^2 = h^2+K^2+b^2-2Kb$

$\rightarrow -2ah+a^2 = -2Kb+b^2$

$\rightarrow -2ah+2Kb+a^2-b^2 = 0 \longrightarrow (i)$

Also $|AO|^2 = |CO|^2$

$(h-a)^2+(K-0)^2 = (h-0)^2+(K-0)^2$

$\rightarrow h^2+a^2-2ah+K^2 = h^2+K^2$

$\rightarrow -2ah+a^2=0$

$\rightarrow a^2 = 2ah \rightarrow a = 2h$

or $h = \frac{a}{2}$ put in (i)

$-2a(\frac{a}{2})+2Kb+a^2-b^2=0$

$\rightarrow -a^2+2Kb+a^2-b^2=0$

$\rightarrow 2Kb = b^2 \rightarrow K = \frac{b}{2}$

As $r = |..O| = \sqrt{h^2+K^2}$

$\rightarrow r = \sqrt{\frac{a^2}{4}+\frac{b^2}{4}} = \frac{\sqrt{a^2+b^2}}{2}$

$\rightarrow r = \frac{\sqrt{a^2+b^2}}{4}$ $\quad \because h = \frac{a}{2}, K = \frac{b}{2}$

Now eq. of circle is

$(x-h)^2+(y-K)^2 = r^2$

$\rightarrow (x-\frac{a}{2})^2+(y-\frac{b}{2})^2 = \frac{a^2+b^2}{4}$

$\rightarrow x^2+\frac{a^2}{4}-ax+y^2+\frac{b^2}{4}-by = \frac{a^2+b^2}{4}$

$\rightarrow 4x^2+a^2-4ax+4y^2+b^2-4by = a^2+b^2$

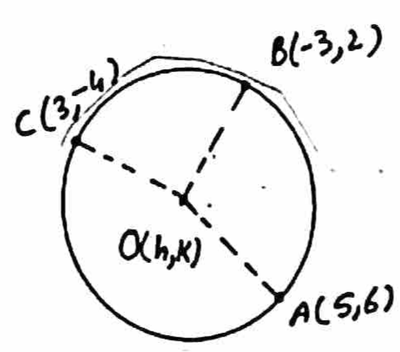
$\rightarrow 4x^2+4y^2-4ax-4by=0$

$\rightarrow x^2+y^2-ax-by=0 \quad (\div by\ 4)$

(d) $A(5,6)$, $B(-3,2)$, $C(3,-4)$

Solution:-

Clearly from fig.,



$|AO|^2 = |BO|^2$

$(h-5)^2+(K-6)^2 = (h+3)^2+(K-2)^2$

$h^2+25-10h+K^2+36-12K = h^2+9+6K+K^2+4-4K$

$-10h-12K+61 = 6h-4K+13$

$\rightarrow -10h-6h-12K+4K+61-13=0$

$\rightarrow -16h-8K+48=0$

$\rightarrow 2h+K-6=0 \quad (\div by -8) \longrightarrow (i)$

Also $|AO|^2 = |CO|^2$

$(h-5)^2+(K-6)^2 = (h-3)^2+(K+4)^2$

$h^2+25-10h+K^2+36-12K = h^2+9-6h+K^2+16+8K$

$-10h+6h-12K-8K+61-25=0$

$-4h-20K+36=0$

$\rightarrow -2h-10K+18=0 \longrightarrow (ii)$

By (i)+(ii) $\rightarrow$

$$\begin{array}{r} 2h+K-6=0 \\ -2h-10K+18=0 \\ \hline -9K+12=0 \end{array}$$

$\rightarrow -9K=-12 \rightarrow K = \frac{4}{3}$ Put in (i)

$2h+\frac{4}{3}-6=0 \rightarrow 6h+4-18=0$

$\rightarrow 6h-14=0 \rightarrow 6h=14 \rightarrow h = 7/3$

As $r = |AO| = \sqrt{(h-5)^2+(K-6)^2}$

$\rightarrow r = \sqrt{(\frac{7}{3}-5)^2+(\frac{4}{3}-6)^2}$

$r = \sqrt{(\frac{7-15}{3})^2+(\frac{4-18}{3})^2}$

$r = \sqrt{(-\frac{8}{3})^2+(-\frac{14}{3})^2} = \sqrt{\frac{64}{9}+\frac{196}{9}}$

$r = \sqrt{\frac{64+196}{9}} \rightarrow r = \sqrt{\frac{260}{9}}$

Now eq. of circle is

$(x-h)^2+(y-K)^2 = r^2$

$\rightarrow (x-\frac{7}{3})^2+(y-\frac{4}{3})^2 = \frac{260}{9}$

$\rightarrow (\frac{3x-7}{3})^2+(\frac{3y-4}{3})^2 = \frac{260}{9}$

$\rightarrow \frac{9x^2+49-42x}{9}+\frac{9y^2+16-24y}{9} = \frac{260}{9}$

$\rightarrow 9x^2+49-42x+9y^2+16-24y-260=0$

$9x^2+9y^2-42x-24y-195=0$

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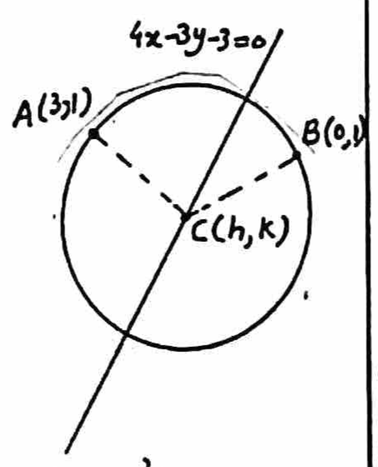
$3x^2+3y^2-14x-8y-65=0 \quad (\div \text{by } 3)$

$\rightarrow 3(x^2+y^2)-14x-8y-65=0$

Q4. In each of the following, find an equation of the circle passing through

(a) $A(3,-1)$, $B(0,1)$ and having centre at $4x-3y-3=0$

Solution:-



$\because C(h,K)$ lies on line so

$4h-3K-3=0 \longrightarrow (i)$

Also from fig

$|AC|^2=|BC|^2$

$\rightarrow (h-3)^2+(K+1)^2 = (h-0)^2+(K-1)^2$

$h^2-6h+9+K^2+1+2K = h^2+K^2+1-2K$

$-6h+2K+10=-2K+1$

$\rightarrow -6h+4K+9=0 \longrightarrow (ii)$

By $4(i)+3(ii)$, we get

$16h-12K-12=0$

$-18h+12K+27=0$

$-2h \quad +15=0 \rightarrow h=\frac{15}{2}$

so $(i) \rightarrow 4\left(\frac{15}{2}\right)-3K-3=0$

$\rightarrow 30-3K-3=0 \rightarrow -3K+27=0$

$-3K=-27 \rightarrow K=9$

Now $r=|AC|=\sqrt{(h-3)^2+(K+1)^2}$

$\rightarrow r=\sqrt{\left(\frac{15}{2}-3\right)^2+(9+1)^2}$

$=\sqrt{\left(\frac{9}{2}\right)^2+(10)^2} = \sqrt{\frac{81}{4}+100}$

$r=\sqrt{\frac{481}{4}}$

Now eq. of circle is

$(x-h)^2+(y-K)^2=r^2$

$\rightarrow \left(x-\frac{15}{2}\right)^2+(y-9)^2=\left(\sqrt{\frac{481}{4}}\right)^2$

$\rightarrow x^2-15x+\frac{225}{4}+y^2+81-18y=\frac{481}{4}$

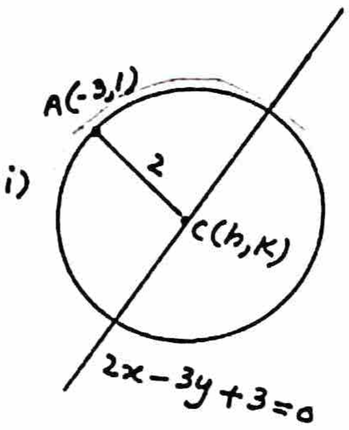
$x^2+y^2-15x-18y+\frac{225-481+324}{4}=0$

$x^2+y^2-15x-18y+\frac{68}{4}=0$

$\rightarrow x^2+y^2-15x-18y+17=0$

(b) $A(-3,1)$ with radius 2 and centre at $2x-3y+3=0$

Solution:-



$\because C(h,K)$ lies on line so

$2h-3K+3=0 \longrightarrow (i)$

Also $|AC|=2$

$\rightarrow |AC|^2=4$

$(h+3)^2+(K-1)^2=4$

$h^2+9+6h+K^2+1-2K=4$

$h^2+K^2+6h-2K+6=0 \longrightarrow (ii)$

From (i) $2h=3K-3$

$\rightarrow h=\frac{3K-3}{2}$ put in (ii)

$\left(\frac{3K-3}{2}\right)^2+K^2+6\left(\frac{3K-3}{2}\right)-2K+6=0$

$\frac{9K^2+9-18K}{4}+K^2+9K-9-2K+6=0$

$9K^2+9-18K+4K^2+36K-36-8K+24=0$

$13K^2+10K-3=0$

$\rightarrow 13K^2+13K-3K-3=0$

$13K(K+1)-3(K+1)=0$

$(13K-3)(K+1)=0$

$\rightarrow 13K-3=0, \quad K+1=0$

$K=\frac{3}{13}, \quad K=-1$

when $K=-1$ then $h=\frac{3(-1)-3}{2}=-3$

when $K=\frac{3}{13}$ then $h=\frac{3\left(\frac{3}{13}\right)-3}{2}$

$h=\frac{9-39}{13}\times\frac{1}{2}=-\frac{30}{13}\times\frac{1}{2}$

$h=-\frac{15}{13}$

when $h=-3$, $K=-1$ then eq. of circle is $(x-h)^2+(y-K)^2=r^2$

$\rightarrow (x+3)^2+(y+1)^2=(2)^2 \quad \because r=2$

$x^2+9+6x+y^2+1+2y=4$

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$x^2+y^2+6x+2y+6=0$

when $h=-\frac{15}{13}$ and $K=\frac{3}{13}$ then

eq. of circle is $(x-h)^2+(y-K)^2=r^2$

$\rightarrow \left(x+\frac{15}{13}\right)^2+\left(y-\frac{3}{13}\right)^2=4 \quad \because r=2$

$\left(\frac{13x+15}{13}\right)^2+\left(\frac{13y-3}{13}\right)^2=4$

$\frac{169x^2+225+390x}{169}+\frac{169y^2-78y+9}{169}=4$

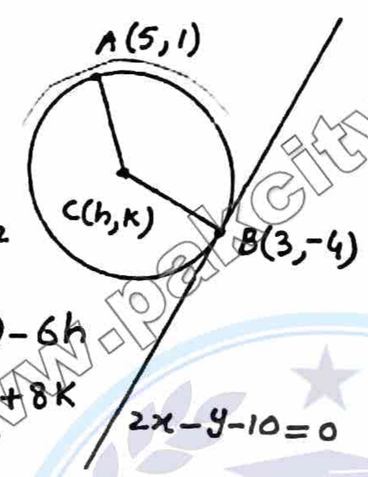
$\rightarrow 169x^2+169y^2+390x-78y+225+9-676=0$

$\rightarrow 169(x^2+y^2)+390x-78y-442=0$

(c) A(5,1) and tangent to the line $2x-y-10=0$ at B(3,-4)

Solution:-

It is clear from figure that



$|AC|^2=|BC|^2$

$(h-5)^2+(K-1)^2=(h-3)^2+(K+4)^2$

$h^2+25-10h+K^2+1-2K=h^2+9-6h+K^2+16+8K$

$-10h-2K+26=-6h+8K+25$

$\rightarrow -4h-10K+1=0 \longrightarrow (i)$

slope of $2x-y-10=0=\frac{-a}{b}=\frac{-2}{-1}=2$

slope of $BC=\frac{K+4}{h-3}$

$\because$ both lines are $\perp$ar so

$(2)\left(\frac{K+4}{h-3}\right)=-1$

$\rightarrow 2K+8=-h+3$

$\rightarrow h+2K+5=0$

$\rightarrow 5h+10K+25=0 \longrightarrow (ii)$

By (i) + (ii) $\rightarrow$
$$\begin{array}{r} 5h+10K+25=0 \\ -4h-10K+1=0 \\ \hline h+26=0 \end{array}$$

$\rightarrow h=-26$ put in (ii)

$-26+2K+5=0$

$\rightarrow 2K-21=0 \rightarrow K=\frac{21}{2}$

$r=|AC|=\sqrt{(h-5)^2+(K-1)^2}$

$\rightarrow r=\sqrt{(-26-5)^2+\left(\frac{21}{2}-1\right)^2}$

$=\sqrt{(-31)^2+\left(\frac{19}{2}\right)^2}$

$=\sqrt{961+\frac{361}{4}}=\sqrt{\frac{3844+361}{4}}$

$r=\frac{\sqrt{4205}}{2}$

Now eq. of circle is

$(x-h)^2+(y-K)^2=r^2$

$\rightarrow (x+26)^2+\left(y-\frac{21}{2}\right)^2=\left(\frac{\sqrt{4205}}{2}\right)^2$

$x^2+52x+676+y^2-21y+\frac{441}{4}=\frac{4205}{4}$

$x^2+y^2+52x-21y+\frac{441}{4}-\frac{4205}{4}=0$

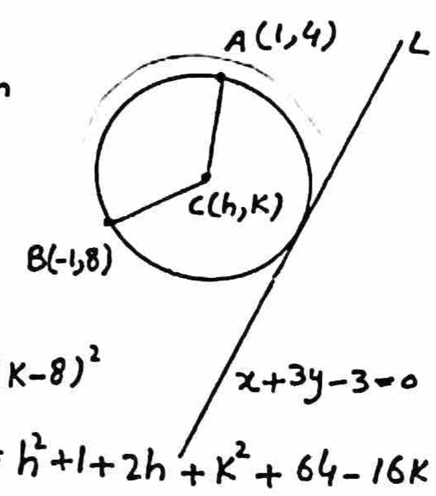
$x^2+y^2+52x-21y+\left(-\frac{1016}{4}\right)=0$

$x^2+y^2+52x-21y-265=0$

(d) A(1,4), B(-1,8) and tangent to the line $x+3y-3=0$

Solution:-

It is clear from fig



$|AC|^2=|BC|^2$

$(h-1)^2+(K-4)^2=(h+1)^2+(K-8)^2$

$h^2+1-2h+K^2+16-8K=h^2+1+2h+K^2+64-16K$

$-2h-8K+16=2h+1-16K+64$

$-4h+8K-48=0$

$\rightarrow h-2K+12=0$

$\rightarrow h=2K-12 \longrightarrow (i)$

Now $|AC|=|CL|$ where L is tangent line

$\sqrt{(h-1)^2+(K-4)^2}=\frac{|h+3K-31|}{\sqrt{(1)^2+(3)^2}}$

$\sqrt{h^2-2h+1+K^2-8K+16}=\frac{|h+3K-31|}{\sqrt{10}}$

squaring both sides

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$$h^2-2h+K^2-8K+17 = \frac{h^2+9K^2+9+6hK-18K-6h}{10}$$

$$\rightarrow 10h^2-20h+10K^2-80K+170 = h^2+9K^2+9+6hK-18K-6h$$

$$\rightarrow 9h^2+K^2-14h-62K+161-6hK=0 \longrightarrow (ii)$$

Put (i) in (ii)

$$9(2K-12)^2+K^2-14(2K-12)-62K+161-6(2K-12)K=0$$

$$9(4K^2+144-48K)+K^2-28K+168-62K+161-12K^2+72K=0$$

$$\rightarrow 36K^2+1296-432K+K^2-28K+168-62K+161-12K^2+72K=0$$

$$\rightarrow 25K^2-450K+1625=0$$

$$\rightarrow K^2-18K+65=0 \quad (\div \text{ by } 25)$$

$$\rightarrow K^2-13K-5K+65=0$$

$$K(K-13)-5(K-13)=0$$

$$(K-13)(K-5)=0$$

$$K-13=0 \quad \text{or} \quad K-5=0$$

$$\rightarrow K=13 \quad , \quad K=5$$

when $K=13$ then $h=2(13)-12=14$

when $K=5$ then $h=2(5)-12=-2$

so $C_1(-2,5)$ and $C_2(14,13)$.

Now

$$r_1=|AC_1|=\sqrt{(1+2)^2+(4-5)^2}=\sqrt{10}$$

$$r_2=|AC_2|=\sqrt{(1-14)^2+(4-13)^2}=\sqrt{250}$$

Now eq. of circle is

$$(x+2)^2+(y-5)^2=(\sqrt{10})^2$$

$$\rightarrow x^2+4+4x+y^2+25-10y-10=0$$

$$\rightarrow x^2+y^2+4x-10y+19=0$$

Also eq. of circle is

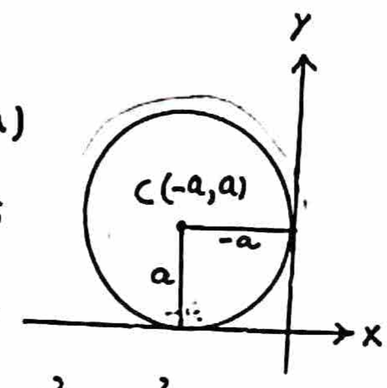
$$(x-14)^2+(y-13)^2=(\sqrt{250})^2$$

$$\rightarrow x^2-28x+196+y^2+169-26y-250=0$$

$$x^2+y^2-28x-26y+115=0$$

Q5. Find an equation of a circle of radius 'a' and lying in the second quadrant such that it is tangent to both the axes.

Solution:-



$\because$ radius $= a$

so centre $=(-a,a)$

$\because$ eq. of circle is

$$(x-h)^2+(y-K)^2=r^2$$

$$\rightarrow (x-(-a))^2+(y-a)^2=a^2$$

$$\rightarrow (x+a)^2+(y-a)^2=a^2$$

$$x^2+a^2+2ax+y^2+a^2-2ay=a^2$$

$$\rightarrow x^2+y^2+2ax-2ay+a^2=0$$

Q6. Show that the lines $3x-2y=0$ and $2x+3y-13=0$ are tangents to the circle

$$x^2+y^2+6x-4y=0$$

Solution:-

$$x^2+y^2+6x-4y=0$$

compare with

$$x^2+y^2+2gx+2fy+c=0$$

$$2g=6 \rightarrow g=3, \quad 2f=-4 \rightarrow f=-2$$

$$c=0$$

$$\because C(-g,-f)=(-3,2)$$

$$r=\sqrt{g^2+f^2-c}=\sqrt{(3)^2+(-2)^2-0}$$

$$r=\sqrt{9+4}=\sqrt{13}$$

Distance of centre from line $3x-2y=0$

$$=\frac{|3(-3)-2(2)+0|}{\sqrt{(3)^2+(-2)^2}}=\frac{|-9-4|}{\sqrt{13}}$$

$$=\frac{13}{\sqrt{13}}=\sqrt{13}=r$$

$\rightarrow$ Hence line $3x-2y=0$ is tangent.

Now Distance of centre from line $2x+3y-13=0$

$$=\frac{|2(-3)+3(2)-13|}{\sqrt{(2)^2+(3)^2}}=\frac{|-6+6-13|}{\sqrt{13}}$$

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$= \frac{13}{\sqrt{13}} = \sqrt{13} = r$

→ Hence line $2x+3y-13=0$ is tangent.

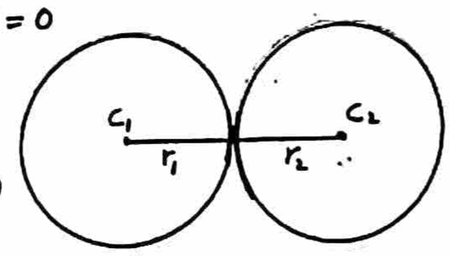
Q7. Show that $x^2+y^2+2x-2y-7=0$ and $x^2+y^2-6x+4y+9=0$ touch externally.

Solution:- For circle

$x^2+y^2+2x-2y-7=0$

compare with

$x^2+y^2+2gx+2fy+c=0$



$2g = 2 \rightarrow g = 1$, $2f = -2 \rightarrow f = -1$

$c = -7$

$C_1(-g,-f) = (-1,1)$, $r_1 = \sqrt{g^2+f^2-c}$

$\rightarrow r_1 = \sqrt{(1)^2+(-1)^2+7} = \sqrt{9} = 3$

For circle

$x^2+y^2-6x+4y+9=0$

$2g = -6 \rightarrow g = -3$, $2f = 4 \rightarrow f = 2$

$C_2(-g,-f) = (3,-2)$, $r_2 = \sqrt{g^2+f^2-c}$

$\rightarrow r_2 = \sqrt{(-3)^2+(2)^2-9} = \sqrt{4} = 2$

Now

$|C_1C_2| = \sqrt{(-1-3)^2+(1+2)^2} = \sqrt{16+9}$

$= \sqrt{25} = 5 = 3+2 = r_1+r_2$

$\rightarrow |C_1C_2| = r_1+r_2$ → Hence touches externally.

Q8. Show that the circles $x^2+y^2+2x-8=0$ and $x^2+y^2-6x+6y-46=0$ touch internally.

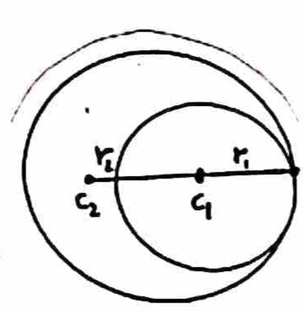
Solution:-

For circle

$x^2+y^2+2x-8=0$

compare with

$x^2+y^2+2gx+2fy+c=0$



$2g = 2 \rightarrow g = 1$, $2f = 0 \rightarrow f = 0$

$c = -8$

For circle

$x^2+y^2-6x+6y-46=0$

compare with

$x^2+y^2+2gx+2fy+c=0$

$2g = -6 \rightarrow g = -3$, $2f = 6 \rightarrow f = 3$

$c = -46$

$C_1(-g,-f) = (-1,0)$, $r_1 = \sqrt{g^2+f^2-c}$

$\rightarrow r_1 = \sqrt{(-1)^2+(0)^2+8}$

$\rightarrow r_1 = \sqrt{9} = 3$

$C_2(-g,-f) = (3,-3)$, $r_2 = \sqrt{g^2+f^2-c}$

$\rightarrow r_2 = \sqrt{(-3)^2+(3)^2+46} = \sqrt{9+9+46}$

$r_2 = \sqrt{64} = 8$

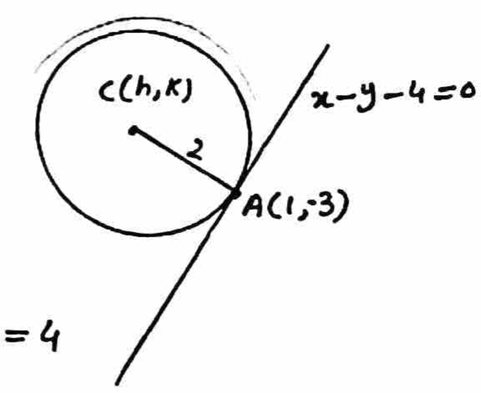
Now $|C_1C_2| = \sqrt{(3+1)^2+(-3-0)^2} = \sqrt{25} = 5$

$\rightarrow |C_1C_2| = 5 = 8-3 = r_2-r_1$

$\rightarrow |C_1C_2| = r_2-r_1$ → Hence touches internally.

Q9. Find equations of circles of radius 2 and tangent to line $x-y-4=0$ at $(1,-3)$

Solution:-



$\because |AC| = 2$

$\rightarrow |AC|^2 = 4$

$(h-1)^2+(K+3)^2 = 4$

$h^2+1-2h+K^2+9+6K = 4$

$\rightarrow h^2+K^2-2h+6K+6 = 0 \longrightarrow (i)$

Slope of $x-y-4=0 = -\frac{a}{b} = \frac{-1}{-1} = 1$

$\therefore$ slope of $AC = \frac{K+3}{h-1}$

$\because$ Both lines are $\perp$ar so

$\left(\frac{K+3}{h-1}\right)(1) = -1 \rightarrow K+3 = -h+1$

$\rightarrow h+K+2 = 0$

or $h = -K-2$ put in (i)

$(-K-2)^2+K^2-2(-K-2)+6K+6 = 0$

$K^2+4+4K+K^2+2K+4+6K+6 = 0$

$2K^2+12K+14 = 0$

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$\rightarrow K^2 + 6K + 7 = 0 \qquad (\div \text{ by } 2)$

$\rightarrow K = \dfrac{-6 \pm \sqrt{36-28}}{2} = \dfrac{-6 \pm \sqrt{8}}{2}$

$K = \dfrac{-6 \pm 2\sqrt{2}}{2} = \dfrac{\cancel{2}(-3 \pm \sqrt{2})}{\cancel{2}}$

$K = -3 + \sqrt{2}$ or $K = -3 - \sqrt{2}$

when $K = -3 + \sqrt{2}$ then

$h = -(-3+\sqrt{2}) - 2 = 3 - \sqrt{2} - 2$

$h = 1 - \sqrt{2}$

when $K = -3 - \sqrt{2}$ then

$h = -(-3-\sqrt{2}) - 2 = 3 - \sqrt{2} - 2$

$h = 1 + \sqrt{2}$

when $h = 1 - \sqrt{2}$, $K = -3 + \sqrt{2}$, $r = 2$

Eq. of circle is

$(x - 1 + \sqrt{2})^2 + (y + 3 - \sqrt{2})^2 = 4$

Also when $h = 1 + \sqrt{2}$, $K = -3 - \sqrt{2}$, $r = 2$

Eq. of circle is

$(x - 1 - \sqrt{2})^2 + (y + 3 + \sqrt{2})^2 = 4$

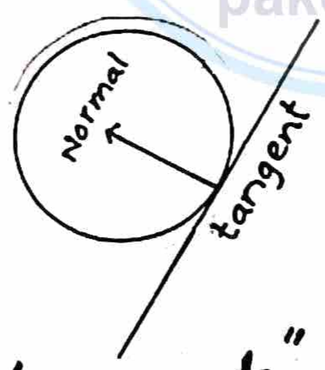
Tangents and Normals

Tangent to a circle:-

"A line which touches the circle without cutting it, is called tangent line."

Normal to a circle:-

"A line $\perp$ar to the tangent line at point of tandency, is called normal line."



For circle $x^2 + y^2 = r^2$

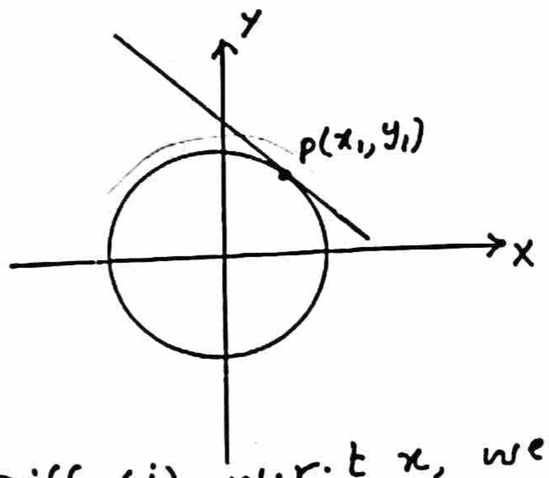
"Equation of tangent"

Given equation of circle is

$x^2 + y^2 = r^2 \longrightarrow (i)$

$\because$ point $P(x_1, y_1)$ lies on the circle so (i) becomes

as $x_1^2 + y_1^2 = r^2 \longrightarrow (ii)$



Diff (i) w.r.t x, we get

$\dfrac{d}{dx}(x^2 + y^2) = \dfrac{d}{dx}(r^2)$

$\rightarrow 2x + 2y\dfrac{dy}{dx} = 0$

$\rightarrow x + y\dfrac{dy}{dx} = 0 \rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$

$\because m =$ slope of tangent line at $P(x_1, y_1)$

so $m = \left.\dfrac{dy}{dx}\right|_P = -\dfrac{x_1}{y_1}$

using $y - y_1 = m(x - x_1)$

$\rightarrow y - y_1 = -\dfrac{x_1}{y_1}(x - x_1)$

$\rightarrow yy_1 - y_1^2 = -xx_1 + x_1^2$

$\rightarrow xx_1 + yy_1 = x_1^2 + y_1^2$

$\boxed{xx_1 + yy_1 = r^2}$ $\quad \because x_1^2 + y_1^2 = r^2$ By (ii)

which is equation of tangent line.

"Equation of Normal"

$\because$ slope of normal line $= \dfrac{y_1}{x_1}$

so eq. of normal line at $P(x_1, y_1)$ is

$y - y_1 = \dfrac{y_1}{x_1}(x - x_1)$

$\rightarrow x_1 y - x_1 y_1 = xy_1 - x_1 y_1$

$\rightarrow x_1 y = xy_1$

$\rightarrow \boxed{\dfrac{y}{y_1} = \dfrac{x}{x_1}}$

which is equation of normal line.

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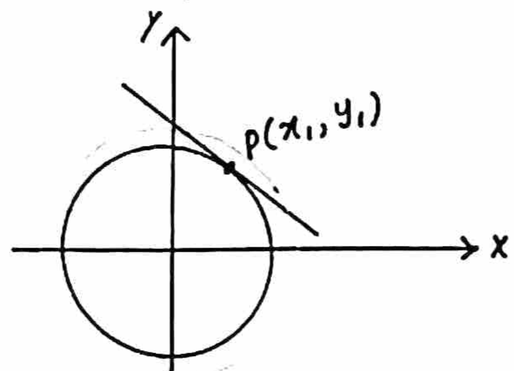
For circle

$x^2+y^2+2gx+2fy+c=0$

"Equation of tangent"



Given equation of circle is

$x^2+y^2+2gx+2fy+c=0 \longrightarrow (i)$

$\because P(x_1,y_1)$ lies on the circle so eq(i) becomes as

$x_1^2+y_1^2+2gx_1+2fy_1+c=0 \longrightarrow (ii)$

Diff (i) w.r.t x, we get

$\frac{d}{dx}(x^2+y^2+2gx+2fy+c)=\frac{d}{dx}(0)$

$\rightarrow 2x+2y\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$

$x+y\frac{dy}{dx}+g+f\frac{dy}{dx}=0$

$(y+f)\frac{dy}{dx}=-(x+g)$

or $\frac{dy}{dx}=-\left(\frac{x+g}{y+f}\right)$

m = slope of tangent line at point $P(x_1,y_1)$

$m=-\left(\frac{x_1+g}{y_1+f}\right)$

using $y-y_1=m(x-x_1)$

$\rightarrow y-y_1=-\left(\frac{x_1+g}{y_1+f}\right)(x-x_1)$

$(y-y_1)(y_1+f)=-(x-x_1)(x_1+g)$

$(y-y_1)y_1+(y-y_1)f=-\{(x-x_1)x_1+g(x-x_1)\}$

$yy_1-y_1^2+yf-y_1f=-\{xx_1-x_1^2+xg-x_1g\}$

$yy_1-y_1^2+fy-fy_1=-xx_1+x_1^2-xg+x_1g$

$xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1$

adding gx_1+fy_1+c on both sides

$xx_1+yy_1+gx+gx_1+fy+fy_1+c$
$=x_1^2+y_1^2+2gx_1+2fy_1+c$

$\rightarrow \boxed{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0}$

$\because x_1^2+y_1^2+2gx_1+2fy_1+c=0$ by (ii)

"Equation of Normal"

Now slope of normal line is $\frac{y_1+f}{x_1+g}$

Equation of normal line is

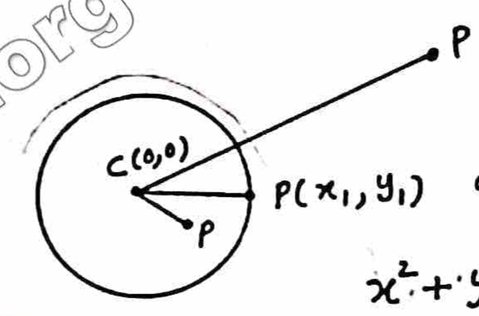
$y-y_1=\left(\frac{y_1+f}{x_1+g}\right)(x-x_1)$

$\rightarrow \boxed{\frac{x-x_1}{x_1+g}=\frac{y-y_1}{y_1+f}}$

Theorem:- The point $P(x_1,y_1)$ lies outside, on or inside the circle, $x^2+y^2=r^2$ according as

$x_1^2+y_1^2-r^2 \gtreqless 0$

Proof:-



Given eq. of circle is

$x^2+y^2=r^2$

centre $=(0,0)$, radius $=r$

Now point $P(x_1,y_1)$ will lie outside, on, inside the circle $x^2+y^2=r^2$

if $|CP| \gtreqless r$

$\rightarrow \sqrt{(x_1-0)^2+(y_1-0)^2} \gtreqless r$

squaring both sides

$\rightarrow x_1^2+y_1^2 \gtreqless r^2$

$\rightarrow x_1^2+y_1^2-r^2 \gtreqless 0$

Hence proved.

Theorem:- The point $P(x_1,y_1)$ lies outside, on or inside the circle $x^2+y^2+2gx+2fy+c=0$ according as

$x_1^2+y_1^2+2gx_1+2fy_1+c \gtreqless 0$

Proof:-

(See on next page)

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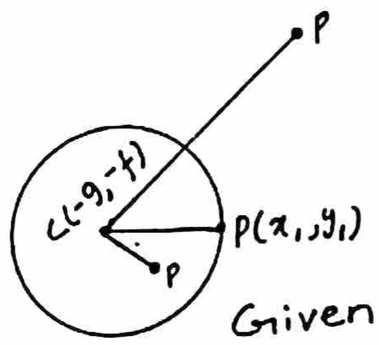


Given eq. of circle is

$$x^2+y^2+2gx+2fy+c=0$$

centre $=(-g,-f)$, radius $=\sqrt{g^2+f^2-c}$

Now point $P(x_1,y_1)$ will lie outside, on or inside the circle $x^2+y^2+2gx+2fy+c=0$ if $|CP| \gtreqless r$

$\rightarrow \sqrt{(x_1-(-g))^2+(y_1-(-f))^2} \gtreqless \sqrt{g^2+f^2-c}$

squaring both sides

$\rightarrow (x_1+g)^2+(y_1+f)^2 \gtreqless g^2+f^2-c$

$\rightarrow x_1^2+g^2+2gx_1+y_1^2+f^2+2fy_1 \gtreqless g^2+f^2-c$

$\rightarrow x_1^2+y_1^2+2gx_1+2fy_1+g^2+f^2-g^2-f^2+c \gtreqless 0$

$\rightarrow x_1^2+y_1^2+2gx_1+2fy_1+c \gtreqless 0$

Hence proved

Example 1. Determine whether the point $P(-5,6)$ lies outside, on or inside the circle $x^2+y^2+4x-6y-12=0$

Solution:-

For Point $(-5,6)$

$x^2+y^2+4x-6y-12$

$=(-5)^2+(6)^2+4(-5)-6(6)-12$

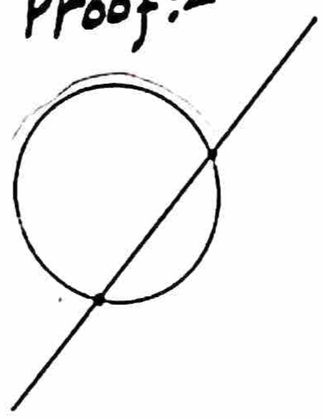
$=25+36-20-36-12$

$=25-32=-7<0$

$\rightarrow$ Point $(-5,6)$ lies inside the circle.

Theorem:- The line $y=mx+c$ intersects the circle $x^2+y^2=a^2$ in at the most two points.

Proof:-



Given equation of circle is

$$x^2+y^2=a^2 \longrightarrow (i)$$

Also given equation of line is

$$y=mx+c \longrightarrow (ii)$$

Put value of y in (i)

$\rightarrow x^2+(mx+c)^2=a^2$

$\rightarrow x^2+m^2x^2+c^2+2mcx-a^2=0$

or $(1+m^2)x^2+2mcx+c^2-a^2=0 \longrightarrow (iii)$

This being quadratic in x, gives two values of x.

Thus the line intersects the circle $x^2+y^2=a^2$ at the most two points.

Nature of points

For nature of points, we examine the discriminant b^2-4ac of eq (iii). The points are

a) Real and distinct if

$$(2mc)^2-4(1+m^2)(c^2-a^2)>0$$

b) Real and coincident if

$$(2mc)^2-4(1+m^2)(c^2-a^2)=0$$

c) Imaginary if

$$(2mc)^2-4(1+m^2)(c^2-a^2)<0$$

Condition of tangency

For condition of tangency;

$$b^2-4ac=0$$

$\rightarrow (2mc)^2-4(1+m^2)(c^2-a^2)=0$

$\rightarrow 4m^2c^2-4(c^2-a^2+m^2c^2-m^2a^2)=0$

or $\cancel{m^2c^2}-c^2+a^2-\cancel{m^2c^2}+m^2a^2=0$

$\rightarrow a^2(1+m^2)=c^2$

or $c=\pm a\sqrt{1+m^2}$

Required condition of tangency.

By (ii) $y=mx\pm a\sqrt{1+m^2}$

which is eq. of tangents to circle $x^2+y^2=a^2$

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Example 2. Find the coordinates of points of intersection of the line $2x+y=5$ and the circle $x^2+y^2+2x-9=0$. Also find the length of the intercepted chord.

Solution:-

Given eq. of circle is

$x^2+y^2+2x-9=0 \longrightarrow (i)$

given line is $2x+y=5$

$\rightarrow y=5-2x \longrightarrow (ii)$ Put in (i)

$\rightarrow x^2+(5-2x)^2+2x-9=0$

$x^2+25+4x^2-20x+2x-9=0$

$5x^2-18x+16=0$

$5x^2-10x-8x+16=0$

$5x(x-2)-8(x-2)=0$

$\rightarrow (x-2)(5x-8)=0$

$\rightarrow x-2=0$ or $5x-8=0$

$x=2$, $x=\frac{8}{5}$

So (ii) $y=5-2(2)=5-4=1$, For $x=2$

and $y=5-2(\frac{8}{5})$

$y=\frac{25-16}{5}=\frac{9}{5}$, For $x=\frac{8}{5}$

so Points of intersection are $P(2,1)$ and $Q(\frac{8}{5},\frac{9}{5})$. Now Length of chord intercepted

$=|PQ|=\sqrt{(\frac{8}{5}-2)^2+(\frac{9}{5}-1)^2}$

$|PQ|=\sqrt{(\frac{8-10}{5})^2+(\frac{9-5}{5})^2}$

$=\sqrt{(-\frac{2}{5})^2+(4/5)^2}$

$=\sqrt{\frac{4+16}{25}}=\sqrt{\frac{20}{25}}$

$|PQ|=\sqrt{\frac{4\times5}{5\times5}}=\frac{2}{\sqrt{5}}$

Theorem:- Two tangents can be drawn to a circle from any point $P(x_1,y_1)$. The tangents are real and distinct, coincident or imaginary according as the point lies outside, on or inside the circle.

Proof:-

consider eq. of circle

$x^2+y^2=a^2 \longrightarrow (i)$

$\because y=mx+a\sqrt{1+m^2} \longrightarrow (ii)$

is eq. of either tangent to the circle $x^2+y^2=a^2$

Let the tangent line passes through point $P(x_1,y_1)$, then equation (ii) becomes

$y_1=mx_1+a\sqrt{1+m^2}$

$\rightarrow y_1-mx_1=a\sqrt{1+m^2}$

squaring both sides

$(y_1-mx_1)^2=(a\sqrt{1+m^2})^2$

$\rightarrow y_1^2+m^2x_1^2-2mx_1y_1=a^2(1+m^2)$

$x_1^2m^2-a^2m^2-2mx_1y_1+y_1^2-a^2=0$

$(x_1^2-a^2)m^2-2x_1y_1m+y_1^2-a^2=0 \longrightarrow (iii)$

This being quadratic in m. It will give two values of m.

Thus two tangents can be drawn to a circle from any point $P(x_1,y_1)$

Nature of tangents

For nature of tangents, we examine the discriminant of eq (iii) i.e.,

The tangents are

a) Real and distinct if $b^2-4ac>0$

b) Real and coincident if $b^2-4ac=0$

c) Imaginary if $b^2-4ac<0$

So $(-2x_1y_1)^2-4(x_1^2-a^2)(y_1^2-a^2)\gtreqless 0$

$\rightarrow 4x_1^2y_1^2-4(x_1^2y_1^2-a^2x_1^2-a^2y_1^2+a^4)\gtreqless 0$

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$\rightarrow x_1^2y_1^2 - x_1^2y_1^2 + a^2x_1^2 + a^2y_1^2 - a^4 \gtreqless 0$

$a^2(x_1^2+y_1^2) - a^4 \gtreqless 0$

$\rightarrow x_1^2 + y_1^2 - a^2 \gtreqless 0$

which is condition for a point $P(x_1, y_1)$ lying outside, on or inside the circle $x^2+y^2=a^2$

Example 3. Write equation of two tangents from (2,3) to the circle $x^2+y^2=9$

Solution:-

Given eq of circle $x^2+y^2=9$

compare with $x^2+y^2=a^2$

$\rightarrow a^2=9 \rightarrow a=3$

$\because y = mx + a\sqrt{1+m^2} \longrightarrow (i)$

Now eq. of tangent to circle $x^2+y^2=9$ from (2,3) is

$3 = 2m + \sqrt{1+m^2}$

$\rightarrow 3-2m = 3\sqrt{1+m^2}$

$\rightarrow (3-2m)^2 = [3\sqrt{1+m^2}]^2$

$\rightarrow 9+4m^2-12m = 9(1+m^2)$

$\rightarrow 9+4m^2-12m = 9+9m^2$

or $9m^2-4m^2+12m=0$

$5m^2+12m=0$

$m(5m+12)=0$

$m=0, \quad 5m+12=0$

$m=0$ or $m=-\frac{12}{5}$

For $a=3$, $m=0$ (i) becomes

$y=3$

For $a=3$, $m=-\frac{12}{5}$ (i) becomes

$\rightarrow y = -\frac{12}{5}x + 3\sqrt{1+(-\frac{12}{5})^2}$

$\rightarrow y = -\frac{12}{5}x + 3\sqrt{\frac{25+144}{25}}$

$\rightarrow y = -\frac{12}{5}x + 3(\frac{13}{5})$

$\rightarrow 5y = -12x + 39$

or $12x+5y-39=0$

Thus $y=3$ and $12x+5y-39=0$ are eqs. of required two tangents.

Example 4. Write equations of tangents to the circle $x^2+y^2-4x+6y+9=0$ at the points on the circle whose ordinate is -2.

Solution:-

Given eq. of circle is

$x^2+y^2-4x+6y+9=0 \longrightarrow (i)$

compare with

$x^2+y^2+2gx+2fy+c=0$

$2g=-4 \rightarrow g=-2$, $2f=6 \rightarrow f=3$, $c=9$

$\because$ ordinate $=-2$ (given)

$\rightarrow y=-2$ put in (i)

$x^2+(-2)^2-4x+6(-2)+9=0$

$\rightarrow x^2+4-4x-12+9=0$

$x^2-4x+1=0$

$\rightarrow x = \frac{-(-4) \pm \sqrt{(-4)^2-4(1)(1)}}{2(1)}$

$= \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2}$

$= \frac{4 \pm 2\sqrt{3}}{2} = \frac{2(2 \pm \sqrt{3})}{2}$

$x = 2 \pm \sqrt{3}$

$\rightarrow x = 2+\sqrt{3}$ or $x = 2-\sqrt{3}$

Now we have two points $P(2+\sqrt{3}, -2)$ and $Q(2-\sqrt{3}, -2)$

we know eq. of tangent to circle $x^2+y^2+2gx+2fy+c=0$ is $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

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for $g=-2,\ f=3,\ c=9$ so

$$xx_1+yy_1-2(x+x_1)+3(y+y_1)+9=0 \longrightarrow \text{(ii)}$$

For $P(2+\sqrt{3},-2)$ so (ii)

$$(2+\sqrt{3})x+(-2)y-2(x+2+\sqrt{3})+3(y+(-2))+9=0$$

$$\rightarrow 2x+\sqrt{3}x-2y-2x-4-2\sqrt{3}+3y-6+9=0$$

$$\rightarrow \sqrt{3}x+y-2\sqrt{3}-1=0$$

For $Q(2-\sqrt{3},-2)$ so (ii)

$$(2-\sqrt{3})x+(-2)y-2(x+2-\sqrt{3})+3(y+(-2))+9=0$$

$$\rightarrow 2x-\sqrt{3}x-2y-2x-4+2\sqrt{3}+3y-6+9=0$$

$$-\sqrt{3}x+y+2\sqrt{3}-1=0$$

or $\sqrt{3}x-y-2\sqrt{3}+1=0$

Thus req. tangents are

$\sqrt{3}x+y-2\sqrt{3}-1=0$ and

$\sqrt{3}x-y-2\sqrt{3}+1=0$

Example 5. Find a joint equation to the pair of tangents drawn from $(5,0)$ to the circle $x^2+y^2=9$

Solution:-

Given eq. of circle $x^2+y^2=9$

Compare with $x^2+y^2=a^2$

$\rightarrow a^2=9 \rightarrow a=3$

Let $P(x_1,y_1)$ be any point on either of two tangents drawn from $A(5,0)$ to the circle $x^2+y^2=9$

then slope of tangent $AP=\dfrac{y_1-0}{x_1-5}=\dfrac{y_1}{x_1-5}$

Now eq. of tangent having slope $\dfrac{y_1}{x_1-5}$ and passing through $A(5,0)$ is using (point-slope form)

$$y-0=\frac{y_1}{x_1-5}(x-5)$$

$$\rightarrow (x_1-5)y_1=y_1x-y_1 5$$

$$\rightarrow y_1x-(x_1-5)y-5y_1=0 \longrightarrow \text{(i)}$$

The line (i) will be tangent to the circle $x^2+y^2=9$, if its distance from centre $(0,0)$ is equal to radius 3. Now

$$3=\frac{|y_1(0)-(x_1-5)(0)-5y_1|}{\sqrt{(y_1)^2+[-(x_1-5)]^2}}$$

$$\rightarrow 3=\frac{|-5y_1|}{\sqrt{y_1^2+(x_1-5)^2}}$$

$$\rightarrow 3\sqrt{y_1^2+(x_1-5)^2}=5y_1$$

$$\rightarrow 9(y_1^2+(x_1-5)^2)=25y_1^2$$

$$\rightarrow 9y_1^2+9(x_1-5)^2=25y_1^2$$

$$\rightarrow 9y_1^2+9(x_1-5)^2-25y_1^2=0$$

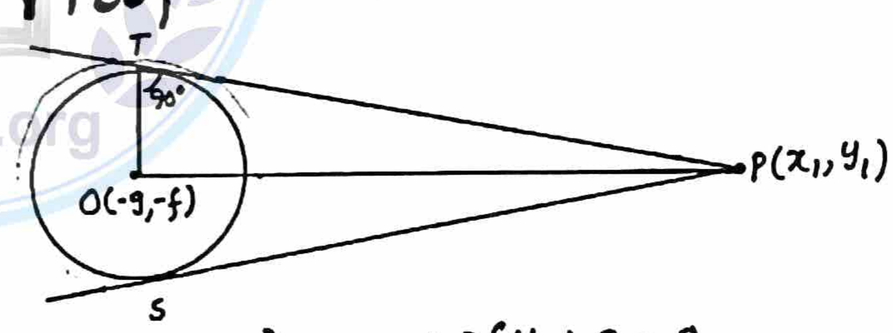
$$9(x_1-5)^2-16y_1^2=0$$

As (x_1,y_1) is point on either of two tangents. so

$9(x-5)^2-16y^2=0$ is req. joint eq. of two tangents.

Theorem:- Find length of tangent to the circle $x^2+y^2+2gx+2fy+c=0$, drawn from point $P(x_1,y_1)$ lying outside the circle.

Proof:- (Tangential Distance)



For $x^2+y^2+2gx+2fy+c=0$

Centre $=O(-g,-f)$

radius $=|TO|=\sqrt{g^2+f^2-c}$

$$\rightarrow |TO|^2=g^2+f^2-c$$

Let PT and PS be two tangents drawn from point $P(x_1,y_1)$ to the given circle

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from figure,

In rt. Δ PTO

$$|PO|^2 = |PT|^2 + |TO|^2$$

$$\rightarrow |PT|^2 = |PO|^2 - |TO|^2 \longrightarrow (i)$$

$$\because |PC| = \sqrt{(x_1-(-g))^2+(y_1-(-f))^2}$$

$$\rightarrow |PC|^2 = (x_1+g)^2+(y_1+f)^2$$

Now (i)

$$|PT|^2 = (x_1+g)^2+(y_1+f)^2-(g^2+f^2-c)$$

$$= x_1^2+g^2+2gx_1+y_1^2+f^2+2fy_1-g^2-f^2+c$$

$$\rightarrow |PT|^2 = x_1^2+y_1^2+2gx_1+2fy_1+c$$

$$\rightarrow |PT| = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

which is required length of tangent.

Example 6. Find the length of the tangent from the point $(-5,10)$ to the circle $5x^2+5y^2+14x+12y-10=0$

Solution:-

Given eq. of circle

$$5x^2+5y^2+14x+12y-10=0$$

$$\rightarrow x^2+y^2+\frac{14}{5}x+\frac{12}{5}y-2=0$$

Compare with

$$x^2+y^2+2gx+2fy+c=0$$

$$2g = \frac{14}{5}, \quad 2f = \frac{12}{5}$$

$$c = -2$$

Here $(x_1, y_1) = (-5, 10)$

$$\rightarrow x_1 = -5, \quad y_1 = 10$$

$\because$ length of tangent

$$= \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

$$= \sqrt{(-5)^2+(10)^2+\left(\frac{14}{5}\right)(-5)+\frac{12}{5}(10)-2}$$

$$= \sqrt{25+100-14+24-2}$$

$$= \sqrt{149-16} = \sqrt{133}$$

Example 7. Write equations of the tangent lines to the circle $x^2+y^2+4x+2y=0$ drawn from $P(-1,2)$. Also find the tangential distance.

Solution:-

Given eq. of circle is

$$x^2+y^2+4x+2y=0 \longrightarrow (i)$$

compare with

$$x^2+y^2+2gx+2fy+c=0$$

$$2g = 4 \rightarrow g = 2, \quad 2f = 2 \rightarrow f = 1, \quad c = 0$$

Centre $= C(-g,-f) = C(-2,-1)$

radius $= r = \sqrt{g^2+f^2-c} = \sqrt{(2)^2+(1)^2-0}$

$$\rightarrow r = \sqrt{5}$$

Now eq. of line through $P(-1,2)$ having slope m is

$$y-2 = m(x+1)$$

$$\rightarrow y-2 = mx+m$$

$$\rightarrow mx-y+m+2=0 \longrightarrow (ii)$$

$\because$ (i) is tangent to the circle (i) if its distance from centre $C(-2,-1)$ is equal to radius $\sqrt{5}$.

$$\rightarrow \sqrt{5} = \frac{|m(-2)+(-1)(-1)+m+2|}{\sqrt{m^2+(-1)^2}}$$

$$\rightarrow \sqrt{5}\sqrt{m^2+1} = |-2m+1+m+2|$$

$$\sqrt{5(m^2+1)} = |3-m|$$

$$\rightarrow 5m^2+5 = (3-m)^2 \quad \text{(squaring)}$$

$$\rightarrow 5m^2+5 = 9+m^2-6m$$

$$\rightarrow 4m^2+6m-4=0$$

or $2m^2+3m-2=0$

$$2m^2+4m-m-2=0$$

$$2m(m+2)-1(m+2)=0$$

$$(m+2)(2m-1)=0$$

$$m+2=0 \quad \text{or} \quad 2m-1=0$$

$$m=-2 \quad \text{or} \quad m=\frac{1}{2}$$

For $m=-2$ (ii) becomes

$$-2x-y+(-2)+2=0 \rightarrow -2x-y=0$$

$$\rightarrow 2x+y=0$$

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For $m=\frac{1}{2}$ eq (ii) becomes.

$\frac{1}{2}x - y + \frac{1}{2} + 2 = 0$

$\rightarrow x - 2y + 1 + 4 = 0$

$\rightarrow x - 2y + 5 = 0$

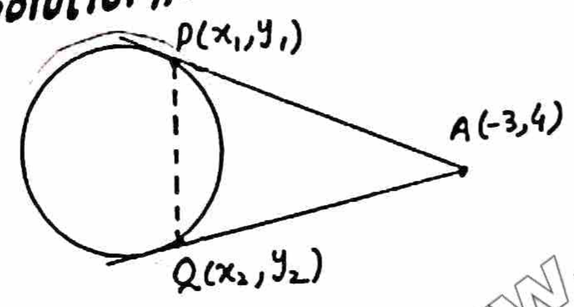
Now tangential distance $= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

$= \sqrt{(-1)^2 + (2)^2 + 4(-1) + 2(2) + 0}$

$= \sqrt{1+4-4+4} = \sqrt{5}$

Example 8. Tangents are drawn from $(-3,4)$ to the circle $x^2+y^2=21$ Find the equation of the line joining the points of contact. (This line is called the **chord of contact**)

Solution:-



Given eq. of circle $x^2+y^2=21$

compare with $x^2+y^2=a^2$

$\rightarrow a^2 = 21$

Let $P(x_1,y_1)$ and $Q(x_2,y_2)$ be points of contact, Eq. of tangent at point $P(x_1,y_1)$ is

$xx_1 + yy_1 = a^2$

$\rightarrow xx_1 + yy_1 = 21$

$\because$ tangent passes through $A(-3,4)$

so $-3x_1 + 4y_1 = 21 \longrightarrow$ (i)

Similarly

Eq. of tangent at point $Q(x_2,y_2)$ is $xx_2 + yy_2 = a^2$

$\rightarrow xx_2 + yy_2 = 21$

$\because$ tangent passes through $A(-3,4)$ so

$-3x_2 + 4y_2 = 21 \longrightarrow$ (ii)

By (i) and (ii), it is clear that $P(x_1,y_1)$ and $Q(x_2,y_2)$ lie on the line $-3x+4y=21$ which is required equation of chord of contact.

Exercise 6.2

Q1. Write down equations of the tangent and normal to the circle; (i) $x^2+y^2=25$ at $(4,3)$

Solution:-

$x^2+y^2=25$

Take derivative w.r.t x

$2x + 2y\frac{dy}{dx} = 0 \rightarrow x + y\frac{dy}{dx} = 0$

or $\frac{dy}{dx} = -\frac{x}{y}$

Slope of tangent $= \frac{dy}{dx}$ at $P(4,3) = -\frac{4}{3}$

Eq. of tangent is

$y-3 = -\frac{4}{3}(x-4)$

$\rightarrow 3y - 9 = -4x + 16$

or $4x + 3y = 25$

Eq. of Normal is

$y-3 = \frac{3}{4}(x-4)$ $\quad \because$ slope of Normal $= \frac{3}{4}$

$\rightarrow 4y - 12 = 3x - 12$

or $3x - 4y = 0$

$x^2+y^2=25$ at $(5\cos\theta, 5\sin\theta)$

Solution:-

$x^2+y^2=25$

Take derivative w.r.t x

$\rightarrow 2x + 2y\frac{dy}{dx} = 0$

$\rightarrow x + y\frac{dy}{dx} = 0$

or $\frac{dy}{dx} = -\frac{x}{y}$

Slope of tangent $= \frac{dy}{dx}$ at $(5\cos\theta, 5\sin\theta)$

$= \frac{-5\cos\theta}{5\sin\theta} = -\frac{\cos\theta}{\sin\theta}$

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Eq. of tangent is

$(y-5\sin\theta) = -\frac{\cos\theta}{\sin\theta}(x-5\cos\theta)$

$\rightarrow y\sin\theta - 5\sin^2\theta = -x\cos\theta + 5\cos^2\theta$

$\rightarrow x\cos\theta + y\sin\theta = 5\cos^2\theta + 5\sin^2\theta.$

$\rightarrow x\cos\theta + y\sin\theta = 5(\cos^2\theta + \sin^2\theta)$

$x\cos\theta + y\sin\theta = 5$

slope of Normal $= \frac{\sin\theta}{\cos\theta}$

Eq. of Normal

$(y-5\sin\theta) = \frac{\sin\theta}{\cos\theta}(x-5\cos\theta)$

$\rightarrow y\cos\theta - 5\cancel{\sin\theta\cos\theta} = x\sin\theta - 5\cancel{\sin\theta\cos\theta}$

$\rightarrow x\sin\theta - y\cos\theta = 0$

(ii) $3x^2+3y^2+5x-13y+2=0$ at $(1, \frac{10}{3})$

Solution:-

$3x^2+3y^2+5x-13y+2=0$

Take derivative w.r.t x

$6x + 6y\frac{dy}{dx} + 5 - 13\frac{dy}{dx} = 0$

$(6y-13)\frac{dy}{dx} = -(6x+5)$

or $\frac{dy}{dx} = -\left(\frac{6x+5}{6y-13}\right)$

slope of tangent $= \left.\frac{dy}{dx}\right|_{(1,\frac{10}{3})} = \frac{-6(1)-5}{6(\frac{10}{3})-13}$

$= \frac{-6-5}{20-13} = \frac{-11}{7}$

Eq. of tangent

$(y-\frac{10}{3}) = \frac{-11}{7}(x-1)$

$\rightarrow 7y - \frac{70}{3} = -11x + 11$

$\rightarrow 21y - 70 = -33x + 33$

$\rightarrow 33x + 21y = 103$

Slope of Normal $= \frac{7}{11}$

Eq. of Normal

$y - \frac{10}{3} = \frac{7}{11}(x-1)$

$\rightarrow 11y - \frac{110}{3} = 7x - 7$

$\rightarrow 33y - 110 = 21x - 21$

$\rightarrow 21x - 33y = 21 - 110$

$\rightarrow 21x - 33y = -89$

Q2. Write down equations of tangent and normal to the circle $4x^2+4y^2-16x+24y-117=0$ at the point on the circle whose abcissa is -4.

Solution:- $4x^2+4y^2-16x+24y-117=0 \rightarrow$ (i)

Abcissa $= x = -4$

Put $x=-4$ in given eq. of circle (i)

i.e., $4(-4)^2+4y^2-16(-4)+24y-117=0$

$\rightarrow 4(16)+4y^2+64+24y-117=0$

$\rightarrow 64+4y^2+64+24y-117=0$

$\rightarrow 4y^2+24y+11=0$

$\rightarrow 4y^2+22y+2y+11=0$

$\rightarrow 2y(2y+11)+1(2y+11)=0$

$(2y+11)(2y+1)=0$

$2y+11=0$ or $2y+1=0$

$y=-\frac{11}{2}$; or $y=-\frac{1}{2}$

$P_1(-4,-\frac{11}{2})$ or $P_2(-4,-\frac{1}{2})$

Take derivative of (i) w.r.t x

$8x + 8y\frac{dy}{dx} - 16 + 24\frac{dy}{dx} = 0$

$\rightarrow \frac{dy}{dx}(8y+24) = -8x+16$

or $\frac{dy}{dx} = -\frac{8x+16}{8y+24} = \frac{8(-x+2)}{8(y+3)}$

$\rightarrow \frac{dy}{dx} = \frac{-x+2}{y+3}$

slope of tangent at P_1 $= \left.\frac{dy}{dx}\right|_{(-4,-\frac{11}{2})} = \frac{-(-4)+2}{-\frac{11}{2}+3}$

$\rightarrow \left.\frac{dy}{dx}\right|_{(-4,-\frac{11}{2})} = \frac{6}{\frac{-11+6}{2}}$

or $\left.\frac{dy}{dx}\right|_{(-4,-\frac{11}{2})} = \frac{-12}{5}$

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Eq. of tangent

$y+\frac{11}{2} = \frac{-12}{5}(x+4)$

$\rightarrow 5y+\frac{55}{2} = -12x-48$

$\rightarrow 10y+55 = -24x-96$

$\rightarrow 24x+10y = -96-55$

or $24x+10y+151=0$

Slope of Normal $= \frac{5}{12}$

Eq. of Normal

$y+\frac{11}{2} = \frac{5}{12}(x+4)$

$\rightarrow 12y+66 = 5x+20$

$\rightarrow 5x-12y-46=0$

Now Slope of tangent at P_2 $= \left.\frac{dy}{dx}\right|_{(-4,-\frac{1}{2})}$

$= \frac{-(-4)+2}{-\frac{1}{2}+3} = \frac{6}{\frac{-1+6}{2}}$

$\left.\frac{dy}{dx}\right|_{(-4,-\frac{1}{2})} = \frac{12}{5}$

Eq. of tangent

$y+\frac{1}{2} = \frac{12}{5}(x+4)$

$\rightarrow 5y+\frac{5}{2} = 12x+48$

$\rightarrow 10y+5 = 24x+96$

or $24x-10y = 5-96$

or $24x-10y = -91$

or $24x-10y+91=0$

slope of Normal $= -\frac{5}{12}$

Eq. of Normal

$y+\frac{1}{2} = -\frac{5}{12}(x+4)$

$\rightarrow 12y+6 = -5x-20$

$\rightarrow 5x+12y = -26$

or $5x+12y+26=0$

Q3. Check the position of the point (5,6) with respect to the circle

i) $x^2+y^2 = 81$

Solution:- $x^2+y^2-81=0$

For (5,6)

$(5)^2+(6)^2-81 = 25+36-81 = -20<0$

$\rightarrow$ Inside circle

ii) $2x^2+2y^2+12x-8y+1=0$

Solution:-

For (5,6)

$2(5)^2+2(6)^2+12(5)-8(6)+1$

$= 50+72+60-48+1 = 135>0$

$\rightarrow$ outside circle

Q4. Find length of tangent drawn from the point (-5,4) to the circle $5x^2+5y^2-10x+15y-131=0$

Solution:-

$5x^2+5y^2-10x+15y-131=0$

$\rightarrow x^2+y^2-2x+3y-\frac{131}{5}=0$ ($\div$ by 5)

compare with

$x^2+y^2+2gx+2fy+c=0$

$2g=-2$, $2f=3$, $c=-\frac{131}{5}$

$x_1=-5$, $y_1=4$

length of tangent

$= \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$

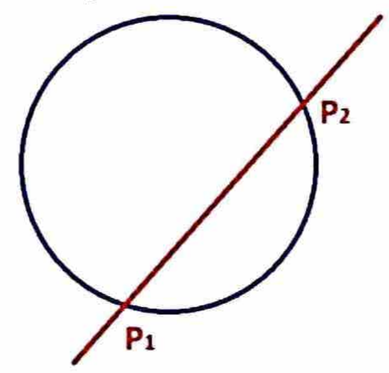
$= \sqrt{(-5)^2+(4)^2+(-2)(-5)+(3)(4)-\frac{131}{5}}$

$= \sqrt{25+16+10+12-\frac{131}{5}}$

$= \sqrt{\frac{125+80+50+60-131}{5}} = \sqrt{\frac{184}{5}}$

Q5. Find the length of the chord cut off from the line $2x+3y=13$ by the circle $x^2+y^2=26$

Soluti



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$|P_1P_2| = ?$

Given line is $2x+3y=13$

$\rightarrow x = \frac{13-3y}{2}$

Put in $x^2+y^2=26$

$\rightarrow \left(\frac{13-3y}{2}\right)^2 + y^2 = 26$

$\frac{169+9y^2-78y}{4} + y^2 = 26$

$\rightarrow 169+9y^2-78y+4y^2 = 104$

or $13y^2-78y+65=0$

$\rightarrow 13y^2-13y-65y+65=0$

$\rightarrow 13y(y-1)-65(y-1)=0$

$(13y-65)(y-1)=0$

$13y-65=0, \quad y-1=0$

$y=\frac{65}{13}=5, \quad y=1$

when $y=1$, $x=\frac{13-3(1)}{2}=\frac{10}{2}$

$\rightarrow x=5$

when $y=5$, $x=\frac{13-3(5)}{2}=\frac{-2}{2}$

$\rightarrow x=-1$

so $P_1(-1,5)$, $P_2(5,1)$ are points of contact.

length of chord $= |P_1P_2|$

$= \sqrt{(5+1)^2+(1-5)^2}$

$= \sqrt{(6)^2+(-4)^2} = \sqrt{36+16}$

$= \sqrt{52} = \sqrt{4\times13} = 2\sqrt{13}$

Q6. Find the coordinates of the points of intersection of the line $x+2y=6$ with the circle $x^2+y^2-2x-2y-39=0$

Solution:- $x^2+y^2-2x-2y-39=0 \longrightarrow (i)$

Given line is $x+2y=6$

$\rightarrow x=6-2y$

Put in (i)

$(6-2y)^2+y^2-2(6-2y)-2y-39=0$

$\rightarrow 36+4y^2-24y+y^2-12+4y-2y-39=0$

$\rightarrow 5y^2-22y-15=0$

$5y^2-25y+3y-15=0$

$5y(y-5)+3(y-5)=0$

$\rightarrow (y-5)(5y+3)=0$

$y-5=0, \quad 5y+3=0$

$y=5, \quad y=-\frac{3}{5}$

when $y=5$, $x=6-2(5)=6-10=-4$

when $y=-\frac{3}{5}$, $x=6-2(-\frac{3}{5})=6+\frac{6}{5}$

$x=\frac{30+6}{5}=\frac{36}{5}$

Points of contacts are

$P_1(-4,5)$ and $P_2\left(\frac{36}{5}, -\frac{3}{5}\right)$

Q7. Find equations of tangent to the circle $x^2+y^2=2$

i) Parallel to line $x-2y+1=0$

Solution:- Given circle is

$x^2+y^2=2=(\sqrt{2})^2$

$\rightarrow r=a=\sqrt{2}$

slope of given line $=-\frac{a}{b}=\frac{-1}{-2}=\frac{1}{2}$

$m=\frac{1}{2}$ ($\because$ parallel)

Eq. of tangent is

$y=mx\pm a\sqrt{1+m^2}$

$\rightarrow y=\frac{1}{2}x\pm\sqrt{2}\sqrt{1+(\frac{1}{2})^2}$

$=\frac{1}{2}x\pm\sqrt{2}\sqrt{1+\frac{1}{4}}$

$y=\frac{1}{2}x\pm\frac{\sqrt{2}\sqrt{5}}{2}$

$\rightarrow 2y=x\pm\sqrt{10}$

$\rightarrow x-2y\pm\sqrt{10}=0$

Hence $x-2y+\sqrt{10}=0$ and $x-2y-\sqrt{10}=0$ are req. tangents.

ii) Perpendicular to line $3x+2y=6$

Solution:-

slope of given $=-\frac{a}{b}=-\frac{3}{2}$

slope of tangent ($\perp$ar to given line) $=m=\frac{2}{3}$

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given circle is

$x^2+y^2 = 2 = (\sqrt{2})^2$

$\rightarrow r = a = \sqrt{2}$

Eq. of tangent is

$y = mx \pm a\sqrt{1+m^2}$

$\rightarrow y = \frac{2}{3}x \pm \sqrt{2}\sqrt{1+\left(\frac{2}{3}\right)^2}$

$y = \frac{2}{3}x \pm \sqrt{2}\sqrt{1+\frac{4}{9}}$

$y = \frac{2}{3}x \pm \sqrt{2}\,\frac{\sqrt{13}}{3}$

$\rightarrow 3y = 2x \pm \sqrt{26}$

$\rightarrow 2x - 3y \pm \sqrt{26} = 0$

Hence $2x - 3y + \sqrt{26} = 0$ and

$2x - 3y - \sqrt{26} = 0$ are req. tangents.

Q8. Find equations of tangent drawn from

i) $(0,5)$ to $x^2+y^2 = 16$

Also find points of contact.

Solution:-

Eq. of circle is $x^2+y^2=16$

$\rightarrow x^2+y^2 = (4)^2 \rightarrow a = 4$

Eq. of tangent is

$y = mx + a\sqrt{1+m^2} \longrightarrow$ (i)

at $(0,5)$ and $a=4$

$5 = m(0) + 4\sqrt{1+m^2}$

$5 = 4\sqrt{1+m^2} \rightarrow 25 = 16(1+m^2)$

$\rightarrow 25 = 16 + 16m^2 \rightarrow 16m^2 = 9$

or $m^2 = \frac{9}{16} \rightarrow m = \pm\frac{3}{4}$

Now (i) for $m = \pm\frac{3}{4}$ and $a=4$

$y = \pm\frac{3}{4}x + 4\sqrt{1+\frac{9}{16}}$

$y = \pm\frac{3}{4}x + 4\sqrt{\frac{25}{16}}$

$\rightarrow y = \pm\frac{3}{4}x + 4\left(\frac{5}{4}\right)$

or $y = \pm\frac{3}{4}x + 5$

$\rightarrow 4y = \pm 3x + 20$

or $\pm 3x - 4y + 20 = 0$

Hence $3x - 4y + 20 = 0$ and

$-3x - 4y + 20 = 0$ or $3x + 4y - 20 = 0$

are req. tangents.

Points of contact

Remember, To find points of contact, solve equation of tangent and circle.

We solve $3x - 4y + 20 = 0$

and $x^2+y^2 = 16$ for points of contact. Now

$x^2+y^2 = 16 \longrightarrow$ (ii)

$3x - 4y + 20 = 0$

$\rightarrow x = \frac{4y-20}{3}$ put in (ii)

$\left(\frac{4y-20}{3}\right)^2 + y^2 = 16$

$\rightarrow \frac{16y^2+400-160y}{9} + y^2 = 16$

$\rightarrow 16y^2 + 400 - 160y + 9y^2 = 144$

$\rightarrow 25y^2 - 160y + 256 = 0$

$25y^2 - 80y - 80y + 256 = 0$

$5y(5y-16) - 16(5y-16) = 0$

$(5y-16)(5y-16) = 0$

$\rightarrow (5y-16)^2 = 0$

$\rightarrow 5y - 16 = 0 \rightarrow y = \frac{16}{5}$

Now $x = \frac{4\left(\frac{16}{5}\right) - 20}{3}$

$= \frac{64-100}{15} = \frac{-36}{15}$

$x = \frac{-12}{5}$

So point of contact is $\left(-\frac{12}{5}, \frac{16}{5}\right)$

Now we solve $3x + 4y - 20 = 0$

and $x^2+y^2 = 16$.

$3x + 4y - 20 = 0$

$\rightarrow x = \frac{20-4y}{3}$ put in (ii)

$\left(\frac{20-4y}{3}\right)^2 + y^2 = 16$

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$$\frac{400+16y^2-160y}{9}+y^2=16$$

$\rightarrow 400+16y^2-160y+9y^2=144$

$25y^2-160y+256=0$

$\rightarrow (5y-16)^2=0$

$5y-16=0 \rightarrow y=\frac{16}{5}$

so $x=\frac{20-4\left(\frac{16}{5}\right)}{3}=\frac{20-\frac{64}{5}}{3}$

$x=\frac{100-64}{15}=\frac{36}{15}=\frac{12}{5}$

Also $\left(\frac{12}{5},\frac{16}{5}\right)$ is point of contact.

(ii) $(-1,2)$ to $x^2+y^2+4x+2y=0$ Also find points of contact.

Solution:-

$x^2+y^2+4x+2y=0 \longrightarrow$ (i)

compare with

$x^2+y^2+2gx+2fy+c=0$

$2g=4 \rightarrow g=2$

$2f=2 \rightarrow f=1$, $c=0$

$r=a=\sqrt{g^2+f^2-c}=\sqrt{(2)^2+(1)^2-0}$

$\rightarrow r=a=\sqrt{5}$

centre $(-g,-f)=(-2,-1)$

Let m be slope of tangent drawn from point $(-1,2)$ to given circle so eq. of tangent is

$y-y_1=m(x-x_1)$

$\rightarrow y-2=m(x+1)$

$\rightarrow y-2=mx+m$

$\rightarrow mx-y+m+2=0 \longrightarrow$ (i)

Distance of line from centre $(-2,-1)=r$

$\rightarrow \frac{|m(-2)-(-1)+m+2|}{\sqrt{m^2+1}}=\sqrt{5}$

$\rightarrow \frac{|-m+3|}{\sqrt{m^2+1}}=\sqrt{5}$

$\rightarrow \frac{m^2-6m+9}{m^2+1}=5$ (squaring)

$\rightarrow 5m^2+5=m^2-6m+9$

$\rightarrow 5m^2+5-m^2+6m-9=0$

or $4m^2+6m-4=0$

$4m^2+8m-2m-4=0$

$4m(m+2)-2(m+2)=0$

$(m+2)(4m-2)=0$

$m+2=0$ or $m=\frac{2}{4}=\frac{1}{2}$

$\rightarrow m=-2$ or $m=\frac{1}{2}$

when $m=-2$ so (i)

$-2x-y-2+2=0$

$\rightarrow -2x-y=0 \rightarrow 2x+y=0$

when $m=\frac{1}{2}$ so (i)

$\frac{1}{2}x-y+\frac{1}{2}+2=0$

$\rightarrow x-2y+1+4=0$

or $x-2y+5=0$

Hence req. tangents are

$2x+y=0$ and $x-2y+5=0$

Points of contact

For $2x+y=0 \rightarrow y=-2x$ put in (i)

$x^2+(-2x)^2+4x+2(-2x)=0$

$x^2+4x^2+4x-4x=0$

$\rightarrow 5x^2=0 \rightarrow x=0$ so $y=-2(0)$

$\rightarrow y=0$

Thus $(0,0)$ is the point of contact.

For $x-2y+5=0 \rightarrow x=2y-5$ put in (i)

$(2y-5)^2+y^2+4(2y-5)+2y=0$

$4y^2-20y+25+y^2+8y-20+2y=0$

$5y^2-10y+5=0$

$\rightarrow y^2-2y+1=0 \rightarrow (y-1)^2=0$

$y-1=0 \rightarrow y=1$

so $x=2(1)-5 \rightarrow x=-3$

Thus $(-3,1)$ is point of contact.

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(iii) $(-7,-2)$ to $(x+1)^2+(y-2)^2=26$

Also find points of contacts

Solution:-

$(x+1)^2+(y-2)^2 = 26 \longrightarrow$ (i)

$\rightarrow (x-(-1))^2+(y-2)^2=(\sqrt{26})^2$

compare with

$(x-h)^2+(y-k)^2 = r^2$

centre $(h,k) = (-1,2)$

$r = \sqrt{26}$

suppose eq. of tangent

$y-y_1 = m(x-x_1)$

$\rightarrow y+2 = m(x+7)$ at $(-7,-2)$

$y+2 = mx+7m$

or $mx-y+7m-2=0 \longrightarrow$ (ii)

Distance of tangent from Centre $(-1,2) = r$

$\rightarrow \dfrac{|m(-1)-2+7m-2|}{\sqrt{m^2+(-1)^2}} = \sqrt{26}$

$\rightarrow \dfrac{|6m-4|}{\sqrt{m^2+1}} = \sqrt{26}$

$\rightarrow \dfrac{36m^2+16-48m}{m^2+1} = 26$ (squaring)

$\rightarrow 36m^2+16-48m = 26m^2+26$

$\rightarrow 36m^2-26m^2-48m+16-26=0$

or $10m^2-48m-10=0$

$10m^2-50m+2m-10=0$

$10m(m-5)+2(m-5)=0$

$(m-5)(10m+2)=0$

$m-5=0$, $10m+2=0$

$m=5$, $m=\frac{-2}{10}=-\frac{1}{5}$

when $m=5$, so (ii)

$5x-y+7(5)-2=0$

$\rightarrow 5x-y+35-2=0$

$5x-y+33=0$

when $m=-\frac{1}{5}$, so (ii)

$-\frac{1}{5}x-y+7(-\frac{1}{5})-2=0$

$-\frac{1}{5}x-y-\frac{7}{5}-2=0$

$\rightarrow -x-5y-7-10=0$

$\rightarrow -x-5y-17=0$

or $x+5y+17=0$

Hence $5x-y+33=0$ and $x+5y+17=0$ are required tangents.

Points of contact

For $x+5y+17=0$

$\rightarrow x=-5y-17$ put in (i)

$(-5y-17+1)^2+(y-2)^2=26$

$\rightarrow (-5y-16)^2+(y-2)^2=26$

$25y^2+160y+256+y^2+4-4y-26=0$

$26y^2+156y+234=0$

$\rightarrow y^2+6y+9=0$ ($\div$ by 26)

or $(y+3)^2=0$

$\rightarrow y+3=0 \rightarrow y=-3$

so $x=-5(-3)-17$

$x = 15-17=-2$

so $(-2,-3)$ is point of contact.

For $5x-y+33=0$

$\rightarrow y=5x+33$ put in (i)

$(x+1)^2+(5x+33-2)^2=26$

$x^2+1+2x+(5x+31)^2=26$

$x^2+1+2x+25x^2+310x+961-26=0$

$26x^2+312x+936=0$

$\rightarrow x^2+12x+36=0$ ($\div$ by 26)

or $(x+6)^2=0 \rightarrow x+6=0$

$x=-6$ so $y=5(-6)+33$

$\rightarrow y=-30+33=3$

so $(-6,3)$ is point of contact.

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Q9. Find an equation of the chord of contact of the tangent drawn from (4,5) to $2x^2+2y^2-8x+12y+21=0$

Solution:-

$2x^2+2y^2-8x+12y+21=0$

$\rightarrow x^2+y^2-4x+6y+\frac{21}{2}=0$

Compare with

$x^2+y^2+2gx+2fy+c=0$

$2g=-4 \rightarrow g=-2$

$2f=6 \rightarrow f=3$, $c=\frac{21}{2}$

Required equation is

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

$\rightarrow 4x+5y-2(x+4)+3(y+5)+\frac{21}{2}=0$

$\rightarrow 4x+5y-2x-8+3y+15+\frac{21}{2}=0$

$\rightarrow 2x+8y+7+\frac{21}{2}=0$

$4x+16y+14+21=0$

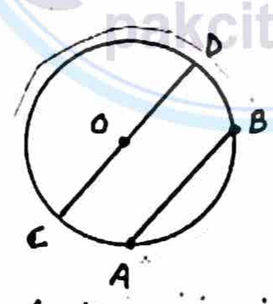
or $4x+16y+35=0$

which is required eq. of chord of contact.

Analytic Proofs of Important Properties of a Circle

"Chord of a circle":-

A line segment, whose end points lie on a circle is called chord of a circle. In figure AB is chord.



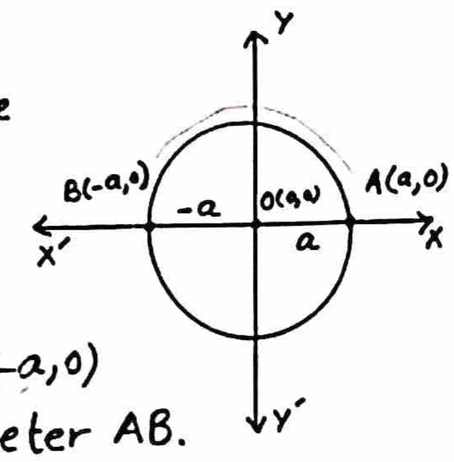
"Diameter of a circle":-

A chord passing through centre of a circle is called diameter of circle. In figure, if O is centre then CD is diametre of circle.

Theorem 1

Length of a diameter of the circle $x^2+y^2=a^2$ is $2a$

Proof:-



Given eq. of circle is $x^2+y^2=a^2$

centre $=O(0,0)$

Radius $=r=a$

Let $A(a,0)$ and $B(-a,0)$ be ends of diameter AB.

then

length of diameter $=|AB|=\sqrt{(a-(-a))^2+(0-0)^2}$

$=|AB|=\sqrt{(2a)^2}$

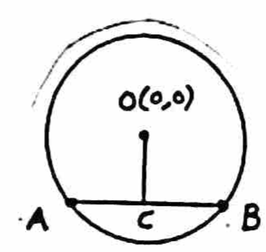
$=2a$

Hence proved.

Theorem 2

Perpendicular dropped from the centre of a circle on a chord bisects the chord.

Proof:-



consider eq. of circle

$x^2+y^2=a^2 \longrightarrow$ (i)

Let $A(x_1,y_1)$ and $B(x_2,y_2)$ be ends of chord AB.

$\because A(x_1,y_1)$ and $B(x_2,y_2)$ lie on the circle (i) so

$x_1^2+y_1^2=a^2 \longrightarrow$ (ii) and

$x_2^2+y_2^2=a^2 \longrightarrow$ (iii)

Let we draw a $\perp$ar OC from centre O on the chord AB

slope of chord AB $=\frac{y_2-y_1}{x_2-x_1}$

slope of OC $=-\left(\frac{x_2-x_1}{y_2-y_1}\right)$ $\because \overline{OC}\perp\overline{AB}$

$\because$ line passing through (x_1,y_1) and having slope m

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Eq. of OC is

$$y - 0 = -\left(\frac{x_2 - x_1}{y_2 - y_1}\right)(x - 0)$$

$$\rightarrow y(y_2 - y_1) = -x(x_2 - x_1)$$

or $(x_2 - x_1)x + (y_2 - y_1)y = 0 \longrightarrow$ (iv)

Midpoint of chord AB is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Now OC will bisect the chord AB if $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ satisfies its equation i-e.,

Put $x = \frac{x_1 + x_2}{2}$, $y = \frac{y_1 + y_2}{2}$ in (iv)

we have

$$(x_2 - x_1)\left(\frac{x_1 + x_2}{2}\right) + (y_2 - y_1)\left(\frac{y_1 + y_2}{2}\right) = 0$$

$$\rightarrow (x_2 - x_1)(x_2 + x_1) + (y_2 - y_1)(y_2 + y_1) = 0$$

$$\rightarrow x_2^2 - x_1^2 + y_2^2 - y_1^2 = 0$$

$$\rightarrow x_2^2 + y_2^2 = x_1^2 + y_1^2$$

$$\rightarrow a^2 = a^2 \quad \text{By (ii) and (iii)}$$

Hence ⊥ar dropped from centre of a circle on a chord bisects the chord

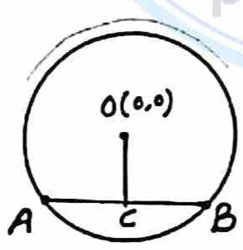
Theorem 3

The perpendicular bisector of any chord of a circle passes through the centre of the circle.

Proof:-

Consider eq. of circle is

$x^2 + y^2 = a^2 \longrightarrow$ (i)



Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be ends of chord AB.

∵ the points A and B lie on the circle (i) then

$x_1^2 + y_1^2 = a^2 \longrightarrow$ (ii) and

$x_2^2 + y_2^2 = a^2 \longrightarrow$ (iii)

Let C be midpoint of chord AB so coordinates of C are

$$C\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

slope of chord AB $= \frac{y_2 - y_1}{x_2 - x_1}$

slope of ⊥ar bisector AB

$$= -\left(\frac{x_2 - x_1}{y_2 - y_1}\right)$$

using $y - y_1 = m(x - x_1)$

$$\rightarrow y - \left(\frac{y_1 + y_2}{2}\right) = -\left(\frac{x_2 - x_1}{y_2 - y_1}\right)\left(x - \frac{x_1 + x_2}{2}\right) \longrightarrow \text{(iv)}$$

This perpendicular bisector will pass through centre (0,0) of circle if (0,0) satisfies eq (iv). For this

Put $x = 0$, $y = 0$ in (iv)

$$0 - \left(\frac{y_1 + y_2}{2}\right) = -\left(\frac{x_2 - x_1}{y_2 - y_1}\right)\left(0 - \frac{x_1 + x_2}{2}\right)$$

$$\rightarrow -\left(\frac{y_1 + y_2}{2}\right) = \left(\frac{x_2 - x_1}{y_2 - y_1}\right)\left(\frac{x_1 + x_2}{2}\right)$$

$$\rightarrow -(y_2 + y_1)(y_2 - y_1) = (x_2 - x_1)(x_2 + x_1)$$

$$\rightarrow -(y_2^2 - y_1^2) = x_2^2 - x_1^2$$

$$\rightarrow y_1^2 - y_2^2 = x_2^2 - x_1^2$$

$$\rightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$$

$$a^2 = a^2 \quad \text{(By (ii) and (iii))}$$

Thus ⊥ar bisector of any chord of a circle passes through centre of the circle.

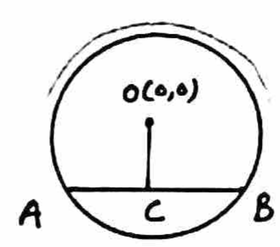
Theorem 4

The line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord

Proof:-

Consider eq. of circle

$x^2 + y^2 = a^2 \longrightarrow$ (i)



Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be ends of chord AB.

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$\because$ the points A and B lie on the circle (i) so

$x_1^2 + y_1^2 = a^2 \longrightarrow$ (ii) and

$x_2^2 + y_2^2 = a^2 \longrightarrow$ (iii)

Let C be midpoint of chord AB then coordinates of C are

$$C\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

slope of chord AB $= \dfrac{y_2-y_1}{x_2-x_1}$

Here OC is line joining centre O of circle to midpoint C of chord AB. then

$$\text{slope of OC} = \frac{\frac{y_1+y_2}{2} - 0}{\frac{x_1+x_2}{2} - 0} = \frac{y_1+y_2}{x_1+x_2}$$

(slope of OC)(slope of chord AB)

$$= \left(\frac{y_1+y_2}{x_1+x_2}\right)\left(\frac{y_2-y_1}{x_2-x_1}\right)$$

$$= \frac{(y_2+y_1)(y_2-y_1)}{(x_2+x_1)(x_2-x_1)}$$

$$= \frac{y_2^2 - y_1^2}{x_2^2 - x_1^2}$$

By (iii) − (ii)

$x_2^2 - x_1^2 + y_2^2 - y_1^2 = a^2 - a^2$

$\rightarrow y_2^2 - y_1^2 = -(x_2^2 - x_1^2)$

Now

(slope of OC)(slope of chord AB)

$$= \frac{-(x_2^2 - x_1^2)}{x_2^2 - x_1^2} = -1$$

$\rightarrow$ OC $\perp$ AB

Hence line joining the centre of a circle to the midpoint of chord is perpendicular to chord.

Congruent chords:-

"Two chords are cogruent if they are equal in length."

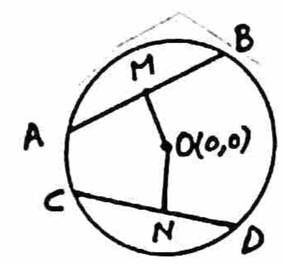
Theorem 5

Congruent chords of a circle are equidistant from the centre.

Proof:-

Consider eq. of circle

$x^2 + y^2 = a^2 \longrightarrow$ (i)



Let AB and CD be two congruent chords.

Let coordinates of the points be $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$. Since these points lie on the circle (i) so

$x_1^2 + y_1^2 = a^2 \longrightarrow$ (ii) $\quad x_2^2 + y_2^2 = a^2 \rightarrow$ (iii)

$x_3^2 + y_3^2 = a^2 \longrightarrow$ (iv) $\quad x_4^2 + y_4^2 = a^2 \rightarrow$ (v)

For congruent chords

so $|AB| = |CD|$

$$\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{(x_4-x_3)^2 + (y_4-y_3)^2}$$

Squaring both sides

$\rightarrow (x_2-x_1)^2 + (y_2-y_1)^2 = (x_4-x_3)^2 + (y_4-y_3)^2$

$x_2^2 + x_1^2 - 2x_1x_2 + y_2^2 + y_1^2 - 2y_1y_2 = x_4^2 + x_3^2 - 2x_3x_4 + y_4^2 + y_3^2 - 2y_3y_4$

from (ii), (iii), (iv) and (v)

$\rightarrow \cancel{a^2} + \cancel{a^2} - 2(x_1x_2 + y_1y_2) = \cancel{a^2} + \cancel{a^2} - 2(x_3x_4 + y_3y_4)$

$\rightarrow -2(x_1x_2 + y_1y_2) = -2(x_3x_4 + y_3y_4)$

or $x_1x_2 + y_1y_2 = x_3x_4 + y_3y_4 \longrightarrow$ (vi)

Let us draw $\perp$ars OM and ON from centre O on the chords AB and CD resp. Now by property of circle, M and N will be midpoints of chords AB and CD respectively.

Now

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coordinates of M are $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Also coordinates of N are $N\left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}\right)$

We are to prove that

$|OM| = |ON| \rightarrow |OM|^2 = |ON|^2$

$$|OM|^2 = \left(\frac{x_1+x_2}{2}\right)^2 + \left(\frac{y_1+y_2}{2}\right)^2$$

$$= \frac{x_1^2+x_2^2+2x_1x_2}{4} + \frac{y_1^2+y_2^2+2y_1y_2}{4}$$

$$= \frac{x_1^2+y_1^2+x_2^2+y_2^2+2(x_1x_2+y_1y_2)}{4}$$

$$= \frac{a^2+a^2+2(x_1x_2+y_1y_2)}{4}$$

By (ii) and (iii)

$$|OM|^2 = \frac{2a^2+2(x_1x_2+y_1y_2)}{4}$$

Also $|ON|^2 = \left(\frac{x_3+x_4}{2}\right)^2 + \left(\frac{y_3+y_4}{2}\right)^2$

$$= \frac{x_3^2+x_4^2+2x_3x_4}{4} + \frac{y_3^2+y_4^2+2y_3y_4}{4}$$

$$= \frac{x_3^2+y_3^2+x_4^2+y_4^2+2(x_3x_4+y_3y_4)}{4}$$

$$= \frac{a^2+a^2+2(x_3x_4+y_3y_4)}{4}$$

$$|ON|^2 = \frac{2a^2+2(x_3x_4+y_3y_4)}{4}$$

By eq (vi)

$$|ON|^2 = \frac{2a^2+2(x_1x_2+y_1y_2)}{4}$$

Hence we conclude that

$|OM|^2 = |ON|^2$

Thus congruent chords of a circle are equidistant from centre.

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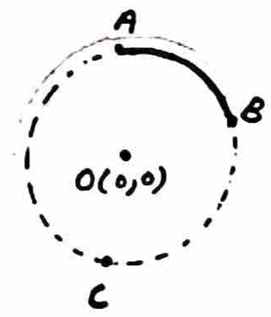
Remember from fig



AB is minor arc

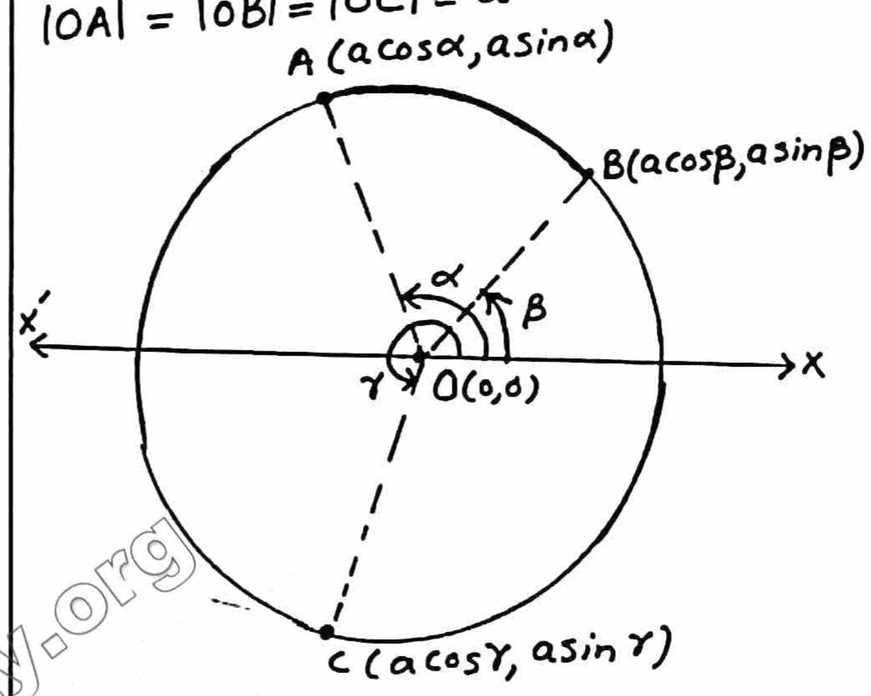
ACB is major arc

O(0,0) is centre of circle

OA, OB, OC are radii

$x^2+y^2=a^2$ and

$|OA| = |OB| = |OC| = a$



from fig

$|OA| = |OB| = |OC| = a$

Let $m\angle XOA = \alpha$, $m\angle XOB = \beta$, $m\angle XOC = \gamma$

coordinates of points are

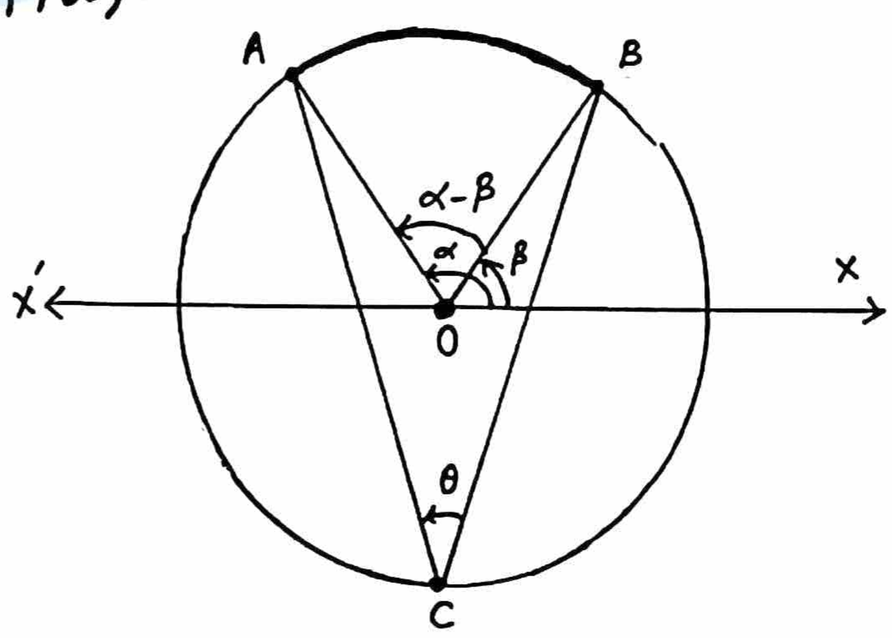
$A(a\cos\alpha, a\sin\alpha)$, $B(a\cos\beta, a\sin\beta)$

and $C(a\cos\gamma, a\sin\gamma)$

Theorem 6

Show that the measure of the central angle of a minor arc is double the measure of the angle in the corresponding major arc

Proof:-



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Consider eq. of circle

$x^2+y^2=a^2 \longrightarrow (i)$ where

a is radius

Let AB be minor arc and ACB be major arc.

Let $m\angle XOA=\alpha$, $m\angle XOB=\beta$

and $m\angle XOC=\gamma$

then

$m\angle BOA=\alpha-\beta$ (Angle subtended by minor arc AB)

Let $m\angle BCA=\theta$ (Angle subtended by major arc ACB)

we are to prove that

$\alpha-\beta=2\theta$ Now

$A(a\cos\alpha, a\sin\alpha)$

$B(a\cos\beta, a\sin\alpha)$

$C(a\cos\gamma, a\sin\gamma)$

Let m_1 = slope of AC

m_2 = slope of BC

$\because m=\dfrac{y_2-y_1}{x_2-x_1}$

$$m_1=\text{slope of AC}=\frac{a\sin\gamma-a\sin\alpha}{a\cos\gamma-a\cos\alpha}$$

$$m_1=\frac{\cancel{a}(\sin\gamma-\sin\alpha)}{\cancel{a}(\cos\gamma-\cos\alpha)}$$

$$m_1=\frac{2\cos\frac{\gamma+\alpha}{2}\sin\frac{\gamma-\alpha}{2}}{-2\sin\frac{\gamma+\alpha}{2}\sin\frac{\gamma-\alpha}{2}}$$

$$\rightarrow m_1=-\cot\left(\frac{\gamma+\alpha}{2}\right) \qquad \because \tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$$

$$\rightarrow m_1=\tan\left(\frac{\pi}{2}+\frac{\gamma+\alpha}{2}\right)$$

$$\rightarrow m_1=\tan\left(\frac{\pi+\gamma+\alpha}{2}\right)$$

Similarly

$$m_2=\text{slope of BC}=\tan\left(\frac{\pi+\gamma+\beta}{2}\right)$$

$\because \theta$ is angle from BC to AC, so

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$$

$$\rightarrow \tan\theta=\frac{\tan\left(\frac{\pi+\gamma+\alpha}{2}\right)-\tan\left(\frac{\pi+\gamma+\beta}{2}\right)}{1+\tan\left(\frac{\pi+\gamma+\alpha}{2}\right)\tan\left(\frac{\pi+\gamma+\beta}{2}\right)}$$

$$\because \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\tan(\alpha-\beta)$$

$$\rightarrow \tan\theta=\tan\left(\frac{\pi+\gamma+\alpha}{2}-\frac{\pi+\gamma+\beta}{2}\right)$$

$$=\tan\left(\frac{\pi+\gamma+\alpha-\pi-\gamma-\beta}{2}\right)$$

$$\tan\theta=\tan\left(\frac{\alpha-\beta}{2}\right)$$

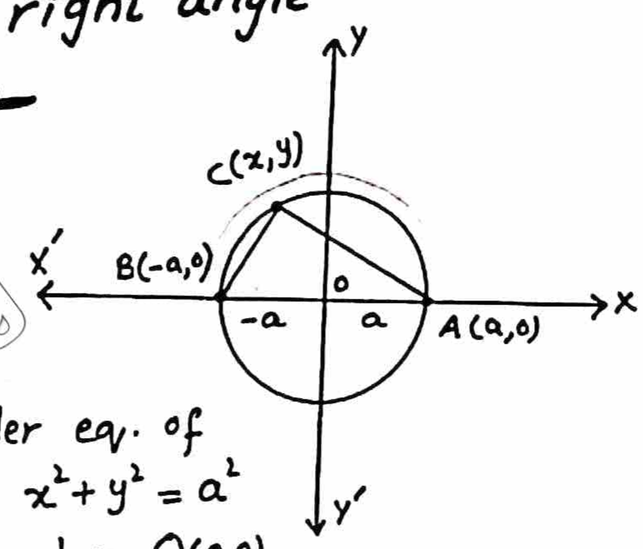
or $\theta=\dfrac{\alpha-\beta}{2} \rightarrow \alpha-\beta=2\theta$

Hence proved

Theorem 7

An angle in a semi circle is a right angle

Proof:-



consider eq. of circle $x^2+y^2=a^2$ with centre $O(0,0)$ and radius a. Let $A(a,0)$ and $B(-a,0)$ be ends of its diameter. Let $C(x,y)$ be any point on the circle. we are to prove that:

$AC \perp BC$

i.e.,

(Slope of AC)(slope of BC) $=-1$

Now

$$\text{slope of AC}=\frac{y-0}{x-a}=\frac{y}{x-a}$$

$$\text{slope of BC}=\frac{y-0}{x-(-a)}=\frac{y}{x+a}$$

so

$$(\text{slope of AC})(\text{slope of BC})=\left(\frac{y}{x-a}\right)\left(\frac{y}{x+a}\right)$$

$$=\frac{y^2}{x^2-a^2}$$

$$=\frac{y^2}{-y^2} \qquad \because x^2+y^2=a^2 \rightarrow x^2-a^2=-y^2$$

$$=-1$$

Hence proved

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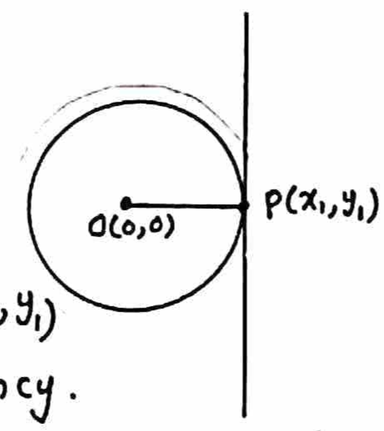
Theorem 8

The tangent to a circle at any point of the circle is perpendicular to the radial segment at that point.

Proof:-



Consider eq. of circle

$x^2+y^2 = a^2 \longrightarrow (i)$

Let the point $P(x_1,y_1)$ be point of tangency.

$\because$ eq. of tangent to the circle (i) at $P(x_1,y_1)$ is

$$xx_1 + yy_1 = a^2$$

$$\rightarrow \quad yy_1 = -x_1x + a^2$$

$$\rightarrow \quad y = \left(-\frac{x_1}{y_1}\right)x + \frac{a^2}{y_1}$$

Compare with $y = mx + c$

slope of tangent line $= -\frac{x_1}{y_1}$

slope of radial segment $OP = \frac{y_1}{x_1}$

(slope of tangent)(slope of OP)

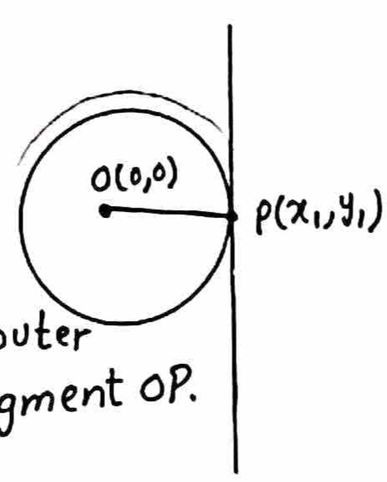
$$= \left(-\frac{x_1}{y_1}\right)\left(\frac{y_1}{x_1}\right) = -1$$

This shows that tangent line to a circle at any point of the circle is perpendicular to the radial segment at that point.

Theorem 9

The perpendicular at the outer end of a radial segment is tangent to the circle.

Proof:-



consider eq. of circle

$x^2+y^2=a^2 \longrightarrow (i)$

Let $P(x_1,y_1)$ be outer end of radial segment OP.

As $P(x_1,y_1)$ lies on the circle (i)

so $x_1^2 + y_1^2 = a^2 \longrightarrow (ii)$

slope of radial segment $OP = \frac{y_1-0}{x_1-0}$

$$= \frac{y_1}{x_1}$$

slope of line $\perp$ar to radial segment $= \frac{-x_1}{y_1}$

Eq. of line $\perp$ar to radial segment and passing through $P(x_1,y_1)$ is

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1) \quad (\because y-y_1 = m(x-x_1))$$

$$\rightarrow yy_1 - y_1^2 = -xx_1 + x_1^2$$

$$\rightarrow xx_1 + yy_1 = x_1^2 + y_1^2$$

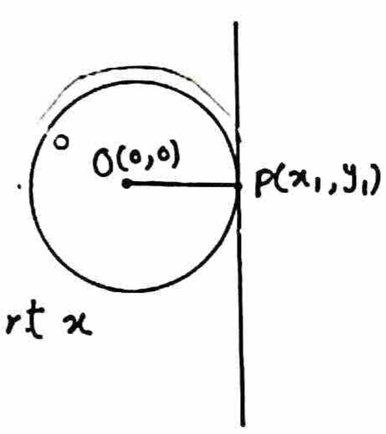
$$\rightarrow xx_1 + yy_1 = a^2 \quad \text{by (ii)}$$

which represents eq. of tangent to the circle (i) at point $P(x_1,y_1)$

Exercise 6.3

Q1. Prove that normal line of a circle pass through the centre of the circle.

Proof:-



Consider eq. of circle

$x^2 + y^2 = a^2 \longrightarrow (i)$

For slope:-

Take derivative w.r.t x of eq (i)

$$2x + 2y\frac{dy}{dx} = 0$$

$$\rightarrow x + y\frac{dy}{dx} = 0 \quad \rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

slope of tangent at P $= m = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = -\frac{x_1}{y_1}$

slope of Normal $= \frac{y_1}{x_1}$

Eq. of Normal through (x_1,y_1) is

$$y - y_1 = \frac{y_1}{x_1}(x - x_1)$$

$$\rightarrow x_1y - x_1y_1 = xy_1 - x_1y_1$$

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$\rightarrow x_1 y = x y_1$

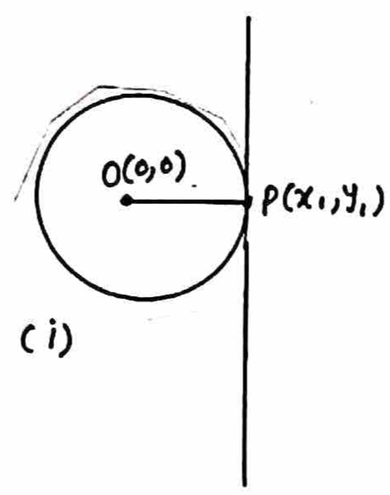
Put centre $(0,0)$

$\rightarrow x_1(0) = 0(y_1) \rightarrow 0 = 0$

Hence normal line pass through centre

Q2. Prove that straight line drawn from the centre of a circle perpendicular to a tangent pass through the point of tangency.

Proof:-



Let eq. of circle

$x^2+y^2=a^2 \longrightarrow (i)$

with centre $O(0,0)$

Take derivative of (i) w.r.t x

$2x + 2y\frac{dy}{dx} = 0$

$\rightarrow \frac{dy}{dx} = -\frac{x}{y}$

slope of tangent at P $= m = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{-x_1}{y_1}$

slope of ⊥ar line $= \frac{-1}{m} = \frac{y_1}{x_1}$

Eq. of ⊥ar line through $O(0,0)$ is

$y - 0 = \frac{y_1}{x_1}(x-0)$

$\rightarrow x_1 y = x y_1$

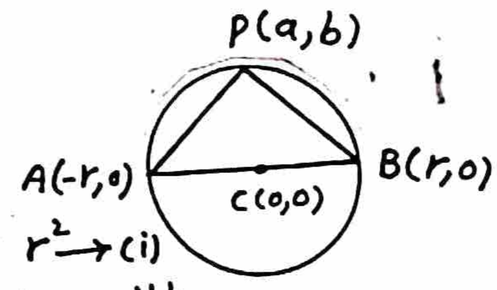
For point $P(x_1,y_1)$

$\rightarrow x_1 y_1 = x_1 y_1$

Hence ⊥ar line pass through point of tandency $P(x_1,y_1)$

Q3. Prove that Mid point of the hypotenuse of a right triangle is the circum centre of the triangle.

Proof:-



Let $x^2+y^2 = r^2 \rightarrow (i)$

is eq. of circle with centre $(0,0)$ and $P(a,b)$ any point lies on (i)

so $a^2+b^2 = r^2 \longrightarrow (ii)$

$|OA| = \sqrt{(-r-0)^2+(0-0)^2} = r$

$|OB| = \sqrt{(r-0)^2+(0-0)^2} = r$

$|OP| = \sqrt{(a-0)^2+(b-0)^2} = \sqrt{a^2+b^2}$

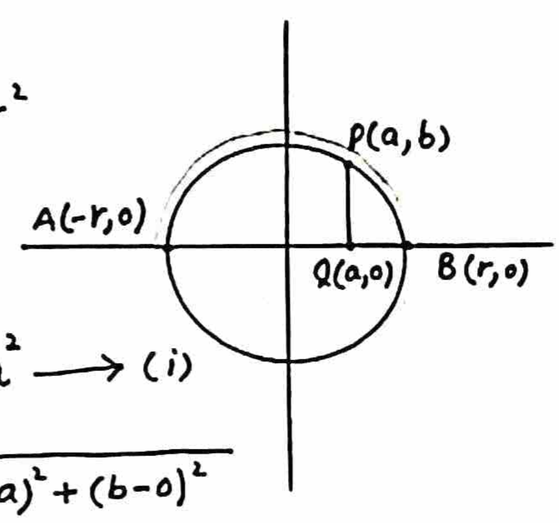
$= \sqrt{r^2} = r$ By (ii)

$\rightarrow |OA| = |OB| = |OP|$ and O is circumcentre

Hence proved

Q4. Prove that perpendicular dropped from a point of a circle on a diameter is mean proportional between the segments into which it divides the diameter.

Proof:-



Let $x^2+y^2=r^2$

$P(a,b)$ lies on it so

$a^2+b^2=r^2$

$\rightarrow b^2 = r^2 - a^2 \longrightarrow (i)$

Now

$|PQ| = \sqrt{(a-a)^2+(b-0)^2}$

$= b$

$|AQ| = |AO| + |OQ| = r+a$

$|BQ| = |OB| - |OQ| = r-a$

$|AQ|\cdot|BQ| = (r+a)(r-a) = r^2 - a^2$

$|AQ|\cdot|BQ| = b^2$ By (i)

$|AQ|\cdot|BQ| = |PQ|^2$

$\rightarrow |AQ|\cdot|BQ| = |PQ||PQ|$

Hence proved

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Parabola

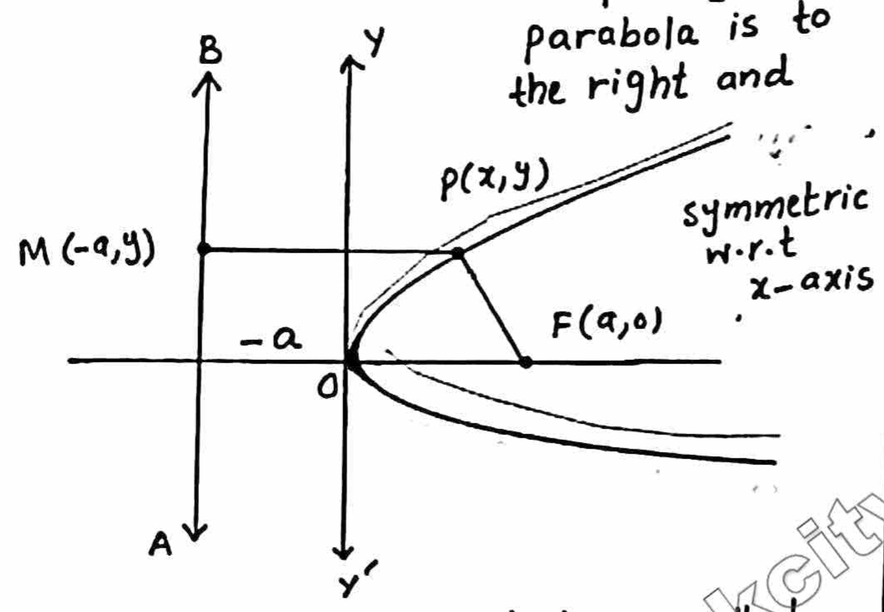
Set of all points which are equidistant from a fixed point and a fixed line.

* fixed point is called Focus of parabola
* Fixed line is called Directrix of parabola.

Equation of parabola

$$y^2 = 4ax$$

opening of parabola is to the right and symmetric w.r.t x-axis



Let AB be a fixed line called directrix drawn parallel to y-axis, such that its equation is $x=-a$ or $x+a=0$. Let $F(a,0)$ be a fixed point called focus not on the line AB.

Let $P(x,y)$ be any point on the parabola. Let $M(-a,y)$ be point on the directrix AB.

By def.,

$$\frac{|FP|}{|PM|} = 1 \rightarrow |FP| = |PM|$$

$$\rightarrow \sqrt{(x-a)^2+(y-0)^2} = \sqrt{(x-(-a))^2+(y-y)^2}$$

$$\rightarrow (x-a)^2+y^2 = (x+a)^2 \quad \text{(By squaring)}$$

$$\rightarrow x^2+a^2-2ax+y^2 = x^2+a^2+2ax$$

or $y^2 = 2ax+2ax$

or $y^2 = 4ax$

This is req. eq. of Parabola.

Definitions

i) The line through the focus and perpendicular to the directrix is called **axis of parabola.**

ii) The point where the axis meets the parabola is called **vertex** of the parabola.

iii) In parabola, fixed point not on the directrix is called **focus of parabola.**

iv) In parabola, fixed line is called **directrix of parabola.**

v) A line passing through vertex and perpendicular to axis of parabola is called **tangent at vertex of parabola.**

vi) A line whose points lie on parabola is called **chord of Parabola**

vii) A chord passing through focus of parabola is called **focal chord.**

viii) A focal chord perpendicular to axis of parabola is called **Letusrectum.**

ix) If $|FP|$ is distance between focus F and point P on parabola and $|PM|$ is $\perp$ar distance of point P from directrix of parabola then $\frac{|FP|}{|PM|}$ is called **eccentricity,** denoted by **e**. For parabola

$$e = 1$$

x) If the point $(at^2, 2at)$ lies on the parabola $y^2=4ax$ for any $t \in \mathbb{R}$ then $x=at^2$, $y=2at$ are called **Parametric equations of Parabola.**

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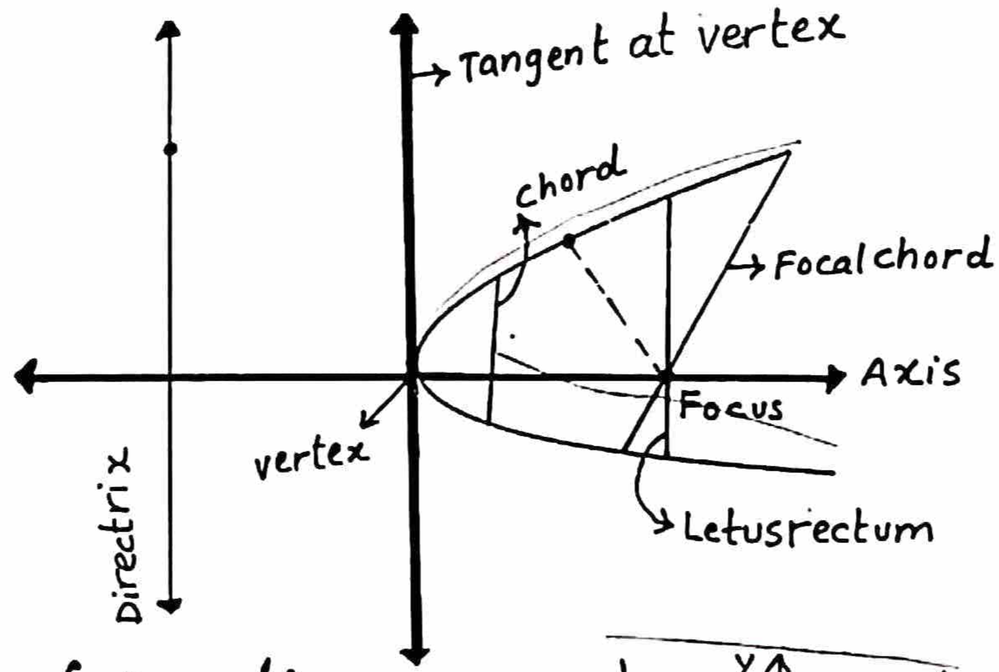

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General form of equation of Parabola

Let $P(x,y)$ be any point on the parabola having $F(h,k)$ as focus and M be point on the directrix $lx+my+c=0$

By def., eq. of parabola is

$$\sqrt{(x-h)^2+(y-k)^2} = \frac{|lx+my+n|}{\sqrt{l^2+m^2}}$$

or $$(x-h)^2+(y-k)^2 = \frac{(lx+my+n)^2}{l^2+m^2}$$

Remember that

Every second degree equation of the form:

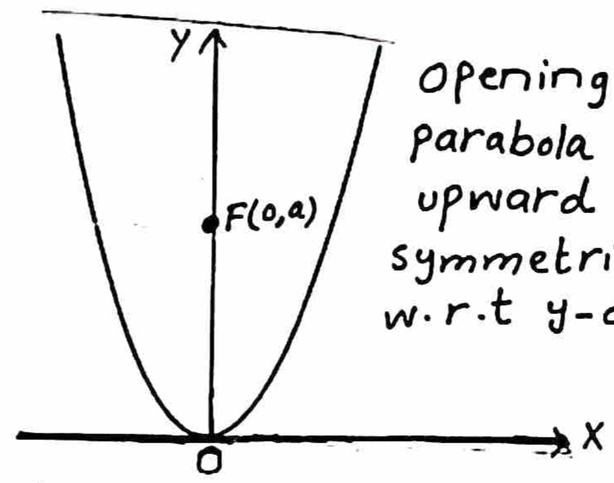
$ax^2+by^2+2gx+2fy+c=0$

will represent parabola if either $a=0$ or $b=0$ but both are not zero simutaneously.

Other Standard parabolas

i) If the focus lies on the y-axis with coordinates $F(0,a)$ and directrix of the parabola is $y=-a$ then eq. of parabola is $x^2=4ay$

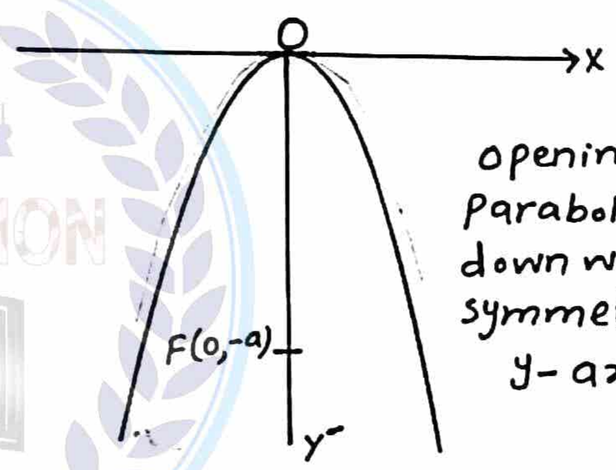
The graph is shown as;



opening of parabola is upward and symmetric w.r.t y-axis

ii) If the focus is $F(0,-a)$ and directrix is the line $y=a$, the eq. of parabola is $x^2=-4ay$

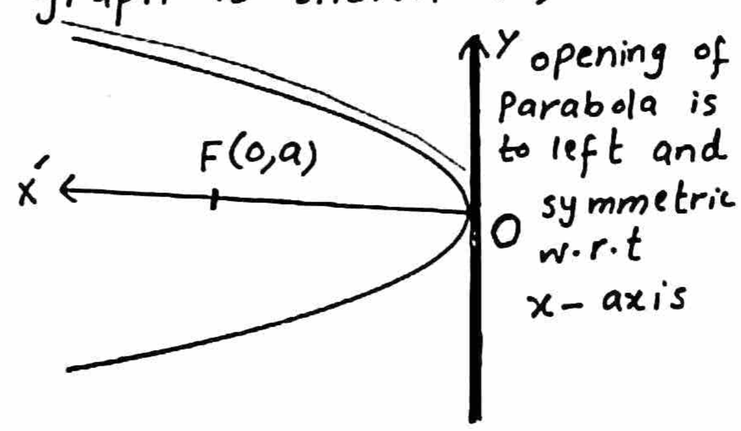
The graph is shown as;



opening of parabola is downward and symmetric w.r.t y-axis

iii) If the focus of parabola is $F(-a,0)$ and is directrix is the line $x=a$, then eq. of parabola is $y^2=-4ax$

The graph is shown as;



opening of parabola is to left and symmetric w.r.t x-axis

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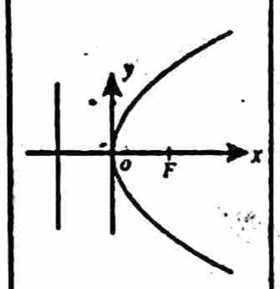
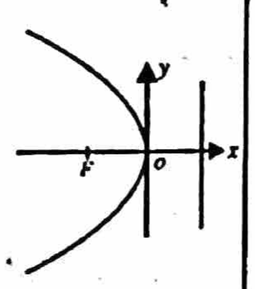
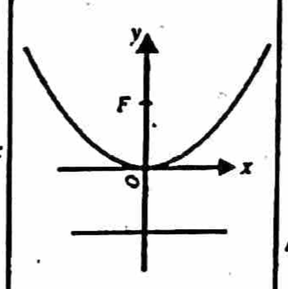
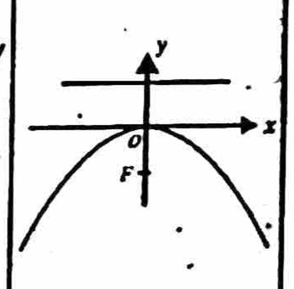
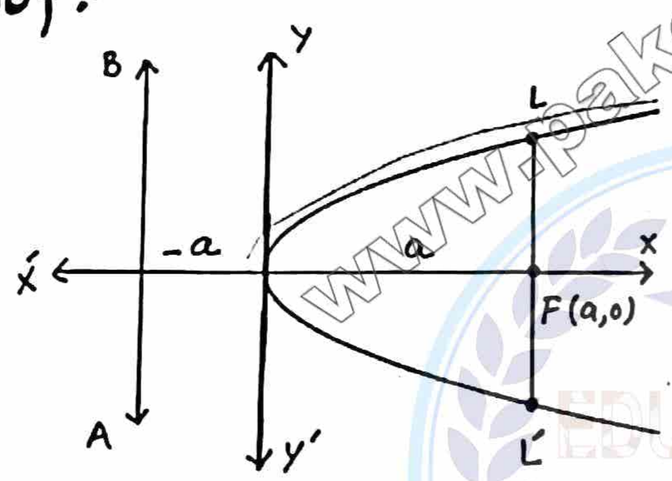
Summary of Standard Parabolas

Sr.No.	1	2	3	4
Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Vertex	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Latusrectum	$x = a$	$x = -a$	$y = a$	$y = -a$
Graph				

Prove that length of letusrectum of parabola is $4a$

Proof:-



Consider eq. of parabola is

$y^2 = 4ax \longrightarrow (i)$

Let LL' be focal chord $\perp$ar to axis of parabola is called letusrectum. Let $x = a$ be eq. of letusrectum LL'.

Put $x = a$ in (i)

$\rightarrow y^2 = 4a(a) \rightarrow y^2 = 4a^2$

or $y = \pm 2a \rightarrow y = 2a$ or $y = -2a$

so coordinates of L and L' are $L(a, 2a)$ and $L'(a, -2a)$

Now,

Length of letusrectum

$= |LL'| = \sqrt{(a-a)^2 + (-2a-2a)^2}$

$= \sqrt{(-4a)^2} = \sqrt{16a^2}$

$= 4a$

Hence proved

Example 1. Analyze the parabola $x^2 = -16y$ and draw its graph.

Solution:-

Given eq. is $x^2 = -16y \longrightarrow (i)$

compare with $x^2 = -4ay$

$\rightarrow 4a = 16 \rightarrow a = 4$

$\because x^2 = -4ay$ represents parabola opening downwards Now

Focus:-

Focus $= F(0, -a) = F(0, -4)$ $\because a = 4$

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Axis:- y-axis is its axis.
so its equation is $x = 0$

Tangent at vertex:-
x-axis is tangent at vertex.
Its equation is $y = 0$

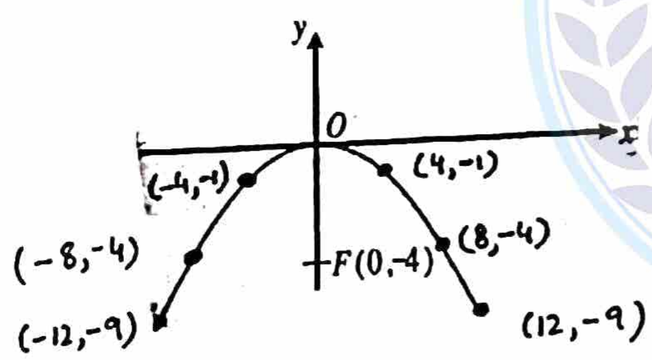
Directrix:- Here directrix will be line parallel (but above) x-axis. Its eq. is $y = a$
$\rightarrow y = 4 \quad \because a = 4$

Vertex:-
vertex lies at origin i-e.,
$V(0,0)$

Length of letus rectum:-
Length of letusrectum $= 4a$
$= 4(4) = 16 \quad \because a = 4$

Sketch:-
$\because$ Given eq. is $x^2 = -16y$
$\rightarrow x = \pm\sqrt{-16y}$, $D_f = (-\infty, -1]$

y	$x = \pm\sqrt{-16y}$	Points
-1	$\pm\sqrt{-16(-1)} = \pm\sqrt{16} = \pm 4$	$(4,-1), (-4,-1)$
0	$\pm\sqrt{-16(0)} = 0$	$(0,0)$
-4	$\pm\sqrt{-16(-4)} = \pm\sqrt{64} = \pm 8$	$(8,-4), (-8,-4)$
-9	$\pm\sqrt{-16(-9)} = \pm\sqrt{144} = \pm 12$	$(12,-9), (-12,-9)$



Example 2. Find an equation of parabola whose focus is $F(-3,4)$ and directrix is $3x - 4y + 5 = 0$

Solution:-
$3x - 4y + 5 = 0$
Let $P(x,y)$ be any point then by def,
$|PF| = |PM|$

$|PF|^2 = |PM|^2$
$\rightarrow (x+3)^2 + (y-4)^2 = \left(\frac{|3x-4y+5|}{\sqrt{(3)^2+(-4)^2}}\right)^2$
$\rightarrow x^2 + 6x + 9 + y^2 - 8y + 16 = \frac{(3x-4y+5)^2}{25}$
$25x^2 + 150x + 225 + 25y^2 - 200y + 400$
$= 9x^2 + 16y^2 + 25 - 24xy + 30x - 40y$
$25x^2 + 150x + 225 + 25y^2 - 200y + 400 - 9x^2$
$- 16y^2 - 25 + 24xy - 30x + 40y = 0$
or $16x^2 + 24xy + 9y^2 + 120x - 160y + 600 = 0$
is required equation

Example 3. Analyze the parabola $x^2 - 4x - 3y + 13 = 0$ and sketch its graph.

Solution:-
Given eq. of parabola is
$x^2 - 4x - 3y + 13 = 0$
$\rightarrow x^2 - 4x + 4 - 3y + 9 = 0$
$\rightarrow (x-2)^2 = 3y - 9$
$(x-2)^2 = 3(y-3) \longrightarrow (i)$
It is of the form;
$X^2 = 4aY$
so Here $X = x - 2$, $Y = y - 3$
and $4a = 3 \rightarrow a = \frac{3}{4}$
$\because X^2 = 4aY$ represents parabola opening upwards then

Focus; $F(X = 0, Y = a)$
$\because X = x - 2, \; Y = y - 3 \; , a = 3/4$
$\rightarrow 0 = x - 2, \quad a = y - 3 \rightarrow \frac{3}{4} = y - 3$
$\rightarrow x = 2 \qquad \rightarrow y = \frac{3}{4} + 3 = \frac{15}{4}$
so $F(2, \frac{15}{4})$

Axis; Axis lies along y-axis
its eq. is $X = 0 \rightarrow x - 2 = 0$
$\rightarrow x = 2$

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Tangent at vertex:-

Line ∥ to x-axis will be tangent at vertex. Its eq. is $Y = 0$

As $Y = y-3 \rightarrow y-3 = 0 \rightarrow y = 3$

Directrix:-

Line ∥ to x-axis will be directrix. i.e., $Y = -a$

$\rightarrow y-3 = -a \rightarrow y-3 = -\frac{3}{4} \quad \because a = \frac{3}{4}$

$\rightarrow y = 3 - \frac{3}{4} \rightarrow y = \frac{9}{4}$

Vertex:- $V(X=0, Y=0)$

$\because X = x-2 \quad , \quad Y = y-3$

$\rightarrow x-2 = 0 \quad , \quad y-3 = 0$

$x = 2 \quad , \quad y = 3$

so Vertex $= V(2,3)$

Length of Letusrectum:-

Length of letusrectum $= 4a$

$= 4\left(\frac{3}{4}\right) = 3$

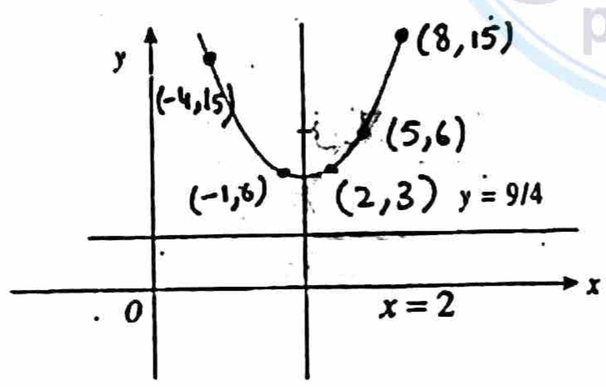
Sketch:-

$\because (x-2)^2 = 3(y-3)$

$\rightarrow x-2 = \pm\sqrt{3(y-3)}$

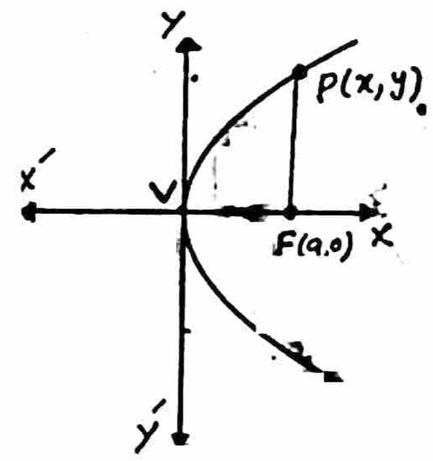
or $x = 2 \pm \sqrt{3(y-3)} \quad , D_f = [3, +\infty)$

y	$x = 2 \pm \sqrt{3(y-3)}$	Points
3	$x = 2 \pm \sqrt{3(3-3)} = 2 \pm 0 = 2$	(2,3)
6	$x = 2 \pm \sqrt{3(6-3)} = 2 \pm 3 = 5, -1$	(5,6), (-1,6)
15	$x = 2 \pm \sqrt{3(15-3)} = 2 \pm 6 = 8, -4$	(8,15), (-4,15)



Theorem:- The point of a parabola which is closest to the focus is the vertex of parabola.

Proof:-



Consider eq. of Parabola

$y^2 = 4ax \longrightarrow (i) \quad , a > 0$

Let $F(a,0)$ be focus and $P(x,y)$ be any point on the parabola. Now

$|PF| = \sqrt{(x-a)^2 + (y-0)^2}$

$= \sqrt{x^2 + a^2 - 2ax + y^2}$

$= \sqrt{x^2 + a^2 - 2ax + 4ax} \quad \because y^2 = 4ax$

$= \sqrt{x^2 + a^2 + 2ax}$

$= \sqrt{(x+a)^2}$

$|PF| = x + a \quad , a > 0, x > 0$

If $V(0,0)$ is vertex then

$|FV| = \sqrt{(a-0)^2 + (0-0)^2} = \sqrt{a^2} = a$

$\rightarrow |FV| < |FP|$

$\rightarrow$ Distance b/w Focus, Vertex

< Distance b/w Focus, Point

Hence the point of parabola which is closest to the focus is vertex of parabola.

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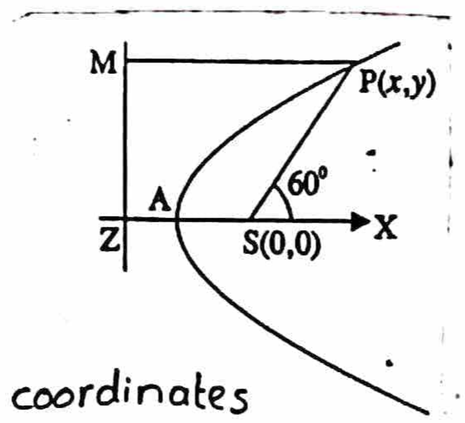
Example 4. A comet has a parabolic orbit with the sun at the focus. When the comet is 100 million km from the sun, the line joining the sun and the commet makes an angle of 60° with the axis of parabola. How close will the comet get to the sun?

Solution:-



Let the comet be at point $p(x,y)$ and sun S be the origin. If the vertex A has coordinates $(-a,0)$ the directrix of parabola is $x=-2a$, $a>0$

By def., $|PS| = |PM|$

$$x^2+y^2 = (x+2a)^2$$

or $y^2 = 4ax+4a^2$ is orbit of the comet.

Now $|PS| = \sqrt{x^2+y^2}$

$$= x+2a = 100{,}000{,}000$$

The comet is closest to the sun when it is at A

Now $x = PS\cos 60°$

$$\rightarrow |x| = \frac{|PS|}{2} = \frac{x+2a}{2}$$

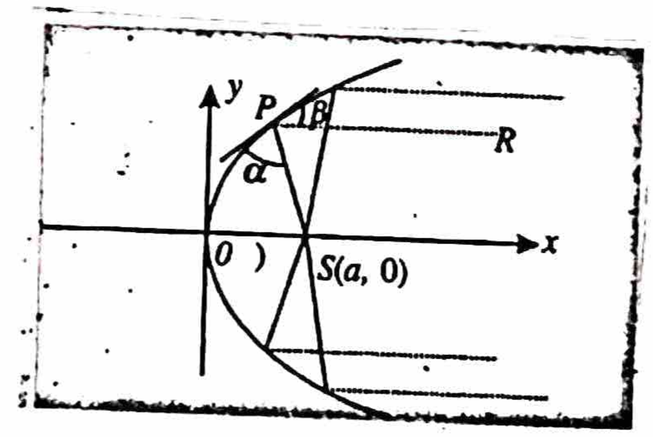
$$\rightarrow \frac{x+2a}{|x|} = \frac{2}{1} \rightarrow \frac{x+2a}{2a} = 2$$

$$\rightarrow \frac{100{,}000{,}000}{2a} = 2 \qquad \left(\begin{aligned}|x| &= |-2a| \\ &= 2a\end{aligned}\right.$$

$$\rightarrow a = 25{,}000{,}000$$

Thus the comet is closest to the sun when it is 25,000,000 km from the sun.

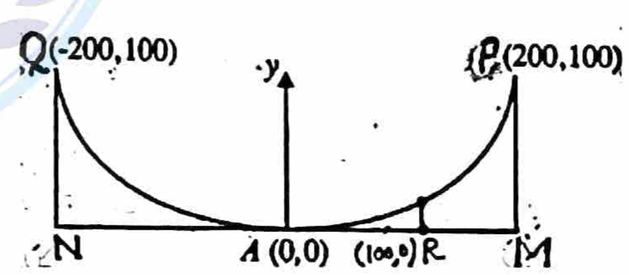
Reflecting Property of Parabola :-



When a light source is placed at the focus of a parabolic reflecting surface, then light ray travelling from focus S to a point P on parabola will be reflected in the direction PR which is parallel to the axes of the parabola shown in figure.

Example 5. A suspension bridge with weight uniformly distributed along the length has two towers of 100m height above the road surface and are 400m apart. The cables are parabolic in shape and are tangent to road surface at the centre of the bridge. Find the height of the cables at a point 100m from the centre.

Solution:-



Let A(0,0) be centre of parabola.

$x^2 = 4ay \longrightarrow$ (i) formed by cables. Let PM and QN be two towers of height 100m each and 200m apart.

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$\because$ P(200, 100) lies on (i) so

$(200)^2 = 4a(100)$

$\rightarrow 400a = 40{,}000$

or $4a = 400$ put in (i)

$x^2 = 400y \longrightarrow$ (ii)

Let R(100, 0) be a point 100m away from centre A(0,0). For height of cables from this point

Put $x = 100$ in (ii)

$(100)^2 = 400y \rightarrow 400y = 10{,}000$

$\rightarrow y = 25$

Hence req. height of cables is 25m.

Exercise 6.4

Q1. Find the focus, vertex and directrix of the parabola. sketch its graph.

Solution:- i) $y^2 = 8x$

Compare with $y^2 = 4ax$

$\rightarrow 4a = 8 \rightarrow a = 2$

Focus; $F(a,0) = F(2,0)$

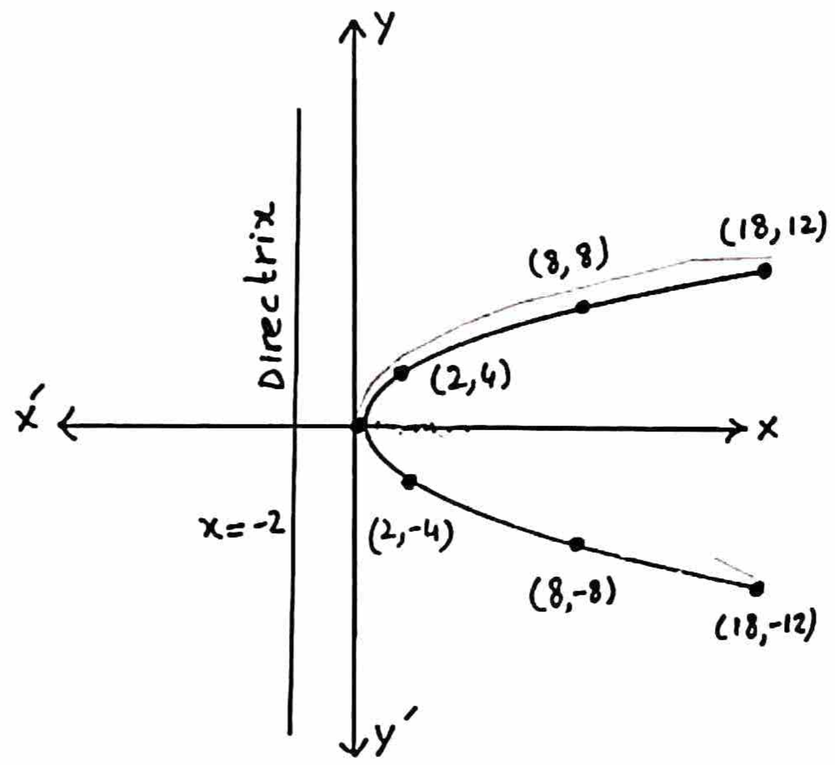
Vertex; $V(0,0) = V(0,0)$

Directrix; $x = -a \rightarrow x = -2$

Sketch:- $\because y^2 = 8x$

$\rightarrow y = \pm\sqrt{8x}$, $D_f = [0, +\infty)$

x	$y = \pm\sqrt{8x}$	Points
0	$y = \pm\sqrt{8(0)} = 0$	(0,0)
2	$y = \pm\sqrt{8(2)} = \pm 4$	(2,4), (2,-4)
8	$y = \pm\sqrt{8(8)} = \pm 8$	(8,8), (8,-8)
18	$y = \pm\sqrt{8(18)} = \pm\sqrt{144} = \pm 12$	(18,12), (18,-12)



ii) $x^2 = -16y$

Compare with $x^2 = -4ay$

$\rightarrow 4a = 16 \rightarrow a = 4$

Focus; $F(0,-a) = F(0,-4)$ $\because a = 4$

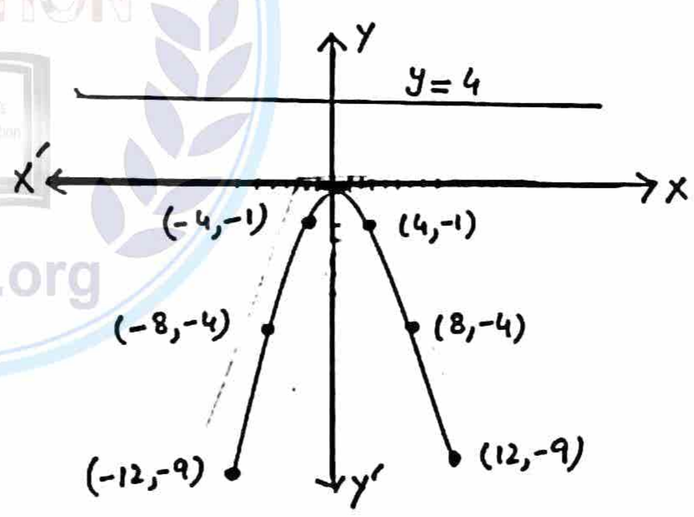
Directrix; $y = a \rightarrow y = 4$

Vertex; $V(0,0) = V(0,0)$

Sketch:- $\because x^2 = -16y$

$\rightarrow x = \pm\sqrt{-16y}$, $D_f = (-\infty, -1]$

y	$x = \pm\sqrt{-16y}$	Points
-1	$x = \pm\sqrt{-16(-1)} = \pm\sqrt{16} = \pm 4$	(4,-1), (-4,-1)
0	$x = \pm\sqrt{-16(0)} = 0$	(0,0)
-4	$x = \pm\sqrt{-16(-4)} = \pm\sqrt{64} = \pm 8$	(8,-4), (-8,-4)
-9	$x = \pm\sqrt{-16(-9)} = \pm\sqrt{144} = \pm 12$	(12,-9), (-12,-9)



iii) $x^2 = 5y$

Compare with $x^2 = 4ay$

$\rightarrow 4a = 5 \rightarrow a = \frac{5}{4}$

Focus; $F(0,a) = F(0, \frac{5}{4})$

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Vertex; $V(0,0) = V(0,0)$

Directrix; $x = -a$

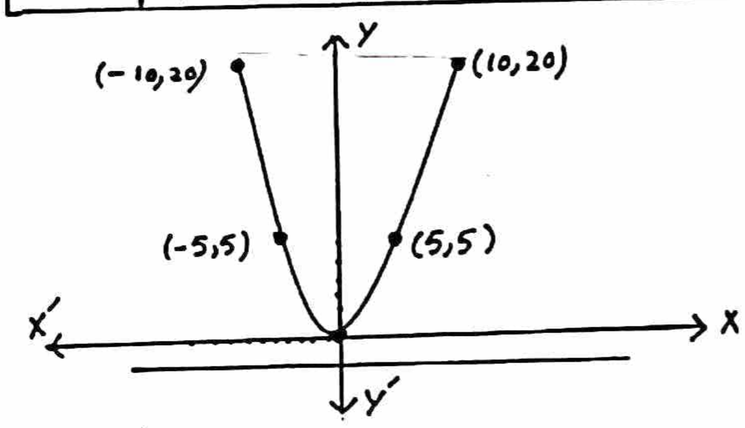
$\rightarrow x = -\frac{5}{4}$ $\quad \because a = 5/4$

or $x + \frac{5}{4} = 0$

Sketch:- $x^2 = 5y \rightarrow x = \pm\sqrt{5y}$

$D_f = [0, +\infty)$

y	$x = \pm\sqrt{5y}$	Points
0	$x = \pm\sqrt{5(0)} = 0$	$(0,0)$
5	$x = \pm\sqrt{5(5)} = \pm 5$	$(5,5), (-5,5)$
20	$x = \pm\sqrt{5(20)} = \pm\sqrt{100} = \pm 10$	$(10,20), (-10,20)$



iv) $y^2 = -12x$

compare with $y^2 = -4ax$

$\rightarrow 4a = 12 \rightarrow a = 3$

Focus; $F(-a,0) = F(-3,0)$ $\because a = 3$

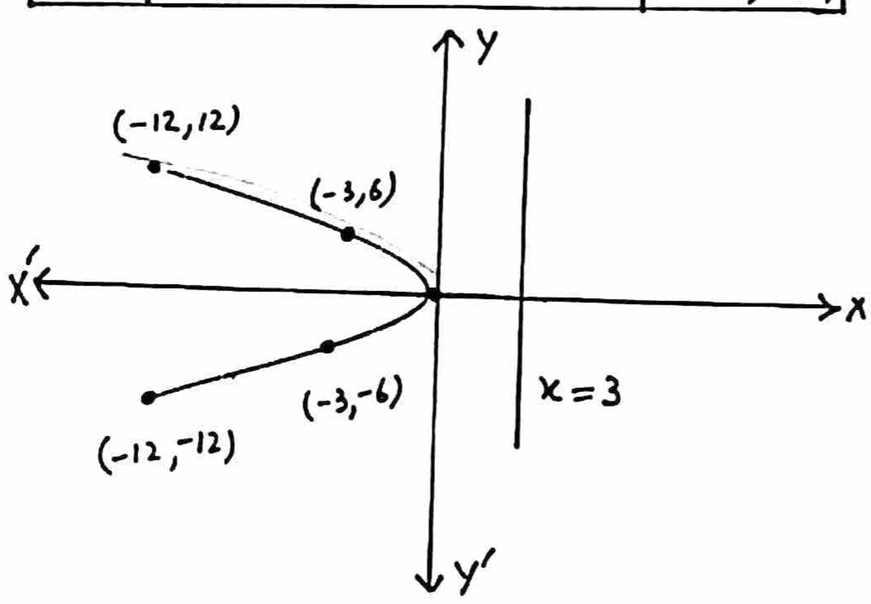
Vertex; $V(0,0) = V(0,0)$

Directrix; $x = a \rightarrow x = 3$

Sketch:- $\because y^2 = -12x$

$\rightarrow y = \pm\sqrt{-12x}$, $D_f = (-\infty, 0]$

x	$y = \pm\sqrt{-12x}$	Points
0	$y = \pm\sqrt{-12(0)} = 0$	$(0,0)$
-3	$y = \pm\sqrt{-12(-3)} = \pm\sqrt{36} = \pm 6$	$(-3,6), (-3,-6)$
-12	$y = \pm\sqrt{(-12)(-12)} = \pm\sqrt{144} = \pm 12$	$(-12,12), (-12,-12)$



v) $x^2 = 4(y-1)$

$\rightarrow (x-0)^2 = 4(y-1)$

Compare with

$X^2 = 4aY$

$\rightarrow X = x - 0 = x, \quad Y = y - 1$

$4a = 4 \rightarrow a = 1$

Focus; $F(X=0, Y=a)$

$\because X = x \quad , Y = y-1$

$\rightarrow X = 0 \quad , \rightarrow a = y-1 \quad \because Y = a$

$\rightarrow x = 0 \quad , \rightarrow 1 = y-1$ or $y = 2$

So focus $= F(0,2)$

Vertex;

$V(X=0, Y=0)$

$\because X = x \rightarrow x = 0 \quad \because X = 0$

$Y = y-1 \rightarrow 0 = y-1 \quad \because Y = 0$

$\rightarrow y = 1$

so vertex $= V(0,1)$

Directrix; $Y = -a$

$\rightarrow y - 1 = -a \quad \because Y = y-1$

$y - 1 = -1 \quad a = 1$

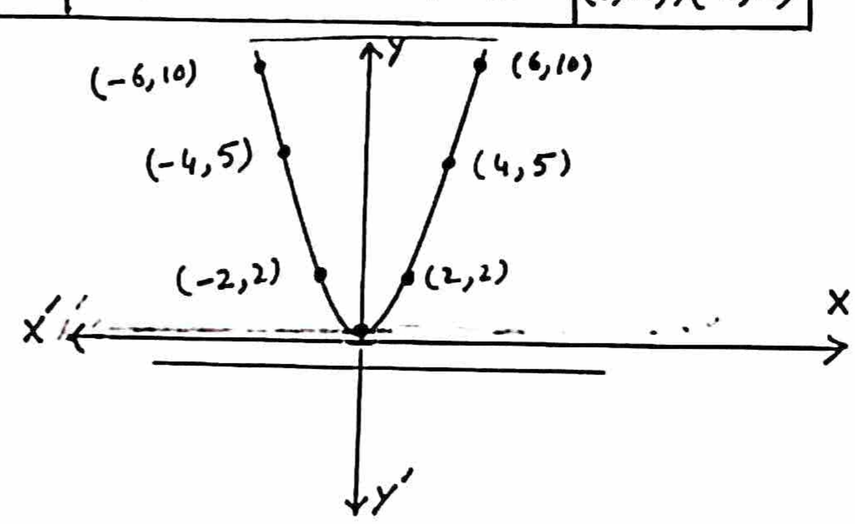
$\rightarrow y = -1 + 1 \rightarrow y = 0$

Sketch:-

$\because x^2 = 4(y-1)$

$\rightarrow x = \pm 2\sqrt{y-1} \quad D_f = [1, +\infty)$

y	$x = \pm 2\sqrt{y-1}$	Points
1	$x = \pm 2\sqrt{1-1} = 0$	$(0,0)$
2	$x = \pm 2\sqrt{2-1} = \pm 2$	$(2,2), (-2,2)$
5	$x = \pm 2\sqrt{5-1} = \pm 2(2) = \pm 4$	$(4,5), (-4,5)$
10	$x = \pm 2\sqrt{10-1} = \pm 2(3) = \pm 6$	$(6,10), (-6,10)$



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vi) $y^2 = -8(x-3)$

$\rightarrow (y-0)^2 = -8(x-3)$

Compare with $Y^2 = -4aX$

$Y = y-0 = y$, $X = x-3$

$4a = 8 \rightarrow a = 2$

Focus; $F(X=-a, Y=0)$

$\because X = x-3 \rightarrow -a = x-3$

or $-2 = x-3 \rightarrow x = -2+3$

or $x = 1$ $(\because a = 2)$

$Y = 0 \rightarrow y = 0$ $(\because Y = y)$

so $F(1, 0)$

Vertex; $(X=0, Y=0)$

$\because X = x-3 \rightarrow 0 = x-3$ $(\because X=0)$

or $x = 3$

also $Y = y \rightarrow 0 = y$ $(\because Y=0)$

or $y = 0$

so Vertex $= V(3, 0)$

Directrix; $X = a$

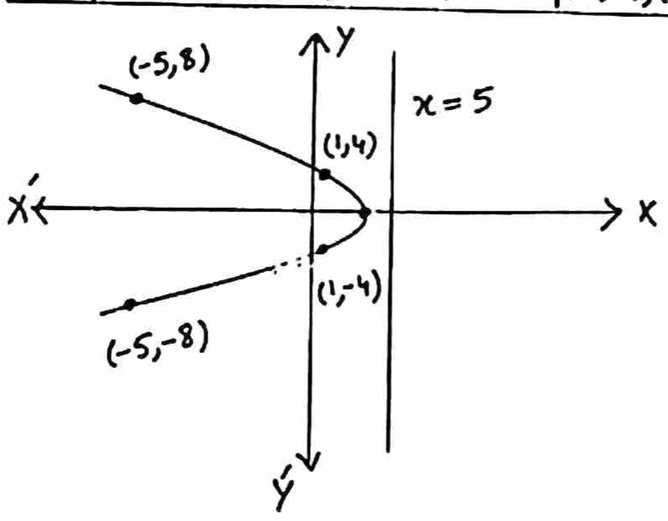
$\rightarrow x-3 = a \rightarrow x-3 = 2$

or $x = 2+3 \rightarrow x = 5$

Sketch:- $\because y^2 = -8(x-3)$

$\rightarrow y = \pm\sqrt{-8(x-3)}$, $D_f = (-\infty, 3]$

x	$y = \pm\sqrt{-8(x-3)}$	Points
3	$y = \pm\sqrt{-8(3-3)} = 0$	$(0,0)$
1	$y = \pm\sqrt{-8(1-3)} = \pm\sqrt{16} = \pm 4$	$(1,4), (1,-4)$
-5	$y = \pm\sqrt{-8(-5-3)} = \pm\sqrt{64} = \pm 8$	$(-5,8), (-5,-8)$



vii) $(x-1)^2 = 8(y+2)$

Compare with $X^2 = 4aY$

$X = x-1$, $Y = y+2$, $4a = 8 \rightarrow a = 2$

Focus; $F(X=0, Y=a)$

$X = x-1 \rightarrow 0 = x-1 \rightarrow x = 1$

$Y = y+2 \rightarrow a = y+2 \rightarrow 2 = y+2$

$\rightarrow y = 0$

so $F(1, 0)$

Vertex; $V(X=0, Y=0)$

$X=0 \rightarrow x-1=0 \rightarrow x=1$

$Y=0 \rightarrow y+2=0 \rightarrow y=-2$

$V(1,-2)$

Directrix; $Y = -a$

$\rightarrow y+2 = -a \rightarrow y+2 = -2$

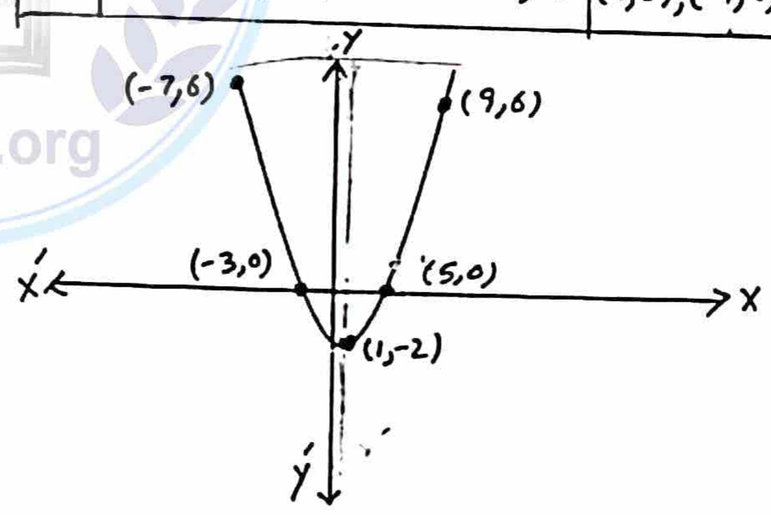
$\rightarrow y = -2-2 \rightarrow y = -4$

Sketch:- $(x-1)^2 = 8(y+2)$

$\rightarrow (x-1) = \pm\sqrt{8(y+2)}$

$\rightarrow x = 1 \pm \sqrt{8(y+2)}$, $D_f = [-2, +\infty)$

y	$x = 1 \pm \sqrt{8(y+2)}$	Points
-2	$x = 1 \pm \sqrt{8(-2+2)} = 1 \pm 0 = 1$	$(1,-2)$
0	$x = 1 \pm \sqrt{8(0+2)} = 1 \pm 4 = 5, -3$	$(5,0), (-3,0)$
6	$x = 1 \pm \sqrt{8(6+2)} = 1 \pm 8 = 9, -7$	$(9,6), (-7,6)$



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viii) $y = 6x^2 - 1$

$\rightarrow 6x^2 = y+1 \rightarrow x^2 = \frac{1}{6}(y+1)$

or $(x-0)^2 = \frac{1}{6}(y+1)$

Compare with $X^2 = 4aY$

$X = x - 0 = x$, $Y = y+1$

$4a = \frac{1}{6} \rightarrow a = \frac{1}{24}$

Focus; $F(X=0, Y=a)$

$X = 0 \rightarrow x = 0$

$Y = a \rightarrow y + 1 = \frac{1}{24}$ $(\because a = \frac{1}{24})$

$\rightarrow y = \frac{1}{24} - 1 \rightarrow y = \frac{-23}{24}$

so $F(0, \frac{-23}{24})$

Vertex; $V(X=0, Y=0)$

$X = 0 \rightarrow x = 0$,

$Y = 0 \rightarrow y + 1 = 0 \rightarrow y = -1$

so $V(0, -1)$

Directrix; $Y = -a$

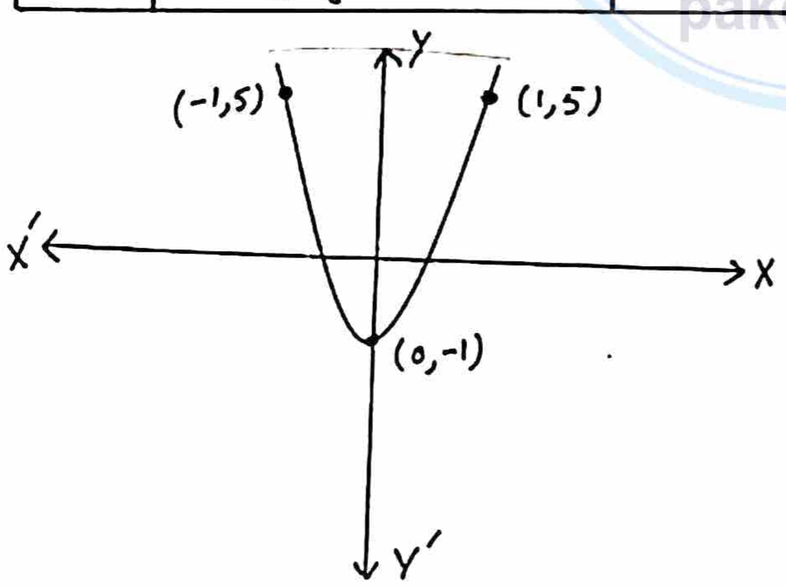
$\rightarrow y + 1 = \frac{-1}{24} \rightarrow y + 1 + \frac{1}{24} = 0$

$\rightarrow y + \frac{25}{24} = 0$

Sketch:- $\because x^2 = \frac{1}{6}(y+1)$

$\rightarrow x = \pm\sqrt{\frac{y+1}{6}}$, $D_f = [-1, +\infty)$

y	$x = \pm\sqrt{\frac{y+1}{6}}$	Points
-1	$x = \pm\sqrt{\frac{-1+1}{6}} = 0$	$(0, -1)$
5	$x = \pm\sqrt{\frac{5+1}{6}} = \pm 1$	$(1,5), (-1,5)$



ix) $x + 8 - y^2 + 2y = 0$

$\rightarrow x + 8 = y^2 - 2y$

$\rightarrow x + 8 + 1 = y^2 - 2y + 1$ (Adding 1 both sides)

$\rightarrow x + 9 = (y-1)^2$

or $(y-1)^2 = 1(x+9)$

Compare with $Y^2 = 4aX$

$\rightarrow Y = y - 1$, $X = x + 9$, $4a = 1$
$\rightarrow a = 1/4$

Focus; $F(X=a, Y=0)$

$X = x + 9 \rightarrow a = x + 9 \rightarrow \frac{1}{4} = x + 9$

$x = \frac{1}{4} - 9 \rightarrow x = -\frac{35}{4}$

$Y = 0 \rightarrow y - 1 = 0 \rightarrow y = 1$

so $F(-\frac{35}{4}, 1)$

Vertex; $V(X=0, Y=0)$

$X = 0 \rightarrow x + 9 = 0 \rightarrow x = -9$

$Y = 0 \rightarrow y - 1 = 0 \rightarrow y = 1$

so $V(-9, 1)$

Directrix; $X = -a$

$\rightarrow X = -\frac{1}{4} \rightarrow x + 9 + \frac{1}{4} = 0$

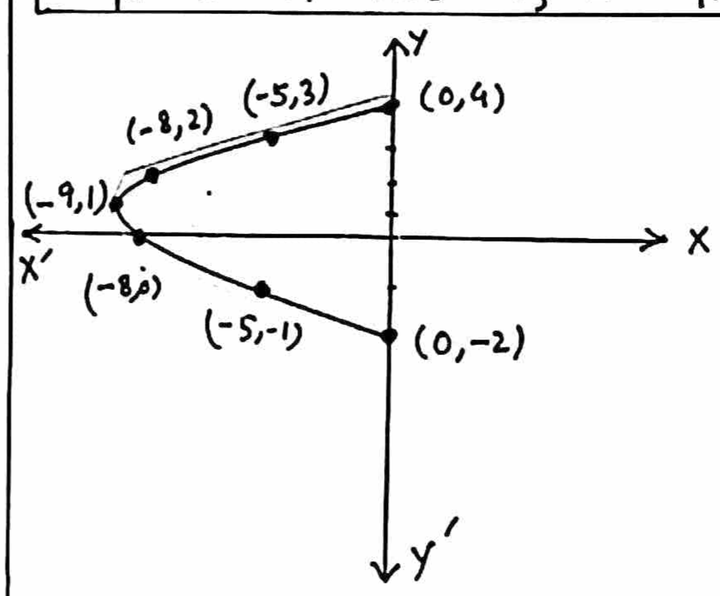
$\rightarrow x + \frac{37}{4} = 0$

Sketch; $\because (y-1)^2 = x + 9$

$\rightarrow y - 1 = \pm\sqrt{x+9}$

or $y = 1 \pm \sqrt{x+9}$ $D_f = [-9, +\infty)$

x	$y = 1 \pm \sqrt{x+9}$	Points
-9	$y = 1 \pm \sqrt{-9+9} = 1 \pm 0 = 1$	$(-9, 1)$
-8	$y = 1 \pm \sqrt{-8+9} = 1 \pm 1 = 2, 0$	$(-8,0), (-8, 2)$
-5	$y = 1 \pm \sqrt{-5+9} = 1 \pm \sqrt{4} = 1 \pm 2 = 3, -1$	$(-5,3), (-5,-1)$
0	$y = 1 \pm \sqrt{0+9} = 1 \pm 3 = 4, -2$	$(0,4), (0,-2)$



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x) $x^2-4x-8y+4=0$

$\rightarrow x^2-4x+4 = 8y$

$\rightarrow (x-2)^2 = 8(y-0)$

Compare with $X^2 = 4aY$

$X = x-2$, $4a = 8 \rightarrow a=2$

$Y = y-0 = y$

Focus; $F(X=0, Y=a)$

$X = x-2 \rightarrow 0 = x-2 \rightarrow x=2$

$Y = a \rightarrow y = 2 \quad (\because a=2)$

so $F(2,2)$

Vertex; $V(X=0, Y=0)$

$X=0 \rightarrow x-2=0 \rightarrow x=2$

$Y=0 \rightarrow y=0$

so $V(2,0)$

Directrix; $Y=-a$

$\rightarrow Y=-2 \rightarrow y=-2$

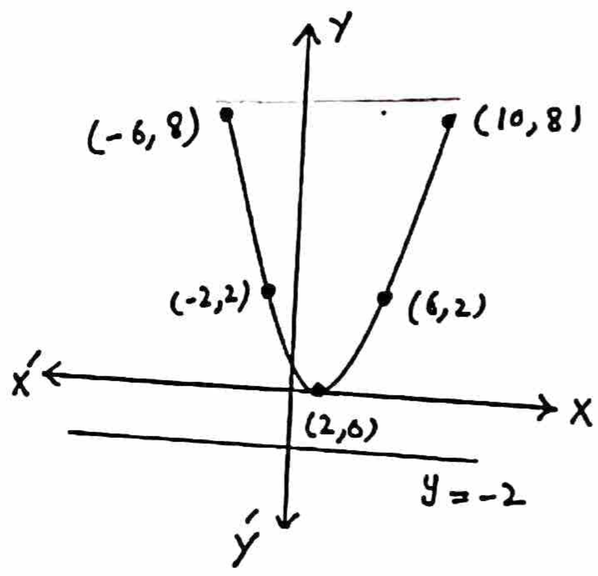
or $y+2=0$

Sketch:- $(x-2)^2 = 8y$

$\rightarrow x-2 = \pm\sqrt{8y}$

or $x = 2 \pm \sqrt{8y}$, $D_f = [0, +\infty)$

y	$x = 2 \pm \sqrt{8y}$	Points
0	$x = 2 \pm \sqrt{8(0)} = 2$	$(2,0)$
2	$x = 2 \pm \sqrt{8(2)} = 2 \pm \sqrt{16} = 6, -2$	$(6,2), (-2,2)$
8	$x = 2 \pm \sqrt{8(8)} = 2 \pm 8 = 10, -6$	$(10,8), (-6,8)$



Q2. Write an equation of the parabola with given elements

i) Focus$(-3,1)$; directrix $x=3$

Solution:- $F(-3,1)$, $M; x-3=0$

Suppose $P(x,y)$ any point on parabola, then by definition

$|PF|^2 = |PM|^2$

$$\left[\sqrt{(x+3)^2+(y-1)^2}\right]^2 = \left[\frac{|x-3|}{\sqrt{(1)^2+(0)^2}}\right]^2$$

$$\rightarrow x^2+9+6x+y^2+1-2y = \frac{x^2+9-6x}{1}$$

$\rightarrow 6x+y^2-2y+1 = -6x$

or $6x-2y+y^2+1+6x=0$

or $y^2-2y+12x+1=0$ is req. equation of Parabola

ii) Focus$(2,5)$; Directrix $y=1$

Solution:- $F(2,5)$, $M; y-1=0$

Take point $P(x,y)$ and $y-1=0$

By def $|PF|^2 = |PM|^2$

(where $|PM|$ is ⊥ar distance of P from $y-1=0$)

$$\left(\sqrt{(x-2)^2+(y-5)^2}\right)^2 = \left(\frac{|y-1|}{\sqrt{(0)^2+(1)^2}}\right)^2$$

$$\rightarrow x^2+4-4x+y^2+25-10y = \frac{y^2+1-2y}{1}$$

$\rightarrow x^2-4x+4+y^2+25-10y = y^2+1-2y$

or $x^2-4x+29-10y-1+2y=0$

or $x^2-4x-8y+28=0$ is req. eq of Parabola

iii) Focus$(-3,1)$; directrix $x-2y-3=0$

Solution:- $F(-3,1)$

Take point $P(x,y)$ and

$M; x-2y-3=0$ By def.,

$|PF|^2 = |PM|^2$

$$\left(\sqrt{(x+3)^2+(y-1)^2}\right)^2 = \left(\frac{|x-2y-3|}{\sqrt{(1)^2+(-2)^2}}\right)^2$$

$$x^2+9+6x+y^2+1-2y = \frac{x^2+4y^2+9-4xy-6x+12y}{5}$$

$\rightarrow 5x^2+45+30x+5y^2+5-10y = x^2+4y^2+9-4xy-6x+12y$

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$5x^2+5y^2+30x-10y+50-x^2-4y^2+4xy+6x-12y-9=0$

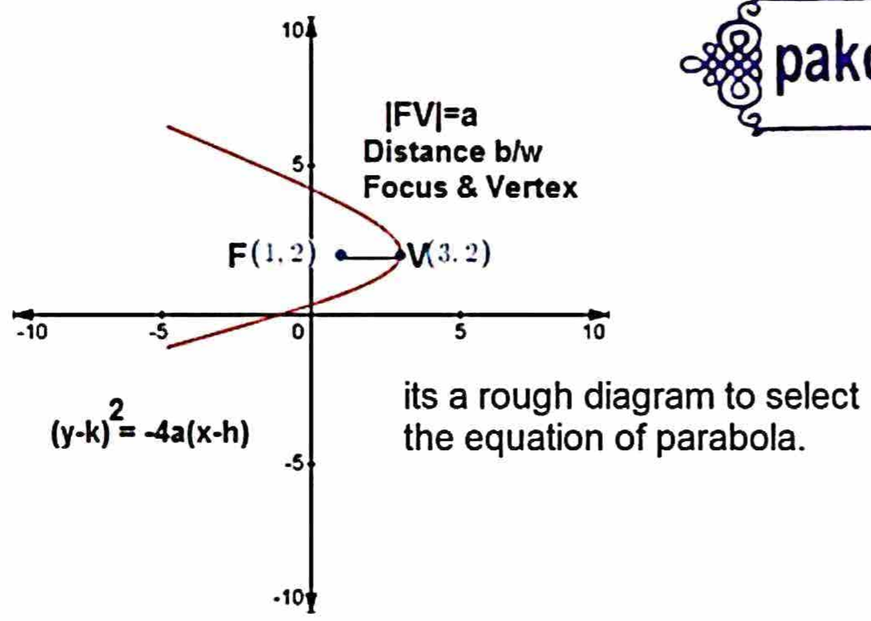
$\rightarrow 4x^2+y^2+36x-22y+4xy+41=0$ is req. eq. of Parabola.

When Focus and Vertex are given

iv) Focus $(1,2)$; Vertex $(3,2)$

Solution:-

$F(1,2)$; $V(3,2)=V(h,k)$



$(y-k)^2=-4a(x-h)$

$\rightarrow (y-2)^2=-4a(x-3) \rightarrow (i)$ $\quad \because h=3,\ k=2$

$a=|FV|=\sqrt{(3-1)^2+(2-2)^2}=\sqrt{4}=2$

Put $a=2$ in (i)

$\rightarrow (y-2)^2=-4(2)(x-3)$

$y^2+4-4y=-8x+24$

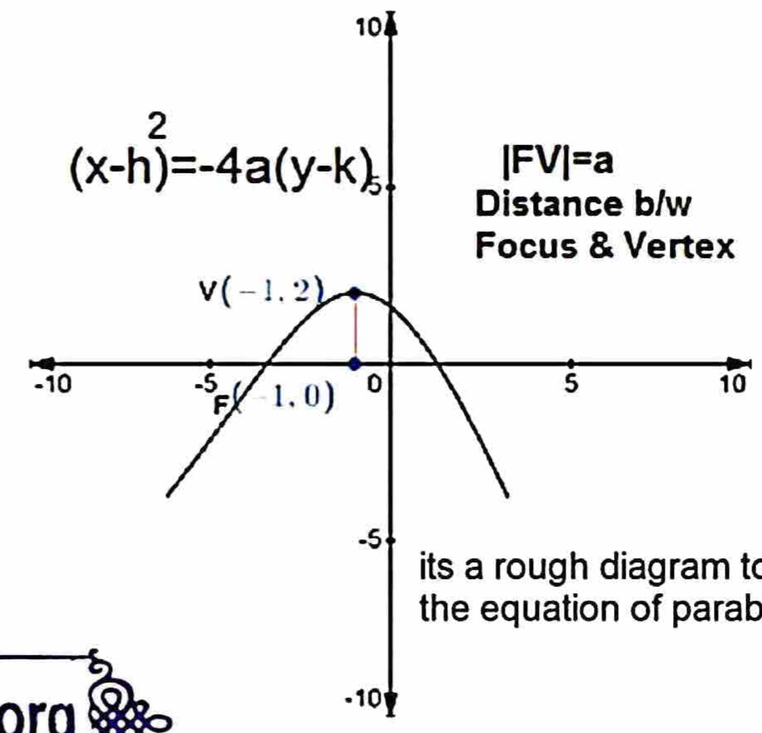
or $y^2+4-4y+8x-24=0$

or $y^2-4y+8x-20=0$ (Req. Parabola)

v) Focus $(-1,0)$; Vertex $(-1,2)$

Solution:-

$F(-1,0)$, $V(-1,2)=V(h,k)$



$(x-h)^2=-4a(y-k)$

$\rightarrow (x+1)^2=-4a(y-2) \rightarrow (i)$ $\quad \because h=-1,\ k=2$

$a=|FV|=\sqrt{(2-0)^2+(-1+1)^2}=2$

Put $a=2$ in (i)

$(x+1)^2=-4(2)(y-2)$

$\rightarrow x^2+1+2x=-8y+16$

or $x^2+2x+1+8y-16=0$

or $x^2+2x+8y-15=0$ (req. parabola)

vi) Directrix $x=-2$; Focus $(2,2)$

Solution:- Take any point $P(x,y)$

$F(2,2)$, $M: x+2=0$

By def.,

$|PF|^2=|PM|^2$

$\rightarrow \left(\sqrt{(x-2)^2+(y-2)^2}\right)^2=\left(\frac{|x+2|}{\sqrt{(1)^2+(0)^2}}\right)^2$

$\rightarrow x^2+4-4x+y^2+4-4y=\frac{x^2+4+4x}{1}$

$\rightarrow x^2-4x+y^2-4y+8-x^2-4-4x=0$

$\rightarrow y^2-4y+4-8x=0$

or $(y-2)^2=8x$ (req. parabola)

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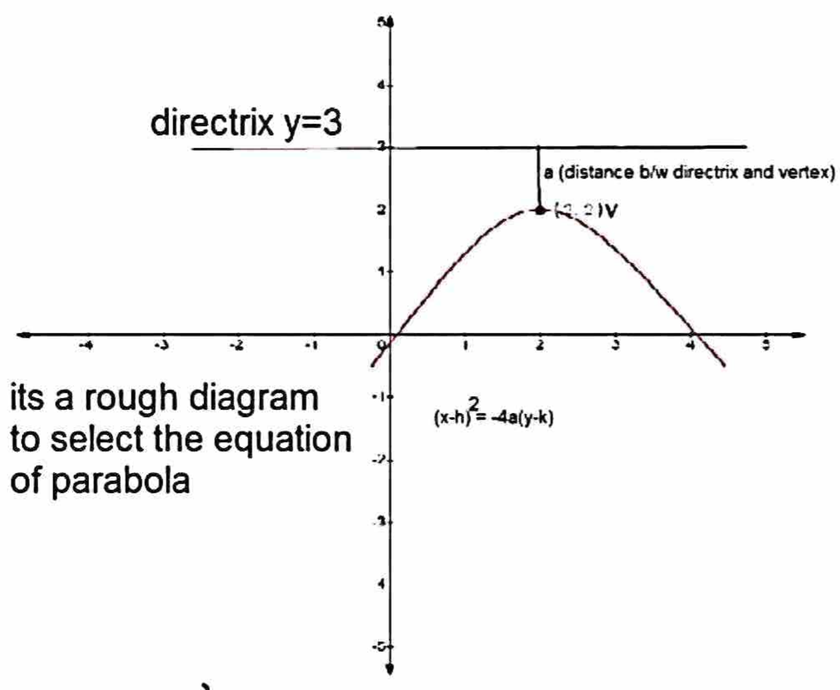
When Vertex and Directrix are given

vii) Directrix $y=3$; Vertex $(2,2)$

Solution:-

$V(2,2)$ and $M;\ y-3=0$



its a rough diagram to select the equation of parabola

$(x-h)^2 = -4a(y-k)$ $\quad \because h=2,\ k=2$

$\rightarrow (x-2)^2 = -4a(y-2) \longrightarrow (i)$

Now

$a = |VM| = \dfrac{|0(2)+1(2)-3|}{\sqrt{(0)^2+(1)^2}} = \dfrac{|2-3|}{1}$

$a = |-1| = 1$

so (i) $\rightarrow (x-2)^2 = -4(1)(y-2)$

$\rightarrow x^2+4-4x = -4y+8$

or $x^2-4x+4y-4=0$

viii) Directrix $y=1$, length of letusrectum is 8. Opens downwards

Solution:-

length of letusrectum $= 4a = 8$

$= a = 2$

$\because$ parabola opens downwards

so eq. is

$(x-h)^2 = -4a(y-k)$

$\rightarrow (x-h)^2 = -8(y-k) \longrightarrow (i)$ $\quad \because a=2$

Compare (i) with $X^2 = -4aY$

$\rightarrow X = x-h,\ Y = y-k$

$\because$ Directrix; $Y=a$

$\rightarrow y-k = 2 \quad (\because a=2)$

$\rightarrow 1-k = 2 \quad (\because y=1)$

$\rightarrow 1-2 = k \rightarrow k=-1$

so (i) $(x-h)^2 = -8(y+1)$

$\rightarrow x^2+h^2-2hx+8y+8=0$

ix) Axis $y=0$, through $(2,1)$ and $(11,-2)$

Solution:-

Let eq. of required parabola is

$(y-k)^2 = 4a(x-h) \longrightarrow (i)$

$\because y=0$ so vertex of parabola lies on x-axis $\rightarrow k=0$

($\because$ On x-axis ordinate is zero i.e. for (h,k), $k=0$)

so (i) $\rightarrow y^2 = 4a(x-h) \longrightarrow (ii)$

put $(2,1)$ in (ii)

$\rightarrow 1 = 4a(2-h)$

$1 = 8a-4ah \longrightarrow (iii)$

put $(11,-2)$ in (ii)

$4 = 4a(11-h)$

$\rightarrow 1 = 11a-ha \longrightarrow (iv)$

By (iii) -4(iv) $\rightarrow$

$1 = 8a-4ah$

$4 = 44a-4ah$

$-\quad -\quad +$

$-3 = -36a \rightarrow 12a = 1 \rightarrow a = \frac{1}{12}$

Now (iii) $\rightarrow 1 = 8\left(\frac{1}{12}\right) - 4\left(\frac{1}{12}\right)h$

$\rightarrow 1 = \frac{2}{3} - \frac{1}{3}h \rightarrow 3 = 2-h$

$\rightarrow h=-1$ so (ii) becomes as

$y^2 = 4\left(\frac{1}{12}\right)(x+1)$

$\rightarrow y^2 = \frac{1}{3}(x+1)$

or $3y^2 = x+1$

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x) Axis parallel to y-axis, the points (0,3), (3,4) and (4,11) lies on the graph

Solution:-

Let eq. of parabola is

$(x-h)^2 = 4a(y-K) \longrightarrow (i)$

Put (0,3) in (i)

$\rightarrow h^2 = 4a(3-K)$

$h^2 = 12a - 4aK \longrightarrow (ii)$

Put (3,4) in (i)

$\rightarrow (3-h)^2 = 4a(4-K)$

$9 + h^2 - 6h = 16a - 4aK \longrightarrow (iii)$

Put (4,11) in (i)

$\rightarrow (4-h)^2 = 4a(11-K)$

$\rightarrow 16 + h^2 - 8h = 44a - 4aK \longrightarrow (iv)$

By (iv) − (iii)

$16 - 8h + h^2 = 44a - 4aK$

$\underline{-9 \mp 6h - h^2 = -12a \mp 4aK}$

$7 - 2h = 28a \longrightarrow (v)$

By (iii) − (ii)

$9 - 6h + h^2 = 16a - 4aK$

$\underline{-h^2 = -12a \mp 4aK}$

$9 - 6h = 4a \longrightarrow (vi)$

Multiplying (v) by 3

$21 - 6h = 84a \longrightarrow (vii)$

By (vii) − (vi)

$21 - 6h = 84a$

$\underline{-9 \mp 6h = -4a}$

$12 = 80a \rightarrow a = \frac{12}{80} = \frac{3}{20}$

Put $a = \frac{3}{20}$ in (vi)

$9 - 6h = 4\left(\frac{3}{20}\right)$

$\rightarrow 9 - 6h = \frac{3}{5}$

$\rightarrow 9 - \frac{3}{5} = 6h$ or $6h = \frac{42}{5}$

$\rightarrow h = \frac{7}{5}$ put in (ii)

$\rightarrow \left(\frac{7}{5}\right)^2 = 12\left(\frac{3}{20}\right) - 4\left(\frac{3}{20}\right)K$

$\rightarrow \frac{49}{25} = \frac{18}{10} - \frac{3}{5}K$

$\rightarrow \frac{3}{5}K = \frac{18}{10} - \frac{49}{25}$

$\frac{3}{5}K = \frac{450 - 490}{250}$

$\rightarrow \frac{3}{5}K = \frac{-40}{250} \rightarrow K = \frac{-4}{25} \times \frac{5}{3}$

$\rightarrow K = -\frac{4}{15}$

Now (i) becomes as

$\left(x - \frac{7}{5}\right)^2 = 4\left(\frac{3}{20}\right)\left(y + \frac{4}{15}\right)$

$\rightarrow \left(x - \frac{7}{5}\right)^2 = \frac{3}{5}\left(y + \frac{4}{15}\right)$

Q3. Find an equation of parabola having its focus at the origin and directrix parallel to (i) x-axis (ii) y-axis

Solution:-

(i) **x-axis** <u>Case I:-</u>

Directrix; $y = a$, Focus $= F(0,0)$

or M; $y - a = 0$, $P(x,y)$ any point

By def.,

$|PF|^2 = |PM|^2$

$\left(\sqrt{(x-0)^2 + (y-0)^2}\right)^2 = \left(\frac{|y-a|}{\sqrt{(0)^2 + (1)^2}}\right)^2$

$x^2 + y^2 = y^2 + a^2 - 2ay$

$\rightarrow x^2 - a^2 + 2ay = 0$

<u>Case II:-</u>

Directrix; $y = -a$ (also ∥ to x-axis)

or M; $y + a = 0$, $F(0,0)$, $P(x,y)$

By def.,

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$|PF|^2 = |PM|^2$

$\rightarrow \left(\sqrt{(x-0)^2+(y-0)^2}\right)^2 = \left(\frac{|y+a|}{\sqrt{(1)^2+(0)^2}}\right)^2$

$\rightarrow x^2+y^2 = y^2 + a^2 + 2ax$

$\rightarrow x^2 - a^2 - 2ax = 0$

(ii) y-axis

Case I:-

Directrix; $x=a$, $F(0,0)$, $P(x,y)$ any point

or M; $x-a=0$

By def., $|PF|^2 = |PM|^2$

$\left(\sqrt{(x-0)^2+(y-0)^2}\right)^2 = \left(\frac{|x-a|}{\sqrt{(1)^2+(0)^2}}\right)^2$

$\rightarrow x^2+y^2 = x^2+a^2-2ax$

or $y^2-a^2+2ax=0$

Case II:-

Directrix; $x=-a$ (also ∥ to y-axis)

or M; $x+a=0$, $F(0,0)$, $P(x,y)$

By def.,

$|PF|^2 = |PM|^2$

$\rightarrow \left(\sqrt{(x-0)^2+(y-0)^2}\right)^2 = \left(\frac{|x+a|}{\sqrt{(1)^2+(0)^2}}\right)^2$

$\rightarrow x^2+y^2 = x^2+a^2+2ax$

or $y^2-a^2-2ax=0$

Q4. Show that the parabola $(x\sin\alpha - y\cos\alpha)^2 = 4a(x\cos\alpha + y\sin\alpha)$ has focus $(a\cos\alpha, a\sin\alpha)$ and its directrix is $x\cos\alpha + y\sin\alpha + a = 0$

Solution:-

Suppose $P(x,y)$ be any point

$F(a\cos\alpha, a\sin\alpha)$

M; $x\cos\alpha + y\sin\alpha + a = 0$

By def.,

$|PF|^2 = |PM|^2$

$\left(\sqrt{(x-a\cos\alpha)^2+(y-a\sin\alpha)^2}\right)^2 = \left(\frac{|x\cos\alpha+y\sin\alpha+a|}{\sqrt{\cos^2\alpha+\sin^2\alpha}}\right)^2$

$x^2+a^2\cos^2\alpha - 2ax\cos\alpha + y^2 + a^2\sin^2\alpha - 2ay\sin\alpha$
$= x^2\cos^2\alpha + y^2\sin^2\alpha + a^2 + 2xy\cos\alpha\sin\alpha + 2ax\cos\alpha + 2ay\sin\alpha$

$x^2+y^2+a^2(\cos^2\alpha+\sin^2\alpha) - 2ax\cos\alpha - 2ay\sin\alpha$
$- x^2\cos^2\alpha - y^2\sin^2\alpha - a^2 = 2xy\cos\alpha\sin\alpha + 2ax\cos\alpha + 2ay\sin\alpha$

$x^2 - x^2\cos^2\alpha + y^2 - y^2\sin^2\alpha + a^2 - a^2 = 2ax\cos\alpha + 2ax\cos\alpha$
$2ay\sin\alpha + 2xy\cos\alpha\sin\alpha \quad + 2ay\sin\alpha$

$\rightarrow x^2(1-\cos^2\alpha) + y^2(1-\sin^2\alpha) = 4ax\cos\alpha + 4ay\sin\alpha + 2xy\cos\alpha\sin\alpha$

$x^2\sin^2\alpha + y^2\cos^2\alpha - 2xy\cos\alpha\sin\alpha = 4a(x\cos\alpha + y\sin\alpha)$

$\rightarrow (x\sin\alpha - y\cos\alpha)^2 = 4a(x\cos\alpha + y\sin\alpha)$

Hence proved

Q5. Show that the ordinate at any point P of the parabola is a mean proportional between the length of letusrectum and the abscissa of P.

Solution:-

Consider eq. of Parabola

$y^2 = 4ax$

$\rightarrow y.y = 4a.x$

$\rightarrow \frac{4a}{y} = \frac{y}{x}$

$\rightarrow \frac{\text{letusrectum}}{\text{ordinate}} = \frac{\text{ordinate}}{\text{abscissa}}$

$\rightarrow$ ordinate is mean proportional between letusrectum and abscissa.

Q6. A comet has parabolic orbit with earth at focus when comet is 150,000 km from the earth, line joining comet and earth make an angle of 30°. How close will the comet to earth.

Solution:-

(See on next page)

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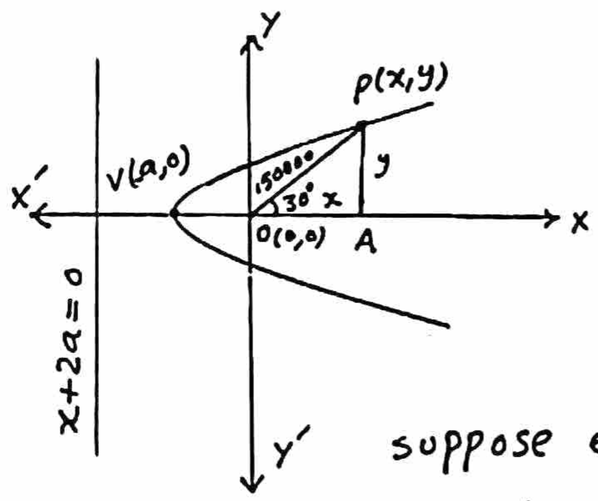


suppose earth be at focus which is origin and $V(-a,0)$ be vertex of parabola. then directrix of parabola is $x = -2a$

$\rightarrow M; x + 2a = 0$

Let comet be at point $P(x,y)$ then by def

$|PF|^2 = |PM|^2$

$\rightarrow \left(\sqrt{(x-0)^2+(y-0)^2}\right)^2 = \left(\frac{|x+2a|}{\sqrt{(1)^2+(0)^2}}\right)^2$

$\rightarrow x^2 + y^2 = \quad \longrightarrow (i)$

In ΔOAP, by pathagoras theorem

$|OA|^2 + |AP|^2 = |OP|^2$

$\rightarrow x^2 + y^2 = (150000)^2 \longrightarrow (ii)$

By (i) and (ii)

$(x+2a)^2 = (150000)^2$

$\rightarrow x + 2a = \pm 150000 \longrightarrow (iii)$

from right ΔOAP

$\cos 30^\circ = \frac{|OA|}{|AP|} \rightarrow \frac{\sqrt{3}}{2} = \frac{x}{150000}$

$\rightarrow x = \frac{\sqrt{3}}{2}(150000)$

By (iii)

$\frac{\sqrt{3}}{2}(150000) + 2a = \pm 150000$

$\rightarrow 2a = \pm 150000 - \frac{\sqrt{3}}{2}(150000)$

$\rightarrow 2a = \pm 150000 - \sqrt{3}(75000)$

$\rightarrow 2a = 75000(\pm 2 - \sqrt{3})$

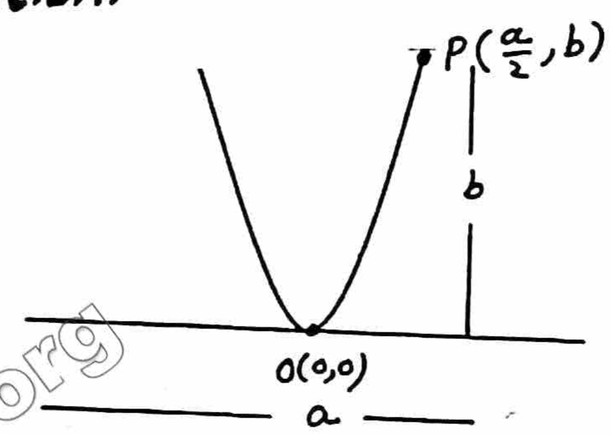
$\rightarrow a = 37500(\pm 2 - \sqrt{3})$

$\because$ a is short distance and can't be −ive so

$a = 37500(2-\sqrt{3})$ Km

Q7. Find an equation of parabola formed by the cables of a suspension bridge whose span is and the vertical height of the supporting towers is bm.

Solution:-



Consider eq. of Parabola with vertex $O(0,0)$ is

$x^2 = 4a'y \longrightarrow (i)$

$\because P(\frac{a}{2}, b)$ lies on parabola

so $\left(\frac{a}{2}\right)^2 = 4a'b \rightarrow \frac{a^2}{4} = 4a'b$

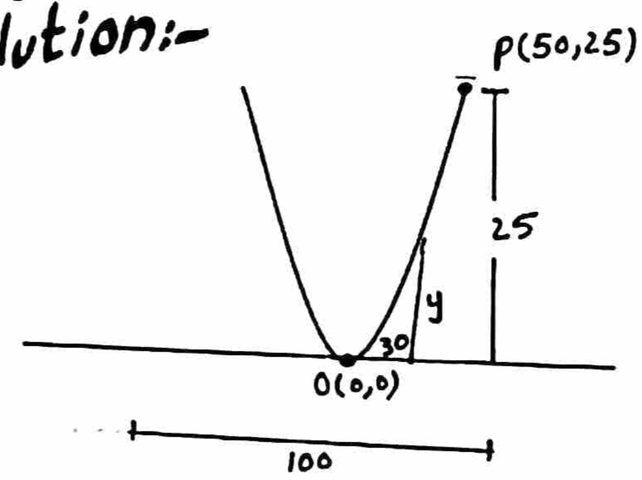
$\rightarrow a' = \frac{a^2}{16b}$ put in (i)

$x^2 = 4\left(\frac{a^2}{16b}\right)y$

$\rightarrow x^2 = \frac{a^2}{4b}y$ is req. equation

Q8. A parabolic arch has 100m base and height 25m. Find height at 30m from centre.

Solution:-



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Consider eq. of Parabola with vertex $O(0,0)$ $\quad x^2 = 4ay \longrightarrow (i)$

$\because$ from fig $P(50,25)$ lies on parabola

$$(50)^2 = 4a(25)$$

$$\rightarrow 2500 = 100a \rightarrow a = 25$$

By (i) $\rightarrow x^2 = 4(25)y \rightarrow x^2 = 100y$

when $x = 30$

$$(30)^2 = 100y \rightarrow y = \frac{900}{100}$$

$$\rightarrow y = 9$$

Hence req. height $= 9m$

Q9. Show that tangent at any point P of a parabola makes equal angles which the line PF and line through P and parallel to x-axis.

Solution:-

Let eq. of parabola is

$$y^2 = 4ax \longrightarrow (i)$$

$m_1 = 0$ ($\because$ line is $\parallel$ to x-axis)

For m_2; Take derivative of (i)

$$2y\frac{dy}{dx} = 4a \rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\rightarrow m_2 = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{2a}{y_1}$$

$$m_3 = \frac{y_1 - 0}{x_1 - a} = \frac{y_1}{x_1 - a}$$

$$\tan\theta_1 = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{\frac{2a}{y_1} - 0}{1 + \left(\frac{2a}{y_1}\right)(0)}$$

$$\tan\theta_1 = \frac{2a}{y_1} \rightarrow \theta_1 = \tan^{-1}\left(\frac{2a}{y_1}\right) \longrightarrow (i)$$

$$\tan\theta_2 = \frac{m_3 - m_2}{1 + m_3 m_2}$$

$$= \frac{\left(\frac{y_1}{x_1 - a}\right) - \left(\frac{2a}{y_1}\right)}{1 + \left(\frac{y_1}{x_1 - a}\right)\left(\frac{2a}{y_1}\right)}$$

$$\tan\theta_2 = \frac{\frac{y_1^2 - 2ax_1 + 2a^2}{(x_1 - a)(y_1)}}{\frac{y_1x_1 - ay_1 + 2ay_1}{y_1(x_1 - a)}}$$

Put $y_1^2 = 4ax_1$ ($\because$ P is on parabola)

$$\rightarrow \tan\theta_2 = \frac{\frac{4ax_1 - 2ax_1 + 2a^2}{y_1(x_1 - a)}}{\frac{y_1x_1 - ay_1 + 2ay_1}{y_1(x_1 - a)}}$$

$$\tan\theta_2 = \frac{2a\cancel{(x_1 - a)}}{y_1\cancel{(x_1 + a)}}$$

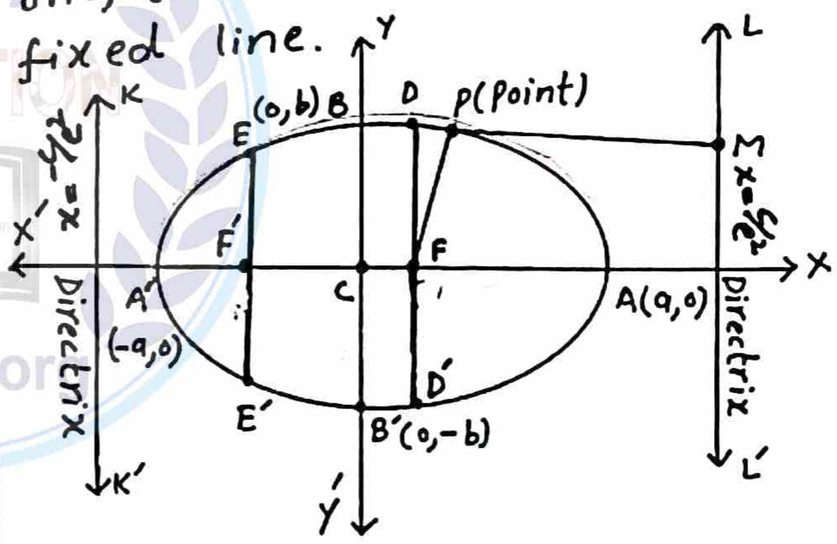
$$\rightarrow \theta_2 = \tan^{-1}\left(\frac{2a}{y_1}\right) \longrightarrow (ii)$$

By (i) and (ii)

$\theta_1 = \theta_2$ Hence proved

Ellipse and its Elements

Ellipse:- "Set of all points in a plane, such that distance of each point from a fixed point bears a constant ratio (less than one) to the distance from a fixed line.



Note:-

* Fixed point is called Focus. Here F is focus
* Fixed line LL' is directrix.
* Constant ratio is called eccentricity, denoted by e.

In fig., $\frac{|FP|}{|PM|} = e$

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→ $|FP| = e|PM|$ where $|FP| < |PM|$
so in ellipse $e < 1$.

Definitions

i) An ellipse has two diameters, called **axes of symmetry**. In fig AA' and BB' are diameters of ellipse

ii) The largest diameter is called **major axis**. In fig AA' is major axis and length of major axis $|AA'|$ is $2a$.

iii) The smallest diameter is called **minor axis**. In fig., BB' is minor axis and length of minor axis $|BB'|$ is $2b$.

iv) An ellipse has two directrix, called **directrices**. In fig., LL' and KK' are directrices.

v) Two fixed points are called **foci** (plural of focus). In fig F and F' are foci. "Foci always lie on major axis."

vi) The end points of major axis called **vertices**. In fig $A(a,0)$ and $A'(-a,0)$ are vertices of ellipse.

vii) The end points of minor axis are called **co-vertices**. In fig $B(0,b)$ and $B'(0,-b)$ are co-vertices of ellipse.

viii) A straight line joining two points of ellipse is called **chord of ellipse**.

ix) A straight line joining two points of ellipse and passing through focus is called **focal chord of ellipse**.

x) Focal chord perpendicular to major axis is called letusrectum. or Focal chord parallel to minor axis is called letusrectum. In fig DD' and EE' are letusrecta (plural of letusrectum)
length of letusrectum is $\frac{2b^2}{a}$

xi) The point of intersection of major and minor axis is called **centre**. Or Midpoint of major axis or minor axis or foci is called centre of ellipse. In fig C is centre.

xii) In fig AC and CA' are semi-major axis.

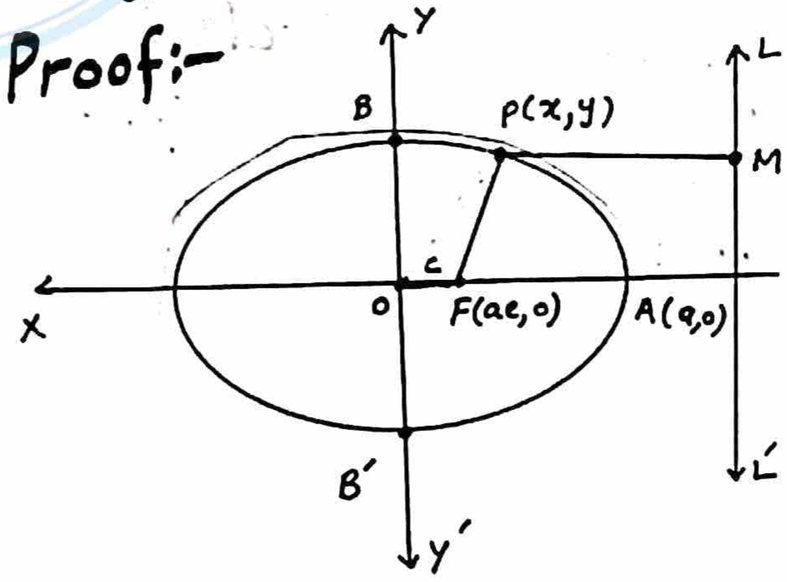
xiii) In fig BC and CB' are semi-minor axis.

xiv) If the point $(a\cos\theta, b\sin\theta)$ lies on ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ wher $\theta \in \mathbb{R}$ then $x = a\cos\theta$, $y = b\sin\theta$ are called **parametric equations** of ellipse.

Standard equation of Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Proof:-



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Let $P(x,y)$ be any point on the ellipse having focus $F(ae,0)$ and directrix $x=\frac{a}{e}$

$\rightarrow ex-a=0$

Let $|PM|$ be $\perp$ar distance of $P(x,y)$ from directrix $ex-a=0$

By def.,

$|FP|^2 = e|PM|^2$

$$\left(\sqrt{(x-ea)^2+(y-0)^2}\right)^2 = e\left(\frac{|ex-a|}{\sqrt{(e)^2+(0)^2}}\right)^2$$

$$\rightarrow x^2+e^2a^2-2eax+y^2 = e\,\frac{(ex-a)^2}{e}$$

$$x^2+e^2a^2-\cancel{2eax}+y^2 = e^2x^2+a^2-\cancel{2eax}$$

$$\rightarrow x^2-e^2x^2+y^2 = a^2-e^2a^2$$

or $x^2(1-e^2)+y^2 = a^2(1-e^2)$

$\div$ by $a^2(1-e^2)$

$$\rightarrow \frac{x^2(1-e^2)}{a^2(1-e^2)}+\frac{y^2}{a^2(1-e^2)} = \frac{a^2(1-e^2)}{a^2(1-e^2)}$$

$$\rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \quad \text{where } a^2(1-e^2) = b^2$$

Note:-

$\because b^2 = a^2(1-e^2)$

$\rightarrow b^2 = a^2-a^2e^2$

$b^2 = a^2-(ae)^2 \quad \because ae = c$

$b^2 = a^2-c^2$

or $c^2 = a^2-b^2$

Q. Prove that circle is special case of ellipse

Proof:-

$\because \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ represents an ellipse having centre at origin. Let $b=a$

then eq. of ellipse becomes as $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1 \rightarrow x^2+y^2=a^2$

which represents eq. of circle having centre at origin.

$\rightarrow$ Circle is a special case of an ellipse.

Note:- In an ellipse,

$b^2 = a^2(1-e^2)$

For a circle $b=a$

$\rightarrow a^2 = a^2(1-e^2)$

$\rightarrow 1 = 1-e^2 \rightarrow e^2 = 1-1$

or $e^2=0 \rightarrow e=0$

so for a circle, eccentricity is $e=0$

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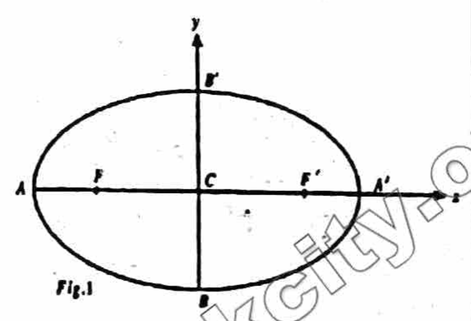
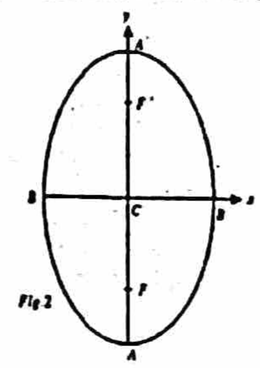
Summary of standard Ellipses

Equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$ $c^2=a^2-b^2$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1, a>b$ $c^2=a^2-b^2$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Directrices	$x=\pm\frac{c}{e^2}$	$y=\pm\frac{c}{e^2}$
Major axis	$y=0$	$x=0$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Covertices	$(0, \pm b)$	$(\pm b, 0)$
Centre	$(0, 0)$	$(0, 0)$
Eccentricity	$e=\frac{c}{a}<1$	$e=\frac{c}{a}<1$
Graph		

Note: In each ellipse

Length of major axis $= 2a$, Length of minor axis $= 2b$

Length of Latusrectum $\approx \frac{2b^2}{a}$, Foci lie on the major axis

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Example 1. Find equation of the ellipse having centre at $(0,0)$, focus at $(0,-3)$ and one vertex at $(0,4)$. Sketch its graph.

Solution:-

$\because$ centre $= O(0,0)$

vertex $= (0,4)$, Focus $= (0,-3)$

$\rightarrow$ vertices $= (0, \pm a) = (0, \pm 4)$

$\rightarrow a = 4$ or $a^2 = 16$

Foci $= (0, \pm c) = (0, \pm 3)$

$\rightarrow c = 3 \rightarrow c^2 = 9$

$\because c^2 = a^2 - b^2 \rightarrow b^2 = a^2 - c^2$

$\rightarrow b^2 = (4)^2 - (3)^2 = 16 - 9 = 7$

$\because$ foci have same abscissa, then major axis is y-axis and eq. of ellipse is

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \rightarrow \frac{x^2}{7}+\frac{y^2}{16}=1$

Sketch:-

$\because$ **Vertices;** $(0, \pm 4)$ i.e., $A(0,4)$, $A'(0,-4)$

Covertices;

$(\pm b, 0) = (\pm\sqrt{7}, 0)$ i.e., $B(\sqrt{7}, 0)$, $B'(-\sqrt{7}, 0)$

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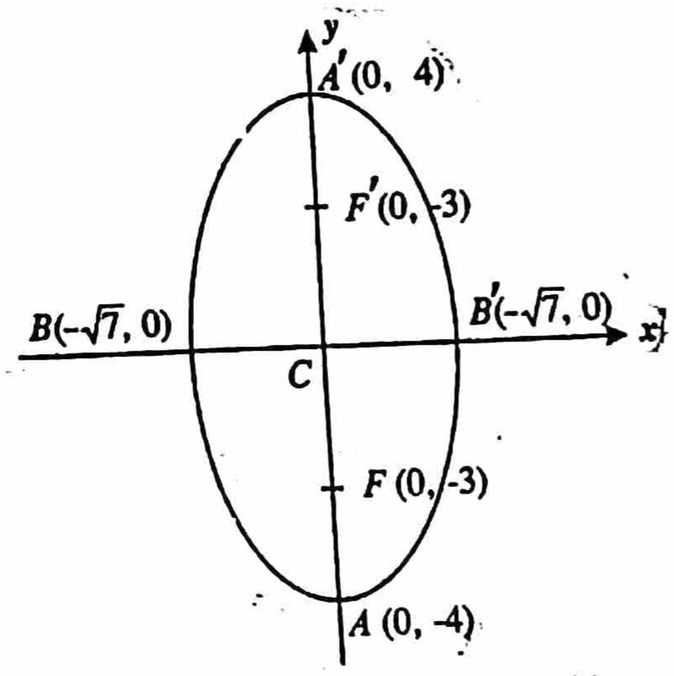
Length of letusrectum;

$$\frac{2b^2}{a} = \frac{2(7)}{4} = \frac{7}{2}$$



Example 2. Analyze the equation $4x^2+9y^2=36$ and sketch its graph.

Solution:-

$$4x^2 + 9y^2 = 36$$

$$\rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1 \quad (\div \text{ by } 36)$$

compare with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a^2 = 9, \quad b^2 = 4$

$\rightarrow a = \pm 3, \quad b = \pm 2$

$\because c^2 = a^2 - b^2 = 9 - 4 = 5$

$\rightarrow c = \pm\sqrt{5}$

Centre:- $O(0,0)$

Foci:- $(\pm c, 0) = (\pm\sqrt{5}, 0)$

Co-vertices:-

$(0, \pm b) = (0, \pm 2)$

Eccentricity:-

$\because e = \frac{c}{a} \rightarrow e = \frac{\sqrt{5}}{3}$

$\because c = \sqrt{5}, \quad a = 3$

Directrices:-

$x = \pm\frac{a}{e} \rightarrow x = \pm\frac{3}{\sqrt{5}/3} \rightarrow x = \pm\frac{9}{\sqrt{5}}$

Vertices:-

$(\pm a, 0) = (\pm 3, 0)$

Length of letusrectum:-

$$\frac{2b^2}{a} = \frac{2(4)}{3} = \frac{8}{3}$$

Sketch:-

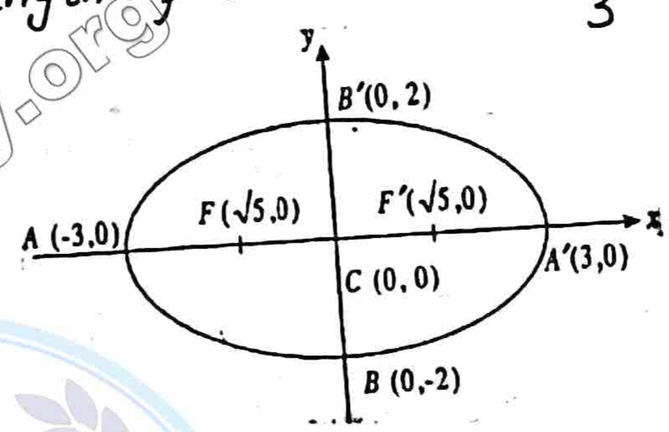
Vertices: $(\pm a, 0) = (\pm 3, 0)$

i-e., $A(3,0)$, $A'(-3,0)$

Co-vertices: $(0, \pm b)$

$= (0, \pm 2)$ i-e., $B(0,2)$, $B'(0,-2)$

Length of letusrectum $= \frac{8}{3}$



Example 3. Show that the equation $9x^2 - 18x + 4y^2 + 8y - 23 = 0$ represents an ellipse. Find its elements and sketch its graph.

Solution:-

$$9x^2 - 18x + 4y^2 + 8y - 23 = 0$$

$$\rightarrow 9(x^2 - 2x) + 4(y^2 + 2y) = 23$$

$$\rightarrow 9(x^2 - 2x + 1 - 1) + 4(y^2 + 2y + 1 - 1) = 23$$

$$\rightarrow 9\{(x-1)^2 - 1\} + 4\{(y+1)^2 - 1\} = 23$$

$$\rightarrow 9(x-1)^2 - 9 + 4(y+1)^2 - 4 = 23$$

or $9(x-1)^2 + 4(y+1)^2 = 36$

$$\rightarrow \frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1 \quad (\div \text{ by } 36)$$

Compare with $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

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$X = x-1, \quad Y = y+1, \quad b^2 = 4, \quad a^2 = 9$

$\rightarrow b = 2, \quad a = 3$

$\because c^2 = a^2 - b^2 \rightarrow c^2 = 9-4 = 5$

$\rightarrow c = \sqrt{5}$

Centre:- $(X=0, Y=0)$

$\because X = x-1 \rightarrow 0 = x-1 \rightarrow x = 1$

$Y = y+1 \rightarrow 0 = y+1 \rightarrow y = -1$

so centre $(1,-1)$

Foci:- $(X=0, Y=\pm c)$

$\because X = x-1 \rightarrow 0 = x-1 \rightarrow x = 1$

$Y = y+1 \rightarrow \pm c = y+1$

$\rightarrow \pm\sqrt{5} = y+1$ or $y = -1 \pm \sqrt{5}$

Hence foci $(1, -1 \pm \sqrt{5})$

Co-vertices:- $(X = \pm b, Y = 0)$

$\because X = x-1 \rightarrow \pm b = x-1 \rightarrow \pm 2 = x-1$

$\rightarrow \pm 2 + 1 = x \rightarrow x = 3, -1$

$Y = y+1 \rightarrow y+1 = 0 \rightarrow y = -1$

So covertices are $(3,-1)$ and $(-1,-1)$

Eccentricity:- $e = \frac{c}{a}$

$\rightarrow e = \frac{\sqrt{5}}{3} \qquad \because c = \sqrt{5}, \ a = 3$

Directrices:- $Y = \pm \frac{a}{e}$

$\rightarrow y+1 = \pm \frac{a}{e} \qquad (\because Y = y+1)$

$\rightarrow y+1 = \pm \frac{3}{\sqrt{5}/3} \rightarrow y+1 = \pm \frac{9}{\sqrt{5}}$

or $y = -1 \pm 9/\sqrt{5}$

Vertices:- $(X=0, Y=\pm a)$

$\because X = x-1 \rightarrow x-1 = 0 \rightarrow x = 1$

$Y = y+1 \rightarrow y+1 = \pm a$

$y+1 = \pm 3 \rightarrow y = -1 \pm 3$

or $y = 2, -4$

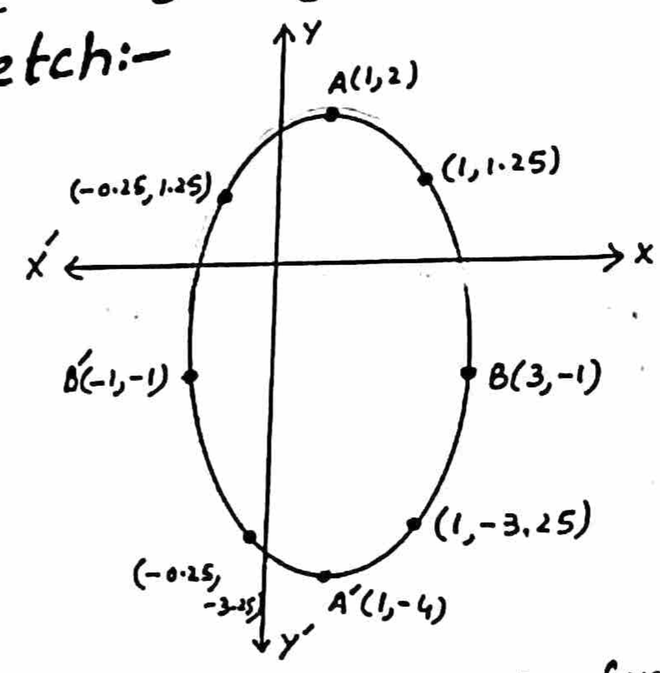
so vertices are

$(1,2)$ and $(1,-4)$

Length of letusrectum:-

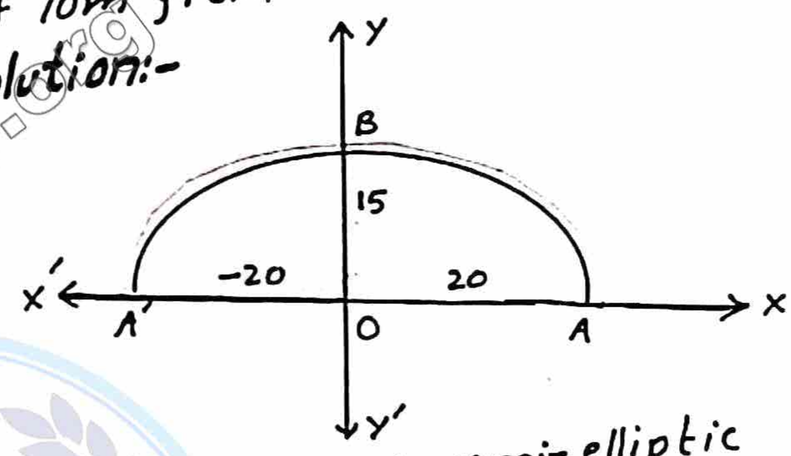
$\frac{2b^2}{a} = \frac{2(4)}{3} = \frac{8}{3}$

Sketch:-



Example 4. An arch is in form of half an ellipse is 40m in wide and 15m high at the centre. Find the height of the arch at a distance of 10m from its centre.

Solution:-



Let AA' be base of semi-elliptic arch having centre at O.

Since the arch is 40m wide at the base, so $|AA'| = 40$. As $|AA'| = 2a$

so $2a = 40 \rightarrow a = 20$ and $a^2 = 400$

Since the arch is 15m high at the centre, so $|OB| = b = 15$ and $b^2 = 225$

$\because$ Eq. of semi-elliptic arch is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow \frac{x^2}{400} + \frac{y^2}{225} = 1 \longrightarrow (i)$

Let h be height of arch at distance $x = 10m$ from centre of base. Put $x = 10, y = h$ in (i)

$\frac{(10)^2}{400} + \frac{h^2}{225} = 1 \rightarrow \frac{100}{400} + \frac{h^2}{225} = 1$

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$\rightarrow \frac{1}{4} + \frac{h^2}{225} = 1 \rightarrow \frac{h^2}{225} = 1 - \frac{1}{4}$

$\rightarrow \frac{h^2}{225} = \frac{3}{4} \rightarrow h^2 = \frac{3 \times 225}{4}$

$\rightarrow h = \sqrt{\frac{3 \times 225}{4}} = \frac{15\sqrt{3}}{2}$

Thus $h = \frac{15\sqrt{3}}{2}$ m is req. height.

Exercise 6.5

Q1. Find an equation of the ellipse with given data and sketch its graph.

(i) Foci $(\pm 3, 0)$ and minor axis of length 10

Solution:-

length of minor axis $= 10$

$c = 3$ and $2b = 10 \rightarrow b = 5$

Now $c^2 = a^2 - b^2 \rightarrow a^2 = c^2 + b^2$

$\rightarrow a^2 = 9 + 25 \rightarrow a^2 = 34$

Now req. eq. of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow \frac{x^2}{34} + \frac{y^2}{25} = 1$$

Sketch:-

Centre = mid point of foci $F(3,0)$ and $F'(-3,0)$

Centre $= \left(\frac{3-3}{2}, \frac{0+0}{2}\right) = (0,0)$

Vertices:- $(\pm a, 0) = (\pm\sqrt{34}, 0)$

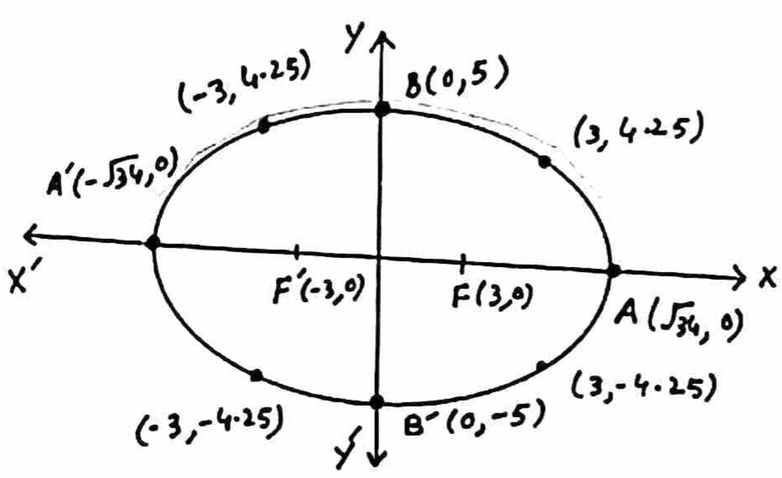
i-e., $A(\sqrt{34}, 0)$, $A'(-\sqrt{34}, 0)$

Co-Vertices:- $(0, \pm b) = (0, \pm 5)$

i-e., $B(0,5)$, $B'(0,-5)$

Length of letusrectum:-

$$\frac{2b^2}{a} = \frac{2(25)}{\sqrt{34}} = \frac{50}{\sqrt{34}}$$



(ii) Foci $(0,-1)$ and $(0,-5)$, major axis of length 6

Solution:-

Centre = midpoint of foci

$(h, k) = \left(\frac{0+0}{2}, \frac{-1-5}{2}\right) = (0, -3)$

$\because 2c =$ Distance b/w foci

$2c = \sqrt{(0-0)^2 + (-5+1)^2} = \sqrt{16} = 4$

$\rightarrow 2c = 4 \rightarrow c = 2 \rightarrow c^2 = 4$

$\because$ length of major axis $= 2a = 6$

$\rightarrow a = 3 \rightarrow a^2 = 9$

$\because c^2 = a^2 - b^2 \rightarrow b^2 = a^2 - c^2 = 9 - 4 = 5$

$\because$ foci have same abscissa, so major axis is along y-axis.

Now eq. of ellipse is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \longrightarrow (i)$$

$\because h = 0,\ k = -3,\ a^2 = 9,\ b^2 = 5$

$$\rightarrow \frac{x^2}{5} + \frac{(y+3)^2}{9} = 1$$

Sketch:-

Vertices: $(X = 0,\ Y = \pm a)$

$\rightarrow X = x \rightarrow x = 0,\ Y = \pm a$

$\rightarrow y + 3 = \pm 3 \rightarrow y = -3 \pm 3 = 0, -6$

so vertices are $A(0,0)$, $A'(0,-6)$

Co-Vertices: $(X = \pm b,\ Y = 0)$

$\because X = x \rightarrow x = \pm b \rightarrow x = \pm\sqrt{5}$

$Y = 0 \rightarrow y + 3 = 0 \rightarrow y = -3$

so co-vertices are $(\sqrt{5}, -3)$, $(-\sqrt{5}, -3)$

Length of letusrectum:-

$$\frac{2b^2}{a} = \frac{2(5)}{3} = \frac{10}{3}$$

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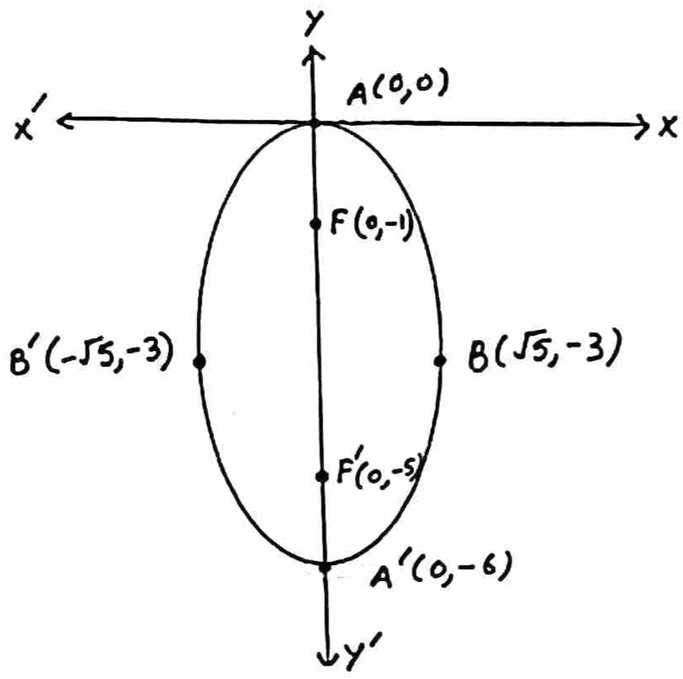


(iii) Foci $(\pm 3\sqrt{3}, 0)$ and vertices $(\pm 6, 0)$

Solution:-

Foci $(\pm c, 0) = (\pm 3\sqrt{3}, 0)$

$\rightarrow c = 3\sqrt{3} \rightarrow c^2 = 27$

centre = midpoint of foci

$= \left(\frac{3\sqrt{3} - 3\sqrt{3}}{2}, \frac{0+0}{2}\right) = (0, 0)$

Vertices $= (\pm a, 0) = (\pm 6, 0)$

$\rightarrow a = 6 \rightarrow a^2 = 36$

$\because c^2 = a^2 - b^2 \rightarrow b^2 = a^2 - c^2$

$\rightarrow b^2 = 36 - 27 = 9$

$\because$ foci $= (\pm 3\sqrt{3}, 0)$

i.e., $F(3\sqrt{3}, 0)$, $F(-3\sqrt{3}, 0)$

$\rightarrow$ same ordinate, so major axis is along x-axis. So ellipse is: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow \frac{x^2}{36} + \frac{y^2}{9} = 1$

Sketch:-

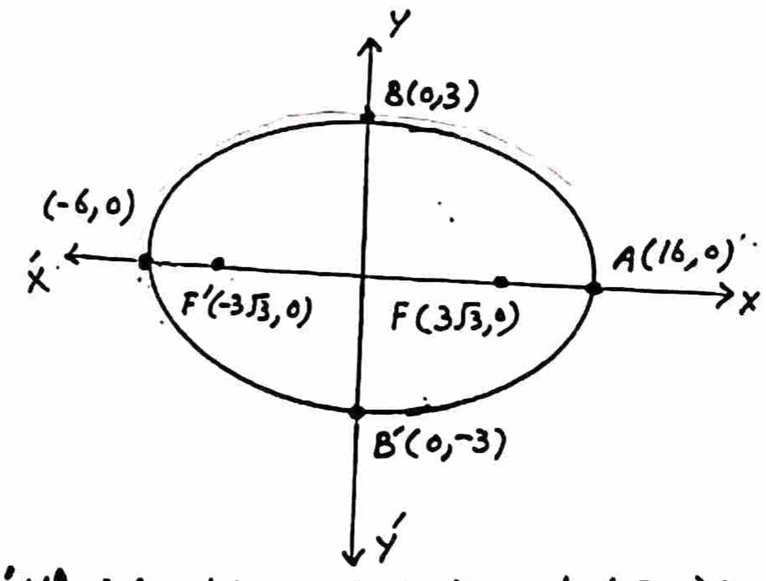
Vertices $= (\pm 6, 0)$ i.e., $A(6,0)$, $A'(-6,0)$

Co-vertices $= (0, \pm b) = (0, \pm 3)$

i.e., $B(0,3)$, $B'(0,-3)$

Length of letusrectum: $\frac{2b^2}{a} = \frac{2(9)}{6}$

$= 3$



(iv) Vertices $(-1,1)$ and $(5,1)$; Foci $(4,1)$ and $(0,1)$

Solution:-

$\because$ Foci: $F(4,1)$, $F'(0,1)$.

Centre = midpoint of foci

$(h, k) = \left(\frac{4+0}{2}, \frac{1+1}{2}\right) = (2, 1)$

$2c =$ Dist. b/w foci

$2c = \sqrt{(4-0)^2 + (1-1)^2} = 4$

$\rightarrow 2c = 4$ or $c = 2 \rightarrow c^2 = 4$

Also Vertices: $(-1,1)$, $(5,1)$

$2a =$ Dist. b/w vertices

$2a = \sqrt{(5+1)^2 + (1-1)^2} = 6$

$\rightarrow 2a = 6 \rightarrow a = 3$ or $a^2 = 9$

$\because c^2 = a^2 - b^2 \rightarrow b^2 = a^2 - c^2$

$\rightarrow b^2 = 9 - 4 \rightarrow b^2 = 5$

$\because$ foci have same ordinate so major axis is along x-axis.

eq. of ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

$\because h = 2, k = 1, a^2 = 9, b^2 = 5$ so

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{5} = 1$$

Sketch:-

Vertices: $A(5,1)$, $A'(-1,1)$

Co vertices: $(X = 0, Y = \pm b)$

$\because X = x - 2 \rightarrow x - 2 = 0 \rightarrow x = 2$

$Y = y - 1 \rightarrow y - 1 = \pm b \rightarrow y = 1 \pm \sqrt{5}$

Co vertices are $B(2, 1+\sqrt{5})$, $B'(2, 1-\sqrt{5})$

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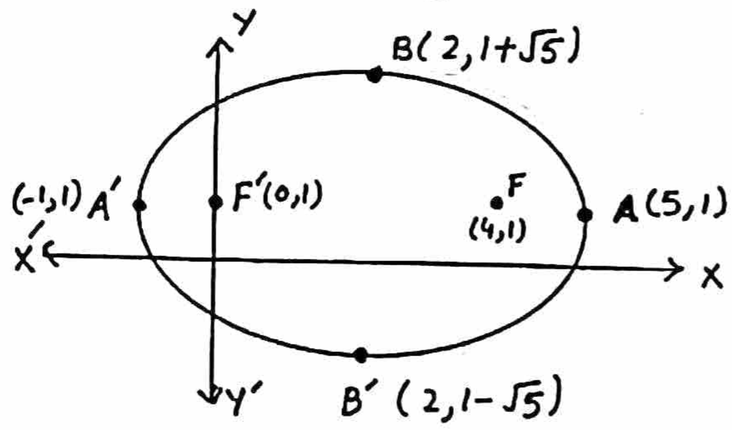
Length of letusrectum: $\frac{2b^2}{a} = \frac{2(5)}{3}$

$= \frac{10}{3}$



(v) Foci $(\pm\sqrt{5},0)$ and passing through the point $(\frac{3}{2},\sqrt{3})$

Solution:-

Foci $= (\pm c, 0) = (\pm\sqrt{5}, 0)$

$\rightarrow c = \sqrt{5} \rightarrow c^2 = 5$

As foci: $F(\sqrt{5},0)$, $F'(-\sqrt{5},0)$

centre = midpoint of foci

$= \left(\frac{\sqrt{5}-\sqrt{5}}{2}, \frac{0+0}{2}\right) = (0,0)$

$\because c^2 = a^2 - b^2 \rightarrow a^2 - b^2 = 5$

$\rightarrow a^2 = b^2 + 5 \longrightarrow (i)$

$\because$ foci have same ordinate. so major axis is x-axis and eq. of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow (ii)$

$\because$ ellipse (ii) passes through $(\frac{3}{2},\sqrt{3})$

so (ii) $\rightarrow \frac{(\frac{3}{2})^2}{a^2} + \frac{(\sqrt{3})^2}{b^2} = 1$

$\rightarrow \frac{9}{4a^2} + \frac{3}{b^2} = 1$

$\rightarrow 9b^2 + 12a^2 = 4a^2b^2$ (x by $4a^2b^2$)

$\rightarrow 9b^2 + 12(b^2+5) = 4(b^2+5)b^2$ ($\because a^2 = b^2+5$)

$\rightarrow 9b^2 + 12b^2 + 60 = 4b^4 + 20b^2$

$\rightarrow 21b^2 - 20b^2 + 60 = 4b^4$

$\rightarrow b^2 + 60 = 4b^4$

or $4b^4 - b^2 - 60 = 0$

$4b^4 - 16b^2 + 15b^2 - 60 = 0$

$4b^2(b^2-4) + 15(b^2-4) = 0$

$(b^2-4)(4b^2+15) = 0$

$b^2 - 4 = 0$, $4b^2 + 15 = 0$

$\rightarrow b^2 = 4$, $b^2 = -\frac{15}{4}$ (Impossible)

$\because b^2 = 4 \rightarrow a^2 = 4+5$ By (i)

$\rightarrow a^2 = 9$ so (ii) becomes

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

Sketch:-

Vertices $= (\pm a, 0) = (\pm 3, 0)$

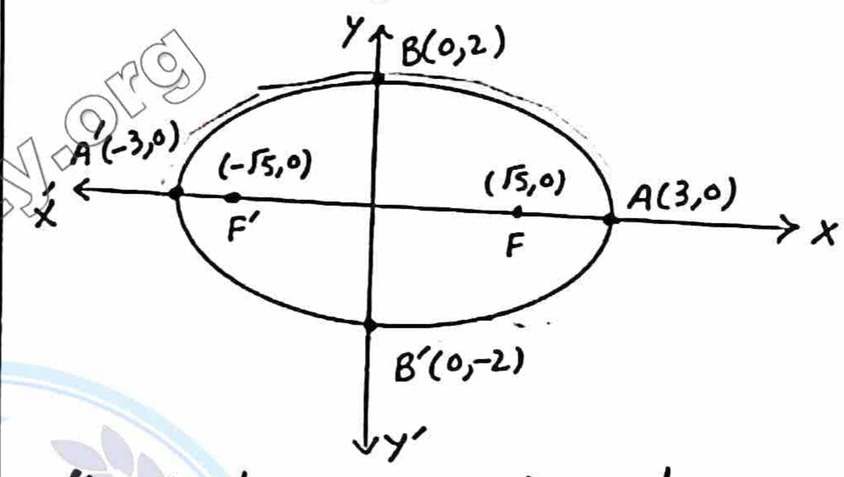
$\rightarrow A(3,0)$, $A'(-3,0)$

Co-Vertices $= (0, \pm b) = (0, \pm 2)$

$\rightarrow B(0,2)$, $B'(0,-2)$

Length of letusrectum: $\frac{2b^2}{a}$

$= \frac{2(4)}{3} = \frac{8}{3}$



(vi) Vertices $(0,\pm 5)$ and eccentricity $\frac{3}{5}$

Solution:-

Vertices $= (0, \pm a) = (0, \pm 5)$

$\rightarrow a = 5$, $a^2 = 25$

Centre = midpoint of vertices

$= \left(\frac{0+0}{2}, \frac{5-5}{2}\right) = (0,0)$

$\because$ eccentricity $= e = \frac{3}{5}$

$\because c = ae \rightarrow c = 5(3/5) = 3$

$\rightarrow c^2 = 9$

$\because c^2 = a^2 - b^2 \rightarrow b^2 = a^2 - c^2 = 25 - 9$

$\rightarrow b^2 = 16$

$\because$ vertices have same abscissa so major axis is y-axis and eq. of ellipse is

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$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \rightarrow \frac{x^2}{16}+\frac{y^2}{25}=1$$

Sketch:-

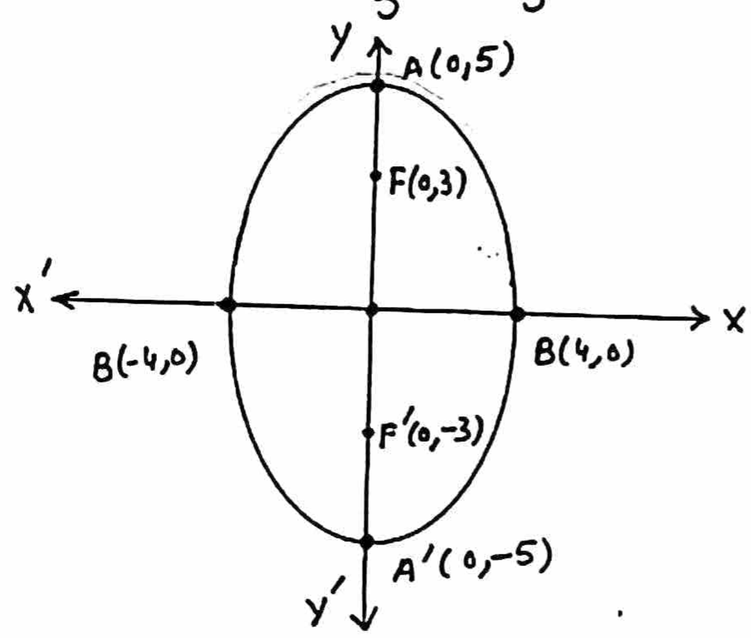
Vertices: $(0,\pm 5) \rightarrow A(0,5)$, $A'(0,-5)$

Covertices; $(\pm b,0)=(\pm 4,0)$

$\rightarrow B(4,0),\ B'(-4,0)$

Length of letusrectum: $\frac{2b^2}{a}$

$= \frac{2(16)}{5} = \frac{32}{5}$



(vii) Centre $(0,0)$, focus $(0,-3)$ and vertex $(0,4)$

Solution:-

Centre $=(0,0)$, focus $=F'(0,-3)$

so other focus $=F(0,3)$

$\rightarrow$ foci $=(0,\pm c)=(0,\pm 3)$

$\rightarrow c=3 \rightarrow c^2=9$

Also vertex $=(0,4)$

other vertex $=(0,-4)$

$\rightarrow$ vertices $=(0,\pm a)=(0,\pm 4)$

$\rightarrow a=4$ or $a^2=16$

$\because c^2=a^2-b^2 \rightarrow b^2=a^2-c^2$

or $b^2=16-9=7$

$\because$ foci have same abscissa so eq. of ellipse is

$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ ($\because$ major axis is y-axis)

$\rightarrow \frac{x^2}{7}+\frac{y^2}{16}=1$

Sketch:-

Vertices $=(0,\pm a)=(0,\pm 4)$

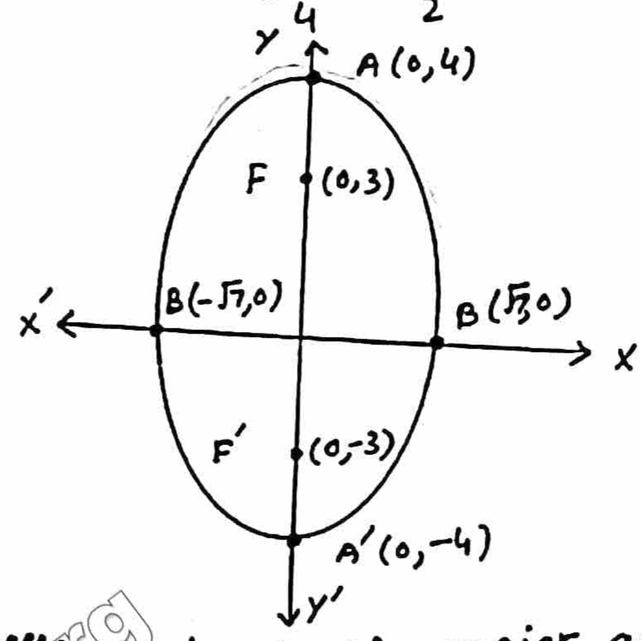
$\rightarrow A(0,4),\ A'(0,-4)$

Co-vertices $=(\pm b,0)=(\pm\sqrt{7},0)$

$\rightarrow B(\sqrt{7},0),\ B'(-\sqrt{7},0)$

Length of letusrectum $=\frac{2b^2}{a}$

$=\frac{2(7)}{4}=\frac{7}{2}$



(viii) Centre $(2,2)$, major axis parallel to y-axis and of length 8 units, minor axis parallel to x-axis and of length 6 units.

Solution:-

Centre $=(h,k)=(2,2)$

Length of major axis $=2a=8$

$\rightarrow a=4 \rightarrow a^2=16$

Length of minor axis $=2b=6$

$\rightarrow b=3 \rightarrow b^2=9$

$\because c^2=a^2-b^2=16-9=7$

$\rightarrow c=\sqrt{7}$

Eq. of ellipse is

$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$ ($\because$ major axis is $\parallel$ to y-axis)

Put $h=2,\ k=2,\ b^2=9,\ a^2=16$

$\rightarrow \frac{(x-2)^2}{9}+\frac{(y-2)^2}{16}=1$

Sketch:-

Vertices: $(x=0,\ y=\pm a)$

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$\because X = x-2 \rightarrow x-2=0 \rightarrow x=2$

$Y = y-2 \rightarrow y-2=\pm a$

$\rightarrow y-2=\pm 4 \rightarrow y=2\pm 4$

or $y = 6, -2$

so vertices are $A(2,6)$, $A'(2,-2)$

Covertices: $(X=\pm b, Y=0)$

$\because X = x-2 \rightarrow x-2=\pm b$

$\rightarrow x-2=\pm 3 \rightarrow x=2\pm 3$

$\rightarrow x=5,-1$

$Y = y-2 \rightarrow y-2=0 \rightarrow y=2$

so covertices are $B(5,2)$, $B'(-1,2)$

Foci: $(X=0, Y=\pm c)$

$X = x-2 \rightarrow x-2=0 \rightarrow x=2$

$Y = y-2 \rightarrow y-2=\pm c$

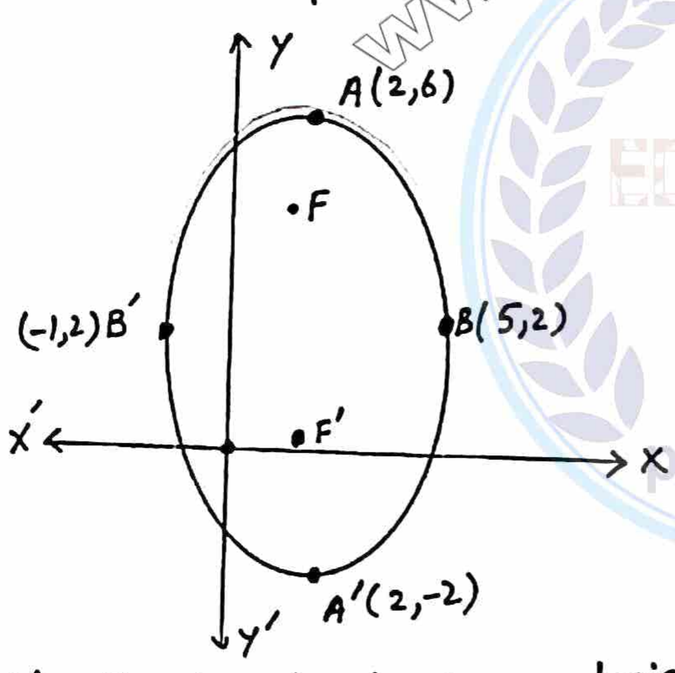
$\rightarrow y-2=\pm\sqrt{7} \rightarrow y=2\pm\sqrt{7}$

so Foci are: $F(2, 2+\sqrt{7})$

$F'(2, 2-\sqrt{7})$

Length of letusrectum: $\frac{2b^2}{a}$

$= \frac{2(9)}{4} = \frac{9}{2}$



(ix) Centre $(0,0)$, symmetric with respect to both the axes and passing through the points $(2,3)$ and $(6,1)$

Solution:-

Centre $=(0,0)$, symmetric w.r.t both axes

Eq. of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \longrightarrow (i)$

$\because$ (i) passes through $(2,3)$ so

$\frac{(2)^2}{a^2}+\frac{(3)^2}{b^2}=1 \rightarrow \frac{4}{a^2}+\frac{9}{b^2}=1$

$\rightarrow 4b^2+9a^2=a^2b^2 \longrightarrow (ii)$

Also (i) passes through $(6,1)$ so

$\frac{(6)^2}{a^2}+\frac{(1)^2}{b^2}=1 \rightarrow \frac{36}{a^2}+\frac{1}{b^2}=1$

$\rightarrow 36b^2+a^2=a^2b^2 \longrightarrow (iii)$

By (ii) and (iii)

$36b^2+a^2=4b^2+9a^2$

$\rightarrow 32b^2=8a^2$

$\rightarrow a^2=4b^2 \longrightarrow$ (iv) put in (ii)

$4b^2+9(4b^2)=(4b^2)b^2$

$\rightarrow 4b^2+36b^2=4b^4$

$\rightarrow 40b^2=4b^4$

or $10=b^2$ $(\div$ by $b^2)$

$\rightarrow b=\sqrt{10}$

(iv) $\rightarrow a^2=4(10) \rightarrow a^2=40$

or $a=\sqrt{40}=2\sqrt{10}$

Req. ellipse (i) becomes as

$\frac{x^2}{40}+\frac{y^2}{10}=1$

Sketch:-

Vertices: $(\pm a,0)=(\pm 2\sqrt{10},0)$

$A(2\sqrt{10},0)$, $A'(-2\sqrt{10},0)$

Covertices: $(0,\pm b)=(0,\pm\sqrt{10})$

$B(0,\sqrt{10})$, $B'(0,-\sqrt{10})$

Length of letusrectum: $\frac{2b^2}{a}$

$=\frac{2(10)}{2\sqrt{10}}=\frac{10}{\sqrt{10}}=\sqrt{10}$

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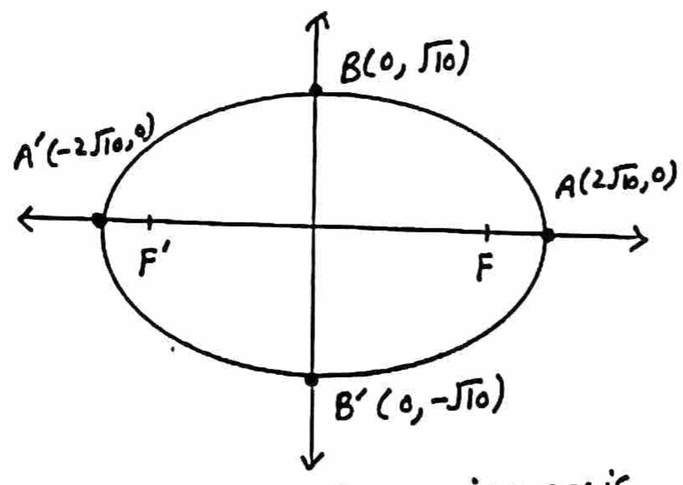


(x) Centre (0,0), major axis horizontal, the points (3,1) and (4,0) lie on the graph

Solution:-

Centre = (0,0)

Major axis horizontal (i.e., x-axis)

Eq. of req. ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \longrightarrow (i)$$

$\because$ (i) passes through (3,1)

so $\frac{(3)^2}{a^2}+\frac{(1)^2}{b^2}=1 \rightarrow \frac{9}{a^2}+\frac{1}{b^2}=1$

$\rightarrow 9b^2+a^2=a^2b^2 \longrightarrow (ii)$

Also (i) passes through (4,0) so

$\frac{(4)^2}{a^2}+\frac{(0)^2}{b^2}=1 \rightarrow \frac{16}{a^2}=1$

$\rightarrow a^2=16 \rightarrow a=4$

Now (ii) $\rightarrow 9b^2+16=16b^2$

$\rightarrow 16=16b^2-9b^2$

$\rightarrow 16=7b^2 \rightarrow b^2=\frac{16}{7}$

or $b=\frac{4}{\sqrt{7}}$

so req ellipse (i) becomes

$$\frac{x^2}{16}+\frac{y^2}{16/7}=1$$

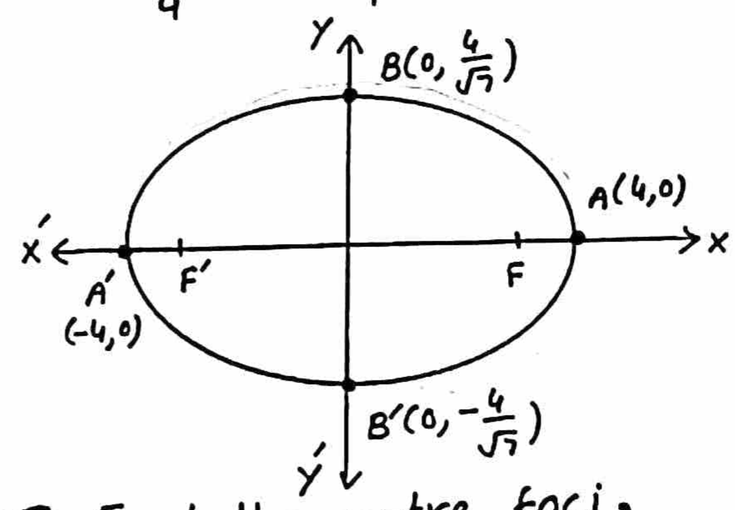
Sketch:-

Vertices: A(4,0), A'(-4,0)

Covertices: $B(0,\frac{4}{\sqrt{7}})$, $B'(0,-\frac{4}{\sqrt{7}})$

length of latusrectum: $\frac{2b^2}{a}$

$= \frac{2(16/7)}{4} = \frac{8}{7}$



Q2. Find the centre, foci, eccentricity, vertices and directrices of the ellipse whose equation is given:

i) $x^2+4y^2=16$

Solution:-

$x^2+4y^2=16$

$\rightarrow \frac{x^2}{16}+\frac{y^2}{4}=1$ (÷ by 16)

Compare with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\rightarrow a^2=16 \rightarrow a=4$

$b^2=4 \rightarrow b=2$

$\because c^2=a^2-b^2=16-4=12$

$\rightarrow c=\sqrt{12} \rightarrow c=2\sqrt{3}$

Centre: (0,0)

Foci: $(\pm c,0)=(\pm 2\sqrt{3},0)$

Eccentricity: $e=\frac{c}{a}=\frac{2\sqrt{3}}{4}$

$\rightarrow e=\frac{\sqrt{3}}{2}$

Vertices: $(\pm a,0)=(\pm 4,0)$

Directrices: $x=\pm\frac{a}{e}$

$\rightarrow x=\pm\frac{4}{\frac{\sqrt{3}}{2}} \rightarrow x=\pm\frac{8}{\sqrt{3}}$

(ii) $9x^2+y^2=18$

Solution:-

$9x^2+y^2=18$

$\rightarrow \frac{x^2}{2}+\frac{y^2}{18}=1$ (÷ by 18)

Compare with $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

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$b^2 = 2 \rightarrow b = \sqrt{2}$ and $a^2 = 18$
$\rightarrow a = \sqrt{18} = 3\sqrt{2}$

$\because c^2 = a^2 - b^2 = 18 - 2 = 16$
$\rightarrow c = 4$

Centre: $(0,0)$

Foci: $(0, \pm c) = (0, \pm 4)$

Eccentricity: $e = \frac{c}{a} = \frac{4}{3\sqrt{2}} = \frac{2 \times 2}{3\sqrt{2}}$
$\rightarrow e = \frac{2\sqrt{2}}{3}$

Vertices: $(0, \pm a) = (0, \pm 3\sqrt{2})$

Directrices: $y = \pm \frac{a}{e}$
$\rightarrow y = \pm \frac{3\sqrt{2}}{\frac{2\sqrt{2}}{3}} = \pm \frac{9}{2} \rightarrow y = \pm \frac{9}{2}$

(iii) $25x^2 + 9y^2 = 225$

Solution:-

$25x^2 + 9y^2 = 225$

$\rightarrow \frac{x^2}{9} + \frac{y^2}{25} = 1$ $(\div \text{ by } 225)$

Compare with $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

$\rightarrow b^2 = 9 \rightarrow b = 3$, $a^2 = 5$
$\rightarrow a = \sqrt{5}$

$\because c^2 = a^2 - b^2 = 25 - 9 = 16$
$\rightarrow c = 4$

Centre: $(0,0)$

Foci: $(0, \pm c) = (0, \pm 4)$

Eccentricity: $e = \frac{c}{a} \rightarrow e = 4/5$

Vertices: $(0, \pm a) = (0, \pm 5)$

Directrices: $y = \pm \frac{a}{e}$
$\rightarrow y = \pm \frac{5}{4/5} = \pm \frac{25}{4} \rightarrow y = \pm \frac{25}{4}$

(iv) $\frac{(2x-1)^2}{16} + \frac{(y+2)^2}{16} = 1$

Solution:-

$\frac{(2x-1)^2}{16} + \frac{(y+2)^2}{16} = 1$

$\rightarrow \frac{\left[2\left(x - \frac{1}{2}\right)\right]^2}{16} + \frac{(y+2)^2}{16} = 1 \longrightarrow (i)$

Compare with $\frac{X^2}{b^2} + \frac{Y^2}{a^2} = 1$

$\rightarrow X = x - \frac{1}{2}$, $Y = y + 2$

$b^2 = 4$, $a^2 = 16$
$\rightarrow b = 2$, $a = 4$

$\because c^2 = a^2 - b^2 = 16 - 4 = 12$

$c = \sqrt{12} = 2\sqrt{3}$

Centre: $(X = 0, Y = 0)$

$\because X = x - \frac{1}{2} \rightarrow x - \frac{1}{2} = 0 \rightarrow x = \frac{1}{2}$

$Y = y + 2 \rightarrow y + 2 = 0 \rightarrow y = -2$

so centre $\left(\frac{1}{2}, -2\right)$

Foci: major axis is along y-axis. so

Foci $(X = 0, Y = \pm c)$

$\because X = x - \frac{1}{2} \rightarrow x - \frac{1}{2} = 0 \rightarrow x = \frac{1}{2}$

$Y = y + 2 \rightarrow y + 2 = \pm c$
$\rightarrow y + 2 = \pm 2\sqrt{3}$
$\rightarrow y = -2 \pm 2\sqrt{3}$

Hence foci $\left(\frac{1}{2}, -2 \pm 2\sqrt{3}\right)$

Eccentricity: $e = \frac{c}{a}$
$\rightarrow e = \frac{2\sqrt{3}}{4} \rightarrow e = \frac{\sqrt{3}}{2}$

Vertices: $(X = 0, Y = \pm a)$

$\because X = x - \frac{1}{2} \rightarrow x - \frac{1}{2} = 0 \rightarrow x = \frac{1}{2}$

$Y = y + 2 \rightarrow y + 2 = \pm a \rightarrow y + 2 = \pm 4$
$\rightarrow y = -2 \pm 4 \rightarrow y = 2, -6$

Hence vertices are $\left(\frac{1}{2}, 2\right), \left(\frac{1}{2}, -6\right)$

Directrices:

$Y = \pm \frac{a}{e} \rightarrow y + 2 = \pm \frac{4}{\frac{\sqrt{3}}{2}}$
$\rightarrow y + 2 = \pm \frac{8}{\sqrt{3}} \rightarrow y = -2 \pm \frac{8}{\sqrt{3}}$

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(v) $x^2+16x+4y^2-16y+76=0$

Solution:-

$x^2+16x+4y^2-16y+76=0$

$\rightarrow x^2+16x+4(y^2-4y)=-76$

$\rightarrow x^2+2(8)(x)+(8)^2+4(y^2-4y+4-4)=-76+(8)^2$

$\rightarrow (x+8)^2+4((y-2)^2-4)=-76+64$

$(x+8)^2+4(y-2)^2-16=-12$

$\rightarrow (x+8)^2+4(y-2)^2=4$

$\rightarrow \frac{(x+8)^2}{4}+\frac{(y-2)^2}{1}=1$

Compare with $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$

$X=x+8,\quad Y=y-2,\quad a^2=4 \rightarrow a=2$

$b^2=1 \rightarrow b=1,\quad \because c^2=a^2-b^2,\ c^2=4-1=3 \rightarrow c=\sqrt{3}$

Centre: $(X=0,\ Y=0)$

$X=x+8 \rightarrow x+8=0 \rightarrow x=-8$

$Y=y-2 \rightarrow y-2=0 \rightarrow y=2$

so centre $(-8,2)$

Foci: As major axis is along x-axis. so foci $(X=\pm c,\ Y=0)$

$X=x+8 \rightarrow x+8=\pm c$

$x+8=\pm\sqrt{3} \rightarrow x=-8\pm\sqrt{3}$

$Y=y-2 \rightarrow y-2=0 \rightarrow y=2$

so foci $(-8\pm\sqrt{3},\ 2)$

Eccentricity: $e=\frac{c}{a} \rightarrow e=\frac{\sqrt{3}}{2}$

Vertices: $(X=\pm a,\ Y=0)$

$X=x+8 \rightarrow x+8=\pm a$

$\rightarrow x+8=\pm 2 \rightarrow x=-8\pm 2$

$\rightarrow x=-6,-10$

$Y=y-2 \rightarrow y-2=0 \rightarrow y=2$

so vertices $(-6,2),(-10,2)$

Directrices: $X=\pm\frac{a}{e}$

$X=x+8 \rightarrow x+8=\pm\frac{a}{e}$

$\rightarrow x+8=\pm\frac{2}{\frac{\sqrt{3}}{2}}$

$\rightarrow x=-8\pm\frac{4}{\sqrt{3}}$

(vi) $25x^2+4y^2-250x-16y+541=0$

Solution:-

$25x^2+4y^2-250x-16y+541=0$

$25(x^2-10x)+4(y^2-4y)=-541$

$25(x^2-10x+25-25)+4(y^2-4y+4-4)=-541$

$25[(x-5)^2-25]+4[(y-2)^2-4]=-541$

$25(x-5)^2-625+4(y-2)^2-16=-541$

$25(x-5)^2+4(y-2)^2=-541+625+16$

$\rightarrow 25(x-5)^2+4(y-2)^2=100$

or $\frac{(x-5)^2}{4}+\frac{(y-2)^2}{25}=1$ $(\div$ by $:100)$

Compare with $\frac{X^2}{b^2}+\frac{Y^2}{a^2}=1$

$X=x-5,\quad Y=y-2$

$b^2=4 \rightarrow b=2,\quad a^2=25 \rightarrow a=5$

$\because c^2=a^2-b^2=25-4 \rightarrow c^2=21 \rightarrow c=\sqrt{21}$

Centre: $(X=0,\ Y=0)$

$\because X=x-5 \rightarrow x-5=0 \rightarrow x=5$

$Y=y-2 \rightarrow y-2=0 \rightarrow y=2$

so centre $(5,2)$

Foci: $\because$ major axis is along y-axis.

Foci $(X=0,\ Y=\pm c)$

$X=x-5 \rightarrow x-5=0 \rightarrow x=5$

$Y=y-2 \rightarrow y-2=\pm c \rightarrow y=2\pm\sqrt{21}$

so foci $(5,\ 2\pm\sqrt{21})$

Eccentricity:

$e=\frac{c}{a} \rightarrow e=\frac{\sqrt{21}}{5}$

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Vertices: $(X = \pm a,\ Y = a)$

$X = x+5 \rightarrow x+5=0 \rightarrow x=-5$

$Y = y-2 \rightarrow y-2 = a \rightarrow y-2 = \pm 5$

$\rightarrow y = 2 \pm 5 \rightarrow y = -3, 7$

so vertices $(5,7)$, $(5,-3)$

Directrices: $Y = \pm \frac{a}{e}$

$Y = y-2 \rightarrow y-2 = \pm \frac{a}{e}$

$\rightarrow y = 2 \pm \frac{5}{\frac{\sqrt{21}}{5}}$ $\left(\because a = 5,\ e = \frac{\sqrt{21}}{5}\right)$

$\rightarrow y = 2 \pm \frac{25}{\sqrt{21}}$

Q3. Let 'a' be a positive number and $0 < c < a$. Let $F(c,0)$ and $F'(-c,0)$ be two given points. Prove that the locus of points $P(x,y)$ such that $|PF| + |PF'| = 2a$ is an ellipse.

Solution:- Given that $0 < c < a$

$|PF'| + |PF| = 2a$ $\quad F'(-c,0)$, $F(c,0)$, $P(x,y)$

$\rightarrow \sqrt{(x+c)^2+(y-0)^2} + \sqrt{(x-c)^2+(y-0)^2} = 2a$

$\rightarrow \sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2} \rightarrow (i)$

squaring both sides

$\rightarrow \cancel{x^2}+\cancel{c^2}+2cx+\cancel{y^2} = 4a^2 + \cancel{x^2}+\cancel{c^2}-2cx+\cancel{y^2} - 4a\sqrt{(x-c)^2+y^2}$

$\rightarrow 2cx+2cx-4a^2 = -4a\sqrt{x^2+c^2-2cx+y^2}$

$\rightarrow 4(cx-a^2) = -4a\sqrt{x^2+y^2+c^2-2cx}$

$\rightarrow cx-a^2 = -a\sqrt{x^2+y^2+c^2-2cx}$

or $a^2 - cx = a\sqrt{x^2+y^2+c^2-2cx}$

Again squaring

$a^4+c^2x^2-2a^2cx = a^2(x^2+y^2+c^2-2cx)$

$a^4+c^2x^2-\cancel{2a^2cx} = a^2x^2+a^2y^2+a^2c^2-\cancel{2a^2cx}$

$\rightarrow c^2x^2-a^2x^2-a^2y^2 = a^2c^2-a^4$

$(a^2-c^2)x^2+a^2y^2 = a^2(a^2-c^2)$

$\div$ by $a^2(a^2-c^2)$

$\rightarrow \frac{(a^2-c^2)x^2}{a^2(a^2-c^2)} + \frac{a^2y^2}{a^2(a^2-c^2)} = \frac{a^2(a^2-c^2)}{a^2(a^2-c^2)}$

$\rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$

$\rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\left(\because a^2-b^2=c^2 \rightarrow a^2-c^2=b^2\right)$

which is an ellipse.

Hence proved

Q4. Let 'a' be a positive number and $0 < c < a$. Let $F(0,0)$ and $F'(1,1)$ be two given points, and locus of points $P(x,y)$ is an ellipse, such that $|PF| + |PF'| = 2$, Find equation of ellipse.

Solution:- $0 < c < a$

$P(x,y)$, $F(0,0)$, $F'(1,1)$

Given that

$|PF| + |PF'| = 2$

$\sqrt{(x-0)^2+(y-0)^2} + \sqrt{(x-1)^2+(y-1)^2} = 2$

$\rightarrow \sqrt{x^2+y^2} + \sqrt{x^2+1-2x+y^2+1-2y} = 2$

$\rightarrow \sqrt{x^2+y^2} + \sqrt{x^2+y^2-2x-2y+2} = 2$

$\rightarrow \sqrt{x^2+y^2-2x-2y+2} = 2 - \sqrt{x^2+y^2}$

Squaring both sides

$x^2+y^2-2x-2y+2 = 4+x^2+y^2-4\sqrt{x^2+y^2}$

$-2x-2y+2 = 4-4\sqrt{x^2+y^2}$

$\rightarrow 4\sqrt{x^2+y^2} = 2+2x+2y$

$2\sqrt{x^2+y^2} = x+y+1$ $\quad (\div$ by $2)$

squaring again

$4(x^2+y^2) = x^2+y^2+1+2xy+2y+2x$

$\rightarrow 4x^2+4y^2-x^2-y^2-2xy-2x-2y-1=0$

or $3x^2+3y^2-2x-2y-2xy-1=0$

which is ellipse

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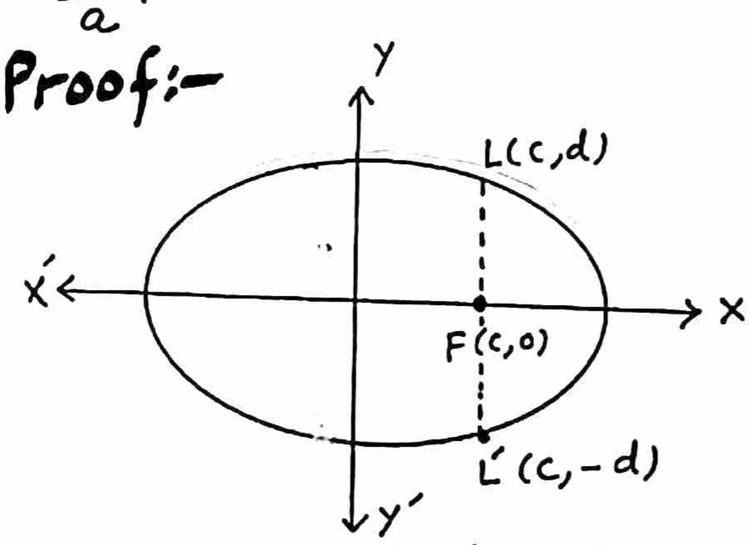
Q5. Prove that the letusrectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{2b^2}{a}$.

Proof:-



Let $L(c,d)$ and $L'(c,d)$ be end points of Letusrectum LL' of given ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \longrightarrow (i)$

$\because$ Length of letusrectum $= |LL'| = 2d$

$\because L(c,d)$ lies on (i) so

$$\frac{c^2}{a^2}+\frac{d^2}{b^2}=1$$

$$\rightarrow c^2b^2+a^2d^2=a^2b^2$$

$$\rightarrow a^2d^2=a^2b^2-b^2c^2$$

$$\rightarrow a^2d^2=b^2(a^2-c^2)$$

$\because c^2=a^2-b^2$

$\rightarrow b^2=a^2-c^2$

$$\rightarrow a^2d^2=b^2(b^2)$$

or $d^2=\frac{b^4}{a^2} \rightarrow d=\sqrt{\frac{b^4}{a^2}}$

$\rightarrow d=\frac{b^2}{a}$ so

Length of letusrectum $= 2d = \frac{2b^2}{a}$

Hence proved

Q6. The major axis of an ellipse in standard form lies along the x-axis and has length $4\sqrt{2}$. The distance between foci equals the length of the minor axis. Write an equation of ellipse.

Solution:-

Given $2a = 4\sqrt{2}$

$\rightarrow a = 2\sqrt{2}$

or $a^2 = 8$

Also given that

$2c = 2b$

$\rightarrow c = b$

Now using $c^2 = a^2 - b^2$

$\rightarrow b^2 = a^2 - b^2$

$\rightarrow 2b^2 = a^2$

$\rightarrow 2b^2 = 8 \rightarrow b^2 = 4$

Thus eq. of ellipse is $\frac{x^2}{8}+\frac{y^2}{4}=1$

Q7. An astroid has elliptic orbit with the sun at one focus. Its distance from the sun ranges 17 millions to 183 millions miles. Write an equation of the orbit of the astroid.

Solution:-

$a - c = 17 \longrightarrow (i)$

$a + c = 183 \longrightarrow (ii)$

By (i) + (ii)

$2a = 200 \rightarrow a = 100$

(ii) $\rightarrow 100 + c = 183 \rightarrow c = 83$

using $c^2 = a^2 - b^2$

$b^2 = a^2 - c^2$

$b^2 = (100)^2 - (83)^2$

$b^2 = 10000 - 6889$

$b^2 = 3111$

$\therefore$ The eq. of elliptic orbit is

$$\frac{x^2}{10{,}000}+\frac{y^2}{3111}=1$$

Q8. An arch in the shape of a semi-ellipse is 90m wide at the base and 30m high at the centre. At what distance from the centre is the arch $20\sqrt{2}$ m high?

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Solution:-

Here $2a = 90 \rightarrow a = 45$

and $b = 30$

$\therefore$ Eq. of ellipse is

$$\frac{x^2}{(45)^2} + \frac{y^2}{(30)^2} = 1 \longrightarrow (i)$$

At the hight $20\sqrt{2}$ m Let x_1 be the distance from the centre then the point $(x_1, 20\sqrt{2})$ lies on ellipse (i)

$$\frac{x_1^2}{(45)^2} + \frac{(20\sqrt{2})^2}{(30)^2} = 1$$

$$\rightarrow \frac{x_1^2}{2025} + \frac{800}{900} = 1$$

$$\rightarrow \frac{x_1^2}{2025} = 1 - \frac{800}{900}$$

$$\rightarrow \frac{x_1^2}{2025} = 1 - \frac{8}{9}$$

$$\rightarrow \frac{x_1^2}{2025} = \frac{1}{9} \rightarrow x_1^2 = \frac{2025}{9}$$

$$\rightarrow x_1^2 = 225 \rightarrow x_1 = \pm 15$$

$\rightarrow x_1 = 15m$ (Neglect -ive value of x_1)

$\therefore$ Req. distance from centre $= 15m$

Q9. The moon orbits the earth in an elliptic path with the earth at one focus. The major and minor axes of the orbit are 7,68,806 km and 7,67,746 km respectively. Find the greatest and least distances.

Solution:-

Let the earth be at F

Given that $2a = 768,806$

$\rightarrow a = 384403$ km

$2b = 767,746$

$\rightarrow b = 383873$ km

using $c^2 = a^2 - b^2$

$\rightarrow c^2 = (a-b)(a+b)$

$\rightarrow c^2 = (530)(768276)$

$c^2 = 407186280$

$c = 20178.86$

Now req. greatest distance

$= a + c = 404582$ Km (Approx)

and Least distance $= a - c$

$= 364224$ km (Approx)

Hyperbola

"A set of all points in a plane such that distance of each point from a fixed point bears a constant ratio (greater than one) to a distance from a fixed line".

Note:-

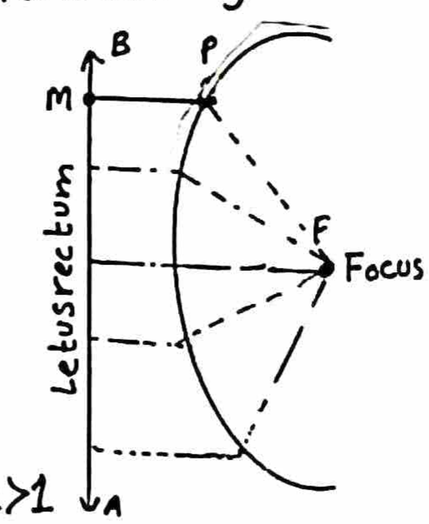
* Fixed point is called focus.
* Fixed line is called directrix
* constant ratio is called eccentricity, denoted by e.
* In fig $\frac{|FP|}{|PM|} = e$

$\rightarrow |FP| = e|PM|$

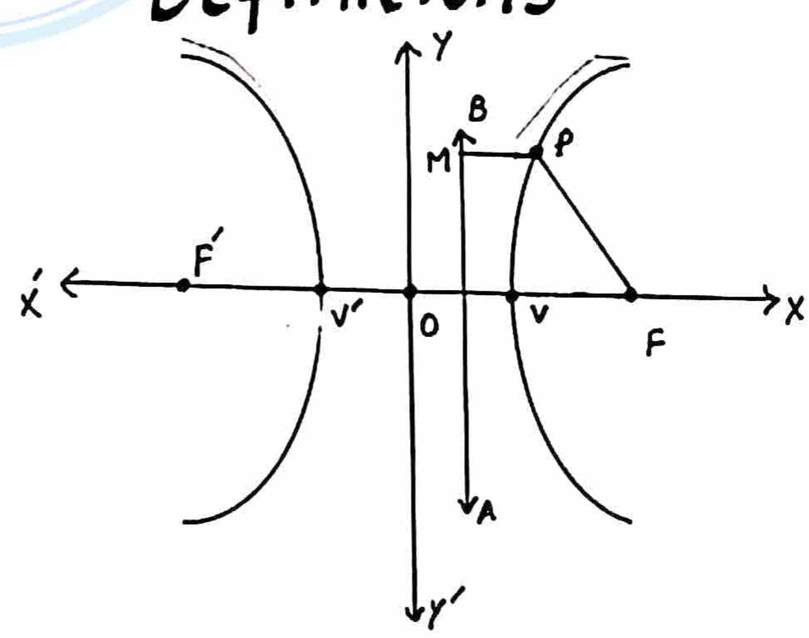
wher $|FP| > |PM|$

$\rightarrow \frac{|FP|}{|PM|} > 1$

so for hyperbola $e > 1$



Definitions



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i) The mid point of line segment joining the foci is called **centre of hyperbola.** In fig F and F′ are foci and O is centre.

ii) The line passing through the foci of hyperbola is called **focal axis or Transverse axis of hyperbola.** In fig; x-axis is transverse axis.

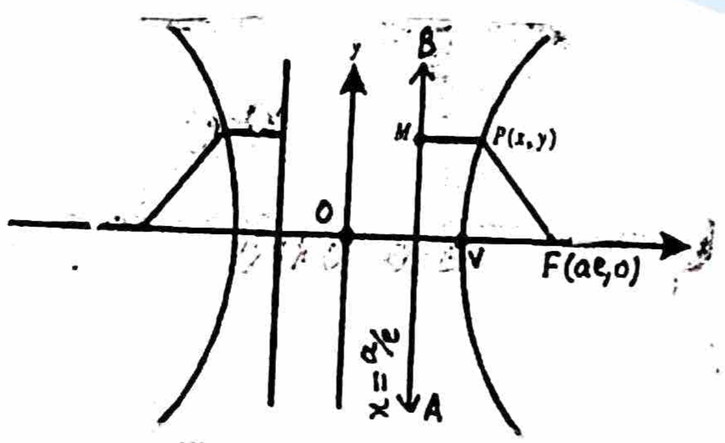
iii) The line passing through centre of hyperbola and perpendicular to transverse axis is called **conjugate axis.** In fig, Y-axis is conjugate axis.

iv) The points where the hyperbola meets its transverse axis are called **vertices of hyperbola.** In fig V and V′ are vertices of hyperbola.

v) Length of letusrectum is $\frac{2b^2}{a}$.

vi) If the point $(a\sec\theta, b\tan\theta)$ lies on hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where $\theta \in \mathbb{R}$ then $x = a\sec\theta$, $y = a\tan\theta$ are parametric equations of hyperbola.

Standard equation of hyperbola



Let $F(ae,0)$ be focus and AB be directrix with equation $x = \frac{a}{e} \rightarrow ex - a = 0$ of req. hyperbola.

Let $|PM|$ be ⊥ar distance of any point $P(x,y)$ on hyperbola to the directrix $ex - a = 0$

By def. of hyperbola

$$\frac{|FP|}{|PM|} = e \rightarrow |FP| = e|PM|$$

$$\rightarrow \sqrt{(x-ae)^2+(y-0)^2} = e\frac{|ex-a|}{\sqrt{(e)^2+(0)^2}}$$

$$\rightarrow \sqrt{x^2+a^2e^2-2aex+y^2} = \cancel{e}\frac{|ex-a|}{\cancel{e}}$$

$$\text{or } x^2+a^2e^2-2aex+y^2 = |ex-a|^2$$

$$\rightarrow x^2+a^2e^2-\cancel{2aex}+y^2 = e^2x^2+a^2-\cancel{2aex}$$

$$\rightarrow a^2e^2-a^2+y^2 = e^2x^2 \quad -x^2$$

$$\rightarrow a^2(e^2-1) = (e^2-1)x^2 \quad -y^2$$

$$\rightarrow (e^2-1)x^2-y^2 = a^2(e^2-1)$$

$$\rightarrow \frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)} = 1 \quad \div \text{by } a^2(e^2-1)$$

$$\text{or } \frac{x^2}{a^2}-\frac{y^2}{b^2} = 1 \quad \text{Let } a^2(e^2-1) = b^2$$

Note:- $\because b^2 = a^2(e^2-1)$

$$\rightarrow b^2 = a^2e^2-a^2 = (ae)^2-a^2$$

$$\rightarrow b^2 = c^2-a^2 \quad \because ae = c$$

$$\text{or } c^2 = a^2+b^2$$

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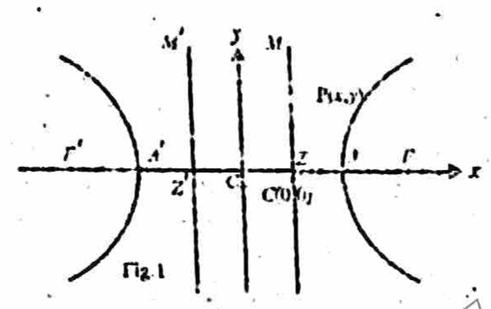
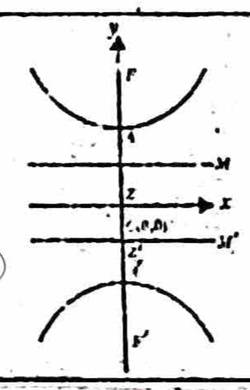
Summary of Standard Hyperbolas

Equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Directrices	$x=\pm\frac{a}{e}$ or $x=\pm\frac{c}{e^2}$	$y=\pm\frac{a}{e}$ or $y=\pm\frac{c}{e^2}$
Transverse axis	$y=0$	$x=0$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Eccentricity	$e=\frac{c}{a}>1$	$e=\frac{c}{a}>1$
Centre	(0, 0)	(0, 0)
Graph		

Example 1. Find an equation of the hyperbola whose foci are $(\pm 4, 0)$ and vertices $(\pm 2, 0)$. Sketch its graph.

Solution:- Given that

Foci $= (\pm c, 0) = (\pm 4, 0)$

$\rightarrow c = 4 \rightarrow c^2 = 16$

centre = mid point of foci

$= \left(\frac{4-4}{2}, \frac{0+0}{2}\right) = (0, 0)$

vertices $= (\pm a, 0) = (\pm 2, 0)$

$\rightarrow a = 2 \rightarrow a^2 = 4$

$\because c^2 = a^2 + b^2 \rightarrow b^2 = c^2 - a^2$

$\rightarrow b^2 = 16 - 4 = 12$

Transverse axis is x-axis. so eq. of req. hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \rightarrow \frac{x^2}{4} - \frac{y^2}{12} = 1$$

Sketch:-

Vertices:

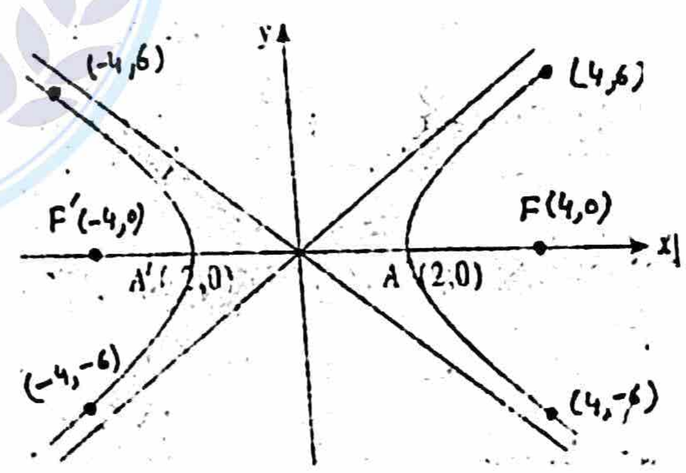
$(\pm a, 0) = (\pm 2, 0) \rightarrow A(2,0)$, $A'(-2,0)$

Foci:

$(\pm c, 0) = (\pm 4, 0) \rightarrow \therefore F(4,0)$, $F'(-4,0)$

Length of letusrectum $= \frac{2b^2}{a}$

$= \frac{2(12)}{2} = 12$



Example 2. Discuss and sketch the graph of the equation

$$25x^2 - 16y^2 = 400$$

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Solution:- $25x^2-16y^2=400$

$\rightarrow \frac{x^2}{16}-\frac{y^2}{25}=1$ ($\div$ by 400)

compare with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$a^2=16 \rightarrow a=4$ $\quad \because c^2=a^2+b^2$

$b^2=25 \rightarrow b=5$ $\quad c^2=16+25$

$\rightarrow c=\sqrt{41}$

Centre:- $(0,0)$

Eccentricity:- $e=\frac{c}{a}=\frac{\sqrt{41}}{4}$

Foci:- $(\pm c,0)=(\pm\sqrt{41},0)$

Asymptotes:- $y=\pm\frac{b}{a}x$

$\rightarrow y=\pm\frac{5}{4}x$

Directrices:- $x=\pm\frac{a}{e}$

$\rightarrow x=\pm\frac{4}{\frac{\sqrt{41}}{4}}=\pm\frac{16}{\sqrt{41}}$

Vertices:- $(\pm a,0)=(\pm 4,0)$

Length of latusrectum:- $\frac{2b^2}{a}$

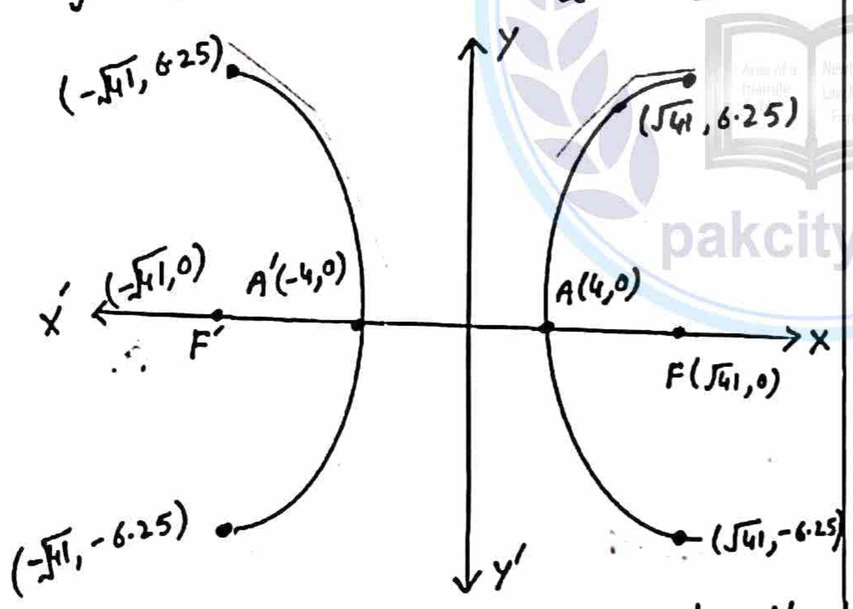
$=\frac{2(25)}{4}=\frac{25}{2}$

Sketch:-

Vertices:- $(\pm 4,0)$ i.e., $A(4,0)$, $A'(-4,0)$

Foci :- $(\pm\sqrt{41},0)$ i.e., $F(\sqrt{41},0)$, $F'(-\sqrt{41},0)$

Length of letusrectum $=\frac{2b^2}{a}=\frac{25}{2}$



Example 3. Find the eccentricity, the coordinates of vertices and foci of hyperbola $\frac{y^2}{16}-\frac{x^2}{49}=1$. Also sketch its graph.

Solution:-

$\frac{y^2}{16}-\frac{x^2}{49}=1$

compare with $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

$\rightarrow a^2=16 \rightarrow a=4$ $\quad \because c^2=a^2+b^2$

$b^2=49 \rightarrow b=7$ $\quad c^2=16+49$

$c=\sqrt{65}$

Vertices:- $(0,\pm a)=(0,\pm 4)$

Foci:- $(0,\pm c)=(0,\pm\sqrt{65})$

Eccentricity:- $e=\frac{c}{a}=\frac{\sqrt{65}}{4}$

Sketch:-

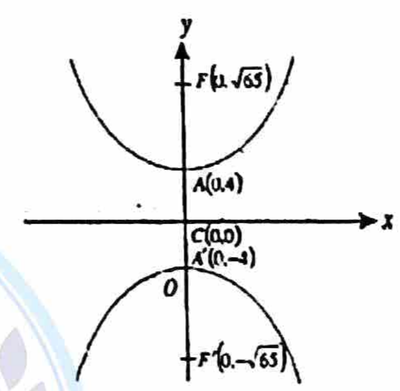
Vertices:- $(0,\pm 4)$ i.e., $A(0,4)$, $A'(0,-4)$

Foci :- $(0,\pm c)=(0,\pm\sqrt{65})$

$F(0,\sqrt{65})$, $F'(0,-\sqrt{65})$

Length of letusrectum:- $\frac{2b^2}{a}=\frac{2(49)}{4}$

$=\frac{49}{2}$



Example 4. Discuss and sketch the graph of the equation

$4x^2-8x-y^2-2y-1=0$

Solution:-

$4x^2-8x-y^2-2y-1=0$

$\rightarrow 4(x^2-2x)-(y^2+2y+1)=0$

$\rightarrow 4(x^2-2x+1-1)-(y+1)^2=0$

$\rightarrow 4((x-1)^2-1)-(y+1)^2=0$

$\rightarrow 4(x-1)^2-4-(y+1)^2=0$

or $4(x-1)^2-(y+1)^2=4$

or $\frac{(x-1)^2}{1}-\frac{(y+1)^2}{4}=1$ ($\div$ by 4)

Compare with

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

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$X = x-1$, $Y = y+1$, $a^2 = 1$, $b^2 = 4$
$a = 1$, $b = 2$

$\because c^2 = a^2 + b^2 = 1 + 4 = 5$
$\rightarrow c = \sqrt{5}$

Centre:- $(X = 0, Y = 0)$

$X = x-1 \rightarrow x-1 = 0 \rightarrow x = 1$
$Y = y+1 \rightarrow y+1 = 0 \rightarrow y = -1$
so centre $(1,-1)$

Eccentricity:- $e = \frac{c}{a} \rightarrow e = \frac{\sqrt{5}}{1}$
$\rightarrow e = \sqrt{5}$

Foci:- $(X = \pm c, Y = 0)$

$X = x-1 \rightarrow x-1 = \pm c \rightarrow x = 1 \pm \sqrt{5}$
$Y = y+1 \rightarrow y+1 = 0 \rightarrow y = -1$
so foci $(1 \pm \sqrt{5}, -1)$

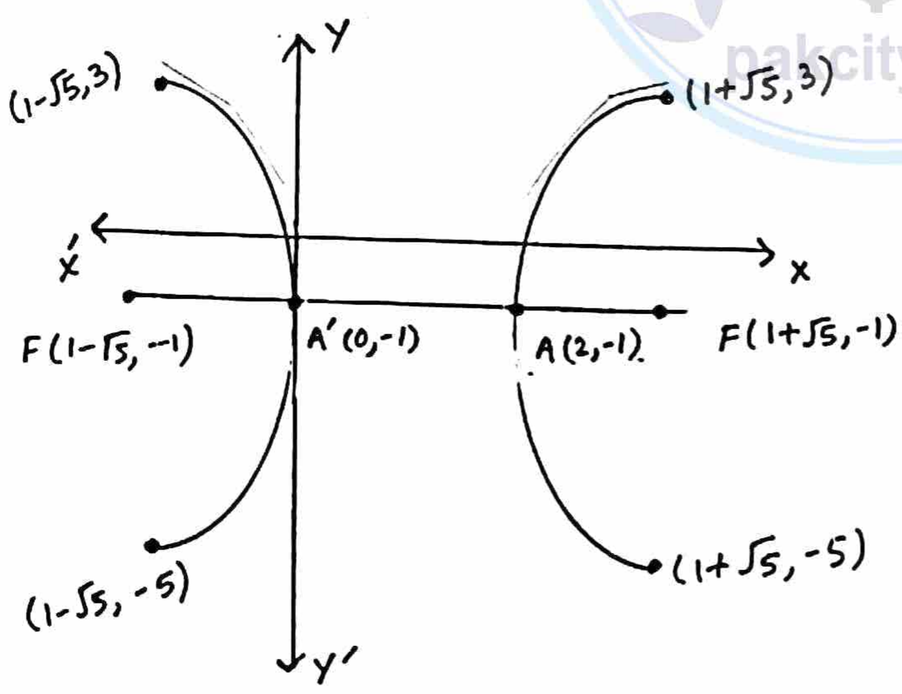
Directrices:- $X = \pm \frac{a}{e}$

$\rightarrow X = x-1 \rightarrow x-1 = \pm \frac{a}{e}$
$\rightarrow x = 1 \pm \frac{1}{\sqrt{5}}$ $(\because a = 1, e = \sqrt{5})$

Vertices:- $(X = \pm a, Y = 0)$

$X = x-1 \rightarrow x-1 = \pm a \rightarrow x = 1 \pm 1$
$\rightarrow x = 2, 0$
$Y = y+1 \rightarrow y+1 = 0 \rightarrow y = -1$
so vertices $(2,-1)$, $(0,-1)$

Sketch:- vertices: $A(2,-1)$, $A'(0,-1)$
Foci:- $F(1+\sqrt{5},-1)$, $F'(1-\sqrt{5},-1)$.
length of letusrectum: $\frac{2b^2}{a} = \frac{2(4)}{1} = 8$
centre: $(1,-1)$



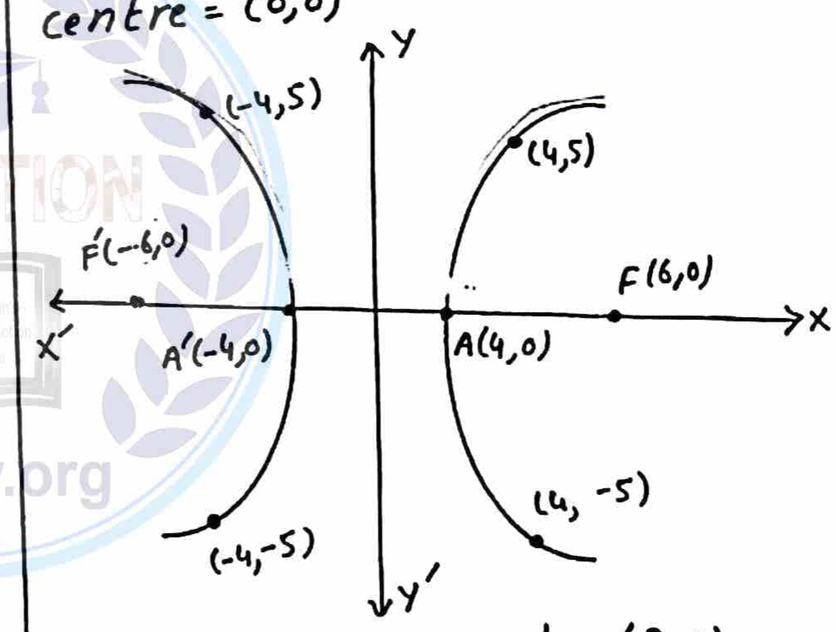
Exercise 6.6

Q1. Find an equation of hyperbola with the given data, and sketch its graph:

(i) Centre $(0,0)$, focus $(6,0)$
Vertex $(4,0)$

Solution:-
centre $(0,0)$, focus $= (6,0)$
so other focus $= (-6,0)$
Foci $(\pm c, 0) = (\pm 6, 0) \rightarrow c = 6$
$\rightarrow c^2 = 36$
vertex $= (4,0)$, other vertex $= (-4,0)$
vertices $= (\pm a, 0) = (\pm 4, 0) \rightarrow a = 4$
$\rightarrow a = 16$
$\because c^2 = a^2 + b^2 \rightarrow b^2 = c^2 - a^2$
$\rightarrow b^2 = 36 - 16 \rightarrow b^2 = 20 \rightarrow b = \sqrt{20}$
clearly from foci and vertices x-axis is transverse axis. so eq. is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \rightarrow \frac{x^2}{16} - \frac{y^2}{20} = 1$

Sketch:- $A(4,0)$, $A'(-4,0)$
$F(6,0)$, $F'(-6,0)$
length of letusrectum $= \frac{2b^2}{a} = \frac{2(20)}{4} = 10$
centre $= (0,0)$



(ii) Foci $(\pm 5, 0)$, vertex $(3,0)$

Solution:-
Foci $(\pm c, 0) = (\pm 5, 0)$ i.e., $F(5,0)$, $F'(-5,0)$
$\rightarrow c = 5$, $c^2 = 25$
centre = midpoint of foci
$= \left(\frac{5-5}{2}, \frac{0+0}{2}\right) = (0,0)$
vertex $= (3,0)$, other vertex $= (-3,0)$

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vertices $= (\pm a, 0) = (\pm 3, 0) \rightarrow a = 3$

$a^2 = 9 \qquad \because c^2 = a^2 + b^2$

$\rightarrow b^2 = c^2 - a^2 = 25 - 9 = 16$

clearly from foci and vertices x-axis is transverse axis.

so eq. is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1$

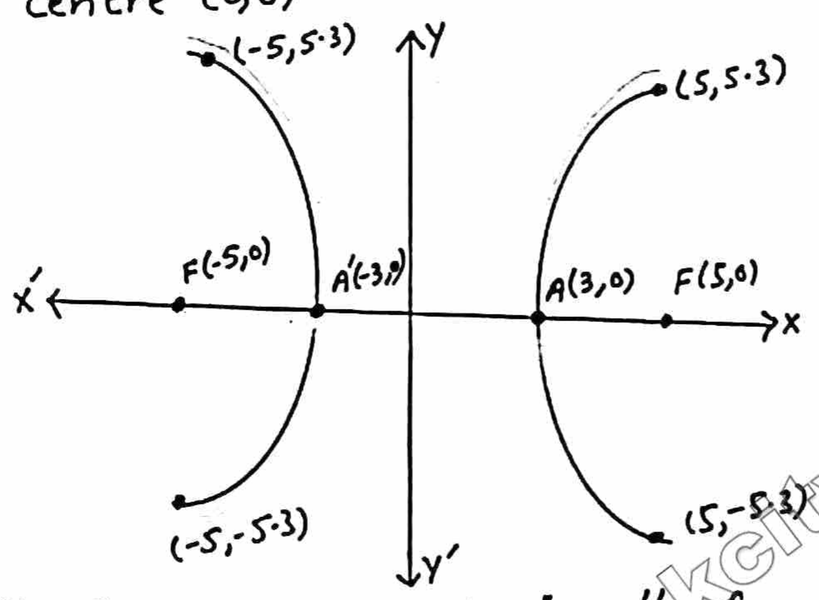
Sketch:-

Vertices:- $A(3,0)$, $A'(-3,0)$

Foci :- $F(5,0)$, $F'(-5,0)$

Length of letusrectum: $\frac{2b^2}{a} = \frac{2(16)}{3} = \frac{32}{3}$

centre $(0,0)$



(iii) Foci $(2 \pm 5\sqrt{2}, -7)$, Length of transverse axis 10

Solution:-

Foci $(2 \pm 5\sqrt{2}, -7) \rightarrow F(2+5\sqrt{2}, -7)$

$F'(2-5\sqrt{2}, -7)$

centre = midpoint of foci

$(h, k) = \left(\frac{2+5\sqrt{2}+2-5\sqrt{2}}{2}, \frac{-7-7}{2}\right) = (2, -7)$

Length of transverse axis $= 2a = 10$

$= a = 5 \rightarrow a^2 = 25$

Now distance b/w focus $F(2+5\sqrt{2}, -7)$ and centre $(2, -7)$ is

$c = \sqrt{(2+5\sqrt{2}-2)^2 + (-7+7)^2} = 5\sqrt{2}$

$\rightarrow c^2 = (5\sqrt{2})^2 = 25(2) \rightarrow c^2 = 50$

$\because c^2 = a^2 + b^2 \rightarrow b^2 = c^2 - a^2 = 50 - 25$

$\rightarrow b^2 = 25$

$\because$ foci have same ordinate

so eq. of hyperbola is

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$\rightarrow \frac{(x-2)^2}{25} - \frac{(y+7)^2}{25} = 1$

Compare with $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

$X = x-2$, $Y = y+7$, $a^2 = 25$, $b^2 = 25$

$\rightarrow a = 5$, $b = 5$

Sketch:-

Vertices $(X = \pm a, Y = 0)$

$X = x-2 \rightarrow x-2 = \pm 5 \rightarrow x = 2 \pm 5$

$\rightarrow x = 7, -3$

$Y = y+7 \rightarrow y+7 = 0 \rightarrow y = -7$

so vertices $(7,-7)$, $(-3,-7)$

Foci $(X = \pm c, Y = 0)$

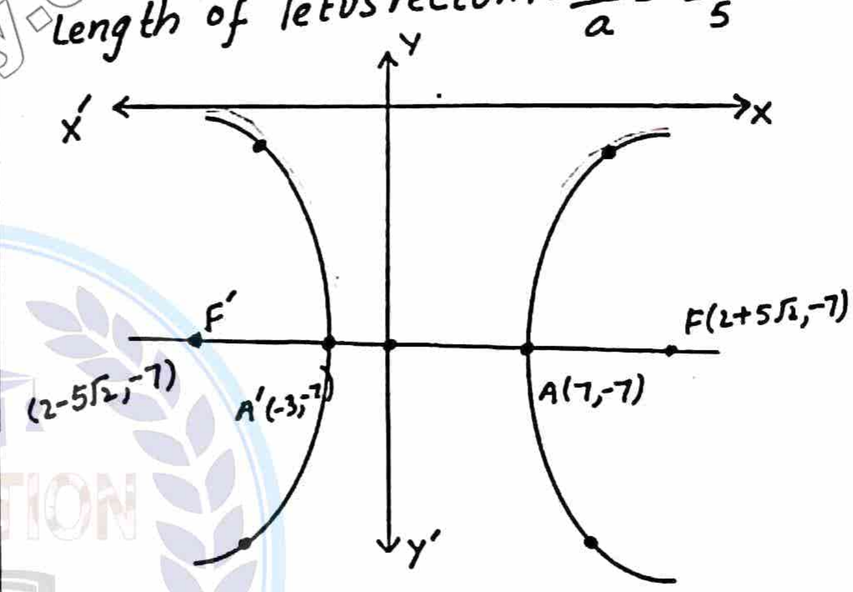
$X = x-2 \rightarrow x-2 = \pm c$

$x = 2 \pm 5\sqrt{2}$

$Y = 0 \rightarrow y+7 = 0 \rightarrow y = -7$

so foci $(2+5\sqrt{2}, -7)$, $F'(2-5\sqrt{2}, -7)$

Length of letus rectum: $\frac{2b^2}{a} = \frac{2(25)}{5} = 10$



(iv) Foci $(0, \pm 6)$, $e = 2$

Solution:-

Foci $= (0, \pm c) = (0, \pm 6)$

$\rightarrow c = 6 \rightarrow c^2 = 36$

Centre = midpoint of foci

$= \left(\frac{0-0}{2}, \frac{6-6}{2}\right) = (0, 0)$

$\because e = 2 \rightarrow e = \frac{c}{a}$ or $a = \frac{c}{e}$

$\rightarrow a = \frac{6}{2} = 3 \rightarrow a^2 = 9$

$\because c^2 = a^2 + b^2 \rightarrow b^2 = c^2 - a^2$

$\rightarrow b^2 = 36 - 9 = 27$

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$\because$ foci have same abscissa.
So y-axis is transverse axis.
so eq. of hyperbola is

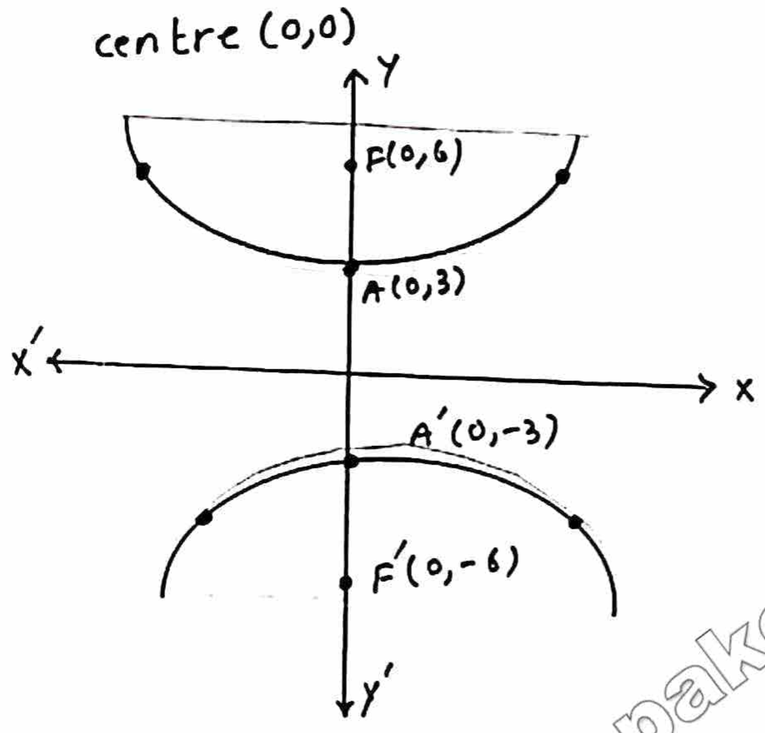
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \rightarrow \frac{y^2}{9} - \frac{x^2}{27} = 1$$

Sketch:-

Vertices: $A(0,3)$, $A'(0,-3)$
Foci : $F(0,6)$, $F'(0,-6)$

Length of letusrectum $= \frac{2b^2}{a} = \frac{2(27)}{3}$
$= 18$

centre $(0,0)$



(V) Foci $(0, \pm 9)$, directrices $y = \pm 4$

Solution:-

Foci $= (0, \pm c) = (0, \pm 9)$
$\rightarrow c = 9 \rightarrow c^2 = 81$

Centre = midpoint of foci
$= \left(\frac{0+0}{2}, \frac{9-9}{2}\right) = (0,0)$

Directrices: $y = \pm 4$

$\because y = \pm \frac{a}{e}$

$\rightarrow \frac{a}{e} = 4 \rightarrow a = 4e \longrightarrow (i)$

Also $c = ae$

$\rightarrow ae = 9 \quad (\because c = 9)$

$\rightarrow e = \frac{9}{a}$ put in (i)

$a = 4\left(\frac{9}{a}\right) \rightarrow a^2 = 36 \rightarrow a = 6$

$\because c^2 = a^2 + b^2 \rightarrow b^2 = c^2 - a^2$

$\rightarrow b^2 = 81 - 36 = 45$

$\because$ foci have same abscissa
so y-axis is transverse axis.
Thus eq. of req. hyperbola is

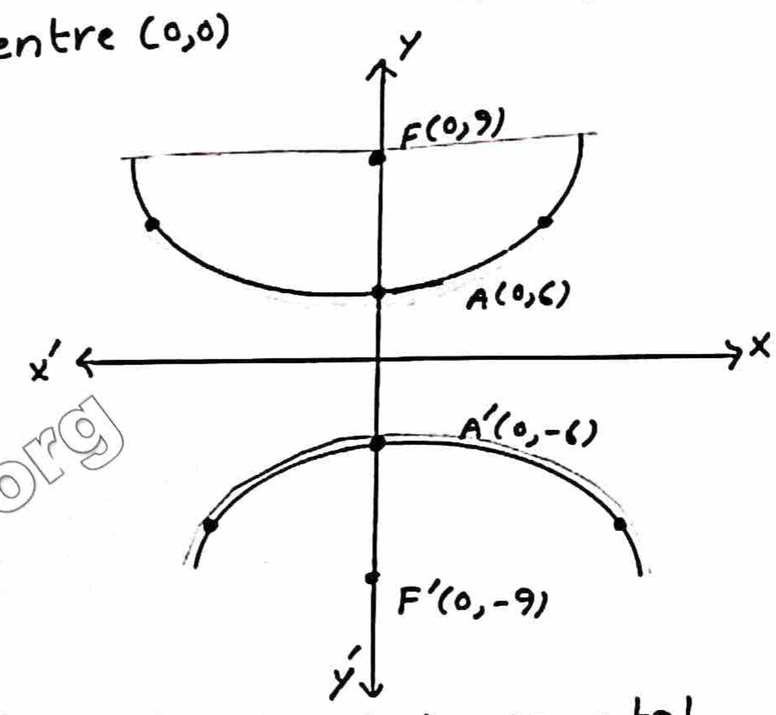
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \rightarrow \frac{y^2}{36} - \frac{x^2}{45} = 1$$

Sketch:-

Vertices: $A(0,6)$, $A'(0,-6)$
Foci : $F(0,9)$, $F'(0,-9)$

Length of letusrectum: $\frac{2b^2}{a} = \frac{2(45)}{6} = 15$

centre $(0,0)$



(vi) centre $(2,2)$, horizontal transverse axis of length 6 and eccentricity $e = 2$

Solution:-

centre $= (h,k) = (2,2)$

Length of transverse axis $= 2a = 6$
$\rightarrow a = 3 \rightarrow a^2 = 9$

$e = 2 \quad \because c = ae \rightarrow c = (3)(2) = 6$
or $c^2 = 36 \quad \because c^2 = a^2 + b^2$
$\rightarrow b^2 = c^2 - a^2 = 36 - 9 = 27$

$\because$ transverse axis is horizontal axis i-e., x-axis. so eq. of req. hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\rightarrow \frac{(x-2)^2}{9} - \frac{(y-2)^2}{27} = 1$$

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Sketch:-

vertices:- $(X = \pm a, Y = 0)$

$X = x-2 \rightarrow x-2 = \pm a$

$\rightarrow x-2 = \pm 3 \rightarrow x = 2 \pm 3$

or $x = 5, -1$

$Y = y-2 \rightarrow y-2 = 0 \rightarrow y = 2$

so vertices $A(5,2), A'(-1,2)$

Foci:- $(X = \pm c, Y = 0)$

$X = x-2 \rightarrow x-2 = \pm 6$

$\rightarrow x = 2 \pm 6 \rightarrow x = 8, -4$

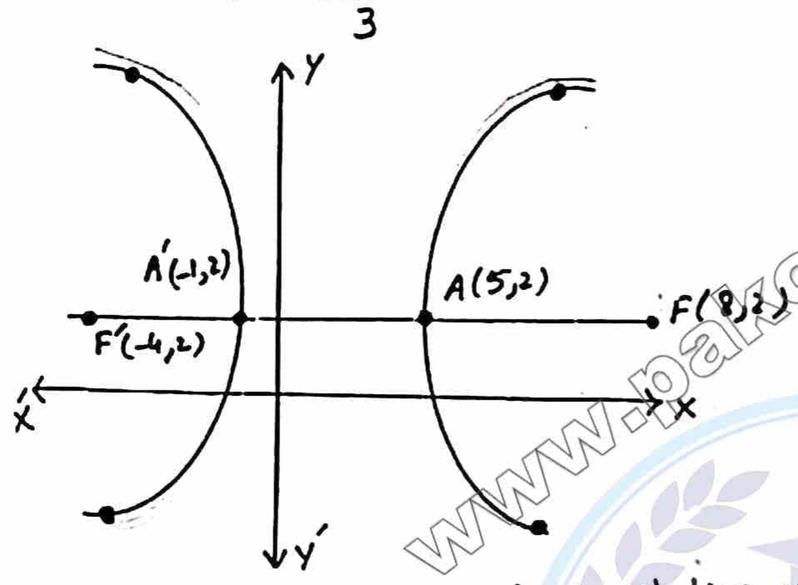
$Y = 0 \rightarrow y-2 = 0 \rightarrow y = 0$

so foci are $F(8,2), F'(-4,2)$

Centre:- $(2, 2)$

Length of letusrectum $= \frac{2b^2}{a}$

$= \frac{2(27)}{3} = 18$



(vii) Vertices $(2, \pm 3)$, $(0,5)$ lies on the curve

Solution:-

Vertices $= (2, \pm 3) \rightarrow A(2,3), A'(2,-3)$

centre = midpoint of vertices

$(h,k) = \left(\frac{2+2}{2}, \frac{3-3}{2}\right) = (2,0)$

$\because$ Distance b/w vertices $= 2a$

$\rightarrow 2a = \sqrt{(2-2)^2 + (-3-3)^2} = \sqrt{36} = 6$

$\rightarrow a = 3 \rightarrow a^2 = 9$

$\because$ vertices have same abscissa so transverse axis is along y-axis. so eq. of hyperbola is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\rightarrow \frac{y^2}{9} - \frac{(x-2)^2}{b^2} = 1 \longrightarrow (i)$$

$\because$ (i) passes through $(0,5)$ so

(i) $\frac{(5)^2}{9} - \frac{(0-2)^2}{b^2} = 1$

$\rightarrow \frac{25}{9} - \frac{4}{b^2} = 1 \rightarrow \frac{4}{b^2} = \frac{25}{9} - 1$

$\rightarrow \frac{4}{b^2} = \frac{16}{9} \rightarrow 36 = 16b^2$

$b^2 = \frac{36}{16} = \frac{9}{4}$

$\because c^2 = a^2 + b^2 \rightarrow c^2 = 9 + \frac{9}{4}$

$\rightarrow c^2 = \frac{45}{4} \rightarrow c = \sqrt{\frac{45}{4}} = \frac{3\sqrt{5}}{2}$

so (i) $\rightarrow \frac{y^2}{9} - \frac{(x-2)^2}{9/4} = 1$

Sketch:-

vertices: $(X = 0, Y = \pm a)$

$X = x-2, \quad x-2 = 0 \rightarrow x = 2$

$Y = y \rightarrow y = \pm a \rightarrow y = \pm 3$

so vertices $A(2,3), A'(2,-3)$

Centre:- $(2,0)$

Foci: $(X = 0, Y = \pm c)$

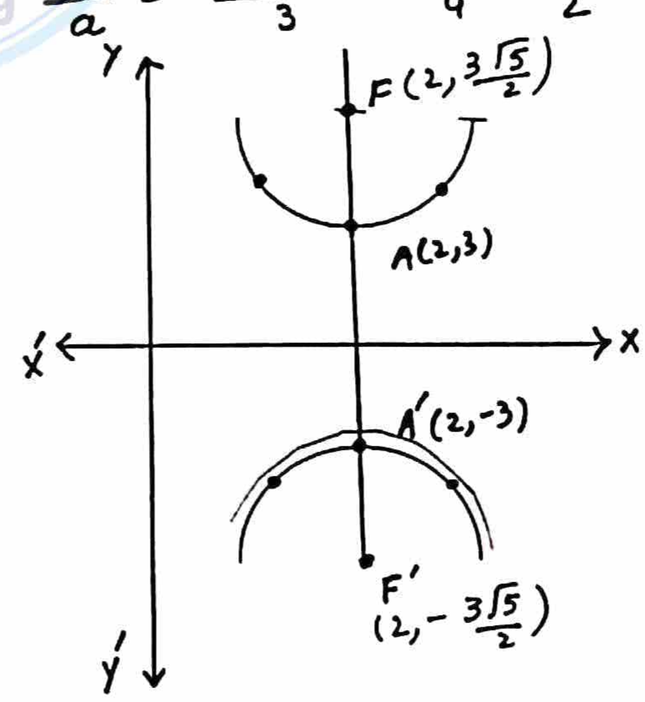
$X = x-2 \rightarrow x-2 = 0, \quad x = 2$

$Y = y \rightarrow y = \pm c \rightarrow y = \pm \frac{3\sqrt{5}}{2}$

so foci $F\left(2, \frac{3\sqrt{5}}{2}\right), F'\left(2, -\frac{3\sqrt{5}}{2}\right)$

length of letusrectum:-

$\frac{2b^2}{a} = \frac{2(9/4)}{3} = \frac{6}{4} = \frac{3}{2}$



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(viii) Foci $(5,-2)$, $(5,4)$ and one vertex $(5,3)$

Solution:-

Foci : $F'(5,-2)$, $F(5,4)$

centre $= (h,k) =$ midpoint of foci

$= \left(\frac{5+5}{2}, \frac{4-2}{2}\right) = (5,1)$

$\because$ one vertex $= (5,3)$

$\because$ Distance b/w centre and vertex $= a$

$a = \sqrt{(5-5)^2+(3-1)^2} = 2 \rightarrow a^2 = 4$

Also distance b/w foci $= 2c$

$2c = \sqrt{(5-5)^2+(4+2)^2} = 6$

$c = 3 \rightarrow c^2 = 9$

$\because c^2 = a^2+b^2 \rightarrow b^2 = c^2 - a^2$

$b^2 = 9-4 = 5$

As foci have same abscissa so transverse axis is along y-axis. so eq. is $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

$\frac{(y-1)^2}{4} - \frac{(x-5)^2}{5} = 1$

Sketch:-

Vertices: $(x=0, y=\pm a)$

$X = x-5 \rightarrow x-5=0 \rightarrow x=5$

$Y = y-1 \rightarrow y-1 = \pm a \rightarrow y-1 = \pm 2$

$y = 1 \pm 2 \rightarrow y = 3, -1$

so vertices $A(5,3)$, $A'(5,-1)$

Foci :- $(X=0, Y=\pm c)$

$X = x-5 \rightarrow x-5=0 \rightarrow x=5$

$Y = y-1 \rightarrow y-1 = \pm c \rightarrow y-1 = \pm 3$

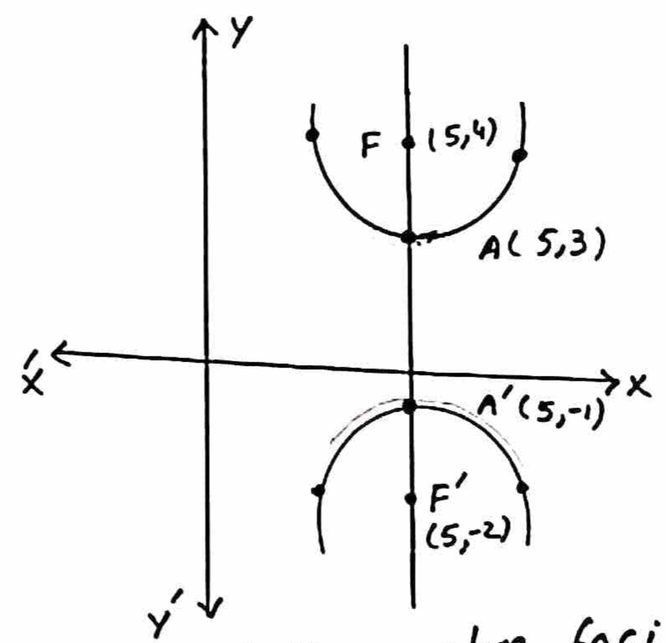
$y = 1 \pm 3 \rightarrow y = 4, -2$

so foci $F(5,4)$, $F'(5,-2)$

Centre: $(5,1)$

Length of Latusrectum: $\frac{2b^2}{a}$

$= \frac{2(5)}{2} = 5$



Q2. Find the centre, foci, eccentricity, vertices and equation of directrices of:

(i) $x^2 - y^2 = 9$

Solution:-

$x^2 - y^2 = 9 \rightarrow \frac{x^2}{9} - \frac{y^2}{9} = 1$

Here $a^2 = 9 \rightarrow a = 3$, $b^2 = 9 \rightarrow b = 3$

$\because c^2 = a^2+b^2 = 9+9 = 18$

$\rightarrow c = \sqrt{18} = 3\sqrt{2}$

Centre:- $(0,0)$

Eccentricity:- $e = \frac{c}{a}$

$\rightarrow e = \frac{3\sqrt{2}}{3} = \sqrt{2}$

Foci :- $(\pm c, 0) = (\pm 3\sqrt{2}, 0)$

Directrices:- $x = \pm \frac{a}{e} = \pm \frac{3}{\sqrt{2}}$

Vertices:- $(\pm a, 0) = (\pm 3, 0)$

(ii) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

Solution:-

$\frac{x^2}{4} - \frac{y^2}{9} = 1$

Here $a^2 = 4 \rightarrow a = 2$, $b^2 = 9 \rightarrow b = 3$

$\because c^2 = a^2+b^2 = 4+9 = 13$

$\rightarrow c = \sqrt{13}$

Centre:- $(0,0)$

Eccentricity:- $e = \frac{c}{a} \rightarrow c = \frac{\sqrt{13}}{2}$

Foci:- $(\pm c, 0) = (\pm\sqrt{13}, 0)$

Directrices:- $x = \pm \frac{a}{e}$

$\rightarrow x = \pm \frac{2}{\sqrt{13}/2} = \pm \frac{4}{\sqrt{13}}$

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Vertices:- $(\pm a, 0) = (\pm 2, 0)$

(iii) $\frac{y^2}{16} - \frac{x^2}{9} = 1$

Solution:-

$\frac{y^2}{16} - \frac{x^2}{9} = 1$ Here $a^2 = 16 \rightarrow a = 4$

$b^2 = 9 \rightarrow b = 3$

$\because c^2 = a^2 + b^2$

$c^2 = 16 + 9 = 25 \rightarrow c = 5$

Centre: $(0,0)$

Eccentricity:- $e = \frac{c}{a} = \frac{5}{4}$

Foci:- $(0, \pm c) = (0, \pm 5)$

Directrices:- $y = \pm \frac{a}{e}$

$\rightarrow y = \pm \frac{4}{5/4} = \pm 16/5$

Vertices:- $(0, \pm a) = (0, \pm 4)$

(iv) $\frac{y^2}{4} - x^2 = 1$

Solution:-

$\frac{y^2}{4} - x^2 = 1 \rightarrow \frac{y^2}{4} - \frac{x^2}{1} = 1$

$a^2 = 4 \rightarrow a = 2$, $b^2 = 1 \rightarrow b = 1$

$\because c^2 = a^2 + b^2 \rightarrow c^2 = 4 + 1 = 5$

$c = \sqrt{5}$

Centre:- $(0,0)$

Eccentricity:- $e = \frac{c}{a} = \frac{\sqrt{5}}{2}$

Foci:- $(0, \pm c) = (0, \pm \sqrt{5})$

Directrices:- $y = \pm \frac{a}{e}$

$\rightarrow y = \pm \frac{2}{\sqrt{5}/2} \rightarrow y = \pm \frac{4}{\sqrt{5}}$

Vertices:- $(0, \pm a) = (0, \pm 2)$

(v) $\frac{(x-1)^2}{2} - \frac{(y-1)^2}{9} = 1$

Solution:-

$\frac{(x-1)^2}{2} - \frac{(y-1)^2}{9} = 1$

Compare with $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

$X = x - 1$, $Y = y - 1$, $a^2 = 2$, $b^2 = 9$

$a = \sqrt{2}$, $b = 3$

$\because c^2 = a^2 + b^2 = 2 + 9 = 11 \rightarrow c = \sqrt{11}$

Centre: $(X = 0, Y = 0)$

$X = x - 1 \rightarrow x - 1 = 0 \rightarrow x = 1$

$Y = y - 1 \rightarrow y - 1 = 0 \rightarrow y = 1$

so centre $(1, 1)$

Eccentricity:-

$e = \frac{c}{a} \rightarrow e = \frac{\sqrt{11}}{\sqrt{2}} = \sqrt{\frac{11}{2}}$

Foci:- $(X = \pm c, Y = 0)$

$\because X = x - 1 \rightarrow x - 1 = \pm c \rightarrow x = 1 \pm \sqrt{11}$

$Y = y - 1 \rightarrow y - 1 = 0 \rightarrow y = 1$

so foci $(1 \pm \sqrt{11}, 1)$

Directrices:- $X = \pm \frac{a}{e}$

$\rightarrow x - 1 = \pm \frac{\sqrt{2}}{\sqrt{11}/\sqrt{2}} \rightarrow x = 1 \pm \frac{2}{\sqrt{11}}$

Vertices:- $(X = \pm a, Y = 0)$

$X = x - 1 \rightarrow x - 1 = \pm a \rightarrow x = 1 \pm \sqrt{2}$

$Y = y - 1 \rightarrow y - 1 = 0 \rightarrow y = 1$

so vertices $(1 \pm \sqrt{2}, 1)$

(vi) $\frac{(y+2)^2}{9} - \frac{(x-2)^2}{16} = 1$

Solution:-

$\frac{(y+2)^2}{9} - \frac{(x-2)^2}{16} = 1$

Compare with $\frac{Y^2}{a^2} - \frac{X^2}{b^2} = 1$

$X = x - 2$, $Y = y + 2$, $a^2 = 9 \rightarrow a = 3$

$b^2 = 16 \rightarrow b = 4$ $\because c^2 = 9 + 16 = 25$

$\rightarrow c = 5$

Centre:- $(X = 0, Y = 0)$

$X = x - 2 \rightarrow x - 2 = 0 \rightarrow x = 2$

$Y = y + 2 \rightarrow y + 2 = 0 \rightarrow y = -2$

so centre $(2, -2)$

Eccentricity:- $\because e = \frac{c}{a} \rightarrow e = \frac{5}{3}$

Foci:- $(X = 0, Y = \pm c)$

$X = x - 2 \rightarrow x - 2 = 0 \rightarrow x = 2$

$Y = y + 2 \rightarrow y + 2 = \pm c \rightarrow y = -2 \pm 5$

$\rightarrow y = -7, 3$

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Foci are $(2,3)$, $(2,-7)$

Directrices:- $Y = \pm \frac{a}{e}$

$\rightarrow y+2 = \pm \frac{3}{5/3} \rightarrow y = -2 \pm \frac{9}{5}$

Vertices:- $(X=0, Y=\pm a)$

$X = x-2 \rightarrow x-2=0 \rightarrow x=2$

$Y = y+2 \rightarrow y+2 = \pm a$

$\rightarrow y = -2 \pm 3 \rightarrow y = -5, 1$

so vertices $(2,-5)$, $(2,1)$

(vii) $9x^2 - 12x - y^2 - 2y + 2 = 0$

Solution:-

$9x^2 - 12x - y^2 - 2y + 2 = 0$

$9(x^2 - \frac{12}{9}x) - (y^2 + 2y) = -2$

$9(x^2 - \frac{4}{3}x) - (y^2 + 2y) = -2$

$9\{x^2 - 2(x)(\frac{2}{3}) + (\frac{2}{3})^2 - (\frac{2}{3})^2\} - \{y^2 + 2y + 1 - 1\} = -2$

$9\{(x - \frac{2}{3})^2 - \frac{4}{9}\} - (y+1)^2 + 1 = -2$

$\rightarrow 9(x - \frac{2}{3})^2 - 4 - (y+1)^2 + 1 = -2$

$\rightarrow 9(x - \frac{2}{3})^2 - (y+1)^2 = -2 + 4 - 1$

or $9(x - \frac{2}{3})^2 - (y+1)^2 = 1$

or $\frac{(x - \frac{2}{3})^2}{1/9} - \frac{(y+1)^2}{1} = 1 \longrightarrow (i)$

compare with $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

$X = x - \frac{2}{3}$, $Y = y+1$, $a^2 = \frac{1}{9}$

$b^2 = 1 \rightarrow b = 1$ $\rightarrow a = \frac{1}{3}$

$\because c^2 = a^2 + b^2 = \frac{1}{9} + 1 = \frac{10}{9}$

$\rightarrow c = \frac{\sqrt{10}}{3}$

Centre:- $(X=0, Y=0)$

$X = x - \frac{2}{3} \rightarrow x - \frac{2}{3} = 0 \rightarrow x = \frac{2}{3}$

$Y = y+1 \rightarrow y+1 = 0 \rightarrow y = -1$

so centre $(\frac{2}{3}, -1)$

Eccentricity:-

$e = \frac{c}{a} \rightarrow e = \frac{\frac{\sqrt{10}}{3}}{1/3} = \sqrt{10}$

Foci:- $(X = \pm c, Y = 0)$

$X = x - \frac{2}{3} \rightarrow x - \frac{2}{3} = \pm c \rightarrow x = \frac{2}{3} \pm \frac{\sqrt{10}}{3}$

$\rightarrow x = \frac{2 \pm \sqrt{10}}{3}$

$Y = y+1 \rightarrow y+1 = 0 \rightarrow y = -1$

so foci $(\frac{2 \pm \sqrt{10}}{3}, -1)$

Directrices:- $X = \pm \frac{a}{e}$

$\rightarrow x - \frac{2}{3} = \pm \frac{1/3}{\sqrt{10}} \rightarrow x = \frac{2}{3} \pm \frac{1}{3\sqrt{10}}$

Vertices:- $(X = \pm a, Y = 0)$

$X = x - \frac{2}{3} \rightarrow x - \frac{2}{3} = \pm a \rightarrow x = \frac{2}{3} \pm \frac{1}{3}$

$\rightarrow x = \frac{2}{3} + \frac{1}{3} \rightarrow x = \frac{3}{3} \rightarrow x = 1$

also $x = \frac{2}{3} - \frac{1}{3} \rightarrow x = \frac{1}{3}$

Hence vertices are $(1,-1)$, $(\frac{1}{3}, -1)$

(viii) $4y^2 + 12y - x^2 + 4x + 1 = 0$

Solution:-

$4y^2 + 12y - x^2 + 4x + 1 = 0$

$4(y^2 + 3y) - (x^2 - 4x) = -1$

$4\{y^2 + 2(y)(\frac{3}{2}) + (\frac{3}{2})^2 - (\frac{3}{2})^2\} - \{x^2 - 4x + 4 - 4\} = -1$

$4(y + \frac{3}{2})^2 - 9 - (x-2)^2 + 4 = -1$

$\rightarrow 4(y + \frac{3}{2})^2 - (x-2)^2 = -1 - 4 + 9$

$\rightarrow 4(y + \frac{3}{2})^2 - (x-2)^2 = 4$

or $\frac{(y + \frac{3}{2})^2}{1} - \frac{(x-2)^2}{4} = 1$

Compare with $\frac{Y^2}{a^2} - \frac{X^2}{b^2} = 1$

$X = x - 2$, $Y = y + \frac{3}{2}$, $a^2 = 1 \rightarrow a = 1$

$b^2 = 4 \rightarrow b = 2$

$\because c^2 = a^2 + b^2 = 1 + 4 \rightarrow c^2 = 5$

or $c = \sqrt{5}$

Centre:- $(X = 0, Y = 0)$

$X = x - 2 \rightarrow x - 2 = 0 \rightarrow x = 2$

$Y = y + \frac{3}{2} \rightarrow y + \frac{3}{2} = 0 \rightarrow y = -\frac{3}{2}$

so centre $(2, -\frac{3}{2})$

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Eccentricity:- $e = \frac{c}{a}$

$\rightarrow e = \frac{\sqrt{5}}{1} \rightarrow e = \sqrt{5}$

Foci:- $(X = 0,\ Y = \pm c)$

$X = x - 2 \rightarrow x - 2 = 0 \rightarrow x = 2$

$Y = y + \frac{3}{2} \rightarrow y + \frac{3}{2} = \pm c \rightarrow y = -\frac{3}{2} \pm \sqrt{5}$

so foci $(2, -\frac{3}{2} \pm \sqrt{5})$

Directrices:- $Y = \pm \frac{a}{e}$

$\rightarrow Y = y + \frac{3}{2} \rightarrow y + \frac{3}{2} = \pm \frac{a}{e}$

$\rightarrow y = -\frac{3}{2} \pm \frac{1}{\sqrt{5}}$

Vertices:- $(X = 0,\ Y = \pm a)$

$X = x - 2 \rightarrow x - 2 = 0,\ x = 2$

$Y = y + \frac{3}{2} \rightarrow y + \frac{3}{2} = \pm a$

$\rightarrow y = -\frac{3}{2} \pm 1 \rightarrow y = -\frac{1}{2}, -\frac{5}{2}$

so vertices $(2, -\frac{1}{2}),\ (2, -\frac{5}{2})$

(ix) $x^2 - y^2 + 8x - 2y - 10 = 0$

Solution:-

$x^2 + 8x - y^2 - 2y - 10 = 0$

$\rightarrow x^2 + (2)(x)(4) + (4)^2 - (4)^2 - (y^2 + 2y + 1 - 1) = 10$

$\rightarrow (x+4)^2 - 16 - (y+1)^2 + 1 = 10$

$\rightarrow (x+4)^2 - (y+1)^2 = 10 + 16 - 1$

$\rightarrow (x+4)^2 - (y+1)^2 = 25$

$\frac{(x+4)^2}{25} - \frac{(y+1)^2}{25} = 1$

Compare with $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

$X = x + 4,\ Y = y + 1,\ a^2 = 25$

$\rightarrow a = 5,\ b^2 = 25 \rightarrow b = 5$

$\because c^2 = a^2 + b^2 = 25 + 25 = 50$

$c = \sqrt{50} = 5\sqrt{2}$

Centre:- $(X = 0,\ Y = 0)$

$X = x + 4 \rightarrow x + 4 = 0 \rightarrow x = -4$

$Y = y + 1 \rightarrow y + 1 = 0 \rightarrow y = -1$

so centre $(-4, -1)$

Eccentricity:- $e = \frac{c}{a}$

$\rightarrow e = \frac{5\sqrt{2}}{5} = \sqrt{2}$

Foci:- $(X = \pm c,\ Y = 0)$

$X = x + 4 \rightarrow x + 4 = \pm c$

$\rightarrow x = -4 \pm 5\sqrt{2}$

$Y = y + 1 \rightarrow y + 1 = 0 \rightarrow y = -1$

so foci $(-4 \pm 5\sqrt{2}, -1)$

Directrices:- $X = \pm \frac{a}{e}$

$\rightarrow x + 4 = \pm \frac{5}{\sqrt{2}} \rightarrow x = -4 \pm \frac{5}{\sqrt{2}}$

Vertices:- $(X = \pm a,\ Y = 0)$

$X = x + 4 \rightarrow x + 4 = \pm a$

$\rightarrow x = -4 \pm 5 \rightarrow x = -9, 1$

$Y = y + 1 \rightarrow y + 1 = 0 \rightarrow y = -1$

so vertices $(1, -1),\ (-9, -1)$

(x) $9x^2 - y^2 - 36x - 6y + 18 = 0$

Solution:-

$9x^2 - 36x - y^2 - 6y = -18$

$9(x^2 - 4x + 4 - 4) - (y^2 + 6y + 9 - 9) = -18$

$9(x-2)^2 - 36 - (y+3)^2 + 9 = -18$

$9(x-2)^2 - (y+3)^2 = -18 - 9 + 36$

$\rightarrow 9(x-2)^2 - (y+3)^2 = 9$

$\rightarrow \frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$

Compare with $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

$X = x - 2,\ Y = y + 3,\ a^2 = 1,\ b^2 = 9$

$\rightarrow a = 1,\ b = 3$

$\because c^2 = a^2 + b^2 = 1 + 9 = 10$

$\rightarrow c = \sqrt{10}$

Centre:- $(X = 0,\ Y = 0)$

$X = x - 2 \rightarrow x - 2 = 0 \rightarrow x = 2$

$Y = y + 3 \rightarrow y = -3$

so centre $(2, -3)$

Foci:- $(X = \pm c,\ Y = 0)$

$X = x - 2 \rightarrow x - 2 = \pm c \rightarrow x - 2 = \pm\sqrt{10}$

$x = 2 \pm \sqrt{10}$

$Y = y + 3 \rightarrow y + 3 = 0 \rightarrow y = -3$

Hence foci are: $(2 \pm \sqrt{10}, -3)$

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Directrices:- $X = \pm \frac{a}{e}$

$\rightarrow x - 2 = \pm \frac{1}{\sqrt{10}} \rightarrow x = 2 \pm \frac{1}{\sqrt{10}}$

Vertices:- $(X = \pm a, Y = 0)$

$X = x - 2 \rightarrow x - 2 = \pm 1$

$\rightarrow x = 2 \pm 1 \rightarrow x = 3, 1$

$Y = 0 \rightarrow y + 3 = 0 \rightarrow y = -3$

so vertices: $(3, -3), (1, -3)$

Q3. Let $0 < c < a$ and $F(c, 0)$, $F'(-c, 0)$ be two fixed points. Show that $\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$

(F and F' are foci of hyperbola)

Solution:-

$P(x, y)$, $F(c, 0)$, $F'(-c, 0)$

By given condition

$|PF| - |PF'| = \pm 2a$

$\sqrt{(x-c)^2 + (y-0)^2} - \sqrt{(x-(-c))^2 + (y-0)^2} = \pm 2a$

$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a$

$\sqrt{x^2 + c^2 - 2cx + y^2} - \sqrt{x^2 + c^2 + 2cx + y^2} = \pm 2a$

$\sqrt{x^2 + c^2 - 2cx + y^2} = \pm 2a + \sqrt{x^2 + c^2 + 2cx + y^2}$

squaring both sides

$\rightarrow x^2 + c^2 - 2cx + y^2 = 4a^2 + x^2 + c^2 + 2cx + y^2 \pm 2(2a)\sqrt{x^2 + 2cx + x^2 + y^2}$

$\rightarrow -4cx = 4a^2 \pm 4a\sqrt{x^2 + 2cx + x^2 + y^2}$

$\rightarrow -cx = a^2 \pm a\sqrt{x^2 + 2cx + x^2 + y^2}$

$\rightarrow -cx - a^2 = \pm a\sqrt{x^2 + 2cx + x^2 + y^2}$

or $-[cx + a^2] = \pm a\sqrt{x^2 + 2cx + x^2 + y^2}$

squaring both sides

$c^2x^2 + a^4 + 2cxa^2 = a^2(x^2 + 2cx + x^2 + y^2)$

$c^2x^2 + a^4 + \cancel{2cxa^2} = a^2x^2 + \cancel{2cxa^2} + a^2x^2 + a^2y^2$

$c^2x^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$

$\rightarrow (c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$

$\rightarrow \frac{x^2}{a^2} - \frac{y^2}{(c^2 - a^2)} = 1 \quad \div \text{ by } a^2(c^2 - a^2)$

Req. eq. of hyperbola

Q4. Let $0 < c < a$ and $F(5, 5)$ $F'(-5, -5)$ be foci. Also the set of points $P(x, y)$ such that $|PF| - |PF'| = \pm 2a$ is the hyperbola with vertices $A(3\sqrt{2}, 3\sqrt{2})$ $A'(-3\sqrt{2}, -3\sqrt{2})$. Find equation of hyperbola.

Solution:-

$\because 2a = |AA'| = \sqrt{(3\sqrt{2} + 3\sqrt{2})^2 + (3\sqrt{2} + 3\sqrt{2})^2}$

$\rightarrow 2a = \sqrt{(6\sqrt{2})^2 + (6\sqrt{2})^2} = \sqrt{72 + 72} = \sqrt{144}$

$\rightarrow 2a = 12 \rightarrow a = 6$

using $|PF| - |PF'| = \pm 2a$

$\rightarrow |PF| = \pm 2a + |PF'|$

$\sqrt{(x-5)^2 + (y-5)^2} = \pm 12 + \sqrt{(x+5)^2 + (y+5)^2}$

squaring both sides

$x^2 + 25 - 10x + y^2 + 25 - 10y = 144 + x^2 + 25 + 10x + y^2 + 25 + 10y \pm 24\sqrt{(x+5)^2 + (y+5)^2}$

$-20x - 20y = 144 \pm 24\sqrt{(x+5)^2 + (y+5)^2}$

$-5x - 5y - 36 = \pm 6\sqrt{(x+5)^2 + (y+5)^2}$

$-[5x + 5y + 36] = \pm 6\sqrt{(x+5)^2 + (y+5)^2}$

squaring both sides

$36(x^2 + 25 + 10x + y^2 + 25 + 10y) = 25x^2 + 25y^2 + 1296 + 50xy - 360x - 360y$

$36x^2 + 36y^2 + 360x + 360y + 1800 = 25x^2 + 25y^2 + 1296 + 50xy + 360x + 360y$

$11x^2 + 11y^2 - 50xy + 504 = 0$

Req. equation of hyperbola

Q5. For any point on the hyperbola the difference of its distances from the points $(2, 2)$ and $(10, 2)$ is 6. Find an equation of hyperbola.

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Solution:- Let $P(x,y)$ be any point on hyperbola. so

$$\sqrt{(x-2)^2+(y-2)^2} - \sqrt{(x-10)^2+(y-2)^2} = 6$$

$$\rightarrow \sqrt{(x-2)^2+(y-2)^2} = 6 + \sqrt{(x-10)^2+(y-2)^2}$$

squaring

$$x^2+4-4x+y^2+4-4y = 36 + x^2+100-20x + y^2+4-4y+12\sqrt{(x-10)^2+(y-2)^2}$$

$$\rightarrow -4x+20x+8-36-104 = 12\sqrt{(x-10)^2+(y-2)^2}$$

$$\rightarrow 16x-132 = 12\sqrt{x^2+100-20x+y^2-4y+4}$$

$$\rightarrow 4x-33 = 3\sqrt{x^2+y^2-20x-4y+104}$$

Again squaring

$$\rightarrow 16x^2+1089-264x = 9(x^2+y^2-20x-4y+104)$$

$$\rightarrow 16x^2+1089-264x = 9x^2+9y^2-180x-36y+936$$

$$\rightarrow 7x^2-9y^2-84x+36y+153=0$$

which is the req. equation

Q6. Two listening posts hear the sound of an enemy gun. The difference in time is one second. If the listening posts are 1400ft apart, write an equation of the hyperbola passing through the position of the enemy gun. (sounds travels at 1080 ft/sec)

Solution:-

Let two listening F_1 and F_2 hear the sound of enemy gun after t and $t-1$ second respectively here listening posts are 1400m apart.

i-e., $2c = 1400 \rightarrow c = 700$

If P is the position of enemy gun.

$\because$ sound travels at 1080 ft/sec so we have

$$|PF_1| - |PF_2| = 2a$$

$$\rightarrow 1080t - 1080(t-1) = 2a$$

$$1080t - 1080t + 1080 = 2a$$

or $2a = 1080$

$\rightarrow a = 540$

using $c^2 = a^2 + b^2$

$\rightarrow b^2 = c^2 - a^2$

$$b^2 = (700)^2 - (540)^2$$

$$b^2 = 490000 - 291600$$

$$b^2 = 198400$$

The eq. of hyperbola is

$$\frac{x^2}{(540)^2} - \frac{y^2}{198400} = 1$$

or $$\frac{x^2}{291600} - \frac{y^2}{198400} = 1$$

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Note:- For Ex 6.7,6.8 & 6.9 Please see other file of this Chapter.

Conic Sections (6.7-6.9)

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