Exercise 4.1

Q #01 Calculate the first, second and third order derivatives of

$$y = \cos^2 x.$$

$$y' = 2\cos x\,(\cos x)' = 2\cos x\,(-\sin x)$$

$$y' = -2\sin x\cos x = -\sin 2x$$

$$y'' = -2\cos 2x$$

$$y''' = -2(-\sin 2x)\,2 = 4\sin 2x.$$

Q# 02 Find the 2nd order derivative of

$$f(x) = \frac{\cos x}{1+\sin x}$$

$$f'(x) = \frac{(1+\sin x)(-\sin x) - \cos x\,(\cos x)}{(1+\sin x)^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$f'(x) = \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2} = \frac{-\sin x - 1}{(1+\sin x)^2}$$

$$= -\frac{\cancel{(1+\sin x)}}{(1+\sin x)^{\cancel{2}}} = -(1+\sin x)^{-1}$$

$$f''(x) = -\left[-(1+\sin x)^{-2}\cos x\right] = \frac{\cos x}{(1+\sin x)^2}$$

Q #03 Find the fourth order derivatives of

(i) $h(t) = 3t^7 - 6t^4 + 8t^3 - 12t + 18$

$$h'(t) = 21t^6 - 24t^3 + 24t^2 - 12$$

$$h''(t) = 126t^5 - 72t^2 + 48t$$

$$h'''(t) = 630t^4 - 144t + 48$$

$$h^{(4)}(t) = 2520t^3 - 144.$$

(ii) $f(x) = \sqrt[3]{x} - \dfrac{1}{8x^2} - \sqrt{x}$

$$f(x) = x^{1/3} - \frac{1}{8}x^{-2} - x^{1/2}$$

$$f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} - \frac{1}{8}\left(-2x^{-3}\right) - \frac{1}{2}x^{\frac{1}{2}-1}$$

$$= \frac{1}{3}x^{-\frac{2}{3}} + \frac{1}{4}x^{-3} - \frac{1}{2}x^{-\frac{1}{2}}$$

$$f''(x) = \frac{1}{3}\left(-\frac{2}{3}x^{-\frac{2}{3}-1}\right) + \frac{1}{4}\left(-3x^{-4}\right) - \frac{1}{2}\left(-\frac{1}{2}x^{-\frac{1}{2}-1}\right)$$

$$= \frac{-2}{9}x^{-5/3} - \frac{3}{4}x^{-4} + \frac{1}{4}x^{-3/2}$$

$$f'''(x) = -\frac{2}{9}\left(-\frac{5}{3}x^{-5/3-1}\right) - \frac{3}{\cancel{4}}\left(-\cancel{4}x^{-5}\right) + \frac{1}{4}\left(-\frac{3}{2}x^{-\frac{3}{2}-1}\right)$$

$$= \frac{10}{27}x^{-8/3} + 3x^{-5} - \frac{3}{8}x^{-\frac{5}{2}}$$

$$f^{(4)}(x) = \frac{10}{27}\left(\frac{-8}{3}x^{-\frac{8}{3}-1}\right) + 3\left(-5x^{-6}\right) - \frac{3}{8}\left(\frac{-5}{2}x^{-\frac{5}{2}-1}\right)$$

$$= \frac{-80}{81}x^{-\frac{11}{3}} - 15x^{-6} + \frac{15}{16}x^{-7/2}$$

$$= \frac{-80}{81}\cdot\frac{1}{x^{\frac{11}{3}}} - \frac{15}{x^6} + \frac{15}{16}\cdot\frac{1}{x^{\frac{7}{2}}}$$

$$= \frac{-80}{81\sqrt[3]{x^{11}}} - \frac{15}{x^6} + \frac{15}{16\sqrt{x^7}}$$

Q # 04 Determine the fourth order derivative of

(i) $r(t) = 3t^2 + 8\sqrt{t}$

$$r'(t) = 3(2t) + 8\left(\frac{1}{2}t^{\frac{1}{2}-1}\right) = 6t + 4t^{-\frac{1}{2}}$$

$$r''(t) = 6 + 4\left(-\frac{1}{2}t^{-\frac{1}{2}-1}\right) = 6 - 2t^{-3/2}$$

$$r'''(t) = -\cancel{2}\left(\frac{-3}{\cancel{2}}t^{-\frac{3}{2}-1}\right) = 3t^{-\frac{5}{2}}$$

$$r^{(4)}(t) = 3\left(-\frac{5}{2}t^{-\frac{5}{2}-1}\right) = -\frac{15}{2}t^{-\frac{7}{2}}$$

$$= \frac{-15}{2t^{\frac{7}{2}}} = \frac{-15}{2\sqrt{t^7}}$$

(ii) $y = \cos x$

$$y' = -\sin x$$

$$y'' = -\cos x$$

$$y''' = -(-\sin x) = \sin x$$

$$y^{(4)} = \cos x$$

(iii) $f(y) = \sin 3y + e^{-2y} + \ln(7y).$

$$f'(y) = 3\cos 3y - 2e^{-2y} + \frac{\cancel{7}}{\cancel{7}y}$$

$$= 3\cos 3y - 2e^{-2y} + y^{-1}$$

$$f''(y) = 3(-\sin 3y)(3) - 2(-2e^{-2y}) + (-y^{-2})$$

$$= -9\sin 3y + 4e^{-2y} - y^{-2}$$

$$f'''(y) = -9(\cos 3y)(3) + 4(-2e^{-2y}) - (-2y^{-3})$$

$$= -27\cos 3y - 8e^{-2y} + 2y^{-3}$$

$$f^{(4)}(y) = -27(-\sin 3y)(3) - 8(-2e^{-2y}) + 2(-3y^{-4})$$

$$= 81\sin 3y + 16e^{-2y} - 6y^{-4}$$

$\boxed{Q \# 05}$ If $x^2+y^2=10$, find y''.

$$x^2+y^2=10$$

$$(x^2+y^2)' = 10'$$

$$2x+2yy' = 0$$

$$2yy' = -2x$$

$$y' = \frac{-\cancel{2}x}{\cancel{2}y}$$

$$y' = -\frac{x}{y}$$

$$y'' = -\left[\frac{yx' - xy'}{y^2}\right]$$

$$= -\frac{1}{y^2}\left(y - x\left(\frac{-x}{y}\right)\right) = \frac{-1}{y^2}\left(y+\frac{x^2}{y}\right)$$

$$= \frac{-1}{y^2}\left(\frac{y^2+x^2}{y}\right) = \frac{-(x^2+y^2)}{y^3}$$

Q # 06 Find $\frac{d^2y}{dx^2}$ if

(i) $2y^2 + 6x^2 = 76$

$$\frac{d}{dx}\left(2y^2 + 6x^2\right) = \frac{d}{dx}(76)$$

$$4y\frac{dy}{dx} + 12x = 0$$

$$\frac{dy}{dx} = -\frac{\cancel{12}^{3}x}{\cancel{4}y} = -3\frac{x}{y}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-3\frac{x}{y}\right)$$

$$\frac{d^2y}{dx^2} = -3\left(\frac{yx' - xy'}{y^2}\right) = -\frac{3}{y^2}\left(y - x\left(-\frac{3x}{y}\right)\right)$$

$$= -\frac{3}{y^2}\left(y - \frac{3x^2}{y}\right) = -\frac{3}{y^2}\left(\frac{y^2 - 3x^2}{y}\right)$$

$$= \frac{-3\left(y^2 - 3x^2\right)}{y^3} = \frac{3\left(3x^2 - y^2\right)}{y^3}.$$

(ii) $x^3 + y^3 = 1$

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(1)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\cancel{3}x^2}{\cancel{3}y^2} = -\frac{x^2}{y^2}$$

$$\frac{d^2y}{dx^2} = -\left(\frac{y^2(2x) - x^2(2yy')}{(y^2)^2}\right)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{y^4}\left(2xy^2 - 2x^2y\left(-\frac{x^2}{y^2}\right)\right)$$

$$= -\frac{1}{y^4}\left(2xy^2 + \frac{2x^4y}{y^2}\right)$$

$$= -\frac{1}{y^4}\left(\frac{2xy^4 + 2x^4y}{y^2}\right)$$

$$= -\frac{2xy\left(y^3 + x^3\right)}{y^6}$$

$$= \frac{-2x\left(x^3 + y^3\right)}{y^5}$$

$x = f(t)$, $y = g(t)$

Q # 07 Find $\dfrac{d^2y}{dx^2}$ if

(i) $x = -5t^3 - 7$, $y = 3t^2 + 16$

$$\frac{dx}{dt} = -15t^2 \quad , \quad \frac{dy}{dt} = 6t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\boxed{\frac{dy'}{dx} = \frac{dy'}{dt} \cdot \frac{dt}{dx}}$$

$$y' = \frac{dy}{dx} = \overset{2}{\cancel{6t}} \cdot \left(\frac{1}{-\underset{15}{\cancel{15}}t^{\cancel{2}}} \right) = -\frac{2}{5} t^{-1}$$

$$\frac{dy'}{dt} = -\frac{2}{5} \left(-t^{-2} \right) = \frac{2}{5t^2}$$

$$\frac{dy'}{dx} = \frac{dy'}{dt} \cdot \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{2}{5t^2} \cdot \left(\frac{1}{-15t^2} \right) = \frac{-2}{75t^4} .$$

(ii) $x = \cos\theta$, $y = \sin\theta$

$$x^2 = \cos^2\theta \quad , \quad y^2 = \sin^2\theta$$

$$x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$$

$$x^2 + y^2 = 1$$

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(1)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y} = -\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = -\left(\frac{yx' - xy'}{y^2} \right) = \frac{-1}{y^2}\left(y - x\left(\frac{-x}{y}\right)\right)$$

$$= \frac{-1}{y^2}\left(y + \frac{x^2}{y} \right) = \frac{-1}{y^2}\left(\frac{y^2 + x^2}{y} \right)$$

$$= -\frac{\left(x^2 + y^2\right)}{y^3}$$

$$= \frac{-1}{\sin^3\theta}$$

Q # 08 The derivative of the function $r(t)$ is given by

$$r'(t) = 6t + 4t^{-1/2} + e^t \quad ;$$

find $r''(t)$, $r'''(t)$ and $r^{(4)}(t)$.

$$r'(t) = 6t + 4t^{-1/2} + e^t$$

$$r''(t) = 6 + 4\left(-\frac{1}{2}t^{-\frac{1}{2}-1}\right) + e^t$$

$$r''(t) = 6 - 2t^{-\frac{3}{2}} + e^t$$

$$r'''(t) = 0 - 2\left(-\frac{3}{2}t^{-\frac{3}{2}-1}\right) + e^t$$

$$r'''(t) = 3t^{-\frac{5}{2}} + e^t$$

$$r^{(4)}(t) = 3\left(-\frac{5}{2}t^{-\frac{5}{2}-1}\right) + e^t$$

$$= -\frac{15}{2}t^{-\frac{7}{2}} + e^t.$$

Q # 09 If $x^2+y^2=25$, then find $\frac{d^2y}{dx^2}$ at $(4,3)$.

$$x^2+y^2=25$$

$$(x^2+y^2)' = 25'$$

$$2x+2yy' = 0$$

$$y' = -\frac{\cancel{2}x}{\cancel{2}y} = -\frac{x}{y}$$

$$y'' = -\left(\frac{yx'-xy'}{y^2}\right) = -\frac{1}{y^2}\left(y - x\left(\frac{-x}{y}\right)\right)$$

$$= -\frac{1}{y^2}\left(y+\frac{x^2}{y}\right) = -\frac{1}{y^2}\left(\frac{y^2+x^2}{y}\right)$$

$$\frac{d^2y}{dx^2} = y'' = -\frac{25}{y^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(4,3)} = \frac{-25}{3^3} = \frac{-25}{27} \quad \text{Ans.}$$

Maclaurin's Theorem:

If $f(x)$ has derivatives of all orders at $x = 0$, then $f(x)$ has series representation

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots$$

Taylor's Theorem:

If $f(x)$ has derivatives of all orders at $x = a$, then $f(x)$ has series representation

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots$$

Exercise 4.2

Q1 Obtain the first three terms of the Maclaurin's series for

(i) $\cos x$

Let $f(x) = \cos x$, $f(0) = \cos 0 = 1$

$f'(x) = -\sin x$, $f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x$, $f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x$, $f'''(0) = \sin 0 = 0$

$f^{(4)}(x) = \cos x$, $f^{(4)}(0) = \cos 0 = 1$

Formula

$$f(x) = f(0) + f'(0)\,x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + \dots$$

$$\cos x = 1 + 0\,x + (-1)\frac{x^2}{2!} + 0\cdot\frac{x^3}{3!} + 1\,\frac{x^4}{4!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

(ii) e^x

Let $f(x) = e^x$, $f(0) = e^0 = 1$

$f'(x) = e^x$, $f'(0) = e^0 = 1$

$f''(x) = e^x$, $f''(0) = e^0 = 1$

Formula

$$f(x) = f(0) + f'(0)\,x + f''(0)\frac{x^2}{2!} + \dots$$

$$e^x = 1 + 1\,x + 1\,\frac{x^2}{2!} + \dots$$

$$= 1 + x + \frac{x^2}{2!} + \dots$$

(iii) $\ln(1+x)$

Let $f(x) = \ln(1+x)$, $\quad f(0) = \ln(1+0) = 0$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$, $\quad f'(0) = \frac{1}{1+0} = 1$

$f''(x) = -1\,(1+x)^{-2}$, $\quad f''(0) = -1(1+0)^{-2} = -1$

$f'''(x) = 2\,(1+x)^{-3}$, $\quad f'''(0) = 2\,(1+0)^{-3} = 2$

Formula

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = 0 + 1x + (-1)\frac{x^2}{2!} + 2\frac{x^3}{3!} + \cdots$$

$$= x - \frac{x^2}{2} + \frac{\cancel{2}x^3}{\cancel{6}3} + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

(iv) $\sin^2 x$

Let $f(x) = \sin^2 x$, $\quad f(0) = \sin^2 0 = 0$

$f'(x) = 2\sin x\cos x = \sin 2x$, $\quad f'(0) = \sin 2(0) = 0$

$f''(x) = 2\cos 2x$, $\quad f''(0) = 2\cos 2(0) = 2$

$f'''(x) = -4\sin 2x$, $\quad f'''(0) = -4\sin 2(0) = 0$

$f^{(4)}(x) = -8\cos 2x$, $\quad f^{(4)}(0) = -8\cos 2(0) = -8$

$f^{(5)}(x) = 16\sin 2x$, $\quad f^{(5)}(0) = 16\sin 2(0) = 0$

$f^{(6)}(x) = 32\cos 2x$, $\quad f^{(6)}(0) = 32\cos 2(0) = 32$

Formula

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + f^{(5)}(0)\frac{x^5}{5!} + f^{(6)}(0)\frac{x^6}{6!} + \cdots$$

$4! = 4\times 3\times 2\times 1$

$$\sin^2 x = 0 + 0x + 2\frac{x^2}{2!} + 0\frac{x^3}{3!} + (-8)\frac{x^4}{4!} + 0\frac{x^5}{5!} + 32\frac{x^6}{6!} + \cdots$$

$$\sin^2 x = \cancel{2}\frac{x^2}{\cancel{2}} + \frac{-\cancel{8}x^4}{\cancel{24}\,3} + \frac{\overset{2\,\cancel{4}\,\cancel{16}}{\cancel{32}}\,x^6}{\underset{3}{\cancel{6}}\times 5\times\cancel{4}\times 3\times\cancel{2}\times 1} + \cdots$$

$$\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \cdots$$

(v) $e^{\sin x}$

$(e^f)' = e^f \cdot f'$

Let $f(x) = e^{\sin x}$, $f(0) = e^{\sin 0} = e^0 = 1$

$f'(x) = e^{\sin x} \cdot \cos x$, $f'(0) = e^{\sin 0} \cdot \cos 0 = e^0 \cdot 1 = 1$

$f''(x) = e^{\sin x}(-\sin x) + \cos x \cdot e^{\sin x} \cos x$, $f''(0) = e^{\sin 0}(-\sin 0) + \cos 0\, e^{\sin 0} \cdot \cos 0$
$= e^0 (0) + 1\, e^0 \cdot 1$
$= 0 + 1 = 1$

Formula

$$f(x) = f(0) + f'(0)\, x + f''(0) \frac{x^2}{2!} + \cdots$$

$$e^{\sin x} = 1 + 1\, x + 1\, \frac{x^2}{2!} + \cdots$$
$$= 1 + x + \frac{x^2}{2!} + \cdots$$

(vi) $x e^{-x}$

Let $f(x) = x e^{-x}$, $f(0) = 0\, e^{-0} = 0$

$f'(x) = x\, e^{-x}(-1) + e^{-x}(1)$
$= -x\, e^{-x} + e^{-x}$
$f'(x) = e^{-x}(-x + 1)$, $f'(0) = e^{-0}(-0 + 1) = 1$

$f''(x) = e^{-x}(-1) + (-x + 1)\, e^{-x}(-1)$
$= -e^{-x} + x e^{-x} - e^{-x}$
$= e^{-x}(-1 + x - 1)$
$f''(x) = e^{-x}(-2 + x)$, $f''(0) = e^{-0}(-2 + 0) = -2$

$f'''(x) = e^{-x}(1) + (-2 + x)\, e^{-x}(-1)$
$= e^{-x} + 2e^{-x} - x e^{-x}$
$= e^{-x}(1 + 2 - x)$
$= e^{-x}(3 - x)$, $f'''(0) = e^{-0}(3 - 0) = 3$

Formula

$$f(x) = f(0) + f'(0)\, x + f''(0) \frac{x^2}{2!} + f'''(0) \frac{x^3}{3!} + \cdots$$

$$x e^{-x} = 0 + 1x + (-2) \frac{x^2}{2!} + 3 \frac{x^3}{3!} + \cdots$$

$$x e^{-x} = x - \cancel{2} \frac{x^2}{\cancel{2}} + \cancel{3} \frac{x^3}{\cancel{6}_2} + \cdots$$

$$x e^{-x} = x - x^2 + \frac{x^3}{2} + \cdots$$

(vii) $\frac{1}{1+x} = (1+x)^{-1}$

Let $f(x) = (1+x)^{-1}$, $f(0) = (1+0)^{-1} = 1$

$f'(x) = -1(1+x)^{-2}$, $f'(0) = -1(1+0)^{-2} = -1$

$f''(x) = 2(1+x)^{-3}$, $f''(0) = 2(1+0)^{-3} = 2$

Formula

$$f(x) = f(0) + f'(0)\,x + f''(0)\,\frac{x^2}{2!} + \cdots$$

$$\frac{1}{1+x} = 1 + (-1)\,x + 2\,\frac{x^2}{2!} + \cdots$$

$$= 1 - x + \cancel{2}\,\frac{x^2}{\cancel{2}} + \cdots$$

$$= 1 - x + x^2 + \cdots$$

(2) Find the first four terms of the Taylor's series for

(i) $\ln x$ centered at $a = 1$.

Let $f(x) = \ln x$, $f(a) = \ln a = \ln 1 = 0$

$f'(x) = \frac{1}{x} = x^{-1}$, $f'(a) = a^{-1} = 1^{-1} = 1$

$f''(x) = -x^{-2}$, $f''(a) = -a^{-2} = -1^{-2} = -1$

$f'''(x) = 2x^{-3}$, $f'''(a) = 2a^{-3} = 2\,1^{-3} = 2$

$f^{(4)}(x) = -6x^{-4}$, $f^{(4)}(a) = -6a^{-4} = -6\,1^{-4} = -6$

Formula

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + f^{(4)}(a)\frac{(x-a)^4}{4!} + \dots$$

$$\ln x = 0 + 1(x-1) + (-1)\frac{(x-1)^2}{2!} + \cancel{2}\frac{(x-1)^3}{\cancel{6}\,3} + (\cancel{-6})\frac{(x-1)^4}{\cancel{24}\,4} + \dots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

(ii) $\frac{1}{x}$ centered at $a = 1$.

Let $f(x) = x^{-1}$, $f(a) = a^{-1} = 1^{-1} = 1$

$f'(x) = -x^{-2}$, $f'(a) = -a^{-2} = -1^{-2} = -1$

$f''(x) = 2x^{-3}$, $f''(a) = 2a^{-3} = 2\,1^{-3} = 2$

$f'''(x) = -6x^{-4}$, $f'''(a) = -6a^{-4} = -6\,1^{-4} = -6$

Formula

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \dots$$

$$\frac{1}{x} = 1 + (-1)(x-1) + \cancel{2}\frac{(x-1)^2}{\cancel{2}} + (\cancel{-6})\frac{(x-1)^3}{\cancel{6}} + \dots$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots$$

(iii) sinx centered at $a = \frac{\pi}{4}$.

Let $f(x) = \sin x$, $f(a) = \sin a = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x$, $f'(a) = \cos a = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x$, $f''(a) = -\sin a = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$f'''(x) = -\cos x$, $f'''(a) = -\cos a = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

Formula

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots$$

$$\sin x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right) + \left(\frac{-1}{\sqrt{2}}\right)\frac{(x-\pi/4)^2}{2!} + \left(-\frac{1}{\sqrt{2}}\right)\frac{(x-\frac{\pi}{4})^3}{3!} + \cdots$$

$$\sin x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right) - \frac{1}{\sqrt{2}\,2!}\left(x-\frac{\pi}{4}\right)^2 - \frac{1}{\sqrt{2}\,3!}\left(x-\frac{\pi}{4}\right)^3 + \cdots$$

(iv) cosx centered at $\alpha = \frac{\pi}{2}$.

Let $f(x) = \cos x$, $f(a) = \cos a = \cos\frac{\pi}{2} = 0$

$f'(x) = -\sin x$, $f'(a) = -\sin a = -\sin\frac{\pi}{2} = -1$

$f''(x) = -\cos x$, $f''(a) = -\cos a = -\cos\frac{\pi}{2} = 0$

$f'''(x) = \sin x$, $f'''(a) = \sin a = \sin\frac{\pi}{2} = 1$

$f^{(4)}(x) = \cos x$, $f^{(4)}(a) = \cos a = \cos\frac{\pi}{2} = 0$

$f^{(5)}(x) = -\sin x$, $f^{(5)}(a) = -\sin a = -\sin\frac{\pi}{2} = -1$

$f^{(6)}(x) = -\cos x$, $f^{(6)}(a) = -\cos a = -\cos\frac{\pi}{2} = 0$

$f^{(7)}(x) = \sin x$, $f^{(7)}(a) = \sin a = \sin\frac{\pi}{2} = 1$

Formula

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + f^{(4)}(a)\frac{(x-a)^4}{4!} + f^{(5)}(a)\frac{(x-a)^5}{5!}$$

$$+ f^{(6)}(a)\frac{(x-a)^6}{6!} + f^{(7)}(a)\frac{(x-a)^7}{7!} + \cdots$$

$$\cos x = 0 + (-1)\left(x-\frac{\pi}{2}\right) + 0 + 1\,\frac{\left(x-\frac{\pi}{2}\right)^3}{3!} + 0 + (-1)\frac{\left(x-\frac{\pi}{2}\right)^5}{5!} + 0 + 1\,\frac{\left(x-\frac{\pi}{2}\right)^7}{7!} + \cdots$$

$$= -\left(x-\frac{\pi}{2}\right) + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5 + \frac{1}{7!}\left(x-\frac{\pi}{2}\right)^7 + \cdots$$

③ Does Maclaurin's series of the functions

$$f(x) = \frac{1}{x}, \quad g(x) = \operatorname{cosec} x \quad \text{and} \quad h(x) = \sqrt{x}$$

exist? If not why? Give appropriate justification.

$f(x) = \frac{1}{x}$, $f(0) = \frac{1}{0}$ is undefined.

f has no Maclaurin series, because f is not defined at 0.

$g(x) = \operatorname{cosec} x = \frac{1}{\sin x}$, $g(0) = \frac{1}{\sin 0} = \frac{1}{0}$ is undefined.

g has no Maclaurin series, because g is not defined at 0.

$h(x) = \sqrt{x} = x^{\frac{1}{2}}$, $h(0) = 0^{\frac{1}{2}} = 0$

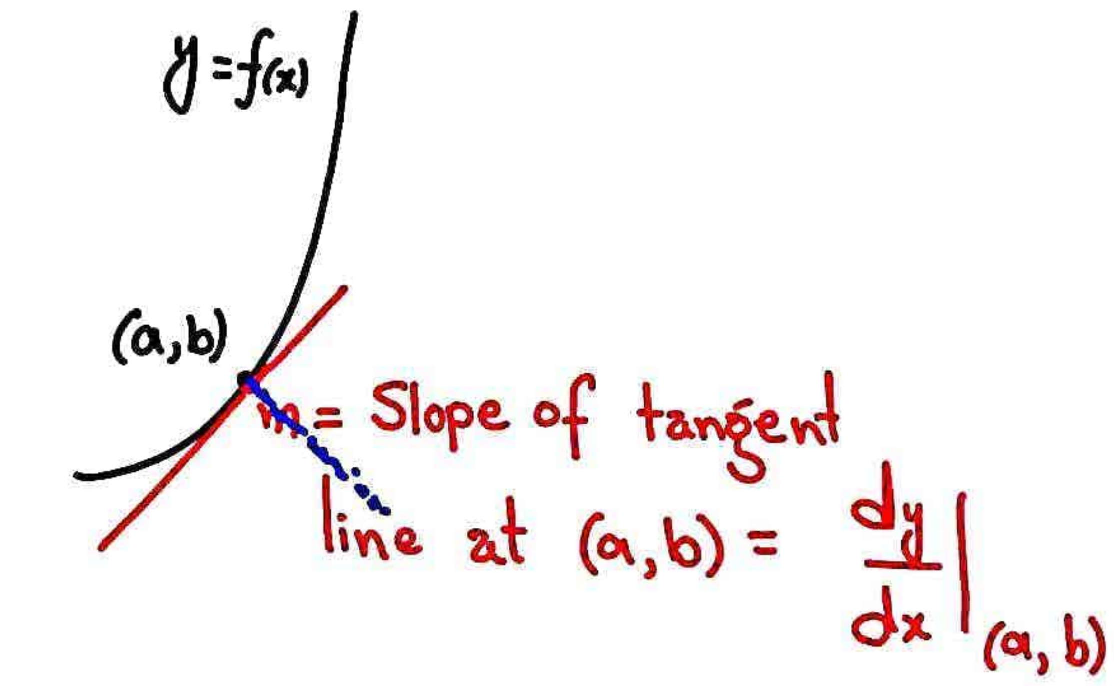
$h'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$, $h'(0) = \frac{1}{2\sqrt{0}} = \frac{1}{0}$ is undefined.

h has no Maclaurin series, because $h'(0)$ is not defined.

Important Points for Exercise 4.3

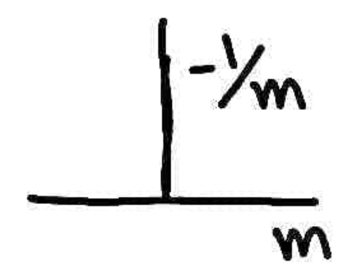
Eq. of tangent at (a,b)

$$y - b = m(x - a)$$

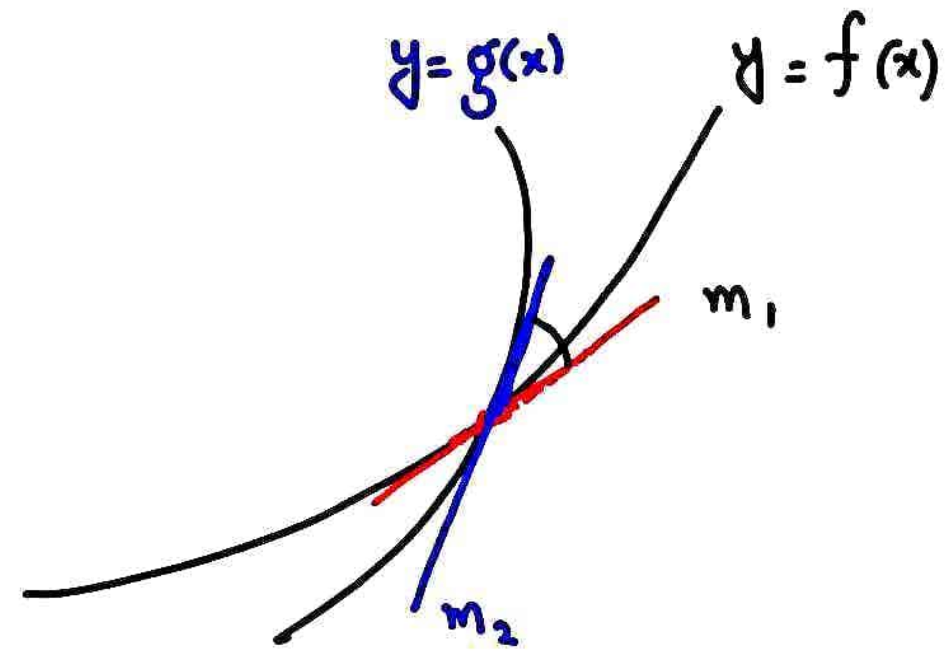


Eq. of normal at (a,b)

$$y - b = -\frac{1}{m}(x - a).$$



Angle b/w two curves at (a,b) is the angle b/w their tangents at (a,b).



Acute angle $\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

Exercise 4.3

(1) Determine the slope of tangent to the curve $y = x^3$ at the point $\left(\frac{3}{2}, \frac{27}{8}\right)$.

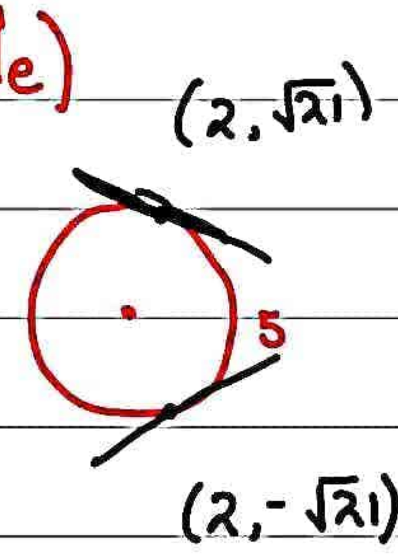
$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

$$\text{Slope at } \left(\frac{3}{2}, \frac{27}{8}\right) = \left.\frac{dy}{dx}\right|_{\left(\frac{3}{2}, \frac{27}{8}\right)} = 3\left(\frac{3}{2}\right)^2 = 3\left(\frac{9}{4}\right) = \frac{27}{4}.$$

(2) Find the slope of tangents to the curve $x^2 + y^2 = 25$ at the point on it whose x-coordinate is 2.

Curve $\quad x^2 + y^2 = 25 \quad - (1)$ (circle)



x - coordinate $= 2$.

Put $x = 2$ in (1)

$$2^2 + y^2 = 25$$

$$y^2 = 25 - 4 = 21$$

$$y = \pm\sqrt{21}$$

We have two points $(2, \sqrt{21})$ and $(2, -\sqrt{21})$.

Given curve $\quad x^2 + y^2 = 25$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

$$\text{Slope of tangent at } (2, \sqrt{21}) = \left.\frac{dy}{dx}\right|_{(2,\sqrt{21})} = -\frac{2}{\sqrt{21}}$$

$$\text{Slope of tangent at } (2, -\sqrt{21}) = \left.\frac{dy}{dx}\right|_{(2,-\sqrt{21})} = \frac{-2}{-\sqrt{21}} = \frac{2}{\sqrt{21}}.$$

(3) Find the equation of tangent and the equation of normal to the curve $y = x + \frac{1}{x}$ at the point where $x = 2$.

$$y = x + \frac{1}{x} = x + x^{-1}$$

Put $x = 2$, $y = 2 + 2^{-1} = 2 + \frac{1}{2} = \frac{5}{2}$

So we have point $(\overset{a}{2}, \overset{b}{\frac{5}{2}})$.

Given $y = x + \frac{1}{x} = x + x^{-1}$

$$\frac{dy}{dx} = 1 - x^{-2} = 1 - \frac{1}{x^2}$$

$$m = \left.\frac{dy}{dx}\right|_{(2,\frac{5}{2})} = 1 - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4}.$$

Eq. of tangent

$$y - b = m\,(x - a)$$

$$y - \frac{5}{2} = \frac{3}{4}(x - 2)$$

$$\frac{2y - 5}{\cancel{2}} = \frac{3}{{}_{2}\cancel{4}}(x - 2)$$

$$2(2y - 5) = 3(x - 2) \Rightarrow 4y - 10 = 3x - 6$$

$$3x - 6 - 4y + 10 = 0$$

$$3x - 4y + 4 = 0.$$

Eq. of normal

$$y - b = -\frac{1}{m}(x - a)$$

$$y - \frac{5}{2} = -\frac{1}{(3/4)}(x - 2)$$

$$\frac{3}{4}\left(y - \frac{5}{2}\right) = -(x - 2)$$

$$\frac{3}{4}y - \frac{15}{8} = -x + 2$$

$$x + \frac{3}{4}y - \frac{15}{8} - 2 = 0$$

Multiply by 8

$$8x + 6y - 15 - 16 = 0$$

$$8x + 6y - 31 = 0.$$

④ Given two curves $y = x^2$ and $y = (x-3)^2$. Find the angle between them.

Given $y = x^2$ — ①, $y = (x-3)^2$ — ②

To find point of intersection, we take

$$x^2 = (x-3)^2$$
$$\cancel{x^2} = \cancel{x^2} + 9 - 6x$$
$$6x = 9$$
$$x = \frac{3}{2}$$

Put in ①

$$y = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

So point of intersection is $\left(\frac{3}{2}, \frac{9}{4}\right)$.

Given $y = x^2$, $y = (x-3)^2$

$$\frac{dy}{dx} = 2x \qquad \frac{dy}{dx} = 2(x-3)$$

$$m_1 = \left.\frac{dy}{dx}\right|_{\left(\frac{3}{2},\frac{9}{4}\right)} = \cancel{2}\left(\frac{3}{\cancel{2}}\right) = 3$$

$$m_2 = \left.\frac{dy}{dx}\right|_{\left(\frac{3}{2},\frac{9}{4}\right)} = 2\left(\frac{3}{2} - 3\right) = \cancel{2}\left(\frac{3-6}{\cancel{2}}\right) = -3$$

Angle $\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\theta = \tan^{-1}\left|\frac{3+3}{1+3(-3)}\right| = \tan^{-1}\left|\frac{6}{-8}\right| = \tan^{-1}\left(\frac{\cancel{6}^{\,3}}{\cancel{8}_{4}}\right)$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 36^\circ 52' \approx 37^\circ.$$

(5) Prove that the tangent lines to the curve $y^2 = 4ax$ at points where $x = a$ are at right angles to each other.

Given curve $y^2 = 4ax$ — ①

Put $x = a$ in ①

$$y^2 = 4a\,a = 4a^2$$

$$y = \pm 2a$$

We have two points

$(a, 2a)$ and $(a, -2a)$.

From ① $y^2 = 4ax$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax)$$

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{\cancel{4}^{2}a}{\cancel{2}y} = \frac{2a}{y}$$

m_1 = Slope of tangent at $(a, 2a) = \left.\frac{dy}{dx}\right|_{(a,2a)} = \frac{2a}{2a} = 1$

m_2 = Slope of tangent at $(a, -2a) = \left.\frac{dy}{dx}\right|_{(a,-2a)} = \frac{2a}{-2a} = -1$.

Since $m_1 = -\frac{1}{m_2}$

So these tangents are at right angle.

Alternative

$$\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \tan^{-1}\left|\frac{1+1}{1+(-1)}\right| = \tan^{-1}\left(\frac{2}{0}\right) = \frac{\pi}{2}.$$

⑥ At what points on the curve $x^2+y^2-2x-4y+1=0$ the tangent is parallel to y-axis?

Given $\quad x^2+y^2-2x-4y+1=0 \quad$ — ①

$$\frac{d}{dx}\left(x^2+y^2-2x-4y+1\right)=\frac{d}{dx}(0)$$

$$2x+2y\frac{dy}{dx}-2-4\frac{dy}{dx}+0=0$$

$$2y\frac{dy}{dx}-4\frac{dy}{dx}=-2x+2$$

$$\frac{dy}{dx}(2y-4)=-2x+2$$

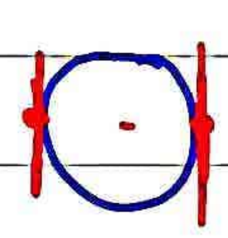
$$\frac{dy}{dx}=\frac{-2x+2}{2y-4}=\frac{\cancel{2}(-x+1)}{\cancel{2}(y-2)}=\frac{-x+1}{y-2}$$

Since slope of y-axis is undefined, so

$\frac{dy}{dx}$ is undefined only when

$$y-2=0$$

$$\boxed{y=2}$$



Put $y=2$ in ①.

$$x^2+2^2-2x-4(2)+1=0$$

$$x^2+4-2x-8+1=0$$

$$x^2-2x-3=0$$

$$x^2-3x+x-3=0$$

$$x(x-3)+1(x-3)=0$$

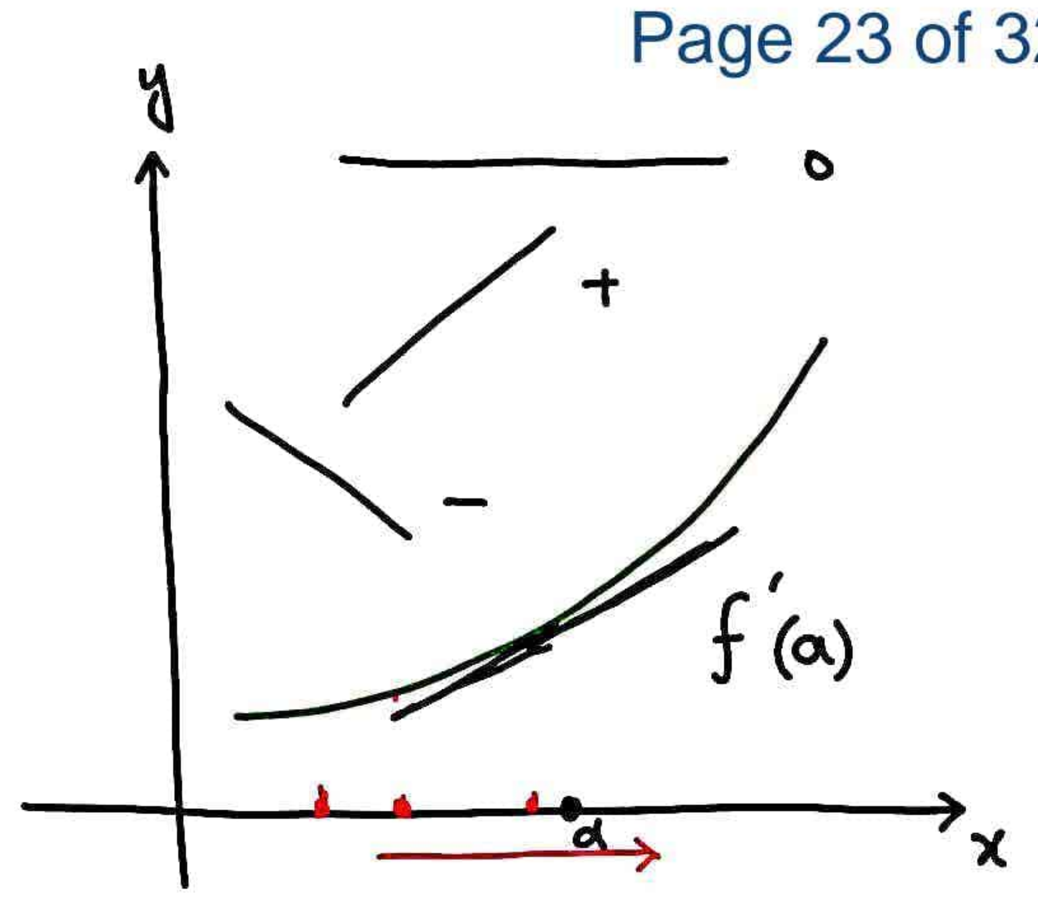
$$(x+1)(x-3)=0$$

$x+1=0 \qquad\qquad x-3=0$

$x=-1 \qquad\qquad x=3$

So required points are $(-1,2)$ and $(3,2)$.

Important Points for Exercise 4.4

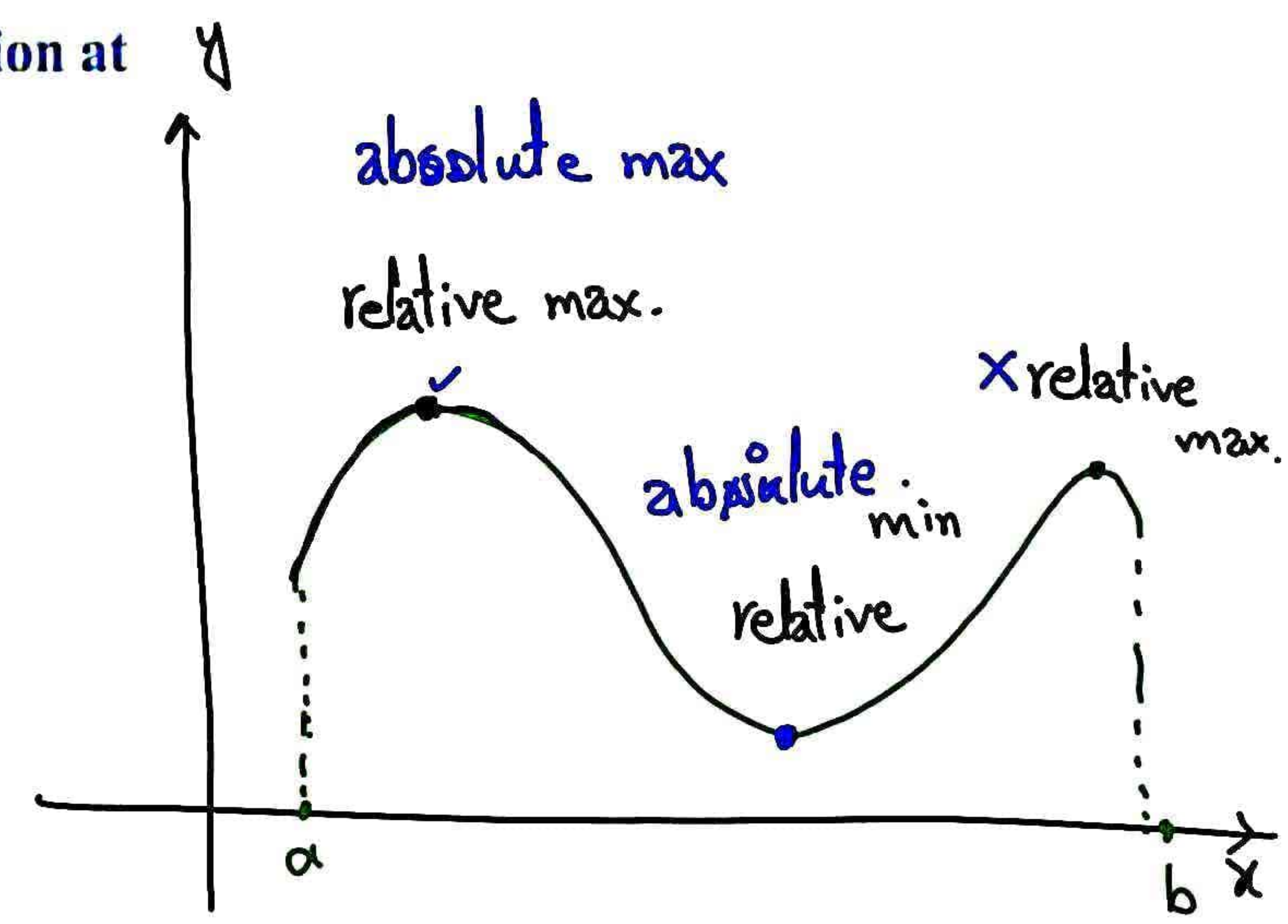


If $f(x)$ is differentiable function on the open interval (a, b) then

- **$f(x)$ is increasing on (a, b) if $f'(x) > 0$ $\forall x \in (a, b)$**
- **$f(x)$ is decreasing on (a, b) if $f'(x) < 0$ $\forall x \in (a, b)$**

maximum / minimum

State the second derivative rule to find the extreme values of a function at a point



Let $y = f(x)$ be a function.

(i) Differentiate w.r.t 'x' and obtain $f'(x)$.

(ii) Put $\boxed{f'(x) = 0}$ solve it and obtain critical ~~function~~ point.

(iii) Differentiate again w.r.t 'x' of obtain $f''(x)$.

(iv) Let $x = a$ be a critical point.

If the $f''(a) < 0 \Rightarrow x = a$ is a point of maxima.

If the $f''(a) > 0 \Rightarrow x = a$ is point of minima.

If the $f''(a) = 0 \Rightarrow$ test fails.

Exercise 4.4

(1) Show that function $f(x) = -x^2 + 10x + 9$ is increasing at $x = 4$.

Given $f(x) = -x^2 + 10x + 9$.

$$f'(x) = -2x + 10$$

$$f'(4) = -2(4) + 10$$
$$= -8 + 10 = 2 > 0$$

So $f(x)$ is increasing at $x = 4$.

(2) Show that $f(x) = \tan^2 x$ is decreasing at $x = \frac{3\pi}{4}$.

$$f(x) = \tan^2 x$$

$$f'(x) = 2\tan x \sec^2 x$$

$\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$

$$f'\left(\frac{3\pi}{4}\right) = 2\tan\frac{3\pi}{4} \cdot \sec^2\frac{3\pi}{4}$$
$$= 2(-1) \cdot (-\sqrt{2})^2$$
$$= -2 \cdot (2) = -4 < 0$$

So $f(x)$ is decreasing at $x = \frac{3\pi}{4}$.

(3) Find the maximum and minimum values, if any, of the function $f: \mathbb{R} \to \mathbb{R}$ in the following cases:

(i) $f(x) = x^2 - 2x + 3$

$$f'(x) = 2x - 2$$

Set $f'(x) = 0$

$$2x - 2 = 0$$
$$2x = 2$$
$$\boxed{x = 1} \quad \text{critical value.}$$

$$f''(x) = 2$$
$$f''(1) = 2 > 0$$

So $f(x)$ has minimum value at $x = 1$, and minimum value is

$$f(1) = 1^2 - 2(1) + 3 = 1 - 2 + 3 = 2.$$

(iii) $f(x) = x^3 - 9x^2 + 15x + 3$

$f'(x) = 3x^2 - 18x + 15$

Critical value

$f'(x) = 0$

$3x^2 - 18x + 15 = 0$

$3(x^2 - 6x + 5) = 0$

$x^2 - 6x + 5 = 0$

$x^2 - 5x - x + 5 = 0$

$x(x-5) - 1(x-5) = 0$

$(x-1)(x-5) = 0$

$x - 1 = 0$, $x - 5 = 0$

$\boxed{x = 1}$ $\boxed{x = 5}$

$f''(x) = 6x - 18$

At $x = 1$

$f''(1) = 6(1) - 18 = 6 - 18 = -12 < 0$

So f has maximum value at $x = 1$ and max. value is

$f(1) = 1 - 9 + 15 + 3 = 10.$

At $x = 5$

$f''(5) = 6(5) - 18 = 30 - 18 = 12 > 0$

So f has min. value at $x = 5$ and min. value is

$f(5) = 5^3 - 9(5)^2 + 15(5) + 3$

$= 125 - 225 + 75 + 3$

$= -100 + 78 = -22.$

(iii) $f(x) = -x^4 + 2x^2$

$f'(x) = -4x^3 + 4x$

Critical value

Put $f'(x) = 0$

$-4x^3 + 4x = 0$

$-4x(x^2 - 1) = 0$

$x(x+1)(x-1) = 0$

$x = 0$, $x+1 = 0$, $x - 1 = 0$

$\boxed{x=0}$, $\boxed{x=-1}$, $\boxed{x=1}$

$f''(x) = -12x^2 + 4$

At $x = 0$

$f''(0) = -12(0)^2 + 4 = 4 > 0$

So f has min. value at $x = 0$

and min. value is

$f(0) = -0^4 + 2(0)^2 = 0.$

At $x = -1$

$f''(-1) = -12(-1)^2 + 4 = -12 + 4 = -8 < 0$

So f has max. value at $x = -1.$

and max. value is

$f(-1) = -(-1)^4 + 2(-1)^2 = -1 + 2 = 1.$

At $x = 1$

$f''(1) = -12(1)^2 + 4 = -12 + 4 = -8 < 0$

So f has max. value at $x = 1$

and max. value is

$f(1) = -1^4 + 2(1^2) = -1 + 2 = 1.$

(iv) $f(x) = e^x \sin x$

$$f'(x) = e^x \cos x + e^x \sin x$$

Critical value

Put $f'(x) = 0$

$$e^x \cos x + e^x \sin x = 0$$

$$e^x (\cos x + \sin x) = 0$$

$e^x \neq 0$, $\cos x + \sin x = 0$

Divide by $\cos x$

$$1 + \tan x = 0$$

$$\tan x = -1$$

✓S	A
T	C✓

$x = \pi - \frac{\pi}{4}$, $x = -\frac{\pi}{4}$

$\boxed{x = \frac{3\pi}{4}}$, $\boxed{x = -\frac{\pi}{4}}$

$$f''(x) = \cancel{-e^x \sin x} + e^x \cos x + e^x \cos x + \cancel{e^x \sin x}$$

$$f''(x) = 2e^x \cos x$$

At $x = -\frac{\pi}{4}$

$$f''\left(-\frac{\pi}{4}\right) = 2e^{-\frac{\pi}{4}} \cos\left(-\frac{\pi}{4}\right) = 2e^{-\frac{\pi}{4}} \cos\frac{\pi}{4} = 2e^{-\pi/4} \cdot \frac{1}{\sqrt{2}} > 0$$

So f has min. value at $x = -\frac{\pi}{4}$.

and the min. value is

$$f\left(-\frac{\pi}{4}\right) = e^{-\frac{\pi}{4}} \sin\left(-\frac{\pi}{4}\right) = -e^{-\frac{\pi}{4}} \sin\frac{\pi}{4} = -\frac{e^{-\pi/4}}{\sqrt{2}}$$

At $x = \frac{3\pi}{4}$

$$f''\left(\frac{3\pi}{4}\right) = 2e^{3\pi/4} \cos\left(\frac{3\pi}{4}\right) = \frac{-2}{\sqrt{2}} e^{\frac{3\pi}{4}} < 0$$

So f has max. value at $x = \frac{3\pi}{4}$,

and the max. value is

$$f\left(\frac{3\pi}{4}\right) = e^{\frac{3\pi}{4}} \sin\left(\frac{3\pi}{4}\right) = \frac{e^{\frac{3\pi}{4}}}{\sqrt{2}}.$$

(v) $f(x) = 2e^{x} + e^{-x}$

$f'(x) = 2e^{x} - e^{-x}$

Critical value

Put $f'(x) = 0$

$2e^{x} - e^{-x} = 0$

$2e^{x} - \frac{1}{e^{x}} = 0$

$\frac{2e^{2x} - 1}{e^{x}} = 0 \qquad , e^{x} \neq 0$

$2e^{2x} - 1 = 0$

$2e^{2x} = 1$

$e^{2x} = \frac{1}{2}$

$\ln e^{2x} = \ln\left(\frac{1}{2}\right) \qquad \boxed{m \ln a = \ln a^{m}}$

$2x = \ln \frac{1}{2}$

$x = \frac{1}{2} \ln\left(\frac{1}{2}\right) = \ln\left(\frac{1}{2}\right)^{\frac{1}{2}} = \ln\left(\sqrt{\frac{1}{2}}\right)$

$\boxed{x = \ln\left(\frac{1}{\sqrt{2}}\right)}$

$f''(x) = 2e^{x} + e^{-x} = f(x)$

At $x = \ln\left(\frac{1}{\sqrt{2}}\right)$

$f''\left(\ln \frac{1}{\sqrt{2}}\right) = 2e^{\ln\left(\frac{1}{\sqrt{2}}\right)} + e^{-\ln\left(\frac{1}{\sqrt{2}}\right)}$

$= 2\left(\frac{1}{\sqrt{2}}\right) + e^{\ln\left(\frac{1}{\sqrt{2}}\right)^{-1}}$

$= \frac{2}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\right)^{-1} = \frac{2}{\sqrt{2}} + \sqrt{2}$

$= \frac{2+2}{\sqrt{2}} = \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancel{4}^{2}\sqrt{2}}{\cancel{2}}$

$= 2\sqrt{2} > 0$

So f has minimum value at $x = \ln\left(\frac{1}{\sqrt{2}}\right)$.

and minimum value is

$f\left(\ln\left(\frac{1}{\sqrt{2}}\right)\right) = f''\left(\ln\left(\frac{1}{\sqrt{2}}\right)\right) = 2\sqrt{2}.$

(vi) $f(x) = 2x - x^2$.

$f'(x) = 2 - 2x$

Critical value

Put $f'(x) = 0$

$2 - 2x = 0$

$2 = 2x$

$\boxed{x = 1}$

$f''(x) = -2$

$f''(1) = -2 < 0$

So f has maximum value at $x = 1$

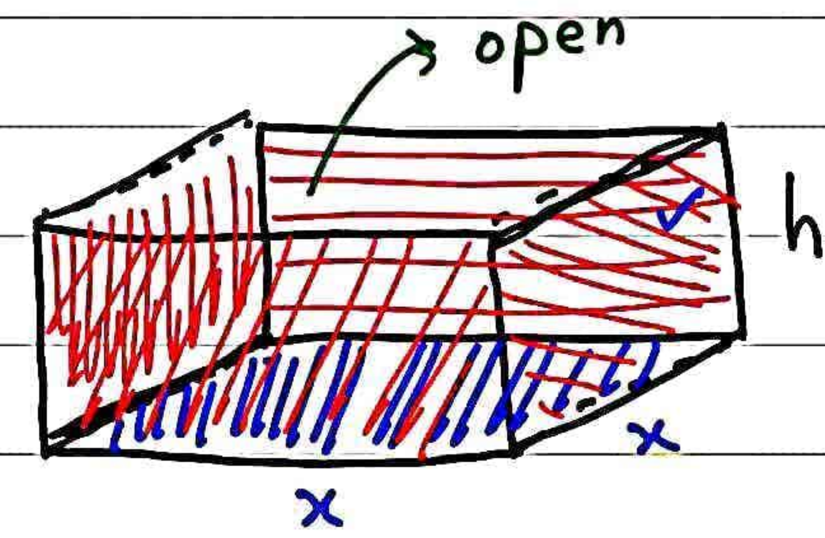
and the max. value is

$f(1) = 2(1) - 1^2 = 2 - 1 = 1.$

4. A rectangular reservoir with a square bottom and open top is to be lined inside with lead. Find the dimensions of the reservoir to hold $\frac{1}{2}a^3$ cubic meters, such that the lead required is minimum.

Let x be the side of bottom.

and h be the height.



Given

$$\text{volume} = \frac{1}{2}a^3$$

$$x \cdot x \cdot h = \frac{1}{2}a^3$$

$$x^2 h = \frac{1}{2}a^3$$

$$h = \frac{a^3}{2x^2} \quad \text{— ①}$$

Surface area $S = x^2 + xh + xh + xh + xh$

$$S = x^2 + 4xh$$

$$S = x^2 + \cancel{4}^{2}\cancel{x}\left(\frac{a^3}{\cancel{2}x^{\cancel{2}}}\right) \quad (\because \text{from ①})$$

$$S = x^2 + \frac{2a^3}{x} = x^2 + 2a^3x^{-1}$$

$$S' = 2x - 2a^3x^{-2} = 2x - \frac{2a^3}{x^2}$$

Put $S' = 0$

$$2x - \frac{2a^3}{x^2} = 0.$$

$$\frac{2x^3 - 2a^3}{x^2} = 0 \Rightarrow x \neq 0, \quad 2x^3 - 2a^3 = 0$$

$$2x^3 = 2a^3$$

$$\boxed{x = a}$$

$$S'' = 2 + 4a^3x^{-3}$$

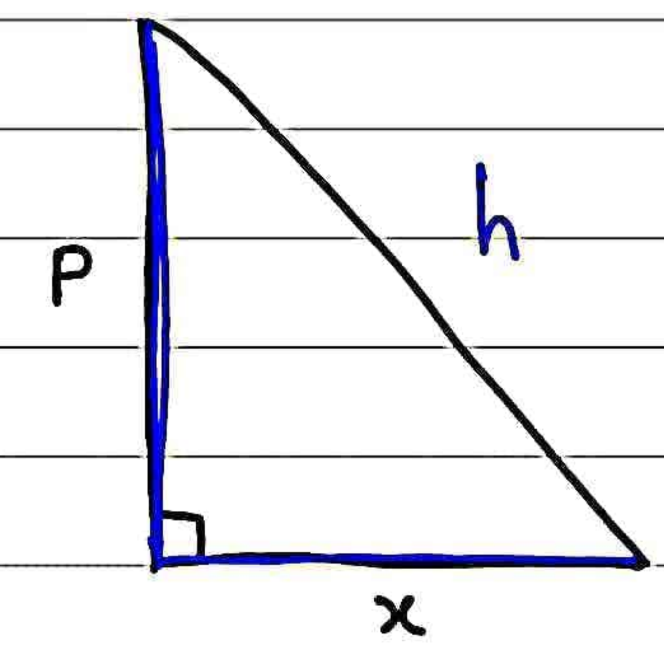
At $x = a$, $S'' = 2 + 4a^3a^{-3} = 2 + 4 = 6 > 0$

So S is minimum at $x = a$.

Put $x = a$ in ①

$$h = \frac{a^3}{2x^2} = \frac{a^{\cancel{3}}}{2\cancel{a^2}} = \frac{1}{2}a.$$

(5) Find a right-angled triangle of maximum area with a hypotenuse of length h.



Let base of $\Delta = x$

then $$p^2 + x^2 = h^2$$
$$p^2 = h^2 - x^2$$
$$p = \sqrt{h^2 - x^2}$$

So Area $$A = \frac{1}{2}(x \cdot p)$$

$$A = \frac{1}{2}\left(x \cdot \sqrt{h^2 - x^2}\right)$$

$$A' = \frac{1}{2}\left[x \cdot \frac{1}{\cancel{2}\sqrt{h^2-x^2}}(-\cancel{2}x) + \sqrt{h^2-x^2} \cdot 1\right]$$

$$A' = \frac{1}{2}\left[\frac{-x^2}{\sqrt{h^2-x^2}} + \sqrt{h^2-x^2}\right] = \frac{1}{2}\left[\frac{-x^2 + h^2 - x^2}{\sqrt{h^2-x^2}}\right]$$

$$A' = \frac{1}{2}\left(\frac{-2x^2+h^2}{\sqrt{h^2-x^2}}\right)$$

Put
$$A' = 0$$
$$\frac{1}{2}\left(\frac{-2x^2+h^2}{\sqrt{h^2-x^2}}\right) = 0 \Rightarrow -2x^2 + h^2 = 0$$
$$h^2 = 2x^2$$
$$\sqrt{2}x = h$$
$$\boxed{x = \frac{h}{\sqrt{2}}}$$

$$A = \frac{1}{2}\left[\frac{h}{\sqrt{2}}\sqrt{h^2 - \left(\frac{h}{\sqrt{2}}\right)^2}\right] = \frac{h}{2\sqrt{2}}\sqrt{h^2 - \frac{h^2}{2}}$$

$$A = \frac{h}{2\sqrt{2}}\sqrt{\frac{h^2}{2}} = \frac{h}{2\sqrt{2}} \cdot \frac{h}{\sqrt{2}} = \frac{h^2}{4}.$$

6. A particle moves so that its distance s at time t is given by

$$s = ut + \frac{1}{2}at^2$$

where u and a are fixed real numbers. Find its speed and magnitude of its acceleration at time t.

$$s = ut + \frac{1}{2}at^2$$

$$\text{Speed} = \frac{ds}{dt} = u + \frac{1}{\cancel{2}}a(\cancel{2}t) = u + at.$$

$$\text{Acceleration} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = 0 + a(1) = a.$$