Unit 5

Differentiation of Vector Functions

5.1 Scalar and Vector Functions

5.1.1 Define scalar and vector function

Scalar function:

A scalar function is a function whose domain and codomain are the subsets of real number.

For example, area of circle is the scalar function of its radius which is defined as $A = \pi r^2$ and temperature is the scalar function of time.

Vector function:

A vector function is a function where each real number in the domain is mapped to either a two or three-dimensional vector. It is donated as $\vec{r}(t)$.

Mathematically, it is written as

$$\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

where $f(t), g(t)$ and $h(t)$ are the components of the vector and they are scalar functions of variable t.

Examples include velocity and acceleration are the vector functions of time.

Let $\vec{F}(t)$ be a vector function. If the initial point of the vector $\vec{F}(t)$ is at the origin, then the graph of vector $\vec{F}(t)$ is the curve traced out by the terminal point of the position vector $\vec{F}(t)$ as t varies over the domain set D. This is shown in the figure 5.1.

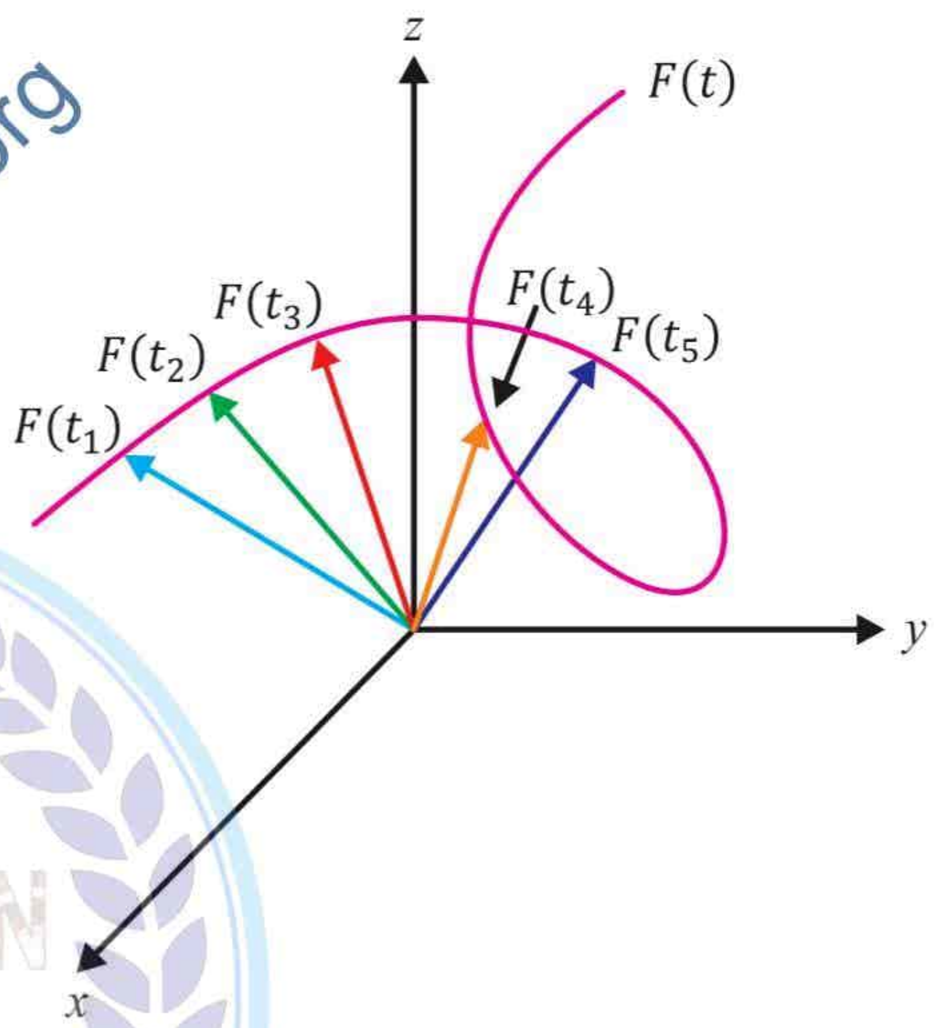

Fig. 5.1

5.1.2 Explain domain and range of a vector function

The domain of the vector function is the set of real numbers and the range of the vector function is the set of the vectors. According to the definition of vector function, it is written as

$$\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

Hence it is function of variable t which is scalar quantity. Therefore, the domain is the set of real numbers. However, the output of the function is a vector. So, its range is the set of vectors.

The intersection of the domains of each components of vector function $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$ is the domain of $\vec{r}(t)$.

i.e., $\text{Dom } \vec{r}(t) = \text{Dom } f(t) \cap \text{Dom } g(t) \cap \text{Dom } h(t)$

Example: Find the domain for the following vector function $\vec{r}(t) = t^2\hat{\imath} + \frac{1}{t}\hat{\jmath} + (t+3)\hat{k}$.

Solution: The vector function is $\vec{r}(t) = t^2\hat{\imath} + \frac{1}{t}\hat{\jmath} + (t+3)\hat{k}$

here $f(t) = t^2, g(t) = \frac{1}{t}$ and $h(t) = t + 3$

$\text{Dom } f = \mathbb{R}$, $\text{Dom } g = \mathbb{R} - \{0\}$, $\text{Dom } h = \mathbb{R}$

$\Rightarrow \text{Dom } \vec{r} = \mathbb{R} - \{0\}$

5.2 Limit and Continuity

5.2.1 Define limit of a vector function and employ the usual technique for algebra of limits of scalar function to demonstrate the following properties of limits of a vector function.

- **The limit of the sum (difference) of two vector functions is the sum (difference) of their limits.**
- **The limit of the dot product of two vector functions is the dot product of their limits.**
- **The limit of the cross product of two vector functions is the cross product of their limits.**
- **The limit of the product of a scalar function and a vector function is the product of their limits.**

Limit of a vector function:

Limit of vector function $\vec{r}(t)$ at $t = t_0$ is the vector $\vec{L}$, such that the values of vector function get close to $\vec{L}$ as long as t becomes close enough to t_0.

i.e., $\lim\limits_{t \to t_0} \vec{r}(t) = \vec{L}$

The limit of $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$ exists at $t = t_0$ if limit of each component of vector function $f(t), g(t)$ and $h(t)$ exists at t_0.

To obtain the limit of $\vec{r}(t)$ at $t = t_0$

Let $\lim\limits_{t \to t_0} \vec{r}(t) = a, \lim\limits_{t \to t_0} g(t) = b$ and $\lim\limits_{t \to t_0} h(t) = c$

then $\lim\limits_{t \to t_0} \vec{r}(t) = \lim\limits_{t \to t_0} \left(f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}\right)$

$= a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

Example 1. Find the limit of vector function $\vec{r}(t) = \frac{e^t - 1}{t}\hat{\imath} + \frac{\sqrt{1+t}-1}{t}\hat{\jmath} + \frac{3}{1+t}\hat{k}$ when $t \to 0$

Solution: Here, $\vec{r}(t) = \frac{e^t - 1}{t}\hat{\imath} + \frac{\sqrt{1+t}-1}{t}\hat{\jmath} + \frac{3}{1+t}\hat{k}$

Now, $$\lim_{t \to 0} \vec{r}(t) = \lim_{t \to 0}\left(\frac{e^t - 1}{e^t}\hat{\imath} + \frac{\sqrt{1+t}-1}{t}\hat{\jmath} + \frac{3}{1+t}\hat{k}\right)$$

$$= \left(\lim_{t\to 0}\frac{e^t - t}{e^t}\right)\hat{\imath} + \left(\lim_{t\to 0}\frac{\sqrt{1+t}-1}{t}\right)\hat{\jmath} + \left(\lim_{t\to 0}\frac{3}{1+t}\right)\hat{k}$$

$$= \left(\frac{1-0}{1}\right)\hat{\imath} + \left(\lim_{t\to 0}\frac{\sqrt{1+t}-1}{t}.\frac{\sqrt{1+t}+1}{\sqrt{1+t}+1}\right)\hat{\jmath} + \left(\frac{3}{1+0}\right)\hat{k}$$

$$= \hat{\imath} + \left(\lim_{t\to 0}\frac{t}{t\left(\sqrt{1+t}+1\right)}\right)\hat{\jmath} + 3\hat{k}$$

$$= \hat{\imath} + \left(\lim_{t\to 0}\frac{1}{\sqrt{1+t}+1}\right)\hat{\jmath} + 3\hat{k} \;=\; \hat{\imath} + \left(\frac{1}{\sqrt{1+0}+1}\right)\hat{\jmath} + 3\hat{k}$$

$$= \hat{\imath} + \frac{1}{2}\hat{\jmath} + 3\hat{k}$$

The limit of the Sum (difference) of two vector functions is the sum of their limits

Limit of the sum or difference of two vector functions $\vec{r}(t)$ and $\vec{s}(t)$ is the sum or difference of the limits of each vector function.

i.e., $$\lim_{t\to t_o}[\vec{r}(t) \pm \vec{s}(t)] = \lim_{t\to t_o}\vec{r}(t) \pm \lim_{t\to t_o}\vec{s}(t)$$

The limit of the dot product of two vector functions is the dot product of their limit functions:

Limit of the dot product of two vector functions $\vec{r}(t)$ and $\vec{s}(t)$ is the dot product of their limits.

i.e., $$\lim_{t\to t_o}[\vec{r}(t)\cdot\vec{s}(t)] = \left[\lim_{t\to t_o}\vec{r}(t)\right]\cdot\left[\lim_{t\to t_o}\vec{s}(t)\right]$$

The limit of the cross product of two vector functions is the cross product of their limits:

Limit of the cross product of two vector functions $\vec{r}(t)$ and $\vec{s}(t)$ is the cross product of the limits of each vector function.

i.e., $$\lim_{t\to t_o}[\vec{r}(t)\times\vec{s}(t)] = \left[\lim_{t\to t_o}\vec{r}(t)\right]\times\left[\lim_{t\to t_o}\vec{s}(t)\right]$$

The limit of the product of a scalar function and a vector function is the product of their limits:

Limit of the product of a scalar function $h(t)$ and a vector function $\vec{s}(t)$ is the product of their limits.

i.e., $$\lim_{t\to t_o}[h(t)\,\vec{s}(t)] = \left(\lim_{t\to t_o}h(t)\right)\left[\lim_{t\to t_o}\vec{s}(t)\right]$$

Example 2. If $\vec{u} = t^3\hat{\imath} - 3\hat{\jmath}$; $\vec{v} = 3t^2\hat{\imath} - \hat{k}$ are vector functions and $h(t) = t + 3$ is scalar function then find the following:

(i) $\lim_{t\to 3}[\vec{u}(t) - \vec{v}(t)]$ (ii) $\lim_{t\to 1}[\vec{u}(t)\cdot\vec{v}(t)]$

(iii) $\lim_{t\to 1}[\vec{u}(t)\times\vec{v}(t)]$ (iv) $\lim_{t\to 0}[h(t)\,\vec{u}(t)]$

Solution:

(i) $\lim\limits_{t\to 3}[\vec{u}(t)-\vec{v}(t)] = \left[\lim\limits_{t\to 3}\vec{u}(t)\right]-\left[\lim\limits_{t\to 3}\vec{v}(t)\right]$

$= \left[\lim\limits_{t\to 3}(t^3\hat{\imath}-3\hat{\jmath})\right]-\left[\lim\limits_{t\to 3}(3t^2\hat{\imath}-\hat{k})\right]$

$= [(3)^3\hat{\imath}-3\hat{\jmath}]-[3(3)^2\hat{\imath}-\hat{k}]$

$= [27\hat{\imath}-3\hat{\jmath}]-[27\hat{\imath}-\hat{k}] = 27\hat{\imath}-27\hat{\imath}-3\hat{\jmath}+\hat{k}$

$= -3\vec{j}+\vec{k}$

(ii) $\lim\limits_{t\to 1}[\vec{u}(t)\cdot\vec{v}(t)] = \left[\lim\limits_{t\to 1}\vec{u}(t)\right]\cdot\left[\lim\limits_{t\to 1}\vec{v}(t)\right]$

$= \left[\lim\limits_{t\to 1}(t^3\hat{\imath}-3\hat{\jmath})\right]\cdot\left[\lim\limits_{t\to 1}(3t^2\hat{\imath}-\hat{k})\right]$

$= [(1)^3\hat{\imath}-3\hat{\jmath}]\cdot[3(1)^2\hat{\imath}-\hat{k}] = [\hat{\imath}-3\hat{\jmath}]\cdot[3\hat{\imath}-\hat{k}] \quad \because [\hat{\imath}\cdot\hat{\imath}=\hat{\jmath}\cdot\hat{\jmath}=\hat{k}\cdot\hat{k}=1]$

$= (1)\cdot(3)(\hat{\imath}\cdot\hat{\imath})+(-3)\cdot(0)(\hat{\jmath}\cdot\hat{\jmath})+(0)\cdot(-1)(\hat{k}\cdot\hat{k}) = 3$

(iii) $\lim\limits_{t\to 1}[\vec{u}(t)\times\vec{v}(t)] = \left[\lim\limits_{t\to 1}\vec{u}(t)\right]\times\left[\lim\limits_{t\to 1}\vec{v}(t)\right]$

$= \left[\lim\limits_{t\to 1}(t^3\hat{\imath}-3\hat{\jmath})\right]\times\left[\lim\limits_{t\to 1}(3t^2\hat{\imath}-\hat{k})\right]$

$= [(1)^3\vec{\imath}-3\vec{\jmath}]\times[3(1)^2\hat{\imath}-\hat{k}]$

$= [\hat{\imath}-3\hat{\jmath}]\times[3\hat{\imath}-\hat{k}]$

$= \begin{vmatrix} i & j & k \\ 1 & -3 & 0 \\ 3 & 0 & -1 \end{vmatrix} = [(-3)(-1)-0]\vec{\imath}-[(1)(-1)-0]\vec{\jmath}+[(1)(0)-(-3)(3)]\vec{k}$

$= 3\vec{\imath}+\vec{\jmath}+9\vec{k}$

(iv) $\lim\limits_{t\to 0}[h(t)\,\vec{u}(t)] = \left[\lim\limits_{t\to 0}h(t)\right]\left[\lim\limits_{t\to 0}\vec{u}(t)\right]$

$= \left[\lim\limits_{t\to 0}(t+3)\right]\left[\lim\limits_{t\to 0}(t^3\hat{\imath}-3\hat{\jmath})\right] = 3\left(\left[\lim\limits_{t\to 0}t^3\right]\hat{\imath}-\left[\lim\limits_{t\to 0}3\right]\hat{\jmath}\right)$

$= -9\hat{\jmath}$

5.2.2 Define continuity of a vector function and demonstrate through examples

A vector function $\vec{r}(t) = f(t)\,\hat{\imath}+g(t)\,\hat{\jmath}+h(t)\hat{k}$ is continuous at $t = t_o$ if the following conditions are satisfied

(i) $t = t_o$ belongs to the domain of a vector function $\vec{r}(t)$

(ii) $\vec{r}(t_o) = \lim\limits_{t\to t_o}\vec{r}(t) = \vec{\mathrm{L}}$

It means value of the vector function $\vec{r}(t)$ at $t = t_o$ is equal to limit of the vector function when t approaches t_o.

If a vector function is continuous at a point then its all components will be continuous at that point.

Example 1. Show that the function $\vec{G}(t) = e^t\hat{\imath} + \cos t\ \hat{\jmath}$ is continuous at $t = 0$

Solution: The components of vector function are $\vec{f}(t) = e^t$; $g(t) = \cos t$;

At $t = 0$ $\quad f(0) = e^0 = 1$; $\quad g(0) = \cos 0 = 1$

Now, $\quad \lim\limits_{t\to 0} \vec{G}(t) = \lim\limits_{t\to 0}(f(t)\,\hat{\imath} + g(t)\,\hat{\jmath})$

$$= \left(\lim_{t\to 0} f(t)\right)\hat{\imath} + \left(\lim_{t\to 0} g(t)\right)\hat{\jmath} = \left(\lim_{t\to 0} e^t\right)\hat{\imath} + \left(\lim_{t\to 0}\cos t\right)\hat{\jmath}$$

$$= e^0\,\hat{\imath} + \cos 0\ \hat{\jmath} = \hat{\imath} + \hat{\jmath}$$

Hence, $\quad \vec{G}(0) = \lim\limits_{t\to 0} \vec{G}(t)$

So, the vector function $\vec{G}(t)$ is continuous at $t = 0$. Hence shown

Example 2. Show that function $\vec{r}(t) = |t|\hat{\imath} + \frac{1}{t+1}\hat{\jmath}$ is continues at $t = 0$.

Solution: The components of vector function are

$f(t) = |t|\hat{\imath} + \frac{1}{t+1}\hat{\jmath}$

Now, $\quad \vec{r}(0) = |0|\hat{\imath} + \frac{1}{0+1}\hat{\jmath} = \hat{\jmath}$

We find limit of function

$$\lim_{t\to 0}(\vec{r}(t)) = \lim_{t\to 0}\left(|t|\,\hat{\imath} + \frac{1}{t+1}\hat{\jmath}\right) = \hat{\jmath}$$

$\because \quad \lim\limits_{t\to 0}\vec{r}(t) = \vec{r}(0)$

$\therefore \quad$ Function is continuous at $t = 0$.

Example 3. Test the continuity of $\vec{r}(t) = \frac{\hat{\imath}}{t^2} + 2t\hat{\jmath} + 3\hat{k}$ at $t = 1$.

Solution: $\quad \vec{r}(1) = \frac{\hat{\imath}}{(1)^2} + 2(1)\hat{\jmath} + 3\hat{k} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$$\lim_{t\to 1}\vec{r}(t) = \lim_{t\to 1}\left(\frac{\hat{\imath}}{t^2} + 2t\hat{\jmath} + 3\hat{k}\right) = \hat{\imath} + 2\hat{\jmath} + 3\hat{k} = \vec{r}(t)$$

$\vec{r}(t)$ is continuous at $t = 1$.

Exercise 5.1

1. Find the domain of the following vector function.

(i) $\vec{r}(t) = 2t\hat{\imath} - 3t\hat{\jmath} + \frac{1}{t}\hat{k}$ $\qquad$ (ii) $\vec{r}(t) = \sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \tan t\,\hat{k}$

(iii) $\vec{r}(t) = (1-t)\hat{\imath} + \sqrt{t}\hat{\jmath} + \frac{1}{t^2}\hat{k}$ $\qquad$ (iv) $\vec{g}(t) = \cos t\,\hat{\imath} - \cot t\,\hat{\jmath} + \operatorname{cosec} t\,\hat{k}$

2. Find the limit of vector function $\vec{r}(t) = (e^{3t} - 1)\,\hat{\imath} + \frac{\sqrt{3+t}-\sqrt{3}}{3t}\hat{\jmath} + \frac{1}{9t+1}\hat{k}$ at $t = 0$.

3. If $\vec{u} = t^2\hat{\imath} - 2\hat{\jmath}$; $\vec{v} = 2t\hat{\imath} - 5\hat{k}$ are vector functions and $h = 3t$ is scalar function then find the following:
   (i) $\lim\limits_{t\to 0} [\vec{u}(t) + \vec{v}(t)]$ (ii) $\lim\limits_{t\to 1} [\vec{u}(t) \cdot \vec{v}(t)]$
   (iii) $\lim\limits_{t\to 1} [\vec{u}(t) \times \vec{v}(t)]$ (iv) $\lim\limits_{t\to 5} [h\,\vec{u}(t)]$
4. Show that function $\vec{R}(t) = sin^2 t\,\hat{\imath} + \tan t\,\hat{\jmath} + \frac{1}{t}\,\hat{k}$ is continuous at $t = \frac{\pi}{4}$.
5. Show that function $\vec{r}(t) = \frac{2}{t}\,\hat{\imath} + \frac{t^3}{2t^3-5}\,\hat{\jmath} + \frac{1}{e^t}\,\hat{k}$ is continuous at $t \to \infty$.
6. For what value of t, following vector functions are continuous
   (i) $\vec{r}(t) = \ln(t+3)\,\hat{\imath} + \frac{1}{t-1}\,\hat{\jmath} + \frac{t+2}{t^2-4}\hat{k}$ (ii) $\vec{r}(t) = \frac{1}{3t+1}\hat{\imath} + \frac{1}{t}\,\hat{\jmath}$

5.3 Derivative of Vector Function

5.3.1 Define derivative of a vector function of a single variable and elaborate the result:

If $\vec{f}(t) = f_1(t)\hat{\imath} + f_2(t)\hat{\jmath} + f_3(t)\hat{k}$, where $f_1(t), f_2(t), f_3(t)$ are differentiable functions of a scalar variable then

$$\frac{d\vec{f}}{dt} = \frac{df_1}{dt}\hat{\imath} + \frac{df_2}{dt}\hat{\jmath} + \frac{df_3}{dt}\hat{k}$$

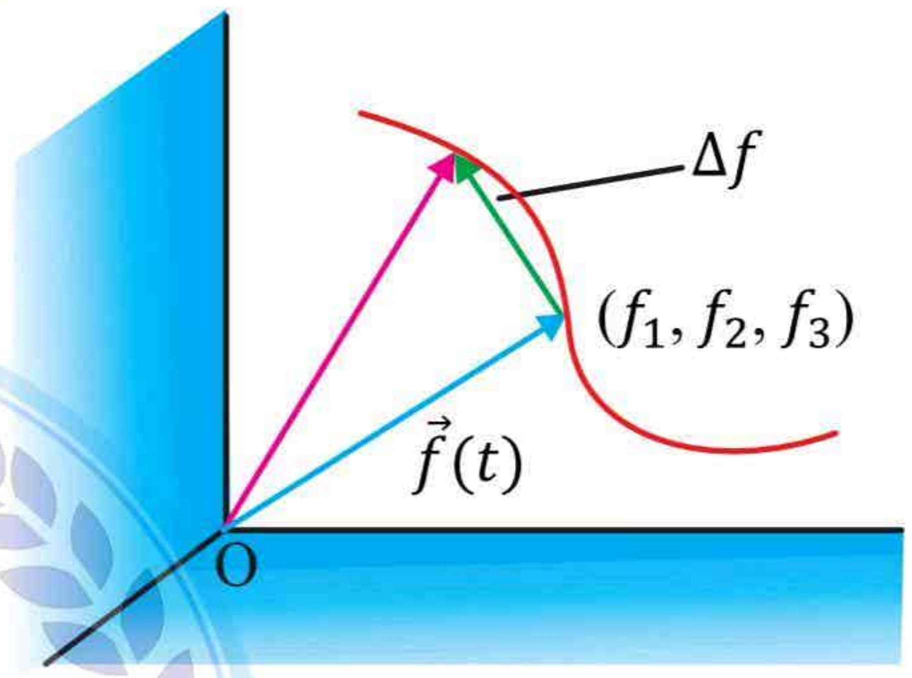

Fig. 5.2

Consider a vector function $\vec{f}(t)$ which is a curve, as the position vector function $\vec{f}(t)$ joining the origin O of a coordinate system at any point (f_1, f_2, f_3), then

$$\vec{f}(t) = f_1(t)\hat{\imath} + f_2(t)\hat{\jmath} + f_3(t)\hat{k}$$

Where, $f_1(t)$, $f_2(t)$, $f_3(t)$ are single variable scalar functions. As t changes, the vector function describes a curve having the following parametric equations.

$$f_1 = f_1(t), \quad f_2 = f_2(t), \quad f_3 = f_3(t)$$

Thus $\lim\limits_{\Delta t\to 0} \frac{\Delta\vec{f}(t)}{\Delta t} = \lim\limits_{\Delta t\to 0} \frac{\vec{f}(t+\Delta t) - \vec{f}(t)}{\Delta t}$

is a vector in the direction of $\Delta\vec{f}$. If $\lim\limits_{\Delta t\to 0} \frac{\Delta\vec{f}(t)}{\Delta t} = \frac{d\vec{f}}{dt}$ exists, the limit will be a vector in the direction of the tangent to the curve $\vec{f}(t)$ at the point (f_1, f_2, f_3) and is given by

$$\frac{d\vec{f}}{dt} = \frac{df_1}{dt}\hat{\imath} + \frac{df_2}{dt}\hat{\jmath} + \frac{df_3}{dt}\hat{k}$$

Here $\frac{df_1}{dt}, \frac{df_2}{dt}$, and $\frac{df_3}{dt}$ are the derivative of scalar function as

$$\frac{df_1}{dt}=\lim_{\Delta t\to 0}\frac{\Delta f_1(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{f_1(t+\Delta t)-f_1(t)}{\Delta t}$$

$$\frac{df_2}{dt}=\lim_{\Delta t\to 0}\frac{\Delta f_2(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{f_2(t+\Delta t)-f_2(t)}{\Delta t}$$

and $$\frac{df_3}{dt}=\lim_{\Delta t\to 0}\frac{\Delta f_3(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{f_3(t+\Delta t)-f_3(t)}{\Delta t}$$

Example: Find $\frac{d\vec{f}}{dt}$ if $\vec{f}(t)=(e^{2t}+1)\hat{\imath}+\sin(2t)\hat{\jmath}+t^3\hat{k}$.

Solution: $\vec{f}(t)=(e^{2t}+1)\hat{\imath}+\sin(2t)\hat{\jmath}+t^3\hat{k}$

By differentiating w.r.t t we get

$$\frac{d\vec{f}(t)}{dt}=\frac{d}{dt}\left[(e^{2t}+1)\hat{\imath}+\sin(2t)\hat{\jmath}+t^3\hat{k}\right]$$

$$=\frac{d}{dt}(e^{2t}+1)\hat{\imath}+\frac{d}{dt}[\sin(2t)]\hat{\jmath}+\frac{d}{dt}[t^3]\hat{k}$$

$$\frac{d\vec{f}}{dt}=2e^{2t}\hat{\imath}+2\cos(2t)\hat{\jmath}+3t^2\hat{k}$$

5.4 Vector Differentiation

5.4.1 Prove the following formulae of differentiation

- $\frac{d\vec{a}}{dt}=0$
- $\frac{d}{dt}[\vec{f}\pm\vec{g}]=\frac{d\vec{f}}{dt}\pm\frac{d\vec{g}}{dt}$
- $\frac{d}{dt}[\emptyset\vec{f}]=\emptyset\frac{d\vec{f}}{dt}+\frac{d\emptyset}{dt}\vec{f}$
- $\frac{d}{dt}[\vec{f}.\vec{g}]=\vec{f}\frac{d\vec{g}}{dt}+\frac{d\vec{f}}{dt}\vec{g}$
- $\frac{d}{dt}[\vec{f}\times\vec{g}]=\vec{f}\times\frac{d\vec{g}}{dt}+\frac{d\vec{f}}{dt}\times\vec{g}$
- $\frac{d}{dt}\left[\frac{f}{\emptyset}\right]=\frac{1}{\emptyset^2}\left[\emptyset\frac{d\vec{f}}{dt}-\vec{f}\frac{d\emptyset}{dt}\right]$

Where *a* is a constant vector function, *f* and *g* are vector functions, and ∅ is a scalar function of t.

In general, the standard rules of differentiation can also be extended to a vector function:

(i) $\dfrac{d\vec{a}}{dt} = 0$

Consider, $\vec{f}(t) = \vec{a}$ is a constant vector function

$$= a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k}$$

$$\frac{d\vec{f}}{dt} = \lim_{\Delta t\to 0}\left[\frac{\left(a_1\,\hat{\imath}+a_2\,\hat{\jmath}+a_3\,\hat{k}\right)-\left(a_1\,\hat{\imath}+a_2\,\hat{\jmath}+a_3\,\hat{k}\right)}{\Delta t}\right]$$

$$\frac{da}{dt} = \lim_{\Delta t\to 0}\left[\frac{a_1-a_1}{\Delta t}\right]\hat{\imath} + \lim_{\Delta t\to 0}\left[\frac{a_2-a_2}{\Delta t}\right]\hat{\jmath} + \lim_{\Delta t\to 0}\left[\frac{a_3-a_3}{\Delta t}\right]\hat{k}$$

$$= \lim_{\Delta t\to 0}[0]\hat{\imath} + \lim_{\Delta t\to 0}[0]\hat{\jmath} + \lim_{\Delta t\to 0}[0]\hat{k} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = 0$$

Hence, $\dfrac{da}{dt} = 0$

(ii) $\dfrac{d}{dt}\left[\vec{f}+\vec{g}\right] = \dfrac{d\vec{f}}{dt} + \dfrac{d\vec{g}}{dt}$

By definition

$$\frac{d}{dt}\left[\vec{f}+\vec{g}\right] = \lim_{\Delta t\to 0}\left[\frac{\left[\left(\vec{f}+\Delta\vec{f}\right)+\left(\vec{g}+\Delta\vec{g}\right)\right]-\left(\vec{f}+\vec{g}\right)}{\Delta t}\right]$$

$$\frac{d}{dt}\left[\vec{f}+\vec{g}\right] = \lim_{\Delta t\to 0}\left[\frac{\left[\vec{f}+\Delta\vec{f}+\vec{g}+\Delta\vec{g}\right]-\left(\vec{f}+\vec{g}\right)}{\Delta t}\right] = \lim_{\Delta t\to 0}\left[\frac{\left[\vec{f}+\Delta\vec{f}+\vec{g}+\Delta\vec{g}-\vec{f}-\vec{g}\right]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\left(\frac{\left(\vec{f}+\Delta\vec{f}\right)-\vec{f}}{\Delta t}\right)+\left(\frac{\left(\vec{g}+\Delta\vec{g}\right)-\vec{g}}{\Delta t}\right)\right] = \lim_{\Delta t\to 0}\frac{\left(\vec{f}+\Delta\vec{f}\right)-\vec{f}}{\Delta t} + \lim_{\Delta t\to 0}\frac{\left(\vec{g}+\Delta\vec{g}\right)-\vec{g}}{\Delta t}$$

$$\frac{d}{dt}\left[\vec{f}+\vec{g}\right] = \frac{d\vec{f}}{dt} + \frac{d\vec{g}}{dt}$$ Hence proved.

Similarly, $\dfrac{d\left[\vec{f}-\vec{g}\right]}{dt} = \dfrac{d\vec{f}}{dt} - \dfrac{d\vec{g}}{dt}$

(iii) $\dfrac{d}{dt}\left[\phi\vec{f}\right] = \phi\dfrac{d\vec{f}}{dt} + \vec{f}\dfrac{d\phi}{dt}$ **where ϕ is scalar function**

$$\frac{d\left[\phi\vec{f}\right]}{dt} = \lim_{\Delta t\to 0}\left[\frac{(\phi+\Delta\phi)\left(\vec{f}+\Delta\vec{f}\right)-\phi\vec{f}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{(\phi+\Delta\phi)\left(\vec{f}+\Delta\vec{f}\right)+\phi\left(\vec{f}+\Delta\vec{f}\right)-\phi\left(\vec{f}+\Delta\vec{f}\right)-\phi\vec{f}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{\left(\vec{f}+\Delta\vec{f}\right)\left[(\phi+\Delta\phi)-\phi\right]+\phi\left[\left(\vec{f}+\Delta\vec{f}\right)-\vec{f}\right]}{\Delta t}\right]$$

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$$= \lim_{\Delta t\to 0}\left[\frac{(\vec{f}+\Delta\vec{f})[(\phi+\Delta\phi)-\phi]}{\Delta t}\right] + \lim_{\Delta t\to 0}\left[\frac{\phi[(\vec{f}+\Delta\vec{f})-\vec{f}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}(\vec{f}+\Delta\vec{f})\left[\lim_{\Delta t\to 0}\frac{[(\phi+\Delta\phi)-\phi]}{\Delta t}\right] + \phi\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f})-\vec{f}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}(\vec{f}+\Delta\vec{f})\left[\lim_{\Delta t\to 0}\frac{[(\phi+\Delta\phi)-\phi]}{\Delta t}\right] + \phi\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f})-\vec{f}]}{\Delta t}\right]$$

$$= \vec{f}\frac{d\phi}{dt} + \phi\frac{d\vec{f}}{dt} = \phi\frac{d\vec{f}}{dt} + \frac{d\phi}{dt}\vec{f}$$

Hence proved. $\left(\because\ \Delta t\longrightarrow 0\ \ \therefore \Delta\vec{f}\longrightarrow 0\right)$

(iv) $\boldsymbol{\frac{d}{dt}(\vec{f}\cdot\vec{g}) = \vec{f}\cdot\frac{d}{dt}(\vec{g}) + \vec{g}\cdot\frac{d}{dt}(\vec{f})}$

$$\frac{d(\vec{f}\cdot\vec{g})}{dt} = \lim_{\Delta t\to 0}\left[\frac{(\vec{f}+\Delta\vec{f})\cdot(\vec{g}+\Delta\vec{g})-\vec{f}\cdot\vec{g}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{(\vec{f}+\Delta\vec{f})\cdot(\vec{g}+\Delta\vec{g})+\vec{f}\cdot(\vec{g}+\Delta\vec{g})-\vec{f}\cdot(\vec{g}+\Delta\vec{g})-\vec{f}\cdot\vec{g}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{\vec{f}\cdot[(\vec{g}+\Delta\vec{g})-\vec{g}]+(\vec{g}+\Delta\vec{g})\cdot[(\vec{f}+\Delta\vec{f})-\vec{f}]}{\Delta t}\right]$$

$$= \vec{f}\cdot\lim_{\Delta t\to 0}\left[\frac{[(\vec{g}+\Delta\vec{g})-\vec{g}]]}{\Delta t}\right] + \lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f})-\vec{f}]}{\Delta t}\right]\cdot\lim_{\Delta t\to 0}(\vec{g}+\Delta\vec{g})$$

$$= \vec{f}\cdot\frac{d\vec{g}}{dt} + \frac{d\vec{f}}{dt}\cdot\vec{g}$$ proved

(v) $\boldsymbol{\frac{d}{dt}(\vec{f}\times\vec{g}) = \vec{f}\times\frac{d\vec{g}}{dt} + \vec{g}\times\frac{d\vec{f}}{dt}}$

$$\frac{d(\vec{f}\times\vec{g})}{dt} = \lim_{\Delta t\to 0}\left[\frac{(\vec{f}+\Delta\vec{f})\times(\vec{g}+\Delta\vec{g})-\vec{f}\times\vec{g}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{(\vec{f}+\Delta\vec{f})\times(\vec{g}+\Delta\vec{g})+\vec{f}\times(\vec{g}+\Delta\vec{g})-\vec{f}\times(\vec{g}+\Delta\vec{g})-\vec{f}\times\vec{g}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{\vec{f}\times[(\vec{g}+\Delta\vec{g})-\vec{g}]+[(\vec{f}+\Delta\vec{f})-\vec{f}]\times(\vec{g}+\Delta\vec{g})}{\Delta t}\right]$$

$$= \vec{f}\times\lim_{\Delta t\to 0}\left[\frac{[(\vec{g}+\Delta\vec{g})-\vec{g}]]}{\Delta t}\right] + \lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f})-\vec{f}]}{\Delta t}\right]\times\lim_{\Delta t\to 0}(\vec{g}+\Delta\vec{g})$$

$$= \vec{f}\times\frac{d\vec{g}}{dt} + \frac{d\vec{f}}{dt}\times\vec{g}$$ Hence proved. $\left(\because\ \Delta t\longrightarrow 0\ \ \therefore \Delta\vec{g}\longrightarrow 0\right)$

(vi) $\dfrac{d}{dt}\left[\dfrac{\vec{f}}{\phi}\right] = \dfrac{\phi\frac{d\vec{f}}{dt} - \vec{f}\frac{d\phi}{dt}}{\phi^2}$ [ϕ is scalar function]

$$\frac{d}{dt}\left[\frac{1}{\phi}\vec{f}\right] = \frac{d}{dt}\left[\phi^{-1}\vec{f}\right] = \lim_{\Delta t\to 0}\left[\frac{(\phi+\Delta\phi)^{-1}(\vec{f}+\Delta\vec{f}) - \phi^{-1}\vec{f}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{(\phi+\Delta\phi)^{-1}(\vec{f}+\Delta\vec{f}) + \phi^{-1}(\vec{f}+\Delta\vec{f}) - \phi^{-1}(\vec{f}+\Delta\vec{f}) - \phi^{-1}\vec{f}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{(\vec{f}+\Delta\vec{f})[(\phi+\Delta\phi)^{-1} - \phi^{-1}] + \phi^{-1}[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{(\vec{f}+\Delta\vec{f})[(\phi+\Delta\phi)^{-1} - \phi^{-1}]}{\Delta t}\right] + \lim_{\Delta t\to 0}\left[\frac{\phi^{-1}[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}(\vec{f}+\Delta\vec{f})\left[\lim_{\Delta t\to 0}\frac{[(\phi+\Delta\phi)^{-1} - \phi^{-1}]}{\Delta t}\right] + \phi^{-1}\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}(\vec{f}+\Delta\vec{f})\left[\lim_{\Delta t\to 0}\frac{[\phi^{-1}(1+\frac{\Delta\phi}{\phi})^{-1} - \phi^{-1}]}{\Delta t}\right] + \phi^{-1}\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}(\vec{f}+\Delta\vec{f})\left[\lim_{\Delta t\to 0}\frac{\left[\phi^{-1}[1-\left(\frac{\Delta\phi}{\phi}\right)+\left(\frac{\Delta\phi}{\phi}\right)^2 - \cdots] - \phi^{-1}\right]}{\Delta t}\right] + \phi^{-1}\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right]$$

[Neglecting the terms involving higher power of $\Delta\phi$]

$$= \lim_{\Delta t\to 0}(\vec{f}+\Delta\vec{f})\left[\lim_{\Delta t\to 0}\frac{\left[\frac{1}{\phi^2}[\phi-\Delta\phi] - \phi\right]}{\Delta t}\right] + \phi^{-1}\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}(\vec{f}+\Delta\vec{f})\left[\frac{1}{\phi^2}\lim_{\Delta t\to 0}\frac{[[\phi-\Delta\phi] - \phi]}{\Delta t}\right] + \phi^{-1}\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right]$$

$$= \frac{1}{\phi^2}\left\{\phi\lim_{\Delta t\to 0}\left[\frac{[(\vec{f}+\Delta\vec{f}) - \vec{f}]}{\Delta t}\right] - \lim_{\Delta f\to 0}(\vec{f}+\Delta\vec{f})\left[\lim_{\Delta t\to 0}\left[\frac{\phi - [\phi-\Delta\phi]}{\Delta t}\right]\right]\right\}$$

$\because\ \Delta t\to 0,\ \therefore\ \Delta\vec{f}\to 0,$ thus

$$= \frac{1}{\phi^2}\left[\phi\frac{d\vec{f}}{dt} - \frac{d\phi}{dt}\vec{f}\right] \quad \text{proved}$$

Example 1. If $\vec{u} = 2t\hat{\imath} - 5\hat{\jmath}$; $\vec{v} = t^2\hat{\imath} - 2t\hat{k}$ are vector functions and $\emptyset(t) = 3t$ is scalar function then find the following:

(i) $\dfrac{d}{dt}\,[\vec{u}(t) + \vec{v}(t)]$ (ii) $\dfrac{d}{dt}\,[\vec{u}(t)\cdot\vec{v}(t)]$

(iii) $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right]$ (iv) $\frac{d}{dt}\left[\emptyset\,\vec{u}(t)\right]$

Solution:

(i) $\frac{d}{dt}\left[\vec{u}(t)+\vec{v}(t)\right] = \left[\frac{d}{dt}(2t\hat{\imath}-5\hat{\jmath})\right]+\left[\frac{d}{dt}(t^2\hat{\imath}-2t\hat{k})\right]$

$= (2\hat{\imath})+(2t\hat{\imath}-2\hat{k}) = (2t+2)\hat{\imath}-2\hat{k}$

(ii) $\frac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)] = \vec{u}(t)\cdot\left[\frac{d}{dt}\vec{v}(t)\right]+\vec{v}(t)\cdot\left[\frac{d}{dt}\vec{u}(t)\right]$

$= (2t\hat{\imath}-5\hat{\jmath})\cdot\left[\frac{d}{dt}(t^2\hat{\imath}-2t\hat{k})\right]+(t^2\hat{\imath}-2t\hat{k})\cdot\left[\frac{d}{dt}(2t\hat{\imath}-5\hat{\jmath})\right]$

$= (2t\vec{\imath}-5\vec{\jmath})\cdot(2t\hat{\imath}-2\hat{k})+(t^2\hat{\imath}-2t\hat{k})\cdot(2\hat{\imath})$ $\because[\hat{\imath}\cdot\hat{\imath}=\hat{\jmath}\cdot\hat{\jmath}=\hat{k}\cdot\hat{k}=1]$

$= (4t^2)+(2t^2) = 6t^2$ $[\hat{\imath}\cdot\hat{\jmath}=\hat{\jmath}\cdot\hat{k}=\hat{k}\cdot\hat{\imath}=0]$

(iii) $\frac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}(t)\times\left[\frac{d}{dt}\vec{v}(t)\right]+\vec{v}(t)\times\left[\frac{d}{dt}\vec{u}(t)\right]$

$= (2t\hat{\imath}-5\hat{\jmath})\times\left[\frac{d}{dt}(t^2\hat{\imath}-2t\hat{k})\right]+(t^2\hat{\imath}-2t\hat{k})\times\left[\frac{d}{dt}(2t\hat{\imath}-5\hat{\jmath})\right]$

$= (2t\hat{\imath}-5\hat{\jmath})\times[2t\,\hat{\imath}-2\,\hat{k}]+(t^2\hat{\imath}-2t\hat{k})\times[2\hat{\imath}]$

$= \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2t & -5 & 0\\ 2t & 0 & -2\end{vmatrix}+\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ t^2 & 0 & -2t\\ 2 & 0 & 0\end{vmatrix}$

$= \{[(-5)(-2)-0]\hat{\imath}-[(2t)(-2)-0]\hat{\jmath}+[0-(-5)(2t)]\,\hat{k}\}+$
$\{[0-0]\hat{\imath}-[0-(-2t)(2)]\hat{\jmath}+[0-0]\,\hat{k}\}$

$= \{[10]\hat{\imath}-[-4t]\hat{\jmath}+[10t]\hat{k}\}+\{-4t]\hat{\jmath}\} = \{[10]\hat{\imath}+[4t]\hat{\jmath}+[10t]\hat{k}\}-\{4t]\hat{\jmath}\}$

$= 10\vec{\imath}+10t\vec{k}$

(iv) $\frac{d}{dt}\left[\emptyset(t)\,\vec{u}(t)\right] = \emptyset(t)\frac{d}{dt}\vec{u}(t)+\vec{u}(t)\frac{d\emptyset}{dt}$

$= (3t)\frac{d}{dt}(2t\hat{\imath}-5\hat{\jmath})+(2t\hat{\imath}-5\hat{\jmath})\frac{d}{dt}(3t)$

$= (3t)(2\hat{\imath}-0)+(2t\hat{\imath}+5\hat{\jmath})(3)$

$= 6t\hat{\imath}+6t\hat{\imath}+15\hat{\jmath}$

$= 12t\hat{\imath}+15\hat{\jmath}$

5.4.2 Apply vector differentiation to calculate velocity and acceleration of a position vector $\vec{r}(t) = x(t)\hat{\imath}+y(t)\hat{\jmath}+z(t)\hat{k}$

Consider $\vec{r}(t) = x(t)\,\hat{\imath}+y(t)\,\hat{\jmath}+z(t)\hat{k}$ is a position vector joining the origin O of the coordinate system at any point (x, y, z) as shown in the figure 5.3.

As t changes, the terminal point $\vec{r}(t)$ describe a curve having parametric equations

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

If $\lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$ exists then the rate of change $\frac{d\vec{r}}{dt}$ will be the velocity $\vec{v}$. We further differentiate velocity $\vec{v}$ with respect to time, we have $\frac{d\vec{v}}{dt}$ i.e., $\frac{d^2\vec{r}}{dt^2}$ which represents acceleration along the curve.

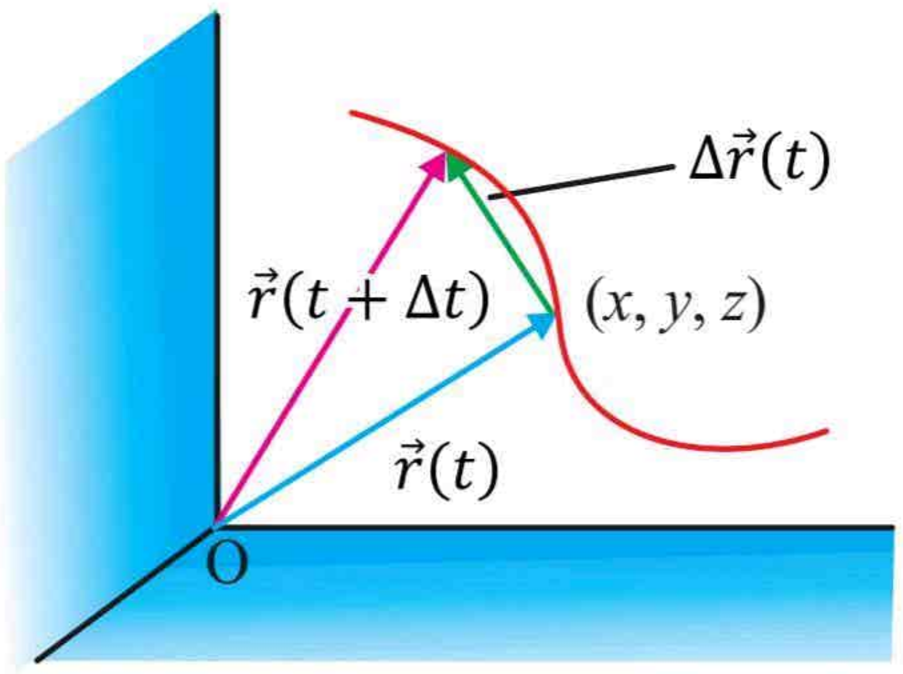


Fig. 5.3

Example 1. A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$ $z = 2\sin 3t$, where t is the time.

(a) Determine its velocity and acceleration at any time.

(b) Find the magnitudes of the velocity and acceleration at $t = 0$.

Solution:

(a) The position vector of the particle is

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\hat{k} = e^{-t}\,\hat{\imath} + 2\cos 3t\;\hat{\jmath} + 2\sin 3t\;\hat{k}$$

The velocity is $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left[e^{-t}\,\hat{\imath} + 2\cos 3t\;\hat{\jmath} + 2\sin 3t\;\hat{k}\right]$

$$= \frac{d}{dt}(e^{-t})\,\hat{\imath} + 2\frac{d}{dt}(\cos 3t)\;\hat{\jmath} + 2\,\frac{d}{dt}(\sin 3t)\;\hat{k}$$

$$\vec{v} = -\,e^{-t}\hat{\imath} - 6\sin 3t\,\hat{\jmath} + 6\cos 3t\,\hat{k}$$

The acceleration is $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[-\,e^{-t}\hat{\imath} - 6\sin 3t\,\hat{\jmath} + 6\cos 3t\,\hat{k}\right]$

$$= \frac{d}{dt}[-\,e^{-t}]\;\hat{\imath} - 6\frac{d}{dt}[\sin 3t]\;\hat{\jmath} + 6\,\frac{d}{dt}[\cos 3t]\;\hat{k}$$

$$\vec{a} = e^{-t}\vec{\imath} - 18\cos 3t\,\vec{\jmath} - 18\sin 3t\,\vec{k}$$

(b) At $t = 0$, the velocity is $\vec{v} = -\,e^{-(0)}\hat{\imath} - 6\sin 3(0)\,\hat{\jmath} + 6\cos 3(0)\,\hat{k}$

$$\vec{v} = -\hat{\imath} + 6\hat{k}$$

The magnitude of $\vec{v}$ i.e., $|\vec{v}| = \sqrt{(-1)^2 + (6)^2} = \sqrt{37}$ units

At $t = 0$, the acceleration is $\vec{a} = e^{-(0)}\hat{\imath} - 18\cos 3(0)\,\hat{\jmath} - 18\sin 3(0)\,\hat{k}$

$$\vec{a} = \hat{\imath} - 18\hat{\jmath}$$

The magnitude of $\vec{a}$ i.e., $|\vec{a}| = \sqrt{(1)^2 + (18)^2} = \sqrt{325}$ units

Example 2. A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction of $\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$.

Solution:

(a) The position vector of the particle is

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\hat{k} = 2t^2\,\hat{\imath} + (t^2 - 4t)\,\hat{\jmath} + (3t - 5)\,\hat{k}$$

The velocity is $\vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}\left[2t^2\,\hat{\imath} + (t^2 - 4t)\,\hat{\jmath} + (3t - 5)\,\hat{k}\right]$

$$= \frac{d}{dt}[2t^2]\,\hat{\imath} + \frac{d}{dt}(t^2 - 4t)\,\hat{\jmath} + \frac{d}{dt}(3t - 5)\,\hat{k}$$

$$\vec{v} = 4t\,\hat{\imath} + (2t - 4)\hat{\jmath} + 3\,\hat{k}$$

The acceleration is $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d}{dt}\left[4t\,\hat{\imath} + (2t - 4)\hat{\jmath} + 3\,\hat{k}\right]$

$$= \left[\frac{d}{dt}[4t]\hat{\imath} + \frac{d}{dt}(2t - 4)\,\hat{\jmath} + \frac{d}{dt}[3]\hat{k}\right]$$

$$\vec{a} = 4\,\hat{\imath} + 2\,\hat{\jmath}$$

(b) At $t = 1$, the velocity is $\vec{v} = 4t\,\hat{\imath} + (2t - 4)\hat{\jmath} + 3\,\hat{k}$

$$\vec{v} = 4\,\hat{\imath} - 2\,\hat{\jmath} + 3\hat{k}$$

At $t = 1$, the acceleration is $\vec{a} = 4\,\hat{\imath} + 2\,\hat{\jmath}$

The component of $\vec{v}$ along the direction of $\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$ is

$$\frac{(4\,\hat{\imath} - 2\,\hat{\jmath} + 3\hat{k}).(\hat{\imath} - 3\hat{\jmath} + 2\hat{k})}{\sqrt{(1)^2 + (-3)^2 + (2)^2}} = \frac{(4)(1) + (-2)(-3) + (3)(2)}{\sqrt{1 + 9 + 4}} = \frac{16}{\sqrt{14}}$$

The component of $\vec{a}$ along the direction of $\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$ is

$$\frac{(4\,\hat{\imath} + 2\,\hat{\jmath}).(\hat{\imath} - 3\hat{\jmath} + 2\hat{k})}{\sqrt{(1)^2 + (-3)^2 + (2)^2}} = \frac{(4)(1) + (2)(-3) + (0)(2)}{\sqrt{1 + 9 + 4}} = \frac{-2}{\sqrt{14}}$$

Exercise 5.2

1. Find the derivative of the following vector functions.
   (i) $\vec{f}(t) = \ln t^2\,\hat{\imath} + e^{2t}\hat{\jmath} + (2t^2 + 1)\hat{k}$
   (ii) $\vec{f}(t) = (t + 1)\hat{\imath} + \ln(t + 2)\,\hat{\jmath}$
   (iii) $\vec{f}(t) = \sec t\,\hat{\imath} + \cos t^2\hat{\jmath} + (t^2 + t + 1)\hat{k}$
2. If $\vec{x} = t\hat{\imath} + 2t\,\hat{\jmath}$; $\vec{y} = 2t\hat{\imath} + 3t\hat{k}$ are vector functions and $\emptyset(t) = 3t$ is scalar function, then find the following:
   (i) $\dfrac{d}{dt}\left[\vec{x}(t) - \vec{y}(t)\right]$ (ii) $\dfrac{d}{dt}\left[\vec{x}(t) \cdot \vec{y}(t)\right]$
   (iii) $\dfrac{d}{dt}\left[\vec{x}(t) \times \vec{y}(t)\right]$ (iv) $\dfrac{d}{dt}\left[\emptyset\,\vec{x}(t)\right]$
3. A particle moves so that its position as a function of time is given by $\vec{r}(t) = \hat{\imath} + 4t^2\hat{\jmath} + t\,\hat{k}$. Write expressions for its:
   (a) velocity (b) acceleration as functions of time.
4. The path of a particle is given for time $t > 0$ by the parametric equations $x = t^2 - 3t$ and $y = \frac{2}{3}t^3$. Find magnitude of velocity and acceleration of particle at $t = 5$.

5. A particle moves along the curve $x = 2t^2$ and $y = 4t$. Find the component of velocity and acceleration at $t = 2$ in the direction of $2\hat{\imath} + \hat{\jmath}$.

Review Exercise 5

1. Choose the correct answer.

(i) The domain of the vector function $\vec{r}(t) = t^3\hat{\imath} + \frac{1}{t-1}\hat{\jmath} + \ln(t-2)\hat{k}$, $t \in R$

(a) $\{t > 2,, t \in R\}$ (b) $\{t < 2,, t \in R\}$
(c) $\{t > 0,, t \in R\}$ (d) $\{t > 1,, t \in R\}$

(ii) Is the Function $\vec{F}(t) = \cos t\,\hat{\imath} + \sin 2t\ \hat{k}$ at $t = \pi$ ----------------

(a) is not continuous (b) is continuous (c) limit does not exist (d) None of these

(iii) What value of t, vector function $\vec{F}(t) = \frac{1}{t-2}\hat{\imath} + \sin t\ \hat{\jmath}$ is continuous at

(a) $t \neq 2,\ t \in R$ (b) $t > 2,\ t \in R$ (c) $t \leq 2,\ t \in R$ (d) $t \in R$

(iv) If $\vec{u} = t^2\hat{\imath} - e^{2t}\hat{\jmath}$ is a vector function and $h(t) = t^2 + 2t - 2$ is scalar function then $\lim_{t\to 0} [h(t)\,\vec{u}(t)] =$ --------------

(a) $-\hat{\jmath}$ (b) $\hat{\jmath}$ (c) $2\hat{\jmath}$ (d) $-2\hat{\jmath}$

(v) If $\vec{u} = t^2\hat{\imath} - e^{2t}\hat{\jmath}$ and $\vec{v} = -2\hat{\imath} - t^2\hat{k}$ are vector functions then $\lim_{t\to 0} [\vec{u}(t) \cdot \vec{v}(t)] =$ ---------------

(a) 0 (b) t^4 (c) $t^2 - e^{2t}$ (d) $-2\,t^2$

(vi) If $\vec{f}(t) = t^2\hat{\imath} - e^{2t}\hat{\jmath}$ then $\vec{f}'(t) =$ ----------------

(a) $2t\hat{\imath} - 2e^{2t}\hat{\jmath}$ (b) $2t\hat{\imath} - e^{2t}\hat{\jmath}$ (c) $2t\hat{\imath} - 2te^{2t}\hat{\jmath}$ (d) None of these

(vii) The velocity function of a particle, whose motion is given by $\vec{r}(t) = \frac{1}{2\pi}t\,\hat{\imath} + 3\cos(t)\,\hat{\jmath}$ at time $t = \frac{\pi}{2}$ is

(a) $\frac{1}{2\pi}\hat{\imath} + 3\hat{\jmath}$ (b) $\frac{1}{2\pi}\hat{\imath}$ (c) $\frac{1}{2\pi}\hat{\imath} - 3\hat{\jmath}$ (d) None of these

(viii) The magnitude of velocity of a particle at $t = \pi$, whose motion is given by $\vec{r}(t) = 4\cos(t)\,\hat{\imath} + 4\sin(t)\,\hat{\jmath} + \frac{3}{2\pi}t^2\hat{k}$

(a) $\sqrt{5}$ (b) 5 (c) $\sqrt{5\pi}$ (d) 5π

(ix) The magnitude of acceleration of a particle at $t = \frac{\pi}{2}$, whose motion is given by $\vec{r}(t) = 4\cos(t)\,\hat{\imath} + 4\sin(t)\,\hat{\jmath} + \frac{3}{2\pi}t^2\hat{k}$ is:

(a) $\frac{\sqrt{16\pi^2+9}}{\pi}$ (b) $\sqrt{16\pi^2+9}$ (c) $\frac{\sqrt{16\pi^2+9}}{\pi^2}$ (d) $\frac{\sqrt{16+9\pi^2}}{\pi}$

(x) Let $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + 5\hat{k}$ then $\dfrac{d\vec{a}}{dt}$ =------------

(a) 10 (b) $\sqrt{38}$ (c) 0 (d) None of these

2. Find the limit of vector function $\vec{r}(t) = 2t\,\hat{\imath} + t^3\,\hat{\jmath} + \hat{k}$ when $t \to 2$.
3. If $\vec{u} = 5\hat{\imath} - 2t\hat{\jmath}$; $\vec{v} = \hat{\imath} - 3t\hat{k}$ are vector functions and $k = t + 1$ is scalar function then find the following:

   (i) $\lim\limits_{t\to 0} [\vec{u}(t) - \vec{v}(t)]$ (ii) $\lim\limits_{t\to 1} [\vec{u}(t) \cdot \vec{v}(t)]$

   (iii) $\lim\limits_{t\to 1} [\vec{u}(t) \times \vec{v}(t)]$ (iv) $\lim\limits_{t\to 5} [k\,\vec{u}(t)]$
4. Check the continuity of the function $\vec{G}(t) = (t+8)\,\hat{\imath} + \frac{6}{t-8}\,\hat{\jmath} + \ln(t+8)\,\hat{k}$ at $t = 0$.
5. For what value of t, following vector functions are continuous

   $\vec{r}(t) = \sqrt{36 - t^2}\hat{\imath} + \ln(t+4)\,\hat{\jmath}$
6. Find $\vec{f}'(t)$ of the following vector functions.

   (i) $\vec{f}(t) = e^{t^4}\hat{\imath} + (\mathrm{t}^3 + 3)\hat{\jmath} + \operatorname{cosec} \mathrm{t}^2\,\hat{k}$

   (ii) $\vec{f}(t) = \frac{1}{t}\hat{\imath} + e^{2t^3}\hat{\jmath} + \sec \mathrm{t}^3\,\hat{k}$
7. A particle moves along the curve whose parametric equations are $x = t^3 + 2t$, $y = 3e^{-2t}$, $z = 2\sin(5\,t)$, where x, y and z show variations of the distance covered by the particle in cm with time in seconds. Find the magnitude of the acceleration of the particle at $t = 0$.
8. The path of a particle is given for time $t > 0$ by the parametric equations $x = t + \frac{2}{t}$ and $y = 3t^2$. Find velocity of particle when time $t = 1$ and acceleration at $t = 2$.

Vector Valued Functions

$f : \mathbb{R} \to \mathbb{R}$ scalar function

$\vec{r} : \mathbb{R} \to \{\text{vectors}\}$ vector function.

$$\vec{r}(t) = \underbrace{f(t)\,\hat{i} + g(t)\,\hat{j}}_{\text{components.}}$$

$$\vec{r}(t) = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$$

$$\text{Dom}\,\vec{r}(t) = \text{Dom}(f) \cap \text{Dom}(g) \cap \text{Dom}(h).$$

$$\lim_{t\to a} \vec{r}(t) = \lim_{t\to a} \left[f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k} \right]$$

$$= \left(\lim_{t\to a} f(t)\right)\hat{i} + \left(\lim_{t\to a} g(t)\right)\hat{j} + \left(\lim_{t\to a} h(t)\right)\hat{k}$$

$\vec{r}(t)$ is continuous at $t = a$ if

$$\vec{r}(a) = \lim_{t\to a} \vec{r}(t) = L$$

Exercise 5.1

Q1 Find the domain of the following functions:

(i) $\vec{r}(t) = 2t\,\hat{i} - 3t\,\hat{j} + \frac{1}{t}\,\hat{k}.$

Let $f(t) = 2t, \quad g(t) = -3t, \quad h(t) = \frac{1}{t}$

$$\text{Dom}\,\vec{r}(t) = \text{Dom}(f) \cap \text{Dom}(g) \cap \text{Dom}(h)$$

$$= \mathbb{R} \cap \mathbb{R} \cap \mathbb{R} - \{0\}$$

$$= \mathbb{R} - \{0\}.$$

(ii) $\vec{r}(t) = \sin t\,\hat{i} + \cos t\,\hat{j} + \tan t\,\hat{k}.$

Let $f(t) = \sin t, \quad g(t) = \cos t, \quad h(t) = \tan t = \frac{\sin t}{\cos t}$

$\text{Dom}(f) = \mathbb{R}, \quad \text{Dom}(g) = \mathbb{R}, \quad \text{Dom}(h) = \mathbb{R} - \left\{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\right\}$

$$\text{Dom}\,\vec{r}(t) = \text{Dom}(f) \cap \text{Dom}(g) \cap \text{Dom}(h)$$

$$= \mathbb{R} \cap \mathbb{R} \cap \mathbb{R} - \left\{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\right\}$$

$$= \mathbb{R} - \left\{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\right\}.$$

(iii) $\vec{r}(t) = (1-t)\,\hat{i} + \sqrt{t}\,\hat{j} + \frac{1}{t^2}\,\hat{k}.$

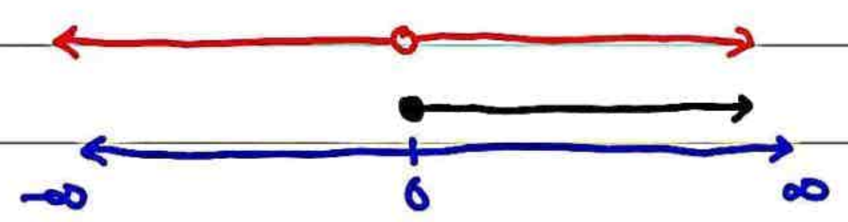
Let $f(t) = 1-t, \quad g(t) = \sqrt{t}, \quad h(t) = \frac{1}{t^2}$

$\text{Dom}(f) = \mathbb{R}, \quad \text{Dom}(g) = \mathbb{R}^+ \cup \{0\}, \quad \text{Dom}(h) = \mathbb{R} - \{0\}.$

$$\text{Dom}\,\vec{r}(t) = \text{Dom}(f) \cap \text{Dom}(g) \cap \text{Dom}(h)$$

$$= \mathbb{R} \cap \mathbb{R}^+ \cup \{0\} \cap \mathbb{R} - \{0\}$$

$$= \left(\mathbb{R}^+ \cup \{0\}\right) \cap \left(\mathbb{R} - \{0\}\right)$$



$$= \mathbb{R}^+.$$

(iv) $\vec{g}(t) = \cos t\,\hat{i} - \cos t\,\hat{j} + \operatorname{cosec} t\,\hat{k}.$

Let $f(t) = \cos t$, $h(t) = -\cos t$, $k(t) = \operatorname{cosec} t = \dfrac{1}{\sin t}$

$\text{Dom}(f) = \mathbb{R}$, $\text{Dom}(h) = \mathbb{R}$, $\text{Dom}(k) = \mathbb{R} - \{n\pi : n \in \mathbb{Z}\}.$

$$\text{Dom}(\vec{g}(t)) = \text{Dom}(f) \cap \text{Dom}(h) \cap \text{Dom}(k)$$
$$= \mathbb{R} \cap \mathbb{R} \cap \mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$$
$$= \mathbb{R} \cap \mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$$
$$= \mathbb{R} - \{n\pi : n \in \mathbb{Z}\}.$$

Q2 Find the limit of the vector function

$$\vec{r}(t) = (e^{3t}-1)\hat{i} + \frac{\sqrt{3+t}-\sqrt{3}}{3t}\hat{j} + \frac{1}{9t+1}\hat{k} \quad \text{at } t=0.$$

$$\lim_{t\to 0} \vec{r}(t) = \lim_{t\to 0}\left[(e^{3t}-1)\hat{i} + \frac{\sqrt{3+t}-\sqrt{3}}{3t}\hat{j} + \frac{1}{9t+1}\hat{k}\right]$$

$$= \left(\lim_{t\to 0} e^{3t}-1\right)\hat{i} + \left(\lim_{t\to 0}\frac{\sqrt{3+t}-\sqrt{3}}{3t}\right)\hat{j} + \left(\lim_{t\to 0}\frac{1}{9t+1}\right)\hat{k}$$

$$= (e^{0}-1)\hat{i} + \left(\lim_{t\to 0}\frac{\sqrt{3+t}-\sqrt{3}}{3t} \times \frac{\sqrt{3+t}+\sqrt{3}}{\sqrt{3+t}+\sqrt{3}}\right) + \left(\frac{1}{9(0)+1}\right)\hat{k}$$

$$= (1-1)\hat{i} + \left(\lim_{t\to 0}\frac{\cancel{3}+\cancel{t}-\cancel{3}}{3\cancel{t}(\sqrt{3+t}+\sqrt{3})}\right)\hat{j} + \left(\frac{1}{1}\right)\hat{k}$$

$$= 0\hat{i} + \left(\frac{1}{3(\sqrt{3}+\sqrt{3})}\right)\hat{j} + \hat{k}$$

$$= 0\hat{i} + \frac{1}{6\sqrt{3}}\hat{j} + \hat{k}$$

$$= \frac{1}{6\sqrt{3}}\hat{j} + \hat{k}.$$

(3) If $\vec{u} = t^2\hat{i} - 2\hat{j}$; $\vec{v} = 2t\hat{i} - 5\hat{k}$ are vector functions and $h = 3t$ is a scalar function, then find the following:

(i) $\lim\limits_{t\to 0}[\vec{u}(t) + \vec{v}(t)]$ (ii) $\lim\limits_{t\to 1}[\vec{u}(t)\cdot\vec{v}(t)]$

(iii) $\lim\limits_{t\to 1}[\vec{u}(t)\times\vec{v}(t)]$ (iv) $\lim\limits_{t\to 5}[h\vec{u}(t)]$.

(i)
$$\lim_{t\to 0}[\vec{u}(t) + \vec{v}(t)] = \lim_{t\to 0}\vec{u}(t) + \lim_{t\to 0}\vec{v}(t)$$
$$= \lim_{t\to 0}(t^2\hat{i} - 2\hat{j}) + \lim_{t\to 0}(2t\hat{i} - 5\hat{k})$$
$$= (0^2\hat{i} - 2\hat{j}) + (0\hat{i} - 5\hat{k})$$
$$= -2\hat{j} - 5\hat{k}.$$

(ii)
$$\lim_{t\to 1}[\vec{u}(t)\cdot\vec{v}(t)] = \lim_{t\to 1}\vec{u}(t)\cdot\lim_{t\to 1}\vec{v}(t)$$
$$= \lim_{t\to 1}(t^2\hat{i} - 2\hat{j})\cdot\lim_{t\to 1}(2t\hat{i} - 5\hat{k})$$
$$= (\hat{i} - 2\hat{j})\cdot(2\hat{i} - 5\hat{k})$$
$$= (\hat{i} - 2\hat{j} + 0\hat{k})\cdot(2\hat{i} + 0\hat{j} - 5\hat{k})$$
$$= 1(2) + (-2)(0) + 0(-5) = 2+0+0 = 2.$$

(iii)
$$\lim_{t\to 1}[\vec{u}(t)\times\vec{v}(t)] = \lim_{t\to 1}\vec{u}(t)\times\lim_{t\to 1}\vec{v}(t)$$
$$= \lim_{t\to 1}(t^2\hat{i} - 2\hat{j})\times\lim_{t\to 1}(2t\hat{i} - 5\hat{k})$$
$$= (\hat{i} - 2\hat{j})\times(2\hat{i} - 5\hat{k})$$
$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 0\\ 2 & 0 & -5\end{vmatrix}$$
$$= \hat{i}\begin{vmatrix}-2 & 0\\ 0 & -5\end{vmatrix} - \hat{j}\begin{vmatrix}1 & 0\\ 2 & -5\end{vmatrix} + \hat{k}\begin{vmatrix}1 & -2\\ 2 & 0\end{vmatrix}$$
$$= \hat{i}(10-0) - \hat{j}(-5-0) + \hat{k}(0+4) = 10\hat{i} + 5\hat{j} + 4\hat{k}.$$

(iv)
$$\lim_{t\to 5}[h\vec{u}(t)] = \left(\lim_{t\to 5}h\right)\left(\lim_{t\to 5}\vec{u}(t)\right) = \left(\lim_{t\to 5}3t\right)\left(\lim_{t\to 5}t^2\hat{i} - 2\hat{j}\right)$$
$$= (3(5))(5^2\hat{i} - 2\hat{j}) = 15(25\hat{i} - 2\hat{j})$$
$$= 375\hat{i} - 30\hat{j}.$$

(4) Show that the function

$$\vec{R}(t) = \sin^2 t\,\hat{i} + \tan t\,\hat{j} + \frac{1}{t}\hat{k}$$

is continuous at $t = \frac{\pi}{4}$.

Value: $$R\left(\frac{\pi}{4}\right) = \sin^2\frac{\pi}{4}\,\hat{i} + \tan\frac{\pi}{4}\,\hat{j} + \frac{1}{(\pi/4)}\hat{k}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2\hat{i} + 1\,\hat{j} + \frac{4}{\pi}\hat{k}$$

$$= \frac{1}{2}\hat{i} + \hat{j} + \frac{4}{\pi}\hat{k}.$$

Limit: $$\lim_{t\to\pi/4}\vec{R}(t) = \left(\lim_{t\to\pi/4}\sin^2 t\right)\hat{i} + \left(\lim_{t\to\pi/4}\tan t\right)\hat{j} + \left(\lim_{t\to\pi/4}\frac{1}{t}\right)\hat{k}$$

$$= \left(\sin^2\frac{\pi}{4}\right)\hat{i} + \left(\tan\frac{\pi}{4}\right)\hat{j} + \left(\frac{1}{\pi/4}\right)\hat{k}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2\hat{i} + 1\,\hat{j} + \frac{4}{\pi}\hat{k}$$

$$= \frac{1}{2}\hat{i} + \hat{j} + \frac{4}{\pi}\hat{k}.$$

So $\vec{R}(t)$ is continuous at $t = \frac{\pi}{4}$.

(5) Show that the function

$$\vec{r}(t) = \frac{2}{t}\hat{i} + \frac{t^3}{2t^3-5}\hat{j} + \frac{1}{e^t}\hat{k}$$

is continuous at $t \to \infty$.

$$\lim_{t\to\infty}\vec{r}(t) = \lim_{t\to\infty}\left[\frac{2}{t}\hat{i} + \frac{t^3}{2t^3-5}\hat{j} + \frac{1}{e^t}\hat{k}\right]$$

$$= \left(\lim_{t\to\infty}\frac{2}{t}\right)\hat{i} + \left(\lim_{t\to\infty}\frac{t^3}{2t^3-5}\right)\hat{j} + \left(\lim_{t\to\infty}\frac{1}{e^t}\right)\hat{k}$$

$$= 0\,\hat{i} + \lim_{t\to\infty}\frac{\left(\frac{t^3}{t^3}\right)}{\left(\frac{2t^3-5}{t^3}\right)}\hat{j} + 0\,\hat{k}$$

$$= 0\hat{i} + \lim_{t\to\infty}\frac{1}{2-\frac{5}{t^3}}\hat{j} + 0\hat{k}$$

$$= 0\hat{i} + \frac{1}{2-0}\hat{j} + 0\hat{k} = 0\hat{i} + \frac{1}{2}\hat{j} + 0\hat{k}.$$

Q6 For what values of t, the following vector functions are continuous?

(i) $\vec{r}(t) = \ln(t+3)\,\hat{i} + \dfrac{1}{t-1}\,\hat{j} + \dfrac{t+2}{t^2-4}\,\hat{k}$

(ii) $\vec{r}(t) = \dfrac{1}{3t+1}\,\hat{i} + \dfrac{1}{t}\,\hat{j}$.

(i) $\vec{r}(t) = \ln(t+3)\,\hat{i} + \dfrac{1}{t-1}\,\hat{j} + \dfrac{t+2}{t^2-4}\,\hat{k}$.

Let $f(t) = \ln(t+3)$, $g(t) = \dfrac{1}{t-1}$, $h(t) = \dfrac{t+2}{t^2-4}$

$f(t)$ is continuous at $t+3>0$, $t>-3$.

$g(t)$ is continuous at $t-1 \neq 0$, $t \neq 1$.

$h(t)$ is continuous at $t^2-4 \neq 0$, $t^2 \neq 4$, $t \neq \pm 2$.

So $\vec{r}(t)$ is continuous at

$$\{t \in \mathbb{R} : t>-3,\ t \neq 1 \text{ and } t \neq \pm 2\}.$$

(ii) $\vec{r}(t) = \dfrac{1}{3t+1}\,\hat{i} + \dfrac{1}{t}\,\hat{j}$

Let $f(t) = \dfrac{1}{3t+1}$, $g(t) = \dfrac{1}{t}$

$f(t)$ is continuous at $3t+1 \neq 0$, $t \neq \dfrac{-1}{3}$

$g(t)$ is continuous at $t \neq 0$.

So $\vec{r}(t)$ is continuous at

$$\{t \in \mathbb{R} : t \neq -\tfrac{1}{3},\ t \neq 0\}.$$

Exercise 5.2

Q1 Find the derivative of the following vector functions:

(i) $\vec{f}(t) = \ln t^2\,\hat{i} + e^{2t}\,\hat{j} + (2t^2+1)\,\hat{k}$

$$\frac{d}{dt}\left(\vec{f}(t)\right) = \frac{d}{dt}\left[\ln t^2\,\hat{i} + e^{2t}\,\hat{j} + (2t^2+1)\,\hat{k}\right]$$

$(\ln f)' = \frac{1}{f}\cdot f'$

$$= \frac{d}{dt}(\ln t^2)\,\hat{i} + \frac{d}{dt}(e^{2t})\,\hat{j} + \frac{d}{dt}(2t^2+1)\,\hat{k}$$

$(e^f)' = e^f \cdot f'$

$$= \frac{1}{t^{\cancel{2}}}(2\cancel{t})\,\hat{i} + (e^{2t})\,2\,\hat{j} + (4t+0)\,\hat{k}$$

$$= \frac{2}{t}\,\hat{i} + 2e^{2t}\,\hat{j} + 4t\,\hat{k}.$$

(ii) $\vec{f}(t) = (t+1)\,\hat{i} + \ln(t+2)\,\hat{j}$

$$\frac{d}{dt}\vec{f}(t) = \frac{d}{dt}\left[(t+1)\,\hat{i} + \ln(t+2)\,\hat{j}\right]$$

$$= \frac{d}{dt}(t+1)\,\hat{i} + \frac{d}{dt}\left[\ln(t+2)\right]\hat{j}$$

$$= 1\,\hat{i} + \frac{1}{t+2}(1)\,\hat{j}$$

$$= \hat{i} + \frac{1}{t+2}\,\hat{j}.$$

(iii) $\vec{f}(t) = \sec t\,\hat{i} + \cos t^2\,\hat{j} + (t^2+t+1)\,\hat{k}$

$$\frac{d}{dt}\vec{f}(t) = \frac{d}{dt}\left[\sec t\,\hat{i} + \cos t^2\,\hat{j} + (t^2+t+1)\,\hat{k}\right]$$

$$= \frac{d}{dt}(\sec t)\,\hat{i} + \frac{d}{dt}(\cos t^2)\,\hat{j} + \frac{d}{dt}(t^2+t+1)\,\hat{k}$$

$$= \sec t\tan t\,\hat{i} + (-\sin t^2)\,2t\,\hat{j} + (2t+1)\,\hat{k}$$

$$= \sec t\tan t\,\hat{i} - 2t\sin t^2\,\hat{j} + (2t+1)\,\hat{k}.$$

(3) If $\vec{x} = t\hat{i} + 2t\hat{j}$; $\vec{y} = 2t\hat{i} + 3t\hat{k}$ and $\phi(t) = 3t$,

then find the following:

(i) $\frac{d}{dt}\left[\vec{x}(t) - \vec{y}(t)\right] = \frac{d}{dt}\vec{x}(t) - \frac{d}{dt}\vec{y}(t)$

$= \frac{d}{dt}(t\hat{i} + 2t\hat{j}) - \frac{d}{dt}(2t\hat{i} + 3t\hat{k})$

$= (1\hat{i} + 2\hat{j}) - (2\hat{i} + 3\hat{k})$

$= \hat{i} + 2\hat{j} - 2\hat{i} - 3\hat{k}$

$= -\hat{i} + 2\hat{j} - 3\hat{k}$.

(ii) $\frac{d}{dt}\left[\vec{x}(t)\cdot\vec{y}(t)\right] = \vec{x}(t)\cdot\frac{d}{dt}\vec{y}(t) + \frac{d}{dt}\vec{x}(t)\cdot\vec{y}(t)$

$= (t\hat{i} + 2t\hat{j})\cdot\left(\frac{d}{dt}(2t\hat{i} + 3t\hat{k})\right) + \frac{d}{dt}(t\hat{i} + 2t\hat{j})\cdot(2t\hat{i} + 3t\hat{k})$

$= (t\hat{i} + 2t\hat{j})\cdot(2\hat{i} + 3\hat{k}) + (\hat{i} + 2\hat{j})\cdot(2t\hat{i} + 3t\hat{k})$

$= 2t + 0 + 0 + 2t + 0 + 0$

$= 4t$

(iii) $\frac{d}{dt}\left[\vec{x}(t)\times\vec{y}(t)\right] = \vec{x}(t)\times\frac{d}{dt}\vec{y}(t) + \frac{d}{dt}\vec{x}(t)\times\vec{y}(t)$

$= (t\hat{i} + 2t\hat{j})\times\left[\frac{d}{dt}(2t\hat{i} + 3t\hat{k})\right] + \left[\frac{d}{dt}(t\hat{i} + 2t\hat{j})\right]\times(2t\hat{i} + 3t\hat{k})$

$= (t\hat{i} + 2t\hat{j})\times(2\hat{i} + 3\hat{k}) + (\hat{i} + 2\hat{j})\times(2t\hat{i} + 3t\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & 2t & 0 \\ 2 & 0 & 3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 2t & 0 & 3t \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 2t & 0 \\ 0 & 3 \end{vmatrix} - \hat{j}\begin{vmatrix} t & 0 \\ 2 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} t & 2t \\ 2 & 0 \end{vmatrix} + \hat{i}\begin{vmatrix} 2 & 0 \\ 0 & 3t \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 0 \\ 2t & 3t \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ 2t & 0 \end{vmatrix}$$

$= \hat{i}(6t) - \hat{j}(3t) + \hat{k}(-4t) + \hat{i}(6t) - \hat{j}(3t) + \hat{k}(-4t)$

$= 6t\hat{i} - 3t\hat{j} - 4t\hat{k} + 6t\hat{i} - 3t\hat{j} - 4t\hat{k}$

$= 12t\hat{i} - 6t\hat{j} - 8t\hat{k}$.

(iv) $\frac{d}{dt}\left[\phi\,\vec{x}(t)\right] = \phi\frac{d}{dt}\vec{x}(t) + \left(\frac{d}{dt}\phi\right)\vec{x}(t)$

$= 3t\frac{d}{dt}(t\hat{i} + 2t\hat{j}) + \left(\frac{d}{dt}(3t)\right)(t\hat{i} + 2t\hat{j})$

$= 3t(\hat{i} + 2\hat{j}) + 3(t\hat{i} + 2t\hat{j}) = 3t\hat{i} + 6t\hat{j} + 3t\hat{i} + 6t\hat{j}$

$= 6t\hat{i} + 12t\hat{j}$.

3. A particle moves so that its position as a function of time is given by $\vec{r}(t) = \hat{i} + 4t^2\hat{j} + t\hat{k}$.

Write expressions for its
(a) velocity (b) acceleration
as functions of time.

$$\vec{r}(t) = \hat{i} + 4t^2\hat{j} + t\hat{k}.$$

(a) velocity $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}\left(\hat{i} + 4t^2\hat{j} + t\hat{k}\right)$

$= 0\hat{i} + 8t\hat{j} + \hat{k}$

(b) acceleration $\vec{a}(t) = \dfrac{d}{dt}\vec{v}(t) = \dfrac{d}{dt}\left(0\hat{i} + 8t\hat{j} + \hat{k}\right)$

$= 0\hat{i} + 8\hat{j} + 0\hat{k} = 8\hat{j}.$

Q4 The path of a particle is given for time >0 by the parametric equations:

$x = t^2 - 3t$, $\quad y = \frac{2}{3}t^3$.

Find magnitude of velocity and acceleration of particle at $t=5$.

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

$$\vec{r}(t) = (t^2-3t)\hat{i} + \left(\frac{2}{3}t^3\right)\hat{j}$$

Velocity $\quad \vec{v} = \frac{d}{dt}\vec{r}(t) = \frac{d}{dt}\left[(t^2-3t)\hat{i} + \left(\frac{2}{3}t^3\right)\hat{j}\right]$

$$= (2t-3)\hat{i} + \left(\frac{2}{\cancel{3}}\cdot\cancel{3}t^2\right)\hat{j}$$

$$= (2t-3)\hat{i} + 2t^2\hat{j}.$$

$$|\vec{v}| = \sqrt{(2t-3)^2 + (2t^2)^2}$$

$$|\vec{v}|\Big|_{t=5} = \sqrt{(2(5)-3)^2 + (2(5)^2)^2} = \sqrt{49 + 2500}$$

$$= \sqrt{2549}$$

Acceleration $\quad \vec{a} = \frac{d}{dt}\vec{v}(t) = \frac{d}{dt}\left[(2t-3)\hat{i} + 2t^2\hat{j}\right]$

$$= 2\hat{i} + 4t\hat{j}$$

$$|\vec{a}| = \sqrt{2^2 + (4t)^2}$$

$$|\vec{a}|\Big|_{t=5} = \sqrt{2^2 + (4(5))^2} = \sqrt{4 + 400} = \sqrt{404}.$$

(5) A particle moves along the curve $x = 2t^2$ and $y = 4t$.
Find the components of velocity and acceleration at $t = 2$ in the direction of $2\hat{i} + \hat{j}$.

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$
$$\vec{r}(t) = 2t^2\hat{i} + 4t\hat{j}$$

Velocity $\vec{v}(t) = \dfrac{d}{dt}\vec{r}(t) = \dfrac{d}{dt}\left(2t^2\hat{i} + 4t\hat{j}\right)$

$$= 4t\hat{i} + 4\hat{j}$$

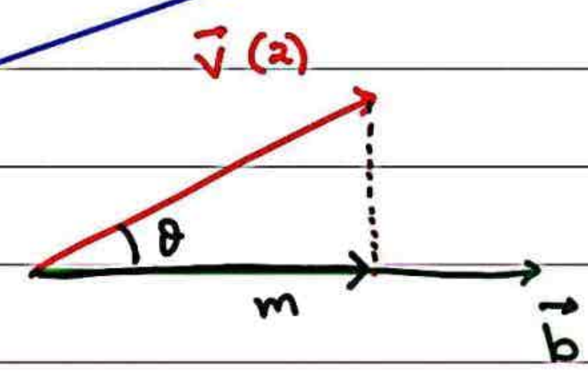
$$\vec{v}(2) = 4(2)\hat{i} + 4\hat{j} = 8\hat{i} + 4\hat{j}$$

Acceleration $\vec{a}(t) = \dfrac{d}{dt}\vec{v}(t) = \dfrac{d}{dt}\left(4t\hat{i} + 4\hat{j}\right)$

$$= 4\hat{i} + 0\hat{j}$$

$$\vec{a}(2) = 4\hat{i} + 0\hat{j}.$$

Let $\vec{b} = 2\hat{i} + \hat{j}$.



$$\vec{v}(2)\cdot\vec{b} = |\vec{v}(2)||\vec{b}|\cos\theta$$

$$\cos\theta = \frac{\vec{v}(2)\cdot\vec{b}}{|\vec{v}(2)||\vec{b}|}$$

$$\cos\theta = \frac{m}{|\vec{v}(2)|}$$

$$\frac{\vec{v}(2)\cdot\vec{b}}{\cancel{|\vec{v}(2)|}|\vec{b}|} = \frac{m}{\cancel{|\vec{v}(2)|}}$$

$$m = \frac{\vec{v}(2)\cdot\vec{b}}{|\vec{b}|}$$

Component of $\vec{v}(2)$ in the direction of $\vec{b}$.

Component of velocity at $t=2$ in the direction of $\vec{b}$

$$= \frac{\vec{v}(2)\cdot\vec{b}}{|\vec{b}|}$$

$$= \frac{(8\hat{i} + 4\hat{j})\cdot(2\hat{i} + \hat{j})}{\sqrt{2^2 + 1^2}} = \frac{16 + 4}{\sqrt{5}} = \frac{20}{\sqrt{5}}$$

Component of acceleration at $t = 2$ in the direction of $\vec{b} = \dfrac{\vec{a}(2)\cdot\vec{b}}{|\vec{b}|}$

$$= \frac{(4\hat{i} + 0\hat{j})\cdot(2\hat{i} + \hat{j})}{\sqrt{2^2 + 1^2}} = \frac{8 + 0}{\sqrt{5}} = \frac{8}{\sqrt{5}}.$$