MATHEMATICS 2nd YEAR

UNIT # 04

INTRODUCTION TO ANALYTIC GEOMETRY

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Contents

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اچھی باتیں

1۔جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔جو دوگے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

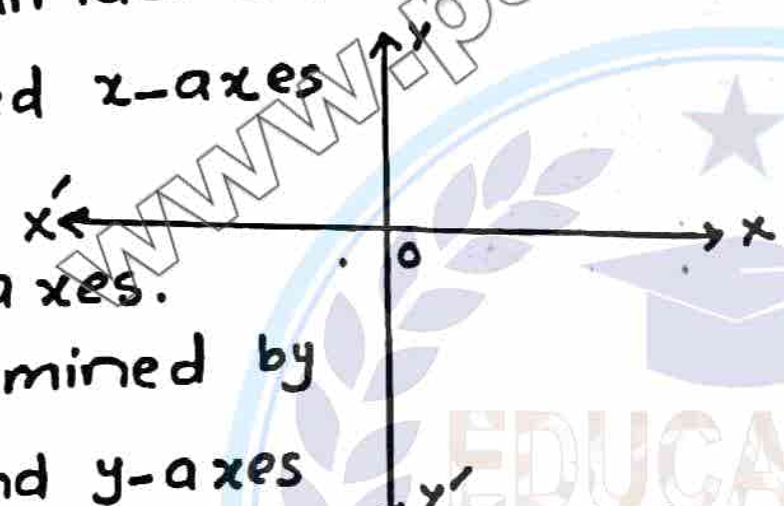
5۔جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Geometry:- The geometry is derived from two Greek words Geo (Earth) and Metron (Measurement). It means knowledge of measurement of earth.

* Geometry is branch of mathematics that deals the shape and size of things.

Analytic geometry:- In analytic geometry or coordinates geometry, points could be represented by numbers, lines and curves represented by equations. A French philosopher and mathematician Rene Descartes (1596-1650 A.D) introduced algebraic methods in geometry named as analytical geometry (or coordinate geometry).

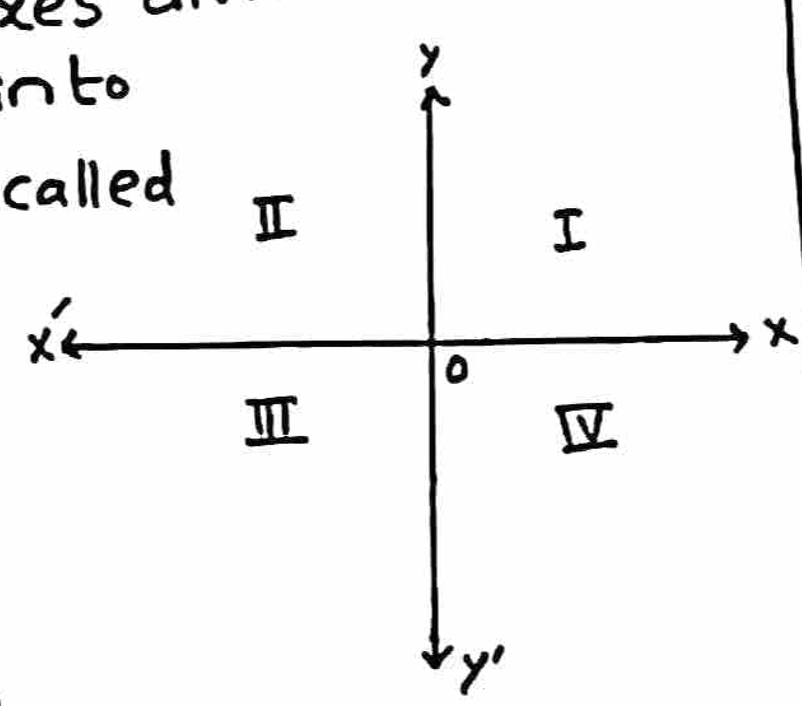
Coordinate System:- Draw in a plane two mutually number lines xx' and yy', one horizontal and the other vertical. Let their point of intersection be O called origin and real number 0 of both lines is represented by O. The two lines are called the coordinate axes. The horizontal line xox' is called x-axes and vertical line yoy' is called y-axes.



The plane determined by both x-axes and y-axes is called xy-plane or cartesian plane.

* If (x, y) are coordinates of a point P then first member of ordered pair (i-e., x) is called x-coordinate or abscissa of point P, and second member of ordered pair (i-e., y) is called y-coordinate or ordinate of point P.

* The coordinate axes divide the coordinate plane into four equal parts, called quadrants.



Quadrant I:-
$\{(x,y) \mid x>0, y>0\}$

Quadrant II:-
$\{(x,y) \mid x<0, y>0\}$

Quadrant III:-
$\{(x,y) \mid x<0, y<0\}$

Quadrant IV:-
$\{(x,y) \mid x>0, y<0\}$

Note:- On x-axis ordinate is zero i-e., $y = 0$

Also on y-axis absissa is zero i-e., $x = 0$

The Distance Formula

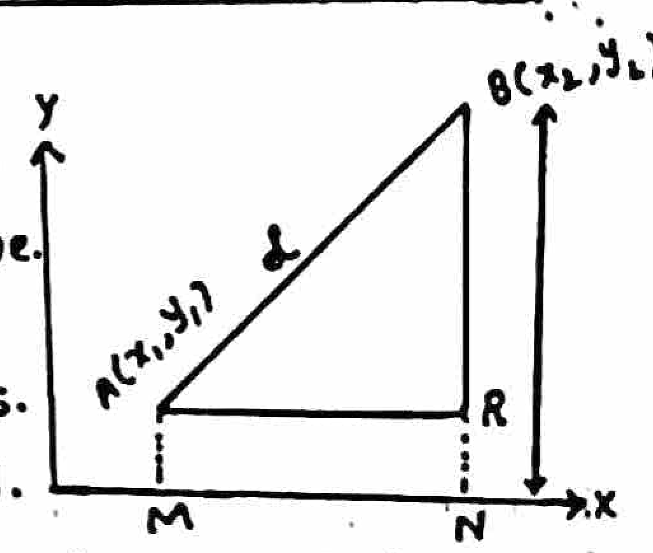
The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in xy-plane is

$$|AB| = d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Note:- AB stands for $m\overline{AB}$ or $|\overline{AB}|$ and d stands for distance

Proof:-

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in xy-plane. Draw $\perp$ars AM and BN from A and B on x-axis. Also draw $\perp$ar AR on BN.



In right $\triangle ABR$, using Pathagoras theorem,

$$|AB|^2 = |AR|^2 + |BR|^2$$

$\because |AR| = |MN| = |ON| - |OM|$

$\rightarrow |AR| = x_2 - x_1$

$\rightarrow |AB|^2 = (x_2-x_1)^2 + (y_2-y_1)^2$

$|BR| = |BN| - |RN| = y_2 - y_1$

$\rightarrow d^2 = |AB|^2 = (x_2-x_1)^2+(y_2-y_1)^2$

$\rightarrow d = |AB| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Example 1. Show that the points $A(-1,2)$, $B(7,5)$ and $C(2,-6)$ are vertices of a right triangle.

Solution:- $\because A(-1,2), B(7,5), C(2,-6)$

$|AB| = \sqrt{(7-(-1))^2+(5-2)^2} = \sqrt{(8)^2+(3)^2}$

$= \sqrt{64+9} = \sqrt{73}$

$|BC| = \sqrt{(2-7)^2+(-6-5)^2} = \sqrt{(-5)^2+(-11)^2} = \sqrt{25+121}$

$= \sqrt{146}$

$|AC| = \sqrt{(2-(-1))^2+(-6-2)^2} = \sqrt{(2+1)^2+(-8)^2}$

$= \sqrt{9+64} = \sqrt{73}$

$\because |AB|^2 = 73,\ |BC|^2 = 146,\ |AC|^2 = 73$

$\rightarrow |AB|^2 + |AC|^2 = 73 + 73 = 146$

$\rightarrow |AB|^2 + |AC|^2 = |BC|^2$ (Pathagoras theorem)

Hence proved.

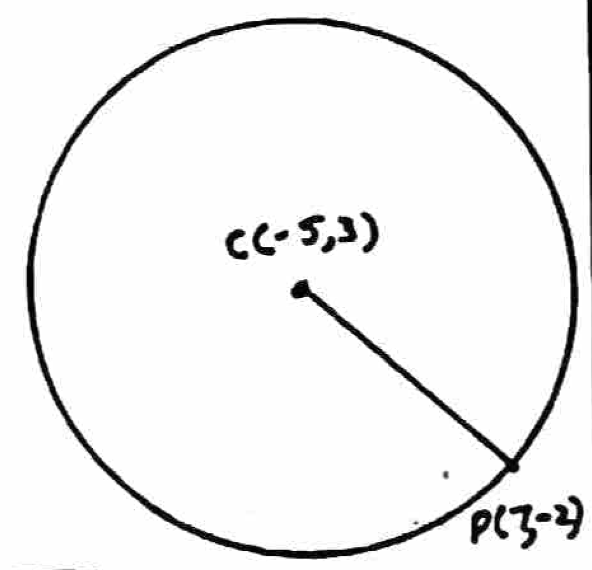
Example 2. The point $C(-5,3)$ is the centre of a circle and $P(7,-2)$ lies on the circle. What is the radius of the circle?

Solution:-

$\because$ distance formula

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

from fig., radius $=|CP|$ so



$$|CP|=\sqrt{(7-(-5))^2+(-2-3)^2}$$

$$=\sqrt{(7+5)^2+(-5)^2}=\sqrt{(12)^2+(5)^2}=\sqrt{144+25}$$

$|CP|=\sqrt{169}=13$ so radius of circle is 13.

Point Dividing the Join of Two points in a given Ratio (Internally).

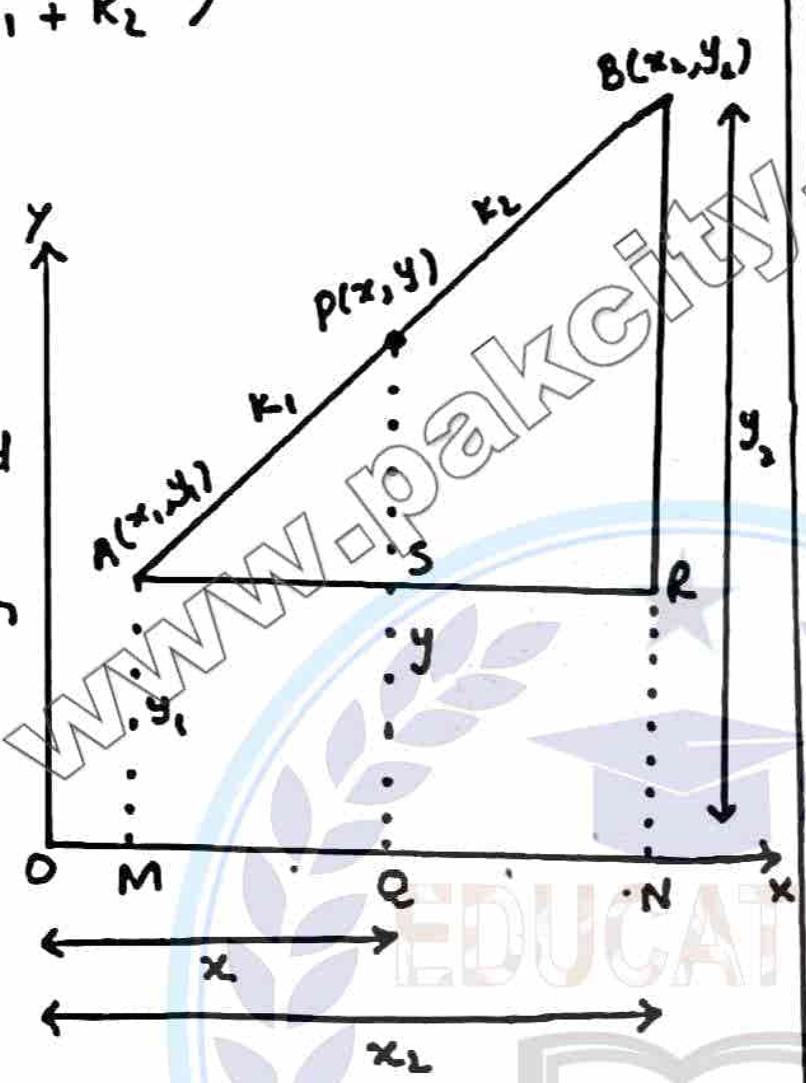
Theorem:- Let $A(x_1,y_1)$ and $B(x_2,y_2)$ be two given points in a plane. The coordinates of the point dividing the line segment AB in the ratio $k_1:k_2$ are

$$\left(\frac{k_1x_2+k_2x_1}{k_1+k_2},\ \frac{k_1y_2+k_2y_1}{k_1+k_2}\right)$$

Proof:-

Let $P(x,y)$ be the point, which divides AB in ratio $k_1:k_2$

Draw $\perp$ars AM, PQ and BN from A, P and B on x-axis as shown in fig. Here ΔAPS and ΔABR are similar triangles. So In ΔABR



$AP:PB=AS:SR$

$\Rightarrow \dfrac{AP}{PB}=\dfrac{AS}{SR}$ ——— (I)

$\because AS=MQ=OQ-OM$
$=x-x_1$

$SR=QN=ON-OQ$
$=x_2-x$

So $\dfrac{k_1}{k_2}=\dfrac{x-x_1}{x_2-x}$ $\quad(\because AP:PB=k_1:k_2)$

$\Rightarrow k_1(x_2-x)=k_2(x-x_1)$

$\Rightarrow k_1x_2-k_1x=k_2x-k_2x_1$

$\Rightarrow k_1x_2+k_2x_1=k_2x+k_1x$

$\Rightarrow k_1x_2+k_2x_1=x(k_2+k_1)$

$\Rightarrow x=\dfrac{k_1x_2+k_2x_1}{k_1+k_2}$

Similarly, by drawing $\perp$ars from A, P and B on y-axis we will get $y=\dfrac{k_1y_2+k_2y_1}{k_1+k_2}$

Thus $P\left(\dfrac{k_1x_2+k_2x_1}{k_1+k_2},\ \dfrac{k_1y_2+k_2y_1}{k_1+k_2}\right)$ is required point.

Note:-

(i) Two geometric figures are similar if one is enlargement of other

(ii) In two triangles, if two corresponding angles are congruent, then triangles are similar

(iii) If the directed distances AP and PB have the same sign, then their ratio is positive and P is said to divide AB internally.

(iv) If the directed distances AP and PB have opposite signs i.e., P is beyond AB, then their ratio is negative and P is said to divide AB externally. $\dfrac{AP}{BP}=\dfrac{k_1}{k_2}$ or $\dfrac{AP}{PB}=-\dfrac{k_1}{k_2}$

In this case we can show that

$$x=\frac{k_1x_2-k_2x_1}{k_1-k_2},\quad y=\frac{k_1y_2-k_2y_1}{k_1-k_2}$$

Thus P is said to divide the line segment AB in ratio $k_1:k_2$ internally or externally according as P lies b/w AB or beyond AB.

(v) If $k_1:k_2=1:1$, then P becomes mid-point of $\overline{AB}$ and coordinates of P are: $x=\dfrac{x_1+x_2}{2}$, $y=\dfrac{y_1+y_2}{2}$

(vi) The above theorem is valid in whichever quadrant A and B lie

Example 1. Find the coordinates of the point that divides the join of $A(-6,3)$ and $B(5,-2)$ in the ratio $2:3$

(i) Internally (ii) Externally

Solution:- Here $k_1=2$, $k_2=3$, $x_1=-6$ and $x_2=5$

(i) We know that if $P(x,y)$ divides AB internally with ratio $k_1:k_2$ so

$$P(x,y)=\left(\frac{k_1x_2+k_2x_1}{k_1+k_2},\ \frac{k_1y_2+k_2y_1}{k_1+k_2}\right)$$

$$=\left(\frac{(2)(5)+(3)(-6)}{5},\ \frac{2(-2)+3(3)}{5}\right)$$

$$P(x,y)=\left(\frac{10-18}{5},\ \frac{-4+9}{5}\right)=\left(-\frac{8}{5},\frac{5}{5}\right)=\left(\frac{-8}{5},1\right)$$

(ii) $\because$ We know that if $P(x,y)$ divides AB externally with ratio $k_1:k_2$ so

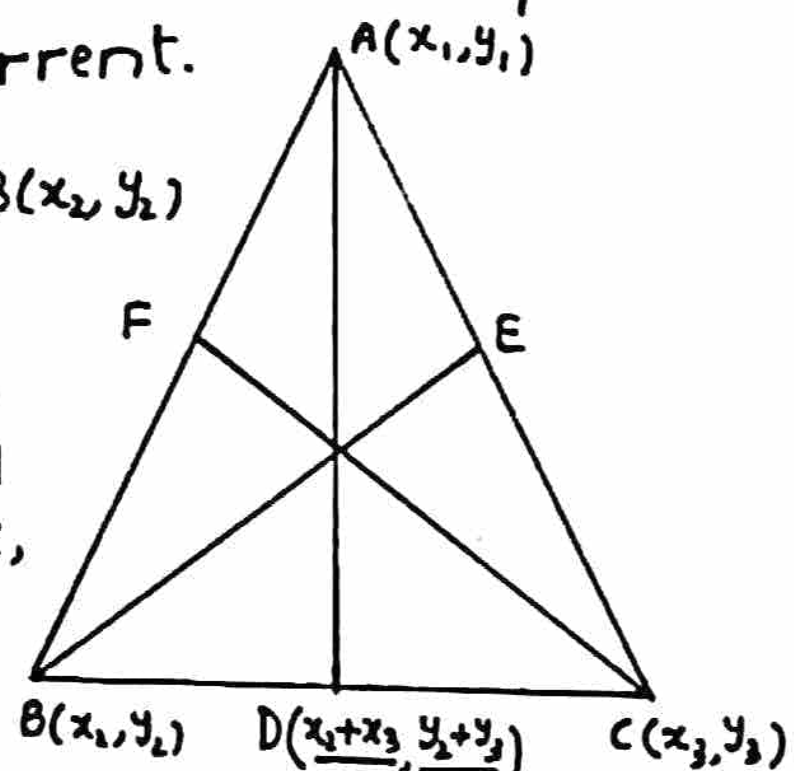
$$P(x,y)=\left(\frac{k_1x_2-k_2x_1}{k_1-k_2},\ \frac{k_1y_2-k_2y_1}{k_1-k_2}\right)$$

$$=\left(\frac{(2)(5)-3(-6)}{2-3},\ \frac{2(-2)-3(3)}{2-3}\right)$$

$$P(x,y)=\left(\frac{10+18}{-1},\ \frac{-4-9}{-1}\right)=(-28,13)$$

Remember, ★ Line segment joining one vertex of a triangle to the mid-point of an opposite side of the triangle is called median
★ A point that divides each median in ratio 2:1 is called centroid.
★ The point of concurrency of medians is called centroid
★ When two or more than two lines meet at a point, then they are said to be concurrent.

Theorem:- Show that medians of a triangle are concurrent.



Proof:- Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ be vertices of a ΔABC.
Let D, E and F be mid points of sides BC, AC and AB resp.
So AD, BE and CF are medians of ΔABC.
$\because$ mid point of BC is $D\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$

Let P be the point dividing BC in ratio 2:1 so using formula $\left(\frac{k_1x_2+k_2x_1}{k_1+k_2}, \frac{k_1y_2+k_2y_1}{k_1+k_2}\right)$

so coordinates of p in the ratio 2:1 are

$$\left(\frac{2\left(\frac{x_2+x_3}{2}\right)+(1)(x_3)}{2+1}, \frac{2\left(\frac{y_2+y_3}{2}\right)+(1)(y_3)}{2+1}\right)$$

$$\Rightarrow P\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

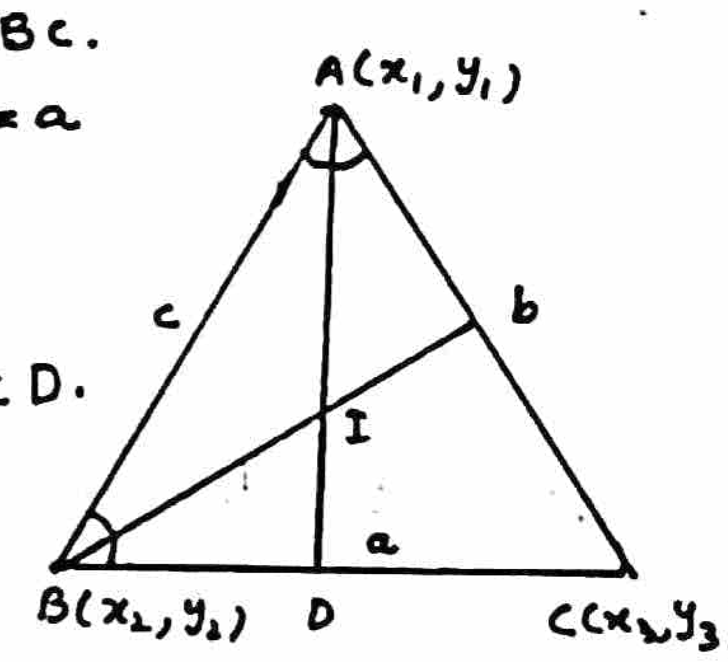
Similarly, it can be proved that coordinates of point that divides medians BE and CF each in 2:1 are $P\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

$\Rightarrow$ Hence medians of a triangle are concurrent.

Remember, ★ A line that divides an angle into two equal parts is called angle bisector.
★ An angle bisector divides line opposite to it into a ratio, equal to ratio of remaining two sides.
★ In figure AD is angle bisector of $\angle A$. The side opposite to $\angle A$ is BC. so
$BD : DC = BA : AC \Rightarrow BD : DC = c : b$ $\left(\because \begin{matrix} BA=c \\ AC=b \end{matrix}\right)$

Theorem:- Bisectors of angles of a triangle are concurrent.

Proof:- Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ be vertices of ΔABC.
then $|AB| = c$, $|BC| = a$
$|AC| = b$



Let bisector $\angle A$ meet BC at point D.
Now

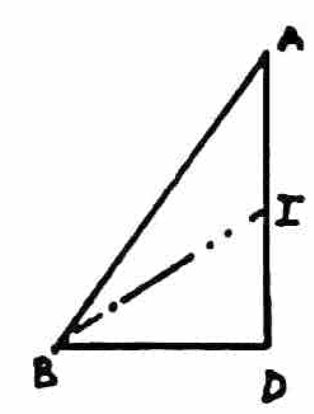
$$\frac{BD}{DC} = \frac{BA}{AC}$$

$\Rightarrow \frac{BD}{DC} = \frac{c}{b}$ ——(I) $(\because |BA| = c, |DC| = b)$

$\Rightarrow BD : DC = c : b$ It means D divides BC in c : b

using ratio formula coordinates of D are $\left(\frac{bx_2+cx_3}{b+c}, \frac{by_2+cy_3}{b+c}\right)$

Let angle bisector of $\angle B$ intersects AD at point I. then



$$\frac{AI}{ID} = \frac{AD}{BD}$$

$\Rightarrow \frac{AI}{ID} = \frac{c}{BD}$ ——(II) $\because |AB| = c$

Now Take reciprocal of eq (I)
i.e., $\frac{DC}{BD} = \frac{b}{c} \Rightarrow 1 + \frac{DC}{BD} = 1 + \frac{b}{c}$

$\Rightarrow \frac{BD+DC}{BD} = \frac{b+c}{c}$ $(\because BD+DC = BC)$

$\Rightarrow \frac{BC}{BD} = \frac{b+c}{c} \Rightarrow \frac{a}{BD} = \frac{b+c}{c}$ $\because |BC| = a$

$\Rightarrow \frac{BD}{a} = \frac{c}{b+c} \Rightarrow BD = \frac{ac}{b+c}$

So (II) $\Rightarrow \frac{AI}{ID} = \frac{c}{\frac{ac}{b+c}} = c\left(\frac{b+c}{ac}\right)$

$\frac{AI}{ID} = \frac{b+c}{a} \Rightarrow AI : ID = (b+c) : a$

By ratio formula,

$$I\left(\frac{(b+c)\left(\frac{bx_2+cx_3}{b+c}\right)+ax_1}{b+c+a}, \frac{(b+c)\left(\frac{by_2+cy_3}{b+c}\right)+ay_1}{b+c+a}\right)$$

$$\Rightarrow I\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

Similarly, it can be prove that bisector of $\angle C$ will also pass through Point I.

$\Rightarrow$ Hence bisectors of angles of a triangle are concurrent.

Exercise 4.1

Q1. Describe the location in the plane of the point $P(x,y)$ for which :

(i) $x>0$ (ii) $x>0$ and $y>0$ (iii) $x=0$

(iv) $y=0$ (v) $x<0$ and $y\geq 0$ (vi) $x=y$

(vii) $|x|=|y|$ (viii) $|x|\geq 3$ (ix) $x>2$ and $y=2$

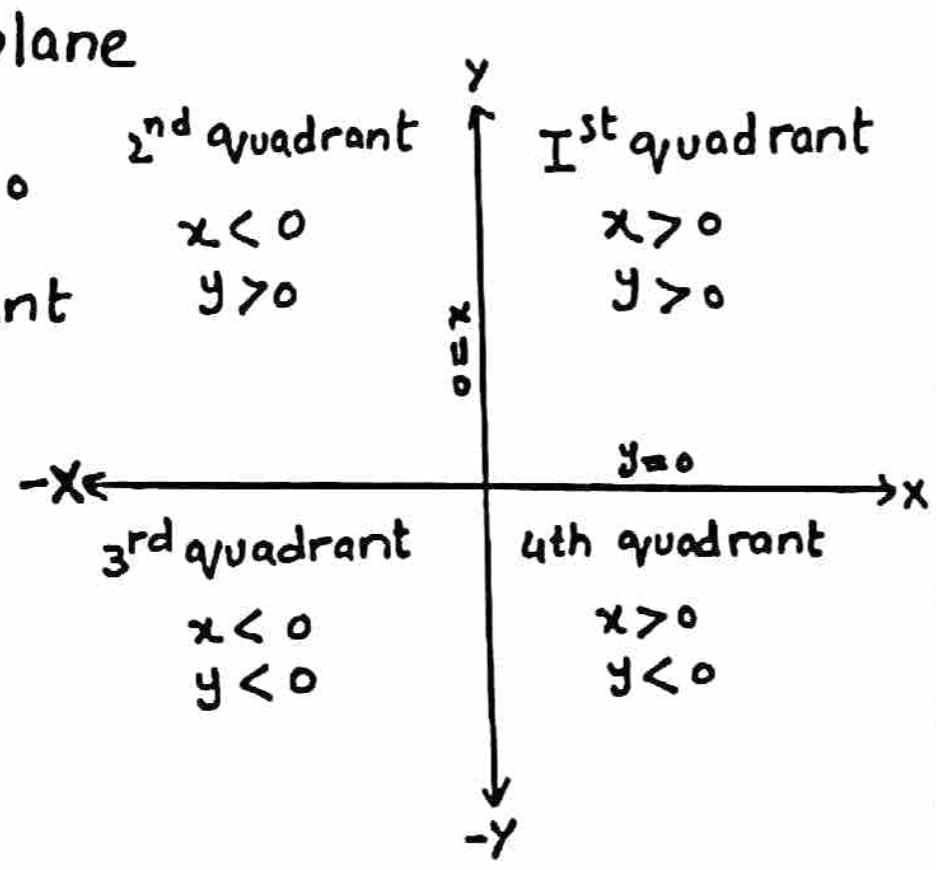
(x) x and y have opposite signs.

Solution:- (i) $x>0$

Right half plane

(ii) $x>0$ and $y>0$

The I^{st} quadrant



(iii) $x=0$

y-axis

(iv) $y=0$

x-axis

(v) $x<0$ and $y\geq 0$

2^{nd} quadrant and negative x-axis
($\because$ $x<0$ and $y>0$ in 2^{nd} quad.)

(vi) $x=y$

It is a line bisecting I^{st} and 3^{rd} quadrant. ($\because$ (1,1), (2,2), (-3,-3), (-4,-4))

(vii) $|x|=|y|$ I^{st} and 3^{rd} quadrant

or 2^{nd} and 4^{th} quadrant

($\because$ (1,1), (-1,-1) for I and III and (1,-1), (-1,1) for II and IV.)

(viii) $|x|\geq 3$

On x-axis less than equal to -3 and greater than equal to 3

($\because$ $|x|\geq 3 \Rightarrow \pm x \geq 3 \Rightarrow x\geq 3$ and $-x\geq 3$

$\Rightarrow x\geq 3$ and $x\leq -3$)

(ix) $x>2$ and $y=2$

In I^{st} quad x greater than 2 and $y=2$

(x) x and y have opposite signs

The II and IV quadrants x and y have opposite sign (In II (-2, 2) and in IV (2,-2))

Q2. Find in each of the following:
(i) the distance between two given points
(ii) midpoint of line segment joining the two points

(a) $A(3,1)$; $B(-2,-4)$ (b) $A(-8,3)$; $B(2,-1)$

(c) $A(-\sqrt{5},-\frac{1}{3})$; $B(-3\sqrt{5},5)$

Solution:- (a) $A(3,1)$; $B(-2,-4)$

(i) $|AB| = \sqrt{(-2-3)^2+(-4-1)^2} = \sqrt{(-5)^2+(-5)^2}$

$|AB| = \sqrt{50} = \sqrt{25\times 2} = 5\sqrt{2}$

(ii) Mid poin of AB $= \left(\frac{3-2}{2}, \frac{1-4}{2}\right)$

$= \left(\frac{1}{2}, \frac{-3}{2}\right)$ $\because$ mid point of AB $= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

(b) $A(-8,3)$, $B(2,-1)$

(i) $|AB| = \sqrt{(2+8)^2+(-1-3)^2} = \sqrt{100+16}$

$= \sqrt{116} = \sqrt{4\times 29} = 2\sqrt{29}$

(ii) Mid point of $AB = \left(\frac{-8+2}{2}, \frac{3-1}{2}\right)$

$= \left(-\frac{6}{2}, \frac{2}{2}\right) = (-4, 1)$

(c) $A(-\sqrt{5}, -\frac{1}{3})$, $B(-3\sqrt{5}, 5)$

(i) $|AB| = \sqrt{(-3\sqrt{5}-(-\sqrt{5}))^2 + (5+\frac{1}{3})^2}$

$= \sqrt{(-3\sqrt{5}+\sqrt{5})^2 + (\frac{15+1}{3})^2}$

$= \sqrt{(-2\sqrt{5})^2 + (\frac{16}{3})^2} = \sqrt{4(5)+\frac{256}{9}}$

$= \sqrt{20+\frac{256}{9}} = \sqrt{\frac{180+256}{9}}$

$= \sqrt{\frac{436}{9}} = \frac{\sqrt{4\times 109}}{3} = \frac{2}{3}\sqrt{109}$

(ii) Mid point of $AB = \left(\frac{-\sqrt{5}-3\sqrt{5}}{2}, \frac{-\frac{1}{3}+5}{2}\right)$

$= \left(-\frac{4\sqrt{5}}{2}, \frac{-1+15}{3\times 2}\right) = \left(-2\sqrt{5}, \frac{14}{6}\right) = \left(-2\sqrt{5}, \frac{7}{3}\right)$

Q3. Which of the following points are at a distance of 15 units from the origin?

(a) $(\sqrt{176}, 7)$ (b) $(10,-10)$ (c) $(1,15)$ (d) $(\frac{15}{2}, \frac{15}{2})$

Solution:- (a) Let $A(\sqrt{176}, 7)$ and $O(0,0)$

$|OA| = \sqrt{(\sqrt{176}-0)^2+(7-0)^2} = \sqrt{176+49} = \sqrt{225} = 15$

$\Rightarrow |OA| = 15$ so A is at a distance of 15 units from origin.

(b) Let $B(10,-10)$ and $O(0,0)$ so

$|OB| = \sqrt{(10-0)^2+(-10-0)^2} = \sqrt{(10)^2+(-10)^2}$

$= \sqrt{100+100} = \sqrt{200} = 10\sqrt{2}$

$\Rightarrow |OB| = 10\sqrt{2}$ so B is not at a distance of 15 units from origin.

(c) Let $C(1,15)$ and $O(0,0)$ so

$|OC| = \sqrt{(1-0)^2+(15-0)^2} = \sqrt{1+225} = \sqrt{226}$

so $|OC| \neq 15$ Thus C is not at a distance of 15 units from origin.

(d) Let $D(\frac{15}{2}, \frac{15}{2})$ and $O(0,0)$ so

$|OD| = \sqrt{(\frac{15}{2}-0)^2+(\frac{15}{2}-0)^2} = \sqrt{\frac{225}{4}+\frac{225}{4}} = \sqrt{\frac{450}{4}}$

$|OD| = \frac{\sqrt{225\times 2}}{2} = \frac{15\sqrt{2}}{2} = \frac{15}{\sqrt{2}} \neq 15$

Thus D is not at distance of 15 units from origin.

Q4. Show that (i) the points $A(0,2)$, $B(\sqrt{3},-1)$ and $C(0,-2)$ are vertices of a right triangle.

Solution:- $\because A(0,2), B(\sqrt{3},-1), C(0,-2)$

$|AB| = \sqrt{(\sqrt{3}-0)^2+(-1-2)^2} = \sqrt{3+9} = \sqrt{12}$

$\rightarrow |AB|^2 = 12$

$|BC| = \sqrt{(0-\sqrt{3})^2+(-2-(-1))^2} = \sqrt{3+(-2+1)^2} = \sqrt{3+1}$

$\rightarrow |BC| = \sqrt{4} \rightarrow |BC|^2 = 4$

$|AC| = \sqrt{(0-0)^2+(-2-2)^2} = \sqrt{0+(-4)^2} = \sqrt{0+16}$

$\rightarrow |AC| = \sqrt{16} \rightarrow |AC|^2 = 16$

$\because |AB|^2+|BC|^2 = 12+4 = 16$

$\rightarrow |AB|^2+|BC|^2 = |AC|^2$ (Pythagoras theorem).

→ Hence given points are vertices of a right triangle.

Remember, (i) A triangle having two sides equal in length (but not equal to third side) is called an isosceles triangle.
(ii) In an isosceles triangle, angles opposite to the equal sides are also equal.

(ii) the points $A(3,1)$, $B(-2,-3)$ and $C(2,2)$ are vertices of an isosceles triangle.

Solution:- $\because A(3,1), B(-2,-3), C(2,2)$ Now

$|AB| = \sqrt{(-2-3)^2+(-3-1)^2} = \sqrt{(-5)^2+(-4)^2} = \sqrt{25+16}$

$\rightarrow |AB| = \sqrt{41}$

$|BC| = \sqrt{(2-(-2))^2+(2-(-3))^2} = \sqrt{(2+2)^2+(2+3)^2} = \sqrt{16+25}$

$\rightarrow |BC| = \sqrt{41}$

$|AC| = \sqrt{(2-3)^2+(2-1)^2} = \sqrt{(-1)^2+(1)^2} = \sqrt{2}$

$\because |AB| = |BC|$ but $|AB| \neq |BC|$ and $|BC| \neq |AC|$

→ given points are vertices of isosceles triangle.

(iii) the points $A(5,2)$, $B(-2,3)$, $C(-3,-4)$ and $D(4,-5)$ are vertices of a parallelogram. Is the parallelogram a square?

Solution:- $\because A(5,2), B(-2,3), C(-3,4), D(4,-5)$

Now $|AB| = \sqrt{(-2-5)^2+(3-2)^2} = \sqrt{(-7)^2+(1)^2} = \sqrt{49+1} = \sqrt{50}$

$|BC| = \sqrt{(-3-(-2))^2+(-4-3)^2} = \sqrt{(-3+2)^2+(-7)^2} = \sqrt{(-1)^2+49} = \sqrt{50}$

$|CD| = \sqrt{(4-(-3))^2+(-5-(-4))^2} = \sqrt{(7)^2+(-1)^2} = \sqrt{49+1} = \sqrt{50}$

$|AD| = \sqrt{(4-5)^2+(-5-2)^2} = \sqrt{(-1)^2+(-7)^2} = \sqrt{1+49} = \sqrt{50}$

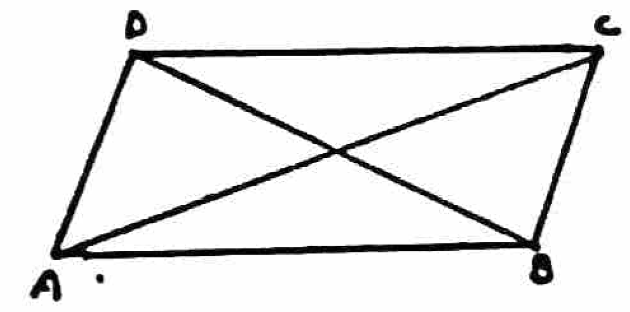
Here $|AB| = |CD|$ and $|AD| = |BC|$

→ so ABCD is ||gram.

Now ABCD will be square if $|AC| = |BD|$

so $|AC| = \sqrt{(-3-5)^2+(-4-2)^2} = \sqrt{(-8)^2+(-6)^2}$

$= \sqrt{64+36} = \sqrt{100} = 10$



$|BD| = \sqrt{(4-(-2))^2+(-5-3)^2}$

$= \sqrt{(6)^2+(-8)^2}$

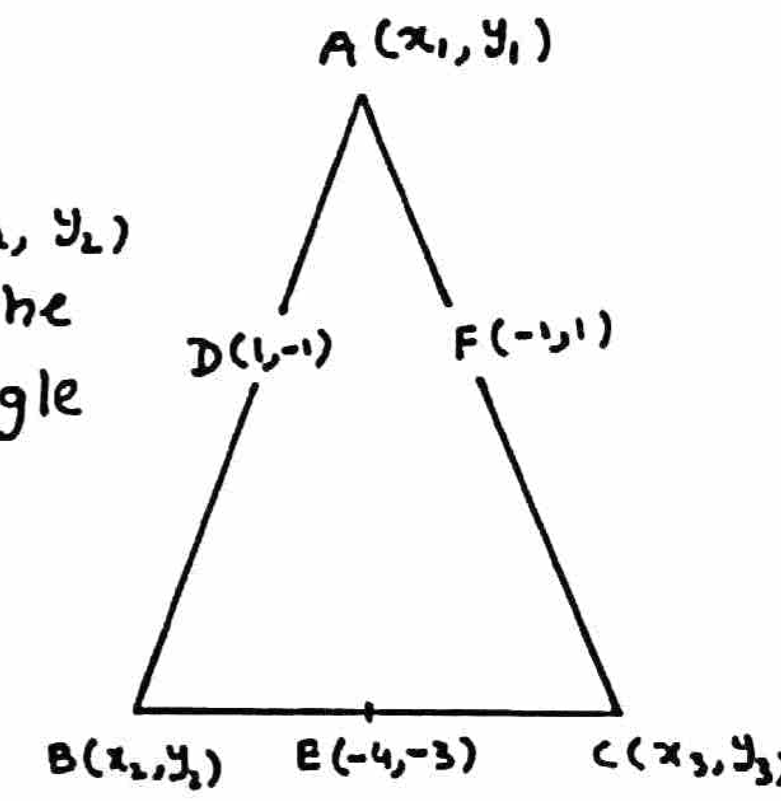
$= \sqrt{36+64} = \sqrt{100}$

$|BD| = 10$

$\because$ diagonals are equal in length, so ABCD is a square.

Q5. The mid points of the sides of a triangle are $(1,-1)$, $(-4,-3)$ and $(-1,1)$. Find coordinates of the vertices of the triangle.

Solution:-



Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ be the vertices of triangle ABC.

$\because$ D is mid point of AB so

$1 = \frac{x_1+x_2}{2}$ and $-1 = \frac{y_1+y_2}{2}$

$\rightarrow x_1+x_2 = 2$ —(i) and $y_1+y_2 = -2$ —(ii)

$\because$ E is mid point of BC so

$-4 = \frac{x_2+x_3}{2}$ and $-3 = \frac{y_2+y_3}{2}$

$\rightarrow x_2+x_3 = -8$ —(iii) and $y_2+y_3 = -6$ —(iv)

$\because$ F is mid point of AC so

$-1 = \frac{x_1+x_3}{2}$ and $1 = \frac{y_1+y_3}{2}$

$\rightarrow x_1+x_3 = -2$ —(v) and $y_1+y_3 = 2$ —(vi)

By (i) − (iii) → $x_1 + \cancel{x_2} = 2$

$\underline{-\cancel{x_2} \pm x_3 = \mp 8}$

$x_1 - x_3 = 10$ —(vii)

By (v) + (vii) → $x_1+x_3 = -2$

$\underline{x_1 - x_3 = 10}$

$2x_1 = 8 \rightarrow \boxed{x_1 = 4}$ put in (i)

so (i) → $4+x_2 = 2 \rightarrow x_2 = 2-4 = -2$

→ $\boxed{x_2 = -2}$ put in (iii) → $-2+x_3 = -8$

→ $x_3 = -8+2 \rightarrow \boxed{x_3 = -6}$

Now by (ii) − (iv) → $y_1+y_2 = -2$

$\underline{\underset{-}{y_2}+\underset{-}{y_3} = \underset{+}{-6}}$

$y_1 - y_3 = 4$ —(viii)

By (vi) + (viii) → $y_1 + y_3 = 2$

$\underline{y_1 - y_3 = 4}$

$2y_1 = 6 \rightarrow \boxed{y_1 = 3}$ put in (ii) so $3+y_2=-2 \rightarrow \boxed{y_2=-5}$ put in (iv)

(iv) $\rightarrow -5+y_3=-6 \rightarrow y_3=-6+5 \rightarrow \boxed{y_3=-1}$

Thus $A(x_1,y_1)=(4,3)$, $B(x_2,y_2)=(-2,-5)$
$C(x_3,y_3)=(-6,-1)$ are required vertices.

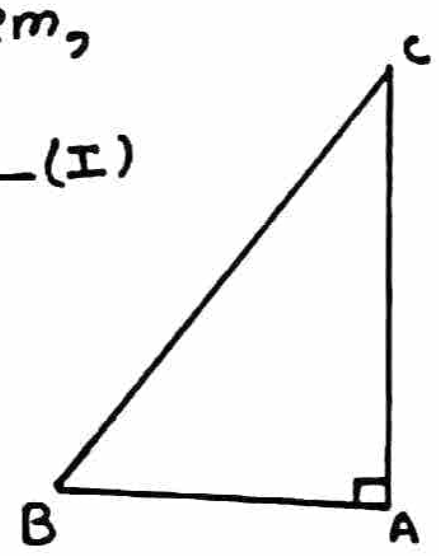
Q6. Find h such that the points $A(\sqrt{3},-1)$, $B(0,2)$ and $C(h,-2)$ are vertices of a right triangle with right angle at the vertex A.

Solution:- $\because A(\sqrt{3},-1), B(0,2), C(h,-2)$

$\because$ given that right angle is at vertex A.

so by pathagoras theorem,

$$|BC|^2=|AC|^2+|AB|^2 \text{ ——— (I)}$$



Now
$$|AB|=\sqrt{(0-\sqrt{3})^2+(2-(-1))^2}$$
$$=\sqrt{(-\sqrt{3})^2+(3)^2}$$
$$|AB|=\sqrt{3+9} \rightarrow |AB|^2=12$$
$$|BC|=\sqrt{(h-0)^2+(-2-2)^2}=\sqrt{h^2+(-4)^2}$$
$$|BC|=\sqrt{h^2+16} \Rightarrow |BC|^2=h^2+16$$

$$|AC|=\sqrt{(h-\sqrt{3})^2+(-2+1)^2}=\sqrt{h^2+3-2h\sqrt{3}+(-1)^2}$$
$$|AC|=\sqrt{h^2+3-2h\sqrt{3}+1}$$
$$\rightarrow |AC|^2=h^2-2h\sqrt{3}+4$$

so (I) becomes as

$$\rightarrow \cancel{h^2}+16=\cancel{h^2}-2h\sqrt{3}+4+12$$
$$\rightarrow 2h\sqrt{3}=16-16 \rightarrow 2h\sqrt{3}=0$$
$$\rightarrow 2\sqrt{3}h=0 \rightarrow h=0 \quad \because 2\sqrt{3}\neq 0$$

Remember, (i) points lying on the same line are called collinear points.
(ii) The points $A(x_1,y_1)$ and $B(x_2,y_2)$ and $C(x_3,y_3)$ collinear if slope of AB = slope of AC (and slope of $AB=\frac{y_2-y_1}{x_2-x_1}$)
(iii) The points $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are collinear if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0$$

Q7. Find h such that $A(-1,h)$, $B(3,2)$ and $C(7,3)$ are collinear.

Solution:- $\because A(-1,h)$, $B(3,2)$ and $C(7,3)$ are collinear $\therefore$ so

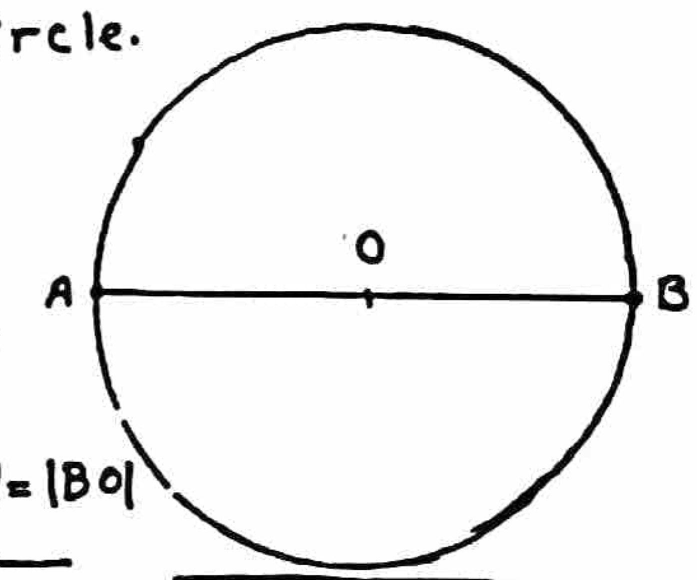
$$\begin{vmatrix} -1 & h & 1 \\ 3 & 2 & 1 \\ 7 & 3 & 1 \end{vmatrix}=0$$

$$\rightarrow -1(2-3)-h(3-7)+1(9-14)=0$$
$$\rightarrow (-1)(-1)-h(-4)+1(-5)=0 \rightarrow 1+4h-5=0$$
$$\rightarrow 4h-4=0 \rightarrow 4h=4 \rightarrow h=1$$

Q8. The points $A(-5,-2)$ and $B(5,-4)$ are ends of a diameter of a circle. Find the centre and radius of the circle.

Solution:-

$\because$ O is mid point of diameter AB is called centre of circle.



so
$$O\left(\frac{-5+5}{2},\frac{-2-4}{2}\right)$$
$$=O\left(\frac{0}{2},\frac{-6}{2}\right)=O(0,-3)$$

In fig., radius $=|AO|=|BO|$

so
$$|BO|=\sqrt{(0-5)^2+(-3-(-4))^2}=\sqrt{(-5)^2+(-3+4)^2}$$
$$|BO|=\sqrt{25+1}=\sqrt{26}$$

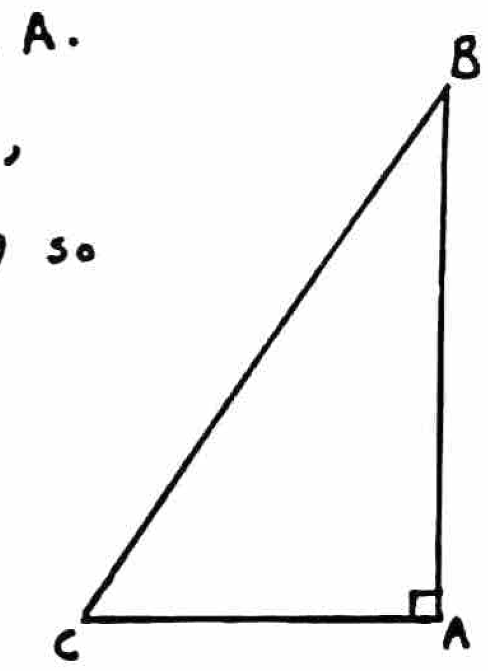
Q9. Find h such that the points $A(h,1)$, $B(2,7)$ and $C(-6,-7)$ are vertices of a right triangle with right angle at the vertex A.

Solution:- $\because A(h,1), B(2,7), C(-6,-7)$

$\because$ right angle is at vertex A.

so by pathagoras theorem,

$$|BC|^2=|AC|^2+|AB|^2 \text{ ——— (I) so}$$



$$|AB|=\sqrt{(2-h)^2+(7-1)^2}$$
$$=\sqrt{4-4h+h^2+36}$$
$$|AB|=\sqrt{40-4h+h^2}$$
$$\rightarrow |AB|^2=40-4h+h^2$$
$$|BC|=\sqrt{(-6-2)^2+(-7-7)^2}=\sqrt{(-8)^2+(-14)^2}=\sqrt{64+196}$$
$$=\sqrt{260} \rightarrow |BC|^2=260$$

$$\rightarrow |AC|=\sqrt{(-6-h)^2+(-7-1)^2}=\sqrt{36+12h+h^2+64}$$
$$|AC|=\sqrt{h^2+12h+100} \rightarrow |AC|^2=h^2+12h+100$$

so (I) becomes as

$$260=h^2+12h+100+40-4h+h^2$$
$$\rightarrow 2h^2+8h+140=260 \rightarrow 2h^2+8h+140-260=0$$
$$\rightarrow 2h^2+8h-120=0$$
$$\rightarrow h^2+4h-60=0 \quad \div \text{ by } 2$$

→ $h^2+10h-6h-60=0$

→ $h(h+10)-6(h+10)=0$ → $(h+10)(h-6)=0$

→ $h+10=0$ or $h-6=0$

→ $h=-10$ or $h=6$

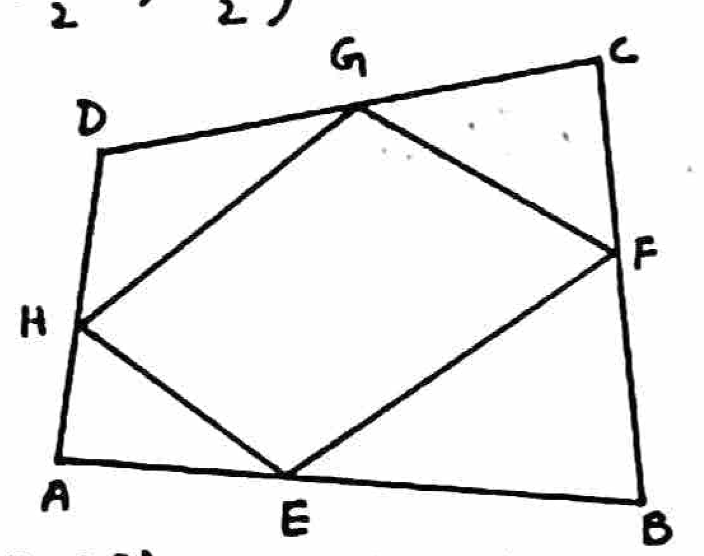
Q10. A quadrilateral has the points $A(9,3)$, $B(-7,7)$, $C(-3,-7)$ and $D(5,-5)$ as its vertices. Find the midpoints of its sides. Show that the figure formed by joining the midpoints consecutively is a parallelogram.

Solution:- ∵ $A(9,3)$, $B(-7,7)$, $C(-3,-7)$, $D(5,-5)$

Midpoint of AB is $E\left(\frac{-7+9}{2}, \frac{7+3}{2}\right)$

$= E\left(\frac{2}{2}, \frac{10}{2}\right)$

$= E(1,5)$



Midpoint of BC is $F\left(\frac{-7+(-3)}{2}, \frac{7+(-7)}{2}\right) = F\left(\frac{-10}{2}, \frac{0}{2}\right)$

$= F(-5,0)$

Midpoint of CD is $G\left(\frac{-3+5}{2}, \frac{-7+(-5)}{2}\right) = G\left(\frac{2}{2}, \frac{-7-5}{2}\right)$

$= G\left(\frac{2}{2}, \frac{-12}{2}\right) = G(1,6)$

Mid point of AD is $H\left(\frac{9+5}{2}, \frac{3-5}{2}\right) = H\left(\frac{14}{2}, \frac{-2}{2}\right)$

$= H(7,-1)$

Now figure formed by midpoints E, F, G and H will be llgram if $|EF|=|HG|$ and $|HE|=|GF|$ so

$|EF| = \sqrt{(-5-1)^2+(0-5)^2} = \sqrt{(-6)^2+(-5)^2} = \sqrt{36+25} = \sqrt{61}$

$|GF| = \sqrt{(-5-1)^2+(0+6)^2} = \sqrt{(-6)^2+(6)^2} = \sqrt{36+36} = \sqrt{72}$

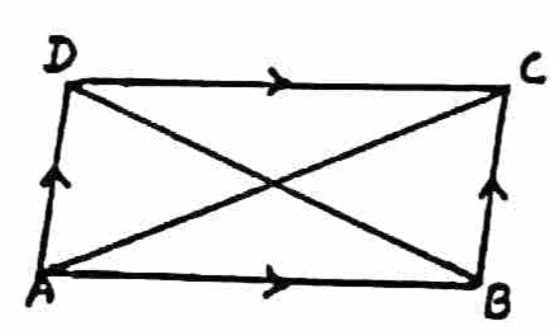
$|HG| = \sqrt{(1-7)^2+(-6+1)^2} = \sqrt{(-6)^2+(-5)^2} = \sqrt{36+25} = \sqrt{61}$

$|HE| = \sqrt{(1-7)^2+(5-(-1))^2} = \sqrt{(-6)^2+(6)^2} = \sqrt{36+36} = \sqrt{72}$

Thus $|EF|=|HG|$ and $|HE|=|GF|$ so EFGH is a llgram.

Q11. Find h such that the quadrilateral with vertices $A(-3,0)$, $B(1,-2)$, $C(5,0)$ and $D(1,h)$ is parallelogram. Is it a square?

Solution:-



∵ $A(-3,0)$, $B(1,-2)$, $C(5,0)$ and $D(1,h)$ are vertices of a llgram. so $|AB|=|DC|$ and $|AD|=|BC|$

Now $|AB| = \sqrt{(1-(-3))^2+(-2-0)^2} = \sqrt{(1+3)^2+(-2)^2} = \sqrt{16+4}$

→ $|AB| = \sqrt{20}$ and

$|CD| = \sqrt{(1-5)^2+(h-0)^2} = \sqrt{(-4)^2+h^2} = \sqrt{h^2+16}$

∵ $|AB|=|CD|$

→ $\sqrt{20} = \sqrt{h^2+16}$ → $h^2+16=20$

→ $h^2 = 20-16$ → $h^2=4$

→ $h=\pm 2$ → $h=2$, $h=-2$

For $h=2$, D is $(1,2)$

Now ABCD will be square if Length of AC = Length of BC

Now $|AC| = \sqrt{(5+3)^2+(0-0)^2} = \sqrt{64} = 8$

and $|BD| = \sqrt{(1-1)^2+(2+2)^2} = \sqrt{0+16} = 4$

∵ $|AC| \neq |BD|$ so ABCD is not square.

Note:- For $h=-2$, D is $(1,-2)$

Here $A(-3,0)$, $B(1,-2)$, $C(5,0)$, $D(1,-2)$

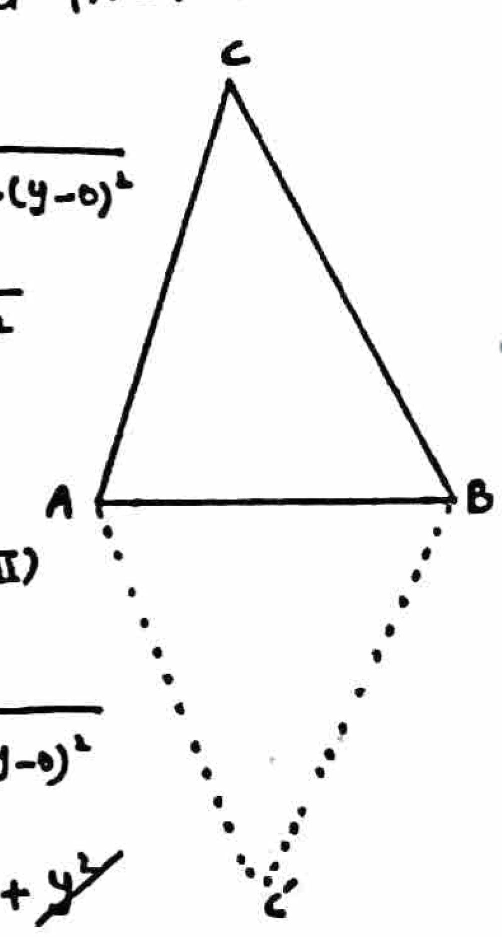
∵ B and D are same points. In this case quadrilateral will reduce to triangle. so we can not take $h=-2$.

> **Remember:** A triangle having all sides equal is called equilateral triangle.

Q12. If two vertices of an equilateral triangle are $A(-3,0)$ and $B(3,0)$, find the third vertex. How many of these triangles are possible?

Solution:- Suppose $C(x,y)$ be the third vertex. ∵ ABC is equilateral triangle, so $|AB|=|AC|=|BC|$

→ $|AB|=|AC|$ ⟶ (I) and $|AC|=|BC|$ ⟶ (II)



(I) → $\sqrt{(3-(-3))^2+(0-0)^2} = \sqrt{(x+3)^2+(y-0)^2}$

→ $\sqrt{(6)^2+(0)^2} = \sqrt{x^2+6x+9+y^2}$

squaring it

→ $36 = x^2+y^2+6x+9$

→ $x^2+y^2+6x-27=0$ —— (III)

From (II) $|AC|=|BC|$

(II) → $\sqrt{(x+3)^2+(y-0)^2} = \sqrt{(x-3)^2+(y-0)^2}$

squaring it,

→ $\cancel{x^2}+6x+\cancel{9}+\cancel{y^2} = \cancel{x^2}+\cancel{9}-6x+\cancel{y^2}$

→ $6x+6x=0$ → $12x=0$ → $x=0$

Put $x=0$ in (III) $(0)^2+y^2+6(0)-27=0$

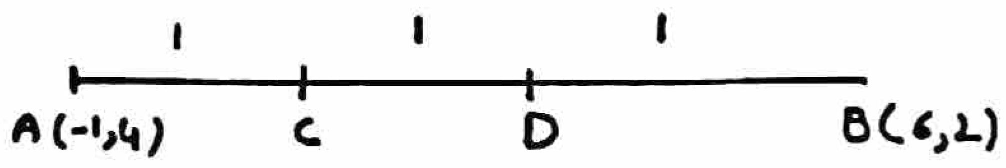
→ $y^2=27$ → $y=\pm\sqrt{27}=\pm 3\sqrt{3}$

so coordinates of C are $(0,3\sqrt{3})$ and $(0,-3\sqrt{3})$

Hence two triangles are possible with vertex $(0,3\sqrt{3})$ and $(0,-3\sqrt{3})$.

Q13. Find the points trisecting the join of A(-1,4) and B(6,2).

Solution:- suppose points $C(x_1, y_1)$ and $D(x_2, y_2)$ trisect line AB



$\because$ C divide AB in ratio 1:2 so coordinates are

$$x_1 = \frac{(1)(6)+2(-1)}{1+2} = \frac{6-2}{3} = \frac{4}{3}$$

$$y_1 = \frac{(1)(2)+2(4)}{1+2} = \frac{2+8}{3} = \frac{10}{3} \quad \text{so } C\left(\frac{4}{3}, \frac{10}{3}\right)$$

Now $D(x_2, y_2)$ divide AB in ratio 2:1

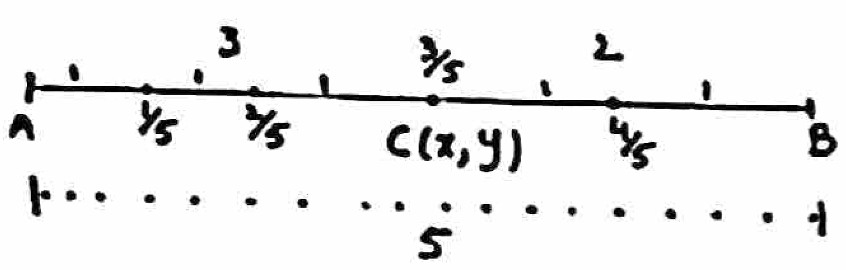
so coordinates are

$$x_2 = \frac{(2)(6)+(1)(-1)}{2+1} = \frac{12-1}{3} = \frac{11}{3}$$

$$y_2 = \frac{(2)(2)+1(4)}{2+1} = \frac{4+4}{3} = \frac{8}{3} \quad \text{so } D\left(\frac{11}{3}, \frac{8}{3}\right)$$

Q14. Find the point three-fifth of the way along the line segment from A(-5,8) to B(5,3).

Solution:- Let $C(x,y)$ be a required point.



$\therefore$ coordinates are

$$x = \frac{(3)(5)+2(-5)}{3+2} = \frac{15-10}{5} = \frac{5}{5} = 1$$

$$y = \frac{(3)(3)+2(8)}{3+2} = \frac{9+16}{5} = \frac{25}{5} = 5$$

So $C(1,5)$ is required point.

Q15. Find the point P on the join of A(1,4) and B(5,6) that is twice as far from A as B is from A and lies

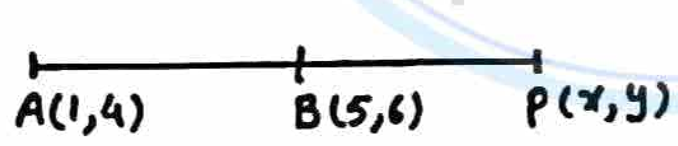
(i) on the same side of A as B does

(ii) on the opposite side of A as B does

Solution:- (i) $A(1,4)$, $B(5,6)$

$\because$ B becomes midpoint of AP. so

$$5 = \frac{1+x}{2}, \quad 6 = \frac{4+y}{2}$$

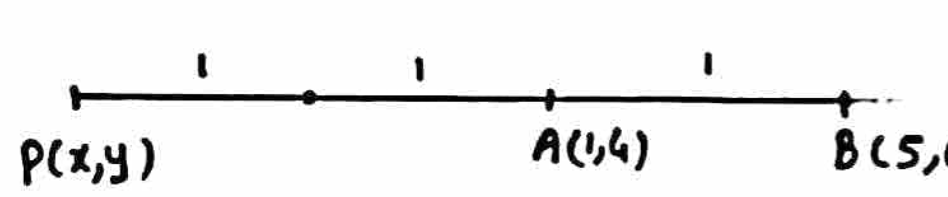


$\Rightarrow 10 = 1+x \quad , \quad 12 = 4+y$

$\Rightarrow x = 10-1 \quad , \quad y = 12-4$

$x = 9 \quad , \quad y = 8$ so $P(9,8)$ is the required point.

(ii) $A(1,4)$, $B(5,6)$



$\because$ A divides PB in ratio 2:1

so

$$1 = \frac{2(5)+1(x)}{2+1} = \frac{10+x}{3}$$

$\Rightarrow \frac{10+x}{3} = 1 \Rightarrow 10+x = 3 \quad , \quad x = 3-10$

$\Rightarrow x = -7$

$$4 = \frac{(2)(6)+1(y)}{2+1} = \frac{12+y}{3}$$

$\Rightarrow \frac{12+y}{3} = 4 \Rightarrow 12+y = 12 \Rightarrow y = 0$

so $P(-7,0)$ is required point.

Q16. Find the point which is equidistant from the points A(5,3), B(-2,2) and C(4,2). What is the radius of the circumference of the $\triangle ABC$?

Solution:- Let $P(x,y)$ be the point, which is equidistant from given points.

so $|AP| = |BP| = |CP|$

As $|AP| = |BP|$

$\Rightarrow \sqrt{(x-5)^2+(y-3)^2} = \sqrt{(x+2)^2+(y-2)^2}$

$\Rightarrow (x-5)^2+(y-3)^2 = (x+2)^2+(y-2)^2 \longrightarrow (I)$

Also $|BP| = |CP|$

$\Rightarrow \sqrt{(x+2)^2+(y-2)^2} = \sqrt{(x-4)^2+(y-2)^2}$

$\Rightarrow (x+2)^2 + \cancel{(y-2)^2} = (x-4)^2 + \cancel{(y-2)^2}$

$\Rightarrow \cancel{x^2}+4+4x = \cancel{x^2}+16-8x$

$\Rightarrow 4x+8x = 16-4 \Rightarrow 12x = 12$

$\Rightarrow x = 1$ put in (I)

$\Rightarrow (1-5)^2+(y-3)^2 = (1+2)^2+(y-2)^2$

$\Rightarrow (-4)^2+y^2-6y+9 = (3)^2+y^2-4y+4$

$\Rightarrow 16+\cancel{y^2}-6y+\cancel{9} = \cancel{9}+\cancel{y^2}-4y+4$

$-6y+4y = 4-16 \Rightarrow -2y = -12$

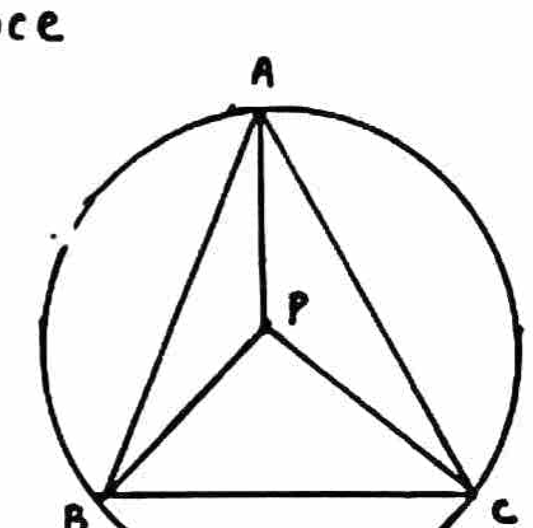
$\Rightarrow y = 6$ Hence $P(1,6)$ is the required point.

Radius of Circumference:-

radius of circumference of $\triangle ABC$

$= |AP| = |BP| = |CP|$

so we find $|AP|$



$|AP| = \sqrt{(1-5)^2+(6-3)^2}$

$= \sqrt{(-4)^2+(3)^2}$

$\Rightarrow |AP| = \sqrt{16+9} = \sqrt{25} = 5$

Q17. The points $(4,-2)$, $(-2,4)$ and $(5,5)$ are the vertices of a triangle. Find in-centre of the triangle.

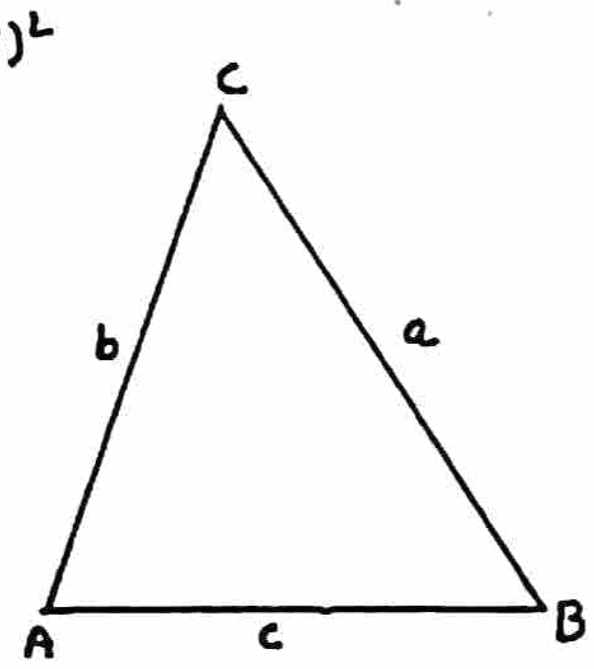
Solution:- Here $|BC| = a$, $|AC| = b$, $|AB| = c$ and $A(x_1,y_1) = (4,-2)$, $B(x_2,y_2) = (-2,4)$, $C(x_3,y_3) = (5,5)$

$a = |BC| = \sqrt{(5+2)^2 + (5-4)^2}$

$\rightarrow a = \sqrt{(7)^2 + (1)^2}$

$a = \sqrt{49+1} = \sqrt{50}$

$\rightarrow a = \sqrt{25\times 2} = 5\sqrt{2}$



$b = |AC| = \sqrt{(5-4)^2 + (5+2)^2} = \sqrt{(1)^2 + (7)^2}$

$\rightarrow b = \sqrt{1+49} = \sqrt{50} = \sqrt{25\times 2} = 5\sqrt{2}$

$c = |AB| = \sqrt{(-2-4)^2 + (4+2)^2} = \sqrt{(-6)^2 + (6)^2}$

$\rightarrow c = \sqrt{36+36} = \sqrt{72} = \sqrt{36\times 2} = 6\sqrt{2}$

$\because$ Incenter (where angles bisecter meet)

$$= \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

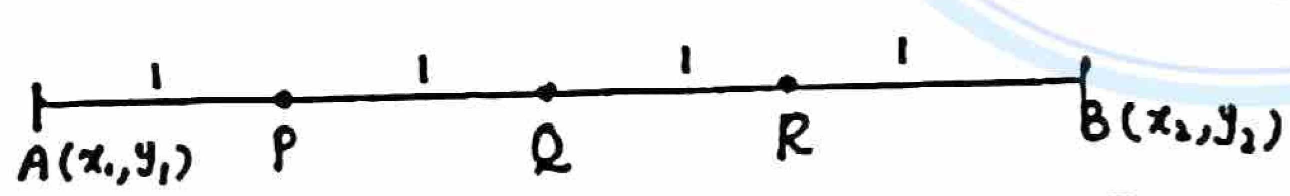
$$= \left(\frac{5\sqrt{2}(4)+5\sqrt{2}(-2)+6\sqrt{2}(6)}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}, \frac{5\sqrt{2}(-2)+5\sqrt{2}(4)+6\sqrt{2}(5)}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}\right)$$

$$= \left(\frac{(20-10+36)\sqrt{2}}{16\sqrt{2}}, \frac{(-10+20+30)\sqrt{2}}{16\sqrt{2}}\right)$$

$$= \left(\frac{40}{16}, \frac{40}{16}\right) = \left(\frac{5}{2}, \frac{5}{2}\right)$$

Q18. Find the points that divide the line segment joining $A(x_1,y_1)$ and $B(x_2,y_2)$ into four equal parts.

Solution:- Let P, Q, R be the required points into four equal parts.



In fig., P divides AB in ratio $1:3$
so coordinates of P are

$$= \left(\frac{1(x_2)+3(x_1)}{1+3}, \frac{1(y_2)+3(y_1)}{1+3}\right) = \left(\frac{3x_1+x_2}{4}, \frac{3y_1+y_2}{4}\right)$$

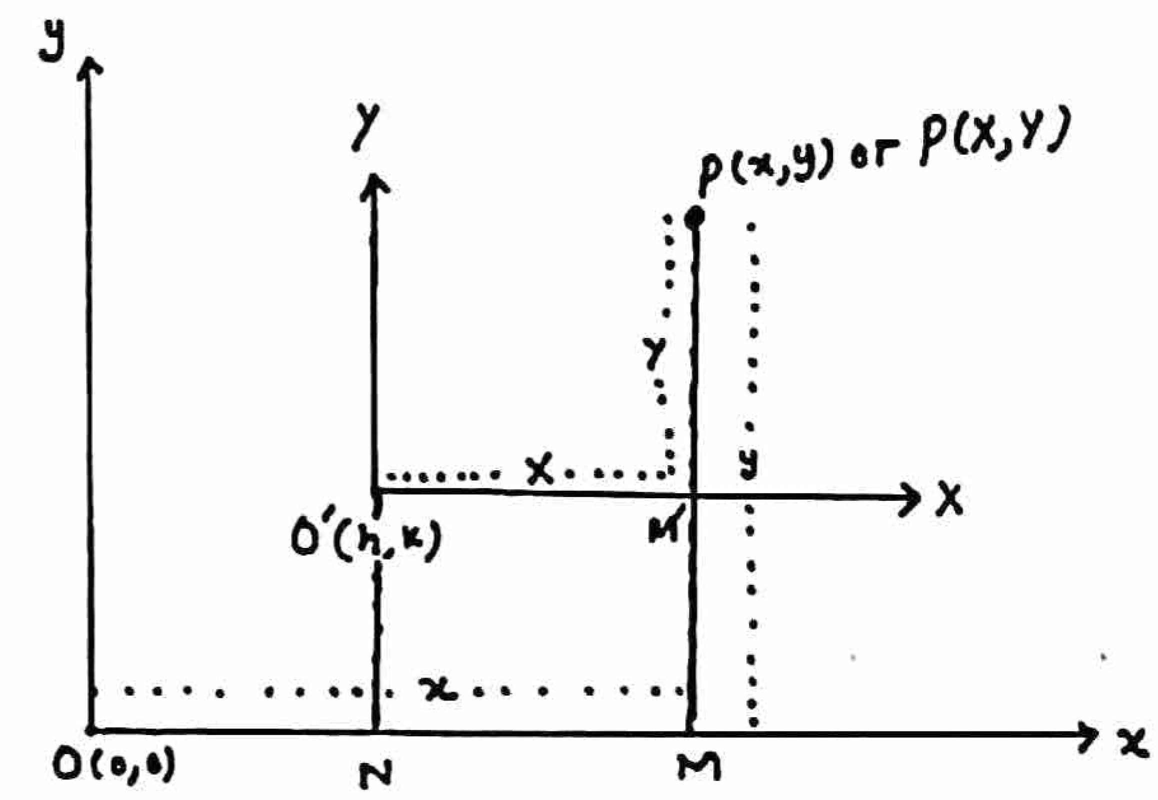
$\because Q$ is midpoint of AB. so $Q\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

R divides AB in ratio $3:1$
so coordinates of R are

$$= \left(\frac{3(x_2)+1(x_1)}{3+1}, \frac{3(y_2)+1(y_1)}{3+1}\right) = \left(\frac{x_1+3x_2}{4}, \frac{y_1+3y_2}{4}\right)$$

Translation and Rotation of Axes

Translation of axes:-



Let $P(x,y)$ be any point in xy-plane. Let we draw two mutually perpendicular lines $O'X$ and $O'Y$ such that they meet at a point $O'(h,k)$ in xy-plane. Here $O'X$ and $O'Y$ are parallel to OX and OY respectively. The new axes $O'X$ and $O'Y$ are called translation of OX and OY-axes through point $O'(h,k)$.

Let $P(X,Y)$ be point in new XY-plane.
Draw $\perp$ars PM and $O'N$ from P and O' on x-axis. In figure., $OM = x$, $PM = y$,
$O'N = M'M = k$, $X = O'M' = NM = OM - ON = x-h$
and $Y = PM' = PM - M'M = PM - O'N = y-k$
Thus coordinates of P in XY-plane are
$(X,Y) = (x-h, y-k)$

Important note:-

As $X = x-h \rightarrow x = X+h$ and
$Y = y-k \rightarrow y = Y+k$

(i) If $P(x,y)$ and $O'(h,k)$ are given in xy-plane and we are to find XY-coordinates of P then we put $X = x-h$ and $Y = y-k$

(ii) If $P(X,Y)$ and $O'(h,k)$ are given in XY-plane and we are to find xy-coordinates of P. then we put $x = X+h$ and $y = Y+k$

Example1: The coordinates of point P are $(-6,9)$. The axes are translated through the point $O'(-3,2)$. Find the coordinates of P reffered to new axes.

Solution:- Here $x = -6$, $y = 7$, $h = -3$, $k = 2$

$P(X,Y) = ?$ As $X = x-h = -6-(-3) = -6+3 = -3$

$Y = y-k = 9-2 = 7$

Thus $P(X,Y) = P(-3,7)$

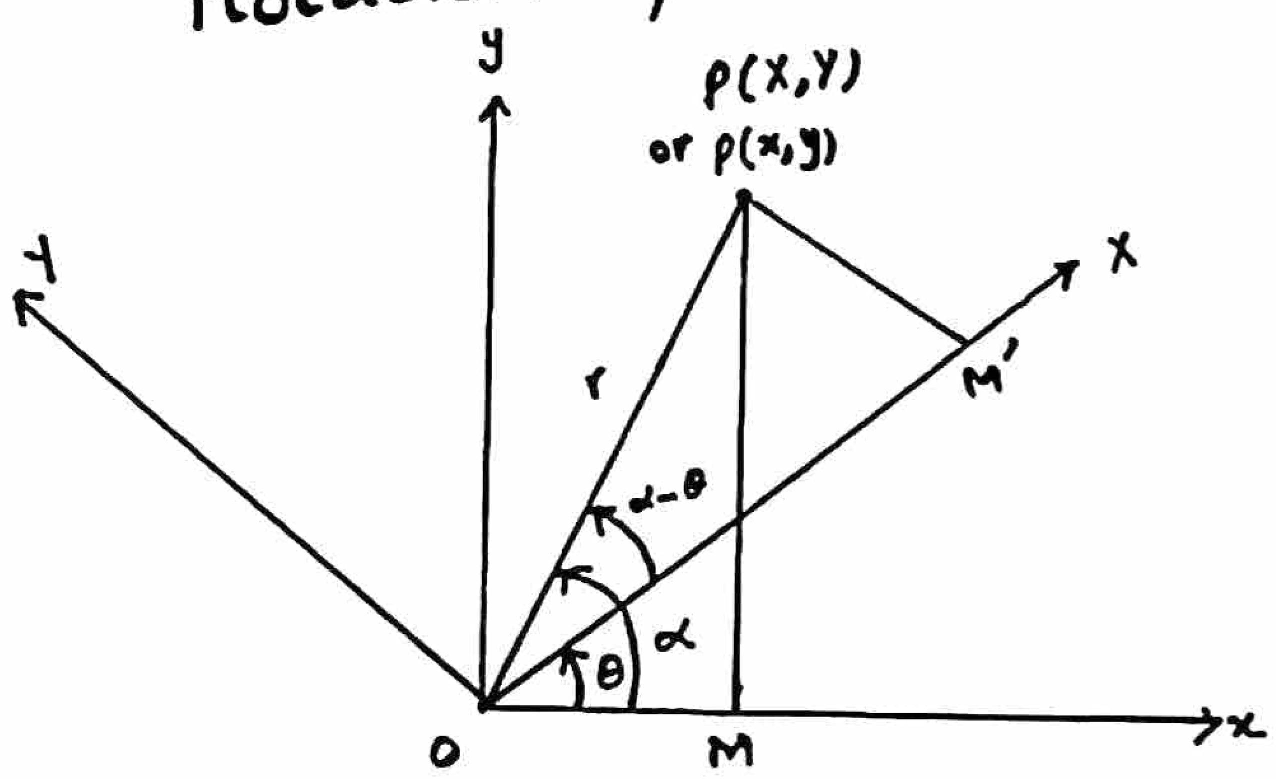
Example 2. The xy-coordinate axes are translated through the point $O'(4,6)$. The coordinates of point P are $(2,-3)$ referred to the new axes. Find the coordinates of P reffered to the orignal axes.

Solution:– Here $X=2,\ Y=-3$ and $h=4,\ k=6$ and $P(x,y)=?$

$\because x = X+h = 2+4 = 6\ ,\quad y = Y+k = -3+6 = 3$

$\rightarrow P(x,y) = P(6,3)$

Rotation of axes:–

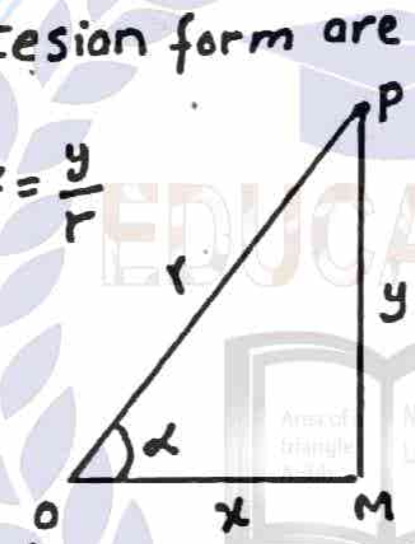


Let $P(x,y)$ be any point in xy-plane. Let we rotate ox and oy through an angle θ, $(0<\theta<90^\circ)$. Let we draw $\perp PM$ from P on x-axis and PM' on OX-axis.

$\because m\angle xOX = \theta\ ,\ m\angle xoP = \alpha$

then $m\angle XoP = \alpha-\theta$

$\because$ coordinates of P in cartesion form are (x,y) so



In $\triangle OPM$ $\quad \cos\alpha = \frac{x}{r}\ ,\ \sin\alpha = \frac{y}{r}$

so $x = r\cos\alpha,\ y = r\sin\alpha$

Let (X,Y) be coordinates of P referred to XY-coordinate system.

In $\triangle OPM'$ $\quad OM' = X = r\cos(\alpha-\theta)$

$\rightarrow X = r\cos\alpha\cos\theta + r\sin\alpha\sin\theta$

and $M'P = Y = r\sin(\alpha-\theta)$

$\rightarrow Y = r\sin\alpha\cos\theta - r\cos\alpha\sin\theta$

$\left(\begin{array}{l}\because \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ \ \ \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta\end{array}\right)$

$\because x = r\cos\alpha,\ y = r\sin\alpha$ so

$X = x\cos\theta + y\sin\theta$

$Y = y\cos\theta - x\sin\theta$

$\rightarrow (X,Y) = (x\cos\theta + y\sin\theta,\ y\cos\theta - x\sin\theta)$

Example 3. The xy-coordinate axes are rotated about the origin through an angle of 30°. If the xy-coordinates of a point are $(5,7)$, find its XY-coordinates where OX and OY are the axes obtained after rotation.

Solution:– Here $x=5,\ y=7,\ \theta=30^\circ$

$P(X,Y) = ?$

$\because X = x\cos\theta + y\sin\theta = 5\cos30^\circ + 7\sin30^\circ$

$= 5\left(\frac{\sqrt3}{2}\right) + 7\left(\frac{1}{2}\right) = \frac{5\sqrt3+7}{\sqrt2}$

$Y = y\cos\theta - x\sin\theta = 7\cos30^\circ - 5\sin30^\circ$

$= 7\left(\frac{\sqrt3}{2}\right) - 5\left(\frac12\right) = \frac{7\sqrt3-5}{2}$

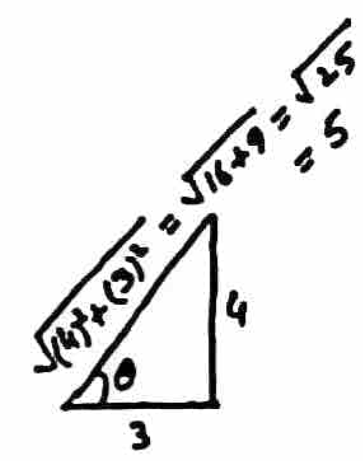
Thus

$P(X,Y) = \left(\frac{5\sqrt3+7}{2},\ \frac{7\sqrt3-5}{2}\right)$

Example 4. The xy-axes are rotated about the origin through an angle of $\arctan\frac43$ lying in the first quadrant. The coordinates of a point P reffered to the new axes OX and OY are $P(-1,-7)$. Find the coordinates of P referred to the xy-coordinate system.

Solution:– Here $X=-1,\ Y=-7$

$\theta = \tan^{-1}\frac43 \rightarrow \tan\theta = \frac43$



$\rightarrow \sin\theta = \frac45$ and $\cos\theta = \frac35$

$P(x,y) = ?$

$\because X = x\cos\theta + y\sin\theta$

$-1 = x\left(\frac35\right) + y\left(\frac45\right) \rightarrow -1 = \frac35x + \frac45y$

$\rightarrow -5 = 3x+4y \rightarrow 3x+4y = -5$ ——— (i)

and $Y = y\cos\theta - x\sin\theta$

$-7 = y\left(\frac35\right) - x\left(\frac45\right) \rightarrow -7 = \frac35y - \frac{4x}{5}$

$\rightarrow -4x+3y = -105$ ——— (ii)

By $4(i)+3(ii) \rightarrow$

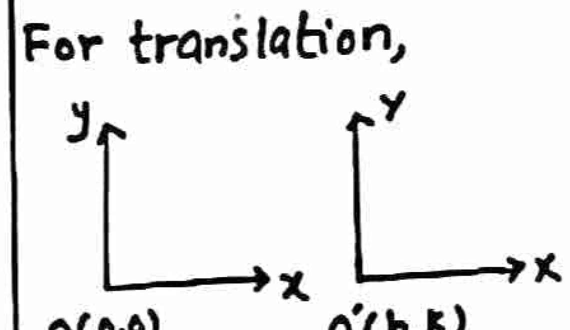
$\cancel{12x} + 16y = -20$
$-\cancel{12x} + 9y = -105$
$25y = -125 \rightarrow y = -5$

Put $y=-5$ in (i) $\rightarrow 3x+4(-5) = -5$

$\rightarrow 3x-20 = -5 \rightarrow 3x = 15 \rightarrow x = 5$

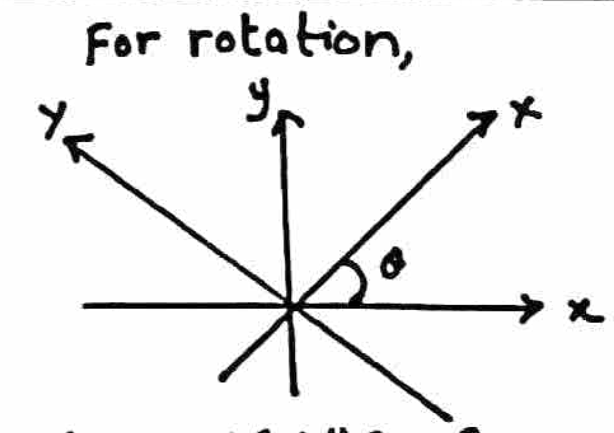
so $P(x,y) = P(5,-5)$.

For translation,



$X = x-h \rightarrow x = X+h$
$Y = y-k \qquad y = Y+k$

For rotation,



$X = x\cos\theta + y\sin\theta$
$Y = y\cos\theta - x\sin\theta$

Exercise 4.2

Q1. The two points P and O' are given in xy-coordinate system. Find the XY-coordinates of P reffered to the translated axes $O'X$ and $O'Y$.

(i) $P(3,2)$; $O'(1,3)$ (ii) $P(-2,6)$; $O'(-3,2)$

(iii) $P(-6,-8)$; $O'(-4,-6)$ (iv) $P(\frac{3}{2},\frac{5}{2})$; $O'(-\frac{1}{2},\frac{7}{2})$

Solution:- (i) Here $x=3, y=2$ and $h=1, k=3$

$P(X,Y)=?$ $\because X = x-h = 3-1 = 2$

and $Y = y-k = 2-3 = -1$ so $P(X,Y)=P(2,-1)$

(ii) Here $x=-2, y=6$ and $h=-3, k=2$

$P(X,Y)=?$ $\because X = x-h = -2-(-3) = -2+3 = 1$

and $Y = y-k = 6-2 = 4$ so $P(X,Y)=P(1,4)$

(iii) Here $x=-6, y=-8, h=-4, k=-6$

$P(X,Y)=?$

$\because X = x-h = -6-(-4) = -6+4 = -2$

$Y = y-k = -8-(-6) = -8+6 = -2$

so $P(X,Y)=P(-2,-2)$

(iv) Here $x=\frac{3}{2}, y=\frac{5}{2}$ and $h=-\frac{1}{2}, k=\frac{7}{2}$

$P(X,Y)=?$

$\because X = x-h = \frac{3}{2}-(-\frac{1}{2}) = \frac{3}{2}+\frac{1}{2} = \frac{4}{2} = 2$

$Y = y-k = \frac{5}{2}-\frac{7}{2} = \frac{5-7}{2} = \frac{-2}{2} = -1$

so $P(X,Y)=P(2,-1)$

Q2. The xy-coordinate axes are translated through the point O' whose coordinates are given in xy-coordinate system. Find the coordinates of P in xy-coordinate system. (i) $P(8,10)$; $O'(3,4)$

(ii) $P(-5,-3)$, $O'(-2,-6)$ (iii) $P(-\frac{3}{4},-\frac{7}{6})$; $O'(\frac{1}{4},-\frac{1}{6})$

(iv) $P(4,-3)$; $O'(-2,3)$

Solution:- (i) Here $X=8, Y=10$ and $h=3, k=4$, $P(x,y)=?$

$\because x = h+X = 3+8 = 11$

$y = Y+k = 10+4 = 14$ so $P(x,y)=P(11,14)$

ii) Here $X=-5, Y=-3, h=-2, k=-6$

$P(x,y)=?$ $\because x = X+h = -5-2 = -7$

$y = Y+k = -3-6 = -9$ so $P(x,y)=P(-7,-9)$

(iii) Here $X=-\frac{3}{4}, Y=-\frac{7}{6}, h=\frac{1}{4}, k=-\frac{1}{6}$

$P(x,y)=?$ $\because x = X+h = -\frac{3}{4}+\frac{1}{4} = \frac{-2}{4} = -\frac{1}{2}$

$y = Y+k = -\frac{7}{6}-\frac{1}{6} = \frac{-7}{6}$

so $P(x,y) = P(-\frac{1}{2},-\frac{7}{6})$

(iv) Here $X=4, Y=-3$ and $h=-2, k=3$

$P(x,y)=?$

$\because x = X+h = 4-2 = 2$

$y = Y+k = -3+3 = 0$ so $P(x,y)=P(2,0)$

Q3. The xy-coordinate axes are rotated about the origin through the indicated angle. The new axes are OX and OY. Find the XY-coordinates of the point P with the given xy-coordinates.

(i) $P(5,3)$; $\theta=45^\circ$ (ii) $P(3,-7)$; $\theta=30^\circ$

(iii) $P(11,-15)$; $\theta=60^\circ$ (iv) $P(15,10)$; $\theta=\arctan\frac{1}{3}$

Solution:- (i) Here $x=5, y=3$ and $\theta=45^\circ$

$P(X,Y)=?$

$\because X = x\cos\theta + y\sin\theta = 5\cos45^\circ + 3\sin45^\circ$

$= 5\cdot\frac{1}{\sqrt{2}} + 3\cdot\frac{1}{\sqrt{2}} = (5+3)\frac{1}{\sqrt{2}} = \frac{8}{\sqrt{2}} = \frac{4\times\sqrt{2}\sqrt{2}}{\sqrt{2}}$

$\Rightarrow X = 4\sqrt{2}$

$Y = y\cos\theta - x\sin\theta = 3\cos45^\circ - 5\sin45^\circ$

$= \frac{3}{\sqrt{2}} - \frac{5}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = \frac{-\sqrt{2}\sqrt{2}}{\sqrt{2}} = -\sqrt{2}$

Thus $P(X,Y) = P(4\sqrt{2},-\sqrt{2})$

(ii) Here $x=3, y=-7, \theta=30^\circ$, $P(X,Y)=?$

$\because X = x\cos\theta + y\sin\theta = 3\cos30^\circ - 7\sin30^\circ$

$= 3(\frac{\sqrt{3}}{2}) - 7(\frac{1}{2}) = \frac{3\sqrt{3}-7}{2}$

$Y = y\cos\theta - x\sin\theta = -7\cos30^\circ - 3\sin30^\circ$

$= \frac{-7\sqrt{3}}{2} - \frac{3}{2} = \frac{-7\sqrt{3}-3}{2}$ so $P(X,Y) = P(\frac{3\sqrt{3}-7}{2},\frac{-7\sqrt{3}-3}{2})$

(iii) Here $x=11, y=-15, \theta=60^\circ$, $P(X,Y)=?$

$\because X = x\cos\theta + y\sin\theta = 11\cos60^\circ - 15\sin60^\circ$

$= 11(\frac{1}{2}) - 15(\frac{\sqrt{3}}{2}) = \frac{11-15\sqrt{3}}{2}$

$Y = y\cos\theta - x\sin\theta = -15\cos60^\circ - 11\sin60^\circ$

$= \frac{-15}{2} - \frac{11\sqrt{3}}{2} = \frac{-15-11\sqrt{3}}{2}$

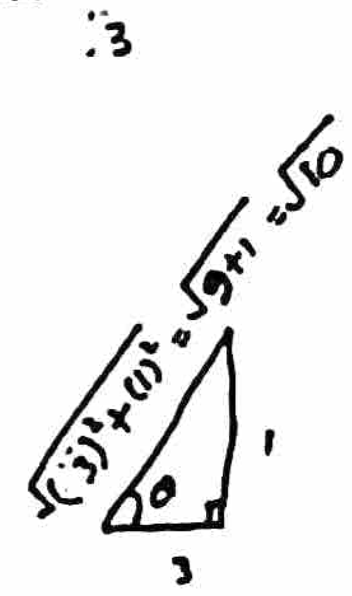
so $P(X,Y) = P\left(\frac{11-15\sqrt{3}}{2}, \frac{-15-11\sqrt{3}}{2}\right)$

(iv) Here $x=15, y=10, \theta=\tan^{-1}\frac{1}{3}$

$P(X,Y)=?$

$\because \theta = \tan^{-1}\frac{1}{3} \Rightarrow \tan\theta = \frac{1}{3}$

$\Rightarrow \cos\theta = \frac{3}{\sqrt{10}}, \quad \sin\theta = \frac{1}{\sqrt{10}}$



$X = x\cos\theta + y\sin\theta$

$= (15)(\frac{3}{\sqrt{10}}) + (10)(\frac{1}{\sqrt{10}})$

$= \frac{45}{\sqrt{10}} + \frac{10}{\sqrt{10}} = \frac{55}{\sqrt{10}}$

$Y = y\cos\theta - x\sin\theta$

$= 10\left(\frac{3}{\sqrt{10}}\right) - 15\left(\frac{1}{\sqrt{10}}\right)$

$\Rightarrow Y = \frac{30}{\sqrt{10}} - \frac{15}{\sqrt{10}} = \frac{15}{\sqrt{10}}$

so $P(X,Y) = \left(\frac{55}{\sqrt{10}}, \frac{15}{\sqrt{10}}\right)$

Q4. The xy-coordinate axes are rotated about the origin through the indicated angle and the new axes are OX and OY. Find the xy-coordinates of P with the given XY-coordinates,

(i) $P(-5,3)$; $\theta = 30°$ (ii) $P(-7\sqrt{2}, 5\sqrt{2})$; $\theta = 45°$

Solution:- (i) Here $X = -5$, $Y = 3$, $\theta = 30°$

and $P(x,y) = ?$

$\because X = x\cos\theta + y\sin\theta$

$-5 = x\cos 30° + y\sin 30°$

$-5 = x\frac{\sqrt{3}}{2} + y\cdot\frac{1}{2} \Rightarrow \sqrt{3}x + y = -10$ —— (I)

and $Y = y\cos\theta - x\sin\theta$

$3 = y\cos 30° - x\sin 30°$

$3 = y\cdot\frac{\sqrt{3}}{2} - x\cdot\frac{1}{2} \Rightarrow -x + \sqrt{3}y = 6$ —— (II)

By (I) + $\sqrt{3}$(II) $\Rightarrow$

$$\begin{array}{r} \sqrt{3}x + y = -10 \\ -\sqrt{3}x + 3y = 6\sqrt{3} \\ \hline 4y = 6\sqrt{3} - 10 \end{array}$$

$\Rightarrow y = \frac{2(3\sqrt{3}-5)}{4}$

$\Rightarrow y = \frac{3\sqrt{3}-5}{2}$ put in (I)

$\sqrt{3}x + \frac{3\sqrt{3}-5}{2} = -10$

$\Rightarrow 2\sqrt{3}x + 3\sqrt{3} - 5 = -20$

$\Rightarrow 2\sqrt{3}x = -20 + 5 - 3\sqrt{3}$

$x = \frac{-15 - 3\sqrt{3}}{2\sqrt{3}}$

$= \frac{-5\times\sqrt{3}\sqrt{3} - 3\sqrt{3}}{2\sqrt{3}}$

$= \frac{\sqrt{3}(-5\sqrt{3} - 3)}{2\sqrt{3}}$

$x = \frac{-3-5\sqrt{3}}{2}$

so $P(x,y) = \left(\frac{-3-5\sqrt{3}}{3}, \frac{3\sqrt{3}-5}{2}\right)$

(ii) Here $X = -7\sqrt{2}$, $Y = 5\sqrt{2}$, $\theta = 45°$

$\because X = x\cos\theta + y\sin\theta$

$-7\sqrt{2} = x\cos 45° + y\sin 45°$

$-7\sqrt{2} = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$ 'x' by $\sqrt{2}$

$\Rightarrow x + y = -7(2)$

$\Rightarrow x + y = -14$ —— (I)

and $Y = y\cos\theta - x\sin\theta$

$\Rightarrow 5\sqrt{2} = y\cos 45° - x\sin 45°$

$5\sqrt{2} = \frac{y}{\sqrt{2}} - \frac{x}{\sqrt{2}}$ 'x' by $\sqrt{2}$

$\Rightarrow -x + y = 5(2)$

$\Rightarrow -x + y = 10$ —— (II)

By (I) + (II) $\Rightarrow$

$$\begin{array}{r} x + y = -14 \\ -x + y = 10 \\ \hline 2y = -4 \end{array}$$

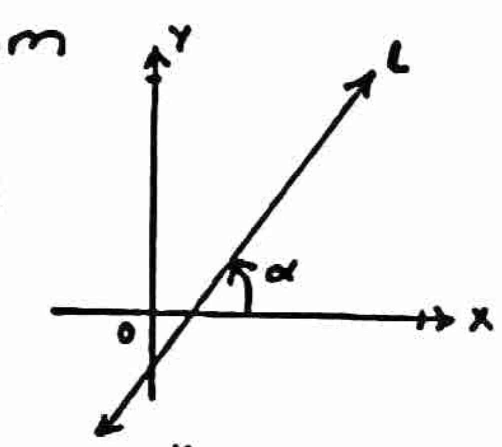
$\Rightarrow y = -2$ put in (I)

$x - 2 = -14 \Rightarrow x = -14 + 2$

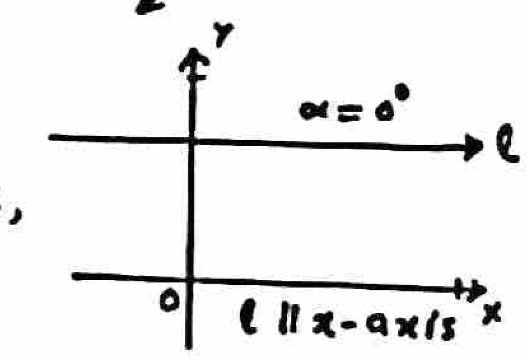
$\Rightarrow x = -12$ so $P(x,y) = (-12, -2)$

Equations of straight Lines

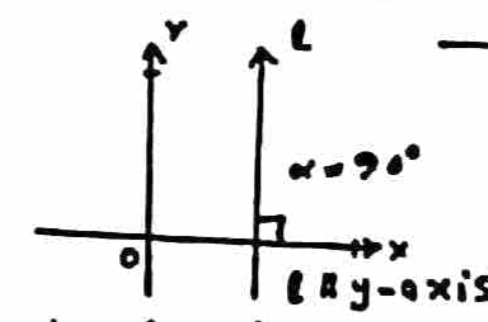
Inclination of a Line:- The angle $(0° < \alpha < 180°)$ measured anti-clockwise from positive x-axis to a non-horizontal straight line l is called inclination of l.



Note:- (i) If l is parallel to x-axis, then $\alpha = 0°$

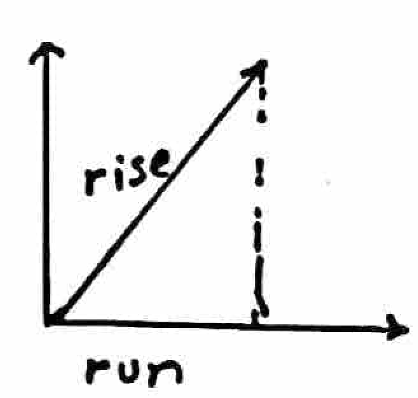


(ii) If l is parallel to y-axis, then $\alpha = 90°$



Slope or gradient of a line:- Let α be inclination of a line, then slope of a line is denoted by m and defined as; $m = \tan\alpha$

* The measure of steepness (ratio of rise to the run) is named as slope or gradient.



Note:- (i) Slope of x-axis or any line parallel to x-axis is zero $(\because \alpha = 0° \Rightarrow \tan 0° = 0)$

(ii) Slope of y-axis or any line parallel to y-axis is undefined. $(\because \alpha = 90° \Rightarrow \tan 90° = \infty)$

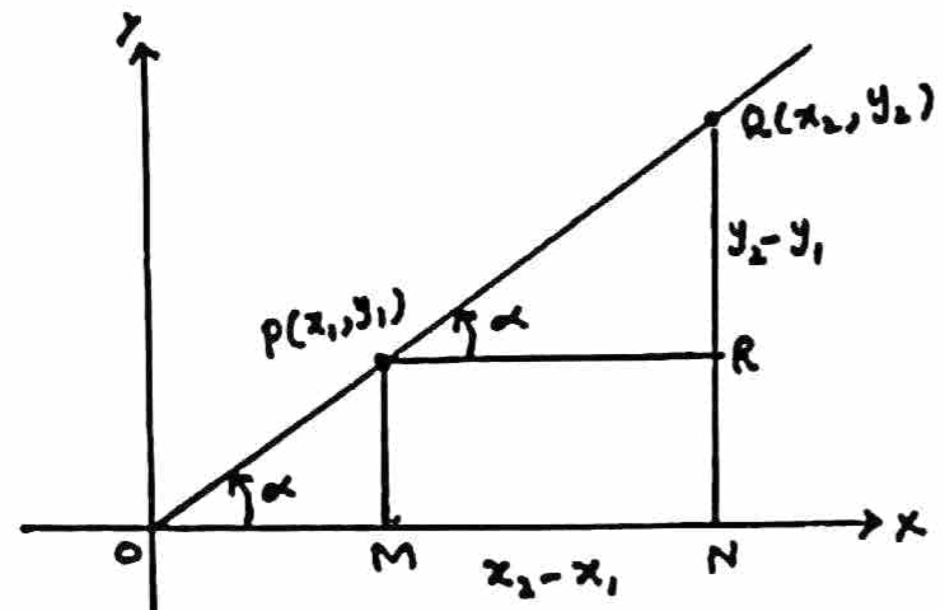
(iii) If $0° < \alpha < 90°$ then m is positive and if $90° < \alpha < 180°$, then m is negative.

Slope of a straight line Joining Two points

Theorem:- If a non-vertical line ℓ with inclination α passes through two points $P(x_1,y_1)$ and $Q(x_2,y_2)$, then the slope or gradient m of ℓ is given by

$$m = \frac{y_2-y_1}{x_2-x_1} = \tan\alpha$$

Proof:- Case (i) When $0<\alpha<\frac{\pi}{2}$



Let us draw $\perp$ars PM and QN from points P and Q on x-axis. Also draw a $\perp$ar PR on QN. we get right angled ΔQPR

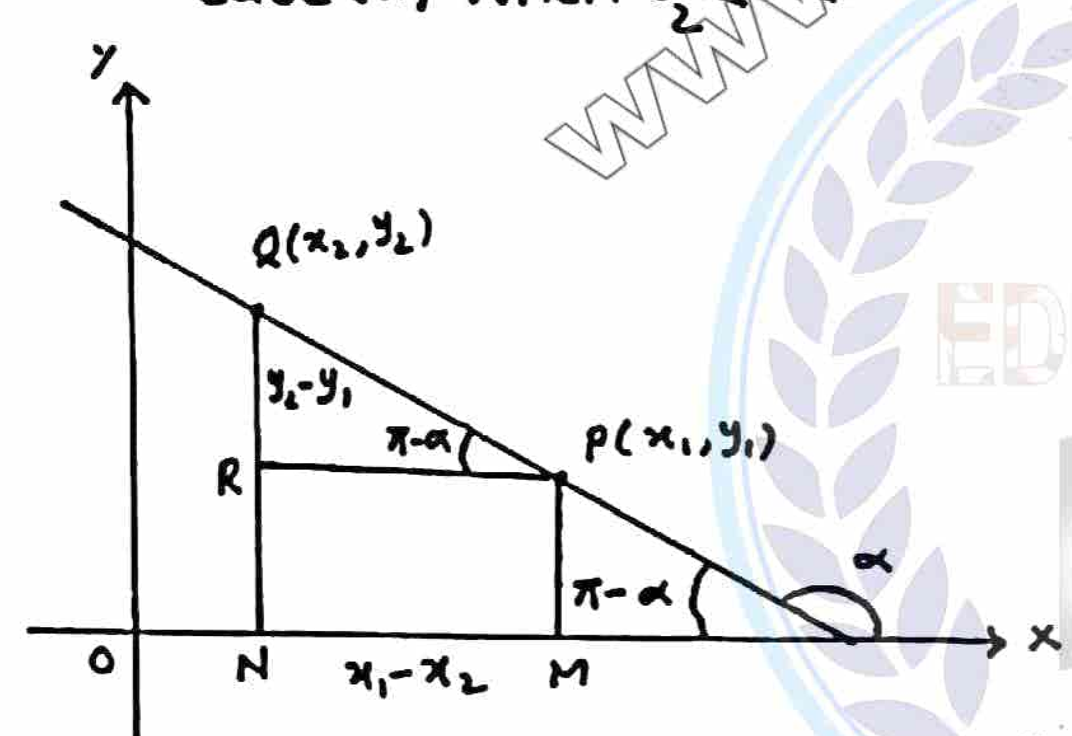
In figure., $|PR| = |MN| = |ON| - |OM| = x_2 - x_1$

and $|QR| = |QN| - |RN| = y_2 - y_1$

In ΔQPR $m = \tan\alpha = \frac{|QR|}{|PR|} = \frac{y_2-y_1}{x_2-x_1}$

Thus $m = \frac{y_2-y_1}{x_2-x_1} = \tan\alpha$

Case (ii) when $\frac{\pi}{2}<\alpha<\pi$



Let us draw $\perp$ars PM and QN from points P and Q on x-axis. Also draw a $\perp$ar PR on QN. we get right-angled ΔQPR

In figure; $|PR| = |MN| = |OM| - |ON| = x_1 - x_2$

also $|QR| = |QN| - |RN| = y_2 - y_1$

In ΔQPR, $m = \tan(\pi-\alpha) = \frac{|QR|}{|PR|} = \frac{y_2-y_1}{x_1-x_2}$

$\Rightarrow m = -\tan\alpha = \frac{y_2-y_1}{x_1-x_2}$

$\Rightarrow m = \tan\alpha = \frac{y_2-y_1}{-(x_1-x_2)} = \frac{y_2-y_1}{x_2-x_1}$

Thus $m = \frac{y_2-y_1}{x_2-x_1} = \tan\alpha$

Hence proved.

Note:- (I) $m \neq \frac{y_2-y_1}{x_1-x_2}$ and $m \neq \frac{y_1-y_2}{x_2-x_1}$

(II) ℓ is horizontal, iff $m=0$ ($\because \alpha = 0°$)

(III) ℓ is verticall, iff m is not defined ($\because \alpha = 90°$)

(IV) If slope of AB = slope of BC, then points A, B and C are collinear.

Theorem:- The two lines ℓ_1 and ℓ_2 with respective slopes m_1 and m_2 are

(i) parallel iff $m_1 = m_2$

(ii) perpendicular iff $m_1 = -\frac{1}{m_2}$ or $m_1 m_2 = -1$

(proof of this theorem is at page #34)

Example 1. Show that the points $A(-3,6)$, $B(3,2)$ and $C(6,0)$ are collinear.

Solution:- $\because$ A, B and C will be collinear if slope of AB = slope of BC so

slope of AB $= \frac{2-6}{3-(-3)} = \frac{-4}{3+3} = \frac{-4}{6} = -\frac{2}{3}$

slope of BC $= \frac{0-2}{6-3} = -\frac{2}{3}$

Thus A, B and C are collinear points.

Example 2. Show that the triangle with vertices $A(1,1)$, $B(4,5)$ and $C(12,-1)$ is a right angle.

Solution:- Slope of AB $= m_1 = \frac{5-1}{4-1} = \frac{4}{3}$

slope of BC $= m_2 = \frac{-1-5}{12-4} = -\frac{6}{8} = \frac{-3}{4}$

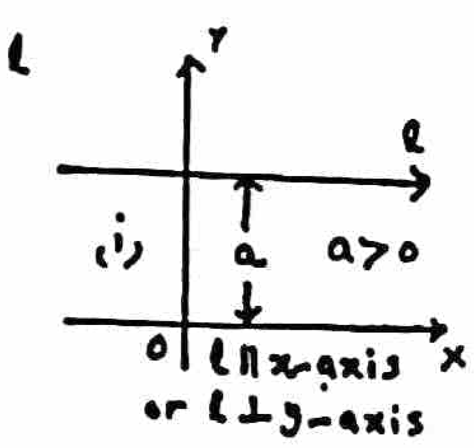
$\because m_1 m_2 = \left(\frac{4}{3}\right)\left(\frac{-3}{4}\right) = -1 \Rightarrow AB \perp BC$

Hence ΔABC is right triangle.

Equation of straight lines

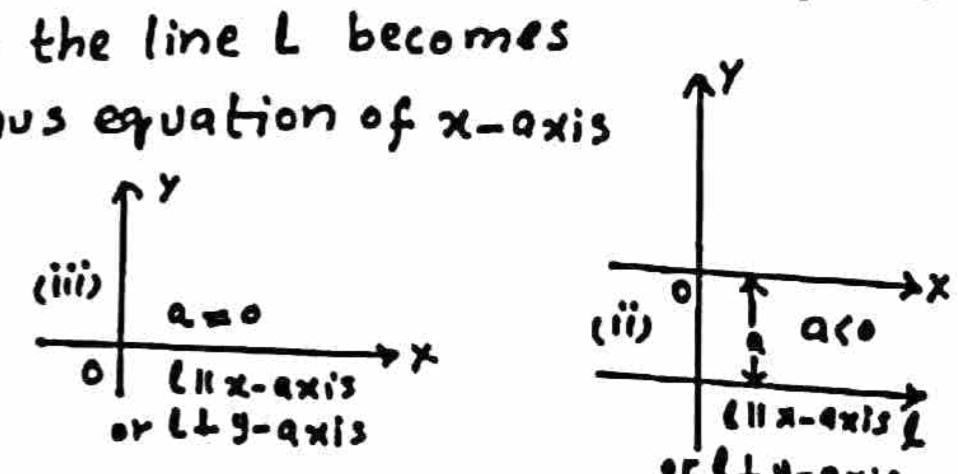
(i) Line parallel to x-axis (or perpendicular to y-axis):- An equation of the form $y=a$ is called equation of line parallel to x-axis.

Note:- (i) If $a>0$, then the line ℓ is above the x-axis.



(ii) If $a<0$, then the line ℓ is below the x-axis.

(iii) If $a=0$, then the line ℓ becomes the x-axis. Thus equation of x-axis is $y=0$



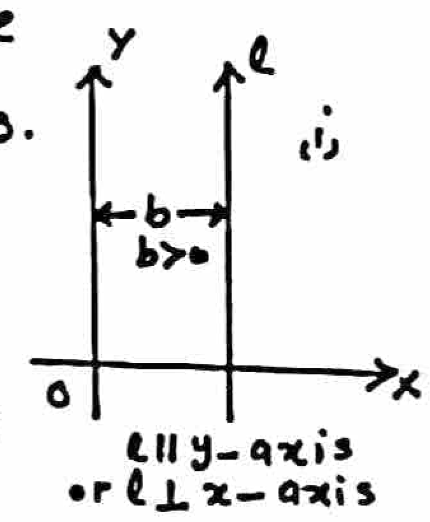
(ii) Line parallel to y-axis (or perpendicular to x-axis):-

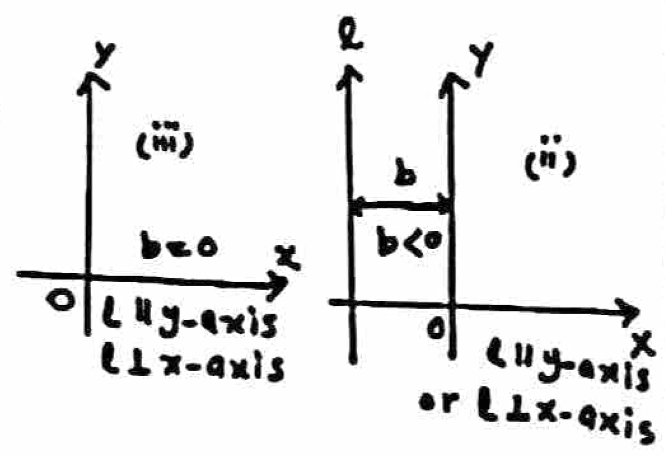
An equation of the form $x=b$ is called equation of line parallel to y-axis.

Note:- (i) If $b>0$ then the line is on the right of the y-axis.

(ii) If $b<0$, then the line is on the left of y-axis.

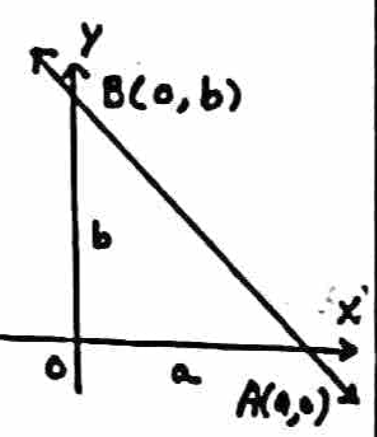
(iii) If $b=0$, then the line becomes the y-axis. Thus the equation of y-axis is $x=0$.





Intercepts:-

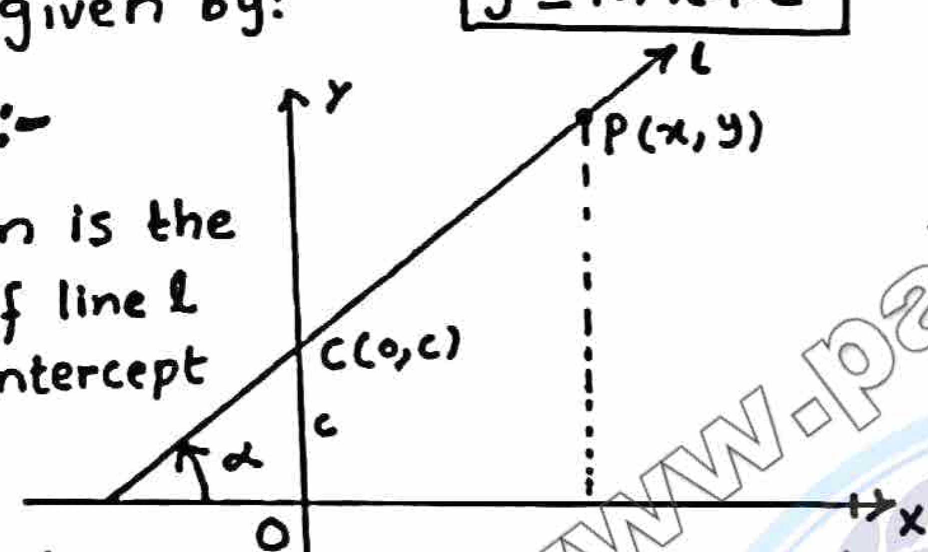
- If a line intersects x-axis at pt. $(a,0)$, then a is called x-intercept of the line.
- If a line intersects y-axis at pt. $(0,b)$, then b is called y-intercept of the line.



Slope-Intercept form

Theorem:- Equation of a non-vertical straight line with slope m and y-intercept c is given by: $\boxed{y=mx+c}$

Proof:- Since m is the slope of line l and y-intercept is c,



So point on y-axis will be $(0,c)$. Let $P(x,y)$ be any point on the line l. ∵ the line l passes through points $C(0,c)$ and $P(x,y)$. So using $m=\frac{y_2-y_1}{x_2-x_1}$

$\Rightarrow m=\frac{y-c}{x-0}=\frac{y-c}{x} \Rightarrow mx=y-c$

$\Rightarrow y=mx+c$ Hence proved.

> **Note:-** The equation of the line for which $c=0$ is $y=mx$. In this case the line passes through the origin.

Example 1. Find an equation of the straight line if (a) its slope is 2 and y-intercept is 5.
(b) it is perpendicular to a line with slope -6 and its y-intercept is $\frac{4}{3}$.

Solution:- (a) Here slope $=m=2$ and y-intercept $=c=5$ so using

$y=mx+c \Rightarrow y=2x+5$ is req. equation.

(b) ∵ Slope of given line $=m_1=-6$

$\Rightarrow$ slope of required line ($\perp$ to given line) is

$m_2=-\frac{1}{m_1}=-\frac{1}{-6}=\frac{1}{6} \Rightarrow m_2=\frac{1}{6}$

Thus required line having slope $m_2=\frac{1}{6}$ (∵ slope of $\perp$ line is -6) and y-intercept $=c=\frac{4}{3}$ so using

$y=mx+c \Rightarrow y=\frac{1}{6}x+\frac{4}{3}$

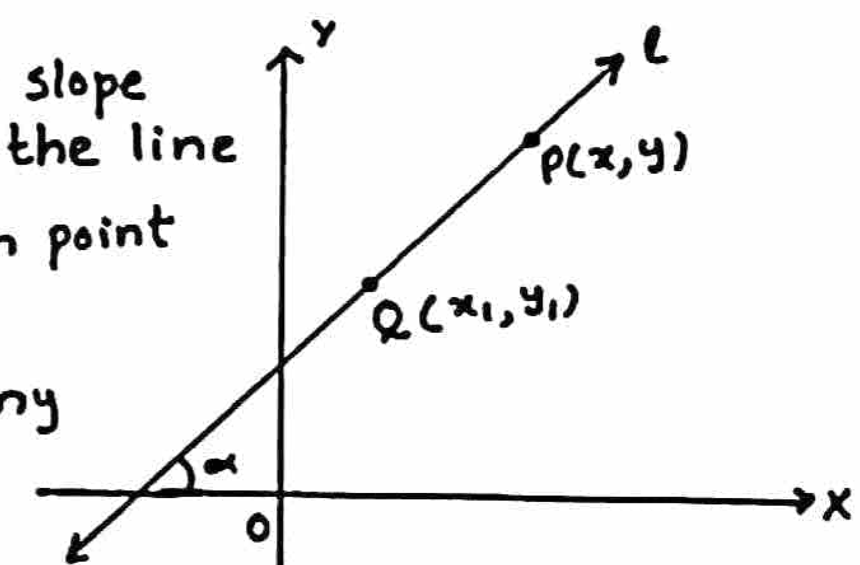
$\Rightarrow 6y=x+8$ is req. equation.

Point-Slope form

Theorem:- Equation of a non-vertical straight line l with slope m and passing through a point $Q(x_1,y_1)$ is

$\boxed{y-y_1=m(x-x_1)}$

Proof:- Since m is the slope of line l and the line passes through point $Q(x_1,y_1)$.
Let $P(x,y)$ be any point on the line l.



Since the line l passes through the points $Q(x_1,y_1)$ and $P(x,y)$ so using $m=\frac{y_2-y_1}{x_2-x_1}$

$\Rightarrow m=\frac{y-y_1}{x-x_1} \Rightarrow m(x-x_1)=y-y_1$

or $y-y_1=m(x-x_1)$ Hence proved.

Symmetric form

We know that

$y-y_1=m(x-x_1)$ $\left(\because m=\tan\alpha=\frac{\sin\alpha}{\cos\alpha}\right)$

$\Rightarrow y-y_1=\frac{\sin\alpha}{\cos\alpha}(x-x_1)$

$\Rightarrow \boxed{\frac{y-y_1}{\sin\alpha}=\frac{x-x_1}{\cos\alpha}=r\ (\text{say})}$ This is called symmetric form of equation of a straight line.

Example 2. Write down an equation of the straight line passing through $(5,1)$ and parallel to a line passing through the points $(0,-1)$, $(7,-15)$.

Solution:- Slope of given line $=\frac{-15-(-1)}{7-0}=\frac{-15+1}{7}$

$=\frac{-14}{7}=-2$ $\left(\because m=\frac{y_2-y_1}{x_2-x_1}\right)$

$\Rightarrow$ Slope of req. line $=-2$

(∵ If two lines are ∥ then they have same slopes)

Thus equation of line having slope $m=-2$ and passing through $(5,1)$ is

$y-1=-2(x-5)$ $(\because y-y_1=m(x-x_1))$

$\Rightarrow y-1=-2x+10$

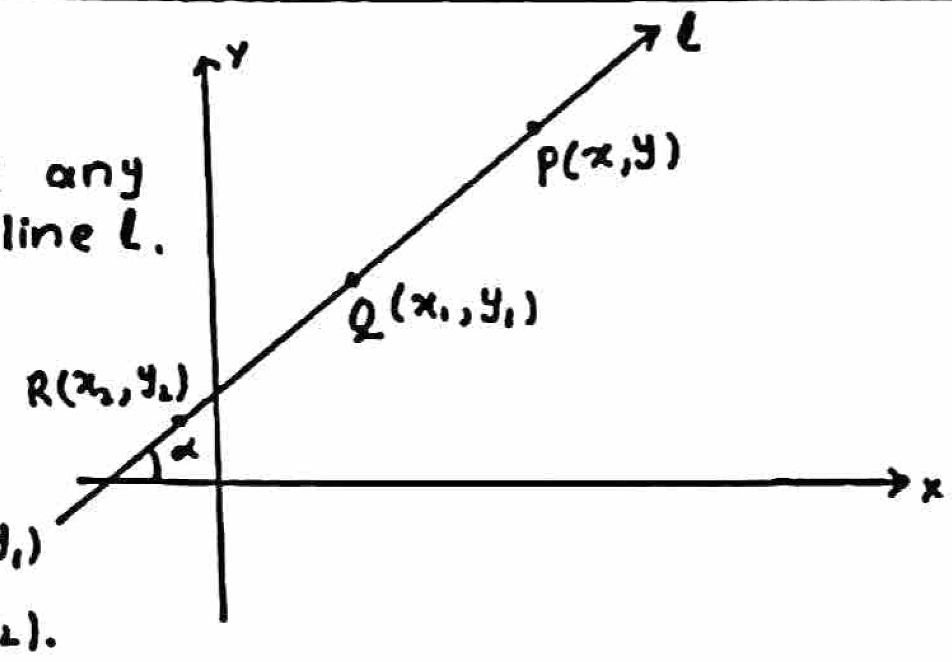
$\Rightarrow -2x+10-y+1=0 \Rightarrow 2x+y-11=0$

Two-point form

Theorem:- Equation of a non-vertical straight line passing through two points $Q(x_1,y_1)$ and $R(x_2,y_2)$ is

$$\boxed{\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \quad \text{or} \quad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0}$$

Proof:-



Let $P(x,y)$ be any point on the line l.
$\because$ Line passes through the points $Q(x_1,y_1)$ and $R(x_2,y_2)$.

As P, Q, R are collinear points. so

slope of QR = slope of QP

$\rightarrow \dfrac{y_2-y_1}{x_2-x_1}=\dfrac{y-y_1}{x-x_1} \rightarrow (y_2-y_1)(x-x_1)=(y-y_1)(x_2-x_1)$

$\rightarrow \boxed{\dfrac{y-y_1}{y_2-y_1}=\dfrac{x-x_1}{x_2-x_1}}$

$\rightarrow y(x_2-x_1)-y_1(x_2-x_1)=x(y_2-y_1)-x_1(y_2-y_1)$

$\rightarrow -x(y_2-y_1)+y(x_2-x_1)+x_1(y_2-y_1)-y_1(x_2-x_1)=0$

$\rightarrow x(y_1-y_2)-y(x_1-x_2)+x_1y_2-\cancel{x_1y_1}-x_2y_1+\cancel{x_1y_1}=0$

$\rightarrow x(y_1-y_2)-y(x_1-x_2)+x_1y_2-x_2y_1=0$

$\rightarrow \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$ Hence proved.

> ★ If $x_1=x_2$, then slope becomes undefined. So, the line is vertical.

Example 3. Find an equation of line through the points $(-2,1)$ and $(6,-4)$.

Solution:- Using two point form

$\dfrac{y-y_1}{y_2-y_1}=\dfrac{x-x_1}{x_2-x_1}$ so

$\rightarrow \dfrac{y-1}{-4-1}=\dfrac{x-(-2)}{6-(-2)} \rightarrow \dfrac{y-1}{-5}=\dfrac{x+2}{6+2}$

$\rightarrow \dfrac{y-1}{-5}=\dfrac{x+2}{8} \rightarrow 8y-8=-5x-10$

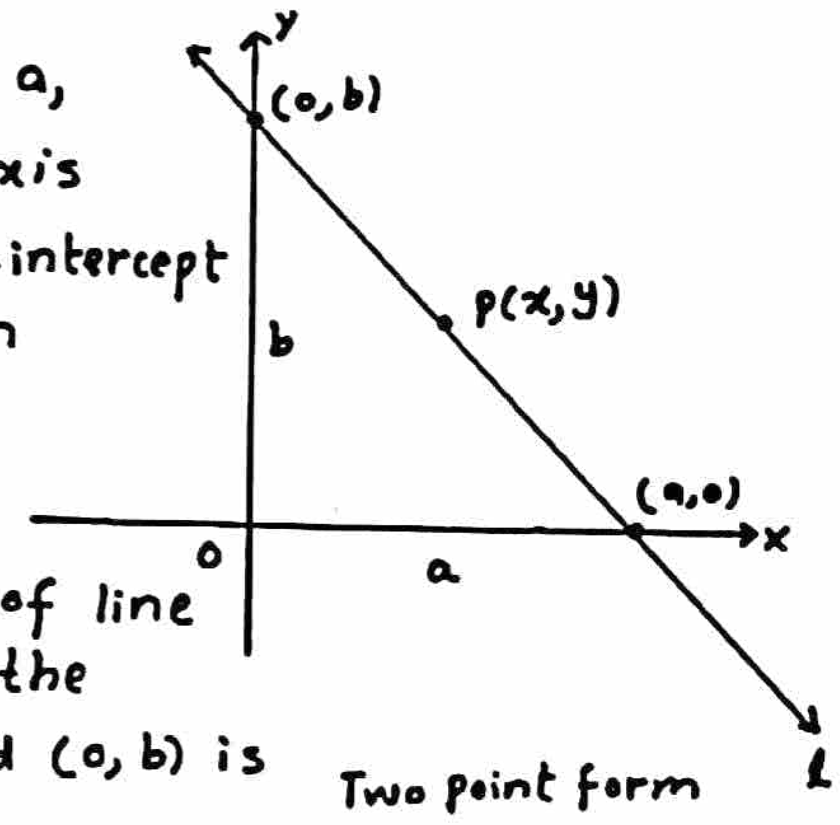
$\rightarrow 5x+8y-8+10=0 \rightarrow 5x+8y+2=0$

Intercept form

Theorem:- Equation of a line whose non-zero x and y-intercepts are a and b resp. is

$$\boxed{\frac{x}{a}+\frac{y}{b}=1}$$

Proof:-



$\because$ x intercept is a, so point on x-axis is $(a,0)$ and y-intercept is b so point on y-axis is $(0,b)$

Hence equation of line passing through the points $(a,0)$ and $(0,b)$ is Two point form

$\dfrac{y-0}{b-0}=\dfrac{x-a}{0-a} \quad \left(\because \dfrac{y-y_1}{y_2-y_1}=\dfrac{x-x_1}{x_2-x_1}\right)$

$\rightarrow \dfrac{y}{b}=\dfrac{x-a}{-a}=\dfrac{x}{-a}+1 \rightarrow \dfrac{y}{b}+\dfrac{x}{a}=1$

$\rightarrow \dfrac{x}{a}+\dfrac{y}{b}=1$ Hence proved.

Example 4. Write down an equation of the line which cuts the x-axis at $(2,0)$ and y-axis at $(0,-4)$.

Solution:- $\because$ line cuts x-axis at $(2,0)$

so x-intercept $= a = 2$

line cuts y-axis at $(0,-4)$ so

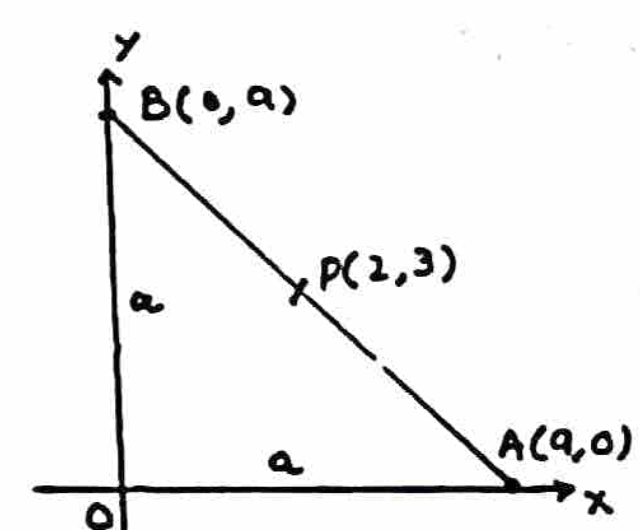
y-intercept $= b = -4$

Eq. of req. line is $\dfrac{x}{2}+\dfrac{y}{-4}=1 \quad \left(\because \dfrac{x}{a}+\dfrac{y}{b}=1\right)$

$\rightarrow 2x-y=4 \rightarrow 2x-y-4=0$

Example 5. Find an equation of the line through the point $P(2,3)$ which forms an isosceles triangle with the coordinate axes in the first quadrant.

Solution:-



Let OAB be an isosceles triangle.
then $|OA|=|OB|$
Let $|OA|=|OB|=a$

Equation of required line is

$\dfrac{x}{a}+\dfrac{y}{a}=1 \quad \left(\because \dfrac{x}{a}+\dfrac{y}{b}=1\right)$

$\rightarrow x+y=a$ —— (I)

$\because$ required line passes through the pt $(2,3)$

so (I) $\rightarrow 2+3=a \rightarrow a=5$

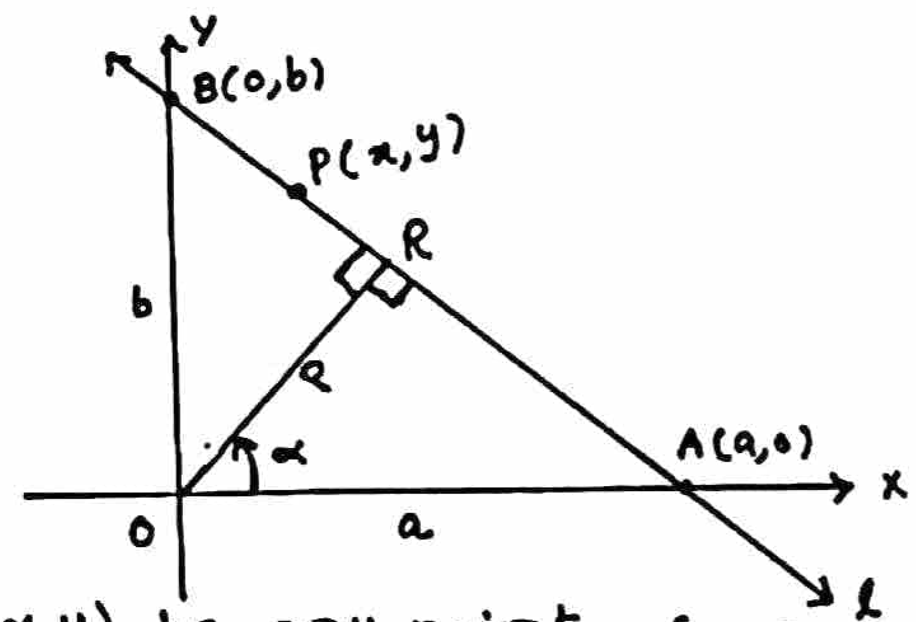
Thus $x+y=5$ is required equation of line.

Normal form

Theorem:- An equation of a non-vertical straight line l, such that length of the perpendicular from the origin to l is P and α is the inclination of this perpendicular is

$$\boxed{x\cos\alpha+y\sin\alpha=P}$$

Proof:-



Let $P(x,y)$ be any point of AB and let OR be $\perp$ar to line l. then $|OR| = p$
Let x-intercept be a and y-intercept be b. so equation of line is

$$\frac{x}{a} + \frac{y}{b} = 1 \longrightarrow (I)$$

In $\triangle AOR$, $\cos\alpha = \frac{OR}{OA} \Rightarrow \cos\alpha = \frac{p}{a}$

$$\Rightarrow a = \frac{p}{\cos\alpha}$$

In $\triangle BOR$, $\sin\alpha = \frac{OR}{OB} = \frac{p}{b}$

$$\Rightarrow b = \frac{p}{\sin\alpha}$$ so (I) becomes as

$$\Rightarrow \frac{x}{p/\cos\alpha} + \frac{y}{p/\sin\alpha} = 1 \Rightarrow \frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1$$

$\Rightarrow x\cos\alpha + y\sin\alpha = p$ Hence proved.

Example 6. The length of perpendicular from origin to a line is 5 units and the inclination of this perpendicular is $120°$. Find the slope and y-intercept of the line.

Solution:- $\because p = 5$ and $\alpha = 120°$
so eq. of required line is:
$x\cos 120° + y\sin 120° = 5$ $\quad (\because x\cos\alpha + y\sin\alpha = p)$ Normal form

$$\Rightarrow x\left(-\tfrac{1}{2}\right) + y\left(\tfrac{\sqrt{3}}{2}\right) = 5$$

$$\Rightarrow -x + \sqrt{3}y = 10 \Rightarrow \sqrt{3}y = x + 10$$

$$\Rightarrow y = \frac{1}{\sqrt{3}}x + \frac{10}{\sqrt{3}} \text{ —— } (I)$$

It is of the form $y = mx + c$ —(II)
i.e., slope-intercept form so
By (I) and (II), slope $= m = \frac{1}{\sqrt{3}}$ and
y-intercept $= c = \frac{10}{\sqrt{3}}$

Linear Equation in two variables

(General equation of straight line)

Theorem:- The linear equation $\boxed{ax + by + c = 0}$ in two variables x and y represents a straight line.

Proof:- Consider general linear equation in x and y

$$ax + by + c = 0 \text{ —— } (I)$$

where a, b, c are constants and $a \neq 0$, $b \neq 0$ simultaneously. so following cases arise:

Case I :- Let $a \neq 0$ but $b = 0$ so
(I) $\Rightarrow ax + 0y + c = 0 \Rightarrow ax + c = 0 \Rightarrow x = -\frac{c}{a}$
which is eq. of line $\parallel$ to y-axis.

Case II :- Let $a = 0$ but $b \neq 0$ so
(I) $\Rightarrow a(0) + by + c = 0 \Rightarrow by + c = 0 \Rightarrow y = -\frac{c}{b}$
which is eq. of line $\parallel$ to x-axis.

Case III :- Let $a \neq 0$, $b \neq 0$ so
(I) $\Rightarrow ax + by + c = 0 \Rightarrow by = -ax - c$
$\Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$ which is of the form $y = mx + c$ (A line in slope-intercept form). Hence in all cases, $ax + by + c = 0$ represents a line.

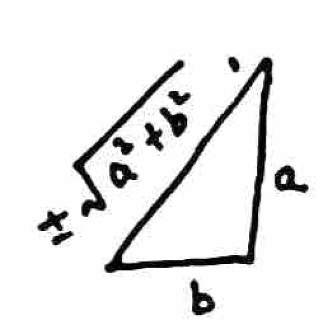
Transform the General Linear Equation to Standard Forms

Theorem:- To transform the equation $ax + by + c = 0$ in the standard form.

1. Slope-intercept form:- $(y = mx + c)$
$\because ax + by + c = 0$
$\Rightarrow by = -ax - c \Rightarrow y = \left(-\frac{a}{b}\right)x + \left(-\frac{c}{b}\right)$
Here slope $= m = -\frac{a}{b}$ and intercept $= -\frac{c}{b}$

2. Point-Slope form:- $(y - y_1 = m(x - x_1))$
A point on the line is $\left(-\frac{c}{b}, 0\right)$ and slope is $-a/b$ so
$y - 0 = -\frac{a}{b}\left(x + \frac{c}{b}\right)$ This is point slope form.

3. Symmetric form:- $\left(\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\sin\alpha}\right)$
$\Rightarrow \sin\alpha = \frac{a}{\pm\sqrt{a^2+b^2}}$, $\cos\alpha = \frac{b}{\pm\sqrt{a^2+b^2}}$ $\because m = \tan\alpha = -\frac{a}{b}$



and point on $ax + by + c = 0$ is $\left(-\frac{c}{a}, 0\right)$ so

$$\frac{x - \left(-\frac{c}{a}\right)}{b/\pm\sqrt{a^2+b^2}} = \frac{y - 0}{b/\pm\sqrt{a^2+b^2}}$$

is required symmetric form and sign of radical to be properly chosen.

4. Two-point form:- $\left(\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}\right)$

We take two points on $ax+by+c=0$ are $\left(-\frac{c}{a},0\right)$ and $\left(0,-\frac{c}{b}\right)$. so required transformed eq. is

$$\frac{y-0}{0+\frac{c}{b}} = \frac{x+\frac{c}{a}}{-\frac{c}{a}-0} \quad \left(\because \frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}\right)$$

5. Intercept form:- $\left(\frac{x}{a}+\frac{y}{b}=1\right)$

$\because ax+by+c=0 \rightarrow ax+by=-c$

$\rightarrow \frac{a}{-c}x+\frac{b}{-c}y = 1$

which is eq. of required two intercepts form.

6. Normal form:- $(x\cos\alpha+y\sin\alpha=p)$

$\because ax+by+c=0$ —— (I)

and $x\cos\alpha+y\sin\alpha=p$ —— (II) (Normal form)

As (I) and (II) are identical so

$\frac{a}{\cos\alpha}=\frac{b}{\sin\beta}=\frac{-c}{p}$ —— (III)

$\because m=\tan\alpha=-\frac{a}{b}$ so $\sin\alpha=\frac{a}{\pm\sqrt{a^2+b^2}}$

$\cos\alpha=\frac{b}{\pm\sqrt{a^2+b^2}}$ so

$\because \frac{p}{-c}=\frac{\cos\alpha}{a}=\frac{\sin\alpha}{b}$

$=\frac{\sqrt{\cos^2\alpha+\sin^2\beta}}{\pm\sqrt{a^2+b^2}}=\frac{1}{\pm\sqrt{a^2+b^2}}$

so (II) $\rightarrow \frac{ax+by}{\pm\sqrt{a^2+b^2}}=\frac{-c}{\pm\sqrt{a^2+b^2}}$ (sign of radical to be properly chosen)

Example 1. Transform the equation $5x-12y+39=0$ into

(i) Slope intercept form (ii) Two intercept form
(iii) Normal form (iv) point slope form
(v) Two point form (vi) Symmetric form

Solution:- $5x-12y+39=0$ —— (I)

(i) Slope intercept form:- $(y=mx+c)$

(I) $\rightarrow -12y=-5x-39 \rightarrow y=\frac{5}{12}x+\frac{39}{12}$

which is required form.

slope $=m=\frac{5}{12}$ and y-intercept $=c=\frac{39}{12}$

(ii) Two-intercept form:- $\left(\frac{x}{a}+\frac{y}{b}=1\right)$

(I) $\rightarrow 5x-12y=-39$

$\rightarrow \frac{5x}{-39}-\frac{12y}{-39}=1$

$\rightarrow \frac{x}{-39/5}+\frac{y}{\frac{39}{12}}=1$ x-intercept $=a=\frac{-39}{5}$, y-intercept $=b=\frac{39}{12}$

(iii) Normal form:- $(x\cos\alpha+y\sin\alpha=p)$

(I) $\rightarrow 5x-12y=-39$

$\rightarrow -5x+12y=39$

Dividing both sides by $\sqrt{(-5)^2+(12)^2}$

$=\sqrt{25+144}=\sqrt{169}=13$

$\rightarrow -\frac{5}{13}x+\frac{12}{13}y=\frac{39}{13}$

$\rightarrow x\left(-\frac{5}{13}\right)+y\left(\frac{12}{13}\right)=3$ $\cos\alpha=-\frac{5}{13}$, $\sin\alpha=\frac{12}{13}$, $p=3$

which is req. form

(iv) Point-slope form:- $(y-y_1=m(x-x_1))$

(I) $\rightarrow 12y=5x+39$

$\rightarrow y=\frac{5x}{12}+\frac{39}{12}$ $m=\frac{5}{12}$

For x-intercept put $y=0$ so

$0=\frac{5}{12}x+\frac{39}{12} \rightarrow \frac{5x}{12}=-\frac{39}{12}$

$\rightarrow 5x=-39 \rightarrow x=-\frac{39}{5}$

so the pt $\left(-\frac{39}{5},0\right)$ and slope $\frac{5}{12}$

so $y-0=\frac{5}{12}\left(x-\left(-\frac{39}{5}\right)\right)$ req. form.

(v) Two point form:- $\left(\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}\right)$

For x-intercept put $y=0$ in (I)

so $5x-12(0)+39=0 \rightarrow x=\frac{-39}{5}$

so pt on x-axis is $\left(-\frac{39}{5},0\right)$

For y-intercept put $x=0$ in (I)

so $5(0)-12y+39=0 \rightarrow y=\frac{39}{12}$

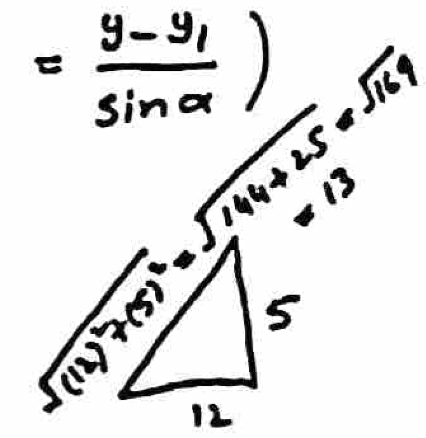
so the pt on y-axis is $\left(0,\frac{39}{12}\right)$

Thus $\frac{y-0}{\frac{39}{12}-0}=\frac{x-(-39/5)}{0-(-39/5)}$ req. form

(vi) Symmetric form:- $\left(\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\sin\alpha}\right)$

$\because m=\tan\alpha=\frac{5}{12}$

so $\sin\alpha=\frac{5}{13}$, $\cos\alpha=\frac{12}{13}$



As $\left(-\frac{39}{5},0\right)$ lies on line so

$\frac{x-(-39/5)}{12/13}=\frac{y-0}{5/13}$ req. form

Example 2. Sketch the line $3x+2y+6=0$

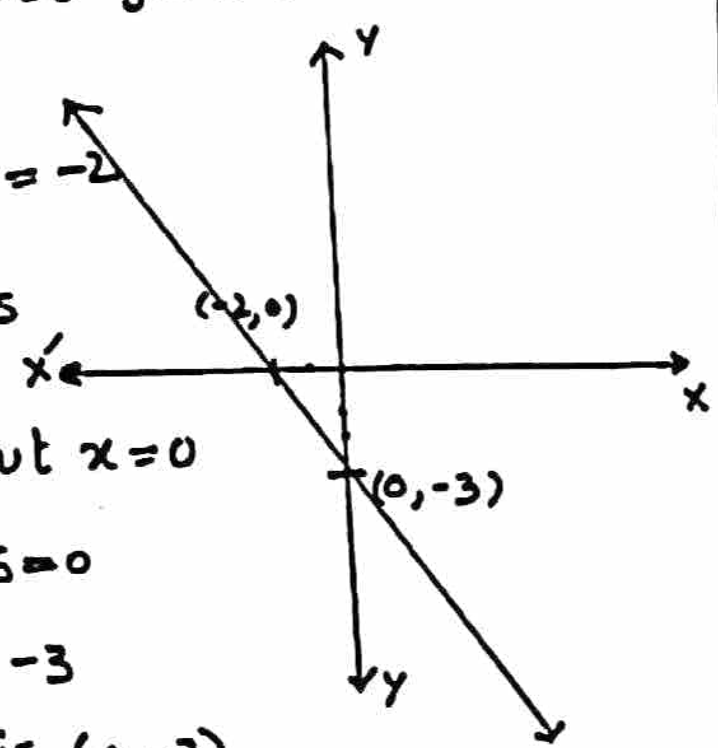
Solution:- $\because 3x+2y+6=0$ —— (I)

For x-intercept; put $y=0$ in (I)

$3x+2(0)+6=0$

$\rightarrow 3x=-6 \rightarrow x=-2$

so point on x-axis is $(-2,0)$



For y-intercept; put $x=0$ in (I) $3(0)+2y+6=0$

$\rightarrow 2y=-6 \rightarrow y=-3$

so pt. on y-axis is $(0,-3)$

By Joining $(-2,0)$ and $(0,-3)$ we sketch $3x+2y+6=0$

Example 3. Find the distance between the parallel lines $2x+y+2=0$, and $6x+3y-8=0$. Sketch the lines. Also find an equation of line parallel to the given lines and lying midway between them.

Solution:- l_1; $2x+y+2=0$, l_2; $6x+3y-8=0$

For l_1, Put $x=0$, $\rightarrow 2(0)+y+2=0 \rightarrow y=-2$

Put $y=0$, $\rightarrow 2x+0+2=0 \rightarrow x=-1$

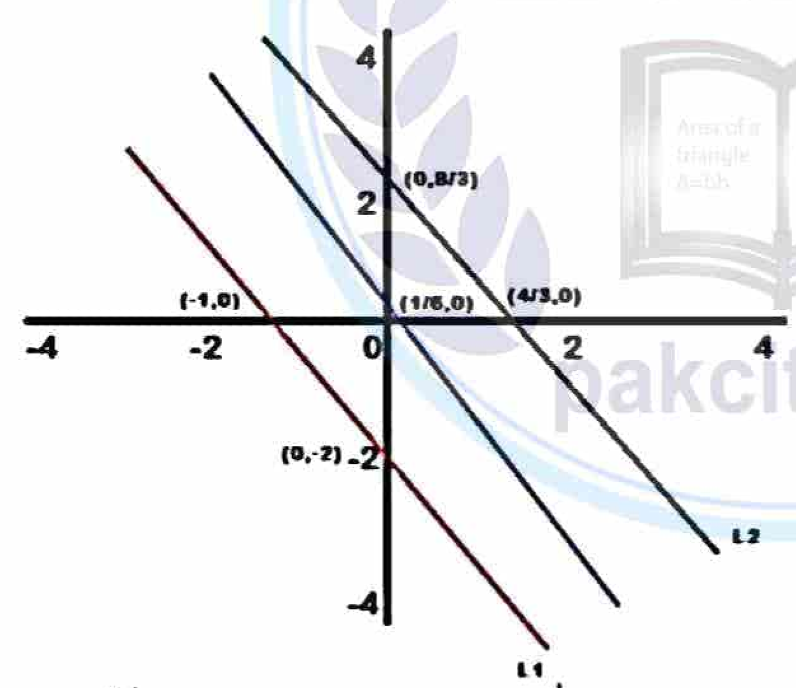
so $(0,-2)$ and $(-1,0)$ on l_1

For l_2, Put $x=0$, $\rightarrow 6(0)+3y-8=0 \rightarrow y=\frac{8}{3}$

Put $y=0$, $\rightarrow 6x+3(0)-8=0 \rightarrow x=\frac{8}{6}=\frac{4}{3}$

so $(0,\frac{8}{3})$ and $(\frac{4}{3},0)$ on l_2

Now distance d from $(-1,0)$ to l_2 is



$$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

$$=\frac{|6(-1)+3(0)-8|}{\sqrt{(6)^2+(3)^2}}$$

$$=\frac{|-6-8|}{\sqrt{36+9}}=\frac{|-14|}{\sqrt{45}}=\frac{14}{\sqrt{9\times5}}$$

$\rightarrow d=\frac{14}{3\sqrt5}$ Thus distance between the parallel lines $\frac{14}{3\sqrt5}$.

Now mid point of $(-1,0)$ and $(\frac{4}{3},0)$ is

$$=\left(\frac{-1+\frac{4}{3}}{2},\frac{0+0}{2}\right)=\left(\frac{\frac{-3+4}{3}}{2},0\right)=\left(\frac{1}{6},0\right)$$

$\because$ slope $=m=-\frac{a}{b}=-\frac{2}{1}=-2$

Now required equation of line passing through point $(\frac{1}{6},0)$ and slope -2 is $(\because y-y_1=m(x-x_1))$

$\rightarrow y-0=-2(x-\frac{1}{6}) \rightarrow y=-2x+\frac{1}{3}$

$\rightarrow 3y=-6x+1 \rightarrow 6x+3y=1$

Position of a point with respect to a line

Theorem:- Let $P(x_1,y_1)$ be a point in the plane not lying on l

$l: ax+by+c=0$ then P lies

a) above the line l if $ax_1+by_1+c>0$

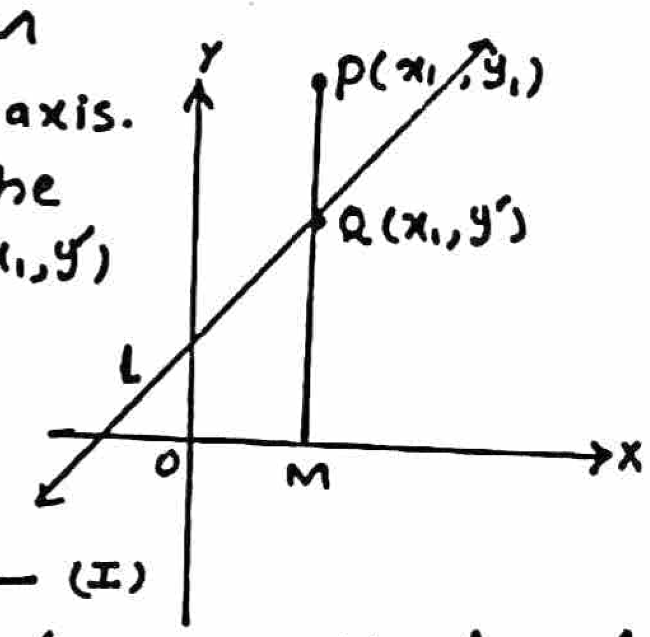
b) below the line l if $ax_1+by_1+c<0$

Proof:- (a)

Let we draw $\perp$ PM from point P on x-axis. s.that it meets the line L at point $Q(x_1,y')$

The point P will lie above line L if

$y_1>y'$ or $y_1-y'>0$ —— (I)



As the point $Q(x_1,y')$ lies on the line l;

l; $ax+by+c=0 \rightarrow ax_1+by'+c=0$

$\rightarrow by'=-ax_1-c \rightarrow y'=-\frac{a}{b}x_1-\frac{c}{b}$ put in (I)

so $y_1-(-\frac{a}{b}x_1-\frac{c}{b})>0$

$\rightarrow y_1+\frac{a}{b}x_1+\frac{c}{b}>0 \rightarrow by_1+ax_1+c>0$

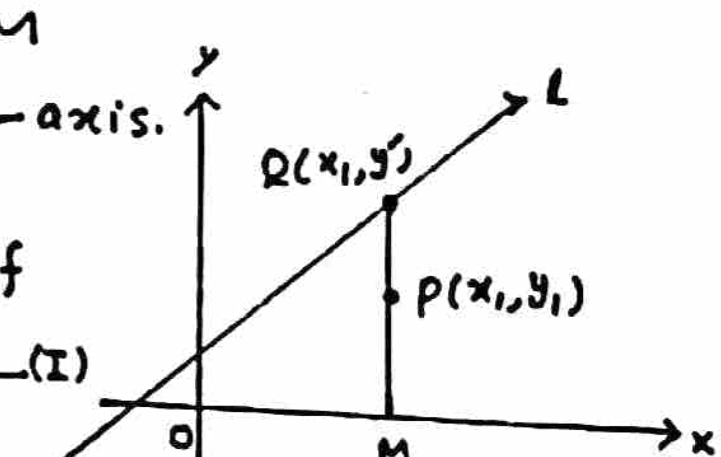
or $ax_1+by_1+c>0$ Hence proved.

(b)

Let us draw $\perp$ QM from point Q on x-axis.

The point P will lie below the line l if

$y'>y_1$ or $y_1-y'<0$ —— (I)



As the point $Q(x_1,y')$ lie on the line l;

l; $ax+by+c=0$

$\rightarrow ax_1+by'+c=0 \rightarrow by'=-ax_1-c$

$\rightarrow y'=-\frac{a}{b}x_1-\frac{c}{b}$ put in (I)

so $y_1-(-\frac{a}{b}x_1-\frac{c}{b})<0$

$\rightarrow y_1+\frac{a}{b}x_1+\frac{c}{b}<0 \rightarrow by_1+ax_1+c<0$

$\rightarrow ax_1+by_1+c<0$ Hence proved.

Corollary 1.

The point P is above or below ℓ respectively if ax_1+by_1+c and b have the same sign or have opposite signs.

Proof:- $\because$ If $P(x_1,y_1)$ above ℓ then

$$y_1 - y' > 0 \Rightarrow y_1 - \left(-\frac{a}{b}x_1 - \frac{c}{b}\right) > 0$$

$$\Rightarrow y_1 + \frac{a}{b}x_1 + \frac{c}{b} > 0 \Rightarrow \frac{ax_1+by_1+c}{b} > 0$$

It is only possible if ax_1+by_1+c and b have same signs.

Similarly, $P(x_1,y_1)$ below ℓ then

$$y_1 - y' < 0 \Rightarrow y_1 - \left(-\frac{a}{b}x_1 - \frac{c}{b}\right) < 0$$

$$\Rightarrow y_1 + \frac{a}{b}x_1 + \frac{c}{b} < 0 \Rightarrow \frac{ax_1+by_1+c}{b} < 0$$

It is only possible if ax_1+by_1+c and b have opposite sign.

Corollary 2.

The point $P(x_1,y_1)$ and origin are

(i) on the same side of ℓ according as ax_1+by_1+c and c have the same sign

(ii) on the opposite side of ℓ according as ax_1+by_1+c and c have opposite sign.

Proof:- (i) The point $P(x_1,y_1)$ and $O(0,0)$ are same side of ℓ if ax_1+by_1+c and $a(0)+b(0)+c$ have same sign.

(ii) The point $P(x_1,y_1)$ and $O(0,0)$ are opposite side of ℓ if ax_1+by_1+c and $a(0)+b(0)+c$ have opposite sign.

Example 1. Check whether the point $(-2,4)$ lies above or below the line

$$4x+5y-3=0$$

Solution:- $4x+5y-3=0$ —— (I)

Comparing (I) with $ax+by+c=0$ we have

$a=4,\ b=5,\ c=-3$ Here $b>0$

Now put $(-2,4)$ in (I) $\Rightarrow 4(-2)+5(4)-3$

$= -8+20-3 = 9 > 0$ Hence the point $(-2,4)$ lies above the line. $4x+5y-3=0$

Example 2. Check whether the origin and the point $P(5,-8)$ lie on the same side or on the opposite sides of the line:

$$3x+7y+15=0$$

Solution:- $3x+7y+15=0$ —— (I)

Comparing (I) with $ax+by+c=0$ we have

$a=3,\ b=7,\ c=15$ Here $c>0$

Put $(5,-8)$ in (I) $\Rightarrow 3(5)+7(-8)+15$

$= 15-56+15 = -26 < 0$

Hence the origin and the point $(5,-8)$ lies on the opposite sides of line (I).

Two and three straight lines

For any two distinct lines ℓ_1, ℓ_2

$\ell_1: a_1x+b_1y+c_1=0$ and $\ell_2: a_2x+b_2y+c_2=0$

one and only one of following holds:

(i) $\ell_1 \parallel \ell_2$ (ii) $\ell_1 \perp \ell_2$ (iii) ℓ_1 and ℓ_2 are not related as (i) or (ii)

slope of $\ell_1 = m_1 = -\dfrac{a_1}{b_1}$

slope of $\ell_2 = m_2 = -\dfrac{a_2}{b_2}$

(i) $\ell_1 \parallel \ell_2$

$\because$ For parallel lines slopes are equal. so

$\Rightarrow$ slope of ℓ_1 = slope of ℓ_2

$$\Rightarrow m_1 = m_2 \Rightarrow -\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$

$$\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} \Rightarrow a_1b_2 = a_2b_1$$

$$\Rightarrow a_1b_2 - a_2b_1 = 0$$

(ii) $\ell_1 \perp \ell_2$

$\because$ For perpendicular lines, product of their slopes equal to -1. so

(slope of ℓ_1)(slope of ℓ_2) $= -1$

$$\Rightarrow m_1m_2 = -1 \Rightarrow \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right) = -1$$

$$\Rightarrow \frac{a_1a_2}{b_1b_2} = -1 \Rightarrow a_1a_2 = -b_1b_2$$

$$\Rightarrow a_1a_2 + b_1b_2 = 0$$

(iii) If ℓ_1 and ℓ_2 are not related as in (i) or (ii), then there is no simple relation of the above forms.

The point of Intersection of Two straight lines

Let $\ell_1: a_1x+b_1y+c_1=0 \longrightarrow$ (I)

$\ell_2: a_2x+b_2y+c_2=0 \longrightarrow$ (II)

be two non-parallel lines

Remember, Two non-parallel lines intersect each other at one and only one point.

Let $P(x_1,y_1)$ be the point of intersection of lines ℓ_1 and ℓ_2

solving (I) and (II) by cross-multiplication method, we have

$$\because \begin{matrix} a_1 & b_1 & c_1 & a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 & c_2 \end{matrix}$$

$$\frac{x_1}{b_1c_2-b_2c_1} = \frac{y_1}{c_1a_2-c_2a_1} = \frac{1}{a_1b_2-a_2b_1}$$

$$\Rightarrow \frac{x_1}{b_1c_2-b_2c_1} = \frac{1}{a_1b_2-a_2b_1} \text{ and } \frac{y_1}{c_1a_2-c_2a_1} = \frac{1}{a_1b_2-a_2b_1}$$

$$\Rightarrow x_1 = \frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1} \text{ and } y_1 = \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$$

Thus $P(x_1,y_1) = \left(\dfrac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \dfrac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)$

Note:- $a_1b_2 - a_2b_1 \neq 0$ otherwise $l_1 \parallel l_2$

Example 1. Find the point of intersection of the lines $5x+7y=35$, $3x-7y=21$

Solution:- $5x+7y=35$ —— (I)
$3x-7y=21$ —— (II)

adding (I) and (II) we get
$\rightarrow 8x=56 \rightarrow x=7$ put in (I)
$\rightarrow 5(7)+7y=35 \rightarrow 35+7y=35$
$\rightarrow 7y=35-35 \rightarrow 7y=0 \rightarrow y=0$

Hence required pt. of intersection is $(7,0)$.

Condition of Concurrency of three straight lines

Three non-parallel lines
$l_1;\ a_1x+b_1y+c_1=0$, $l_2;\ a_2x+b_2y+c_2=0$
$l_3;\ a_3x+b_3y+c_3=0$ are concurrent iff
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Proof:- we know that point of intersection of lines l_1 and l_2 is
$P\left(\dfrac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \dfrac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)$ $\because$ the lines are concurrent so l_3 will also pass through this point. then l_3 becomes

$$\rightarrow a_3\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)+c_3=0$$
$$\rightarrow a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0$$

It can be written in determinant form,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is a necessary and sufficient condition of concurrency of the given three lines.

Example 1. Check whether the following lines are concurrent or not. If concurrent find the point of concurrency. $3x-4y-3=0$, $5x+12y+1=0$, $32x+4y-17=0$

Solution:- we know that three lines are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

so,
$$\begin{vmatrix} 3 & -4 & -3 \\ 5 & 12 & 1 \\ 32 & 4 & -17 \end{vmatrix}$$
$= 3(-204-4)-5(68+12)+32(-4+36)$
$= 3(-208)-5(80)+32(32)$
$= -624-400+1024 = -1024+1024 = 0$

Thus given lines are concurrent.

Now we find point of concurrency of any two given lines so
$3x-4y-3=0$ —— ①, $32x+4y-17=0$ —— ②

① + ② $\rightarrow$
$$\begin{array}{r} 3x-\cancel{4y}-3=0 \\ 32x+\cancel{4y}-17=0 \\ \hline 35x \quad -20=0 \end{array}$$
$\rightarrow x=\frac{20}{35}=\frac{4}{7}$

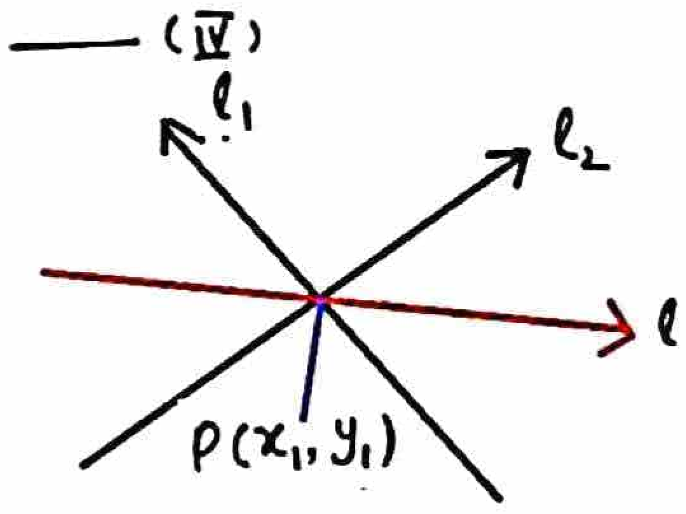
put $x=\frac{4}{7}$ in ① $\rightarrow 3\left(\frac{4}{7}\right)-4y-3=0$
$\rightarrow \frac{12}{7}-4y-3=0 \rightarrow -4y = 3-\frac{12}{7} = \frac{21-12}{7}$
$\rightarrow -4y = \frac{9}{7} \rightarrow y = -\frac{9}{28}$ so pt. of intersection is $\left(\frac{4}{7}, -\frac{9}{28}\right)$

Equation of Lines through the point of intersection of two lines

Consider $l_1;\ a_1x+b_1y+c_1=0$ —— (I)
$l_2;\ a_2x+b_2y+c_2=0$ —— (II)

Let $P(x_1,y_1)$ be the point of intersection of lines l_1 and l_2 so (I) and (II) becomes
as, $a_1x_1+b_1y_1+c_1=0$ —— (III)
and $a_2x_1+b_2y_1+c_2=0$ —— (IV)

Consider
$$\boxed{l;\ l_1+Kl_2=0}$$



$\rightarrow a_1x+b_1y+c_1+K(a_2x+b_2y+c_2)=0$ —— (V)
$\rightarrow a_1x+b_1y+c_1+Ka_2x+Kb_2y+Kc_2=0$
$\rightarrow a_1x+Ka_2x+b_1y+Kb_2y+c_1+Kc_2=0$
$\rightarrow (a_1+Ka_2)x+(b_1+Kb_2)y+(c_1+Kc_2)=0$

which is of the form $ax+by+c=0$
Hence (V) represents a straight line.
For different values of K, (V) represents different lines. so it is also called family of lines.

Note:- Now line (V) will pass through the point $P(x_1,y_1)$ if it satisfies the eq. of line (V) i.e., $a_1x_1+b_1y_1+c_1+K(a_2x_1+b_2y_1+c_2)=0$
$\because a_1x_1+b_1y_1+c_1=0$ and $a_2x_1+b_2y_1+c_2=0$
so L.H.S $= a_1x_1+b_1y_1+c_1+K(a_2x_1+b_2y_1+c_2)$
$= 0+K(0) = 0 =$ R.H.S

Example 2. Find the family of lines through the point of intersection of the lines $3x-4y-10=0$, $x+2y-10=0$. Find the member of the faimily which is (i) parallel to a line with slope $-\frac{2}{3}$ (ii) perpendicular to the line $l: 3x-4y+1=0$

Solution:- we know that a family of lines passing through two given lines is

$$l_1+kl_2=0$$

For given lines we have,

$3x-4y-10+k(x+2y-10)=0$ (req. family of lines) —(I)

$\rightarrow 3x-4y-10+kx+2ky-10k=0$

$\rightarrow (3+k)x+(2k-4)y-(10+10k)=0$

$\rightarrow (2k-4)y=-(3+k)x+(10+10k)$

$\rightarrow y=-\left(\frac{3+k}{2k-4}\right)x+\frac{10+10k}{2k-4}$

Comparing with $y=mx+c$ so

slope of req. eq $=m=-\left(\frac{3+k}{2k-4}\right)$

slope of given line $=-\frac{2}{3}$

(i) For $\parallel$ lines, slopes are equal. so

$-\left(\frac{3+k}{2k-4}\right)=-\frac{2}{3} \rightarrow 3(3+k)=2(2k-4)$

$\rightarrow 9+3k=4k-8 \rightarrow 3k-4k=-8-9$

$\rightarrow -k=-17 \rightarrow \boxed{k=17}$ so (I)

$\rightarrow 3x-4y-10-17(x+2y-10)=0$

$\rightarrow 3x-4y-10-17x-34y+170=0$

$\rightarrow 20x+30y-180=0$ ($\div$ by 10)

$\rightarrow 2x+3y-18=0$

(ii) For $\perp$ar lines, product of slopes $=-1$

Given line is $3x-4y+1=0$

$\rightarrow y=\frac{3}{4}x+\frac{1}{4} \rightarrow$ slope $=\frac{3}{4}$ so

$-\left(\frac{3+k}{2k-4}\right)\left(\frac{3}{4}\right)=-1 \rightarrow \frac{9+3k}{8k-16}=1$

$\rightarrow 9+3k=8k-16 \rightarrow 8k-3k=9+16$

$\rightarrow 5k=25 \rightarrow \boxed{k=5}$ so (I)

$3x-4y-10+5(x+2y-10)=0$

$\rightarrow 3x-4y-10+5x+10y-50=0$

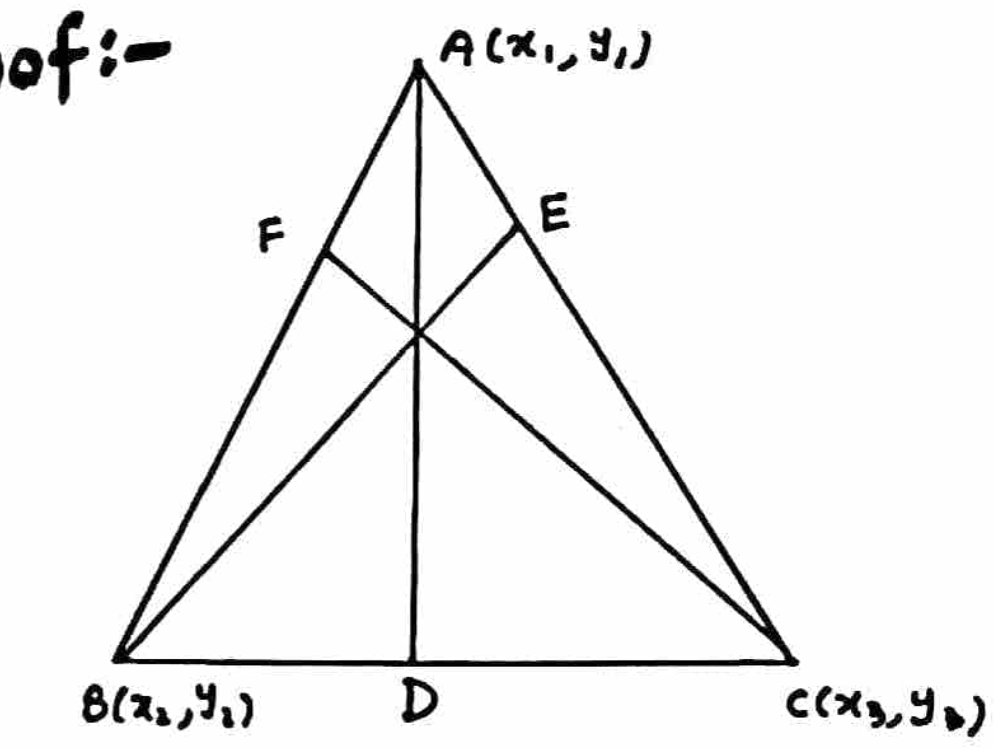
$\rightarrow 8x+6y-60=0$ $\div$ by 2

$\rightarrow 4x+3y-30=0$

*An infinite number of lines can pass through a point.

Theorem:- Altitudes of a triangle are concurrent.

Proof:-



Let $A(x_1,y_1)$, $B(x_2,y_2)$, and $C(x_3,y_3)$ be vertices of ΔABC.
Draw $\perp$ars AD, BE and CF from vertices A, B and C on the sides BC, AC and AB resp. AD, BE and CF are altitudes of ΔABC.

$\because$ slope of side $BC=\frac{y_3-y_2}{x_3-x_2}$

$\rightarrow$ slope of altitude $AD=-\left(\frac{x_3-x_2}{y_3-y_2}\right)$ ($\because AD\perp BC$)

so eq. of altitude AD is

$y-y_1=-\left(\frac{x_3-x_2}{y_3-y_2}\right)(x-x_1)$ (point-slope form)

$\rightarrow (y-y_1)+\left(\frac{x_3-x_2}{y_3-y_2}\right)(x-x_1)=0$

$\rightarrow (y-y_1)(y_3-y_2)+(x_3-x_2)(x-x_1)=0$

$\rightarrow x(x_3-x_2)+y(y_3-y_2)-x_1(x_3-x_2)-y_1(y_3-y_2)=0$

so eqs. of altitudes BE and CF respectively (By symmetry)

$x(x_3-x_1)+y(y_3-y_1)-x_2(x_3-x_1)-y_2(y_3-y_1)=0$

$x(x_2-x_1)+y(y_2-y_1)-x_3(x_2-x_1)-y_3(y_2-y_1)=0$

Now altitudes will be concurrent if

$$\begin{vmatrix} x_3-x_2 & y_3-y_2 & -x_1(x_3-x_2)-y_1(y_3-y_2) \\ x_3-x_1 & y_3-y_1 & -x_2(x_3-x_1)-y_2(y_3-y_1) \\ x_2-x_1 & y_2-y_1 & -x_3(x_2-x_1)-y_3(y_2-y_1) \end{vmatrix}=0$$

Now taking (-1) as common from R_2

$$=(-1)\begin{vmatrix} x_3-x_2 & y_3-y_2 & -x_1(x_3-x_2)-y_1(y_3-y_2) \\ x_1-x_3 & y_1-y_3 & x_2(x_3-x_1)+y_2(y_3-y_1) \\ x_2-x_1 & y_2-y_1 & -x_3(x_2-x_1)-y_3(y_2-y_1) \end{vmatrix}$$

By $R_1+(R_2+R_3)$

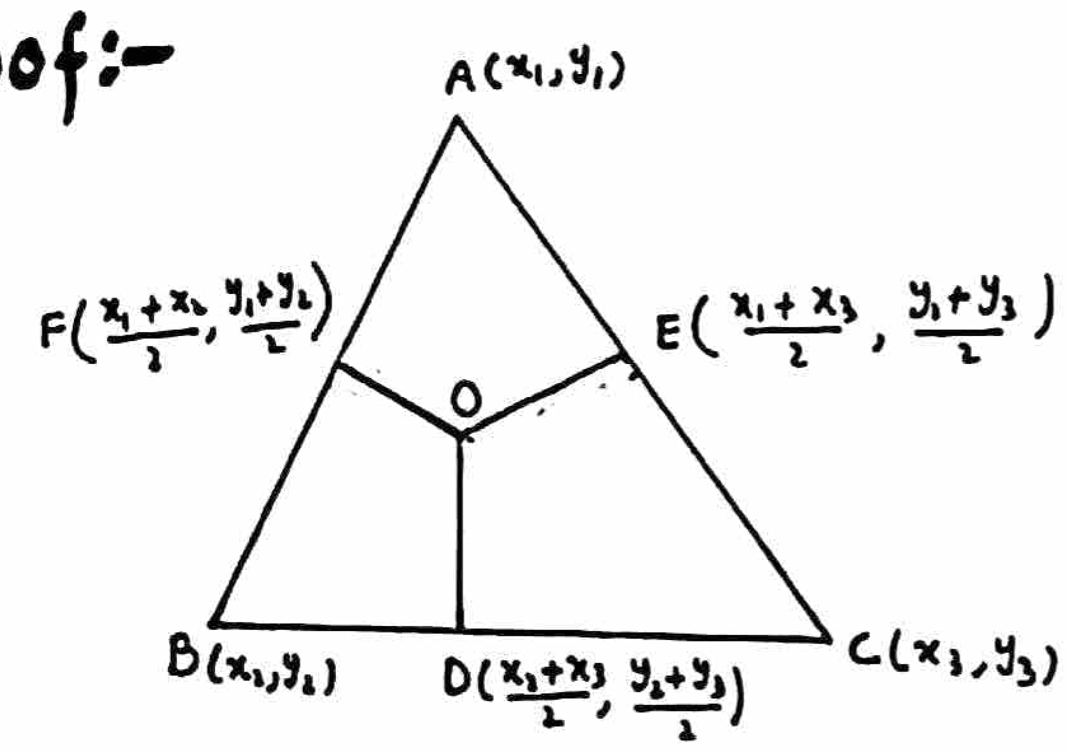
$$=(-1)\begin{vmatrix} 0 & 0 & 0 \\ x_1-x_3 & y_1-y_3 & x_2(x_3-x_1)+y_2(y_3-y_1) \\ x_2-x_1 & y_2-y_1 & -x_3(x_2-x_1)-y_3(y_2-y_1) \end{vmatrix}$$

$=0$ ($\because R_1=0$)

Thus altitudes of a triangle are concurrent.

Theorem:- Right bisectors of a triangle are concurrent.

Proof:-



Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ be vertices of $\triangle ABC$. Let D, E and F be mid-points of the sides BC, AC and AB resp. so OD, OE and OF are right bisectors.

coordinates of D are $\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$

coordinates of E are $\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$

coordinates of F are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$\because$ slope of side BC $= \frac{y_3-y_2}{x_3-x_2}$ $(\because OD \perp BC)$

$\Rightarrow$ slope of right bisector OD $= -\left(\frac{x_3-x_2}{y_3-y_2}\right)$

so eq. of right bisector OD is:

$$y-\left(\frac{y_2+y_3}{2}\right) = -\left(\frac{x_3-x_2}{y_3-y_2}\right)\left(x-\left(\frac{x_2+x_3}{2}\right)\right) \quad \text{(point-slope form)}$$

$$\Rightarrow y-\left(\frac{y_2+y_3}{2}\right)+\left(\frac{x_3-x_2}{y_3-y_2}\right)\left(x-\left(\frac{x_2+x_3}{2}\right)\right)=0$$

$$\Rightarrow y(y_3-y_2)-(y_3-y_2)\left(\frac{y_2+y_3}{2}\right)+(x_3-x_2)\left(x-\left(\frac{x_2+x_3}{2}\right)\right)=0$$

$$\Rightarrow x(x_3-x_2)+y(y_3-y_2)-\frac{1}{2}(x_3-x_2)(x_3+x_2) -\frac{1}{2}(y_3+y_2)(y_3-y_2)=0$$

$$\Rightarrow x(x_3-x_2)+y(y_3-y_2)-\frac{1}{2}(x_3^2-x_2^2)-\frac{1}{2}(y_3^2-y_2^2)=0$$

Equations of the other two right bisectors OE and OF are (By symmetry)

$$\Rightarrow x(x_3-x_1)+y(y_3-y_1)-\frac{1}{2}(x_3^2-x_1^2)-\frac{1}{2}(y_3^2-y_1^2)=0$$

and

$$x(x_2-x_1)+y(x_2-x_1)-\frac{1}{2}(x_2^2-x_1^2)-\frac{1}{2}(y_2^2-y_1^2)=0$$

Right bisectors will be concurrent if

$$\begin{vmatrix} x_3-x_2 & y_3-y_2 & -\frac{1}{2}(x_3^2-x_2^2)-\frac{1}{2}(y_3^2-y_2^2) \\ x_3-x_1 & y_3-y_1 & -\frac{1}{2}(x_3^2-x_1^2)-\frac{1}{2}(y_3^2-y_1^2) \\ x_2-x_1 & y_2-y_1 & -\frac{1}{2}(x_2^2-x_1^2)-\frac{1}{2}(x_2^2-x_1^2) \end{vmatrix}=0$$

Now taking (-1) as common from R_2

$$=(-1)\begin{vmatrix} x_3-x_2 & y_3-y_2 & -\frac{1}{2}(x_3^2-x_2^2)-\frac{1}{2}(y_3^2-y_2^2) \\ x_1-x_3 & y_1-y_3 & -\frac{1}{2}(x_3^2-x_1^2)+\frac{1}{2}(y_3^2-y_1^2) \\ x_2-x_1 & y_2-y_1 & -\frac{1}{2}(x_2^2-x_1^2)-\frac{1}{2}(y_2^2-y_1^2) \end{vmatrix}$$

By $R_1+(R_2+R_3)$ so

$$=(-1)\begin{vmatrix} 0 & 0 & 0 \\ x_1-x_3 & y_1-y_3 & -\frac{1}{2}(x_3^2-x_1^2)+\frac{1}{2}(y_3^2-y_1^2) \\ x_2-x_1 & y_2-y_1 & -\frac{1}{2}(x_2^2-x_1^2)-\frac{1}{2}(y_2^2-y_1^2) \end{vmatrix}$$

$= 0 \quad (\because R_1 = 0)$

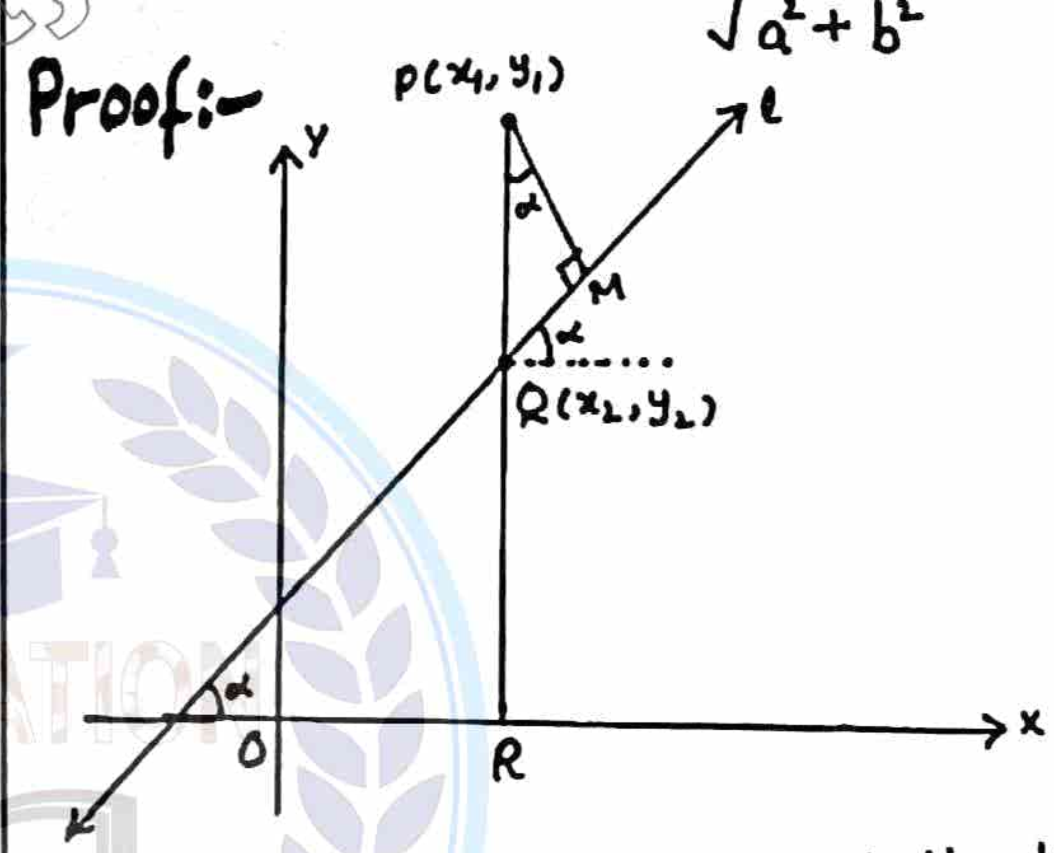
Thus right bisectors of triangle are concurrent.

> **Note:-** If equations of sides of the triangle are given, then intersection of any two lines gives a vertex of the triangle.

Distance of a point from a line

Theorem:- The distance d from the point $P(x_1,y_1)$ to the line $l: ax+by+c=0$ is given by $d = \left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

Proof:-



Let α be the inclination of the line $l; ax+by+c=0$. Draw $\perp$ars PR from point P on x-axis such that it meets line l at point $Q(x_2,y_2)$. Also draw a $\perp$r PM on line l. In $\triangle PQM$, $m\angle QPM = \alpha$

$|PM| = d$, $|PQ| = |y_1-y_2|$

$\cos\alpha = \frac{|PM|}{|PQ|} \Rightarrow |PM| = |PQ|\cos\alpha$

$\Rightarrow d = |y_1-y_2|\cos\alpha$ ——(I)

$\because Q(x_2,y_2)$ lies on line $l: ax+by+c=0$

so $ax_2+by_2+c=0 \Rightarrow by_2 = -ax_2-c$

$\Rightarrow y_2 = -\frac{a}{b}x_2-\frac{c}{b}$

given eq. of line is $ax+by+c=0$

$\Rightarrow y = -\frac{a}{b}x-\frac{c}{b}$

→ slope of given line $= m = -\frac{a}{b}$

$\because m = \tan\alpha \rightarrow \tan\alpha = -\frac{a}{b}$

$\because 1+\tan^2\alpha = \sec^2\alpha \rightarrow 1+\left(-\frac{a}{b}\right)^2 = \sec^2\alpha$

$\rightarrow 1+\frac{a^2}{b^2} = \sec^2\alpha \rightarrow \sec\alpha = \frac{\sqrt{a^2+b^2}}{b}$

$\rightarrow \cos\alpha = \frac{b}{\sqrt{a^2+b^2}}$ Put all values in (I)

$$\rightarrow d = \left|y_1 + \left(\frac{ax_1+c}{b}\right)\right|\left(\frac{b}{\sqrt{a^2+b^2}}\right)$$

$$\rightarrow d = \left|\frac{ax_1+by_1+c}{b}\right|\left(\frac{b}{\sqrt{a^2+b^2}}\right)$$

$$d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$ Hence proved.

Distance between two parallel Lines

*The distance between two parallel lines is the distance from any point on one of the lines to the other line.

Example: Find the distance between the parallel lines l_1; $2x-5y+13=0$

l_2; $2x-5y+6=0$

Solution:- $2x-5y+13=0$ ——— (I)

$2x-5y+6=0$ ——— (II)

For x-intercept, put $y=0$ in (II)

$2x-5(0)+6=0 \rightarrow 2x=-6$

$\rightarrow x=-3$ so the point $(-3,0)$

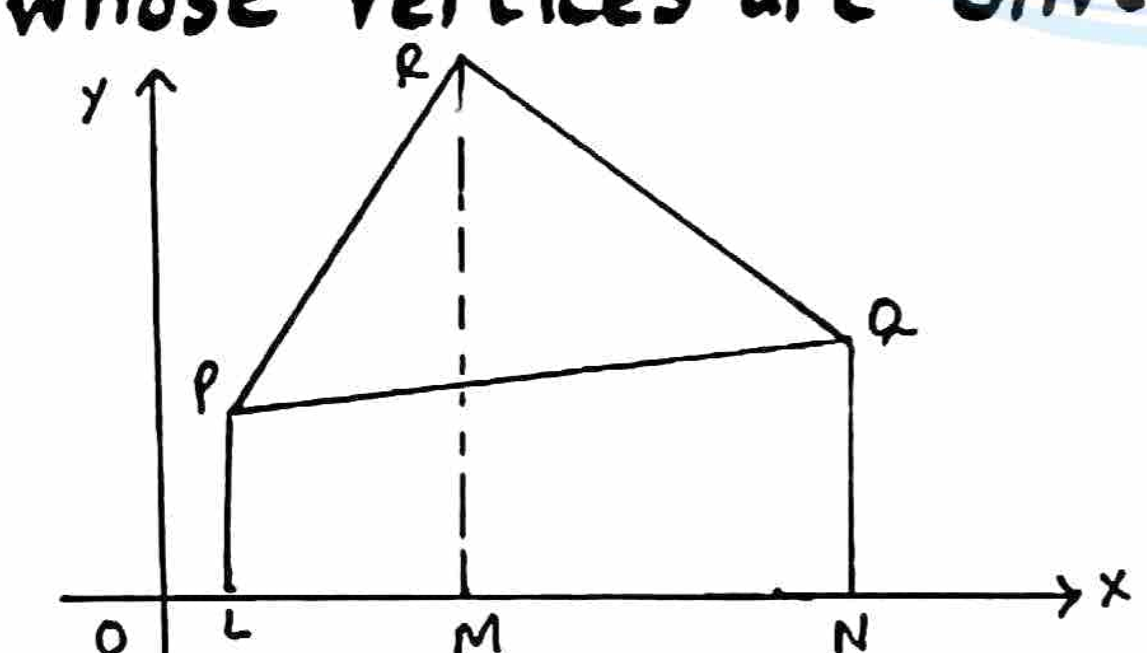
Now distance of $(-3,0)$ from $2x-5y+13=0$ will be distance b/w two ∥ lines

so $d = \frac{|2(-3)+(-5)(0)+13|}{\sqrt{(2)^2+(-5)^2}} = \frac{|-6+13|}{\sqrt{4+25}}$

$\because a=2,\ b=-5,\ c=13$

$\rightarrow d = \frac{7}{\sqrt{29}}$ $\because d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

Area of Triangular Region whose Vertices are Given



Let $P(x_1,y_1)$, $Q(x_2,y_2)$ and $R(x_3,y_3)$ be the vertices.

we draw ⊥ars PL, RM and QN on x-axis.

Area of ΔPQR = Area of trapezium PLMR

+ Area of trapezium RMNQ

− Area of trapezium PLNQ

$= \frac{1}{2}(|PL|+|PM|)(|LM|) + \frac{1}{2}(|RM|+|QN|)(|MN|)$

$- \frac{1}{2}(|PL|+|QN|)(|LN|)$

$= \frac{1}{2}[(y_1+y_3)(x_3-x_1)+(y_3+y_2)(x_2-x_3)-(y_1+y_2)(x_2-x_1)]$

$= \frac{1}{2}(x_3y_1+x_3y_3-x_1y_1-x_1y_3+x_2y_3+x_2y_2-x_3y_3$

$-x_3y_2-x_2y_1-x_2y_2+x_1y_1+x_1y_2)$

$= \frac{1}{2}(x_3y_1-x_1y_3+x_2y_3-x_3y_2-x_2y_1+x_1y_2)$

$\Delta = \frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$

$$\rightarrow \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$ req. area of ΔPQR.

Corollary:- If the points P, Q, R are collinear then $\Delta = 0$

Example 1. Find the area of the region bounded by the triangle with vertices $(a, b+c)$, $(a, b-c)$, $(-a, c)$

Solution:- $\because \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

so required area of triangle $= \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ a & b-c & 1 \\ -a & c & 1 \end{vmatrix}$

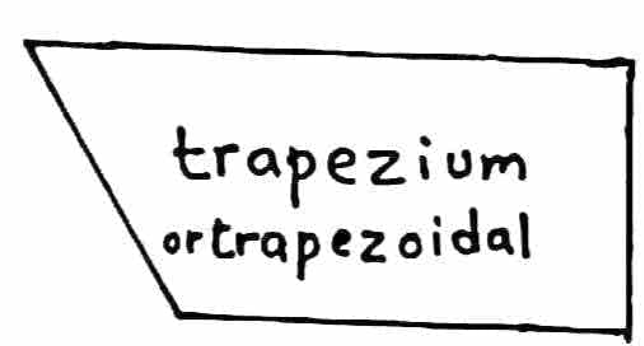
$= \frac{1}{2}\{a(b-2c)-ab-a(\cancel{b}+c-\cancel{b}+c)\}$

$= \frac{1}{2}(\cancel{ab}-2ac-\cancel{ab}-2ac)$

$\Delta = \frac{1}{2}(-4ac) = -2ac = +2ac$

$\because$ Area is always positive so $\Delta = 2ac$

*Trapezium:- A quadrilateral having two sides parallel and two non-parallel is called trapezium. Its area is $\frac{1}{2}$(sum of length of ∥ sides)(distance b/w ∥ sides)



Example 2. By considering the area of the region bounded by the triangle with vertices $A(1,4)$, $B(2,-3)$ and $C(3,-10)$. check whether the points collinear or not.

Solution:- $\because$ area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

so area of req. triangle $= \frac{1}{2}\begin{vmatrix} 1 & 4 & 1 \\ 2 & -3 & 1 \\ 3 & -10 & 1 \end{vmatrix}$

$= \frac{1}{2}\left[1(-3+10) - 2(4+10) + 3(4+3)\right]$

$\Delta = \frac{1}{2}(7 - 28 + 21) = \frac{1}{2}(28-28) = 0$

Thus $\Delta = 0 \rightarrow$ given points are collinear.

Exercise 4.3

Q1. Find the slope and inclination of the line joining the points:

(i) $(-2,4)$; $(5,11)$ (ii) $(3,-2)$; $(2,7)$

(iii) $(4,6)$; $(4,8)$

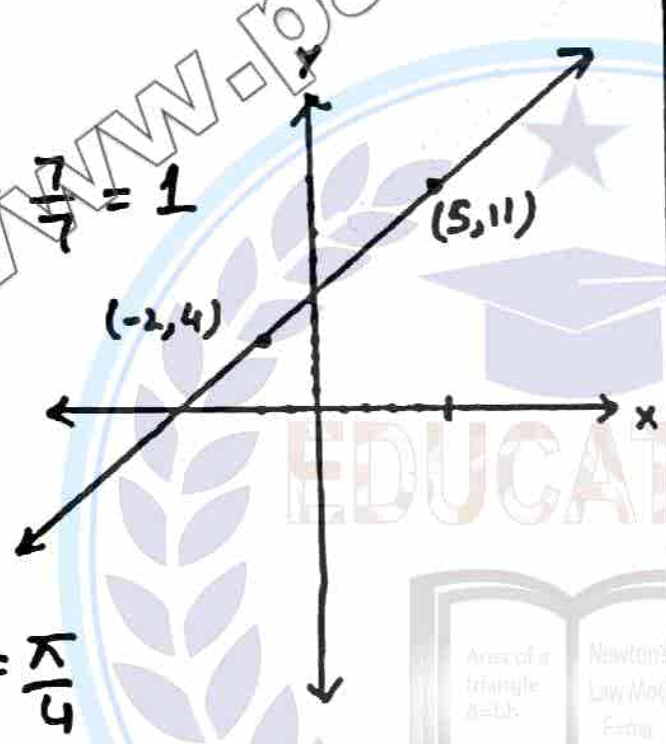
sketch each line in plane

Solution:- (i) $(-2,4)$; $(5,11)$

Here $x_1 = -2$, $y_1 = 4$, $x_2 = 5$, $y_2 = 11$

$\because$ slope $= m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{11-4}{5-(-2)} = \frac{11-4}{5+2} = \frac{7}{7} = 1$



$\because \tan\alpha = m$

$\rightarrow \tan\alpha = 1$

$\alpha = \tan^{-1}(1) = \frac{\pi}{4}$

so inclination $= \alpha = \frac{\pi}{4}$

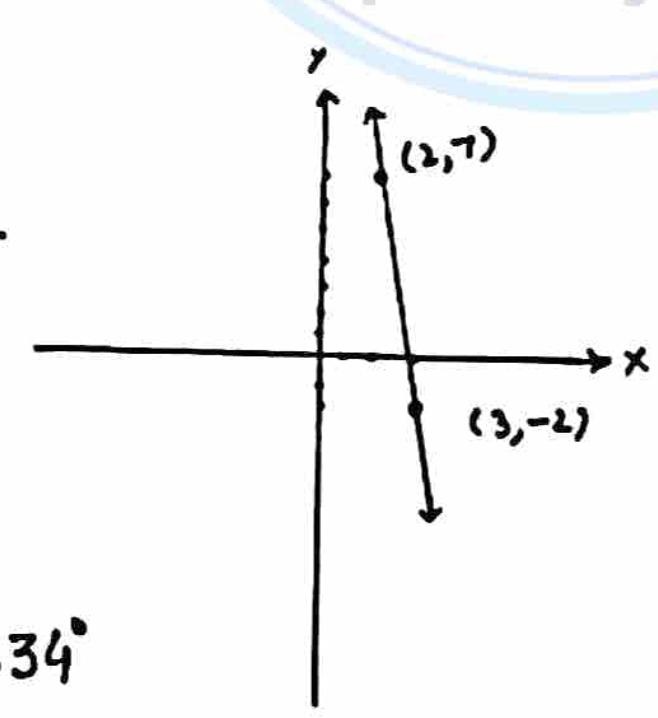
(ii) $(3,-2)$; $(2,7)$

Here $x_1 = 3$, $y_1 = -2$, $x_2 = 2$, $y_2 = 7$

$\because$ slope $= m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{7-(-2)}{2-3} = \frac{7+2}{-1}$

$m = \frac{9}{-1} = -9$



so $\tan\alpha = m$

$\rightarrow \tan\alpha = -9$

$\alpha = \tan^{-1}(-9) = 96.34^\circ$

(iii) $(4,6)$; $(4,8)$

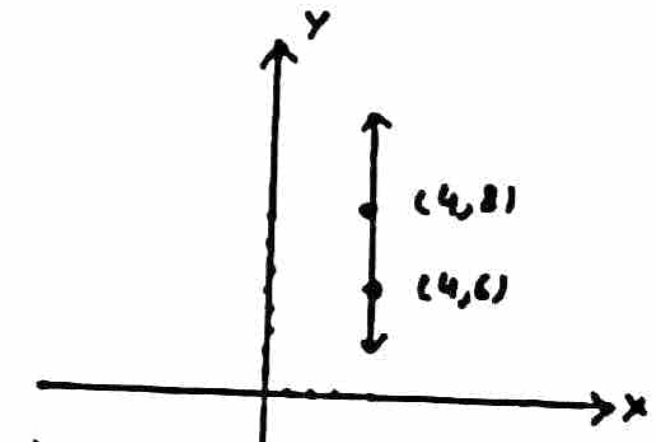
Here $x_1 = 4$, $y_1 = 6$, $x_2 = 4$, $y_2 = 8$

$\because$ slope $= m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-6}{4-4} = \frac{2}{0} = \infty$

Now $\tan\alpha = m = \infty$

$\rightarrow \alpha = \tan^{-1}(\infty)$

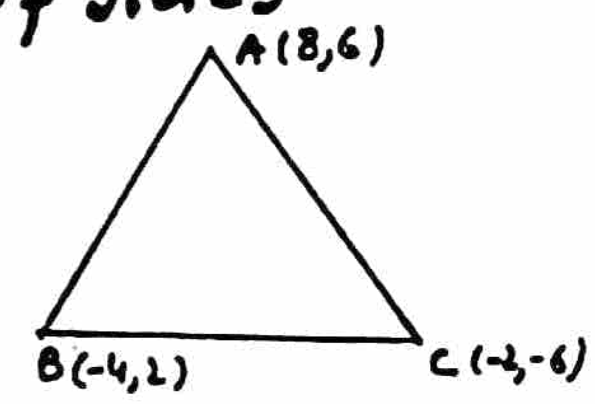
$\rightarrow \alpha = 90^\circ$



Q2. In the triangle $A(8,6)$, $B(-4,2)$, $C(-2,-6)$, find the slope of (i) each side of the triangle (ii) each median of the triangle (iii) each altitude of the triangle.

Solution:- (i) **Slope of sides**

Here AB, BC and CA are three sides of triangle



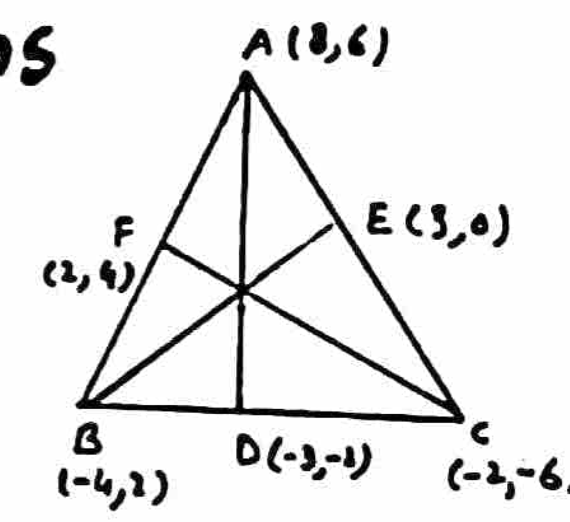
slope of $AB = \frac{2-6}{-4-8} = \frac{-4}{-12} = \frac{1}{3}$

slope of $BC = \frac{-6-2}{-2-(-4)} = \frac{-8}{2} = -4$

slope of $AC = \frac{-6-6}{-2-8} = \frac{-12}{-10} = \frac{6}{5}$

(ii) **Slope of medians**

Let D, E and F be the midpoints of sides of a $\triangle ABC$. so



coordinates of D are $= \left(\frac{-4-2}{2}, \frac{2-6}{2}\right) = \left(-\frac{6}{2}, -\frac{4}{2}\right) = (-3,-2)$

coordinates of E are $= \left(\frac{8-2}{2}, \frac{6-6}{2}\right) = \left(\frac{6}{2}, \frac{0}{2}\right)$

$= (3,0)$

coordinates of F are $= \left(\frac{8-4}{2}, \frac{6+2}{2}\right) = \left(\frac{4}{2}, \frac{8}{2}\right) = (2,4)$

Now

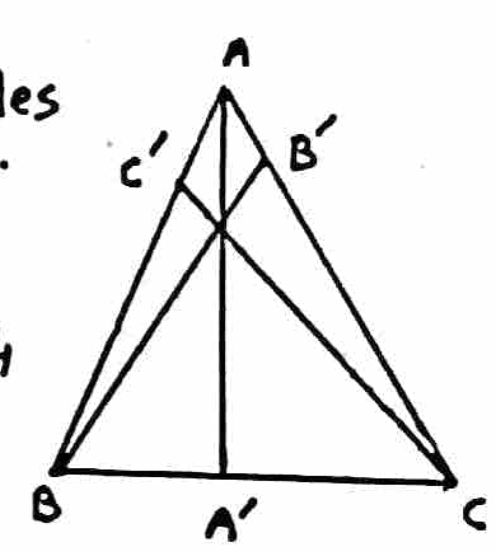
slope of $AD = \frac{-2-6}{-3-8} = \frac{-8}{-11} = \frac{8}{11}$

slope of $BE = \frac{0-2}{3-(-4)} = \frac{-2}{3+4} = \frac{-2}{7}$

slope of $CF = \frac{4-(-6)}{2-(-2)} = \frac{4+6}{2+2} = \frac{10}{4} = \frac{5}{2}$

(iii) **Slope of Altitudes**

$\because AA'$, BB' and CC' are altitudes of sides BC, CA and AB resp.



Now

slope of side BC

$= \frac{-6-2}{-2-(-4)} = \frac{-8}{2} = -4$

$(\because AA' \perp BC)$ so

$\rightarrow$ slope of $AA' = \frac{1}{4}$

slope of side $AC = \frac{-6-6}{-2-8} = \frac{6}{5}$

$\rightarrow$ slope of $BB' = -\frac{5}{6}$ $(\because BB' \perp AB)$

slope of $AB = \frac{2-6}{-4-8} = \frac{-4}{-12} = \frac{1}{3}$

$\rightarrow$ slope of $CC' = -3$ $(\because CC' \perp AB)$

Q3. By means of slopes show that the following points lie on the same line. (a) $(-1,-3)$; $(1,5)$; $(2,9)$

Solution:- (a) Let $A(-1,-3)$, $B(1,5)$, $C(2,9)$

$$\text{slope of } AB = \frac{5-(-3)}{1-(-1)} = \frac{8}{2} = 4$$

$$\text{slope of } BC = \frac{9-5}{2-1} = \frac{4}{1} = 4$$

$\because$ Slope of AB = slope of BC

$\rightarrow$ A, B and C lie on same line or collinear.

(b) $(4,-5)$; $(7,5)$; $(10,15)$

Solution:- (b) Let $A(4,-5)$, $B(7,5)$, $C(10,15)$

$$\text{slope of } AB = \frac{5+5}{7-4} = \frac{10}{3}$$

$$\text{slope of } BC = \frac{15-5}{10-7} = \frac{10}{3}$$

$\because$ slope of AB = slope of BC

$\rightarrow$ A, B and C lie on same line.

(c) $(-4,6)$; $(3,8)$; $(10,10)$

Solution:- Let $A(-4,6)$, $B(3,8)$, $C(10,10)$

$$\text{slope of } AB = \frac{8-6}{3+4} = \frac{2}{7}$$

$$\text{slope of } BC = \frac{10-8}{10-3} = \frac{2}{7}$$

$\because$ slope of AB = slope of BC

$\rightarrow$ A, B and C lie on same line.

(d) $(a,2b)$, $(c,a+b)$, $(2c-a,2a)$

Solution:- Let $A(a,2b)$, $B(c,a+b)$, $C(2c-a,2a)$

$$\text{slope of } AB = \frac{a+b-2b}{c-a} = \frac{a-b}{c-a}$$

$$\text{slope of } BC = \frac{2a-a-b}{2c-a-c} = \frac{a-b}{c-a}$$

$\because$ slope of AB = slope of BC

$\rightarrow$ A, B and C lie on same line.

Q4. Find K so that the line joining $A(7,3)$; $B(K,-6)$ and the line joining $C(-4,5)$; $D(-6,4)$ are (i) parallel (ii) perpendicular

Solution:- $\text{slope of } AB = \frac{-6-3}{K-7} = \frac{-9}{K-7}$

$$\text{slope of } CD = \frac{4-5}{-6-(-4)} = \frac{-1}{-6+4} = \frac{-1}{-2} = \frac{1}{2}$$

(i) $\because$ AB is $\parallel$ to CD so

slope of AB = slope of CD

$$\rightarrow -\frac{9}{K-7} = \frac{1}{2} \rightarrow -18 = K-7$$

$\rightarrow K = -11$

(ii) $\because$ AB is $\perp$ar to CD so

$$(\text{slope of } AB)(\text{slope of } CD) = -1$$

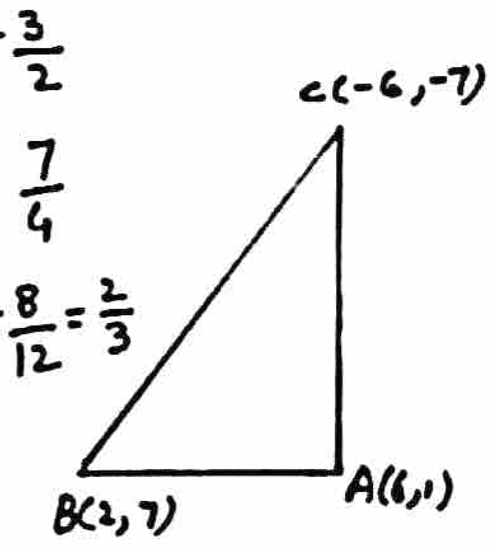
$$\rightarrow (\tfrac{1}{2})(\cancel{-}\tfrac{9}{K-7}) = \cancel{-}1$$

$$\rightarrow \frac{9}{2K-14} = 1 \rightarrow 2K-14 = 9 \rightarrow 2K = 9+14$$

$\rightarrow K = \frac{23}{2}$

Q5. Using slopes, show that the triangle with vertices $A(6,1)$, $B(2,7)$ and $C(-6,-7)$ is a right triangle.

Solution:-



$$m_1 = \text{slope of } AB = \frac{7-1}{2-6} = \frac{6}{-4} = -\frac{3}{2}$$

$$m_2 = \text{slope of } BC = \frac{-7-7}{-6-2} = \frac{-14}{-8} = \frac{7}{4}$$

$$m_3 = \text{slope of } CA = \frac{1-(-7)}{6-(-6)} = \frac{1+7}{6+6} = \frac{8}{12} = \frac{2}{3}$$

so

$$m_1 m_3 = (-\tfrac{3}{2})(\tfrac{2}{3}) = -1$$

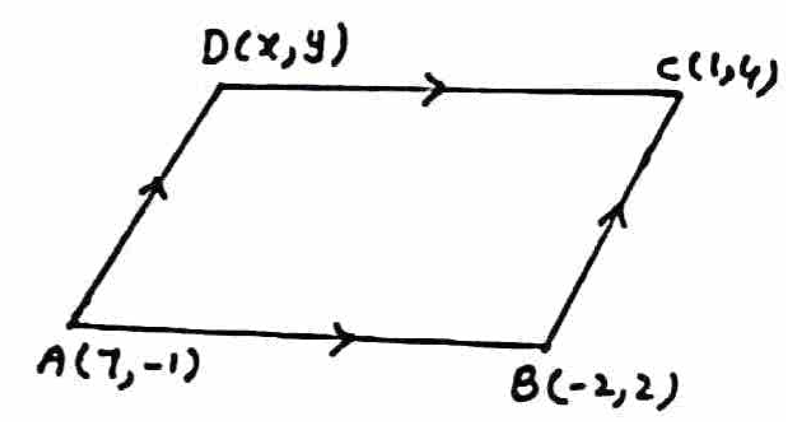
$\rightarrow$ This proves that $AB \perp AC$

$\rightarrow$ Hence $\triangle ABC$ is right triangle.

Q6. The three points $A(7,-1)$, $B(-2,2)$ and $C(1,4)$ are consecutive vertices of a parallelogram. Find the forth vertex.

Solution:-

Let $D(x,y)$ be the forth vertex. given figure is llgram. so



slope of AB = slope of DC

$$\rightarrow \frac{2+1}{-2-7} = \frac{4-y}{1-x} \rightarrow \frac{3}{-9} = \frac{4-y}{1-x} \rightarrow \frac{-1}{3} = \frac{4-y}{1-x}$$

$\rightarrow -1+x = 12-3y \rightarrow x+3y-13 = 0$ —— (i)

Also, slope of AD = slope of BC

$$\rightarrow \frac{y-(-1)}{x-7} = \frac{4-2}{1-(-2)} \rightarrow \frac{y+1}{x-7} = \frac{2}{3}$$

$\rightarrow 3y+3 = 2x-14 \rightarrow 2x-3y-17 = 0$ —— (ii)

By (i) + (ii) $\rightarrow$

$$\begin{array}{r} x+3y-13=0 \\ 2x-3y-17=0 \\ \hline 3x \quad -30=0 \end{array}$$

$\rightarrow x = 10$ put in (i)

so (i) $\rightarrow 10+3y-13=0 \rightarrow 3y=3 \rightarrow y=1$

so required point is $D(x,y) = D(10,1)$

Q7. The points $A(-1,2)$, $B(3,-1)$ and $C(6,3)$ are consecutive vertices of a rhombus. Find the fourth vertex and show that the diagonals of the rhombus are perpendicular to each other.

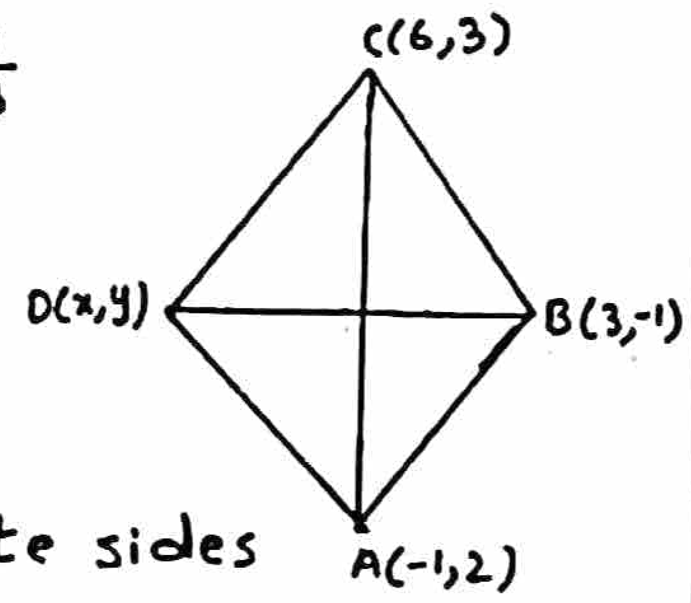
Solution:- Let $D(x,y)$ be the forth vertex.

then slope of $AB = \frac{-1-2}{3+1} = \frac{-3}{4}$

slope of $BC = \frac{3+1}{6-3} = \frac{4}{3}$

slope of $CD = \frac{y-3}{x-6}$

slope of $DA = \frac{2-y}{-1-x}$



In rhombus opposite sides are parallel. so

slope of AB = slope of CD

$-\frac{3}{4} = \frac{y-3}{x-6} \rightarrow -3x+18 = 4y-12$

$\rightarrow -3x-4y+18+12=0$

$\rightarrow 3x+4y-30=0$ ——— (i)

Also slope of BC = slope of DA

$\rightarrow \frac{4}{3} = \frac{2-y}{-1-x} \rightarrow -4-4x = 6-3y$

$\rightarrow -4x+3y-4-6=0$

$\rightarrow 4x-3y+10=0$ ——— (ii)

By $3(i)+4(ii) \rightarrow$

$$\begin{array}{r} 9x+12y-90=0 \\ 16x-12y+40=0 \\ \hline 25x \quad -50=0 \end{array}$$

$\rightarrow 25x=50 \rightarrow x=2$ put in (i) so

(i) $\rightarrow 3(2)+4y-30=0 \rightarrow 6+4y-30=0$

$\rightarrow 4y-24=0 \rightarrow 4y=24 \rightarrow y=6$

Thus fourth vertex is $D(x,y) = D(2,6)$

Now slope of diagonal $AC = \frac{3-2}{6-(-1)} = \frac{1}{6+1} = \frac{1}{7}$

slope of diagonal $BD = \frac{6-(-1)}{2-3} = \frac{6+1}{-1} = -7$

Now (slope of AC)(slope of BD) $= \frac{1}{7}(-7) = -1$

Hence diagonals AC and BD are ⊥ar.

Q8. Two pairs of points are given. Find whether the two lines determined by these points are (i) Parallel (ii) Perpendicular (iii) None

(a) $(1,-2),(2,4)$ and $(4,1),(-8,2)$

(b) $(-3,4),(6,2)$ and $(4,5),(-2,-7)$

Solution:- (a)

Let $A(1,-2)$, $B(2,4)$ and $C(4,1)$, $D(-8,2)$

slope of $AB = \frac{4-(-2)}{2-1} = \frac{4+2}{1} = 6$

slope of $CD = \frac{2-1}{-8-4} = \frac{1}{-12}$

∵ slope of AB ≠ slope of CD

so lines are not parallel.

∵ (slope of AB)(slope of CD) $= 6(-\frac{1}{12}) \neq -1$

so lines are not perpendicular

(b) Let $A(-3,4)$, $B(6,2)$, $C(4,5)$, $D(-2,-7)$

slope of $AB = \frac{2-4}{6-(-3)} = \frac{-2}{6+3} = -\frac{2}{9}$

slope of $CD = \frac{-7-5}{-2-4} = \frac{-12}{-6} = 2$

∵ slope of AB ≠ slope of CD

so lines are not parallel.

Also (slope of AB)(slope of CD) $= (-\frac{2}{9})(2) \neq -1$

so lines are not perpendicular.

Q9. Find an equation of

(a) the horizontal line through $(7,-9)$

(b) The vertical line through $(-5,3)$

(c) The line bisecting first and third quadrants

(d) The line bisecting second and fourth quadrants.

Solution:- (a) ∵ slope of horizontal line $= m = 0$

so eq. of horizontal line through $(7,-9)$

$y-(-9) = 0(x-7)$ (∵ $y-y_1 = m(x-x_1)$ point-slope form)

$\rightarrow y+9=0 \rightarrow y=-9$

(b) ∵ slope of vertical line $= m = \infty = \frac{1}{0}$

so eq. of vertical line through $(-5,3)$ is

$y-3 = \frac{1}{0}(x+5)$ (∵ $y-y_1 = m(x-x_1)$)

$\rightarrow 0 = x+5 \rightarrow x=-5$

(c) The line bisecting the first and third quadrant makes angle $45°$ with x-axis.

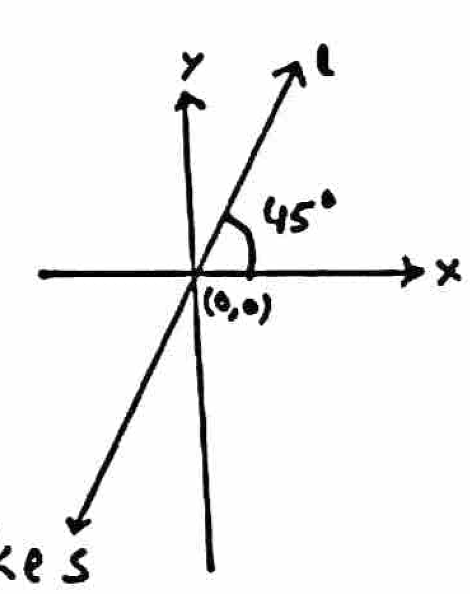
so slope $= m = \tan 45° = 1$

∵ it passes through origin $(0,0)$

so equation is

$y-0 = (1)(x-0)$

$\rightarrow y = x \rightarrow y-x=0$



(d) The line bisecting the 2^{nd} and 4^{th} quadrant makes an angle of $135°$ with x-axis

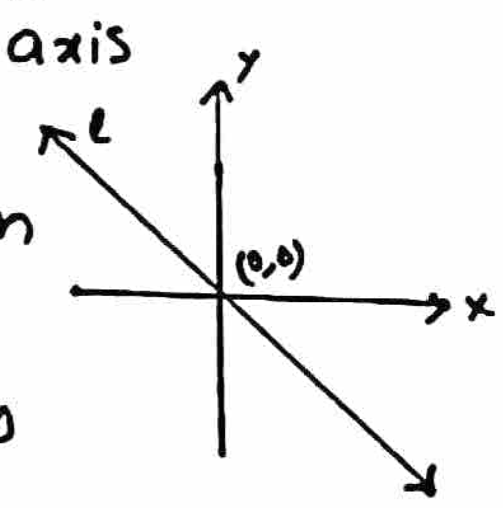
so slope $= m = \tan 135° = -1$

∵ it passes through origin

so eq. is

$y-0 = (-1)(x-0)$

$\rightarrow y = -x \rightarrow y+x=0$



Q10. Find an equation of the line (a) through A(−6,5) having slope 7

Solution:- Let $A(x_1,y_1)=A(-6,5)$, slope $=m=7$

Eq. of line is $y-y_1=m(x-x_1)$ (point-slope form)

$\rightarrow y-5=7(x-(-6)) \rightarrow y-5=7(x+6)$

$\rightarrow y-5=7x+42 \rightarrow 7x-y+47=0$

(b) through (8,−3) having slope 0

Solution:- Let $A(x_1,y_1)=(8,-3)$ and slope $=m=0$ so

eq. of line is $y-y_1=m(x-x_1)$ (point-slope form)

$\rightarrow y-(-3)=0(x-8) \rightarrow y+3=0$

(c) through (−8,5) having slope undefined

Solution:- Let $A(x_1,y_1)=(-8,5)$, slope $=m=\infty$

eq. of line is $y-y_1=m(x-x_1)$ (point-slope form)

$\rightarrow y-5=\infty(x-(-8))$

$\rightarrow y-5=\frac{1}{0}(x+8) \rightarrow 0=x+8 \rightarrow x+8=0$

(d) through (−5,−3) and (9,−1)

Solution:- Let $A(x_1,y_1)=(-5,-3)$, $B(x_2,y_2)=(9,1)$ so eq. of line is

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \quad \text{(Two-point form)}$$

$$\rightarrow \frac{y-(-3)}{-1-(-3)}=\frac{x-(-5)}{9-(-5)} \rightarrow \frac{y+3}{-1+3}=\frac{x+5}{9+5}$$

$$\rightarrow \frac{y+3}{\cancel{2}\,1}=\frac{x+5}{\cancel{14}\,7} \rightarrow 7y+21=x+5$$

$\rightarrow x-7y-21+5=0 \rightarrow x-7y-16=0$

(e) y-intercept: −7 and slope: −5

Solution:- $\because$ y-intercept $=c=-7$

Slope $=m=-5$ so eq. of line is

$y=mx+c$ (slope-intercept form)

$\rightarrow y=-5x-7 \rightarrow 5x+y+7=0$

(f) x-intercept: −3 and y-intercept: 4

Solution:- x-intercept $=a=-3$

y-intercept $=b=4$ so eq. of line is

$\frac{x}{a}+\frac{y}{b}=1$ (Two-intercept form)

$\rightarrow \frac{x}{-3}+\frac{y}{4}=1 \rightarrow 12\left(\frac{x}{-3}\right)+12\left(\frac{y}{4}\right)=12(1)$

$\rightarrow -4x+3y=12$ or $4x-3y+12=0$

(g) x-intercept: −9 and slope: −4

Solution:- x-intercept $=-9$ so pt. on x-axis is $(-9,0)$ and Let $A(x_1,y_1)=A(-9,0)$

and slope $=m=-4$

eq. of line is $y-y_1=m(x-x_1)$ (point-slope form)

$\rightarrow y-0=-4(x-(-9)) \rightarrow y=-4(x+9)$

$\rightarrow y=-4x-36 \rightarrow 4x+y+36=0$

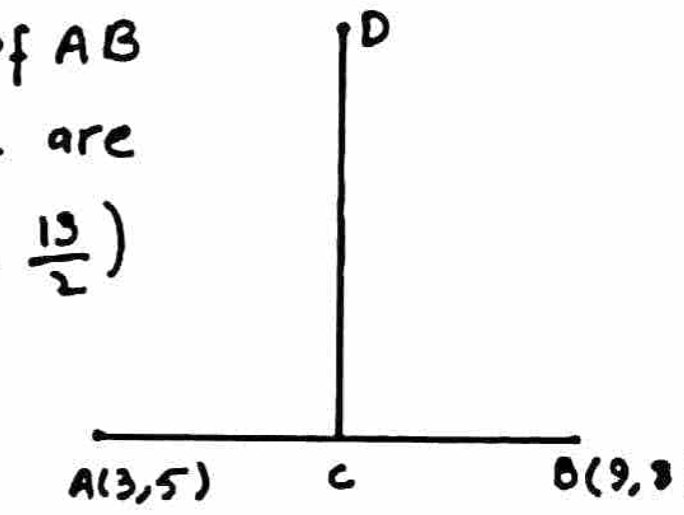
Rhombus: A llgram having equal sides is called rhombus.

Q11. Find an equation of the perpendicular bisector of the segment joining the points A(3,5) and B(9,8).

Solution:-

Let C is midpoint of AB so coordinates of C are

$=\left(\frac{3+9}{2},\frac{5+8}{2}\right)=\left(6,\frac{13}{2}\right)$



slope of line AB

$=\frac{8-5}{9-3}=\frac{3}{6}=\frac{1}{2}$

Slope of perpendicular bisector $CD=-2$

($\because CD \perp AB$)

Now eq. of $\perp$ bisector CD through $C\left(6,\frac{13}{2}\right)$ is $y-y_1=m(x-x_1)$ (Point-slope form)

$\rightarrow y-\frac{13}{2}=-2(x-6)$

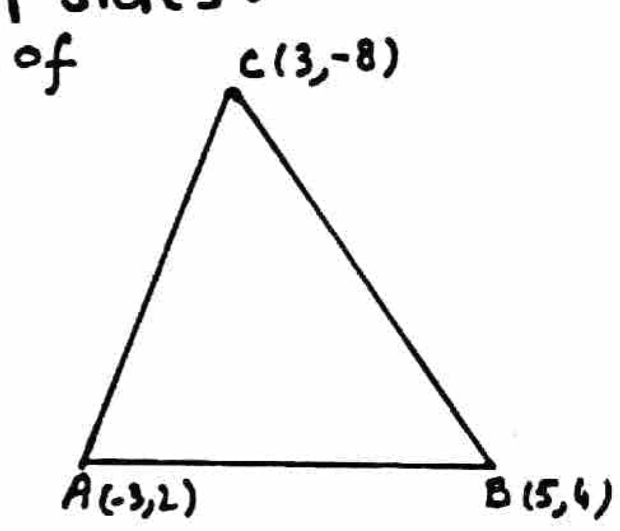
$\rightarrow 2y-13=-4x+24 \rightarrow 4x+2y-13-24=0$

$\rightarrow 4x+2y-37=0$

Q12. Find equations of the sides, altitudes and medians of the triangle whose vertices are A(−3,2), B(5,4) and C(3,−8)

Solution:- Equation of sides:-

$\because$ AB, BC and AC be sides of $\triangle ABC$.



Eq. of side AB is;

$$\frac{y-2}{4-2}=\frac{x-(-3)}{5-(-3)}$$

$$\rightarrow \frac{y-2}{\cancel{2}\,1}=\frac{x+3}{\cancel{8}\,4}$$

$\rightarrow 4y-8=x+3 \rightarrow x-4y+8+3=0$

$\rightarrow x-4y+11=0$ $\left(\because \frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}\right.$ Two-point form$\left.\right)$

Eq. of side BC is;

$$\frac{y-4}{-8-4}=\frac{x-5}{3-5} \rightarrow \frac{y-4}{\cancel{-12}\,6}=\frac{x-5}{\cancel{-2}\,1}$$

$\rightarrow y-4=6x-30 \rightarrow 6x-y+4-30=0$

$\rightarrow 6x-y-26=0$

Eq. of side AC is;

$$\frac{y-2}{-8-2}=\frac{x-(-3)}{3-(-3)} \rightarrow \frac{y-2}{\cancel{-10}\,5}=\frac{x+3}{\cancel{6}\,3}$$

$\rightarrow 3y-6=-5x-15$

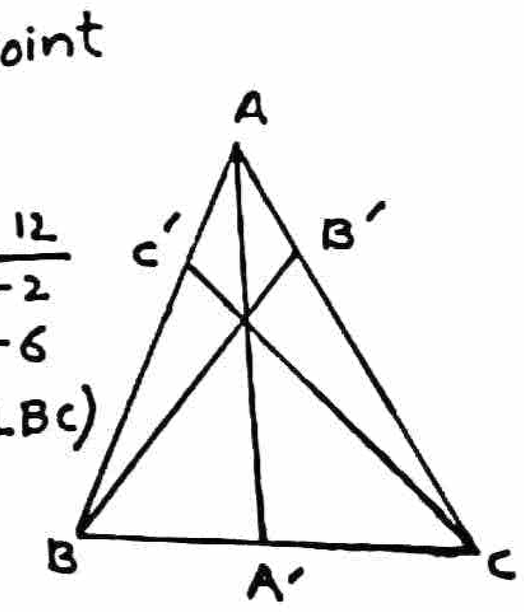
$\rightarrow 5x+3y-6+15=0 \rightarrow 5x+3y+9=0$

Equation of altitudes:-

$\because AA'$ be altitude from point A on side BC

slope of side $BC=\frac{-8-4}{3-5}=\frac{-12}{-2}=-6$

slope of altitude $AA'=\frac{-1}{6}$ ($\because AA' \perp BC$)



Eq. of altitude AA′ is; (from pt $A(-3,2)$ having slope $1/6$)

$y-y_1=m(x-x_1)$ (point-slop form)

$\Rightarrow y-2=\frac{-1}{6}(x-(-3)) \Rightarrow 6y-12=-x-3$

$\Rightarrow x+6y-12+3=0 \Rightarrow x+6y-9=0$

∵ BB′ be altitude from point B on side AC so

slope of $AC=\frac{-8-2}{3-(-3)}=\frac{-10}{3+3}=\frac{-10}{6}=-\frac{5}{3}$

slope of $BB'=\frac{3}{5}$ (∵ $BB'\perp AC$)

Eq. of altitude BB′ is; (from pt $B(5,4)$ having slope $3/5$)

$\Rightarrow y-4=\frac{3}{5}(x-5)$ ∵ $y-y_1=m(x-x_1)$ (Point-slope form)

$\Rightarrow 5y-20=3x-15$

$\Rightarrow 3x-5y+20-15=0 \Rightarrow 3x-5y+5=0$

∵ CC′ be altitude from point C on side AB so

slope of $AB=\frac{4-2}{5-(-3)}=\frac{2}{5+3}=\frac{2}{8}=\frac{1}{4}$

slope of altitude $CC'=-4$ (∵ $CC'\perp AB$)

Eq. of altitude CC′ is; (from pt $C(3,-8)$ and having slope -4)

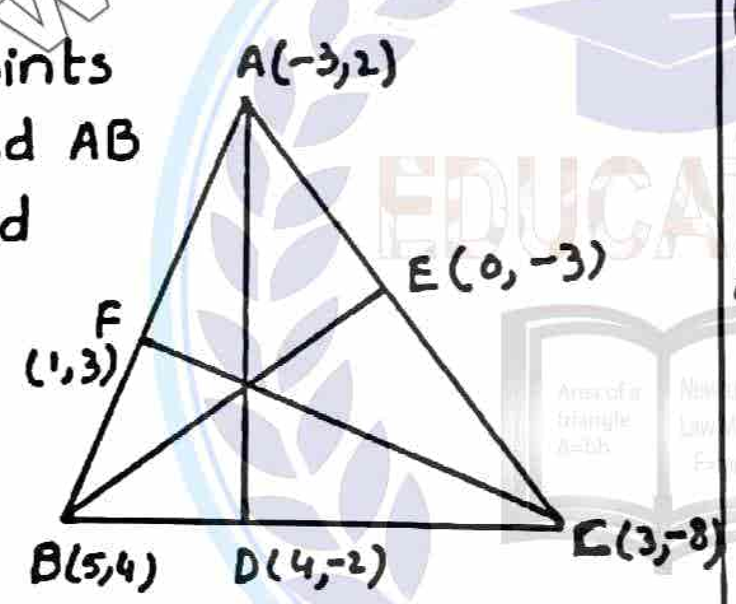
$y+8=-4(x-3)$ ∵ $y-y_1=m(x-x_1)$

$\Rightarrow y+8=-4x+12 \Rightarrow 4x+y-12+8=0$

$\Rightarrow 4x+y-4=0$

Equation of medians

∵ D, E, F are mid-points of sides BC, AC and AB resp. so AD, BE and CF are medians.



∵ D is midpoint of BC so

coordinates of D are $\left(\frac{5+3}{2},\frac{4-8}{2}\right)$

$=D\left(\frac{8}{2},\frac{-4}{2}\right)=D(4,-2)$

Eq. of median AD is:

$\frac{y-2}{-2-2}=\frac{x-(-3)}{4-(-3)}$ $\left(∵ \frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}\right)$

$\Rightarrow \frac{y-2}{-4}=\frac{x+3}{4+3} \Rightarrow \frac{y-2}{-4}=\frac{x+3}{7}$

$\Rightarrow 7y-14=-4x-12 \Rightarrow 4x+7y-14+12=0$

$\Rightarrow 4x+7y-2=0$

∵ E is mid-point of side AC

so coordinates of E are $\left(\frac{-3+3}{2},\frac{2-8}{2}\right)$

$=E\left(\frac{0}{2},-\frac{6}{2}\right)=E(0,-3)$

Eq. of median BE is;

$\frac{y-4}{-3-4}=\frac{x-5}{0-5} \Rightarrow \frac{y-4}{-7}=\frac{x-5}{-5}$

$\Rightarrow -5y+20=-7x+35$

$\Rightarrow 7x-5y+20-35=0 \Rightarrow 7x-5y-15=0$

∵ F is mid-point of side AB

so coordinates of F are $\left(\frac{5-3}{2},\frac{4+2}{2}\right)$

$=F\left(\frac{2}{2},\frac{6}{2}\right)=F(1,3)$

eq. of median CF is

$\frac{y+8}{3-(-8)}=\frac{x-3}{1-3} \Rightarrow \frac{y+8}{11}=\frac{x-3}{-2}$

$\Rightarrow -2y-16=11x-33 \Rightarrow 11x+2y+16-33=0$

$\Rightarrow 11x+2y-17=0$

Q13. Find an equation of line through $(-4,-6)$ and perpendicular to a line having slope $-\frac{3}{2}$

Solution:- slope of given line $=-\frac{3}{2}$

$\Rightarrow$ slope of required line $=\frac{2}{3}$

(req. line is $\perp$ to given line)

so eq. of req. line is;

$y-(-6)=\frac{2}{3}(x-(-4))$ $(∵ y-y_1=m(x-x_1))$

$\Rightarrow y+6=\frac{2}{3}(x+4) \Rightarrow 3y+18=2x+8$

$\Rightarrow 2x-3y+8-18=0 \Rightarrow 2x-3y-10=0$

Q14. Find an equation of the line through $(11,-5)$ and parallel to a line with slope -24.

Solution:- slope of given line $=-24$

slope of required line $=-24$

(∵ req. line is parallel to given line)

Eq. of required line is;

$y-(-5)=-24(x-11)$ $(∵ y-y_1=m(x-x_1))$

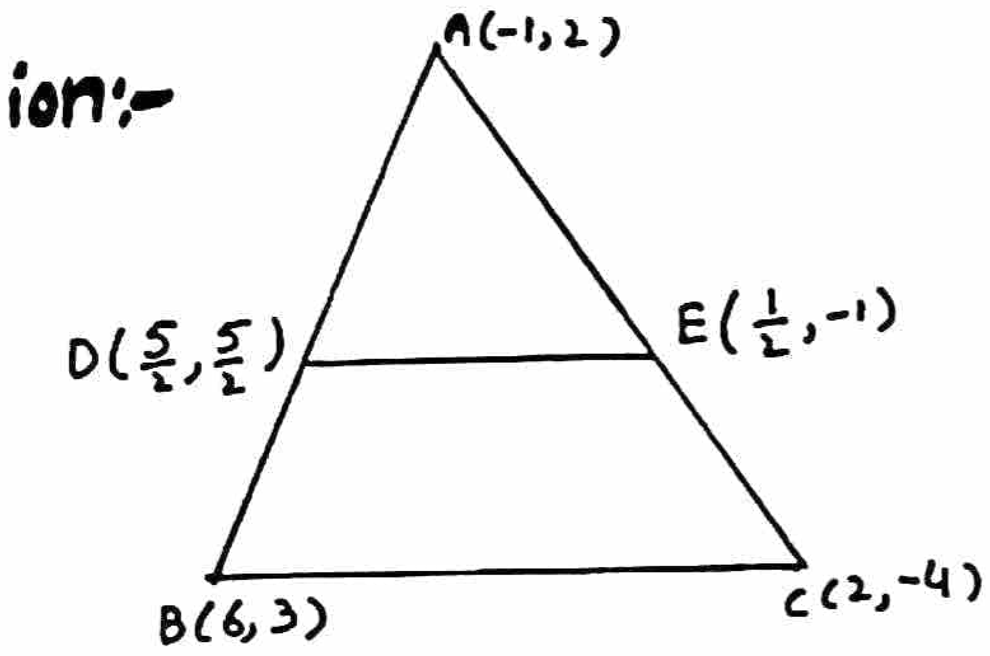
$\Rightarrow y+5=-24x+264$

$\Rightarrow 24x+y+5-264=0$

$\Rightarrow 24x+y-259=0$

Q15. The points $A(-1,2)$, $B(6,3)$ and $C(2,-4)$ are vertices of a triangle. Show that the line joining the mid point D of AB and the midpoint E of AC is parallel to BC and $DE=\frac{1}{2}BC$

Solution:-



$\because$ D is mid point of AB. so coordinates of D are $\left(\frac{-1+6}{2}, \frac{2+3}{2}\right) = D\left(\frac{5}{2}, \frac{5}{2}\right)$

Also E is midpoint of AC. so coordinates of E are $\left(\frac{-1+2}{2}, \frac{2-4}{2}\right) = E\left(\frac{1}{2}, \frac{-1}{2}\right)$

slope of BC $= \frac{-4-3}{2-6} = \frac{-7}{-4} = \frac{7}{4}$

slope of DE $= \dfrac{-1-\frac{5}{2}}{\frac{1}{2}-\frac{5}{2}} = \dfrac{\frac{-2-5}{2}}{\frac{1-5}{2}} = \dfrac{\frac{-7}{\cancel{2}}}{\frac{-4}{\cancel{2}}} = \frac{7}{4}$

$\because$ slope of BC = slope of DE

$\rightarrow$ DE is parallel to BC. Now

$|BC| = \sqrt{(2-6)^2+(-4-3)^2} = \sqrt{(-4)^2+(-7)^2} = \sqrt{16+49}$
$= \sqrt{65}$

and $|DE| = \sqrt{\left(\frac{1}{2}-\frac{5}{2}\right)^2+\left(-\frac{2}{2}-\frac{5}{2}\right)^2} = \sqrt{\left(\frac{1-5}{2}\right)^2+\left(\frac{-2-5}{2}\right)^2}$
$= \sqrt{\left(\frac{-4}{2}\right)^2+\left(\frac{-7}{2}\right)^2} = \sqrt{\frac{16}{4}+\frac{49}{4}} = \sqrt{\frac{65}{4}}$

$|DE| = \frac{\sqrt{65}}{2}$ $\rightarrow$ $|DE| = \frac{1}{2}|BC|$ Hence proved. both results

Q21. Convert each of the following equation into
(i) Slope intercept form (ii) Two intercepts form
(iii) Normal form
(a) $2x-4y+11=0$ (b) $4x+7y-2=0$
(c) $15y-8x+3=0$
Also find the length of the perpendicular from (0,0) to each line.

Solution:- (a) $2x-4y+11=0$

(i) Slope-intercept form:- $(y=mx+c)$

$2x-4y+11=0 \rightarrow -4y = -2x-11$
$\rightarrow 4y = 2x+11 \rightarrow y = \frac{2}{4}x+\frac{11}{4}$
$\rightarrow y = \frac{1}{2}x+\frac{11}{4}$ it is of the form
$y = mx+c$ where $m=\frac{1}{2}$, $c=\frac{11}{4}$

(ii) Two-intercepts form:- $\left(\frac{x}{a}+\frac{y}{b}=1\right)$

$2x-4y+11=0 \rightarrow -2x-4y=-11$
$\rightarrow \frac{2x}{-11}+\frac{-4y}{-11}=1$
$\rightarrow \frac{x}{\left(-\frac{11}{2}\right)}+\frac{y}{\left(\frac{11}{4}\right)}=1$ This is of the form
$\frac{x}{a}+\frac{y}{b}=1$ where $a=\frac{-11}{2}$, $b=\frac{11}{4}$

(iii) Normal form $(x\cos\alpha+y\sin\alpha=p)$

$2x-4y+11=0$
$\rightarrow 2x-4y=-11$
'$\div$' both sides by $\sqrt{(2)^2+(-4)^2} = \sqrt{4+16} = \sqrt{20}$
$\rightarrow \frac{2x}{\sqrt{20}} - \frac{4y}{\sqrt{20}} = \frac{-11}{\sqrt{20}}$ $\rightarrow \frac{-2x}{\sqrt{20}}+\frac{4}{\sqrt{20}}y = \frac{11}{\sqrt{20}}$

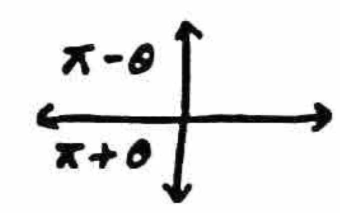
$x\left(-\frac{2}{\sqrt{20}}\right)+y\left(\frac{4}{\sqrt{20}}\right) = \frac{11}{\sqrt{20}}$ This is of the form $x\cos\alpha+y\sin\alpha=p$ where
$\cos\alpha = \frac{-2}{\sqrt{20}}$, $\sin\alpha = \frac{4}{\sqrt{20}}$, $p=\frac{11}{\sqrt{20}}$

Now $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ $\because \cos\alpha<0$, $\sin\alpha>0$

$\rightarrow \tan\alpha = \dfrac{4/\sqrt{20}}{-2/\sqrt{20}} = \frac{4}{-2} = -2$ $\rightarrow \alpha$ lies in II quad.

$\rightarrow \tan\alpha=-2 \rightarrow \alpha = \tan^{-1}(-2) = -63.43°$

Thus $\alpha = 180°-63.43° = 116.57°$
($\because 180°-\theta$ lies in II quad)



Thus $x\cos 116.57° + y\sin 116.57° = \frac{11}{\sqrt{20}}$

Also length of perpendicular from (0,0) to given line is $p=\frac{11}{\sqrt{20}}$ or $p=\frac{11}{2\sqrt{5}}$

(b) $4x+7y-2=0$

(i) Slope-intercept form:- $(y=mx+c)$

$4x+7y-2=0 \rightarrow 7y=-4x+2$
$\rightarrow y = -\frac{4}{7}y+\frac{2}{7} \longrightarrow y=mx+c$
where $m=-\frac{4}{7}$, $c=\frac{2}{7}$

(ii) Two intercept form:- $\left(\frac{x}{a}+\frac{y}{b}=1\right)$

$4x+7y-2=0 \rightarrow 4x+7y=2$
$\rightarrow \frac{4x}{2}+\frac{7y}{2}=1 \rightarrow \frac{x}{2/4}+\frac{y}{2/7}=1$
$\rightarrow \frac{x}{1/2}+\frac{y}{2/7}=1 \longrightarrow \frac{x}{a}+\frac{y}{b}=1$
where $a=\frac{1}{2}$, $b=\frac{2}{7}$

(iii) Normal form:- $(x\cos\alpha+y\sin\alpha=p)$

$4x+7y-2=0 \rightarrow 4x+7y=2$
($\div$) both sides by $\sqrt{(4)^2+(7)^2} = \sqrt{16+49} = \sqrt{65}$
$\rightarrow \frac{4}{\sqrt{65}}x+\frac{7}{\sqrt{65}}y = \frac{2}{\sqrt{65}}$
$\rightarrow x\left(\frac{4}{\sqrt{65}}\right)+y\left(\frac{7}{\sqrt{65}}\right) = \frac{2}{\sqrt{65}} \longrightarrow x\cos\alpha+y\sin\alpha=p$
where $\cos\alpha=\frac{4}{\sqrt{65}}$, $\sin\alpha=\frac{7}{\sqrt{65}}$, $p=\frac{2}{\sqrt{65}}$

$\because \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \dfrac{7/\sqrt{65}}{4/\sqrt{65}} = 7/4$ $\because \cos\alpha>0$, $\sin\alpha>0$ $\rightarrow \alpha$ lies in I quad

$\rightarrow \alpha = \tan^{-1}(7/4) = 60.26°$

so $x\cos 60.26°+y\sin 60.26° = \frac{2}{\sqrt{65}}$

Also length of $\perp$ar from (0,0) to given line is $p = 2/\sqrt{65}$

(c) $15y-8x+3=0$

(i) Slope-intercept form:- $(y=mx+c)$

$\because 15y-8x+3=0 \rightarrow 15y=8x-3$

$\rightarrow y=\frac{8}{15}x-\frac{3}{15} \longrightarrow y=mx+c$

where $m=\frac{8}{15}$, $c=\frac{-3}{15}=-\frac{1}{5}$

(ii) Intercept form:- $\left(\frac{x}{a}+\frac{y}{b}=1\right)$

$\because 15y-8x+3=0 \rightarrow 15y-8x=-3$

$\rightarrow \frac{15y}{-3}-\frac{8x}{-3}=1 \rightarrow -5y+\frac{8}{3}x=1$

$\rightarrow \frac{x}{(3/8)}+\frac{y}{(-1/5)}=1 \longrightarrow \frac{x}{a}+\frac{y}{b}=1$

where $a=3/18$, $b=-1/5$

(iii) Normal line:- $(x\cos\alpha+y\sin\alpha=p)$

$\because 15y-8x+3=0 \rightarrow 15y-8x=-3$

$\rightarrow -8x+15y=-3$

'÷' by $\sqrt{(-8)^2+(15)^2}=\sqrt{64+225}=\sqrt{289}=17$

$\rightarrow -\frac{8}{17}x+\frac{15}{17}y=\frac{-3}{17} \rightarrow \frac{8}{17}x-\frac{15}{17}y=\frac{3}{17}$

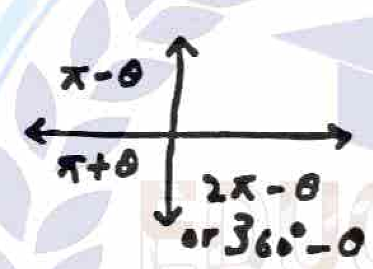
$\rightarrow x\left(\frac{8}{17}\right)+y\left(-\frac{15}{17}\right)=\frac{3}{17} \longrightarrow x\cos\alpha+y\sin\alpha=p$

where $\cos\alpha=\frac{8}{17}$, $\sin\alpha=\frac{-15}{17}$

$\tan\alpha=\frac{\sin\alpha}{\cos\alpha}=\frac{-15/17}{8/17}$ $\because \cos\alpha>0$, $\sin\alpha<0$ $\rightarrow \alpha$ lies in IV quad.

$\rightarrow \tan\alpha=\frac{-15}{8} \rightarrow \alpha=\tan^{-1}\left(\frac{-15}{8}\right)=-61.93^\circ$

$\rightarrow \alpha=360^\circ-61.93^\circ=298.07^\circ$



Thus $x\cos 298.07^\circ+y\sin 298.07^\circ=\frac{3}{17}$

Also length of $\perp$ from $(0,0)$ is $p=\frac{3}{17}$

Q22. In each of the following check whether the two lines are

(i) parallel (ii) perpendicular (iii) neither parallel nor perpendicular

(a) $2x+y-3=0$; $4x+2y+5=0$

Solution:- $2x+y-3=0$, $4x+2y+5=0$

$\because m_1=-\frac{a}{b}=\frac{-2}{1}=-2$, $m_2=-\frac{a}{b}=\frac{-4}{2}=-2$

$\because$ slopes $m_1=m_2 \rightarrow$ given lines are $\parallel$.

(b) $3y=2x+5$; $3x+2y-8=0$

Solution:- $3y=2x+5$; $3x+2y-8=0$

$\rightarrow 2x-3y+5=0$, $3x+2y-8=0$

$\because m_1=-\frac{a}{b}=\frac{-2}{-3}=\frac{2}{3}$, $m_2=-\frac{a}{b}=-\frac{3}{2}$

$\because m_1m_2=\left(\frac{2}{3}\right)\left(\frac{-3}{2}\right)=-1$

$\rightarrow$ given lines are $\perp$ar.

(c) $4y+2x-1=0$; $x-2y-7=0$

Solution:- $2x+4y-1=0$, $x-2y-7=0$

$\because m_1=-\frac{a}{b}=-\frac{2}{4}=-\frac{1}{2}$, $m_2=-\frac{a}{b}=\frac{-1}{-2}=\frac{1}{2}$

$\because m_1\neq m_2$ so given lines are neither $\parallel$ nor $\perp$ar

(d) $4x-y+2=0$; $12x-3y+1=0$

Solution:- $4x-y+2=0$, $12x-3y+1=0$

$\because m_1=-\frac{a}{b}=\frac{-4}{-1}=4$, $m_2=-\frac{a}{b}=\frac{-12}{-3}=4$

$\because m_1=m_2$ so given lines are parallel.

(e) $12x+35y-7=0$; $105x-36y+11=0$

Solution:- $12x+35y-7=0$; $105x-36y+11=0$

$\because m_1=-\frac{a}{b}=-\frac{12}{35}$, $m_2=-\frac{a}{b}=\frac{-105}{-36}=\frac{35}{12}$

$\because m_1m_2=\left(-\frac{12}{35}\right)\left(\frac{35}{12}\right)=-1$

so given lines are perpendicular.

Q23. Find the distance between the given parallel lines. Sketch the parallel lines. Also find an equation of the parallel line lying midway between them.

(a) $3x-4y+3=0$; $3x-4y+7=0$

Solution:- l_1; $3x-4y+3=0$

l_2; $3x-4y+7=0$

For l_1; put $x=0$, $3(0)-4y+3=0 \rightarrow -4y=-3$

$\rightarrow y=3/4$

put $y=0$, $3x-4(0)+3=0 \rightarrow 3x=-3$

$\rightarrow x=-1$ so $(0,\frac{3}{4})$ and $(-1,0)$ on l_1

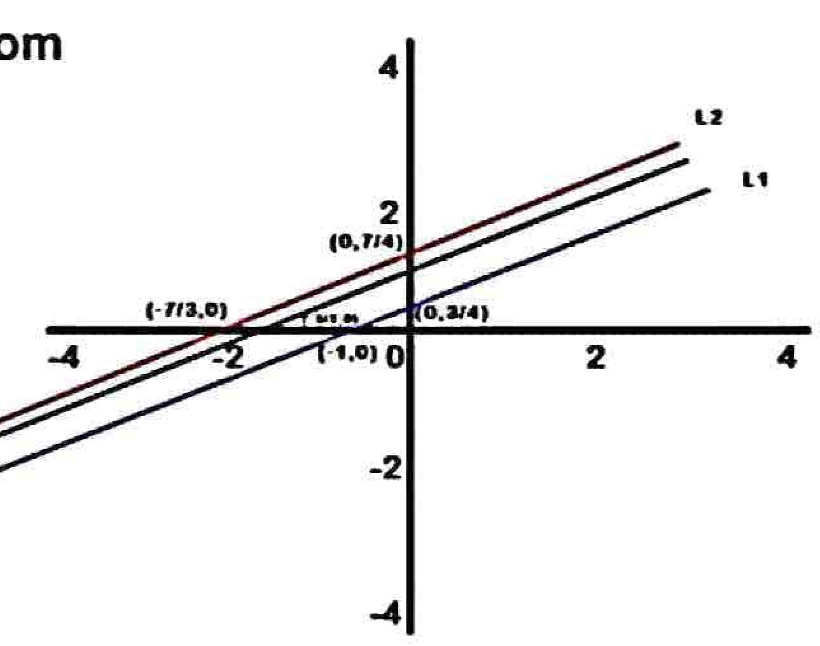
For l_2, put $x=0$, $3(0)-4y+7=0 \rightarrow -4y=-7$

$\rightarrow y=7/4$

put $y=0$, $3x-4(0)+7=0 \rightarrow 3x=-7$

$\rightarrow x=-7/3$ so $(0,\frac{7}{4})$ and $(-\frac{7}{3},0)$ on l_2

Now distance d from $(-1,0)$ to l_2 is



$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

$=\frac{|3(-1)-4(0)+7|}{\sqrt{(3)^2+(-4)^2}}$

$d=\frac{|-3+7|}{\sqrt{9+16}}=\frac{4}{\sqrt{25}}=\frac{4}{5}$

$\Rightarrow d = \frac{4}{5}$ Thus distance between the parallel lines $\frac{4}{5}$

Now mid-point of $(-1,0)$ and $(-\frac{7}{3},0)$ is

$= \left(\frac{-1-\frac{7}{3}}{2}, \frac{0+0}{2}\right) = \left(\frac{-3-7}{6}, 0\right) = \left(\frac{-10}{6}, 0\right)$

$= \left(-\frac{5}{3}, 0\right)$

$\because$ slope $= m = -\frac{a}{b} = \frac{-3}{-4} = \frac{3}{4}$

Now required equation of line passing through point $(-\frac{5}{3},0)$ and slope $\frac{3}{4}$ is

$y - 0 = \frac{3}{4}\left(x + \frac{5}{3}\right)$ $\quad (\because y - y_1 = m(x-x_1))$

$\Rightarrow 4y = 3x+5 \Rightarrow 3x - 4y + 5 = 0$

(b) $12x+5y-6=0$; $12x+5y+13=0$

Solution:- l_1; $12x+5y-6=0$, l_2; $12x+5y+13=0$

For l_1; put $x=0 \Rightarrow 12(0)+5y-6=0$

$\Rightarrow 5y = 6 \Rightarrow y = 6/5$

put $y=0 \Rightarrow 12x+5(0)-6=0$

$\Rightarrow 12x = 6 \Rightarrow x = 6/12 \Rightarrow x = 1/2$

so $(0, 6/5)$ and $(\frac{1}{2}, 0)$ on l_1

For l_2; put $x=0 \Rightarrow 12(0)+5y+13=0$

$\Rightarrow 5y+13=0 \Rightarrow 5y=-13 \Rightarrow y = -\frac{13}{5}$

put $y=0 \Rightarrow 12x+5(0)+13=0$

$\Rightarrow 12x = -13 \Rightarrow x = -\frac{13}{12}$

so $(0, -\frac{13}{5})$ and $(-\frac{13}{12}, 0)$ on l_2

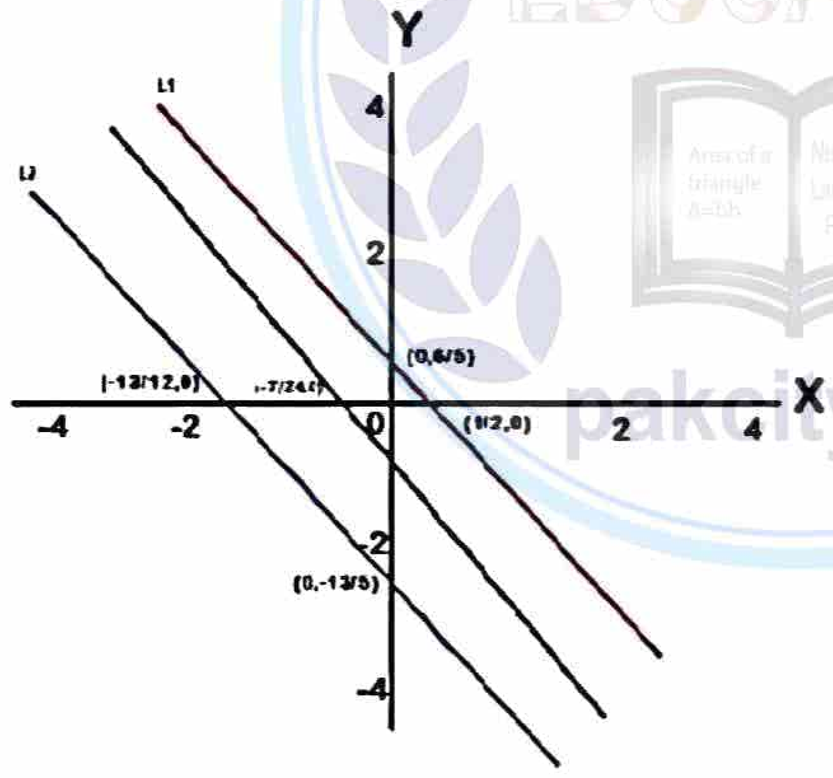
Now distance d from $(1/2, 0)$ from l_2 is

$d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

$= \frac{|12(\frac{1}{2})+5(0)+13|}{\sqrt{(12)^2+(5)^2}}$

$d = \frac{|6+13|}{\sqrt{144+25}}$

$= \frac{19}{\sqrt{169}} = \frac{19}{13}$



$\Rightarrow d = \frac{19}{13}$ Thus distance between the parallel lines $\frac{19}{13}$

Now midpoint of $(1/2, 0)$ and $(-\frac{13}{12}, 0)$ is

$= \left(\frac{\frac{1}{2}-\frac{13}{12}}{2}, \frac{0+0}{2}\right) = \left(\frac{\frac{6-13}{12}}{2}, 0\right)$

$= \left(-\frac{7}{24}, 0\right)$

$\because m = \text{slope} = -\frac{a}{b} = -\frac{12}{5}$

Now required equation of line passing through point $(-\frac{7}{24}, 0)$ and slope $-\frac{12}{5}$ is

$y - 0 = -\frac{12}{5}\left(x + \frac{7}{24}\right)$ $\quad (\because y - y_1 = m(x-x_1))$

$\Rightarrow 5y = -12x - \frac{7}{2} \Rightarrow 12x + 5y + \frac{7}{2} = 0$

(c) $x+2y-5=0$; $2x+4y=1$

Solution:- l_1; $x+2y-5=0$, l_2; $2x+4y=1$

For l_1; put $x=0 \Rightarrow 0+2y-5=0$

$\Rightarrow 2y = 5 \Rightarrow y = \frac{5}{2}$

put $y=0 \Rightarrow x+2(0)-5=0 \Rightarrow x=5$

so $(0, 5/2)$ and $(5,0)$ on l_1

For l_2; put $x=0$, $2(0)+4y=1 \Rightarrow y = 1/4$

put $y=0$, $2x+4(0)=1 \Rightarrow x = 1/2$

so $(0, \frac{1}{4})$ and $(\frac{1}{2}, 0)$ on l_2

Now distance d from $(5,0)$ to l_2 is

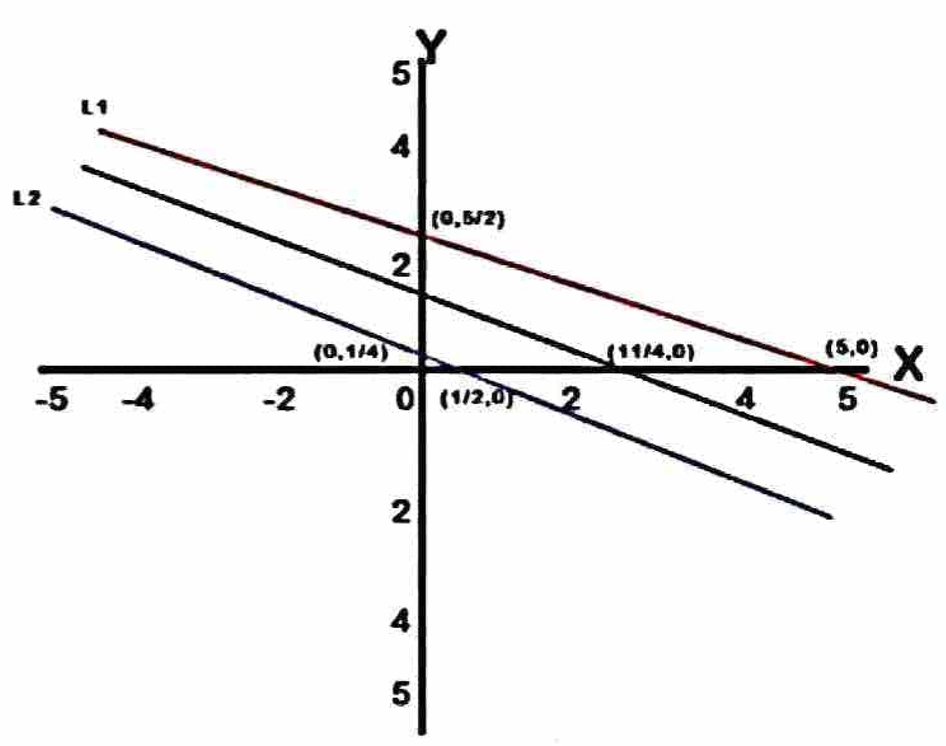
$d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

$= \frac{|2(5)+4(0)}{\sqrt{(2)^2+(4}$

$d = \frac{|10-1|}{\sqrt{4+16}}$

$= \frac{9}{\sqrt{20}}$

$= \frac{9}{2\sqrt{5}}$



$\Rightarrow d = \frac{9}{2\sqrt{5}}$ Thus distance between the parallel lines $\frac{9}{2\sqrt{5}}$

Now midpoint of $(5,0)$ and $(1/2, 0)$

$= \left(\frac{5+\frac{1}{2}}{2}, \frac{0+0}{2}\right) = \left(\frac{10+1}{4}, 0\right) = \left(\frac{11}{4}, 0\right)$

$\because$ slope $= m = -\frac{a}{b} = -\frac{1}{2}$

Now required equation of line passing through point $(\frac{11}{4}, 0)$ and slope $-\frac{1}{2}$ is

$y - 0 = -\frac{1}{2}\left(x - \frac{11}{4}\right)$ $\quad (\because y - y_1 = m(x-x_1))$

$\Rightarrow 2y = -x + \frac{11}{4} \Rightarrow x + 2y - \frac{11}{4} = 0$

Q24. Find an equation of the line through $(-4,7)$ and parallel to the line $2x-7y+4=0$

Solution:- Given line $2x-7y+4=0$

$\because$ slope of given line $\therefore m = -\frac{a}{b} = \frac{-2}{-7} = \frac{2}{7}$

$\rightarrow$ slope of required line $= \frac{2}{7}$

($\because$ req. line is $\parallel$ to given line then slopes are equal)

Thus eq. of required line through $(-4,7)$ having slope $\frac{2}{7}$ is

$\rightarrow y-7 = \frac{2}{7}(x+4)$ $\quad \because y-y_1 = m(x-x_1)$ point-slope form

$\rightarrow 7y-49 = 2x+8$

$\rightarrow 2x-7y+49+8=0 \rightarrow 2x-7y+57=0$

Q25. Find an equation of the line through $(5,-8)$ and perpendicular to the join of $A(-15,-8)$, $B(10,7)$

Solution:- slope of AB (given line) $= \frac{7-(-8)}{10+15}$

$= \frac{15}{25} = \frac{3}{5}$

$\rightarrow$ slope of req. line $= -\frac{5}{3}$

($\because$ req. line is $\perp$ar to given line)

Thus eq. of req. line through $(5,-8)$ and having slope $-\frac{5}{3}$ is

$y+8 = -\frac{5}{3}(x-5)$ $\quad (\because y-y_1 = m(x-x_1))$

$\rightarrow 3y+24 = -5x+25$

$\rightarrow 5x+3y+24-25=0$

$\rightarrow 5x+3y-1=0$

Q26. Find equations of two parallel lines perpendicular to $2x-y+3=0$ such that the product of the x- and y-intercepts of each is 3.

Solution:- Given line l; $2x-y+3=0$

slope of given line $= -\frac{a}{b} = \frac{-2}{-1} = 2$

$\because$ req. line is $\perp$ar to l so

slope of req. line $= -\frac{1}{2}$

Let y-intercept of req. line $= c$

so eq. of required line having slope $-\frac{1}{2}$ and y-intercept c is

$y = -\frac{1}{2}x + c$ ___ (i) $\quad \because y = mx+c$ (slope intercept form)

$\rightarrow \frac{1}{2}x + y = c$

$\rightarrow \frac{x}{2c} + \frac{y}{c} = 1 \longrightarrow \frac{x}{a} + \frac{y}{b} = 1$

This is two intercepts form of eq. of line with x-intercept $= 2c$ and y-intercept $= c$

$\because$ product of intercepts $= 3$

$\rightarrow (c)(2c) = 3$

$\rightarrow 2c^2 = 3 \rightarrow c^2 = \frac{3}{2} \rightarrow c = \pm\sqrt{\frac{3}{2}}$

so (i) $\rightarrow y = -\frac{1}{2}x \pm \sqrt{\frac{3}{2}} \rightarrow \frac{1}{2}x + y = \pm\sqrt{\frac{3}{2}}$

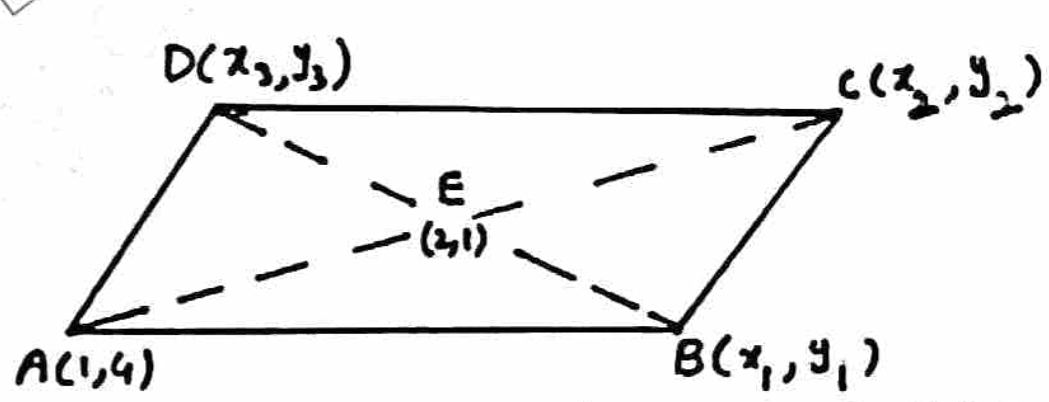
$\rightarrow \frac{1}{2}x + y \mp \sqrt{\frac{3\times 2}{2\times 2}} = 0$

$\rightarrow \frac{1}{2}x + y \mp \frac{\sqrt{6}}{2} = 0 \rightarrow x+2y \mp \sqrt{6} = 0$

Thus required two parallel lines are $x+2y-\sqrt{6}=0$ and $x+2y+\sqrt{6}=0$

Q27. One vertex of a parallelogram is $(1,4)$, the diagonals intersect at $(2,1)$ and the sides have slopes 1 and $-\frac{1}{7}$. Find the other three vertices.

Solution:-



Let $(1,4)$ be given vertex and $B(x_1,y_1)$ $C(x_2,y_2)$ and $D(x_3,y_3)$ be required vertices.

$\because$ E is midpoint of AC. so

$(2,1) = \left(\frac{1+x_2}{2}, \frac{4+y_2}{2}\right)$

$\rightarrow 2 = \frac{1+x_2}{2}$, $1 = \frac{4+y_2}{2}$

$\rightarrow 1+x_2 = 4$, $4+y_2 = 2 \rightarrow \boxed{x_2=3}, \boxed{y_2=-2}$

so $C(x_2,y_2) = (3,-2)$ Now

slope of AD $= \frac{y_3-4}{x_3-1} \rightarrow 1 = \frac{y_3-4}{x_3-1}$

$\rightarrow x_3-1 = y_3-4 \rightarrow x_3-y_3-1+4=0$

$\rightarrow x_3-y_3+3=0$ ——— (I)

slope of BC $= \frac{-2-y_1}{3-x_1} \rightarrow 1 = \frac{-2-y_1}{3-x_1}$

$\rightarrow 3-x_1 = -2-y_1 \rightarrow x_1-3-2-y_1=0$

$\rightarrow x_1-y_1-5=0$ ——— (II)

slope of AB $= \frac{y_1-4}{x_1-1} \rightarrow -\frac{1}{7} = \frac{y_1-4}{x_1-1}$

$\rightarrow -x_1+1 = 7y_1-28 \rightarrow x_1+7y_1-1-28=0$

$\rightarrow x_1+7y_1-29=0$ ——— (III)

Slope of DC $= \frac{-2-y_3}{3-x_3} \Rightarrow -\frac{1}{7} = \frac{-2-y_3}{3-x_3}$

$\Rightarrow -3+x_3 = -14-7y_3$

$\Rightarrow -3+x_3+14+7y_3 = 0$

$\Rightarrow x_3+7y_3+11 = 0$ ——— (IV)

By IV − I $\Rightarrow$

$$\begin{array}{r} \cancel{x_3}+7y_3+11=0 \\ -\cancel{x_3} \mp y_3 \pm 3 = 0 \\ \hline 8y_3+8=0 \end{array}$$

$\Rightarrow 8y_3 = -8 \Rightarrow \boxed{y_3=-1}$ put in (I)

$x_3-(-1)+3=0 \Rightarrow x_3+1+3=0$

$\Rightarrow x_3+4=0 \Rightarrow \boxed{x_3=-4}$

By III − II $\Rightarrow$

$$\begin{array}{r} \cancel{x_1}+7y_1-29=0 \\ \cancel{x_1} \mp y_1 \mp 5=0 \\ \hline 8y_1-24=0 \end{array}$$

$\Rightarrow 8y_1 = 24 \Rightarrow \boxed{y_1=3}$ put in II

$x_1-3-5=0 \Rightarrow x_1-8=0$

$\Rightarrow \boxed{x_1=8}$

Hence required vertices are

$B(x_1,y_1) = B(8,3)$, $C(x_2,y_2) = C(3,-2)$

$D(x_3,y_3) = D(-4,-1)$

Remember, Above line:- If sign y in given equation and our answer is same. Below line:- If sign y in given equation and our answer is different.

Q28. Find whether the given point lies above or below the given line. (a) (5,8); $2x-3y+6=0$

(b) (−7,6); $4x+3y-9=0$

Solution:- (a) (5,8); $2x-3y+6=0$

sign of coefficient of $y = -3 = -$ive

Now for (5,8), $2x-3y+6 = 2(5)-3(8)+6$

$= 10-24+6 = -8 = -$ive

Thus (5,8) lies above the given line.

(b) (−7,6); $4x+3y-9=0$

sign of coefficient of $y = 3 = +$ive

Now for (−7,6), $4x+3y-9 = 4(-7)+3(6)-9$

$= -28+18-9 = -19 = -$ive

So (−7,6) is below the given line.

Q29. Check whether the given points are on the same or opposite sides of the given line.

(a) (0,0) and (−4,7); $6x-7y+70=0$

(b) (2,3) and (−2,3); $3x-5y+8=0$

Solution:- (a) (0,0) and (−4,7); $6x-7y+70=0$

sign of coefficient of $y = -7 = -$ive

For (0,0), $6x-7y+70 = 6(0)-7(0)+70 = 70 = +$ive

so (0,0) is below the line.

For (−4,7), $6x-7y+70 = 6(-4)-7(7)+70$

$= -24-49+70 = -69+70 = 1 = +$ive

so (−4,7) is below the line

∵ both points are below so both points are on same sides.

(b) (2,3) and (−2,3); $3x-5y+8=0$

sign of coefficient of $y = -5 = -$ive

For (2,3); $3x-5y+8 = 3(2)-5(3)+8$

$= 6-15+8 = 14-15 = -1 = -$ive

so (2,3) is above the line.

For (−2,3); $3x-5y+8 = 3(-2)-5(3)+8$

$= -6-15+8 = -15+2 = -13 = -$ive

so (−2,3) is above the line.

∵ both points are above so both points are on same sides.

Q30. Find the distance from the point P(6,−1) to the line $6x-4y+9=0$.

Solution:- Here $x_1 = 6$, $y_1 = -1$

$a = 6$, $b = -4$, $c = 9$

so $d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

$d = \frac{|6(6)+(-4)(-1)+9|}{\sqrt{(6)^2+(-4)^2}} = \frac{|36+4+9|}{\sqrt{36+16}}$

$\Rightarrow d = \frac{49}{\sqrt{52}} = \frac{49}{\sqrt{4\times 13}} = \frac{49}{2\sqrt{13}}$

Q31. Find the area of the triangular region whose vertices are $A(5,3)$, $B(-2,2)$, $C(4,2)$

Solution:- Here $x_1 = 5$, $y_1 = 3$, $x_2 = -2$, $y_2 = 2$, $x_3 = 4$, $y_3 = 2$

$\because$ Area of triangle $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$= \frac{1}{2}\begin{vmatrix} 5 & 3 & 1 \\ -2 & 2 & 1 \\ 4 & 2 & 1 \end{vmatrix}$

$= \frac{1}{2}[5(2-2) - 3(-2-4) + 1(-4-8)]$

$= \frac{1}{2}[5(0) - 3(-6) + 1(-12)]$

$= \frac{1}{2}(0 + 18 - 12) = \frac{1}{2}(6) = 3$ sq. units

Q32. The coordinates of three points are $A(2,3)$, $B(-1,1)$ and $C(4,-5)$. By computing the area bounded by ABC check whether the points are collinear.

Solution:- Here $x_1 = 2$, $y_1 = 3$, $x_2 = -1$, $y_2 = 1$, $x_3 = 4$, $y_3 = -5$

$\because$ Area of triangle $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$= \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 4 & -5 & 1 \end{vmatrix}$

$= \frac{1}{2}[2(1+5) - 3(-1-4) + 1(5-4)]$

$A = \frac{1}{2}[12 + 15 + 1] = \frac{28}{2} = 14$ sq. units

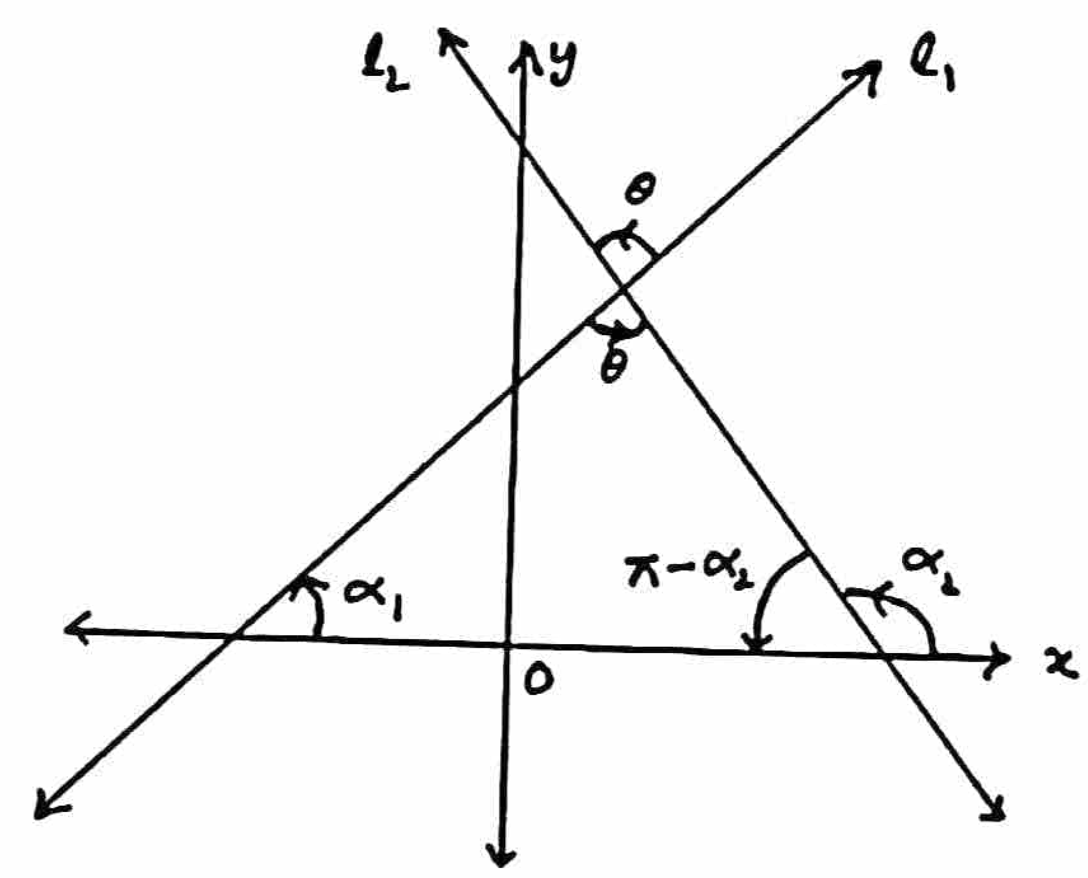
$\because A = 14 \neq 0$ so points are not collinear.

Angle between two Lines

Theorem:- Let l_1 and l_2 be two non-vertical lines such that they are not perpendicular to each other. If m_1 and m_2 are the slopes of l_1 and l_2 respectively, then the angle θ from l_1 to l_2 is given by;

$$\boxed{\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}}$$

Proof:-



$\because$ sum of all three angles is equal to $180°$

so $\alpha_1 + \theta + 180° - \alpha_2 = 180°$

$\Rightarrow \alpha_1 - \alpha_2 + \theta = 180° - 180°$

$\Rightarrow \alpha_1 - \alpha_2 + \theta = 0 \Rightarrow \theta = \alpha_2 - \alpha_1$

$\Rightarrow \tan\theta = \tan(\alpha_2 - \alpha_1)$

$\tan\theta = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2}$ $\quad \because \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$\Rightarrow \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$ $\quad \because m_1 = \tan\alpha_1 =$ slope of l_1, $m_2 = \tan\alpha_2 =$ slope of l_2

Corollary 1. If two lines are parallel then their slopes are equal.

i.e., $l_1 \parallel l_2$ if and only if $m_1 = m_2$

Proof:- Let m_1 and m_2 be slopes of lines l_1 and l_2 respectively.

Let θ be angle from l_1 to l_2.

$\because$ lines are $\parallel$ so $\theta = 0$

we know that $\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$

$\Rightarrow \tan 0 = \frac{m_2 - m_1}{1 + m_1 m_2} \Rightarrow 0 = \frac{m_2 - m_1}{1 + m_1 m_2}$

$\Rightarrow m_2 - m_1 = 0 \Rightarrow m_1 = m_2$

Hence proved.

Corollary 2. If two lines are perpendicular then product of their slopes is equal to -1.

i.e., $l_1 \perp l_2$ iff $1 + m_1 m_2 = 0$

Proof:- Let m_1 and m_2 be slopes of l_1 and l_2 respectively.

Let θ be angle from ℓ_1 and ℓ_2

$\because$ lines are $\perp$ar so $\theta = 90^\circ$

we know that

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\Rightarrow \tan 90^\circ = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\Rightarrow \infty = \frac{m_2 - m_1}{1 + m_1 m_2} \Rightarrow \frac{1}{0} = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\Rightarrow 1 + m_1 m_2 = 0 \Rightarrow m_1 m_2 = -1$$

Hence proved.

Example 1. Find angle from the line with slope $-7/3$ to the line with slope $\frac{5}{2}$.

Solution:- Here $m_1 = -\frac{7}{3}$, $m_2 = \frac{5}{2}$

If θ is required angle, then

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\frac{5}{2} - (-\frac{7}{3})}{1 + (\frac{5}{2})(-\frac{7}{3})}$$

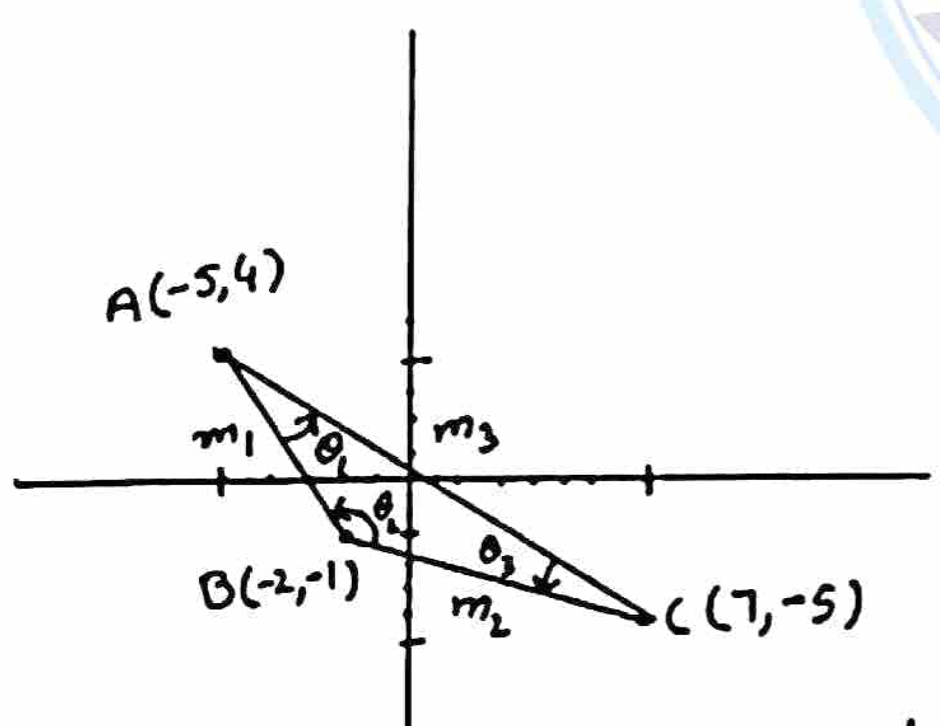
$$\tan\theta = \frac{\frac{5}{2} + \frac{7}{3}}{1 + (-\frac{35}{6})} = \frac{\frac{15+14}{\cancel{6}}}{\frac{6-35}{\cancel{6}}}$$

$$\Rightarrow \tan\theta = \frac{29}{-29} \Rightarrow \tan\theta = -1$$

$$\Rightarrow \theta = \tan^{-1}(-1) = 135^\circ$$

Example 2. Find the angles of the triangle whose vertices are $A(-5,4)$, $B(-2,-1)$, $C(7,-5)$

Solution:-
$A(-5,4)$, $B(-2,-1)$, $C(7,-5)$



$$\text{slope of } AB = m_1 = \frac{-1-4}{-2+5} = -\frac{5}{3}$$

$$\text{slope of } BC = m_2 = \frac{-5+1}{7+2} = -\frac{4}{9}$$

$$\text{slope of } CA = m_3 = \frac{4+5}{-5-7} = \frac{9}{-12} = -\frac{3}{4}$$

Now $\tan\theta_1 = \dfrac{m_3 - m_1}{1 + m_3 m_1}$ ($\because \theta_1$ is measured from m_1 to m_3)

$$\Rightarrow \tan\theta_1 = \frac{-\frac{3}{4} + \frac{5}{3}}{1 + (-\frac{3}{4})(-\frac{5}{3})} = \frac{\frac{-9+20}{12}}{\frac{12+15}{12}} = \frac{11}{27}$$

$$\Rightarrow \tan\theta_1 = \frac{11}{27} \Rightarrow \theta_1 = \tan^{-1}\left(\frac{11}{27}\right)$$

$$\theta_1 = 22.16^\circ = 22.2^\circ$$

$\tan\theta_2 = \dfrac{m_1 - m_2}{1 + m_1 m_2}$ ($\because \theta_2$ is measured from m_2 to m_1)

$$\tan\theta_2 = \frac{-\frac{5}{3} - (-\frac{4}{9})}{1 + (-\frac{5}{3})(-\frac{4}{9})} = \frac{-\frac{5}{3} + \frac{4}{9}}{1 + \frac{20}{27}}$$

$$\tan\theta_2 = \frac{\frac{-45+12}{27}}{\frac{27+20}{27}} = \frac{-33}{47}$$

$$\theta_2 = \tan^{-1}\left(-\frac{33}{47}\right)$$

$\boxed{\tan^{-1}(-x) = -\tan^{-1}x}$

$$\theta_2 = -\tan^{-1}\left(\frac{33}{47}\right)$$

$\theta_2 = -35.07^\circ$ (Clock-Wise)

$= -35.07^\circ + 180^\circ$

$\theta_2 = 144.9^\circ$

Since inclination of line is measured counter-clock wise (anti-clock wise) and $0^\circ < \theta < 180^\circ$

$\tan\theta_3 = \dfrac{m_2 - m_3}{1 + m_2 m_3}$ ($\because \theta_3$ is measured from m_3 to m_2)

$$= \frac{-\frac{4}{9} - (-\frac{3}{4})}{1 + (-\frac{4}{9})(-\frac{3}{4})} = \frac{-\frac{4}{9} + \frac{3}{4}}{1 + \frac{12}{36}}$$

$$\tan\theta_3 = \frac{\frac{-16+27}{36}}{\frac{36+12}{36}} = \frac{11}{48}$$

$$\Rightarrow \theta_3 = \tan^{-1}\left(\frac{11}{48}\right) = 12.9^\circ$$

Equation of a Straight Line in Matrix form

One Linear equation:-

A linear equation $\ell; ax + by + c = 0$ in two variables x and y has its matrix form as

$$[ax + by] = [-c]$$

or
$$\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [-c]$$

$$\Rightarrow AX = B \qquad A = [a \;\; b],\; X = \begin{bmatrix} x \\ y \end{bmatrix}, \; B = [-c]$$

A system of two Linear Equations:-

A system of two linear equations

l_1: $a_1x + b_1y + c_1 = 0$

l_2: $a_2x + b_2y + c_2 = 0$ in two variables x and y can be written in matrix form as:

$$\begin{bmatrix} a_1x & b_1y \\ a_2x & b_2y \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}$$

$$AX = C$$

where $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $C = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}$

A system of three Linear Equations:-

A system of three linear equations

l_1: $a_1x + b_1y + c_1 = 0$

l_2: $a_2x + b_2y + c_2 = 0$

l_3; $a_3x + b_3y + c_3 = 0$ in two variables x and y takes the form as;

$$\begin{bmatrix} a_1x + b_1y + c_1 \\ a_2x + b_2y + c_2 \\ a_3x + b_3y + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 1. Express the system $3x + 4y - 7 = 0$, $2x - 5y + 8 = 0$, $x + y - 3 = 0$ in matrix form and check whether the three lines are concurrent.

Solution:- The matrix form of the system is

$$\begin{bmatrix} 3 & 4 & -7 \\ 2 & -5 & 8 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $A = \begin{bmatrix} 3 & 4 & -7 \\ 2 & -5 & 8 \\ 1 & 1 & -3 \end{bmatrix}$

Now $\det A = \begin{vmatrix} 3 & 4 & -7 \\ 2 & -5 & 8 \\ 1 & 1 & -3 \end{vmatrix}$

$= 3(15-8) - 4(-6-8) - 7(2+5)$

$= 3(7) - 4(-14) - 7(7)$

$\det|A| = 21 + 56 - 49 = 28 \neq 0$

so given lines are not concurrent.

Example 2. Find a system of linear equations corresponding to the matrix form

$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 5 & 1 \\ 2 & 7 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Are the lines represented by the system concurrent?

Solution:-

$x + 2y + 5 = 0$

$3x + 5y + 1 = 0$

$2x + 7y + 6 = 0$

This is required system of eqs.

Now $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 5 & 1 \\ 2 & 7 & 6 \end{bmatrix}$

$\det A = 1(30-7) - 3(12-35) + 4(2-25)$

$= 23 - 3(-23) + 4(-23)$

$\det A = 23 + 69 - 92 = 0$

so system of given lines is concurrent.

Exercise 4.4

Q1. Find the point of intersection of the lines

(i) $x - 2y + 1 = 0$ and $2x - y + 2 = 0$

Solution:- $x - 2y + 1 = 0 \longrightarrow (i)$

$2x - y + 2 = 0 \longrightarrow (ii)$

By $2(ii) - (i) \rightarrow$

$$\begin{array}{r} 4x - \cancel{2y} + 4 = 0 \\ -x \mp \cancel{2y} - 1 = 0 \\ \hline 3x + 3 = 0 \end{array}$$

$\rightarrow 3x = -3 \rightarrow x = -1$ put in (i)

so (i) $\rightarrow -1 - 2y + 1 = 0 \rightarrow -2y = 0$

$\rightarrow y = 0$ Thus $(-1, 0)$ is the point of intersection.

(ii) $3x + y + 12 = 0$ and $x + 2y - 1 = 0$

Solution:- $3x + y + 12 = 0 \longrightarrow (i)$

$x + 2y - 1 = 0 \longrightarrow (ii)$

By $2(i) - (ii) \rightarrow$

$$\begin{array}{r} 6x + 2y + 24 = 0 \\ -x - 2y + 1 = 0 \\ \hline 5x + 25 = 0 \end{array}$$

$\rightarrow 5x = -25 \rightarrow x = -5$ put in (ii)

so (ii) $\rightarrow -5 + 2y - 1 = 0 \rightarrow 2y - 6 = 0$

$\rightarrow 2y = 6 \rightarrow y = 3$ so $(-5, 3)$ is the point of intersection.

(iii) $x+4y-12=0$ and $x-3y+3=0$

Solution:- $x+4y-12=0 \longrightarrow$ (i)

$x-3y+3=0 \longrightarrow$ (ii)

By (i) − (ii) →

$$\begin{array}{r} x+4y-12=0 \\ \underline{\underset{-}{x}\ \underset{+}{-}\ 3y\ \underset{-}{+}\ 3=0} \\ 7y-15=0 \end{array}$$

$\rightarrow 7y=15 \rightarrow y=\frac{15}{7}$ put in (i)

so (i) $\rightarrow x+4\left(\frac{15}{7}\right)-12=0$

$\rightarrow x+\frac{60}{7}-12=0 \rightarrow x+\frac{60-84}{7}=0$

$\rightarrow x+\left(-\frac{24}{7}\right)=0 \rightarrow x=\frac{24}{7}$

Thus $\left(\frac{24}{7},\frac{15}{7}\right)$ is the point of intersection.

Q2. Find an equation of the line through (i) the point $(2,-9)$ and the intersection of lines $2x+5y-8=0$ and $3x-4y-6=0$

Solution:- Let $l_1; 2x+5y-8=0 \longrightarrow$ (i)

$l_2; 3x-4y-6=0 \longrightarrow$ (ii)

($\because$ A line 'l' passing through l_1 & l_2 is $l; l_1+Kl_2=0$)

so eq. of line passing through point of intersection of l_1 & l_2 is

$l;\ l_1+K\,l_2=0$

$\rightarrow 2x+5y-8+K(3x-4y-6)=0 \longrightarrow$ (I)

$\because (2,-9)$ lies on (i)

Therefore put $x=2$ and $y=-9$ in (I)

(I) $\rightarrow 2(2)+5(-9)-8+K(3(2)-4(-9)-6)=0$

$\rightarrow 4-45-8+K(6+36-6)=0$

$-49+36K=0 \rightarrow \boxed{K=\frac{49}{36}}$

Put value of K in (I)

$2x+5y-8+\frac{49}{36}(3x-4y-6)=0$

$72x+180y-288+147x-196y-294=0$

$\rightarrow 219x-16y-582=0$ req. equation

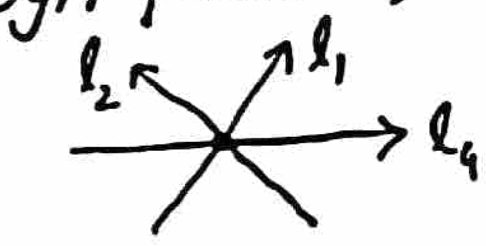
(ii) the intersection of the lines $x-y-4=0$ and $7x+y+20=0$ and (a) parallel (b) perpendicular to the line $6x+y-14=0$

Solution:- Let $l_1; x-y-4=0 \longrightarrow$ (i)

$l_2; 7x+y+20=0 \longrightarrow$ (ii)

$l_3;\ 6x+y-14=0$

($\because$ A line l_4 passing through point of intersection of l_1 & l_2 is $l_4; l_1+Kl_2=0$)



$\rightarrow x-y-4+K(7x+y+20)=0 \longrightarrow$ (I)

$x-y-4+7Kx+7yK+20K=0$

$\rightarrow (1+7K)x+(-1+K)y+(-4+20K)=0$

Slope of $l_4 = m_1 = -\frac{1+7K}{-1+K}$

slope of $l_3 = m_2 = -\frac{6}{1} = -6$

(a) If l_3 & l_4 are parallel then

$m_1 = m_2 \rightarrow -\frac{1+7K}{-1+K} = -6$

$\rightarrow 1+7K=6(-1+K) \rightarrow 1+7K=-6+6K$

$\rightarrow \boxed{K=-7}$ so (I) becomes

$x-y-4-7(7x+y+20)=0$

$x-y-4-49x-7y-140=0$

$-48x-8y-144=0 \rightarrow 6x+y+18=0$

(b) If l_3 & l_4 are $\perp$ then $m_1m_2=-1$

$\left(-\frac{1+7K}{-1+K}\right)(-6)=-1 \rightarrow 6(1+7K)=-(-1+K)$

$\rightarrow 6+42K=1-K \rightarrow \boxed{K=-\frac{5}{43}}$

so (I) $x-y-4-\frac{5}{43}(7x+y+20)=0$

$\rightarrow 43x-43y-172-35x-5y-100=0$

$\rightarrow 8x-48y-272=0$

$\rightarrow x-6y-34=0$

(iii) through the intersection of the lines $x+2y+3=0$, $3x+4y+7=0$ and making equal intercepts on the axes.

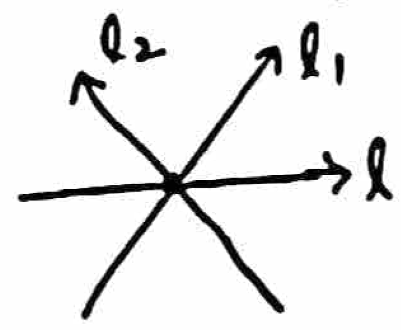
Solution:- Any line through intersection of $l_1; x+2y+3=0$ and $l_2; 3x+4y+7=0$ is

$l;\ l_1+K\,l_2=0$

$x+2y+3+K(3x+4y+7)=0$ ——— (I)

$\rightarrow x+2y+3+3kx+4ky+7k=0$ (∵ a line passing through intersection of l_1 and l_2 is l; $l_1+kl_2=0$)

$\rightarrow (3k+1)x+(2+4k)y+3+7k=0$



For x-intercept, $y=0$

so $(3k+1)x+3+7k=0$

$\rightarrow x = \dfrac{-(3+7k)}{3k+1}$

For y-intercept, $x=0$

$\rightarrow (2+4k)y+3+7k=0$

$\rightarrow y = \dfrac{-(3+7k)}{2+4k}$

∵ Both intercepts are equal so

x-intercept = y-intercept

$\dfrac{-(3+7k)}{3k+1} = \dfrac{-(3+7k)}{2+4k}$

$\rightarrow \dfrac{1}{3k+1} = \dfrac{1}{2+4k}$

$\rightarrow 3k+1 = 2+4k$

$\rightarrow 4k-3k+2-1=0 \rightarrow k+1=0$

$\rightarrow \boxed{k=-1}$ so (I) becomes

as $x+2y+3+(-1)(3x+4y+7)=0$

$\rightarrow x+2y+3-3x-4y-7=0$

$\rightarrow -2x-2y-4=0 \rightarrow 2x+2y+4=0$

$\rightarrow x+y+2=0$ (÷ by 2)

Q3. Find an equation of the line through the intersection of $16x-10y-33=0$; $12x+14y+29=0$ and the intersection of $x-y+4=0$; $x-7y+2=0$

Solution:- First we find intersection of

$16x-10y-33=0 \longrightarrow (i)$

$12x+14y+29=0 \longrightarrow (ii)$

By $14(i)+10(ii) \rightarrow$

$$\begin{array}{r} 224x-140y-462=0 \\ 120x+140y+290=0 \\ \hline 344x-172=0 \end{array}$$

$\rightarrow x=\dfrac{172}{344}=\dfrac{1}{2} \rightarrow x=\dfrac{1}{2}$ put in (i)

(i) $\rightarrow 16\left(\frac{1}{2}\right)-10y-33=0 \rightarrow 8-10y-33=0$

$\rightarrow -10y-25=0 \rightarrow -10y=25 \rightarrow y=\dfrac{25}{-10}$

$\rightarrow y=-\dfrac{5}{2}$ so point of intersection is $\left(\frac{1}{2},-\frac{5}{2}\right)$

Now we find intersection of

$x-y+4=0 \longrightarrow (iii)$ and $x-7y+2=0 \longrightarrow (iv)$

By (iii) − (iv) $\rightarrow$

$$\begin{array}{r} \cancel{x}-y+4=0 \\ \cancel{x}-7y+2=0 \\ - \quad + \quad - \\ \hline 6y+2=0 \end{array}$$

$\rightarrow 6y=-2 \rightarrow y=-\dfrac{2}{6} \rightarrow y=-\dfrac{1}{3}$ put in (iii)

so (iii) $\rightarrow x-\left(-\frac{1}{3}\right)+4=0 \rightarrow x+\frac{1}{3}+4=0$

$\rightarrow x+\dfrac{1+12}{3}=0 \rightarrow x=-\dfrac{13}{3}$

so point of intersection is $\left(-\frac{13}{3},-\frac{1}{3}\right)$

Slope through $\left(\frac{1}{2},-\frac{5}{2}\right)$ and $\left(-\frac{13}{3},-\frac{1}{3}\right)$ is

$m = \dfrac{-\frac{1}{3}-\left(-\frac{5}{2}\right)}{-\frac{13}{3}-\frac{1}{2}} = \dfrac{-\frac{1}{3}+\frac{5}{2}}{-\frac{13}{3}-\frac{1}{2}} = \dfrac{\frac{-2+15}{6}}{\frac{-26-3}{6}}$

$m = \dfrac{13}{-29}$

so equation of req. line through $\left(\frac{1}{2},-\frac{5}{2}\right)$ and slope $m=-\frac{13}{29}$ is

$y-\left(-\frac{5}{2}\right) = -\frac{13}{29}\left(x-\frac{1}{2}\right) \rightarrow y+\frac{5}{2} = \frac{-13}{29}\left(x-\frac{1}{2}\right)$

$\rightarrow 29y+\frac{145}{2} = -13x+\frac{13}{2}$

$\rightarrow 13x+29y+\frac{145}{2}-\frac{13}{2}=0$

$\rightarrow 13x+29y+\frac{132}{2}=0$

$\rightarrow 13x+29y+66=0$ (req. line)

Q4. Find the condition that the lines $y=m_1x+c_1$; $y=m_2x+c_2$ and $y=m_3x+c_3$ are concurrent.

Solution:- Arranging given lines

$m_1x-y+c_1=0$, m_2x-y+c_2, $m_3x-y+c_3=0$

∵ given lines are concurrent so

$\begin{vmatrix} m_1 & -1 & c_1 \\ m_2 & -1 & c_2 \\ m_3 & -1 & c_3 \end{vmatrix}=0$

(∵ Three lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$, $a_3x+b_3y+c_3=0$ are said to be concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$)

$\rightarrow \begin{vmatrix} m_1 & -1 & c_1 \\ m_2-m_1 & 0 & c_2-c_1 \\ m_3-m_1 & 0 & c_3-c_1 \end{vmatrix}=0$ By R_2-R_1, R_3-R_1

Expanding by C_2

$-(-1)[(m_2-m_1)(c_3-c_1)-(m_3-m_1)(c_2-c_1)]+0-0=0$

$\rightarrow (m_2-m_1)(c_3-c_1)-(m_3-m_1)(c_2-c_1)=0$

$\rightarrow (m_2-m_1)(c_3-c_1)=(m_3-m_1)(c_2-c_1)$

Required condition.

Q5. Determine the value of p such that the lines $2x-3y-1=0$; $3x-y-5=0$ and $3x+py+8=0$ meet at a point.

Solution:- ∵ given lines meet at a point so lines are concurrent.

$$\begin{vmatrix} 2 & -3 & -1 \\ 3 & -1 & -5 \\ 3 & p & 8 \end{vmatrix} = 0$$

$\rightarrow 2(-8+5p)-(-3)(24+15)+(-1)(3p+3)=0$

$\rightarrow -16+10p+3(39)-3p-3=0$

$\rightarrow -16+10p+117-3p-3=0$

$\rightarrow 7p+98=0 \rightarrow p=-\frac{98}{7} \rightarrow p=-14$

Q6. Show that the lines $4x-3y-8=0$, $3x-4y-6=0$ and $x-y-2=0$ are concurrent and the third-line bisects the angle fromed by the first two lines.

Solution:- Give lines are

$l_1;\ 4x-3y-8=0$

$l_2;\ 3x-4y-6=0$

$l_3;\ x-y-2=0$

$$\begin{vmatrix} 4 & -3 & -8 \\ 3 & -4 & -6 \\ 1 & -1 & -2 \end{vmatrix}$$

For concurrency $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$

$=4(8-6)-(-3)(-6+6)-8(-3+4)$

$=4(2)+3(0)-8(1)=8-8=0$

so given lines are concurrent.

Slope of $l_1 = m_1 = -\frac{a}{b} = \frac{4}{3}$

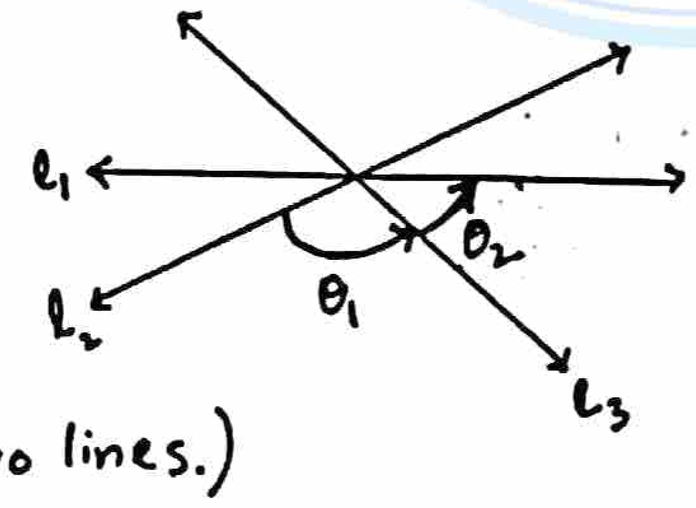
Slope of $l_2 = m_2 = -\frac{a}{b} = \frac{3}{4}$

Slope of $l_3 = m_3 = -\frac{a}{b} = 1$

Now the condition to be prove

$\theta_1 = \theta_2$

(i.e., third line bisects the angle made by first two lines.)



$\tan\theta_1 = \frac{m_3-m_2}{1+m_3m_2}$ (∵ θ_1 is angle from l_2 to l_3)

$\tan\theta_1 = \frac{1-\frac{3}{4}}{1+(1)(\frac{3}{4})} = \frac{\frac{4-3}{4}}{\frac{4+3}{4}}$

$\rightarrow \tan\theta_1 = \frac{1}{7} \rightarrow \theta_1 = \tan^{-1}(\frac{1}{7})$

$\tan\theta_2 = \frac{m_1-m_3}{1+m_1m_3}$ (∵ θ_2 is angle from l_3 to l_1)

$= \frac{\frac{4}{3}-1}{1+(\frac{4}{3})(1)} = \frac{\frac{4-3}{3}}{\frac{3+4}{3}} = \frac{1}{7}$

$\rightarrow \theta_2 = \tan^{-1}(\frac{1}{7})$

$\rightarrow \theta_1 = \theta_2$

Q7. The vertices of a triangle are $A(-2,3)$ $B(-4,1)$ and $C(3,5)$. Find coordinates of (i) centroid (ii) Orthocenter (iii) Circum-centre

Are these three points collinear?

Solution:-

> **Centroid:-** The point of intersection of medians is called centroid. If (x_1,y_1), (x_2,y_2) and (x_3,y_3) are vertices of triangle then centroid is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

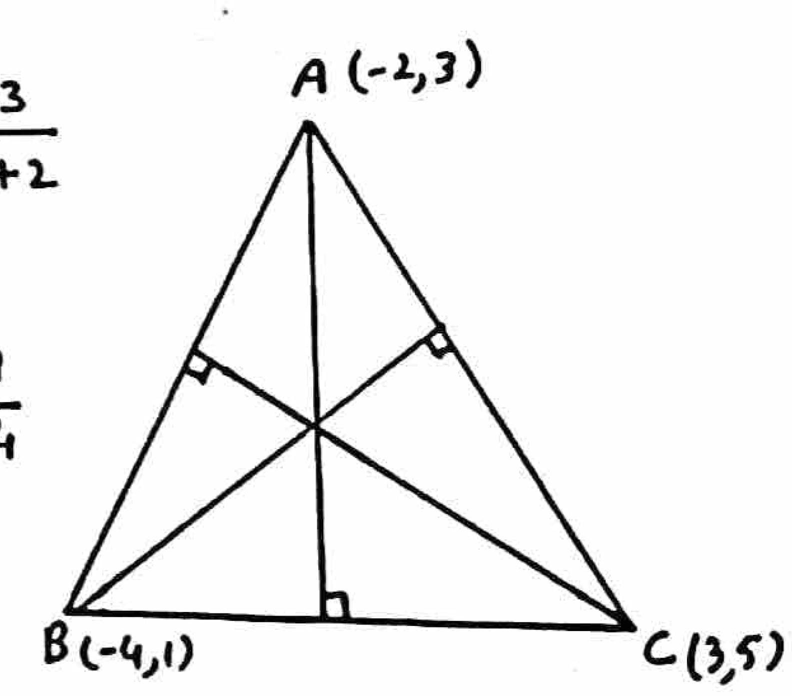
(i) $x_1=-2,\ y_1=3,\ x_2=-4,\ y_2=1,\ x_3=3$
$y_3=5$

centroid $=\left(\frac{-2-4+3}{3}, \frac{3+1+5}{3}\right)$

$=\left(-\frac{3}{3}, \frac{9}{3}\right) = (-1,3)$

> **Orthocentre:-** The point of intersection of altitudes is called orthocentre

(ii)



Slope of $AB = m_1 = \frac{1-3}{-4+2}$

$= \frac{-2}{-2} = 1$

Slope of $BC = m_2 = \frac{5-1}{3+4}$

$= \frac{4}{7}$

∵ altitudes are ⊥ar to sides therefore

slope of altitude on $AB = -\frac{1}{m_1} = \frac{-1}{1} = -1$

slope of altitude on $BC = \frac{-1}{m_2} = \frac{-1}{4/7} = \frac{-7}{4}$

Now

eq. of altitude on AB with slope -1 from $C(3,5)$

$y-5=-1(x-3)$ ∵ $y-y_1=m(x-x_1)$

$\rightarrow y-5=-x+3$

$\rightarrow y+x-5-3=0$

$\Rightarrow x + y - 8 = 0 \longrightarrow (i)$

Eq. of altitude on BC with slope $-\frac{7}{4}$ from $A(-2,3)$

$y - 3 = -\frac{7}{4}(x+2)$

$\Rightarrow 4y - 12 = -7x - 14$

$\Rightarrow 7x + 4y - 12 + 14 = 0$

$\Rightarrow 7x + 4y + 2 = 0 \longrightarrow (ii)$

By $7(i) - (ii) \Rightarrow$

$$\begin{array}{r} 7x + 7y - 56 = 0 \\ \underline{-7x \mp 4y \mp 2 = 0} \\ 3y - 58 = 0 \end{array}$$

$\Rightarrow y = \frac{58}{3}$ put in (i)

(i) $\Rightarrow x + \frac{58}{3} - 8 = 0 \Rightarrow x + \frac{58-24}{3} = 0$

$\Rightarrow x + \frac{34}{3} = 0 \Rightarrow x = -\frac{34}{3}$

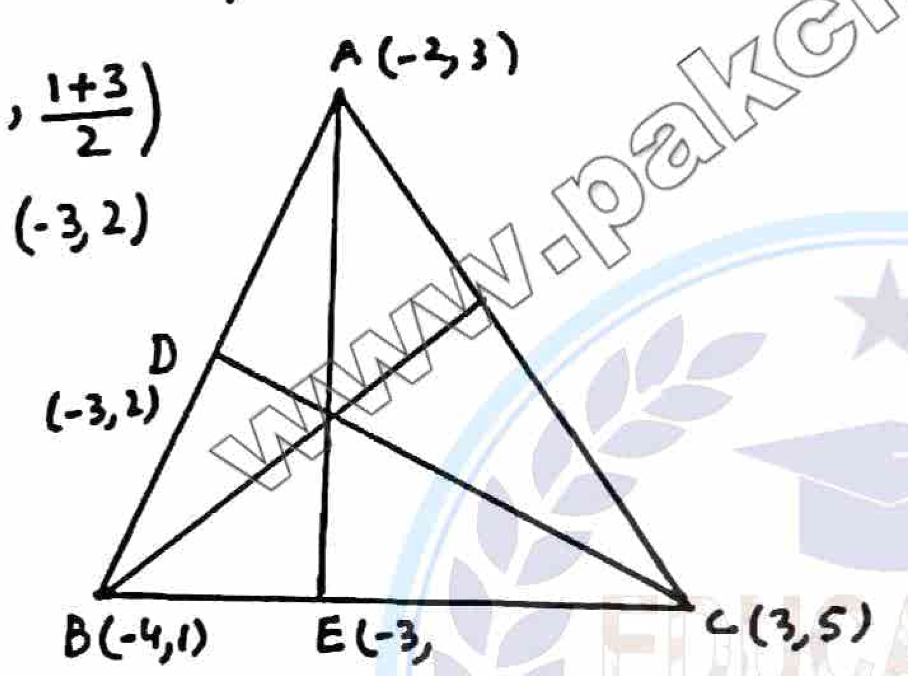
so orthocentre is $\left(-\frac{34}{3}, \frac{58}{3}\right)$

Circum-centre:- The point of intersection of right bisectors is called circum-centre

(iii)

$\because$ D is midpoint of AB. so coordinates of D are $\left(\frac{-4-2}{2}, \frac{1+3}{2}\right)$

$= \left(-\frac{6}{2}, \frac{4}{2}\right) = (-3, 2)$



E is midpoint of BC so coordinates of E are $\left(\frac{-4+3}{2}, \frac{1+5}{2}\right) = \left(-\frac{1}{2}, \frac{6}{2}\right) = \left(-\frac{1}{2}, 3\right)$

slope of $AB = m_1 = \frac{1-3}{-4+2} = \frac{-2}{-2} = 1$

slope of $BC = m_2 = \frac{5-1}{3+4} = \frac{4}{7}$

slope of $\perp$ bisector on $AB = \frac{-1}{m_1} = \frac{-1}{1} = -1$

slope of $\perp$ bisector on $BC = \frac{-1}{m_2} = \frac{-1}{4/7} = \frac{-7}{4}$

Now eq. of $\perp$ bisector with slope -1 through $D(-3,2)$

$y - 2 = -1(x+3)$

$\Rightarrow y - 2 = -x - 3$

$\Rightarrow y + x - 2 + 3 = 0$

$x + y + 1 = 0 \longrightarrow (i)$

Now eq. of $\perp$ bisector with slope $-\frac{7}{4}$ through $E\left(-\frac{1}{2}, 3\right)$

$y - 3 = -\frac{7}{4}\left(x + \frac{1}{2}\right) \Rightarrow 4y - 12 = -7x - \frac{7}{2}$

$\Rightarrow 7x + 4y - 12 + \frac{7}{2} = 0$

$\Rightarrow 14x + 8y - 24 + 7 = 0$

$\Rightarrow 14x + 8y - 17 = 0 \longrightarrow (ii)$

By $8(i) - (ii) \Rightarrow$

$$\begin{array}{r} 8x + \cancel{8y} + 8 = 0 \\ \underline{-14x \mp \cancel{8y} \pm 17 = 0} \\ -6x + 25 = 0 \end{array}$$

$\Rightarrow -6x = -25 \Rightarrow x = \frac{25}{6}$ put in (i)

so (i) $\Rightarrow \frac{25}{6} + y + 1 = 0$

$\Rightarrow y + \frac{25+6}{6} = 0 \Rightarrow y + \frac{31}{6} = 0$

$\Rightarrow y = -\frac{31}{6}$

so circumcentre is $\left(\frac{25}{6}, -\frac{31}{6}\right)$

Now we check $(-1,3), \left(-\frac{34}{3}, \frac{58}{3}\right)$ and $\left(\frac{25}{6}, -\frac{31}{6}\right)$ are collinear. so

$$\begin{vmatrix} -1 & 3 & 1 \\ -\frac{34}{3} & \frac{58}{3} & 1 \\ \frac{25}{6} & -\frac{31}{6} & 1 \end{vmatrix}$$

$= -1\left(\frac{58}{3} + \frac{31}{6}\right) - 3\left(-\frac{34}{3} - \frac{25}{6}\right) + 1\left[\left(-\frac{34}{3}\right)\left(-\frac{31}{6}\right) - \left(\frac{25}{6}\right)\left(\frac{58}{3}\right)\right]$

$= -\frac{58}{3} - \frac{31}{6} + 34 + \frac{25}{2} + \frac{1054}{18} - \frac{1450}{18}$

$= \frac{-348 - 93 + 612 + 225 + 1054 - 1450}{18}$

$= \frac{1891 - 1891}{18} = \frac{0}{18} = 0$

Hence points are collinear.

Q8. Check whether the lines $4x - 3y - 8 = 0$, $3x - 4y - 6 = 0$; and $x - y - 2 = 0$ are concurrent? If so, find the point where they meet.

Solution:-

$$\begin{vmatrix} 4 & -3 & -8 \\ 3 & -4 & -6 \\ 1 & -1 & -2 \end{vmatrix}$$

$= 4(8-6) - (-3)(-6+6) - 8(-3+4)$

$= 4(2) + 3(0) - 8(1) = 8 - 8 = 0$

so given lines are concurrent.

Now we find point where lines meet. $4x - 3y - 8 = 0 \longrightarrow (i)$

$x - y - 2 = 0 \longrightarrow (ii)$

By (i) − 4(ii) → $\cancel{4x} - 3y - \cancel{8} = 0$

$\cancel{4x} - 4y \cancel{-8} = 0$

$y = 0$ put in (ii)

so (ii) → $x - 0 - 2 = 0$ → $x = 2$

so $(2, 0)$ is point of concurrency.

Q9. Find the coordinates of the vertices of the triangle formed by the lines $x - 2y - 6 = 0$; $3x - y + 3 = 0$; $2x + y - 4 = 0$. Also find measures of the angles of the triangle.

Solution:- $x - 2y - 6 = 0$ —— (i)

$3x - y + 3 = 0$ —— (ii), $2x + y - 4 = 0$ —— (iii)

Solving (i) and (ii) $\quad x - 2y - 6 = 0$, $3x - y + 3 = 0$

$$\frac{x}{-6-6} = \frac{y}{-18-3} = \frac{1}{-1+6} \qquad \begin{array}{c|cccc|c} 1 & -2 & -6 & 1 & -2 & -6 \\ 3 & -1 & 3 & 3 & -1 & 3 \end{array}$$

$\rightarrow \frac{x}{-12} = \frac{y}{-21} = \frac{1}{5}$

$\rightarrow \frac{x}{-12} = \frac{1}{5}$ and $\frac{y}{-21} = \frac{1}{5}$

$\rightarrow x = -\frac{12}{5}$ and $y = -\frac{21}{5}$

Solving (ii) and (iii) $\quad 3x - y + 3 = 0$, $2x + y - 4 = 0$

$$\frac{x}{4-3} = \frac{y}{6+12} = \frac{1}{3+2} \qquad \begin{array}{c|cccc|c} 3 & -1 & 3 & 3 & -1 & 3 \\ 2 & 1 & -4 & 2 & 1 & -4 \end{array}$$

$\rightarrow \frac{x}{1} = \frac{y}{18} = \frac{1}{5}$

$\rightarrow \frac{x}{1} = \frac{1}{5}$ and $\frac{y}{18} = \frac{1}{5}$

$\rightarrow x = \frac{1}{5}$ and $y = \frac{18}{5}$

Solving (i) and (iii) $\quad x - 2y - 6 = 0$, $2x + y - 4 = 0$

$$\frac{x}{8+6} = \frac{y}{-12+4} = \frac{1}{1+4} \qquad \begin{array}{c|cccc|c} 1 & -2 & -6 & 1 & -2 & -6 \\ 2 & 1 & -4 & 2 & 1 & -4 \end{array}$$

$\rightarrow \frac{x}{14} = \frac{y}{-8} = \frac{1}{5}$

$\rightarrow \frac{x}{14} = \frac{1}{5}$ and $\frac{y}{-8} = \frac{1}{5}$

$\rightarrow y = \frac{14}{5}$ and $y = -\frac{8}{5}$

So vertices of triangle are

$A\left(\frac{14}{5}, -\frac{8}{5}\right)$, $B\left(\frac{1}{5}, \frac{18}{5}\right)$, $C\left(-\frac{12}{5}, -\frac{21}{5}\right)$

Now

$m_1 = \text{slope of } AB = \dfrac{\frac{18}{5} - \left(-\frac{8}{5}\right)}{\frac{1}{5} - \left(\frac{14}{5}\right)} = \dfrac{\frac{18+8}{5}}{\frac{1-14}{5}}$

$\rightarrow m_1 = \frac{26}{-13} = -2$

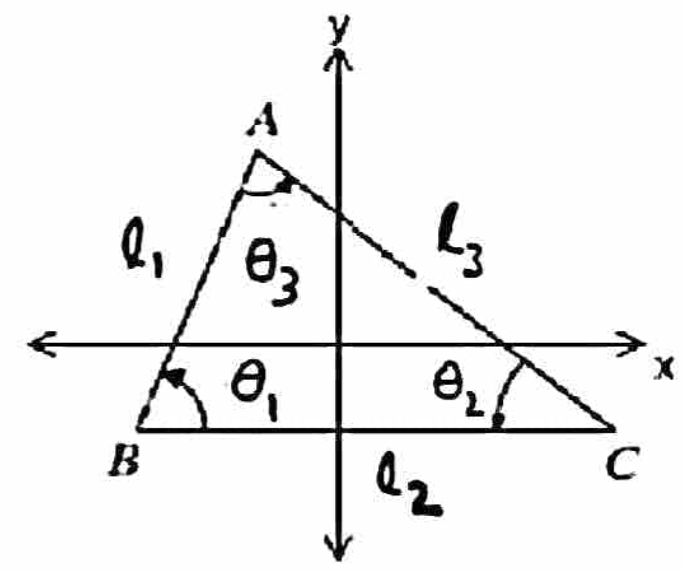
$m_2 = \text{slope of } BC = \dfrac{\frac{-21}{5} - \frac{18}{5}}{\frac{-12}{5} - \frac{1}{5}} = \dfrac{\frac{-39}{5}}{\frac{-13}{5}} = \frac{-39}{-13}$

$\rightarrow m_2 = 3$

$m_3 = \text{slope of } CA = \dfrac{\frac{-8}{5} + \frac{21}{5}}{\frac{14}{5} + \frac{12}{5}} = \dfrac{\frac{13}{5}}{\frac{26}{5}} = \frac{13}{26} = \frac{1}{2}$

$\rightarrow m_2 = \frac{1}{2}$

$\tan\theta_1 = \dfrac{m_1 - m_2}{1 + m_1 m_2}$



($\because \theta_1$ is the angle from l_2 to l_1)

$= \dfrac{-2 - (-3)}{1 + (-2)(3)} = \dfrac{-5}{1-6} = \frac{-5}{-5} = 1$

$\rightarrow \tan\theta_1 = 1 \rightarrow \theta_1 = \tan^{-1}(1) = 45°$

$\tan\theta_2 = \dfrac{m_2 - m_3}{1 + m_2 m_3}$ ($\because \theta_2$ is the angle from l_3 to l_2)

$\tan\theta_2 = \dfrac{3 - \frac{1}{2}}{1 + 3\left(\frac{1}{2}\right)} = \dfrac{\frac{6-1}{2}}{\frac{2+3}{2}} = \frac{5}{5} = 1$

$\rightarrow \tan\theta_2 = 1 \rightarrow \theta_2 = \tan^{-1}(1) = 45°$

$\tan\theta_3 = \dfrac{m_3 - m_1}{1 + m_3 m_1} = \dfrac{\frac{1}{2} - (-2)}{1 + \left(\frac{1}{2}\right)(-2)} = \dfrac{\frac{1}{2} + 2}{1 - 1} = \infty$

$\rightarrow \tan\theta_3 = \infty \rightarrow \theta_3 = \tan^{-1}(\infty) = 90°$

($\because \theta_3$ is the angle from l_1 to l_3)

Q10. Find the angle measured from the line l_1 to the line l_2 where

(a) l_1: Joining $(2,7)$ and $(7,10)$

l_2: Joining $(1,1)$ and $(-5,3)$

(b) l_1: Joining $(3,-1)$ and $(5,7)$

l_2: Joining $(2,4)$ and $(-8,2)$

(c) l_1: Joining $(1,-7)$ and $(6,-4)$
l_2: Joining $(-1,2)$ and $(-6,-1)$

(d) l_1: Joining $(-9,-1)$ and $(3,-5)$
l_2; Joining $(2,7)$ and $(-6,-7)$
Also find acute angle in each case.

Solution:- (a) l_1; $(2,7), (7,10)$
l_2; $(1,1), (-5,3)$

$m_1 = \text{slope of } l_1 = \frac{10-7}{7-2} = \frac{3}{5}$

$m_2 = \text{slope of } l_2 = \frac{3-1}{-5-1} = \frac{2}{-6} = -\frac{1}{3}$

$\because$ Angle θ from l_1 to l_2 then

$$\text{Tan}\theta = \frac{m_2-m_1}{1+m_2m_1} = \frac{-\frac{1}{3}-\frac{3}{5}}{1+(-\frac{1}{3})(\frac{3}{5})} = \frac{\frac{-5-9}{15}}{\frac{5-1}{5}}$$

$$\text{Tan}\theta = \frac{-14}{15} \times \frac{5}{4} = \frac{-7}{6}$$

$\Rightarrow \theta = \text{Tan}^{-1}(-7/6)$

$(\because \text{Tan}^{-1}(-x) = -\text{Tan}^{-1}(x))$

$\theta = \text{Tan}^{-1}(-\frac{7}{6}) = -\text{Tan}^{-1}\frac{7}{6} = -49°23'$

$= -49°23' + 180°$ $\boxed{\text{for inclination which is counterclock wise and } 0 < \theta < 180°}$

$\Rightarrow \theta = 130°36'$

Acute angle $= 180° - 130°36' = 49°23'$

(b) l_1; $(3,-1),(5,7)$, l_2; $(2,4),(-8,2)$

$m_1 = \text{slope of } l_1 = \frac{7-(-1)}{5-3} = \frac{8}{2} = 4$

$m_2 = \text{slope of } l_2 = \frac{2-4}{-8-2} = \frac{-2}{-10} = \frac{1}{5}$

$\because$ Angle θ from l_1 to l_2 then

$$\text{Tan}\theta = \frac{m_2-m_1}{1+m_2m_1} = \frac{\frac{1}{5}-4}{1+(\frac{1}{5})(4)} = \frac{\frac{1-20}{5}}{1+\frac{4}{5}}$$

$\text{Tan}\theta = \frac{-19}{9} \Rightarrow \theta = \text{Tan}^{-1}(-\frac{19}{5})$

$(\because \text{Tan}^{-1}(-x) = -\text{Tan}^{-1}(x))$

$\theta = \text{Tan}^{-1}(-\frac{19}{5}) = -\text{Tan}^{-1}(\frac{19}{5}) = -64.6°$

$\theta = -64.6° + 180°$ $\boxed{\text{for inclination which is counterclock wise and } 0 < \theta < 180°}$

$\theta = 115.4°$

Acute angle $= 180° - 115.4° = 64.6°$

(c) l_1; $(3,-7),(6,-4)$, l_2; $(-1,2),(-6,-1)$

$m_1 = \text{slope of } l_1 = \frac{-4-(-7)}{6-1} = \frac{-4+7}{5} = \frac{3}{5}$

$m_2 = \text{slope of } l_2 = \frac{-1-2}{-6-(-1)} = \frac{-3}{-6+1} = \frac{3}{5}$

$\because$ angle θ is from l_1 to l_2 then

$$\text{Tan}\theta = \frac{m_2-m_1}{1+m_2m_1} = \frac{\frac{3}{5}-\frac{3}{5}}{1+(\frac{3}{5})(\frac{3}{5})} = 0$$

$\Rightarrow \theta = \text{Tan}^{-1}(0) = 0°$

This is also acute angle.

(d) l_1; $(-9,-1), (3,-5)$, l_2; $(2,7), (-6,-7)$

$m_1 = \text{slope of } l_1 = \frac{-5-(-1)}{3-(-9)} = \frac{-5+1}{3+9}$

$m_1 = \frac{-4}{12} = -\frac{1}{3}$

$m_2 = \text{slope of } l_2 = \frac{-7-7}{-6-2} = \frac{-14}{-8} = \frac{7}{4}$

$\because$ Angle θ is from l_2 to l_1 then

$$\text{Tan}\theta = \frac{m_2-m_1}{1+m_2m_1} = \frac{\frac{7}{4}-(-\frac{1}{3})}{1+(\frac{7}{4})(-\frac{1}{3})} = \frac{\frac{7}{4}+\frac{1}{3}}{1-\frac{7}{12}}$$

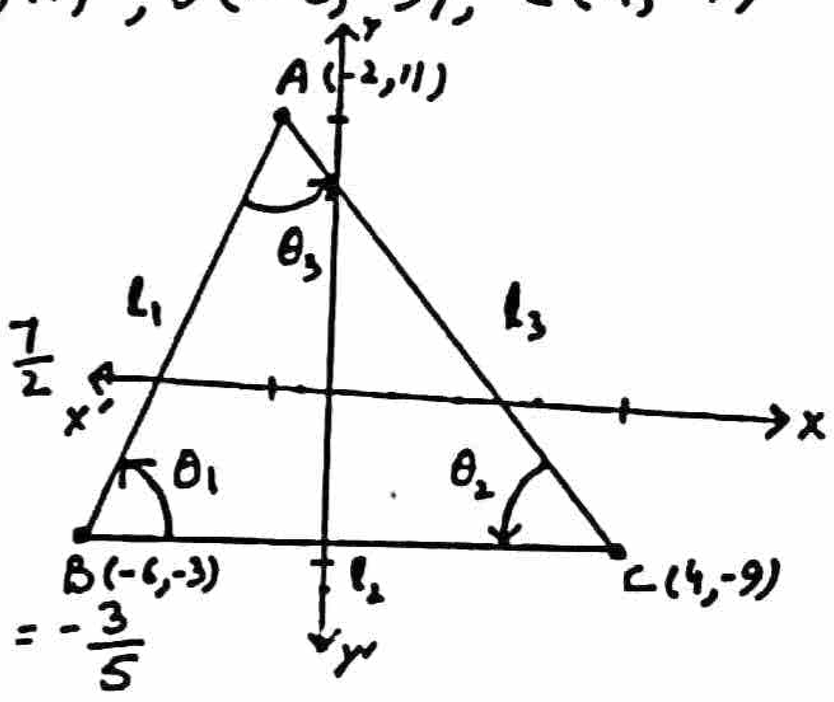
$$\text{Tan}\theta = \frac{\frac{21+4}{12}}{\frac{12-7}{12}} = \frac{25}{5} = 5$$

$\Rightarrow \theta = \text{Tan}^{-1}(5) = 78.69°$

This is also acute angle.

Q11. Find the interior angles of the triangle whose vertices are (a) $A(-2,11)$, $B(-6,-3)$, $C(4,-9)$

Solution:-



Slope of AB(l_1)
$m_1 = \frac{-3-11}{-6+2} = \frac{-14}{-4} = \frac{7}{2}$

slope of BC(l_2)
$m_2 = \frac{-9+3}{4+6} = \frac{-6}{10} = -\frac{3}{5}$

slope of CA(l_3) $= m_3 = \frac{11+9}{-2-4} = \frac{20}{-6} = \frac{-10}{3}$

$\text{Tan}\theta_1 = \frac{m_1-m_2}{1+m_1m_2}$ ($\because \theta_1$ is the angle from l_2 to l_1)

$$= \frac{\frac{7}{2}-(\frac{-3}{5})}{1+(\frac{7}{2})(\frac{-3}{5})} = \frac{\frac{7}{2}+\frac{3}{5}}{\frac{10-21}{10}} = \frac{\frac{35+6}{10}}{\frac{-11}{10}}$$

$\text{Tan}\theta_1 = \frac{41}{-11} \Rightarrow \theta_1 = \text{Tan}^{-1}(-\frac{41}{11})$

$(\because \text{Tan}^{-1}(-x) = \pi - \text{Tan}^{-1}x)$

$\theta_1 = \text{Tan}^{-1}(-\frac{41}{11}) = -\text{Tan}^{-1}\frac{41}{11} = -74.98°$

$\theta_1 = -74.98° + 180°$ $\boxed{\text{for inclination which is counterclock wise and } 0 < \theta < 180°}$

$\theta_1 = 105.02°$

$\text{Tan}\theta_2 = \frac{m_2-m_3}{1+m_2m_3}$ ($\because \theta_2$ is the angle from l_3 to l_2)

$$\rightarrow \tan\theta_2 = \frac{-\frac{3}{5} - \left(-\frac{10}{3}\right)}{1+\left(-\frac{3}{5}\right)\left(-\frac{10}{3}\right)} = \frac{-\frac{3}{5} + \frac{10}{3}}{1 + \frac{30}{15}}$$

$$\tan\theta_2 = \frac{\frac{-9+50}{15}}{\frac{15+30}{15}} = \frac{41}{45}$$

$$\rightarrow \theta_2 = \tan^{-1}\left(\frac{41}{45}\right) = 42^\circ 20'$$

$$\tan\theta_3 = \frac{m_3 - m_1}{1+m_3 m_1} = \frac{-\frac{10}{3} - \frac{7}{2}}{1+\left(-\frac{10}{3}\right)\left(\frac{7}{2}\right)}$$

$$\tan\theta_3 = \frac{\frac{-20-21}{6}}{\frac{6-70}{6}} = \frac{-41}{-64} = \frac{41}{64}$$

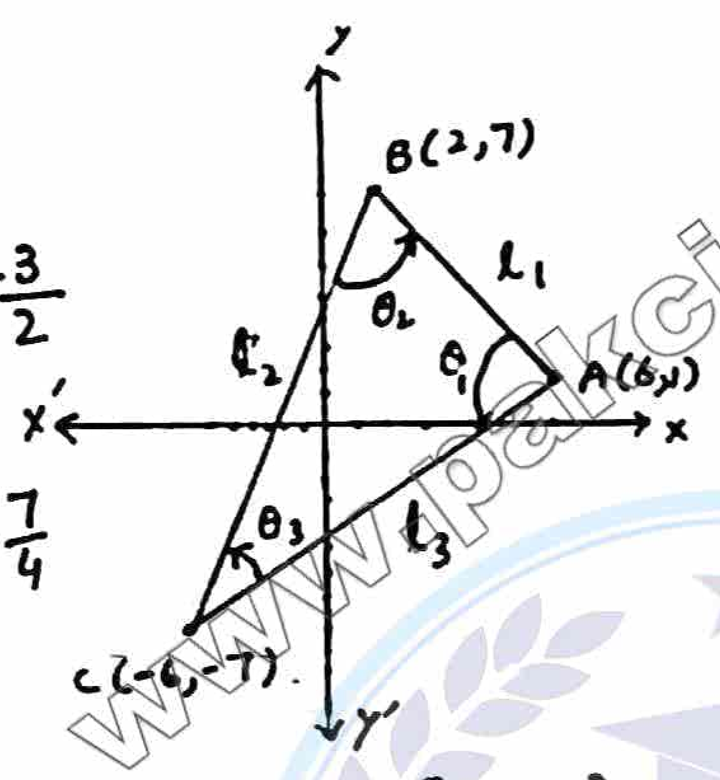
$$\rightarrow \theta_3 = \tan^{-1}\left(\frac{41}{64}\right) = 32^\circ 38'$$

So interior angles are $\theta_1 = 105.02^\circ$

$\theta_2 = 42^\circ 20'$, $\theta_3 = 32^\circ 38'$

(b) A(6,1), B(2,7), C(−6,−7)

Solution:-



slope of AB(l_1)

$$= m_1 = \frac{7-1}{2-6} = \frac{6}{-4} = \frac{-3}{2}$$

slope of BC(l_2)

$$= m_2 = \frac{-7-7}{-6-2} = \frac{-14}{-8} = \frac{7}{4}$$

slope of CA(l_3)

$$= m_3 = \frac{1-(-7)}{6-(-6)} = \frac{1+7}{6+6} = \frac{8}{12} = \frac{2}{3}$$

$$\tan\theta_1 = \frac{m_3 - m_1}{1+m_3 m_1} \quad (\because \theta_1 \text{ is angle from } l_1 \text{ to } l_3)$$

$$\tan\theta_1 = \frac{\frac{2}{3} - \left(-\frac{3}{2}\right)}{1+\left(\frac{\cancel{2}}{\cancel{3}}\right)\left(\frac{-\cancel{3}}{\cancel{2}}\right)} = \frac{\frac{2}{3} + \frac{3}{2}}{1-1} = \infty$$

$$\rightarrow \theta_1 = \tan^{-1}(\infty) = 90^\circ$$

$$\tan\theta_2 = \frac{m_1 - m_2}{1+m_1 m_2} \quad (\because \theta_2 \text{ is angle from } l_2 \text{ to } l_1)$$

$$= \frac{-\frac{3}{2} - \frac{7}{4}}{1+\left(-\frac{3}{2}\right)\left(\frac{7}{4}\right)} = \frac{\frac{-6-7}{4}}{\frac{8-21}{8}} = \frac{-13}{4} \times \frac{8}{-13}$$

$$\tan\theta_2 = \frac{8}{4} = 2$$

$$\rightarrow \theta_2 = \tan^{-1}(2) = 63.4^\circ$$

$$\tan\theta_3 = \frac{m_2 - m_3}{1+m_2 m_3} \quad (\because \theta_3 \text{ is angle from } l_3 \text{ to } l_2)$$

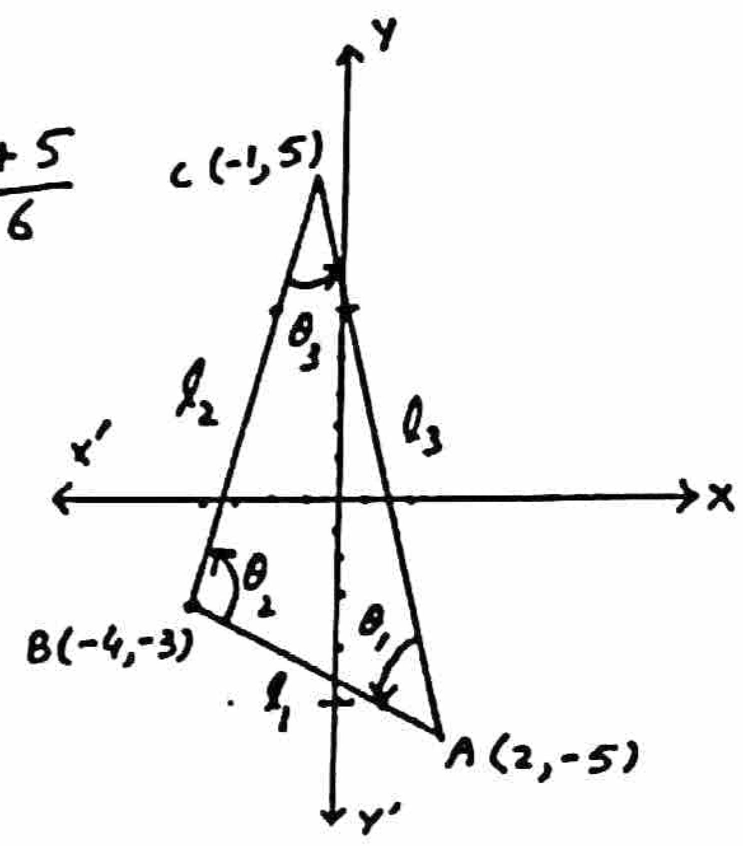
$$= \frac{\frac{7}{4} - \frac{2}{3}}{1+\left(\frac{7}{4}\right)\left(\frac{2}{3}\right)} = \frac{\frac{21-8}{12}}{\frac{12+14}{12}} = \frac{13}{26} = \frac{1}{2}$$

$$\rightarrow \tan\theta_3 = \frac{1}{2} \rightarrow \theta_3 = \tan^{-1}\left(\frac{1}{2}\right)$$

$$\rightarrow \theta_3 = 26.6^\circ$$

(c) A(2,−5), B(−4,−3), C(−1,5)

Solution:-



slope of AB(l_1)

$$= m_1 = \frac{-3-(-5)}{-4-2} = \frac{-3+5}{-6}$$

$$= \frac{2}{-6} = -\frac{1}{3}$$

slope of BC(l_2)

$$= m_2 = \frac{5-(-3)}{-1-(-4)}$$

$$= \frac{5+3}{-1+4} = \frac{8}{3}$$

$$\text{slope of CA}(l_3) = m_3 = \frac{5-(-5)}{-1-2} = \frac{5+5}{-3} = \frac{-10}{3}$$

$$\tan\theta_1 = \frac{m_1 - m_3}{1+m_1 m_3} \quad (\because \theta_1 \text{ is angle from } l_3 \text{ to } l_1)$$

$$= \frac{-\frac{1}{3} - \left(-\frac{10}{3}\right)}{1+\left(-\frac{1}{3}\right)\left(-\frac{10}{3}\right)} = \frac{-\frac{1}{3} + \frac{10}{3}}{1 + \frac{10}{9}} = \frac{9}{3} \times \frac{9}{19}$$

$$\tan\theta_1 = \frac{27}{19} \rightarrow \theta_1 = \tan^{-1}\left(\frac{27}{19}\right) = 54.87^\circ$$

$$\tan\theta_2 = \frac{m_2 - m_1}{1+m_2 m_1} \quad (\because \theta_2 \text{ is the angle from } l_1 \text{ to } l_2)$$

$$= \frac{\frac{8}{3} - \left(-\frac{1}{3}\right)}{1+\left(\frac{8}{3}\right)\left(-\frac{1}{3}\right)} = \frac{\frac{8+1}{3}}{1 - \frac{8}{9}} = \frac{\frac{9}{3}}{\frac{9-8}{9}}$$

$$\tan\theta_2 = \frac{9}{3} \times \frac{9}{1} = 27$$

$$\rightarrow \theta_2 = \tan^{-1}(27) = 87.9^\circ$$

$$\tan\theta_3 = \frac{m_3 - m_2}{1+m_3 m_2} \quad (\because \theta_3 \text{ is angle from } l_2 \text{ to } l_3)$$

$$= \frac{-\frac{10}{3} - \frac{8}{3}}{1+\left(-\frac{10}{3}\right)\left(\frac{8}{3}\right)} = \frac{\frac{-18}{3}}{1 - \frac{80}{9}} = \frac{\frac{-18}{3}}{\frac{9-80}{9}}$$

$$\tan\theta_3 = \frac{-18}{3} \times \frac{9}{-71} = \frac{54}{71}$$

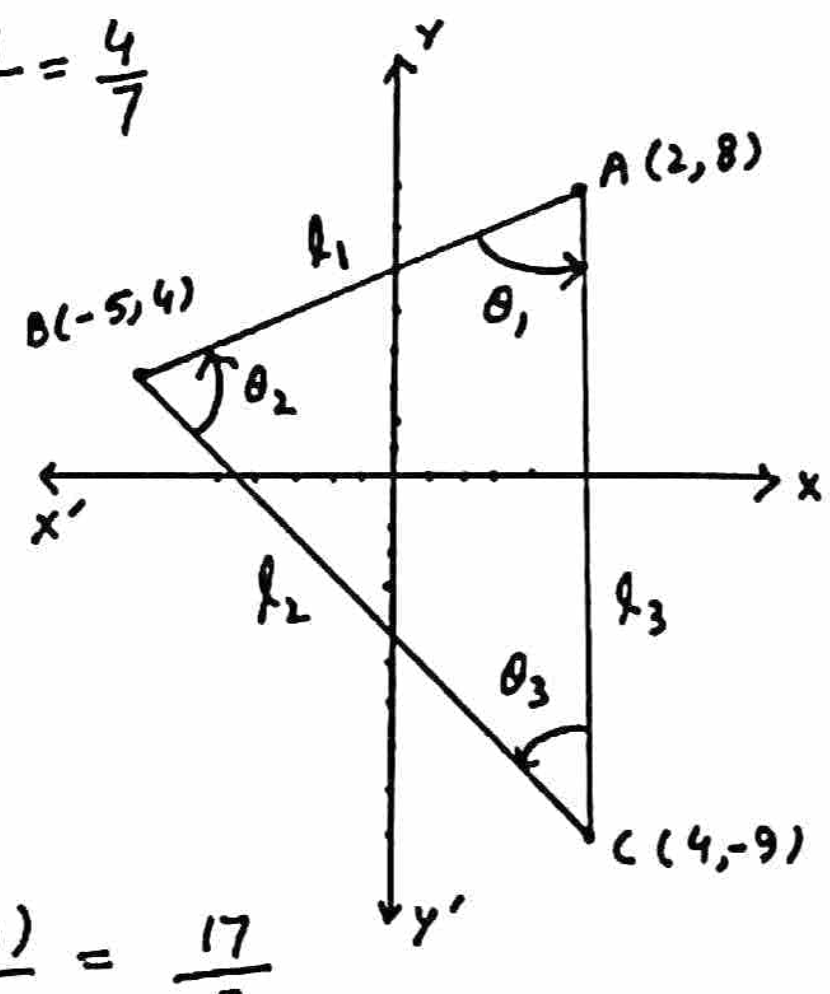
$$\rightarrow \theta_3 = \tan^{-1}\left(\frac{54}{71}\right) = 37.2^\circ$$

(d) $A(2,8)$, $B(-5,4)$, $C(4,-9)$

Solution:-

slope of $AB(l_1)$

$$= m_1 = \frac{4-8}{-5-2} = \frac{-4}{-7} = \frac{4}{7}$$



Slope of $BC(l_2)$

$$= m_2 = \frac{-9-4}{4-(-5)}$$

$$m_2 = \frac{-13}{9}$$

Slope of $CA(l_3)$

$$= m_3 = \frac{8-(-9)}{2-4} = \frac{17}{-2}$$

$$\tan\theta_1 = \frac{m_3 - m_1}{1+m_3 m_1} \quad (\because \theta_1 \text{ is angle from } l_1 \text{ to } l_3)$$

$$= \frac{\frac{-17}{2} - \frac{4}{7}}{1+\left(-\frac{17}{2}\right)\left(\frac{4}{7}\right)} = \frac{\frac{-119-8}{14}}{1 - \frac{68}{14}}$$

$$\tan\theta_1 = \frac{\frac{-127}{14}}{\frac{14-68}{14}} = \frac{-127}{-54} = \frac{127}{54}$$

$$\Rightarrow \theta_1 = \tan^{-1}\left(\frac{127}{54}\right) = 67°$$

$$\tan\theta_2 = \frac{m_1 - m_2}{1+m_1 m_2} \quad (\because \theta_2 \text{ is angle from } l_2 \text{ to } l_1)$$

$$= \frac{\frac{4}{7} - \left(-\frac{13}{9}\right)}{1+\left(\frac{4}{7}\right)\left(-\frac{13}{9}\right)} = \frac{\frac{4}{7} + \frac{13}{9}}{1 - \frac{52}{63}}$$

$$\tan\theta_2 = \frac{\frac{36+91}{63}}{\frac{63-52}{63}} = \frac{127}{11}$$

$$\Rightarrow \theta_2 = \tan^{-1}\left(\frac{127}{11}\right) = 85°$$

$$\tan\theta_3 = \frac{m_2 - m_3}{1+m_2 m_3} \quad (\because \theta_3 \text{ is angle from } l_3 \text{ to } l_2)$$

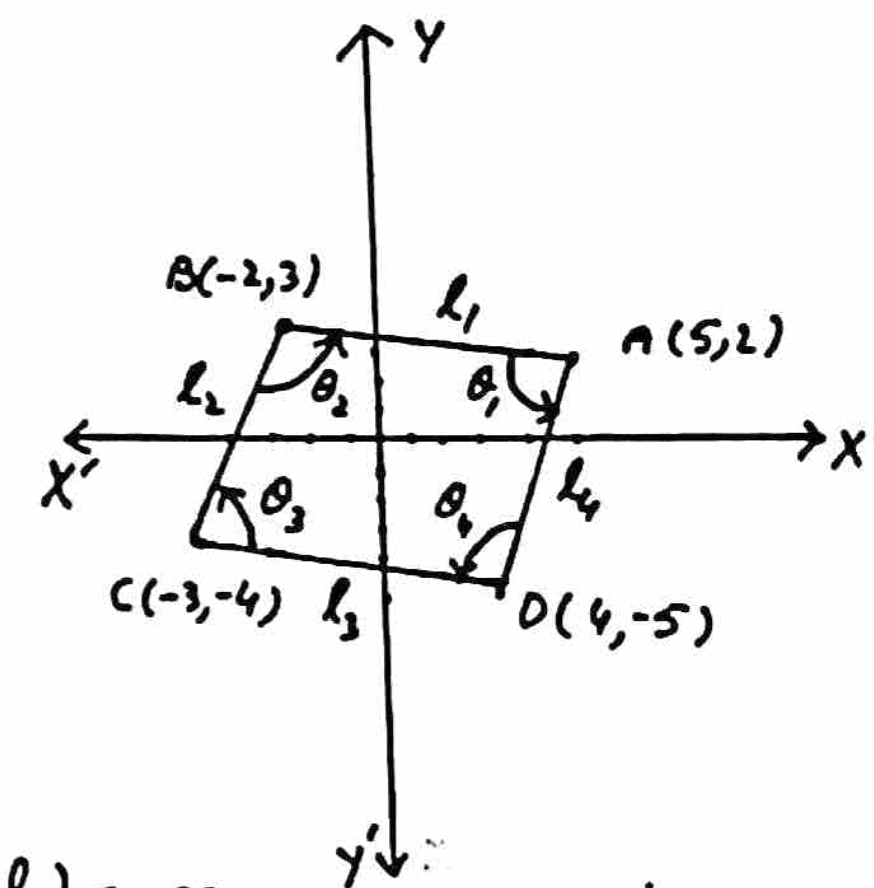
$$= \frac{\frac{-13}{9} - \left(\frac{-17}{2}\right)}{1+\left(\frac{-13}{9}\right)\left(\frac{-17}{2}\right)} = \frac{\frac{-13}{9} + \frac{17}{2}}{1 + \frac{221}{18}}$$

$$\tan\theta_3 = \frac{\frac{-26+153}{18}}{\frac{18+221}{18}} = \frac{127}{239}$$

$$\Rightarrow \theta_3 = \tan^{-1}\left(\frac{127}{239}\right) = 28°$$

Q12. Find the interior angles of quadrilateral whose vertices are $A(5,2)$, $B(-2,3)$, $C(-3,-4)$ and $D(4,-5)$

Solution:-



slope of $AB(l_1) = m_1 = \dfrac{3-2}{-2-5} = -\dfrac{1}{7}$

slope of $BC(l_2) = m_2 = \dfrac{-4-3}{-3+2} = \dfrac{-7}{-1} = 7$

slope of $CD(l_3) = m_3 = \dfrac{-5+4}{4+3} = -\dfrac{1}{7}$

slope of $DA(l_4) = m_4 = \dfrac{2+5}{5-4} = \dfrac{7}{1} = 7$

$$\tan\theta_1 = \frac{m_4 - m_1}{1+m_4 m_1} \quad (\because \theta_1 \text{ is angle from } l_1 \text{ to } l_4)$$

$$= \frac{7 - \left(-\frac{1}{7}\right)}{1+7\left(-\frac{1}{7}\right)} = \frac{7+\frac{1}{7}}{1-1} = \infty$$

$$\Rightarrow \theta_1 = \tan^{-1}(\infty) = 90°$$

$$\tan\theta_2 = \frac{m_1 - m_2}{1+m_1 m_2} = \frac{-\frac{1}{7} - 7}{1+\left(-\frac{1}{7}\right)(7)} = \frac{-\frac{1}{7} - 7}{1-1} = \infty$$

$(\because \theta_2$ is angle from l_2 to $l_1)$

$$\tan\theta_3 = \frac{m_2 - m_3}{1+m_2 m_3} \quad (\because \theta_3 \text{ is angle from } l_3 \text{ to } l_2)$$

$$= \frac{7 - \left(-\frac{1}{7}\right)}{1+(7)\left(-\frac{1}{7}\right)} = \frac{7+\frac{1}{7}}{1-1} = \infty$$

$$\Rightarrow \theta_3 = \tan^{-1}(\infty) = 90°$$

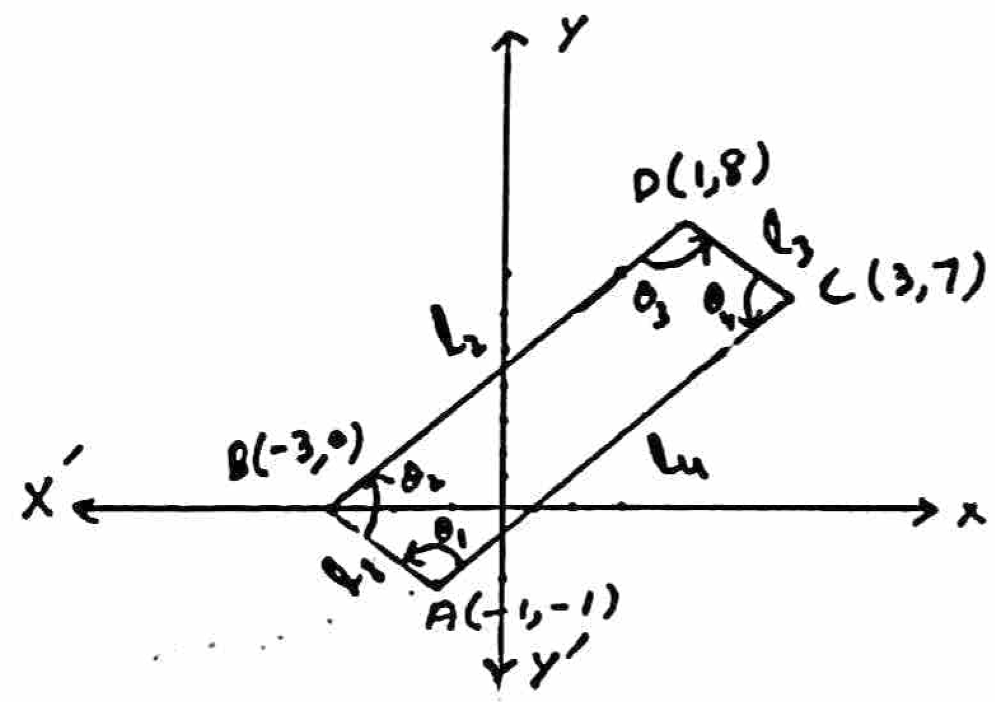
$$\tan\theta_4 = \frac{m_3 - m_4}{1+m_3 m_4} = \frac{-\frac{1}{7} - 7}{1+\left(-\frac{1}{7}\right)(7)} = \frac{-\frac{1}{7} - 7}{1-1}$$

$$\tan\theta_4 = \infty \quad (\because \theta_4 \text{ is angle from } l_4 \text{ to } l_3)$$

$$\Rightarrow \theta_4 = \tan^{-1}(\infty) = 90°$$

Q13. Show that the points $A(-1,-1)$, $B(-3,0)$, $C(3,7)$ and $D(1,8)$ are the vertices of Rectangle. Find its interior angles.

Solution:–



slope of $AB(l_1) = m_1 = \dfrac{0-(-1)}{-3-(-1)} = \dfrac{1}{-3+1} = -\dfrac{1}{2}$

slope of $BD(l_2) = m_2 = \dfrac{8-0}{1+3} = \dfrac{8}{4} = 2$

slope of $DC(l_3) = m_3 = \dfrac{7-8}{3-1} = -\dfrac{1}{2}$

slope of $CA(l_4) = m_4 = \dfrac{-1-7}{-1-3} = \dfrac{-8}{-4} = 2$

$\because m_1 = m_3$ and $m_2 = m_4$

i.e, l_1 is $\parallel$ to l_3 and l_1 is $\parallel$ l_4

Hence A, B, C and D are vertices of rectangle. Now we find interior angles

$\tan\theta_1 = \dfrac{m_4 - m_1}{1+m_4 m_1}$ ($\because \theta_1$ is angle from l_4 to l_1)

$= \dfrac{2-(-\frac{1}{2})}{1+\cancel{2}(-\frac{1}{\cancel{2}})} = \dfrac{2+\frac{1}{2}}{1-1} = \infty$

$\rightarrow \theta_1 = \tan^{-1}(\infty) = 90°$

$\tan\theta_2 = \dfrac{m_2 - m_1}{1+m_2 m_1}$ ($\because \theta_2$ is angle from l_1 to l_2)

$= \dfrac{2-(-\frac{1}{2})}{1+\cancel{2}(-\frac{1}{\cancel{2}})} = \dfrac{2+\frac{1}{2}}{1-1} = \infty$

$\rightarrow \theta_2 = \tan^{-1}(\infty) = 90°$

$\tan\theta_3 = \dfrac{m_3 - m_2}{1+m_3 m_2} = \dfrac{-\frac{1}{2}-2}{1+(-\frac{1}{\cancel{2}})(\cancel{2})} = \dfrac{-\frac{1}{2}-2}{1-1}$

$\tan\theta_3 = \infty$ ($\because \theta_3$ is angle from l_2 to l_3)

$\rightarrow \theta_3 = \tan^{-1}(\infty)$

$\theta_3 = 90°$

$\tan\theta_4 = \dfrac{m_4 - m_3}{1+m_4 m_3}$ ($\because \theta_4$ is angle from l_3 to l_4)

$= \dfrac{2-(-\frac{1}{2})}{1+2(-\frac{1}{2})} = \dfrac{2+\frac{1}{2}}{1-1} = \infty$

$\rightarrow \tan\theta_4 = \infty \rightarrow \theta_4 = \tan^{-1}(\infty) = 90°$

Q14. Find the area of the region bounded by the triangle whose sides are $7x-y-10=0$; $10x+y-41=0$; $3x+2y+3=0$

Solution:– $7x-y-10=0 \longrightarrow$ (i)

$10x+y-41=0 \longrightarrow$ (ii) , $3x+2y+3=0 \longrightarrow$ (iii)

Solving (i) and (ii) $\quad 7x-y-10=0$
$\quad 10x+y-41=0$

$\dfrac{x}{41+10} = \dfrac{y}{-100+287} = \dfrac{1}{7+10}$

$\rightarrow \dfrac{x}{51} = \dfrac{y}{187} = \dfrac{1}{17}$

$\rightarrow \dfrac{x}{51} = \dfrac{1}{17}$ and $\dfrac{y}{187} = \dfrac{1}{17}$

$\rightarrow x = \dfrac{51}{17}$, $y = \dfrac{187}{17}$

$\rightarrow x = 3$, $y = 11$

Solving (ii) and (iii) $\quad 10x+y-41=0$
$\quad 3x+2y+3=0$

$\dfrac{x}{3+82} = \dfrac{y}{-30-123} = \dfrac{1}{20-3}$

$\rightarrow \dfrac{x}{85} = \dfrac{y}{-153} = \dfrac{1}{17}$

$\rightarrow \dfrac{x}{85} = \dfrac{1}{17}$ and $\dfrac{y}{-153} = \dfrac{1}{17}$

$\rightarrow x = \dfrac{85}{17}$, $y = -\dfrac{153}{17}$

$\rightarrow x = 5$, $y = -9$

Solving (iii) and (i) $\quad 3x+2y+3=0$
$\quad 7x-y-10=0$

$\dfrac{x}{-20+3} = \dfrac{y}{21+30} = \dfrac{1}{-3-14}$

$\rightarrow \dfrac{x}{-17} = \dfrac{y}{51} = \dfrac{1}{-17}$

$\rightarrow \dfrac{x}{-17} = \dfrac{1}{-17}$ and $\dfrac{y}{51} = \dfrac{1}{-17}$

$\rightarrow x = \dfrac{-17}{-17}$, $y = \dfrac{51}{-17}$

$\rightarrow x = 1$, $y = -3$

So vertices are $A(3,11)$, $B(5,-9)$
$C(1,-3)$

Now area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$= \frac{1}{2}\begin{vmatrix} 3 & 11 & 1 \\ 5 & -9 & 1 \\ 1 & -3 & 1 \end{vmatrix} = \frac{1}{2}\left[3(-9+3) - 11(5-1) + 1(-15+9)\right]$

$= \frac{1}{2}\left[3(-6) - 11(4) + (-6)\right]$

$= \frac{1}{2}(-18-44-6) = \frac{1}{2}(-68) = -34$

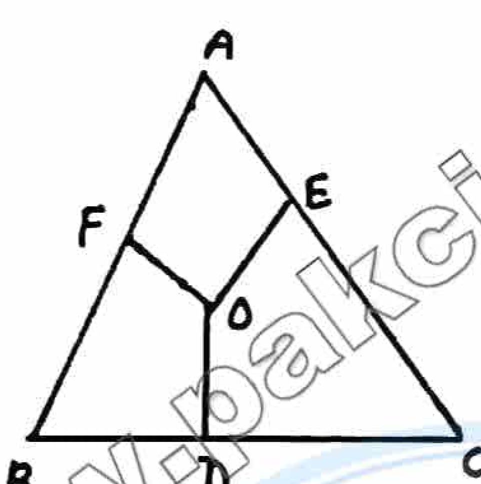
Area of $\Delta ABC = 34$ square units

($\because$ Area is always +ive)

Q15. The vertices of a triangle are $A(-2,3)$, $B(-4,1)$ and $C(3,5)$. Find the centre of the circumcircle of the triangle.

> **Circum centre** (Centre of circumcircle)
> "The point of intersection of right bisectors is called circum-centre

Solution:-



Slope of $BC = \frac{5-1}{3+4}$

$= \frac{4}{7}$

slope of right bisector $OD = -\frac{7}{4}$ ($\because OD \perp BC$)

$\because D$ is mid point of BC so

coordinates of D are $= \left(\frac{-4+3}{2}, \frac{1+5}{2}\right) = \left(-\frac{1}{2}, 3\right)$

Equation of OD is;

$y - 3 = -\frac{7}{4}\left(x - \left(-\frac{1}{2}\right)\right)$ $\quad \because y - y_1 = m(x - x_1)$

$\rightarrow 4y - 12 = -7x - \frac{7}{2}$

$\rightarrow 8y - 24 = -14x - 7$

$\rightarrow 14x + 8y - 24 + 7 = 0$

$\rightarrow 14x + 8y - 17 = 0 \longrightarrow (i)$

Slope of $AC = \frac{5-3}{3+2} = \frac{2}{5}$

slope of right bisector $OE = -\frac{5}{2}$ ($\because OE \perp AC$)

$\because E$ is midpoint of AC so

coordinates of E are $= \left(\frac{-2+3}{2}, \frac{3+5}{2}\right) = \left(\frac{1}{2}, 4\right)$

Equation of AC is

$y - 4 = -\frac{5}{2}\left(x - \frac{1}{2}\right)$ $\quad \because y - y_1 = m(x - x_1)$

$\rightarrow 2y - 8 = -5x + \frac{5}{2}$

$\rightarrow 4y - 16 = -10x + 5$

$\rightarrow 10x + 4y - 16 - 5 = 0$

$\rightarrow 10x + 4y - 21 = 0 \longrightarrow (ii)$ Now

By $2(ii) - (i) \rightarrow$

$$\begin{array}{r} 20x + 8y - 42 = 0 \\ \underset{-}{14x} \underset{-}{+} 8y \underset{+}{-} 17 = 0 \\ \hline 6x - 25 = 0 \end{array}$$

$\rightarrow 6x = 25 \rightarrow x = \frac{25}{6}$ put in (i)

(i) $\rightarrow 14\left(\frac{25}{6}\right) + 8y - 17 = 0$

$\rightarrow 7\left(\frac{25}{3}\right) + 8y - 17 = 0$

$\rightarrow \frac{175}{3} + 8y - 17 = 0 \rightarrow 8y = 17 - \frac{175}{3}$

$\rightarrow 8y = \frac{51 - 175}{3} \rightarrow 8y = -\frac{124}{3}$

$\rightarrow y = -\frac{\cancel{124}^{31}}{3 \times \cancel{8}_2} = -\frac{31}{6} \rightarrow y = -\frac{31}{6}$

Hence $\left(\frac{25}{6}, -\frac{31}{6}\right)$ is circumcentre.

Q16. Express the given system of equations in matrix form. Find in each case whether the lines are concurrent.

(a) $x + 3y - 2 = 0$; $2x - y + 4 = 0$; $x - 11y + 14 = 0$

Solution:-

In matrix form

$$\begin{bmatrix} x + 3y - 2 \\ 2x - y + 4 \\ x - 11y + 14 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$AX = B$

$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$ for concurrent

$|A| = \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{vmatrix}$

$= 1(-14+44) - 3(28-4) + (-2)(-22+1)$

$= 1(30) - 3(24) - 2(-21)$

$= 30 - 72 + 42$

$|A| = 72 - 72 = 0$

So given lines are concurrent.

(b) $2x+3y+4=0$; $x-2y-3=0$; $3x+y-8=0$

Solution:- In matrix form

$$\begin{bmatrix} 2x+3y+4 \\ x-2y-3 \\ 3x+y-8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Or $\begin{bmatrix} 2 & 3 & 4 \\ 1 & -2 & -3 \\ 3 & 1 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$AX = B$

$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -2 & -3 \\ 3 & 1 & -8 \end{bmatrix}$ for concurrent

$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 1 & -2 & -3 \\ 3 & 1 & -8 \end{vmatrix}$

$= 2(16+3) - 3(-8+9) + 4(1+6)$

$= 2(19) - 3(1) + 4(7)$

$|A| = 38 - 3 + 28 = 63 \neq 0$

so given lines are not concurrent.

(c) $3x-4y-2=0$; $x+2y-4=0$; $3x-2y+5=0$

Solution:- In matrix form

$$\begin{bmatrix} 3x-4y-2 \\ x+2y-4 \\ 3x-2y+5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Or $\begin{bmatrix} 3 & -4 & -2 \\ 1 & 2 & -4 \\ 3 & -2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 3 & -4 & -2 \\ 1 & 2 & -4 \\ 3 & -2 & 5 \end{bmatrix}$ for concurrent

$|A| = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & -4 \\ 3 & -2 & 5 \end{vmatrix}$

$= 3(10-8) - (-4)(5+12) + (-2)(-2-6)$

$= 3(2) + 4(17) - 2(-8)$

$|A| = 6 + 68 + 16 = 90 \neq 0$

So given lines are not concurrent.

Q17. Find a system of linear equations corresponding to the given matrix form. check whether the lines represented by the system are concurrent.

(a) $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution:- $x+0-1=0$

$2y+0+1=0$

$0-y+2=0$

or $x-1=0$; $2y+1=0$; $-y+2=0$

This is req. system of equations.

For concurrency

$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{vmatrix}$

$= 1(0+1) - 0(-1)(-2-0) = 1-0+2 = 3 \neq 0$

So given lines are not concurrent.

(b) $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution:- $x+y+2=0$

$2x+4y-3=0$

$3x+6y-5=0$

For concurrency

$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{vmatrix}$

$= 1(-20+18) - 1(-10+19) + 2(12-12)$

$= -2+1+0 = -1 \neq 0$

so given lines are not concurrent.

Homogeneous Equation of the Second degree in two variables

suppose two straight lines

$a_1x + b_1y + c_1 = 0 \longrightarrow (i)$ and $a_2x + b_2y + c_2 = 0 \longrightarrow (ii)$

So by (i) and (ii) $(a_1x+b_1y+c_1)(a_2x+b_2y+c_2) = 0 \longrightarrow (iii)$

It is second degree equation in x and y.

Eq (iii) is called Joint equation of the pair of lines (i) and (ii).

General Homogeneous Equation:-

$ax^2 + 2hxy + by^2 = 0$ (where a, h and b non-zero) is called general Homogeneous quadratic equation.

Note:- Let $y = m_1 x$ and $y = m_2 x$ be two lines passing through origin. Their joint equation is:

$$(y - m_1 x)(y - m_2 x) = 0$$

or $$y^2 - m_2 xy - m_1 xy + m_1 m_2 x^2 = 0$$

$$\Rightarrow y^2 - (m_1 + m_2)xy + m_1 m_2 x^2 = 0$$

This is special type of second degree homogeneous equation

Homogeneous Equation

Let $f(x, y) = 0 \longrightarrow (I)$ be any eq. in variables x and y is called homogeneous equation of degree n (a positive integer) if

$$f(kx, ky) = k^n f(x, y) \quad \text{for } k \in R$$

For example, $y^2 - (m_1 + m_2)xy + m_1 m_2 y^2 = 0$

Replacing x and y by kx and ky

$$\Rightarrow (ky)^2 - (m_1 + m_2)(kx)(ky) + m_1 m_2 (ky)^2 = 0$$

$$\Rightarrow k^2 (y^2 - (m_1 + m_2)xy + m_1 m_2 y^2) = 0$$

$\Rightarrow k^2 f(x, y) = 0$ Thus it is Homo. eq. of degree 2.

* A general second degree homogeneous equation can be written as:

$$ax^2 + 2hxy + by^2 = 0$$

where a, h and b are simultaneously not zero.

Theorem:- Every homogeneous equation of second degree $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines through the origin. The lines are

(i) real and distinct, if $h^2 > ab$

(ii) real and coincident, if $h^2 = ab$

(iii) Imaginary, if $h^2 < ab$

Proof:- $\because ax^2 + 2hxy + by^2 = 0$

$$\Rightarrow by^2 + 2hxy + ax^2 = 0$$

(quadratic eq. in y)

using quadratic formula,

$$y = \frac{-2hx \pm \sqrt{(2hx)^2 - 4(b)(ax^2)}}{2b}$$

$$\Rightarrow y = \frac{-2hx \pm \sqrt{4h^2x^2 - 4abx^2}}{2b}$$

$$\Rightarrow y = \frac{-2hx \pm \sqrt{4x^2(h^2 - ab)}}{2b}$$

$$\Rightarrow y = \frac{-2hx \pm 2x\sqrt{h^2 - ab}}{2b}$$

$$\Rightarrow y = 2\left(\frac{-hx \pm x\sqrt{h^2 - ab}}{2b}\right)$$

$$\Rightarrow y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x$$

Clearly, this represents a pair of lines passing through origin. The lines are

(i) real and distinct if $h^2 > ab$

(ii) real and coincident if $h^2 = ab$

(iii) imaginary if $h^2 < ab$

Example 1. Find an equation of each of the lines represented by

$$20x^2 + 17xy - 24y^2 = 0$$

Solution:- $20x^2 + 17xy - 24y^2 = 0$

$$\Rightarrow 20x^2 - 15xy + 32xy - 24y^2 = 0$$

$$\Rightarrow 5x(4x - 3y) + 8y(4x - 3y) = 0$$

$$\Rightarrow (4x - 3y)(5x + 8y) = 0$$

This is req. pair of lines.

To find measure of the angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$

We know that every homogeneous equation represents a pair of lines through origin is

$$y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x$$

$$\Rightarrow y = \left(\frac{-h + \sqrt{h^2 - ab}}{b}\right)x, \quad y = \left(\frac{-h - \sqrt{h^2 - ab}}{b}\right)x$$

Slope of $l_1 = m_1 = \dfrac{-h + \sqrt{h^2 - ab}}{b}$

slope of $l_2 = m_2 = \dfrac{-h - \sqrt{h^2 - ab}}{b}$

$$\because m_1 + m_2 = \frac{-h + \sqrt{h^2 - ab}}{b} + \frac{-h - \sqrt{h^2 - ab}}{b}$$

$$= \frac{-h + \sqrt{h^2 - ab} - h - \sqrt{h^2 - ab}}{b}$$

$$m_1 + m_2 = \frac{-2h}{b}$$

and $m_1m_2 = \left(\frac{-h+\sqrt{h^2-ab}}{b}\right)\left(\frac{-h-\sqrt{h^2-ab}}{b}\right)$

$$= \frac{(-h)^2-\left(\sqrt{h^2-ab}\right)^2}{b^2} = \frac{h^2-(h^2-ab)}{b^2}$$

$$m_1m_2 = \frac{\cancel{h^2}-\cancel{h^2}+ab}{b^2} \rightarrow m_1m_2 = \frac{a}{b}$$

If θ is measured from l_1 to l_2 so

$$\tan\theta = \frac{m_2-m_1}{1+m_2m_1}$$

$$\tan\theta = \frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{1+m_2m_1}$$

$\left(\because (a+b)^2-(a-b)^2 = 4ab \rightarrow \sqrt{(a+b)^2-4ab} = a-b\right)$

$$\rightarrow \tan\theta = \frac{\sqrt{\left(-\frac{2h}{b}\right)^2-\frac{4a}{b}}}{1+\frac{a}{b}}$$

$$= \frac{\sqrt{\frac{4h^2}{b^2}-\frac{4a}{b}}}{\frac{b+a}{b}}$$

$$= \frac{\sqrt{\frac{4h^2-4ab}{b^2}}}{\frac{b+a}{b}} = \frac{\frac{\sqrt{4(h^2-ab)}}{b}}{\frac{a+b}{b}}$$

$$\boxed{\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}}$$

Note:- (i) The two lines are parallel if $\theta = 0$, so $\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$

$\rightarrow \tan 0 = \frac{2\sqrt{h^2-ab}}{a+b}$

$\rightarrow 0 = \frac{2\sqrt{h^2-ab}}{a+b}$

$\rightarrow 2\sqrt{h^2-ab} = 0 \rightarrow h^2-ab = 0$

or $h^2 = ab$

Thus lines will be parallel if $h^2 = ab$

(ii) The two lines are perpendicular if $\theta = 90°$, so $\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$

$\rightarrow \tan 90° = \frac{2\sqrt{h^2-ab}}{a+b}$

$\rightarrow \frac{1}{0} = \frac{2\sqrt{h^2-ab}}{a+b}$

$\rightarrow a+b = (0)(2\sqrt{h^2-ab})$

$\rightarrow a+b = 0$

Thus lines will be perpendicular if $a+b = 0$

Example 2. Find the measure of the angle between the lines represented by $x^2-xy-6y^2=0$

Solution:- $x^2-xy-6y^2=0$

compare with $ax^2+2hxy+by^2=0$

$a=1$, $2h=-1 \rightarrow h=-\frac{1}{2}$, $b=-6$

$$\because \tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$$

$$= \frac{2\sqrt{(-\frac{1}{2})^2-(1)(-6)}}{1-6} = \frac{2}{-5}\sqrt{\frac{1}{4}+6}$$

$$\tan\theta = \frac{2}{-5}\sqrt{\frac{25}{4}} = \frac{2}{-5}\times\frac{5}{2} = -1$$

$\rightarrow \theta = \tan^{-1}(-1) = 135°$

For acute angle $= 180°-135° = 45°$

so $\theta = 45°$

Example 3. Find a joint equation of the straight lines through the origin perpendicular to the lines represented by

$$x^2+xy-6y^2=0$$

Solution:- $x^2+xy-6y^2=0$

Let $y = m_1x$ and $y = m_2x$ be two lines passing through the origin

Now slopes of lines $\perp$ar to given lines are

$-\frac{1}{m_1}$ and $-\frac{1}{m_2}$, then their equations are

$y = -\frac{1}{m_1}x$

& $y = -\frac{1}{m_2}x$

$\Rightarrow x+m_1y = 0$ & $x+m_2y = 0$

Their joint equation:

$(x+m_1y)(x+m_2y) = 0$

$\rightarrow x^2+(m_1+m_2)xy+m_1m_2y^2 = 0$ ——(1)

$\boxed{m_1+m_2 = -\frac{2h}{b}}$ $= -\frac{1}{6}$ | $a=1$, $2h=1$, $b=-6$

$\rightarrow m_1+m_2 =$

& $\boxed{m_1m_2 = \frac{a}{b}}$ $= \frac{1}{-6} \rightarrow m_1m_2 = -\frac{1}{6}$

So (1)

$x^2+\frac{1}{6}xy+\frac{1}{-6}y^2=0$

$\rightarrow 6x^2+xy-y^2=0$

or $\rightarrow y^2-xy-6x^2=0$

Exercise 4.5

Find the lines represented by each of the following and also find measure of angle between them. (problems 1-6)

Q1. $10x^2 - 23xy - 5y^2 = 0$

Solution:- $\because 10x^2 - 23xy - 5y^2 = 0 \longrightarrow (I)$

$\Rightarrow 10x^2 + 2xy - 25xy - 5y^2 = 0$

$\Rightarrow 2x(5x+y) - 5y(5x+y) = 0$

$\Rightarrow (2x-5y)(5x+y) = 0$

Req. pair of lines.

Now compare (I) with $ax^2 + 2hxy + by^2 = 0$

$\Rightarrow a = 10,\ 2h = -23 \Rightarrow h = -\frac{23}{2},\ b = -5$

$\because \tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$

$= \frac{2\sqrt{\left(-\frac{23}{2}\right)^2 - (10)(-5)}}{10-5} = \frac{2\sqrt{\frac{529}{4} + 50}}{5}$

$= \frac{2}{5}\sqrt{\frac{529+200}{4}} = \frac{2}{5}\sqrt{\frac{729}{4}}$

$\tan\theta = \frac{2}{5}\left(\frac{27}{2}\right) = \frac{27}{5} \Rightarrow \theta = \tan^{-1}\left(\frac{27}{5}\right)$

$\Rightarrow \theta = 79.51^\circ$

Q2. $3x^2 + 7xy + 2y^2 = 0$

Solution:- $\because 3x^2 + 7xy + 2y^2 = 0 \longrightarrow (I)$

$\Rightarrow 3x^2 + 6xy + xy + 2y^2 = 0$

$\Rightarrow 3x(x+2y) + y(x+2y) = 0$

$\Rightarrow (x+2y)(3x+y) = 0$

$x+2y=0$ and $3x+y=0$ req. pair of lines

Now compare (I) with $ax^2 + 2hxy + by^2 = 0$

$a = 3,\ 2h = 7 \Rightarrow h = \frac{7}{2},\ b = 2$

$\because \tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$

$= \frac{2\sqrt{\left(\frac{7}{2}\right)^2 - 3(2)}}{3+2}$

$= \frac{2}{5}\sqrt{\frac{49}{4} - 6}$

$= \frac{2}{5}\sqrt{\frac{49-24}{4}} = \frac{2}{5}\sqrt{\frac{25}{4}} = \frac{2}{5}\left(\frac{5}{2}\right)$

$\tan\theta = 1 \Rightarrow \theta = \tan^{-1}(1) = 45^\circ$

Q3. $9x^2 + 24xy + 16y^2 = 0$

Solution:- $\because 9x^2 + 24xy + 16y^2 = 0$

$\Rightarrow 9x^2 + 12xy + 12xy + 16y^2 = 0$

$\Rightarrow 3x(3x+4y) + 4y(3x+4y) = 0$

$\Rightarrow (3x+4y)(3x+4y) = 0$

$3x+4y=0$ and $3x+4y=0$ req. pair of lines

$\because$ The lines are real and coincident. so for coincident lines,

$\theta = 0^\circ$

Q4. $2x^2 + 3xy - 5y^2 = 0 \longrightarrow (I)$

Solution:- $\because 2x^2 + 3xy - 5y^2 = 0$

$\Rightarrow 2x^2 - 2xy + 5xy - 5y^2 = 0$

$\Rightarrow 2x(x-y) + 5y(x-y) = 0$

$\Rightarrow (x-y)(2x+5y) = 0$

$x - y = 0$ and $2x+5y=0$ req. pair of lines

Now compare (I) with $ax^2 + 2hxy + by^2 = 0$

$a = 2,\ 2h = 3 \Rightarrow h = \frac{3}{2},\ b = -5$

$\because \tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} = \frac{2\sqrt{\left(\frac{3}{2}\right)^2 - (2)(5)}}{2-5}$

$= \frac{2}{-3}\sqrt{\frac{9}{4} + 10} = \frac{2}{-3}\sqrt{\frac{9+40}{4}}$

$\tan\theta = \frac{2}{-3}\sqrt{\frac{49}{4}} = -\frac{2}{3}\left(\frac{7}{2}\right) = -\frac{7}{3}$

$\Rightarrow \theta = \tan^{-1}\left(-\frac{7}{3}\right) = -66.80^\circ$

$\left(\because \tan^{-1}(-x) = -\tan^{-1}(x)\right)$

for inclination which is counterclock wise and $0 < \theta < 180^\circ$

$\Rightarrow \theta = -66.80^\circ + 180^\circ = 113.2^\circ$

Q5. $6x^2 - 19xy + 15y^2 = 0$

Solution:- $\because 6x^2 - 19xy + 15y^2 = 0 \longrightarrow (I)$

$\Rightarrow 6x^2 - 9xy - 10xy + 15y^2 = 0$

$\Rightarrow 3x(2x-3y) - 5y(2x-3y) = 0$

$(2x-3y)(3x-5y) = 0$

$\Rightarrow 2x-3y=0$ and $3x-5y=0$ req. pair of lines

Compare (I) with $ax^2+2hxy+by^2=0$

$a=6$, $2h=-19 \Rightarrow h=-\frac{19}{2}$, $b=15$

$\because \tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$

$= \frac{2\sqrt{(-\frac{19}{2})^2-(6)(15)}}{6+15} = \frac{2}{21}\sqrt{\frac{361}{4}-90}$

$\tan\theta = \frac{2}{21}\sqrt{\frac{361-360}{4}} = \frac{2}{21}\sqrt{\frac{1}{4}} = \frac{2}{21}\left(\frac{1}{2}\right)$

$\Rightarrow \tan\theta = \frac{1}{21} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{21}\right)$

$\Rightarrow \theta = 2.73^\circ$

Q6. $x^2+2xy\sec\alpha+y^2=0 \longrightarrow$ (I)

Solution:- $\because x^2+2xy\sec\alpha+y^2=0$

$\Rightarrow y^2+(2x\sec\alpha)y+x^2=0$

using quadratic formula,

$y = \frac{-2x\sec\alpha \pm \sqrt{(2x\sec\alpha)^2-4(1)(x^2)}}{2}$

$= \frac{-2x\sec\alpha \pm \sqrt{4x^2\sec^2\alpha-4x^2}}{2}$

$y = \frac{-2x\sec\alpha \pm 2x\sqrt{\sec^2\alpha-1}}{2}$

$\Rightarrow y = \frac{-2x\sec\alpha \pm 2x\sqrt{\tan^2\alpha}}{2}$

$y = \frac{-2x\sec\alpha \pm 2x\tan\alpha}{2}$

$= \frac{-2x(\sec\alpha \pm \tan\alpha)}{2}$

$y = -x(\sec\alpha \pm \tan\alpha)$

$\Rightarrow y = -x\left(\frac{1}{\cos x} \pm \frac{\sin\alpha}{\cos\alpha}\right)$

$y = -x\left(\frac{1 \pm \sin\alpha}{\cos\alpha}\right)$

$\Rightarrow y = -x\left(\frac{1+\sin\alpha}{\cos\alpha}\right)$ and $y = -x\left(\frac{1-\sin\alpha}{\cos\alpha}\right)$

$y\cos\alpha = -x(1+\sin\alpha)$

$\Rightarrow x(1+\sin\alpha)+y\cos\alpha=0$ and

$y\cos\alpha = -x(1-\sin\alpha)$

$\Rightarrow x(1-\sin\alpha)-y\cos\alpha=0$

Required pair of lines.

Compare (I) with $ax^2+2hxy+by^2=0$

$\Rightarrow a=1,\ b=1,\ h=\sec\alpha$

$\because \tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$

$= \frac{2\sqrt{\sec^2\alpha-(1)(1)}}{2} =$

$\tan\theta = \sqrt{\sec^2\alpha-1} = \sqrt{\tan^2\alpha} = \tan\alpha$

$\Rightarrow \tan\theta = \tan\alpha \Rightarrow \boxed{\theta=\alpha}$

Q7. Find a joint equation of the lines through the origin and perpendicular to the lines:

$x^2-2xy\tan\alpha-y^2=0$

Solution:-

Let $y=m_1x$ and $y=m_2x$ be two lines passing through the origin

Now slopes of lines $\perp$ar to given lines are

$-\frac{1}{m_1}$ and $-\frac{1}{m_2}$, then their equations are

$$y = -\frac{1}{m_1}x$$

$$\& \quad y = -\frac{1}{m_2}x$$

$$\Rightarrow x+m_1y = 0 \quad \& \quad x+m_2y = 0$$

Their joint equation:

$$(x+m_1y)(x+m_2y) = 0$$

$$\Rightarrow x^2+(m_1+m_2)xy+m_1m_2y^2 = 0 \text{ ——(1)}$$

$$\boxed{m_1+m_2 = -\frac{2h}{b}} = -\frac{-2\tan\alpha}{-1}$$

$a=1$, $2h=-2\tan\alpha \Rightarrow h=-\tan\alpha$, $b=-1$

$$\Rightarrow m_1+m_2 = -2\tan\alpha$$

$$\& \quad \boxed{m_1m_2 = \frac{a}{b}} = \frac{1}{-1} \Rightarrow m_1m_2 = -1$$

So (1)

$$\Rightarrow x^2+(-2\tan\alpha)xy+(-1)y^2 = 0$$

$$\Rightarrow x^2-2xy\tan\alpha-y^2 = 0$$

Q8. Find a joint equation of the lines through the origin and perpendicular to the lines $ax^2+2hxy+by^2=0$

Solution:-

Let $y=m_1x$ and $y=m_2x$ be two lines passing through the origin

Now slopes of lines $\perp$ar to given lines are

$-\frac{1}{m_1}$ and $-\frac{1}{m_2}$, then their equations are

$$y=-\frac{1}{m_1}x$$

$$\& \quad y=-\frac{1}{m_2}x$$

$$\Rightarrow x+m_1y=0 \quad \& \quad x+m_2y=0$$

Their joint equation:

$$(x+m_1y)(x+m_2y)=0$$

$$\Rightarrow x^2+(m_1+m_2)xy+m_1m_2y^2=0 \text{ ——— (1)}$$

$$\boxed{m_1+m_2=-\frac{2h}{b}} \qquad \begin{aligned} a&=a\\ 2h&=2h\\ b&=b \end{aligned}$$

$$\& \quad \boxed{m_1m_2=\frac{a}{b}}$$

So (1)

$$\Rightarrow x^2+\left(-\frac{2h}{b}\right)xy+\left(\frac{a}{b}\right)y^2=0$$

$$\Rightarrow bx^2-2hxy+ay^2=0$$

Q9. Find the area of the origin bounded by $10x^2-xy-21y^2=0$ and $x+y+1=0$

Solution:- $\because 10x^2-xy-21y^2=0$

$\rightarrow 10x^2-15xy+14xy-21y^2=0$

$\rightarrow 5x(2x-3y)+7y(2x-3y)=0$

$\rightarrow (2x-3y)(5x+7y)=0$

$\rightarrow 2x-3y=0 \longrightarrow (i)$ $\quad 5x+7y=0 \longrightarrow (ii)$

and $x+y+1=0$ (given) $\longrightarrow (iii)$

By $3(iii)+(i) \rightarrow$
$$\begin{aligned} 3x+3y+3&=0\\ 2x-3y&=0\\ \hline 5x+3&=0 \end{aligned}$$

$\rightarrow x=-\frac{3}{5}$ put in (iii)

So (iii) $\rightarrow -\frac{3}{5}+y+1=0$

$\rightarrow y=\frac{3}{5}-1 \rightarrow y=\frac{3-5}{5}$

$\rightarrow y=-\frac{2}{5}$

Point of intersection of (i) and (iii) is $\left(-\frac{3}{5}, -\frac{2}{5}\right)$

Now by $7(iii)-(ii)$

$$\begin{aligned} \rightarrow 7x+7y+7&=0\\ 5x+7y&=0\\ \hline 2x+7&=0 \end{aligned}$$

$\rightarrow x=-\frac{7}{2}$ put in (ii)

So (ii) $\rightarrow 5\left(-\frac{7}{2}\right)+7y=0$

$\rightarrow -\frac{35}{2}+7y=0$

$\rightarrow 7y=\frac{35}{2} \rightarrow y=\frac{5}{2}$

So point of intersection of (ii) and (iii) is $\left(-\frac{7}{2}, \frac{5}{2}\right)$

$\because$ point of intersection of (i) and (ii) is $(0,0)$. Now we find area

$$\text{Area}=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1 \end{vmatrix}$$

$$=\frac{1}{2}\begin{vmatrix} 0 & 0 & 1\\ -3/5 & -2/5 & 1\\ -\frac{7}{2} & \frac{5}{2} & 1 \end{vmatrix}$$

$$=\frac{1}{2}\left[0-0+1\left(\left(-\frac{3}{5}\right)\left(\frac{5}{2}\right)-\left(-\frac{2}{5}\right)\left(-\frac{7}{2}\right)\right)\right]$$

$$=\frac{1}{2}\left(-\frac{3}{2}-\frac{7}{5}\right)=\frac{1}{2}\left(\frac{-15-14}{10}\right)$$

$$\text{Area}=\frac{-29}{20}$$

$\rightarrow$ Area $=\frac{29}{20}$ sq. units (Always +ive)