MATHEMATICS 2nd YEAR

UNIT

03

INTEGRATION

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Contents

Sherazi Mathematics

اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Integration:- The technique or method to find such a function whose derivative is given involves the inverse process of differentiation, called anti-derivation or integration.

Differential of variables:-

Let f be a differentiable function defined as $y = f(x)$ $\rightarrow y + \delta y = f(x+\delta x)$

$\rightarrow \delta y = f(x+\delta x) - y$ $\rightarrow \delta y = f(x+\delta x) - f(x)$

Now $\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim\limits_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$

$\rightarrow \frac{dy}{dx} = f'(x)$

$\because$ before the limit is reached, $\frac{\delta y}{\delta x}$ differs from $f'(x)$ by a very small real number ϵ.

i-e., $\frac{\delta y}{\delta x} = f'(x) + \epsilon$ $\rightarrow \delta y = f'(x)\delta x + \epsilon \delta x$

As ϵ is very small so neglecting $\epsilon \delta x$, so $\delta y = f'(x)\delta x$. Now $f'(x)$ is called differential of dependent variable y. so we denote differential of y as dy. so

$dy = f'(x)\delta x \rightarrow dy = f'(x)dx$

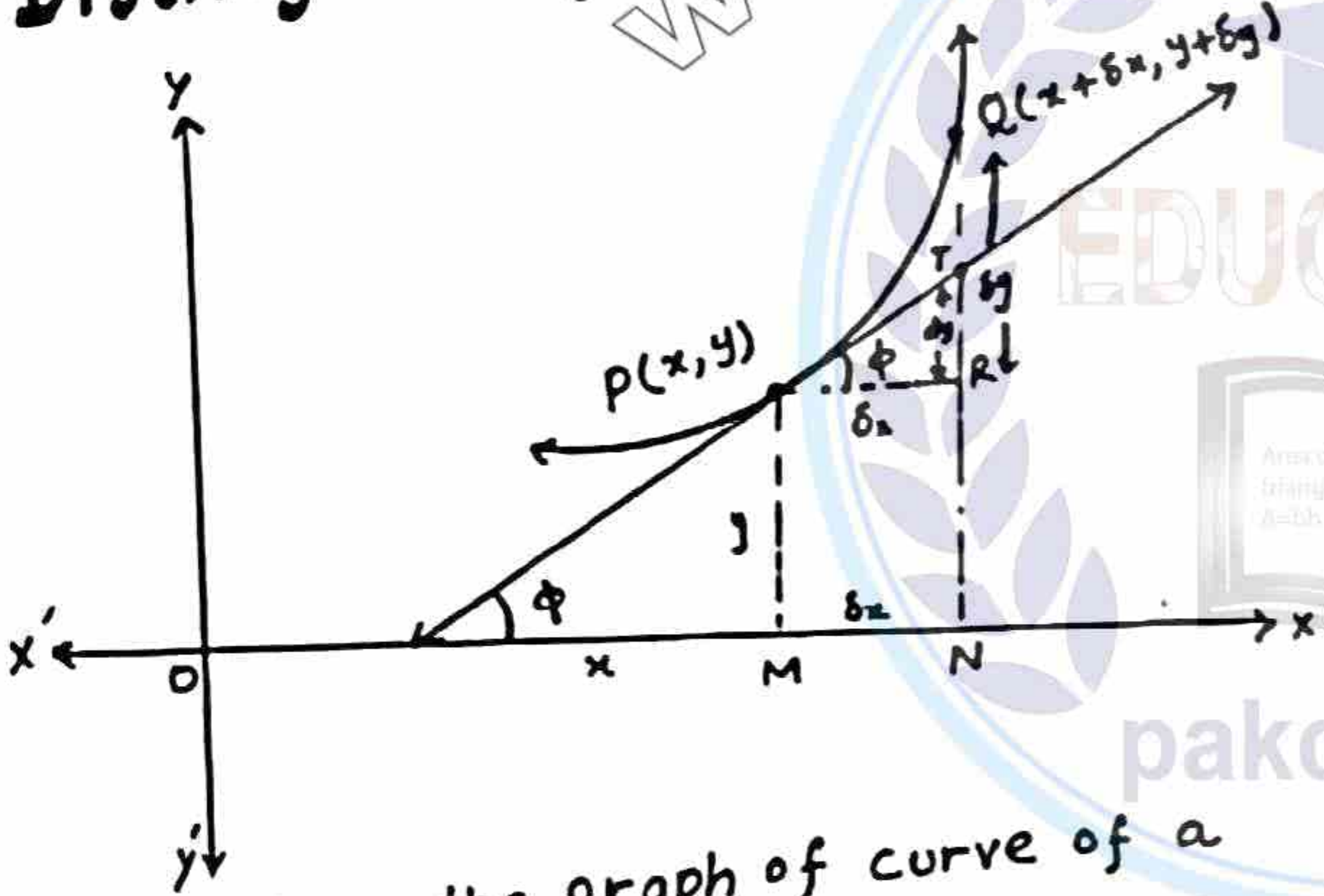
Note:- 1. The differential of x is denoted by dx and defined as $dx = \delta x$

i-e., For $y = x$ $\rightarrow dy = \frac{d}{dx}(x)\delta x$

$\rightarrow dy = 1.\delta x \rightarrow dx = \delta x$ $\because y = x$

2. $f'(x)$ is called differential coefficient.

Distinguishing between dy and δy



Let us draw the graph of curve of a function $y = f(x)$

Let $P(x,y)$ and $Q(x+\delta x, y+\delta y)$ be two neighboring points on the curve.

Draw a tangent line to the curve at point $P(x,y)$ s.that it makes an angle ϕ with x-axis. Draw ⊥ar PM and QN on x-axis. Also draw ⊥ar PR on QN.

In fig., $|PR| = \delta x$, $|QR| = \delta y$

$\boxed{\because \delta x = dx}$ so $|PR| = dx$

$|QR| = |QT| + |TR|$

$\rightarrow \delta y = |QT| + |TR| \longrightarrow (i)$

In ΔTPR, $\tan\phi = \frac{|TR|}{|PR|} = \frac{|TR|}{dx}$

$\rightarrow |TR| = \tan\phi \, dx$

so (i) $\rightarrow$ $\delta y = \tan\phi \, dx + |QT|$ $\quad \because \frac{dy}{dx} = \tan\phi$

$\rightarrow \delta y = \left(\frac{dy}{dx}\right)dx + |QT|$

$\rightarrow \delta y = dy + |QT|$ $\quad \because |QT|$ is very small

so by neglecting $|QT|$ $\boxed{\rightarrow \delta y \approx dy}$

As we know that $y = f(x)$, $dy = f'(x)dx$

$$y + \delta y = f(x+\delta x)$$
$$\text{Or } \delta y = f(x+\delta x) - y$$
$$\delta y = f(x+\delta x) - f(x)$$
$$f(x+\delta x) = f(x) + \delta y$$
$$f(x+\delta x) \approx f(x) + dy \quad \because \delta y \approx dy$$
$$f(x+\delta x) \approx f(x) + f'(x)dx$$

Example: Find δy and dy of the function defined as $f(x) = x^2$, when $x = 2$ and $dx = 0.01$

Solution:- Let $y = f(x)$, $dy = ?$

$\rightarrow y = x^2$ $\rightarrow \frac{dy}{dx} = 2x$ $\rightarrow dy = 2x\,dx$

Take $x = 2$ and $dx = 0.01$

$\rightarrow dy = 2(2)(0.01) = 0.04$

Now we find δy, $\quad y + \delta y = (x+\delta x)^2$

$\rightarrow \delta y = (x+\delta x)^2 - y$, $\quad y = (x)^2 = (2)^2 = 4$

$= (2+0.01)^2 - 4$ $\quad \because dx = \delta x = 0.01$

$\rightarrow \delta y = 4.0401 - 4 = 0.0401$

Example: Use differentials find $\frac{dy}{dx}$ when $\frac{y}{x} - \ln x = \ln c$

Solution:- $\frac{y}{x} - \ln x = \ln c$

$\rightarrow d\left(\frac{y}{x} - \ln x\right) = d(\ln c)$

$\rightarrow d\left(\frac{y}{x}\right) - d(\ln x) = 0$

$\rightarrow \frac{x\,dy - y\,dx}{x^2} - \frac{1}{x}dx = 0$ $\rightarrow \frac{x\,dy - y\,dx}{x^2} = \frac{1}{x}dx$

$\rightarrow x\,dy - y\,dx = x\,dx$ $\rightarrow x\,dy = x\,dx + y\,dx$

$\rightarrow dy = \frac{(x+y)dx}{x}$ $\rightarrow \frac{dy}{dx} = \frac{x+y}{x}$

Example 1. Use differentials to approximate the value of $\sqrt{17}$.

Solution:-

$$\text{Let } f(x) = \sqrt{x} \quad , \quad f'(x) = \frac{1}{2\sqrt{x}}$$
$$\Rightarrow f(x+\delta x) = \sqrt{x+\delta x}$$

Take $x = 16$ & $dx = \delta x = 1$ so $f(x) = \sqrt{16} = 4$

And $f(x+\delta x) = \sqrt{16+1} = \sqrt{17}$

$$f'(x) = \frac{1}{2\sqrt{16}} = \frac{1}{8} = 0.125$$

Since we know that

$$f(x+\delta x) \approx f(x) + f'(x)dx$$
$$\Rightarrow \sqrt{17} \approx 4 + 0.125\ (1) = 4.125$$

Example 2. Use differentials to approximate the value of $\sqrt[3]{8.6}$

Solution:-

Let $f(x) = \sqrt[3]{x} = (x)^{\frac{1}{3}}$

$f'(x) = \frac{1}{3x^{\frac{2}{3}}}$

And $f(x+\delta x) = \sqrt[3]{x+\delta x}$

Take $x = 8$ & $dx = \delta x = 0.6$

$f(x) = (8)^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2$

$f'(x) = \frac{1}{3(8)^{\frac{2}{3}}} = \frac{1}{3(4)} = \frac{1}{12} = 0.0833$

$f(x+\delta x) = \sqrt[3]{8+0.6} = \sqrt[3]{8.6}$

As we know $f(x+\delta x) \approx f(x) + f'(x)dx$

So $\sqrt[3]{8.6} \approx 2 + (0.0833)(0.6) = 2.05$

Example 3. Find differentials, find the approximate value of $\sin 46^\circ$

Solution:-

Let $f(x) = \sin x$

$f'(x) = \cos x$

And $f(x+\delta x) = \sin(x+\delta x)$

Take $x = 45^\circ$ & $dx = \delta x = 1^\circ = 0.01745$

$f(x) = \sin 45^\circ = 0.7071$

$f'(x) = \cos 45^\circ = 0.7071$

$f(x+\delta x) = \sin(45^\circ + 1^\circ) = \sin 46^\circ$

As we know $f(x+\delta x) \approx f(x) + f'(x)dx$

So $\sin 46^\circ \approx 0.7071 + (0.7071)(0.01745)$
$= 0.7194$

Example 4. The side of a cube is measured to be 20cm with a maximum error of 0.12cm in its measurement. Find the maximum error in the calculated volume of the cube.

Solution:- For a cube length of each side be x and v be the volume.

so Volume $= x \times x \times x$

$\rightarrow V = x^3$

$\rightarrow dV = d(x^3) = 3x^2 dx$

Take $x = 20, \; dx = 0.12$

$\rightarrow dV = 3(20)^3(0.12) = 3(400)(0.12)$

$\rightarrow dV = 144 \text{ cm}^3$

Exercise 3.1

Q1. Find δy and dy in the following cases:

(i) $y = x^2 - 1$ when x changes from 3 to 3.02

Solution:- $y = x^2 - 1$

$\rightarrow dy = d(x^2 - 1) = 2x\,dx$

Take $x = 3$, $dx = 0.02$

$\rightarrow dy = 2(3)(0.02) = 0.12$

Now $y + \delta y = (x+\delta x)^2 - 1$

$\rightarrow \delta y = (x+\delta x)^2 - 1 - y$ $\quad \because y = x^2 - 1$
$= (3+0.02)^2 - 1 - 8$ $\quad = (3)^2 - 1 = 9 - 1$
$\quad \rightarrow y = 8$

$\delta y = 9.1204 - 9 = 0.1204$

(ii) $y = x^2 + 2x$ when x changes from 2 to 1.8.

Solution:- $y = x^2 + 2x$

$\rightarrow dy = d(x^2 + 2x) = (2x + 2)dx$

$dy = 2(x+1)dx$

Take $x = 2$, $dx = -0.2$

$\rightarrow dy = 2(2+1)(-0.2) = 2(3)(-0.2) = -1.2$

Now $y + \delta y = (x+\delta x)^2 + 2(x+\delta x)$ $\quad \because y = (2)^2 + 2(2)$

$\rightarrow \delta y = (x+\delta x)\{x+\delta x+2\} - y$ $\quad y = 8$

$= (2-0.2)(2-0.2+2) - 8$ $\quad \because dx = \delta x$

$= (1.8)(3.8) - 8$

$\delta y = 6.84 - 8 = -1.16$

(iii) $y = \sqrt{x}$ when x changes from 4 to 4.41

Solution:- $y = \sqrt{x} = x^{1/2}$

$\rightarrow dy = d(x^{1/2}) = \frac{1}{2}x^{-1/2}dx = \frac{1}{2\sqrt{x}}dx$

Take $x = 4$, $dx = 0.41$

$dy = \frac{1}{2\sqrt{4}}(0.41) = \frac{0.41}{4} = 0.1025$

$\rightarrow dy = 0.1025$

Now $y + \delta y = \sqrt{x+\delta x}$ $\quad \because y = \sqrt{x}$

$\rightarrow \delta y = \sqrt{x+\delta x} - y$ $\quad \rightarrow y = \sqrt{4} = 2$

$\rightarrow \delta y = \sqrt{4+0.41} - 2$

$\rightarrow \delta y = \sqrt{4.41} - 2 = 2.1 - 2 = 0.1$

Q2. Using differentials find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in the following equations

(i) $xy + x = 4$

Solution:- $xy + x = 4$

$\rightarrow d(xy + x) = d(4)$ $\rightarrow d(xy) + d(x) = 0$

$\rightarrow x\,dy + y\,dx + dx = 0$ $\rightarrow x\,dy + (y+1)dx = 0$

$\rightarrow x\,dy = -(y+1)dx$ $\rightarrow \frac{dy}{dx} = \frac{-(y+1)}{x}$

Taking reciprocal

$\rightarrow \frac{dx}{dy} = \frac{-x}{y+1}$

(ii) $x^2 + 2y^2 = 16$

Solution:- $x^2 + 2y^2 = 16$

$\rightarrow d(x^2 + 2y^2) = d(16)$ $\rightarrow d(x^2) + 2d(y^2) = 0$

$\rightarrow 2x\,dx + 2(2y\,dy) = 0$ $\rightarrow 2x\,dx + 4y\,dy = 0$

$\rightarrow x\,dx + 2y\,dy = 0$ $\rightarrow 2y\,dy = -x\,dx$

$\rightarrow \frac{dy}{dx} = \frac{-x}{2y}$ Taking reciprocal

$\rightarrow \frac{dx}{dy} = -\frac{2y}{x}$

(iii) $x^4 + x^2 = xy^2$

Solution:- $x^4 + x^2 = xy^2$

$\rightarrow d(x^4 + x^2) = d(xy^2)$

$\rightarrow d(x^4) + d(x^2) = x\,d(y^2) + y^2 d(x)$

$\rightarrow 4x^3dx + 2ydy = x(2ydy) + y^2dx$

$\rightarrow 2ydy - 2xydy = y^2dx - 4x^3dx$

$\rightarrow (2y - 2xy)dy = (y^2 - 4x^3)dx$

$\rightarrow \dfrac{dy}{dx} = \dfrac{y^2 - 4x^3}{2(y - xy)}$

Taking reciprocal

$\rightarrow \dfrac{dx}{dy} = \dfrac{2(y - xy)}{y^2 - 4x^3} = \dfrac{2y(1-x)}{y^2 - 4x^3}$

(iv) $xy - \ln x = c$

Solution:- $xy - \ln x = c$

$\rightarrow d(xy - \ln x) = d(c)$

$\rightarrow d(xy) - d(\ln x) = 0$

$\rightarrow xdy + ydx - \frac{1}{x}dx = 0 \rightarrow xdy = \frac{1}{x}dx - ydx$

$\rightarrow xdy = (\frac{1}{x} - y)dx \rightarrow xdy = (\frac{1-xy}{x})dx$

$\rightarrow \dfrac{dy}{dx} = \dfrac{1-xy}{x^2}$ Taking reciprocal

$\rightarrow \dfrac{dx}{dy} = \dfrac{x^2}{1-xy}$

Q3. Use differentials to approximate the values of

(i) $\sqrt[4]{17}$

Solution:-

Let $f(x) = \sqrt[4]{x} = (x)^{\frac{1}{4}}$

$f'(x) = \dfrac{1}{4x^{\frac{3}{4}}}$

And $f(x + \delta x) = \sqrt[4]{x + \delta x}$

Take $x = 16$ & $dx = \delta x = 1$

$f(x) = (16)^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = 2$

$f'(x) = \dfrac{1}{4(16)^{\frac{3}{4}}} = \dfrac{1}{32} = 0.03125$

$f(x + \delta x) = \sqrt[4]{16 + 1} = \sqrt[4]{17}$

As we know $f(x + \delta x) \approx f(x) + f'(x)dx$

So $\sqrt[4]{17} \approx 2 + (0.03125)(1)$

$= 2.03125$

(ii) $(31)^{1/5}$

Solution:-

Let $f(x) = (x)^{\frac{1}{5}}$

$f'(x) = \dfrac{1}{5x^{\frac{4}{5}}}$

And $f(x + \delta x) = (x + \delta x)^{\frac{1}{5}}$

Take $x = 32$ & $dx = \delta x = -1$

$f(x) = (32)^{\frac{1}{5}} = (2^5)^{\frac{1}{5}} = 2$

$f'(x) = \dfrac{1}{5(32)^{\frac{4}{5}}} = \dfrac{1}{80} = 0.0125$

$f(x + \delta x) = (32 - 1)^{\frac{1}{5}} = (31)^{\frac{1}{5}}$

As we know $f(x + \delta x) \approx f(x) + f'(x)dx$

So $(31)^{\frac{1}{5}} \approx 2 + (0.0125)(-1)$

$= 1.9875$

(iii) $\cos 29^\circ$

Solution:-

Let $f(x) = cosx$

$f'(x) = -sinx$

And $f(x + \delta x) = cos(x + \delta x)$

Take $x = 30^\circ$ & $dx = \delta x = -1^\circ = -0.01745$

$f(x) = cos30^\circ = 0.866$

$f'(x) = -sin30^\circ = 0.5$

$f(x + \delta x) = sin(30 - 1^\circ) = cos29^\circ$

As we know $f(x + \delta x) \approx f(x) + f'(x)dx$

So $cos29^\circ \approx 0.866 + (-0.5)(-0.01745)$

$= 0.874725$

(iv) $\sin 61^\circ$

Solution:-

Let $f(x) = sinx$

$f'(x) = cosx$

And $f(x + \delta x) = sin(x + \delta x)$

Take $x = 60^\circ$ & $dx = \delta x = 1^\circ = 0.01745$

$f(x) = sin60^\circ = 0.866$

$f'(x) = cos60^\circ = 0.5$

$f(x + \delta x) = sin(60 + 1^\circ) = sin61^\circ$

As we know $f(x + \delta x) \approx f(x) + f'(x)dx$

So $sin61^\circ \approx 0.866 + (0.5)(0.01745)$

$= 0.874725$

Q4. Find the approximate increase in the volume of a cube if the length of its each edge changes from 5 to 5.02

Solution:- Let x be the length of each edge and v be the volume,

so Volume $= x \cdot x \cdot x$

$V = x^3 \rightarrow dv = d(x^3)$

$\rightarrow dv = 3x^2dx$ Take $x = 5$ and $dx = 0.02$

$\rightarrow dv = 3(5)^2(0.02) = 1.5$ cube units.

Q5. Find the approximate increase in the area of circular disc if its diameter is increased from 44cm to 44.4cm

Solution:- Diameter = 44cm

$\rightarrow$ radius $= r = \frac{44}{2} = 22$cm

change in diameter = 0.4cm

$\rightarrow$ change in radius $= dr = \frac{0.4}{2} = 0.2$ cm

$\because$ Area $= \pi r^2 \rightarrow A = \pi r^2$

$\rightarrow dA = \pi d(r^2) = \pi(2r)dr$

$\rightarrow dA = 2\pi r dr = 2(3.14)(22)(0.2)$

$\rightarrow dA = 27.646$ sq. units

Integration as Anti-derivative (Inverse of derivative)

The inverse process of differentiation i.e., the process of finding such a function whose derivative is given is called anti-differentiation or integration.

Consider $F(x)$ is anti-derivative of a function $f(x)$ if $F'(x) = f(x)$

then $\int f(x)dx = \int(F'(x))dx = \int(\frac{d}{dx}F(x))dx$

$\int f(x)dx = F(x) + c$

$\because \frac{d}{dx}$ and $\int dx$ are inverse operations of each other.

* The symbol $\int \dots dx$ indicates that integrand is to be integrated w.r.t 'x'.
* The antiderivative of a function is also called indefinite integral.
* The function which is to be integrated is called integrand of the integral.

Some Standard Formulae for Anti-Derivatives

$\int 1dx = x+c$, $\int x^n dx = \frac{x}{n+1} + c\ (n \neq -1)$

$\int \sin x dx = -\cos x + c$, $\int \cos x dx = \sin x + c$

$\int \sec^2 x dx = \tan x + c$, $\int \operatorname{cosec}^2 x dx = -\cot x + c$

$\int \sec x \tan x dx = \sec x + c$, $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$

$\int e^x dx = e^x + c$. $\int a^x dx = \frac{1}{\ln a} \cdot a^x + c\ (a \neq 0, a \neq 1)$

$\int \frac{1}{x} dx = \ln|x| + c, x \neq 0$, $\int \tan x dx = \ln|\sec x| + c = -\ln|\cos x| + c$

$\int \cot x dx = \ln|\sin x| + c$,

$\int \sec x dx = \ln|\sec x + \tan x| + c$

$\int \operatorname{cosec} x dx = \ln|\operatorname{cosec} x - \cot x| + c$

Here c is constant of integration.

These formulae can be varified by showing that the derivative of right hand side of each w.r.t 'x' is equal to the corresponding integrand.

Examples:

1. $\int x^5 dx$ $\qquad \because \int x^n dx = \frac{x^{n+1}}{n+1} + c$

$= \frac{x^{5+1}}{5+1} + c = \frac{x^6}{6} + c = \frac{1}{6} x^6 + c$

2. $\int \frac{1}{\sqrt{x^3}} dx$

$= \int \frac{1}{x^{3/2}} dx = \int x^{-3/2} dx = \frac{x^{-3/2+1}}{-\frac{3}{2}+1} + c = \frac{x^{-1/2}}{-1/2} + c$

$= -\frac{2}{\sqrt{x}} + c$

3. $\int \frac{1}{(2x+3)^4} dx$ $\qquad \because \int (ax+b)^n dx = \frac{1}{a} \cdot \frac{(ax+b)^{n+1}}{n+1} + c$

$= \int (2x+3)^{-4} dx$

$= \frac{1}{2} \cdot \frac{(2x+3)^{-4+1}}{-4+1} + c = \frac{-1}{6(2x+3)^3} + c$

4. $\int \cos 2x dx$ $\qquad \because \int \cos ax dx = \frac{\sin ax}{a} + c$

$= \frac{\sin 2x}{2} + c$

5. $\int \sin 3x dx$ $\qquad \because \int \sin ax dx = -\frac{\cos ax}{a} + c$

$= \frac{-\cos 3x}{3} + c$

6. $\int \operatorname{cosec}^2 x dx$

$= -\cot x + c$

7. $\int \sec 5x \tan 5x dx$ $\qquad \because \int \sec ax \tan ax dx = \frac{\sec ax}{a} + c$

$= \frac{\sec 5x}{5} + c$

8. $\int e^{ax+b} dx$ $\qquad \because \int e^{ax} dx = \frac{e^{ax}}{a} + c$

$= \frac{e^{ax+b}}{a} + c$

9. $\int 3^{\lambda x} dx$ $\qquad \because \int a^{bx} dx = \frac{a^{bx}}{b \ln a} + c$

$= \frac{3^{\lambda x}}{\lambda \ln 3} + c$

10. $\int \frac{1}{(ax+b)} dx$

$= \int (ax+b)^{-1} dx = \frac{1}{a} \ln(ax+b) + c$

Theorems on Anti-Derivatives

(1) $\int a f(x) dx = a \int f(x) dx$

(2) $\int [f_1(x) \pm f_2(x)] dx = \int f_1(x) dx + \int f_2(x) dx$

Prove that

$$\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c, (n \neq -1)$$

Proof:- we know that

$\frac{d}{dx} (f^{n+1}(x)) = (n+1) f^n(x) \cdot \frac{d}{dx} f(x)$

$\rightarrow \frac{d}{dx} (f^{n+1}(x)) = (n+1) f^n(x) \cdot f'(x)$

Taking integration,

$\int \frac{d}{dx} (f^{n+1}(x)) dx = (n+1) \int f^n(x) \cdot f'(x) dx$

$\rightarrow f^{n+1}(x) = (n+1) \int f^n(x) f'(x) dx$

$\rightarrow \int f^n(x) f'(x) dx = \frac{f^{n+1}(x)}{n+1} + c$ (By def.) ($\int f(x) dx = F(x) + c$)

Hence proved

Prove that

$$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$$

Proof:- we know that

$\frac{d}{dx} [\ln f(x)] = \frac{1}{f(x)} \cdot f'(x)$

Taking integration

$\int \frac{d}{dx} [\ln f(x)] dx = \int \frac{f'(x)}{f(x)} dx$

$\rightarrow \ln f(x) = \int \frac{f'(x)}{f(x)} dx$

$\rightarrow \int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$ (By definition) ($\int f(x) dx = F(x) + c$)

Hence proved

Examples: Evaluate

(i) $\int (x+1)(x-3)\,dx$

Solution:- $\int (x+1)(x-3)\,dx$

$= \int (x^2-3x+x-3)\,dx = \int (x^2-2x-3)\,dx$

$= \int x^2dx - 2\int x\,dx - 3\int dx = \dfrac{x^{2+1}}{2+1} - 2\cdot\dfrac{x^{1+1}}{1+1} - 3x + c$

$= \dfrac{x^3}{3} - 2\dfrac{x^2}{2} - 3x + c$

(ii) $\int x\sqrt{x^2-1}\,dx$

Solution:- $\int x\sqrt{x^2-1}\,dx$

$= \int (x^2-1)^{\frac{1}{2}}\cdot x\,dx = \frac{1}{2}\int (x^2-1)^{\frac{1}{2}}(2x)\,dx$

$= \dfrac{1}{2}\cdot\dfrac{(x^2-1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c \qquad \because \int f^n(x)f'(x)\,dx = \dfrac{f^{n+1}(x)}{n+1} + c$

$= \dfrac{1}{2}\ \dfrac{(x^2-1)^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{1}{2}\cdot\frac{2}{3}(x^2-1)^{\frac{3}{2}} + c = \frac{1}{3}(x^2-1)^{\frac{3}{2}} + c$

(iii) $\int \dfrac{x}{x+2}\,dx,\ (x>-2)$

Solution:- $\int \dfrac{x}{x+2}\,dx$

$= \int \left(1 - \dfrac{2}{x+2}\right)dx$

$$\begin{array}{r|l} & 1 \\ x+2 & x \\ & \underline{-x \pm 2} \\ & -2 \end{array}$$

$= \int 1\,dx - 2\int \dfrac{1}{x+2}\,dx$

$= x - 2\ln|x+2| + c \qquad \because \int \dfrac{f'(x)}{f(x)}\,dx = \ln f(x) + c$

(iv) $\int \dfrac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx$

Solution:- $\int \dfrac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx$

$= \int \left(\dfrac{1}{\sqrt{x}+1}\cdot\dfrac{1}{\sqrt{x}}\right)dx = 2\int \dfrac{1}{\sqrt{x}+1}\cdot\dfrac{1}{2\sqrt{x}}\,dx$

$= 2\int \dfrac{\frac{1}{2\sqrt{x}}}{\sqrt{x}+1}\,dx = 2\ln(\sqrt{x}+1) + c$

(v) $\int \dfrac{dx}{\sqrt{x+1}-\sqrt{x}}$

Solution:- $\int \dfrac{dx}{\sqrt{x+1}-\sqrt{x}}$

$= \int \dfrac{dx}{\sqrt{x+1}-\sqrt{x}} \times \dfrac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}$

$= \int \dfrac{\sqrt{x+1}+\sqrt{x}}{(\sqrt{x+1})^2-(\sqrt{x})^2}\,dx = \int \dfrac{(x+1)^{\frac{1}{2}}+(x)^{\frac{1}{2}}}{x+1-x}\,dx$

$= \int (x+1)^{\frac{1}{2}}dx + \int x^{\frac{1}{2}}dx$

$= \dfrac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} + \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + c \qquad \because \int f^n(x)f'(x)\,dx$

$= \frac{2}{3}(x+1)^{\frac{3}{2}} + \frac{2}{3}x^{\frac{3}{2}} + c \qquad = \dfrac{f^{n+1}(x)}{n+1} + c$

(vi) $\int \dfrac{\sin x + \cos^3 x}{\cos^2 x\sin x}\,dx$

Solution:- $\int \dfrac{\sin x + \cos^3 x}{\cos^2 x\sin x}\,dx$

$= \int \dfrac{\sin x}{\cos^2 x\sin x}\,dx + \int \dfrac{\cos^3 x}{\cos^2 x\sin x}\,dx$

$= \int \dfrac{1}{\cos^2 x}\,dx + \int \dfrac{\cos x}{\sin x}\,dx = \int \sec^2 x\,dx + \int \cot x\,dx$

$= \tan x + \ln|\sin x| + c$

(vii) $\int \dfrac{3-\cos 2x}{1+\cos 2x}\,dx,\ (\cos 2x \neq -1)$

Solution:- $\int \dfrac{3-\cos 2x}{1+\cos 2x}\,dx$

$= \int \left(-1 + \dfrac{4}{1+\cos 2x}\right)dx$

$$\begin{array}{r|l} & -1 \\ 1+\cos 2x & 3-\cos 2x \\ & \underline{\overset{+}{-1} \overset{+}{-} \cos 2x} \\ & 4 \end{array}$$

$= -\int dx + 4\int \dfrac{1}{1+\cos 2x}\,dx$

$= -x + \overset{2}{\cancel{4}}\int \dfrac{1}{\cancel{2}\cos^2 x}\,dx \qquad \because 1+\cos 2x = 2\cos^2 x$

$= -x + 2\int \sec^2 x\,dx$

$= -x + 2\tan x + c \qquad \because \int \sec^2 x\,dx = \tan x + c$

Exercise 3.2

Q1. Evaluate the following indefinite integrals

(i) $\int (3x^2-2x+1)\,dx$

Solution:- $\int (3x^2-2x+1)\,dx$

$= 3\int x^2dx - 2\int x\,dx + \int 1\,dx$

$= 3\cdot\dfrac{x^3}{3} - 2\cdot\dfrac{x^2}{2} + x + c = x^3 - x^2 + x + c$

(ii) $\int \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)dx$

Solution:- $\int \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)dx \quad ,\ (x>0)$

$= \int \sqrt{x}\,dx + \int \dfrac{1}{\sqrt{x}}\,dx = \int x^{\frac{1}{2}}dx + \int x^{-\frac{1}{2}}dx$

$= \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \dfrac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c = \frac{2}{3}x^{\frac{3}{2}} + 2\sqrt{x} + c$

(iii) $\int x(\sqrt{x}+1)\,dx \quad ,\ (x>0)$

Solution:- $\int x(x^{\frac{1}{2}}+1)\,dx$

$= \int (x\cdot x^{\frac{1}{2}} + x)\,dx = \int (x^{\frac{3}{2}} + x)\,dx$

$= \int x^{\frac{3}{2}}dx + \int x\,dx = \dfrac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \dfrac{x^2}{2} + c$

$= \dfrac{x^{\frac{5}{2}}}{\frac{5}{2}} + \dfrac{x^2}{2} + c = \frac{2}{5}x^{\frac{5}{2}} + \dfrac{x^2}{2} + c$

(iv) $\int (2x+3)^{\frac{1}{2}} dx$

Solution:- $\int (2x+3)^{\frac{1}{2}} dx$

$= \frac{1}{2}\int (2x+3)^{\frac{1}{2}} \cdot 2dx \qquad \because \int f^{n}(x) f'(x) dx = \frac{f^{n+1}(x)}{n+1} + c$

$= \frac{1}{2} \frac{(2x+3)^{\frac{1}{2}+1}}{\frac{1}{3}+1} + c$

$= \frac{1}{2} \frac{(2x+3)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{1}{2}\cdot\frac{2}{3}(2x+3)^{\frac{3}{2}} + c$

$= \frac{1}{3}(2x+3)^{\frac{3}{2}} + c$

(v) $\int (\sqrt{x}+1)^2 dx \quad , (x>0)$

Solution:- $\int (\sqrt{x}+1)^2 dx$

$= \int ((\sqrt{x})^2 + (1)^2 + 2(\sqrt{x})(1)) dx$

$= \int (x + 1 + 2\sqrt{x}) dx = \int x dx + \int dx + 2\int x^{\frac{1}{2}} dx$

$= \frac{x^2}{2} + x + 2\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{x^2}{2} + x + \frac{4}{3}x^{\frac{3}{2}} + c$

(vi) $\int (\sqrt{x} - \frac{1}{\sqrt{x}})^2 dx \quad , (x>0)$

Solution:- $\int (\sqrt{x} - \frac{1}{\sqrt{x}})^2 dx$

$= \int ((\sqrt{x})^2 + (\frac{1}{\sqrt{x}})^2 - 2(\sqrt{x})(\frac{1}{\sqrt{x}})) dx$

$= \int (x + \frac{1}{x} - 2) dx = \int x dx + \int \frac{1}{x} dx - 2\int dx$

$= \frac{x^2}{2} + \ln x - 2x + c$

(vii) $\int \frac{3x+2}{\sqrt{x}} dx \quad , (x>0)$

Solution:- $\int \frac{3x+2}{\sqrt{x}} dx$

$= \int (\frac{3x}{\sqrt{x}} + \frac{2}{\sqrt{x}}) dx = \int (\frac{3\sqrt{x}\sqrt{x}}{\sqrt{x}} + \frac{2}{\sqrt{x}}) dx$

$= \int (3x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}) dx = 3\int x^{\frac{1}{2}} dx + 2\int x^{-\frac{1}{2}} dx$

$= 3\cdot\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 2\cdot\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = 3\cdot\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 2\cdot\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 3\cdot\frac{2}{3}x^{\frac{3}{2}} + 2\cdot\frac{2}{1}x^{\frac{1}{2}} + c = 2x^{\frac{3}{2}} + 4\sqrt{x} + c$

(viii) $\int \frac{\sqrt{y}(y+1)}{y} dy \quad , (y>0)$

Solution:- $\int \frac{\sqrt{y}(y+1)}{y} dy$

$= \int \frac{\cancel{\sqrt{y}}(y+1)}{\cancel{\sqrt{y}}\sqrt{y}} dy = \int (\frac{y+1}{\sqrt{y}}) dy$

$= \int (\frac{y}{\sqrt{y}} + \frac{1}{\sqrt{y}}) dy = \int (\frac{\cancel{\sqrt{y}}\sqrt{y}}{\cancel{\sqrt{y}}} + \frac{1}{\sqrt{y}}) dy$

$= \int (y^{\frac{1}{2}} + y^{-\frac{1}{2}}) dy = \int y^{\frac{1}{2}} dy + \int y^{-\frac{1}{2}} dy$

$= \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + \frac{y^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}y^{\frac{3}{2}} + 2y^{\frac{1}{2}} + c$

$= \frac{2}{3}y^{\frac{3}{2}} + 2\sqrt{y} + c$

(ix) $\int \frac{(\sqrt{\theta}-1)^2}{\sqrt{\theta}} d\theta \quad , (\theta > 0)$

Solution:- $\int \frac{(\sqrt{\theta}-1)^2}{\sqrt{\theta}} d\theta$

$= \int \frac{((\sqrt{\theta})^2 + (1)^2 - 2(\sqrt{\theta})(1))}{\sqrt{\theta}} d\theta$

$= \int (\frac{\theta + 1 - 2\sqrt{\theta}}{\sqrt{\theta}}) d\theta = \int (\frac{\theta}{\sqrt{\theta}} + \frac{1}{\sqrt{\theta}} - \frac{2\sqrt{\theta}}{\sqrt{\theta}}) d\theta$

$= \int (\sqrt{\theta} + \theta^{-\frac{1}{2}} - 2) d\theta = \int \theta^{\frac{1}{2}} d\theta + \int \theta^{-\frac{1}{2}} d\theta - 2\int d\theta$

$= \frac{\theta^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{\theta^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - 2\theta + c = \frac{\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{\theta^{\frac{1}{2}}}{\frac{1}{2}} - 2\theta + c$

$= \frac{2}{3}\theta^{\frac{3}{2}} + 2\theta^{\frac{1}{2}} - 2\theta + c$

(x) $\int \frac{(1-\sqrt{x})^2}{\sqrt{x}} dx, \ (x>0)$

Solution:- $\int \frac{(1-\sqrt{x})^2}{\sqrt{x}} dx \quad , (x>0)$

$= \int \frac{((1)^2 + (\sqrt{x})^2 - 2(1)(\sqrt{x}))}{\sqrt{x}} dx$

$= \int (\frac{1 + x - 2\sqrt{x}}{\sqrt{x}}) dx = \int (\frac{1}{\sqrt{x}} + \frac{x}{\sqrt{x}} - \frac{2\sqrt{x}}{\sqrt{x}}) dx$

$= \int (x^{-\frac{1}{2}} + x^{\frac{1}{2}} - 2) dx = \int x^{-\frac{1}{2}} dx + \int x^{\frac{1}{2}} dx - 2\int dx$

$= \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 2x + c = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 2x + c$

$= 2x^{\frac{1}{2}} + \frac{2}{3}x^{\frac{3}{2}} - 2x + c$

(xi) $\int \frac{e^{2x} + e^{x}}{e^{x}} dx$

Solution:- $\int \frac{e^{2x} + e^{x}}{e^{x}} dx$

$= \int (\frac{e^{2x}}{e^{x}} + \frac{e^{x}}{e^{x}}) dx = \int (e^{x} + 1) dx$

$= \int e^{x} dx + \int dx = e^{x} + x + c$

Q2. Evaluate

(i) $\int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}} \quad \begin{pmatrix} x+a > 0 \\ x+b > 0 \end{pmatrix}$

Solution:- $\int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}}$

$= \int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}} \times \frac{\sqrt{x+a} - \sqrt{x+b}}{\sqrt{x+a} - \sqrt{x+b}}$

$= \int (\frac{\sqrt{x+a} - \sqrt{x+b}}{(\sqrt{x+a})^2 - (\sqrt{x+b})^2}) dx$

$= \int (\frac{(x+a)^{\frac{1}{2}} - (x+b)^{\frac{1}{2}}}{x+a-x-b}) dx = \int (\frac{(x+a)^{\frac{1}{2}} - (x+b)^{\frac{1}{2}}}{a-b}) dx$

$= \frac{1}{a-b}\left[\int (x+a)^{\frac{1}{2}} dx - \int (x+b)^{\frac{1}{2}} dx\right]$

$$= \frac{1}{a-b}\left\{\int (x+a)^{\frac{1}{2}}dx - \int (x+b)^{\frac{1}{2}}dx\right\}$$

$$= \frac{1}{a-b}\left\{\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right\} + c$$

$$= \frac{1}{a-b}\left\{\frac{2}{3}(x+a)^{\frac{3}{2}} - \frac{2}{3}(x+b)^{\frac{3}{2}}\right\} + c$$

$$= \frac{2}{3(a-b)}\left\{(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right\} + c$$

(ii) $\int \frac{1-x^2}{1+x^2}dx$

Solution:- $\int \frac{1-x^2}{1+x^2}dx$

$$= \int\left(-1 + \frac{2}{1+x^2}\right)dx$$

$$\begin{array}{r|l} & -1 \\ \hline 1+x^2 & 1-x^2 \\ & -1-x^2 \\ & + \quad + \\ \hline & 2 \end{array}$$

$$= -\int 1\,dx + 2\int \frac{1}{1+x^2}dx$$

$$= -x + 2\,\text{Tan}^{-1}x + c$$

$\because \int \frac{1}{1+x^2}dx = \text{Tan}^{-1}x + c$

(iii) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$, $(x>0,\ a>0)$

Solution:- $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$

$$= \int \frac{dx}{\sqrt{x+a}+\sqrt{x}} \times \frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}$$

$$= \int\left(\frac{\sqrt{x+a}-\sqrt{x}}{(\sqrt{x+a})^2-(\sqrt{x})^2}\right)dx = \int\left(\frac{(x+a)^{\frac{1}{2}}-x^{\frac{1}{2}}}{\cancel{x}+a-\cancel{x}}\right)dx$$

$$= \frac{1}{a}\int\left((x+a)^{\frac{1}{2}} - x^{\frac{1}{2}}\right)dx = \frac{1}{a}\left\{\int (x+a)^{\frac{1}{2}}dx - \int x^{\frac{1}{2}}dx\right\}$$

$$= \frac{1}{a}\left\{\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right\} + c$$

$$= \frac{1}{a}\left\{\frac{2}{3}(x+a)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right\} + c = \frac{2}{3a}\left\{(x+a)^{\frac{3}{2}} - x^{\frac{3}{2}}\right\} + c$$

(iv) $\int (a-2x)^{\frac{3}{2}}dx$

Solution:- $\int (a-2x)^{\frac{3}{2}}dx$

$$= -\frac{1}{2}\int (a-2x)^{\frac{3}{2}}\cdot(-2)\,dx$$

$$= -\frac{1}{2}\,\frac{(a-2x)^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$$

$\because \int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + c$

$$= -\frac{1}{2}\,\frac{(a-2x)^{\frac{5}{2}}}{\frac{5}{2}} + c = -\frac{1}{2}\times\frac{2}{5}(a-2x)^{\frac{5}{2}} + c$$

$$= -\frac{1}{5}(a-2x)^{\frac{5}{2}} + c$$

(v) $\int \frac{(1+e^x)^3}{e^x}dx$

Solution:- $\int \frac{(1+e^x)^3}{e^x}dx$

$$= \int \frac{1+e^{3x}+3e^x+3e^{2x}}{e^x}dx$$

$\because (a+b)^3 = a^3+b^3+3a^2b+3ab^2$

$\rightarrow (1+e^x)^3 = (1)^3+(e^x)^3+3(1)^2e^x+3(1)(e^x)^2$

$= 1+e^{3x}+3e^x+3e^{2x}$

$$= \int\left(\frac{1}{e^x} + \frac{e^{3x}}{e^x} + \frac{3e^x}{e^x} + \frac{3e^{2x}}{e^x}\right)dx$$

$$= \int e^{-x}dx + \int e^{2x}dx + 3\int dx + 3\int e^x dx$$

$$= \frac{e^{-x}}{-1} + \frac{e^{2x}}{2} + 3x + 3e^x + c$$

$$= -e^{-x} + \frac{1}{2}e^{2x} + 3x + 3e^x + c$$

(vi) $\int \sin(a+b)x\,dx$

Solution:- $\int \sin(a+b)x\,dx$

$$= -\frac{\cos(a+b)x}{a+b} + c$$

$\because \int \sin ax\,dx = -\frac{\cos ax}{a} + c$

(vii) $\int \sqrt{1-\cos 2x}\,dx$, $(1-\cos 2x > 0)$

Solution:- $\int \sqrt{1-\cos 2x}\,dx$

$$= \int \sqrt{2\sin^2 x}\,dx$$

$$= \sqrt{2}\int \sin x\,dx$$

$$= \sqrt{2}(-\cos x) + c$$

$$= -\sqrt{2}\cos x + c$$

$\because \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$

$\rightarrow \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$

$\rightarrow 2\sin^2\frac{\theta}{2} = 1-\cos\theta$

$\rightarrow 2\sin^2\theta = 1-\cos 2\theta$

(viii) $\int (\ln x)\cdot\frac{1}{x}dx$

Solution:- $\int (\ln x)^1\cdot\frac{1}{x}dx$ $(x>0)$

$$= \frac{(\ln x)^2}{2} + c$$

$\because \int f^n(x)\cdot f'(x)\,dx = \frac{f^{n+1}(x)}{n+1} + c$

(ix) $\int \sin^2 x\,dx$

Solution:- $\int \sin^2 x\,dx$

$$= \int \frac{1-\cos 2x}{2}dx$$

$$= \frac{1}{2}\int (1-\cos 2x)\,dx$$

$$= \frac{1}{2}\left\{\int dx - \int \cos 2x\,dx\right\}$$

$$= \frac{1}{2}\left\{x - \frac{\sin 2x}{2}\right\} + c = \frac{x}{2} - \frac{\sin 2x}{4} + c$$

$\because \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$

$\rightarrow \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$

$\rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$

(x) $\int \frac{1}{1+\cos x}dx$, $\left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$

Solution:- $\int \frac{1}{1+\cos x}dx$

$$= \int \frac{1}{2\cos^2\frac{x}{2}}dx$$

$$= \frac{1}{2}\int \sec^2\frac{x}{2}\,dx$$

$$= \frac{1}{2}\,\frac{\tan\frac{x}{2}}{\frac{1}{2}} + c = \frac{1}{\cancel{2}}\times\frac{\cancel{2}}{1}\tan\frac{x}{2} + c$$

$$= \tan\frac{x}{2} + c$$

$\because \cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$

$\rightarrow \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$

$\rightarrow 2\cos^2\frac{\theta}{2} = 1+\cos\theta$

(xi) $\int \frac{ax+b}{ax^2+2bx+c}\,dx$

Solution:- $\int \frac{ax+b}{ax^2+2bx+c}\,dx$

$= \frac{1}{2}\int \frac{2ax+2b}{ax^2+2bx+c}\,dx \qquad \because \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

$= \frac{1}{2}\ln|ax^2+2bx+c| + c$

(xii) $\int \cos 3x \sin 2x\,dx$

Solution:- $\int \cos 3x \sin 2x\,dx$

$= \frac{1}{2}\int 2\cos 3x \sin 2x\,dx$

$\left(\because 2\cos\alpha \sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)\right)$

$= \frac{1}{2}\int \left(\sin(3x+2x) - \sin(3x-2x)\right)dx$

$= \frac{1}{2}\int (\sin 5x - \sin x)\,dx$

$= \frac{1}{2}\left(\int \sin 5x\,dx - \int \sin x\,dx\right)$

$= \frac{1}{2}\left(-\frac{\cos 5x}{5} - (-\cos x)\right) + c$

$= \frac{1}{2}\left(-\frac{1}{5}\cos 5x + \cos x\right) + c$

$= -\frac{1}{2}\left(\frac{\cos 5x}{5} - \cos x\right) + c$

(xiii) $\int \frac{\cos 2x - 1}{1+\cos 2x}\,dx$, $(1+\cos 2x \neq 0)$

Solution:- $\int \frac{\cos 2x - 1}{1+\cos 2x}\,dx$

$= \int \left(1 - \frac{2}{1+\cos 2x}\right)dx$

$$1+\cos 2x \,\overline{\smash{)}\,\cos 2x - 1}\overset{1}{} \qquad \begin{array}{r} \cos 2x - 1 \\ \underline{\overset{-}{\cos 2x} \overset{-}{+} 1} \\ -2 \end{array}$$

$= \int 1\,dx - 2\int \frac{1}{1+\cos 2x}\,dx \qquad \left(\begin{array}{l} \because 1+\cos 2x = 2\cos^2 x \\ 1-\cos 2x = 2\sin^2 x \end{array}\right.$

$= x - \cancel{2}\int \frac{1}{\cancel{2}\cos^2 x}\,dx$

$= x - \int \sec^2 x\,dx = x - \tan x + c$

(xiv) $\int \tan^2 x\,dx$

Solution:- $\int (\tan^2 x)\,dx$

$= \int (\sec^2 x - 1)\,dx \qquad \because 1+\tan^2\theta = \sec^2\theta \rightarrow \tan^2\theta = \sec^2\theta - 1$

$= \int \sec^2 x\,dx - \int 1\,dx$

$= \tan x - x + c$

Integration by method of substitution

sometimes it is possible to convert an integral into standard form by a suitable change of a variable. This is called substitution method.

i-e., Evaluate $\int f(x)\,dx$ by method of substitution. Let $x = \phi(t) \rightarrow dx = \phi'(t)\,dt$

so $\int f(x)\,dx = \int f(\phi(t))\,\phi'(t)\,dt$

Example 1. Evaluate $\int \frac{a\,dt}{2\sqrt{at+b}}$, $(at+b>0)$

Solution:- $\int \frac{a\,dt}{2\sqrt{at+b}}$

Let $at+b = u \rightarrow a\,dt = du$

$= \int \frac{du}{2\sqrt{u}} = \frac{1}{2}\int u^{-\frac{1}{2}}\,du = \frac{1}{2}\cdot \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{1}{2}\cdot \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{1}{2}\times\frac{2}{1}u^{\frac{1}{2}} + c$

$= \sqrt{u} + c = \sqrt{at+b} + c \qquad \because u = at+b$

Example 2. Evaluate $\int \frac{x}{\sqrt{4+x^2}}\,dx$

Solution:- $\int \frac{x}{\sqrt{4+x^2}}\,dx$

Put $4+x^2 = t \rightarrow 2x\,dx = dt$

$\rightarrow x\,dx = \frac{1}{2}dt$

$= \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\int t^{-\frac{1}{2}}\,dt = \frac{1}{2}\cdot\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \frac{1}{2}\,\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{1}{\cancel{2}}\times\frac{\cancel{2}}{1}\sqrt{t} + c = \sqrt{4+x^2} + c$

Example 3. Evaluate $\int x\sqrt{x-a}\,dx$ $(x>a)$

Solution:- $\int x\sqrt{x-a}\,dx$

Put $x-a = t \rightarrow x = a+t$

$\rightarrow dx = dt$

$= \int (a+t)\sqrt{t}\,dt = \int \left(a t^{\frac{1}{2}} + t^{\frac{3}{2}}\right)dt$

$= a\int t^{\frac{1}{2}}\,dt + \int t^{\frac{3}{2}}\,dt = a\cdot\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{t^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$

$= a\,\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{5}{2}}}{\frac{5}{2}} + c = \frac{2}{3}a t^{\frac{3}{2}} + \frac{2}{5}t^{\frac{5}{2}} + c$

$= \frac{2}{3}a\,t^{\frac{3}{2}} + \frac{2}{5}t\cdot t^{\frac{3}{2}} = 2t^{\frac{3}{2}}\left(\frac{a}{3} + \frac{t}{5}\right) + c$

$= 2t^{\frac{3}{2}}\left(\frac{5a+3t}{15}\right) + c = \frac{2}{15}(x-a)^{\frac{3}{2}}\left(5a+3(x-a)\right) + c$

$= \frac{2}{15}(x-a)^{\frac{3}{2}}(5a+3x-3a) + c$

$= \frac{2}{15}(x-a)^{\frac{3}{2}}(2a+3x) + c$

Example 4. $\int \frac{\cot\sqrt{x}}{\sqrt{x}}dx, \quad (x>0)$

Solution:- $\int \cot\sqrt{x}\cdot\frac{1}{\sqrt{x}}dx$

Put $\sqrt{x} = z \rightarrow x^{\frac{1}{2}} = z$

$\rightarrow d(x^{\frac{1}{2}}) = dz \rightarrow \frac{1}{2}x^{-\frac{1}{2}}dx = dz$

$\rightarrow \frac{1}{2\sqrt{x}}dx = dz \rightarrow \frac{1}{\sqrt{x}}dx = 2dz$

$= \int \cot z(2dz) = 2\int \cot z\,dz$

$\because \int \cot x\,dx = \ln|\sin x| + c$

$= 2\ln|\sin z| + c$

$= 2\ln|\sin\sqrt{x}| + c$

Example 5. Evaluate (i) $\int \operatorname{cosec}x\,dx$ (ii) $\int \sec x\,dx$

Solution:- (i) $\int \operatorname{cosec}x\,dx$

$= \int \frac{\operatorname{cosec}x(\operatorname{cosec}x-\cot x)}{\operatorname{cosec}x-\cot x}dx$ ('x'ing and ÷ing by $\operatorname{cosec}x-\cot x$)

$= \int \frac{(\operatorname{cosec}^2x - \operatorname{cosec}x\cot x)}{\operatorname{cosec}x-\cot x}dx$

Put $\operatorname{cosec}x - \cot x = t$

$\rightarrow d(\operatorname{cosec}x - \cot x) = dt$

$\rightarrow (-\operatorname{cosec}x\cot x - (-\operatorname{cosec}^2x))dx = dt$

$\rightarrow \operatorname{cosec}^2x - \operatorname{cosec}x\cot x = dt$

$= \int \frac{dt}{t} = \ln|t| + c = \ln|\operatorname{cosec}x - \cot x| + c$

(ii) $\int \sec x\,dx$

$= \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}dx$ ('x'ing and ÷ing by $(\sec x+\tan x)$)

$= \int \frac{(\sec^2x + \sec x\tan x)}{\sec x+\tan x}dx$

Put $\sec x + \tan x = t$

$\rightarrow d(\sec x+\tan x) = dt$

$\rightarrow (\sec x\tan x + \sec^2x)dx = dt$

$= \int \frac{dt}{t} = \ln|t| + c = \ln|\sec x+\tan x| + c$

Example 6. Evaluate $\int \cos^3x\sqrt{\sin x}\,dx$, $(\sin x>0)$

Solution:- $\int \cos^3x\sqrt{\sin x}\,dx$

$= \int \cos^2x\sqrt{\sin x}\cos x\,dx$

$\because \cos^2\theta + \sin^2\theta = 1$
$\rightarrow \cos^2\theta = 1 - \sin^2\theta$

$= \int (1-\sin^2x)\sqrt{\sin x}\cos x\,dx$

Put $\sin x = t \rightarrow d(\sin x) = dt$

$\rightarrow \cos x\,dx = dt$

$= \int (1-t^2)\sqrt{t}\,dt = \int (1-t^2)t^{\frac{1}{2}}dt$

$= \int (t^{\frac{1}{2}} - t^{2}\cdot t^{\frac{1}{2}})dt = \int (t^{\frac{1}{2}} - t^{\frac{5}{2}})dt$

$= \int t^{\frac{1}{2}}dt - \int t^{\frac{5}{2}}dt = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \frac{t^{\frac{7}{2}}}{\frac{7}{2}} + c$

$= \frac{2}{3}t^{\frac{3}{2}} - \frac{2}{7}t^{\frac{7}{2}} + c$

$= \frac{2}{3}\sin^{\frac{3}{2}}x - \frac{2}{7}\sin^{\frac{7}{2}}x + c$

Example 7. Evaluate $\int \sqrt{1+\sin x}\,dx$, $(-\frac{\pi}{2} < x < \frac{\pi}{2})$

Solution:- $\int \sqrt{1+\sin x}\,dx$

$= \int \sqrt{1+\sin x} \times \frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}dx$ ('x' and '÷' by $\sqrt{1-\sin x}$)

$= \int \frac{\sqrt{1-\sin^2x}}{\sqrt{1-\sin x}}dx = \int \frac{\cos x}{\sqrt{1-\sin x}}dx$ $\quad \because \cos x = \sqrt{1-\sin^2x}$

Put $1-\sin x = t \rightarrow d(1-\sin x) = dt$

$\rightarrow -\cos x\,dx = dt \rightarrow \cos x\,dx = -dt$

$= \int \frac{-dt}{\sqrt{t}} = -\int t^{-\frac{1}{2}}dt = -\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= -\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = -2t^{\frac{1}{2}} + c = -2\sqrt{1-\sin x} + c$

Example 8. Find $\int \frac{dx}{x(\ln 2x)^3}$, $(x>0)$

Solution:- $\int \frac{dx}{x(\ln 2x)^3}$

$= \int (\ln 2x)^{-3}\cdot\frac{1}{x}dx$

Put $\ln 2x = t$

$\rightarrow d(\ln 2x) = dt \rightarrow \frac{1}{2x}dx = dt$

$\rightarrow \frac{1}{x}dx = 2dt$

$= \int t^{-3}dt = \frac{t^{-3+1}}{-3+1} + c = \frac{t^{-2}}{-2} + c = -\frac{1}{2t^2} + c$

$= -\frac{1}{2(\ln 2x)^2} + c$

Example 9. Find $\int a^{x^2}x\,dx$, $(a>0, a\neq 1)$

Solution:- $\int a^{x^2}x\,dx$

Put $x^2 = t \rightarrow 2x\,dx = dt$

$\rightarrow x\,dx = \frac{1}{2}dt$

$\because \int a^x dx = \frac{a^x}{\ln a} + c$

$= \frac{1}{2}\int a^t dt = \frac{1}{2}\cdot\frac{a^t}{\ln a} + c$

$= \frac{1}{2}\cdot\frac{a^{x^2}}{\ln a} + c$

Some useful substitutions

Expression involving	suitable substitution
(i) $\sqrt{a^2-x^2}$	Put $x = a\sin\theta$ $(\because 1-\sin^2\theta = \cos^2\theta)$
(ii) $\sqrt{x^2-a^2}$	Put $x = a\sec\theta$ $(\because \sec^2\theta - 1 = \tan^2\theta)$
(iii) $\sqrt{a^2+x^2}$	Put $x = a\tan\theta$ $(\because 1+\tan^2\theta = \sec^2\theta)$
(iv) $\sqrt{x+a}$ (or $\sqrt{x-a}$)	Put $\sqrt{x+a} = t$ (or $\sqrt{x-a} = t$)
(v) $\sqrt{2ax-x^2}$	Put $x-a = a\sin\theta$
(vi) $\sqrt{2ax+x^2}$	Put $x+a = a\sec\theta$

Example 10. Evaluate

(i) $\int \frac{1}{\sqrt{a^2-x^2}}dx$, $(-a<x<a)$

Solution:- $\int \frac{1}{\sqrt{a^2-x^2}}dx$

Put $x = a\sin\theta$ for $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$dx = a\cos\theta d\theta$

$$= \int \frac{a\cos\theta d\theta}{\sqrt{a^2-(a\sin\theta)^2}} = a\int \frac{\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$$

$$= a\int \frac{\cos\theta\, d\theta}{\sqrt{a^2(1-\sin^2\theta)}} = \frac{a}{a}\int \frac{\cos\theta d\theta}{\sqrt{\cos^2\theta}}$$

$$= \int \frac{\cos\theta}{\cos\theta}d\theta = \int 1 d\theta = \theta + c$$

$$= \sin^{-1}\frac{x}{a} + c \qquad \because x = a\sin\theta \rightarrow \frac{x}{a} = \sin\theta \rightarrow \sin^{-1}\frac{x}{a} = \theta$$

(ii) $\int \frac{1}{x\sqrt{x^2-a^2}}dx$, $(x>a$ or $x<-a)$

Solution:- $\int \frac{1}{x\sqrt{x^2-a^2}}dx$

Put $x = a\sec\theta$ for $0<\theta<\frac{\pi}{2}$ or $\frac{\pi}{2}<\theta<\pi$

$dx = a\sec\theta\tan\theta d\theta$

$$= \int \frac{\cancel{a}\cancel{\sec\theta}\tan\theta d\theta}{(\cancel{a}\cancel{\sec\theta})\sqrt{(a\sec\theta)^2-a^2}}$$

$$= \int \frac{\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}}d\theta$$

$$= \int \frac{\tan\theta}{\sqrt{a^2(\sec^2\theta-1)}}d\theta = \frac{1}{a}\int \frac{\tan\theta d\theta}{\sqrt{\tan^2\theta}}$$

$$= \frac{1}{a}\int \frac{\tan\theta}{\tan\theta}d\theta = \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$$

$(\because \frac{x}{a} = \sec\theta)$

Example 1. Evaluate $\int \frac{1}{\sqrt{a^2+x^2}}dx$, $(a>0)$

Solution:- $\int \frac{1}{\sqrt{a^2+x^2}}dx$

Put $x = a\tan\theta$

$\rightarrow dx = a\,d(\tan\theta) = a\sec^2\theta d\theta$

$$= \int \frac{a\sec^2\theta d\theta}{\sqrt{a^2+(a\tan\theta)^2}} = \int \frac{a\sec^2\theta d\theta}{\sqrt{a^2(1+\tan^2\theta)}}$$

$$= \frac{\cancel{a}}{\cancel{a}}\int \frac{\sec^2\theta\, d\theta}{\sqrt{\sec^2\theta}} = \int \frac{\sec^2\theta d\theta}{\sec\theta}$$

$$= \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + c_1 \xrightarrow{\quad} (I)$$

$\because x = a\tan\theta \rightarrow \tan\theta = \frac{x}{a}$

and $1+\tan^2\theta = \sec^2\theta$

$\rightarrow 1 + (\frac{x}{a})^2 = \sec^2\theta$

$\rightarrow \sec\theta = \sqrt{1+\frac{x^2}{a^2}} = \sqrt{\frac{a^2+x^2}{a^2}}$

$\rightarrow \sec\theta = \frac{\sqrt{a^2+x^2}}{a}$

so by (I) $= \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + c_1$

$= \ln\left|\frac{\sqrt{a^2+x^2}+x}{a}\right| + c_1 = \ln|\sqrt{a^2+x^2}+x| - \ln a + c_1$

$= \ln|\sqrt{a^2+x^2} - x| + c \qquad \because c = c_1 - \ln a$

Example 2. Evaluate $\int \frac{dx}{\sqrt{2x+x^2}}$, $(x>0)$

Solution:- $\int \frac{dx}{\sqrt{2x+x^2}}$

$$= \int \frac{dx}{\sqrt{x^2+2x+1-1}} = \int \frac{dx}{\sqrt{(x+1)^2-(1)^2}}$$

Put $x+1 = \sec\theta$ $(0<\theta<\frac{\pi}{2})$

$\rightarrow dx = \sec\theta\tan\theta d\theta$

$$= \int \frac{\sec\theta\tan\theta d\theta}{\sqrt{\sec^2\theta - 1}} = \int \frac{\sec\theta\tan\theta d\theta}{\sqrt{\tan^2\theta}}$$

$$= \int \frac{\sec\theta\tan\theta}{\tan\theta}d\theta = \int \sec\theta d\theta$$

$= \ln|\sec\theta + \tan\theta| + c$

$= \ln|(x+1) + \sqrt{x^2+2x}| + c$

$\because x+1 = \sec\theta$

$\tan^2\theta = \sec^2\theta - 1$

$= (x+1)^2 - 1$

$\tan^2\theta = x^2+2x+1-1$

$\rightarrow \tan\theta = \sqrt{x^2+2x}$

Exercise 3.3

Evaluate the following integrals:

Q1. $\int \frac{-2x}{\sqrt{4-x^2}}dx$

Solution:- $\int \frac{-2x}{\sqrt{4-x^2}}dx = \int (4-x^2)^{-\frac{1}{2}}(-2x)dx$

Put $4-x^2 = t \rightarrow -2x\,dx = dt$

$= \int t^{-\frac{1}{2}}dt = \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c = \frac{t^{\frac{1}{2}}}{\frac{1}{2}}+c = 2\sqrt{t}+c$

$= 2\sqrt{4-x^2}+c \quad (\because t = 4-x^2)$

Q2. $\int \frac{dx}{x^2+4x+13}$

Solution:- $\int \frac{dx}{x^2+4x+13}$

$= \int \frac{dx}{x+4x+4+9} = \int \frac{dx}{(x+2)^2+(3)^2}$

Put $x+2 = 3\tan\theta$ $\quad \because d(\tan\theta) = \sec^2\theta d\theta$

$\rightarrow dx = 3\sec^2\theta d\theta$

$= \int \frac{3\sec^2\theta d\theta}{(3\tan\theta)^2+(3)^2} = \int \frac{3\sec^2\theta d\theta}{9\tan^2\theta+9}$

$= \frac{3}{9}\int \frac{\sec^2\theta\, d\theta}{\tan^2\theta+1} = \frac{1}{3}\int \frac{\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}}d\theta = \frac{1}{3}\int d\theta$

$= \frac{1}{3}\theta + c$ $\quad \because x+2 = 3\tan\theta \rightarrow \frac{x+2}{3} = \tan\theta \rightarrow \tan^{-1}(\frac{x+2}{3}) = \theta$

$= \frac{1}{3}\tan^{-1}(\frac{x+2}{3})+c$

Q3. $\int \frac{x^2}{4+x^2}dx$

$= \int (1 - \frac{4}{4+x^2})dx$ $\qquad \begin{array}{r} 1 \\ 4+x^2 \overline{)\, x^2} \\ \underline{\underset{-}{x^2}\underset{-}{+}4} \\ -4 \end{array}$

$= \int 1dx - 4\int \frac{1}{4+x^2}dx$

$= x - 4\int \frac{1}{(2)^2+x^2}dx$

$= x - 4I \longrightarrow (i)$

where $I = \int \frac{1}{(2)^2+x^2}dx$

Put $x = 2\tan\theta \rightarrow dx = 2\sec^2\theta d\theta$

$= \int \frac{2\sec^2\theta d\theta}{(2)^2+(2\tan\theta)^2} = \frac{2}{4}\int \frac{\sec^2\theta d\theta}{1+\tan^2\theta}$

$= \frac{1}{2}\int \frac{\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}}d\theta = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + c$

$I = \frac{1}{2}\tan^{-1}\frac{x}{2}+c$ $\quad \because x = 2\tan\theta \Rightarrow \frac{x}{2} = \tan\theta \Rightarrow \tan^{-1}\frac{x}{2} = \theta$

Thus (i) becomes as

$= x - 4(\frac{1}{2}\tan^{-1}\frac{x}{2}) + c$

$= x - 2\tan^{-1}\frac{x}{2} + c$

Q4. $\int \frac{1}{x\ln x}dx$

Solution:- $\int \frac{1}{x\ln x}dx$

$= \int \frac{1}{\ln x}\cdot\frac{1}{x}dx = \int \frac{x^{-1}}{\ln x}dx = \int \frac{1/x}{\ln x}dx$

Put $\ln x = t \rightarrow d(\ln x) = dt$

$\rightarrow \frac{1}{x}dx = dt$

$= \int \frac{dt}{t} = \ln|t|+c = \ln|\ln x|+c$

Q5. $\int \frac{e^x}{e^x+3}dx$

Solution:- $\int \frac{e^x}{e^x+3}dx$

Put $e^x+3 = t \rightarrow d(e^x+3) = dt$

$\rightarrow e^x dx = dt$

$= \int \frac{dt}{t} = \ln|t|+c = \ln|e^x+3|+c$

Q6. $\int \frac{x+b}{(x^2+2bx+c)^{\frac{1}{2}}}dx$

Solution:- $\int \frac{x+b}{(x^2+2bx+c)^{\frac{1}{2}}}dx$

$= \int (x^2+2bx+c)^{-\frac{1}{2}}(x+b)dx$

$= \frac{1}{2}\int (x^2+2bx+c)^{-\frac{1}{2}}(2x+2b)dx$

Put $x^2+2bx+c = t$

$\rightarrow (2x+2b)dx = dt$

$= \frac{1}{2}\int t^{-\frac{1}{2}}dt = \frac{1}{2}\cdot\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$

$= \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}}+c = \frac{1}{\cancel{2}}\times\frac{\cancel{2}}{1}\sqrt{t}+c$

$= \sqrt{x^2+2bx+c}+c$

Q7. $\int \frac{\sec^2x}{\sqrt{\tan x}}dx$

Solution:- $\int \frac{\sec^2x}{\sqrt{\tan x}}dx = \int (\tan x)^{-\frac{1}{2}}(\sec^2x)dx$

put $\tan x = t \rightarrow d(\tan x) = dt$

$\rightarrow \sec^2x\,dx = dt$

$= \int t^{-\frac{1}{2}}dt = \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c = \frac{t^{\frac{1}{2}}}{\frac{1}{2}}+c = 2\sqrt{t}+c$

$= 2\sqrt{\tan x}+c$

Q8 (a) Show that

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln(x+\sqrt{x^2-a^2})+c$$

Solution:- $\int \frac{dx}{\sqrt{x^2-a^2}}$

Put $x = a\sec\theta \rightarrow dx = a\,d(\sec\theta)$

$\rightarrow dx = a\sec\theta\tan\theta\, d\theta$

$$= \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{(a\sec\theta)^2 - a^2}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2(\sec^2\theta - 1)}}$$

$$= \frac{a}{a}\int \frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}} d\theta = \int \frac{\sec\theta . \cancel{\tan\theta}}{\cancel{\tan\theta}} d\theta$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + c_1$$

As $x = a\sec\theta \rightarrow \sec\theta = \frac{x}{a}$ and

$$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{x}{a}\right)^2 - 1}$$

$$= \sqrt{\frac{x^2}{a^2} - 1} = \sqrt{\frac{x^2-a^2}{a^2}}$$

$$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$$ Thus

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + c_1$$

$$= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c_1$$

$$= \ln|x+\sqrt{x^2-a^2}| - \ln a + c_1$$

$$= \ln|x+\sqrt{x^2-a^2}| + c \quad \because c = c_1 - \ln a$$

Hence proved.

(b) Show that

$$\int \sqrt{a^2-x^2}\, dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c$$

Solution:- $\int \sqrt{a^2-x^2}\, dx$

Put $x = a\sin\theta \rightarrow dx = a\cos\theta\, d\theta$

$$= \int \sqrt{a^2-(a\sin\theta)^2} . a\cos\theta\, d\theta$$

$$= \int \sqrt{a^2-a^2\sin^2\theta} . a\cos\theta\, d\theta$$

$$= \int \sqrt{a^2(1-\sin^2\theta)} . a\cos\theta\, d\theta$$

$$= a^2\int \sqrt{\cos^2\theta} . \cos\theta\, d\theta = a^2\int \cos^2\theta\, d\theta$$

$$= a^2\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta \quad \because \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \frac{a^2}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\int 1\, d\theta + \frac{a^2}{2}\int \cos 2\theta\, d\theta$$

$$= \frac{a^2}{2}.\theta + \frac{a^2}{2}\frac{\sin 2\theta}{2} + c$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\frac{2\sin\theta\cos\theta}{2} + c \quad \because \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\sin\theta\sqrt{1-\sin^2\theta} + c \quad \because \cos\theta = \sqrt{1-\sin^2\theta}$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{a^2}{2}\left(\frac{x}{a}\right)\sqrt{1-\left(\frac{x}{a}\right)^2} + c \quad \because x = a\sin\theta$$

$\rightarrow \frac{x}{a} = \sin\theta$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{a^2}{2}.\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}} + c \quad \rightarrow \sin^{-1}\frac{x}{a} = \theta$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{\cancel{a^2}^{\,x}}{2}.\frac{x}{\cancel{a^2}}\sqrt{a^2-x^2} + c$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c$$

Hence proved.

Evaluate the following integrals:

Q9. $\int \frac{dx}{(1+x^2)^{3/2}}$

Solution:- $\int \frac{dx}{(1+x^2)^{3/2}}$

Put $x = \tan\theta \rightarrow dx = \sec^2\theta\, d\theta$

$$= \int \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}} = \int \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}} d\theta = \int \frac{\sec^2\theta}{\sec^3\theta} d\theta$$

$$= \int \frac{1}{\sec\theta} d\theta = \int \cos\theta\, d\theta = \sin\theta + c$$

$$= \frac{x}{\sqrt{1+x^2}} + c$$

$\because \tan\theta = x$

$\rightarrow \tan\theta = \frac{x}{1}$

$(Hyp)^2 = (Base)^2 + (Perp)^2$

$\rightarrow Hyp = \sqrt{1+x^2}$ from fig

$\sin\theta = \frac{x}{\sqrt{1+x^2}}$

Q10. $\int \frac{1}{(1+x^2)\tan^{-1}x} dx$

Solution:- $\int \frac{1}{\tan^{-1}x} . \frac{1}{1+x^2} dx$

$$= \int \frac{1}{\tan^{-1}x}(1+x^2)^{-1} dx = \int \frac{\frac{1}{1+x^2}}{\tan^{-1}x} dx$$

Put $\tan^{-1}x = t \rightarrow d(\tan^{-1}x) = dt$

$\rightarrow \frac{1}{1+x^2} dx = dt$

$$= \int \frac{dt}{t} = \ln|t| + c = \ln|\tan^{-1}x| + c$$

Q11. $\int \sqrt{\frac{1+x}{1-x}} dx$

Solution:- $\int \sqrt{\frac{1+x}{1-x}} dx$

$$= \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}} dx = \int \frac{(\sqrt{1+x})^2}{\sqrt{(1-x)(1+x)}} dx$$

$$= \int \frac{(1+x)\, dx}{\sqrt{1-x^2}}$$

Put $x = \sin\theta \rightarrow dx = \cos\theta\, d\theta$

$$= \int \frac{(1+\sin\theta)\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = \int \frac{(1+\sin\theta)\cos\theta\, d\theta}{\sqrt{\cos^2\theta}}$$

$$= \int \frac{(1+\sin\theta)\cancel{\cos\theta}}{\cancel{\cos\theta}} d\theta = \int (1+\sin\theta)\, d\theta$$

$= \int 1 d\theta + \int \sin\theta d\theta = \theta - \cos\theta + c$

$\because x = \sin\theta \quad$ so $\cos\theta = \sqrt{1-\sin^2\theta}$

$\Rightarrow \sin^{-1}x = \theta \quad \Rightarrow \cos\theta = \sqrt{1-x^2}$

$= \sin^{-1}x - \sqrt{1-x^2} + c$

Q12. $\int \frac{\sin\theta}{1+\cos^2\theta} d\theta$

Solution:- $\int \frac{\sin\theta}{1+\cos^2\theta} d\theta$

Put $\cos\theta = t$

$\Rightarrow d(\cos\theta) = dt \Rightarrow -\sin\theta d\theta = dt$

$\Rightarrow \sin\theta d\theta = -dt$

$= \int \frac{-dt}{1+t^2} = -\int \frac{1}{1+t^2} dt = -\tan^{-1}t + c$

$\because \int \frac{1}{1+x^2} dx = \tan^{-1}x + c$

$= -\tan^{-1}(\cos\theta) + c$

Q13. $\int \frac{ax}{\sqrt{a^2-x^4}} dx$

Solution:- $\int \frac{ax}{\sqrt{a^2-x^4}} dx$

$= \int \frac{ax}{\sqrt{a^2-(x^2)^2}} dx \quad$ Put $x^2 = a\sin\theta$

$d(x^2) = a\, d(\sin\theta)$

$\Rightarrow 2x dx = a\cos\theta d\theta$

$\Rightarrow x dx = \frac{1}{2} a\cos\theta d\theta$

$= \int \frac{a \cdot \frac{a}{2}\cos\theta d\theta}{\sqrt{a^2-(a\sin\theta)^2}}$

$= \frac{a^2}{2} \int \frac{\cos\theta d\theta}{\sqrt{a^2(1-\sin^2\theta)}} = \frac{a^2}{2} \int \frac{\cos\theta d\theta}{a\sqrt{\cos^2\theta}}$

$= \frac{a}{2} \int \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}} d\theta = \frac{a}{2} \int 1 d\theta = \frac{a}{2}\theta + c$

$= \frac{a}{2} \sin^{-1}\frac{x^2}{2} + c \quad \because x^2 = a\sin\theta$

$\Rightarrow \frac{x^2}{a} = \sin\theta$

$\Rightarrow \sin^{-1}\left(\frac{x^2}{a}\right) = \theta$

Q14. $\int \frac{dx}{\sqrt{7-6x-x^2}}$

Solution:- $\int \frac{dx}{\sqrt{7-6x-x^2}}$

$= \int \frac{dx}{\sqrt{7-6x-x^2-9+9}} = \int \frac{dx}{\sqrt{16-(x^2+6x+9)}}$

$= \int \frac{dx}{\sqrt{(4)^2-(x+3)^2}} \quad$ Put $x+3 = 4\sin\theta$

$dx = 4\cos\theta d\theta$

$= \int \frac{4\cos\theta d\theta}{\sqrt{(4)^2-(4\sin\theta)^2}} = 4 \int \frac{\cos\theta d\theta}{\sqrt{(4)^2(1-\sin^2\theta)}}$

$= \frac{\cancel{4}}{\cancel{4}} \int \frac{\cos\theta d\theta}{\sqrt{\cos^2\theta}} = \int \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}} d\theta$

$= \int 1 d\theta = \theta + c = \sin^{-1}\left(\frac{x+3}{4}\right) + c$

$\because x+3 = 4\sin\theta$

$\Rightarrow \frac{x+3}{4} = \sin\theta \Rightarrow \sin^{-1}\left(\frac{x+3}{4}\right) = \theta$

Q15. $\int \frac{\cos x}{\sin x \ln\sin x} dx$

Solution:- $\int \frac{\cos x}{\sin x \ln\sin x} dx$

$= \int \frac{\cos x}{\sin x} \cdot \frac{1}{\ln\sin x} dx = \int \cot x \cdot \frac{1}{\ln\sin x} dx$

$= \int \frac{\cot x}{\ln\sin x} dx \quad$ Put $\ln\sin x = t$

$d(\ln\sin x) = dt$

$= \int \frac{dt}{t} = \ln|t| + c \quad \Rightarrow \frac{1}{\sin x}(\cos x dx) = dt$

$\because t = \ln\sin x \quad \Rightarrow \cot x dx = dt$

$= \ln|\ln\sin x| + c$

Q16. $\int \cos x \left(\frac{\ln\sin x}{\sin x}\right) dx$

Solution:- $\int \cos x \left(\frac{\ln\sin x}{\sin x}\right) dx$

$= \int \frac{\cos x}{\sin x} (\ln\sin x) dx = \int \cot x \cdot \ln(\sin x) dx$

$= \int \ln\sin x \cdot \cot x dx \quad$ Put $\ln\sin x = t$

$\Rightarrow d(\ln\sin x) = dt$

$= \int t dt = \frac{t^2}{2} + c \quad \Rightarrow \frac{1}{\sin x}(\cos x dx) = dt$

$= \frac{(\ln\sin x)^2}{2} + c \quad \Rightarrow \cot x dx = dt$

$\because t = \ln\sin x$

Q17. $\int \frac{x dx}{4+2x+x^2}$

Solution:- $\int \frac{x dx}{x^2+2x+4}$

$= \frac{1}{2} \int \frac{2x dx}{x^2+2x+4} = \frac{1}{2} \int \frac{2x+2-2}{x^2+2x+4} dx$

$= \frac{1}{2} \int \left(\frac{2x+2}{x^2+2x+4} - \frac{2}{x^2+2x+4} \right) dx$

$= \frac{1}{2} \int \frac{2x+2}{x^2+2x+4} dx - \frac{1}{\cancel{2}} \int \frac{\cancel{2}}{x^2+2x+4} dx$

$= \frac{1}{2} \ln|x^2+2x+4| - \int \frac{1}{x^2+2x+4} dx$

$= \frac{1}{2} \ln|x^2+2x+4| - I$ —— (i)

$I = \int \frac{1}{x^2+2x+1+3} dx = \int \frac{1}{(x+1)^2+(\sqrt{3})^2} dx$

Put $x+1 = \sqrt{3}\tan\theta \Rightarrow dx = \sqrt{3}\sec^2\theta d\theta$

$I = \int \frac{\sqrt{3}\sec^2\theta d\theta}{(\sqrt{3}\tan\theta)^2+(\sqrt{3})^2} = \frac{\sqrt{3}}{(\sqrt{3})^2} \int \frac{\sec^2\theta}{\tan^2\theta+1} d\theta$

$= \frac{1}{\sqrt{3}} \int \frac{\sec^2\theta}{\sec^2\theta} d\theta = \frac{1}{\sqrt{3}} \int 1 d\theta$

$= \frac{1}{\sqrt{3}}\theta + c = \frac{1}{\sqrt{3}} \tan^{-1}\frac{x+1}{\sqrt{3}} + c$

$\because x+1 = \sqrt{3}\tan\theta$

$\Rightarrow \frac{x+1}{\sqrt{3}} = \tan\theta$

$\Rightarrow \tan^{-1}\frac{x+1}{\sqrt{3}} = \theta$

so by (i)

$= \frac{1}{2} \ln|x^2+2x+4| - \frac{1}{\sqrt{3}} \tan^{-1}\frac{x+1}{\sqrt{3}} + c$

Q18.

See at page # 60

Q19. $\int\left[\cos\left(\sqrt{x}-\frac{x}{2}\right)\right]\times\left(\frac{1}{\sqrt{x}}-1\right)dx$

Solution:- $\int\left[\cos\left(\sqrt{x}-\frac{x}{2}\right)\right]\times\left(\frac{1}{\sqrt{x}}-1\right)dx$

put $\sqrt{x}-\frac{x}{2}=t \rightarrow x^{1/2}-\frac{x}{2}=t$

$\rightarrow d\left(x^{1/2}-\frac{x}{2}\right)=dt \rightarrow \left(\frac{1}{2}x^{-1/2}-\frac{1}{2}\right)dx=dt$

$\rightarrow\left(\frac{1}{2\sqrt{x}}-\frac{1}{2}\right)dx=dt \rightarrow \frac{1}{2}\left(\frac{1}{\sqrt{x}}-1\right)dx=dt$

$\rightarrow\left(\frac{1}{\sqrt{x}}-1\right)dx=2dt$

$=\int \cos t\cdot 2dt = 2\int \cos t\,dt = 2(-\sin t)+c$

$=-2\sin\left(\sqrt{x}-\frac{x}{2}\right)+c \qquad \because t=\sqrt{x}-\frac{x}{2}$

Q20. $\int \frac{x+2}{\sqrt{x+3}}dx$

Solution:- $\int \frac{x+2}{\sqrt{x+3}}dx$

$=\int \frac{x+2+1-1}{\sqrt{x+3}}dx = \int \frac{x+3-1}{\sqrt{x+3}}dx$

$=\int \frac{x+3}{\sqrt{x+3}}dx - \int \frac{1}{\sqrt{x+3}}dx$

$=\int \frac{\cancel{(\sqrt{x+3})}(\sqrt{x+3})}{\cancel{\sqrt{x+3}}}dx - \int\frac{1}{\sqrt{x+3}}dx$

$=\int (x+3)^{1/2}dx - \int (x+3)^{-1/2}dx$

$=\frac{(x+3)^{3/2}}{3/2}-\frac{(x+3)^{1/2}}{1/2}+c = \frac{3}{2}(x+3)^{3/2}-2(x+3)^{1/2}+c$

$=\frac{3}{2}(x+3)^{3/2}-2\sqrt{x+3}+c$

Q21. $\int \frac{\sqrt{2}}{\sin x+\cos x}dx$

Solution:- $\int \frac{\sqrt{2}}{\sin x+\cos x}dx$

$=\int \frac{1}{\frac{\sin x+\cos x}{\sqrt{2}}}dx = \int \frac{1}{\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x}dx$

$=\int \frac{1}{\sin x\sin\frac{\pi}{4}+\cos x\cos\frac{\pi}{4}}dx \qquad \because \sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}, \quad \cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

$=\int \frac{1}{\cos\left(x-\frac{\pi}{4}\right)}dx \qquad \because \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$=\int \sec\left(x-\frac{\pi}{4}\right)dx$

$=\ln\left|\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right|+c$

Q22. $\int \frac{dx}{\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x}$

$=\int \frac{dx}{\sin x\cos\frac{\pi}{3}+\cos x\sin\frac{\pi}{3}} \qquad \because \sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}, \quad \cos\frac{\pi}{3}=\frac{1}{2}$

$=\int \frac{dx}{\sin\left(x+\frac{\pi}{3}\right)}$

$=\int \operatorname{cosec}\left(x+\frac{\pi}{3}\right)dx$

$=\ln\left|\operatorname{cosec}\left(x+\frac{\pi}{3}\right)-\cot\left(x+\frac{\pi}{3}\right)\right|+c$

Integration by Parts

We know that for two functions f and g

$$\frac{d}{dx}(f(x)g(x))=f'(x)g(x)+f(x)g'(x)$$

$$\rightarrow f(x)g'(x)=\frac{d}{dx}(f(x)g(x))-f'(x)g(x)$$

Taking integration w.r.t 'x' we get

$$\int f(x)g'(x)dx=\int\left[\frac{d}{dx}(f(x)g(x))-f'(x)g(x)\right]dx$$

$$=\int\left(\frac{d}{dx}(f(x)g(x)\right)-\int f'(x)g(x)dx$$

$$\int f(x)g'(x)dx=f(x)\,g(x)-\int g(x)f'(x)dx$$

or $\int f(x)g'(x)dx=f(x)\int g'(x)dx-\int\left(\int g'(x)\,dx\right)\cdot f'(x)\,dx$

In other words,

$$\boxed{\begin{aligned}&\int(I^{st}\text{ function})(2^{nd}\text{ function})dx\\&=(I^{st}\text{func.})\int(2^{nd}\text{func.})dx-\int\left(\underset{\text{func}}{\text{integrated}}\right)\cdot\frac{d}{dx}(I^{st}\text{func.})dx\end{aligned}}$$

This is called "Integration by parts".

"Some basic rules for Integration by parts"

- ✡ chose the function as 2^{nd} function whose integration is known or possible.
- ✡ If integration of both given functions are known but one of the given function is polynomial function then chose polynomial function as I^{st} function.
- ✡ If integration of both given functions are known but no one is polynomial function then we may chose any function as I^{st}.
- ✡ If we are given only 1 function whose integration is unknown or cannot be easily find (i-e., $\sin^{-1}x$, $\cos^{-1}x$, $\sqrt{a^2-x^2}$, $\frac{1}{\sqrt{x^2-a^2}}$ etc) then we take 1 as 2^{nd} function.

"Review above rules"

	1st Function	2nd Function
$\int x^n\cos x\,dx$	x^n	$\cos x$
$\int x^n\sin x\,dx$	x^n	$\sin x$
$\int x^n\sin^{-1}x\,dx$	$\sin^{-1}x$	x^n
$\int x^n\tan^{-1}x\,dx$	$\tan^{-1}x$	x^n
$\int e^x\sin x\,dx$	e^x	$\sin x$
or	$\sin x$,	e^x
$\int \ln x\cdot x^n dx$	x^n	$\ln x$
$\int \tan^{-1}x\,dx$	$\tan^{-1}x$	1
$\int \sqrt{a^2+x^2}\,dx$	$\sqrt{a^2+x^2}$	1

You may remember the word "ILATE"
I = Inverse functions, L = Logarithmic functions
A = Algebraic functions, T = trigonometric functions
E = Exponential functions.

Example 1. Find $\int x\cos x\,dx$

Solution:- $\int \underset{I}{x}\ \underset{II}{\cos x}\,dx$

$= x\sin x - \int \sin x \cdot 1\,dx$

$= x\sin x - (-\cos x) + c$

$= x\sin x + \cos x + c$

$\because \int \sin x\,dx = -\cos x$

$\int \cos x\,dx = \sin x$

Example 2. Find $\int x e^x dx$

Solution:- $\int \underset{I}{x}\,\underset{II}{e^x}dx$

$= x\cdot e^x - \int e^x (1)dx = xe^x - e^x + c$

Example 3. Evaluate $\int x\tan^2 x\,dx$

Solution:- $\int x\tan^2 x\,dx$

$= \int x(\sec^2 x - 1)dx$

$\because 1+\tan^2 x = \sec^2 x$
$\Rightarrow \tan^2 x = \sec^2 x - 1$

$= \int x\sec^2 x\,dx - \int x\,dx$

$= \int \underset{I}{x}\ \underset{II}{\sec^2 x}\,dx - \frac{x^2}{2}$

$\because \int \sec^2 x\,dx = \tan x + c$

$= x\cdot\tan x - \int \tan x\,(1)dx - \frac{x^2}{2}$

$= x\tan x + \ln|\cos x| - \frac{x^2}{2} + c$

$\because \int \tan x\,dx = -\ln|\cos x| + c$
or $= \ln|\sec x| + c$

Example 4. Evaluate $\int x^5 \ln x\,dx$

Solution:- $\int \underset{I}{\ln x}\cdot \underset{II}{x^5}\,dx$

$= \ln x\cdot\frac{x^6}{6} - \int \frac{x^6}{6}\cdot\frac{1}{x}dx$

$= \frac{x^6}{6}\ln x - \frac{1}{6}\int x^5 dx = \frac{x^6}{6}\ln x - \frac{1}{6}\cdot\frac{x^6}{6} + c$

$= \frac{x^6}{6}\ln x - \frac{x^6}{36} + c$

Example 5. Evaluate $\int \ln(x+\sqrt{x^2+1})\,dx$

Solution:- $\int \underset{I}{\ln(x+\sqrt{x^2+1})}\cdot\underset{II}{1}\,dx$

$= \ln(x+\sqrt{x^2+1})\cdot x - \int x\cdot\frac{d}{dx}\ln(x+(x^2+1)^{\frac{1}{2}})\,dx$

$= \ln(x+\sqrt{x^2+1})\cdot x - \int x\cdot\frac{1}{x+\sqrt{x^2+1}}\,\frac{d}{dx}(x+(x^2+1)^{\frac{1}{2}})\,dx$

$= x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{x+\sqrt{x^2+1}}\left(1+\frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)\right)dx$

$= x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)dx$

$= x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{\cancel{x+\sqrt{x^2+1}}}\left(\frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}}\right)dx$

$= x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{\sqrt{x^2+1}}\,dx$

$= x\ln(x+\sqrt{x^2+1}) - \frac{1}{2}\int (x^2+1)^{-\frac{1}{2}}(2x)\,dx$

$= x\ln(x+\sqrt{x^2+1}) - \frac{1}{\cancel{2}}\cdot\frac{(x^2+1)^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} + c$

$= x\ln(x+\sqrt{x^2+1}) - \sqrt{x^2+1} + c$

Example 6. Evaluate $\int x^2\cdot a\,e^{ax}dx$

Solution:- $a\int \underset{I}{x^2}\,\underset{II}{e^{ax}}dx$

$= a\left[x^2\cdot\frac{e^{ax}}{a} - \int\frac{e^{ax}}{a}(2x)\,dx\right]$

$= a\left(x^2\frac{e^{ax}}{a} - \frac{2}{a}\int x e^{ax}dx\right)$

$= x^2e^{ax} - 2\int \underset{I}{x}\,\underset{II}{e^{ax}}\,dx$

$= x^2e^{ax} - 2\left(x\cdot\frac{e^{ax}}{a} - \int\frac{e^{ax}}{a}(1)\,dx\right]$

$= x^2e^{ax} - \frac{2}{a}xe^{ax} + \frac{2}{a}\int e^{ax}dx$

$= x^2e^{ax} - \frac{2}{a}xe^{ax} + \frac{2}{a}\cdot\frac{e^{ax}}{a} + c$

$= x^2e^{ax} - \frac{2}{a}xe^{ax} + \frac{2}{a^2}e^{ax} + c$

Example 7. Find $\int e^{ax}\cos bx\,dx$

Solution:- $I = \int \underset{II}{e^{ax}}\,\underset{I}{\cos bx}\,dx$

$I = \cos bx\cdot\frac{e^{ax}}{a} - \int\frac{e^{ax}}{a}(-\sin bx\cdot b)dx$

$= \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int \underset{II}{e^{ax}}\,\underset{I}{\sin bx}\,dx$

$= \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\left[\sin bx\cdot\frac{e^{ax}}{a} - \int\frac{e^{ax}}{a}(\cos bx\cdot b)\,dx\right]$

$I = \frac{e^{ax}\cos bx}{a} + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}\int e^{ax}\cos bx\,dx$

$\rightarrow I = \frac{e^{ax}\cos bx}{a} + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}I + c_1$

$\rightarrow I + \frac{b^2}{a^2}I = \frac{e^{ax}\cos bx}{a} + \frac{b}{a^2}e^{ax}\sin bx + c_1$

$I\left(\frac{a^2+b^2}{a^2}\right) = e^{ax}\left(\frac{\cos bx}{a} + \frac{b\sin bx}{a^2}\right) + c_1$

$I\left(\frac{a^2+b^2}{a^2}\right) = e^{ax}\left(\frac{a\cos bx + b\sin bx}{a^2}\right) + c_1$

$\rightarrow I = \frac{\cancel{a^2}e^{ax}}{a^2+b^2}\left(\frac{a\cos bx + b\sin bx}{\cancel{a^2}}\right) + \frac{a^2c_1}{a^2+b^2}$

$I = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx) + c$

$\because c = \frac{a^2c_1}{a^2+b^2}$

Put $a = r\cos\theta$, $b = r\sin\theta$

$\rightarrow a^2+b^2 = r^2\cos^2\theta + r^2\sin^2\theta$
$= r^2(\cos^2\theta+\sin^2\theta) = r^2(1)$

$\rightarrow a^2+b^2 = r^2$ $\quad\because \cos^2\theta + \sin^2\theta = 1$

and $\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \rightarrow \theta = \tan^{-1}\frac{b}{a}$

so $I = \frac{e^{ax}}{r^2}(\sin bx\cdot r\sin\theta + r\cos\theta\cdot\cos bx) + c$

$$I = \frac{e^{ax}}{r^2}.r\left(\cos bx\cos\theta + \sin bx\sin\theta\right) + c$$

$\because \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)$

$$= \frac{e^{ax}}{r}\cos(bx-\theta) + c$$

$$I = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos(bx-\theta) + c$$

Thus

$$\int e^{ax}\cos bx\,dx = \frac{1}{\sqrt{a^2+b^2}}e^{ax}\cos\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

*Remember Useful formulas

(1) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

(2) $\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left|x+\sqrt{x^2-a^2}\right| + c$

(3) $\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left|x+\sqrt{x^2+a^2}\right| + c$

Example 8. Evaluate $\int\sqrt{a^2+x^2}\,dx$

Solution:- $I = \int \underset{I}{\sqrt{a^2+x^2}}\cdot\underset{II}{1}\,dx$

$$I = \sqrt{a^2+x^2}\cdot x - \int x\frac{d}{dx}(a^2+x^2)^{\frac{1}{2}}dx$$

$$= x\sqrt{a^2+x^2} - \int x\cdot\frac{1}{\cancel{2}}(a^2+x^2)^{-\frac{1}{2}}(\cancel{2}x)dx$$

$$= x\sqrt{a^2+x^2} - \int\frac{x^2}{\sqrt{a^2+x^2}}dx$$

$$= x\sqrt{a^2+x^2} - \int\frac{a^2+x^2-a^2}{\sqrt{a^2+x^2}}dx$$

$$= x\sqrt{a^2+x^2} - \int\frac{a^2+x^2}{\sqrt{a^2+x^2}}dx + a^2\int\frac{1}{\sqrt{a^2+x^2}}dx$$

$$I = x\sqrt{a^2+x^2} - \int\sqrt{a^2+x^2}\,dx + a^2\ln\left|x+\sqrt{a^2+x^2}\right|$$

$$I = x\sqrt{a^2+x^2} - I + a^2\ln|x+\sqrt{a^2+x^2} + c_1$$

$$2I = x\sqrt{a^2+x^2} + a^2\ln\left|x+\sqrt{a^2+x^2}\right| + c_1$$

$$\rightarrow I = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| + \frac{c_1}{2}$$

Thus

$$\int\sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| + c$$

$\left(\because c = \frac{c_1}{2}\right)$

$\left(\because \int\frac{1}{\sqrt{a^2+x^2}}dx = \ln\left|x+\sqrt{a^2+x^2}\right| + c\right)$

Example 9. Evaluate $\int\sin^4 x\,dx$

Solution:- $I = \int\sin^4 x\,dx$

$$= \int\sin^2 x\cdot\sin^2 x\,dx$$

$$= \int\sin^2 x(1-\cos^2 x)dx$$

$\because \sin^2 x + \cos^2 x = 1$

$1-\cos^2 x \;\therefore = \sin^2 x$

$$= \int(\sin^2 x - \sin^2 x\cos^2 x\,dx$$

$$= \int\sin^2 x\,dx - \int\sin^2 x\cos^2 x\,dx$$

$$= \int\frac{1-\cos 2x}{2}dx - \int\cos x(\sin^2 x\cos x)dx$$

$$= \frac{1}{2}\int 1dx - \frac{1}{2}\int\cos 2x\,dx - \int\underset{I}{\cos x}\underset{II}{(\sin^2 x\cos x)}dx$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} - \left[\cos x\cdot\frac{\sin^3 x}{3} - \int\frac{\sin^3 x}{3}(-\sin x)dx\right]$$

$$\left(\because \int\sin^2 x\cos x\,dx = \frac{\sin^3 x}{3} + c\right)$$

$$I = \frac{x}{2} - \frac{\sin 2x}{4} - \frac{1}{3}\sin^3 x\cos x - \frac{1}{3}\int\sin^4 x\,dx$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} - \frac{1}{3}\sin^3 x\cos x - \frac{1}{3}I$$

$$I + \frac{1}{3}I = \frac{x}{2} - \frac{\sin 2x}{4} - \frac{1}{3}\sin^3 x\cos x + c_1$$

$$\frac{4}{3}I = \frac{x}{2} - \frac{\sin 2x}{4} - \frac{1}{3}\sin^3 x\cos x + c_1$$

$$\rightarrow I = \frac{3}{8}x - \frac{3}{16}\sin 2x - \frac{1}{4}\sin^3 x\cos x + \frac{3}{4}c_1$$

Thus $\int\sin^4 x\,dx = \frac{3}{8}x - \frac{3}{16}\sin 2x - \frac{1}{4}\sin^3 x\cos x + c$

$\left(\because \frac{3c_1}{4} = c\right)$

Prove that $\int e^x(f(x)+f'(x))dx = e^x f(x) + c$

Proof:- Let $\int\underset{II}{e^x}\underset{I}{f(x)}\,dx$

$$= f(x)e^x - \int e^x f'(x)dx$$

$$\rightarrow \int e^x f(x)dx + \int e^x f'(x)dx = e^x f(x)$$

$$\rightarrow \int e^x(f(x)+f'(x))dx = e^x f(x) + c$$

Hence proved.

Example 10. Evaluate $\int\frac{e^x(1+\sin x)}{1+\cos x}dx$

Solution:- $\int\frac{e^x(1+\sin x)}{1+\cos x}dx$

$\because \sin 2\theta = 2\sin\theta\cos\theta$

$\rightarrow \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\because \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$

$\rightarrow 2\cos^2\frac{\theta}{2} = 1+\cos\theta$

$$= \int\frac{e^x\left(1+2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{2\cos^2\frac{x}{2}}dx$$

$$= \int e^x\left(\frac{1}{2\cos^2\frac{x}{2}} + \frac{\cancel{2}\sin\frac{x}{2}\cancel{\cos\frac{x}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{x}{2}}\right)dx$$

$$= \int e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)dx$$

$$= e^x\tan\frac{x}{2} + c$$

$\because \int e^x(f(x)+f'(x)dx = e^x f(x)+c$

and $f(x) = \tan\frac{x}{2}$

$f'(x) = \frac{1}{2}\sec^2\frac{x}{2}$

Example 11. Show that

$$\int e^{ax}[af(x)+f'(x)]dx = e^{ax}f(x)+c$$

Solution:-

$$\int e^{ax}(af(x)+f'(x))\,dx$$

$$= a\int e^{ax}f(x)dx + \int \underset{\text{I}}{e^{ax}}\,\underset{\text{II}}{f'(x)}dx \qquad \because \int f'(x)dx = f(x)$$

$$= \int ae^{ax}f(x)dx + e^{ax}f(x) - \int f(x).e^{ax}.a\,dx$$

$$= \int ae^{ax}\cancel{f(x)}\,dx + e^{ax}f(x) - \int ae^{ax}\cancel{f(x)}\,dx$$

$$= e^{ax}f(x)+c \quad \text{Hence proved.}$$

Exercise 3.4

Evaluate the following integrals by parts add a word representing all the functions are defined.

Q1. (i) $\int x\sin x\,dx$

Solution:- $\int \underset{\text{I}}{x}\,\underset{\text{II}}{\sin x}\,dx$

$$= x(-\cos x) - \int(-\cos x)\cdot 1\,dx$$

$$= -x\cos x + \int \cos x\,dx = -x\cos x+\sin x+c$$

(ii) $\int \ln x\,dx$

Solution:- $\int \underset{\text{I}}{\ln x}\cdot\underset{\text{II}}{1}\,dx$

$$= \ln x\cdot x - \int x\cdot\frac{1}{x}dx = x\ln x - \int 1dx$$

$$= x\ln x - x + c$$

(iii) $\int x\ln x\,dx$

Solution:- $\int \underset{\text{II}}{x}\,\underset{\text{I}}{\ln x}\,dx$

$$= \ln x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{x}dx$$

$$= \ln x\cdot\frac{x^2}{2} - \frac{1}{2}\int x\,dx = \ln x\cdot\frac{x^2}{2} - \frac{1}{2}\cdot\frac{x^2}{2}+c$$

$$= \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + c = \frac{x^2}{2}(\ln x - \frac{1}{2})+c$$

(iv) $\int x^2\ln x\,dx$

Solution:- $\int \underset{\text{II}}{x^2}\,\underset{\text{I}}{\ln x}\,dx$

$$= \ln x\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{x}dx = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\cdot\frac{x^3}{3}+c = \frac{x^3}{3}\ln x - \frac{x^3}{9}+c$$

(v) $\int x^3\ln x\,dx$

Solution:- $\int \underset{\text{II}}{x^3}\,\underset{\text{I}}{\ln x}\,dx$

$$= \ln x\cdot\frac{x^4}{4} - \int\frac{x^4}{4}\cdot\frac{1}{x}dx$$

$$= \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3dx = \frac{x^4}{4}\ln x - \frac{1}{4}\cdot\frac{x^4}{4}+c$$

$$= \frac{x^4}{4}\ln x - \frac{x^4}{16}+c = \frac{x^4}{4}(\ln x - \frac{1}{4})+c$$

(vi) $\int x^4\ln x\,dx$

Solution:- $\int \underset{\text{II}}{x^4}\,\underset{\text{I}}{\ln x}\,dx$

$$= \ln x\cdot\frac{x^5}{5} - \int\frac{x^5}{5}\cdot\frac{1}{x}dx$$

$$= \frac{x^5}{5}\ln x - \frac{1}{5}\int x^4dx = \frac{x^5}{5}\ln x - \frac{1}{5}\cdot\frac{x^5}{5}+c$$

$$= \frac{x^5}{5}\ln x - \frac{x^5}{25}+c = \frac{x^5}{5}(\ln x - \frac{1}{5})+c$$

(vii) $\int \text{Tan}^{-1}x\,dx$

Solution:- $\int \underset{\text{I}}{\text{Tan}^{-1}x}\cdot\underset{\text{II}}{1}\,dx$

$$= \text{Tan}^{-1}x\cdot x - \int x\cdot\frac{d}{dx}(\text{Tan}^{-1}x)dx$$

$$= x\text{Tan}^{-1}x - \int x\cdot\frac{1}{1+x^2}dx$$

$$= x\text{Tan}^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}dx$$

$$= x\text{Tan}^{-1}x - \frac{1}{2}\ln|1+x^2|+c \qquad \because \int\frac{f'(x)}{f(x)}dx = \ln|f(x)|+c$$

(viii) $\int x^2\sin x\,dx$

Solution:- $\int \underset{\text{I}}{x^2}\,\underset{\text{II}}{\sin x}\,dx$

$$= x^2(-\cos x) - \int(-\cos x)(2x)dx$$

$$= -x^2\cos x + 2\int \underset{\text{I}}{x}\,\underset{\text{II}}{\cos x}\,dx$$

$$= -x^2\cos x + 2[x\sin x - \int\sin x(1)dx]$$

$$= -x^2\cos x + 2x\sin x - 2\int\sin x\,dx$$

$$= -x^2\cos x + 2x\sin x - 2(-\cos x)+c$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

(ix) $\int x^2\,\text{Tan}^{-1}x\,dx$

Solution:- $\int \underset{\text{II}}{x^2}\,\underset{\text{I}}{\text{Tan}^{-1}x}\,dx$

$$= \text{Tan}^{-1}x\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{1+x^2}dx$$

$$= \frac{x^3}{3}\text{Tan}^{-1}x - \frac{1}{3}\int\frac{x^3}{1+x^2}dx$$

$$= \frac{x^3}{3}\text{Tan}^{-1}x - \frac{1}{3}\int(x - \frac{x}{1+x^2})dx \qquad \begin{array}{r} x \\ 1+x^2\overline{)\;x^3} \\ \underline{\mp x^3 \mp x} \\ -x \end{array}$$

$$= \frac{x^3}{3}\text{Tan}^{-1}x - \frac{1}{3}\int x\,dx + \frac{1}{3}\int\frac{x}{1+x^2}dx$$

$$= \frac{x^3}{3}\text{Tan}^{-1}x - \frac{1}{3}\cdot\frac{x^2}{2} + \frac{1}{6}\int\frac{2x}{1+x^2}dx$$

$$= \frac{x^3}{3}\text{Tan}^{-1}x - \frac{x^2}{6} + \frac{1}{6}\ln|1+x^2|+c$$

$$\because \int\frac{f'(x)}{f(x)}dx = \ln|f(x)|+c$$

(x) $\int x \operatorname{Tan}^{-1}x\,dx$

Solution:- $\int \underset{II}{x}\, \underset{I}{\operatorname{Tan}^{-1}x}\,dx$

$= \operatorname{Tan}^{-1}x \cdot \frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{1}{1+x^2}\,dx$

$= \frac{x^2}{2}\operatorname{Tan}^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$

$= \frac{x^2}{2}\operatorname{Tan}^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$

$$\begin{array}{r|l} & 1 \\ \hline 1+x^2 & x^2 \\ & \underset{-}{x^2} \underset{-}{+1} \\ \hline & -1 \end{array}$$

$= \frac{x^2}{2}\operatorname{Tan}^{-1}x - \frac{1}{2}\int 1\,dx + \frac{1}{2}\int\frac{1}{1+x^2}\,dx$

$= \frac{x^2}{2}\operatorname{Tan}^{-1}x - \frac{1}{2}\cdot x + \frac{1}{2}\tan^{-1}x + c$

$= \frac{x^2}{2}\operatorname{Tan}^{-1}x - \frac{x}{2} + \frac{1}{2}\operatorname{Tan}^{-1}x + c$

$= \frac{x^2}{2}\operatorname{Tan}^{-1}x + \frac{1}{2}\operatorname{Tan}^{-1}x - \frac{x}{2} + c$

$= \frac{1}{2}\operatorname{Tan}^{-1}x\,(x^2+1) - \frac{x}{2} + c$

(xi) $\int x^3\tan^{-1}x\,dx$

Solution:- $\int \underset{II}{x^3}\,\underset{I}{\tan^{-1}x}\,dx$

$= \tan^{-1}x\cdot\frac{x^4}{4} - \int\frac{x^4}{4}\cdot\frac{1}{1+x^2}\,dx$

$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int\frac{x^4}{1+x^2}\,dx$

$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int\left((x^2-1) + \frac{1}{1+x^2}\right)dx$

$$\begin{array}{r|l} & x^2-1 \\ \hline 1+x^2 & x^4 \\ & \underset{-}{x^4} \underset{-}{+x^2} \\ \hline & -x^2 \\ & \underset{+}{-x^2} \underset{+}{-1} \\ \hline & 1 \end{array}$$

$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int x^2dx + \frac{1}{4}\int dx - \frac{1}{4}\int\frac{dx}{1+x^2}$

$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\cdot\frac{x^3}{3} + \frac{1}{4}x - \frac{1}{4}\tan^{-1}x + c$

$= \frac{1}{4}\left[x^4\tan^{-1}x - \frac{x^3}{3} + x - \tan^{-1}x\right] + c$

(xii) $\int x^3\cos x\,dx$

Solution:- $\int \underset{I}{x^3}\,\underset{II}{\cos x}\,dx$

$= x^3(\sin x) - \int \sin x\cdot 3x^2\,dx$

$= x^3\sin x - 3\int \underset{I}{x^2}\,\underset{II}{\sin x}\,dx$

$= x^3\sin x - 3\left[x^2(-\cos x) - \int(-\cos x)(2x)dx\right]$

$= x^3\sin x + 3x^2\cos x - 6\int \underset{I}{x}\,\underset{II}{\cos x}\,dx$

$= x^3\sin x + 3x^2\cos x - 6\left[x\sin x - \int \sin x(1)\,dx\right]$

$= x^3\sin x + 3x^2\cos x - 6x\sin x + 6\int\sin x\,dx$

$= x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x + c$

$= (x^3 - 6x)\sin x + (3x^2-6)\cos x + c$

(xiii) $\int \sin^{-1}x\,dx$

Solution:- $\int \underset{I}{\sin^{-1}x}\cdot\underset{II}{1}\,dx$

$= \sin^{-1}x\cdot x - \int x\cdot\frac{1}{\sqrt{1-x^2}}\,dx$

$= x\sin^{-1}x - \int (1-x^2)^{-1/2}\cdot x\,dx$

$= \sin^{-1}x\cdot x + \frac{1}{2}\int(1-x^2)^{-1/2}(-2x)\,dx$

$= x\sin^{-1}x + \frac{1}{\cancel{2}}\cdot\frac{(1-x^2)^{1/2}}{\frac{1}{\cancel{2}}} + c$

$\left(\because \int f^n(x)f'(x)dx = \frac{f^{n+1}(x)}{n+1} + c\right)$

$= x\sin^{-1}x + \sqrt{1-x^2} + c$

(xiv) $\int x\sin^{-1}x\,dx$

Solution:- $\int \underset{II}{x}\,\underset{I}{\sin^{-1}x}\,dx$

$= \sin^{-1}x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}\,dx$

$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}\,dx$

$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int\frac{1-x^2}{\sqrt{1-x^2}}\,dx - \frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}\,dx$

$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int\sqrt{1-x^2}\,dx - \frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}\,dx$

$\left(\begin{array}{l}\because \int\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c \\ \rightarrow \int\sqrt{1-x^2}\,dx = \frac{1}{2}\sin^{-1}x + \frac{x}{2}\sqrt{1-x^2} + c\end{array}\right)$

and $\int\frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c$

$\rightarrow \int\frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\sin^{-1}x + \frac{1}{2}\left(\frac{1}{2}\sin^{-1}x + \frac{x}{2}\sqrt{1-x^2}\right) + c$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\sin^{-1}x + \frac{1}{4}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} + c$

$= \frac{x^2}{2}\sin^{-1}x + \left(\frac{1}{4} - \frac{1}{2}\right)\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} + c$

$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} + c$

(xv) $\int e^x\sin x\cos x\,dx$

Solution:- $\int e^x\sin x\cos x\,dx$

$= \frac{1}{2}\int e^x(2\sin x\cos x)dx = \frac{1}{2}\int e^x\sin 2x\,dx \longrightarrow (i)$

∴ Let $I = \int \underset{II}{e^x}\,\underset{I}{\sin 2x}\,dx$

$= \sin 2x\cdot e^x - \int e^x\cos 2x(2)\,dx$

$= e^x\sin 2x - 2\int \underset{II}{e^x}\,\underset{I}{\cos 2x}\,dx$

$= e^x\sin 2x - 2\left[e^x\cos 2x - \int e^x(-\sin 2x)(2)\,dx\right]$

$= e^x\sin 2x - 2e^x\cos 2x - 4\int e^x\sin 2x\,dx$

$\rightarrow I = e^x\sin 2x - 2e^x\cos 2x - 4I$

$\rightarrow 5I = e^x(\sin 2x - 2\cos 2x)$

$$\rightarrow I = \frac{1}{5}e^{x}(\sin 2x - 2\cos 2x)$$

or $\int e^{x}\sin 2x\,dx = \frac{1}{5}e^{x}(\sin 2x - 2\cos 2x)$

Thus (i) becomes as

$$= \frac{1}{2}\left[\frac{1}{5}e^{x}(\sin 2x - 2\cos 2x)\right]$$

$$= \frac{1}{5}e^{x}\left(\frac{1}{2}\sin 2x - \cos 2x\right)$$

$\because \cos 2\theta = 1 - 2\sin^2\theta$

$$= \frac{1}{5}e^{x}\left(\frac{1}{2}\sin 2x - (1 - 2\sin^2 x)\right)$$

$$= \frac{1}{5}e^{x}\left(\frac{1}{2}\sin 2x - 1 + 2\sin^2 x\right) + c$$

(xiv) $\int x\sin x\cos x\,dx$

Solution:- $\int x\sin x\cos x\,dx$

$$= \frac{1}{2}\int x(2\sin x\cos x)\,dx$$

$$= \frac{1}{2}\int x\sin 2x\,dx \longrightarrow (i)$$

$\because \sin 2\theta = 2\sin\theta\cos\theta$

Let $I = \int \underset{I}{x}\,\underset{II}{\sin 2x}\,dx$

$$= x\left(-\frac{\cos 2x}{2}\right) - \int\left(-\frac{\cos 2x}{2}\right)(1)\,dx$$

$$= -\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x\,dx$$

$$I = -\frac{x\cos 2x}{2} + \frac{1}{2}\,\frac{\sin 2x}{2} + c$$

Thus (i) becomes

$$= \frac{1}{2}\left(-\frac{x\cos 2x}{2} + \frac{1}{2}\frac{\sin 2x}{2}\right) + c$$

$$= -\frac{x\cos 2x}{4} + \frac{1}{8}\sin 2x + c$$

$$= -\frac{x\cos 2x}{4} + \frac{\cancel{2}\sin x\cos x}{\cancel{8}4} + c$$

$$= \frac{1}{4}(-x\cos 2x + \sin x\cos x) + c$$

(xvii) $\int x\cos^2 x\,dx$

Solution:- $\int x\cos^2 x\,dx$

$$= \int x\left(\frac{1+\cos 2x}{2}\right)dx$$

$\because \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$

$\rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \frac{1}{2}\int (x + x\cos 2x)\,dx$$

$$= \frac{1}{2}\int x\,dx + \frac{1}{2}\int \underset{I}{x}\,\underset{II}{\cos 2x}\,dx$$

$$= \frac{1}{2}\cdot\frac{x^2}{2} + \frac{1}{2}\left[x\,\frac{\sin 2x}{2} - \int\frac{\sin 2x}{2}(1)\,dx\right]$$

$$= \frac{x^2}{4} + \frac{1}{2}\left[\frac{x\sin 2x}{2} - \frac{1}{2}\int \sin 2x\,dx\right]$$

$$= \frac{x^2}{4} + \frac{1}{4}x\sin 2x - \frac{1}{4}\left(-\frac{\cos 2x}{2}\right) + c$$

$$= \frac{x^2}{4} + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + c$$

$$= \frac{1}{4}\left(x^2 + x\sin 2x + \frac{1}{2}\cos 2x\right) + c$$

(xviii) $\int x\sin^2 x\,dx$

Solution:- $\int x\left(\frac{1-\cos 2x}{2}\right)dx$

$$= \frac{1}{2}\int x(1-\cos 2x)\,dx$$

$\because \sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$= \frac{1}{2}\int (x - x\cos 2x)\,dx$$

$$= \frac{1}{2}\int x\,dx - \frac{1}{2}\int \underset{I}{x}\,\underset{II}{\cos 2x}\,dx$$

$$= \frac{1}{2}\cdot\frac{x^2}{2} - \frac{1}{2}\left[x\frac{\sin 2x}{2} - \int\frac{\sin 2x}{2}(1)\,dx\right]$$

$$= \frac{x^2}{4} - \frac{1}{2}\left[\frac{x\sin 2x}{2} - \frac{1}{2}\int \sin 2x\,dx\right]$$

$$= \frac{x^2}{4} - \frac{1}{4}x\sin 2x + \frac{1}{4}\left(\frac{-\cos 2x}{2}\right) + c$$

$$= \frac{1}{4}\left(x^2 - x\sin 2x - \frac{1}{2}\cos 2x\right) + c$$

(xix) $\int (\ln x)^2\,dx$

Solution:- $\int \underset{I}{(\ln x)^2}\cdot\underset{II}{1}\,dx$

$$= (\ln x)^2\cdot x - \int \cancel{x}\cdot 2(\ln x)\cdot\frac{1}{\cancel{x}}\,dx$$

$$= (\ln x)^2\cdot x - 2\int \underset{I}{(\ln x)}\cdot\underset{II}{1}\,dx$$

$$= x(\ln x)^2 - 2\left[(\ln x)\cdot x - \int \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx\right]$$

$$= x(\ln x)^2 - 2[x(\ln x) - x] + c$$

$$= x(\ln x)^2 - 2x\ln x + 2x + c$$

$$= (x\ln x)(\ln x - 2) + 2x + c$$

(xx) $\int \ln(\tan x)\cdot\sec^2 x\,dx$

Solution:- $\int \underset{I}{\ln(\tan x)}\cdot\underset{II}{\sec^2 x}\,dx$

$$= \ln(\tan x)\cdot\tan x - \int \cancel{\tan x}\cdot\frac{1}{\cancel{\tan x}}(\sec^2 x)\,dx$$

$(\because \int \sec^2 x\,dx = \tan x + c)$

$$= \tan x\ln(\tan x) - \int \sec^2 x\,dx$$

$$= \tan x\ln(\tan x) - \tan x + c$$

(xxi) $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Solution:- $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

$$= \int \sin^{-1}x\cdot\frac{x}{\sqrt{1-x^2}}\,dx$$

$$= -\frac{1}{2}\int \underset{I}{\sin^{-1}x}\cdot\underset{II}{(1-x^2)^{-1/2}(-2x)}\,dx$$

$$= -\frac{1}{2}\left[\sin^{-1}x\cdot\frac{(1-x^2)^{1/2}}{1/2} - \int\frac{(1-x^2)^{1/2}}{1/2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx\right]$$

$$= -\frac{1}{2}\left[2\sin^{-1}x\sqrt{1-x^2} - 2\int\frac{\cancel{\sqrt{1-x^2}}}{\cancel{\sqrt{1-x^2}}}\,dx\right]$$

$$= -\frac{1}{2}\left[2\sin^{-1}x\sqrt{1-x^2} - 2\int dx\right]$$

$$= -\frac{1}{2}\left[2\sin^{-1}x\sqrt{1-x^2} - 2x\right] + c$$

$$= -\sin^{-1}x\sqrt{1-x^2} + x + c$$

$$= x - \sqrt{1-x^2}\,\sin^{-1}x + c$$

Q2. Evaluate the following integral.

(i) $\int \tan^4 x\,dx$

Solution:- $\int \tan^4 x\,dx$

$= \int \tan^2 x \cdot \tan^2 x\,dx$

$= \int (\sec^2 x - 1)\tan^2 x\,dx$

$= \int (\tan^2 x \sec^2 x - \tan^2 x)\,dx$

$= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$

$\because \int f^n(x) f'(x)\,dx = \frac{f^{n+1}(x)}{n+1} + c$

$= \frac{\tan^3 x}{3} - \int (\sec^2 x - 1)\,dx$

$= \frac{\tan^3 x}{3} - \int \sec^2 x + \int 1\,dx = \frac{\tan^3 x}{3} - \tan x + x + c$

(ii) $\int \sec^4 x\,dx$

Solution:- $\int \sec^4 x\,dx$

$= \int \sec^2 x \cdot \sec^2 x\,dx$

$= \int \sec^2 x\,(1 + \tan^2 x)\,dx$

$= \int (\sec^2 x + \tan^2 x \sec^2 x)\,dx$

$= \int \sec^2 x\,dx + \int \tan^2 x \sec^2 x\,dx$

$= \tan x + \frac{\tan^3 x}{3} + c \qquad \because \int f^n(x) f'(x)\,dx = \frac{f^{n+1}(x)}{n+1} + c$

(iii) $\int e^x \sin 2x \cos x\,dx$

Solution:— $\int e^x \sin 2x \cos x\,dx$

$= \frac{1}{2}\int e^x (2\sin 2x \cos x)\,dx$

$[\because 2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$= \frac{1}{2}\int e^x(\sin 3x + \sin x)\,dx$

$= \frac{1}{2}\int e^x \sin 3x\,dx + \frac{1}{2}\int e^x \sin x\,dx$

$= \frac{1}{2} I_1 + \frac{1}{2} I_2 \longrightarrow (i)$

Let $I_1 = \int \underset{\text{II}}{e^x}\, \underset{\text{I}}{\sin 3x}\,dx$

$= \sin 3x \cdot e^x - \int e^x \cos 3x \cdot 3\,dx$

$= e^x \sin 3x - 3\int \underset{\text{II}}{e^x}\, \underset{\text{I}}{\cos 3x}\,dx$

$= e^x \sin 3x - 3[\cos 3x \cdot e^x - \int e^x(-3\sin 3x)\,dx]$

$I_1 = e^x \sin 3x - 3e^x \cos 3x - 9\int e^x \sin 3x\,dx$

$I_1 = e^x \sin 3x - 3e^x \cos 3x - 9I_1$

$\rightarrow I_1 + 9I_1 = e^x(\sin 3x - 3\cos 3x)$

$10 I_1 = e^x(\sin 3x - 3\cos 3x)$

$I_1 = \frac{1}{10} e^x(\sin 3x - 3\cos 3x)$

and $I_2 = \int \underset{\text{II}}{e^x}\, \underset{\text{I}}{\sin x}\,dx$

$= \sin x \cdot e^x - \int \underset{\text{II}}{e^x}\, \underset{\text{I}}{\cos x}\,dx$

$= e^x \sin x - [\cos x \cdot e^x - \int e^x(-\sin x)\,dx]$

$= e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$

$I_2 = e^x \sin x - e^x \cos x - I_2$

$\rightarrow I_2 + I_2 = e^x \sin x - e^x \cos x$

$\rightarrow 2I_2 = e^x(\sin x - \cos x)$

$I_2 = \frac{1}{2} e^x(\sin x - \cos x)$

Thus (i) becomes

$= \frac{1}{2}\left[\frac{e^x}{10}(\sin 3x - 3\cos 3x) + \frac{e^x}{2}(\sin x - \cos x)\right] + c$

$= \frac{1}{2}\cdot\frac{e^x}{2}\left[\frac{1}{5}(\sin 3x - 3\cos 3x) + \sin x - \cos x\right] + c$

$= \frac{e^x}{4}\left[\frac{1}{5}\sin 3x - \frac{3}{5}\cos 3x + \sin x - \cos x\right] + c$

(iv) $\int \tan^3 x \sec x\,dx$

Solution:- $\int \tan^2 x \cdot \sec x \tan x\,dx$

$= \int (\sec^2 x - 1)\sec x \tan x\,dx$

$= \int (\sec^2 x)(\sec x \tan x)\,dx - \int \sec x \tan x\,dx$

$= \frac{\sec^3 x}{3} - \sec x + c \qquad \because \int f^n(x) f'(x)\,dx = \frac{f^{n+1}(x)}{n+1} + c$

(v) $\int x^2 e^{5x}\,dx$

Solution:- $\int \underset{\text{I}}{x^3}\, \underset{\text{II}}{e^{5x}}\,dx$

$= x^3 \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5}(3x^2)\,dx$

$= \frac{1}{5}x^3 e^{5x} - \frac{3}{5}\int \underset{\text{I}}{x^2}\, \underset{\text{II}}{e^{5x}}\,dx$

$= \frac{1}{5}x^3 e^{5x} - \frac{3}{5}\left[x^2 \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5}(2x)\,dx\right]$

$= \frac{1}{5}x^3 e^{5x} - \frac{3}{25}x^2 e^{5x} + \frac{6}{25}\int \underset{\text{I}}{x}\, \underset{\text{II}}{e^{5x}}\,dx$

$= \frac{1}{5}x^3 e^{5x} - \frac{3}{25}x^2 e^{5x} + \frac{6}{25}\left[x \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5}(1)\,dx\right]$

$= \frac{1}{5}x^3 e^{5x} - \frac{3}{25}x^2 e^{5x} + \frac{6xe^{5x}}{125} - \frac{6}{125}\int e^{5x}\,dx$

$= \frac{1}{5}x^3 e^{5x} - \frac{3}{25}x^2 e^{5x} + \frac{6}{125}xe^{5x} - \frac{6}{125}\cdot\frac{e^{5x}}{5} + c$

$= \frac{e^{5x}}{5}\left[x^3 - \frac{3}{25}x^2 + \frac{6}{125}x - \frac{6}{125}\right] + c$

(vi) $\int e^{-x} \sin 2x\,dx$

Solution:- $I = \int \underset{\text{II}}{e^{-x}}\, \underset{\text{I}}{\sin 2x}\,dx$

$= \sin 2x \cdot \frac{e^{-x}}{-1} - \int \frac{e^{-x}}{-1} \cdot 2\cos 2x\,dx$

$= -e^{-x}\sin 2x + 2\int \underset{\text{II}}{e^{-x}}\, \underset{\text{I}}{\cos 2x}\,dx$

$= -e^{-x}\sin 2x + 2\left[\cos 2x \cdot \frac{e^{-x}}{-1} - \int \frac{e^{-x}}{-1}(-\sin 2x \cdot 2)\,dx\right]$

$= -e^{-x}\sin 2x + 2\left[-e^{-x}\cos 2x - 2\int e^{-x}\sin 2x\,dx\right]$

$I = -e^{-x}\sin 2x - 2e^{-x}\cos 2x - 4I$

$\rightarrow I + 4I = -2e^{-x}\left(\frac{1}{2}\sin 2x + \cos 2x\right) + c_1$

$$\Rightarrow 5I = -2e^{-x}\left(\frac{1}{2}\sin 2x+\cos 2x\right)+c_1$$

$$\Rightarrow I = -\frac{2}{5}e^{-x}\left(\frac{1}{2}\sin 2x+\cos 2x\right)+\frac{c_1}{5}$$

Thus $\int e^{-x}\sin 2x\,dx = -\frac{2}{5}e^{-x}\left(\frac{1}{2}\sin 2x+\cos 2x\right)+c$

$\because \frac{c_1}{5} = c$

(vii) $\int e^{2x}\cos 3x\,dx$

Solution:- $I = \int \underset{II}{e^{2x}}\,\underset{I}{\cos 3x}\,dx$

$$= \cos 3x\cdot\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}(-\sin 3x)(3)\,dx$$

$$= \frac{e^{2x}}{2}\cos 3x+\frac{3}{2}\int \underset{II}{e^{2x}}\,\underset{I}{\sin 3x}\,dx$$

$$= \frac{e^{2x}}{2}\cos 3x+\frac{3}{2}\left[\sin 3x\cdot\frac{e^{2x}}{2} - \int\frac{e^{2x}}{2}\cos 3x\cdot 3\,dx\right]$$

$$I = \frac{e^{2x}}{2}\cos 3x+\frac{3}{4}e^{2x}\sin 3x-\frac{9}{4}\int e^{2x}\cos 3x\,dx$$

$$\Rightarrow I = \frac{e^{2x}}{2}\cos 3x+\frac{3}{4}e^{2x}\sin 3x-\frac{9}{4}I$$

$$I+\frac{9}{4}I = e^{2x}\left(\frac{1}{2}\cos 3x+\frac{3}{4}\sin 3x\right)$$

$$\Rightarrow \frac{13}{\cancel{4}}I = e^{2x}\left(\frac{2\cos 3x+3\sin 3x}{\cancel{4}}\right)$$

$$\Rightarrow I = \frac{e^{2x}}{13}(2\cos 3x+3\sin 3x)+c$$

$$I = \frac{3e^{2x}}{13}\left(\frac{2}{3}\cos 3x+\sin 3x\right)+c$$

(viii) $\int \operatorname{cosec}^3 x\,dx$

Solution:- $I = \int \operatorname{cosec}^3 x\,dx$

$$I = \int \underset{II}{\operatorname{cosec}^2 x}\cdot\underset{I}{\operatorname{cosec} x}\,dx$$

$$= \operatorname{cosec} x(-\cot x) - \int(-\cot x)\frac{d}{dx}(\operatorname{cosec} x)\,dx$$

$$= -\operatorname{cosec} x\cot x+\int\cot x(-\operatorname{cosec} x\cot x)\,dx$$

$$= -\operatorname{cosec} x\cot x-\int\cot^2 x\operatorname{cosec} x\,dx$$

$$(\because 1+\cot^2\theta = \operatorname{cosec}^2\theta \Rightarrow \cot^2\theta = \operatorname{cosec}^2\theta-1)$$

$$= -\operatorname{cosec} x\cot x-\int(\operatorname{cosec}^2 x-1)\operatorname{cosec} x\,dx$$

$$I = -\operatorname{cosec} x\cot x-\int\operatorname{cosec}^3 x\,dx+\int\operatorname{cosec} x\,dx$$

$$I = -\operatorname{cosec} x\cot x - I+\int\operatorname{cosec} x\,dx$$

$$\Rightarrow 2I = -\operatorname{cosec} x\cot x+\ln|\operatorname{cosec} x-\cot x|$$

$$\Rightarrow I = -\frac{1}{2}(\operatorname{cosec} x\cot x-\ln|\operatorname{cosec} x-\cot x|)+c$$

Q3. Show that

$$\int e^{ax}\sin bx\,dx = \frac{1}{\sqrt{a^2+b^2}}e^{ax}\sin\left(bx-\operatorname{Tan}^{-1}\frac{b}{a}\right)+c$$

Solution:- Let $I = \int \underset{II}{e^{ax}}\,\underset{I}{\sin bx}\,dx$

$$= \sin bx\cdot\frac{e^{ax}}{a} - \int\frac{e^{ax}}{a}\cos bx\cdot b\,dx$$

$$= \frac{e^{ax}}{a}\sin bx-\frac{b}{a}\int\underset{II}{e^{ax}}\,\underset{I}{\cos bx}\,dx$$

$$= \frac{e^{ax}}{a}\sin bx-\frac{b}{a}\left[\cos bx\cdot\frac{e^{ax}}{a}-\int\frac{e^{ax}}{a}(-\sin bx\cdot b)\,dx\right]$$

$$I = \frac{e^{ax}}{a}\sin bx-\frac{b}{a^2}e^{ax}\cos bx-\frac{b^2}{a^2}\int e^{ax}\sin bx$$

$$I = \frac{e^{ax}}{a}\sin bx-\frac{b}{a^2}e^{ax}\cos bx-\frac{b^2}{a^2}I$$

$$I+\frac{b^2}{a^2}I = e^{ax}\left(\frac{1}{a}\sin bx-\frac{b}{a^2}\cos bx\right)$$

$$I\left(\frac{a^2+b^2}{\cancel{a^2}}\right) = e^{ax}\left(\frac{a\sin bx-b\cos bx}{\cancel{a^2}}\right)$$

$$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}(\sin bx\cdot a-\cos bx\cdot b)$$

Let $a = r\cos\theta$ and $b = r\sin\theta$

$$\Rightarrow a^2+b^2 = r^2\cos^2\theta+r^2\sin^2\theta$$

$$a^2+b^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2(1) = r^2$$

$$(\because \cos^2\theta+\sin^2\theta = 1),\ \frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \Rightarrow \theta = \tan^{-1}\frac{b}{a}$$

$$I = \frac{e^{ax}}{r^2}(\sin bx\cdot r\cos\theta-\cos bx\cdot r\sin\theta)$$

$$= \frac{e^{ax}\cdot r}{r^2}(\sin bx\cos\theta-\cos bx\sin\theta)$$

$$= \frac{e^{ax}}{r}\sin(bx-\theta)$$

$\because \sin\alpha\cos\beta-\cos\alpha\sin\beta = \sin(\alpha-\beta)$

$$I = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)+c$$

Hence proved.

Q4. Evaluate the following indefinite integrals.

(i) $\int\sqrt{a^2-x^2}\,dx$

Solution:- Let $I = \int\underset{I}{\sqrt{a^2-x^2}}\cdot\underset{II}{1}\,dx$

$$= \sqrt{a^2-x^2}\cdot x-\int x\cdot\frac{d}{dx}(a^2-x^2)^{\frac{1}{2}}\,dx$$

$$= x\sqrt{a^2-x^2}-\int x\cdot\frac{1}{\cancel{2}}(a^2-x^2)^{-\frac{1}{2}}(-\cancel{2}x)\,dx$$

$$= x\sqrt{a^2-x^2}+\int\frac{x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2}-\int\frac{-x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2}-\int\frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int\left(\frac{a^2-x^2}{\sqrt{a^2-x^2}} - \frac{a^2}{\sqrt{a^2-x^2}}\right)dx$$

$$= x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,dx + a^2\int\frac{1}{\sqrt{a^2-x^2}}dx$$

$$I = x\sqrt{a^2-x^2} - I + a^2\int\frac{1}{\sqrt{a^2-x^2}}dx$$

$$\rightarrow 2I = x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}$$

$$\rightarrow I = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

$$\left(\because \int\frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + c\right)$$

(ii) $\int\sqrt{x^2-a^2}\,dx$

Solution:- Let $I = \int\underset{\text{I}}{\sqrt{x^2-a^2}}\cdot\underset{\text{II}}{1}\,dx$

$$I = \sqrt{x^2-a^2}\cdot x - \int x\cdot\frac{d}{dx}(x^2-a^2)^{\frac{1}{2}}dx$$

$$= x\sqrt{x^2-a^2} - \int x\cdot\frac{1}{\cancel{2}}(x^2-a^2)^{-\frac{1}{2}}(\cancel{2}x)\,dx$$

$$= x\sqrt{x^2-a^2} - \int\frac{x^2}{\sqrt{x^2-a^2}}dx$$

$$I = x\sqrt{x^2-a^2} - \int\frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}dx$$

$$= x\sqrt{x^2-a^2} - \int\frac{x^2-a^2}{\sqrt{x^2-a^2}}dx - a^2\int\frac{1}{\sqrt{x^2-a^2}}dx$$

$$I = x\sqrt{x^2-a^2} - \int\sqrt{x^2-a^2}\,dx - a^2\int\frac{1}{\sqrt{x^2-a^2}}dx$$

$$\rightarrow I = x\sqrt{x^2-a^2} - I - a^2\ln|x+\sqrt{x^2-a^2}|$$

$$\rightarrow 2I = x\sqrt{x^2-a^2} - a^2\ln|x+\sqrt{x^2+a^2}|$$

$$\rightarrow I = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln|x+\sqrt{x^2+a^2}| + c$$

(iii) $\int\sqrt{4-5x^2}\,dx$

Solution:- $I = \int\sqrt{4-5x^2}\cdot 1\,dx$

$$= \sqrt{4-5x^2}\cdot x - \int x\cdot\frac{d}{dx}\sqrt{4-5x^2}\,dx$$

$$= \sqrt{4-5x^2}\cdot x - \int x\cdot\frac{1}{\cancel{2}}(4-5x^2)^{-\frac{1}{2}}(-\overset{5}{\cancel{10}}x)\,dx$$

$$= x\sqrt{4-5x^2} - \int\frac{-5x^2}{\sqrt{4-5x^2}}dx$$

$$= x\sqrt{4-5x^2} - \int\frac{4-5x^2-4}{\sqrt{4-5x^2}}dx$$

$$= x\sqrt{4-5x^2} - \int\frac{4-5x^2}{\sqrt{4-5x^2}}dx + 4\int\frac{1}{\sqrt{4-5x^2}}dx$$

$$I = x\sqrt{4-5x^2} - \int\sqrt{4-5x^2}\,dx + 4\int\frac{1}{\sqrt{5(\frac{4}{5}-x^2)}}dx$$

$$I = x\sqrt{4-5x^2} - I + \frac{4}{\sqrt5}\int\frac{1}{\sqrt{(\frac{2}{\sqrt5})^2-x^2}}dx$$

$$\rightarrow 2I = x\sqrt{4-5x^2} + \frac{4}{\sqrt5}\sin^{-1}\frac{x}{(\frac{2}{\sqrt5})}$$

$$\rightarrow I = x\sqrt{4-5x^2} + \frac{2}{\sqrt5}\sin^{-1}\left(\frac{\sqrt5\,x}{2}\right) + c$$

(iv) $\int\sqrt{3-4x^2}\,dx$

Solution:- $I = \int\underset{\text{I}}{\sqrt{3-4x^2}}\cdot\underset{\text{II}}{1}\,dx$

$$= x\sqrt{3-4x^2} - \int x\cdot\frac{d}{dx}(3-4x^2)^{\frac{1}{2}}dx$$

$$= x\sqrt{3-4x^2} - \int x\cdot\frac{1}{\cancel{2}}(3-4x^2)^{-\frac{1}{2}}(-\overset{4}{\cancel{8}}x)\,dx$$

$$= x\sqrt{3-4x^2} - \int\frac{-4x^2}{\sqrt{3-4x^2}}dx$$

$$= x\sqrt{3-4x^2} - \int\frac{3-4x^2-3}{\sqrt{3-4x^2}}dx$$

$$= x\sqrt{3-4x^2} - \int\frac{3-4x^2}{\sqrt{3-4x^2}}dx + 3\int\frac{1}{\sqrt{3-4x^2}}dx$$

$$= x\sqrt{3-4x^2} - \int\sqrt{3-4x^2} + 3\int\frac{1}{\sqrt{4(\frac{3}{4}-x^2)}}dx$$

$$= x\sqrt{3-4x^2} - I + \frac{3}{2}\int\frac{1}{\sqrt{(\frac{\sqrt3}{2})^2-x^2}}dx$$

$$I+I = x\sqrt{3-4x^2} + \frac{3}{2}\sin^{-1}\frac{x}{\frac{\sqrt3}{2}}$$

$$\rightarrow 2I = x\sqrt{3-4x^2} + \frac{3}{2}\sin^{-1}\frac{2x}{\sqrt3}$$

$$\rightarrow I = \frac{x\sqrt{3-4x^2}}{2} + \frac{3}{4}\sin^{-1}\frac{2x}{\sqrt3} + c$$

(v) $\int\sqrt{x^2+4}\,dx$

Solution:- Let $I = \int\underset{\text{I}}{\sqrt{x^2+4}}\cdot\underset{\text{II}}{1}\,dx$

$$= \sqrt{x^2+4}\cdot x - \int x\,\frac{d}{dx}(x^2+4)^{\frac{1}{2}}dx$$

$$= x\sqrt{x^2+4} - \int x\cdot\frac{1}{2}(x^2+4)^{-\frac{1}{2}}(2x)\,dx$$

$$= x\sqrt{x^2+4} - \int\frac{x^2}{\sqrt{x^2+4}}dx$$

$$= x\sqrt{x^2+4} - \int\frac{x^2+4-4}{\sqrt{x^2+4}}dx$$

$$= x\sqrt{x^2+4} - \int\frac{x^2+4}{\sqrt{x^2+4}}dx + 4\int\frac{1}{\sqrt{x^2+4}}dx$$

$$I = x\sqrt{x^2+4} - \int\sqrt{x^2+4}\,dx + 4\ln(x+\sqrt{x^2+4})$$

$$\rightarrow I = x\sqrt{x^2+4} - I + 4\ln(x+\sqrt{x^2+4})$$

$$\rightarrow 2I = x\sqrt{x^2+4} + 4\ln(x+\sqrt{x^2+4})$$

$$\rightarrow I = \frac{x\sqrt{x^2+4}}{2} + 2\ln(x+\sqrt{x^2+4}) + c$$

$$\left(\because \int\frac{1}{\sqrt{x^2+a^2}}dx = \ln|x+\sqrt{x^2+a^2}| + c\right)$$

(vi) $\int x^2 e^{ax} dx$

Solution:- $\int \underset{I}{x^2}\, \underset{II}{e^{ax}}\, dx$

$= x^2 \cdot \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a}\cdot (2x) dx$

$= \frac{x^2 e^{ax}}{a} - \frac{2}{a}\int \underset{I}{x}\, \underset{II}{e^{ax}} dx$

$= \frac{x^2 e^{ax}}{a} - \frac{2}{a}\left[x\cdot \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a}(1) dx\right]$

$= \frac{x^2 e^{ax}}{a} - \frac{2}{a^2} x e^{ax} + \frac{2}{a^2}\int e^{ax} dx$

$= \frac{x^2 e^{ax}}{a} - \frac{2}{a^2} x e^{ax} + \frac{2}{a^2}\cdot \frac{e^{ax}}{a} + c$

$= \frac{x^2 e^{ax}}{a} - \frac{2}{a^2} x e^{ax} + \frac{2 e^{ax}}{a^3} + c$

Q5. Evaluate the following integrals.

(i) $\int e^x \left(\frac{1}{x} + \ln x\right) dx$

Solution:- $\int e^x \left(\frac{1}{x} + \ln x\right) dx$

$= \int e^x \left(\ln x + \frac{1}{x}\right) dx$

$= \int e^x \left(\ln x + \frac{d}{dx}(\ln x)\right) dx$

$= e^x \ln x + c$ $\because \int e^x (f(x) + f'(x)) = e^x f(x) + c$

(ii) $\int e^x (\cos x + \sin x) dx$

Solution:- $\int e^x (\cos x + \sin x) dx$

$= \int e^x (\sin x + \cos x) dx$

$= \int e^x \left(\sin x + \frac{d}{dx}(\sin x)\right) dx$

$= e^x \sin x + c$

(iii) $\int e^{ax}\left[a \sec^{-1} x + \frac{1}{x\sqrt{x^2-1}}\right] dx$

Solution:- $\int e^{ax}\left[a \sec^{-1} x + \frac{1}{x\sqrt{x^2-1}}\right]$

$= e^{ax} \sec^{-1} x + c$

$\left(\because \int e^{ax}(a f(x) + f'(x)) dx = e^{ax} f(x) + c\right)$

(iv) $\int e^{3x}\left(\frac{3\sin x - \cos x}{\sin^2 x}\right) dx$

Solution:- $\int e^{3x}\left(\frac{3\sin x - \cos x}{\sin^2 x}\right) dx$

$= \int e^{3x}\left(\frac{3\sin x}{\sin^2 x} - \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}\right) dx$

$= \int e^{3x}\left(\frac{3}{\sin x} - \operatorname{cosec} x \cot x\right) dx$

$= \int e^{3x}\left(3 \operatorname{cosec} x - \operatorname{cosec} x \cot x\right) dx$

$= \int e^{3x}\left[3 \operatorname{cosec} x + (-\operatorname{cosec} x \cot x)\right] dx$

$= e^{3x} \operatorname{cosec} x + c$

$\because \int e^{ax}(a f(x) + f'(x)) dx = e^{ax} f(x) + c$

(v) $\int e^{2x}[-\sin x + 2\cos x] dx$

Solution:- $\int e^{2x}[-\sin x + 2\cos x] dx$

$= \int e^{2x}[2\cos x + (-\sin x)] dx$

$= \int e^{2x}\left[2\cos x + \frac{d}{dx}(\cos x)\right] dx$

$= e^{2x} \cos x + c$

(vi) $\int \frac{x e^x}{(1+x)^2} dx$

Solution:- $\int \frac{x e^x}{(1+x)^2} dx$

$= \int e^x \cdot \frac{1+x-1}{(1+x)^2} dx$

$= \int e^x \cdot \left[\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2}\right] dx$

$= \int e^x \left[\frac{1}{1+x} + \left(\frac{-1}{(1+x)^2}\right)\right] dx$

$= \int e^x \left[\frac{1}{1+x} + \frac{d}{dx}\left(\frac{1}{1+x}\right)\right] dx$

$\left(\because \frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{d}{dx}(1+x)^{-1} = (-1)(1+x)^{-1-1} = \frac{-1}{(1+x)^2}\right.$

$= e^x \left(\frac{1}{1+x}\right) + c = \frac{e^x}{1+x} + c$

(vii) $\int e^{-x}(\cos x - \sin x) dx$

Solution:- $\int e^{-x}(\cos x - \sin x) dx$

$= \int e^{-x}(-\sin x + \cos x) dx$

$= \int e^{(-1)x}((-1)\sin x + \cos x) dx$

$= e^{-x} \sin x + c$ $\because \int e^{ax}(a f(x) + f'(x)) dx = e^{ax} f(x) + c$

(viii) $\int \frac{e^{m\tan^{-1} x}}{(1+x^2)} dx$

Solution:- $\int \frac{e^{m\tan^{-1} x}}{1+x^2} dx$

$= \int e^{m\tan^{-1} x} \cdot \frac{1}{1+x^2} dx$

Put $t = \tan^{-1} x \rightarrow dt = d(\tan^{-1} x)$

$\rightarrow dt = \frac{1}{1+x^2} dx$

$= \int e^{mt} dt = \frac{e^{mt}}{m} + c = \frac{e^{m\tan^{-1} x}}{m} + c$

(ix) $\int \frac{2x}{1-\sin x}\,dx$

Solution:- $\qquad \because \cos(\frac{\pi}{2}-\theta)=\sin\theta$

$$= \int \frac{2x}{1-\cos(\frac{\pi}{2}-x)}\,dx$$

$$= \int \frac{2x}{2\sin^2\frac{(\frac{\pi}{2}-x)}{2}}\,dx \qquad \because 1-\cos\theta = 2\sin^2\frac{\theta}{2}$$

$$= \int \frac{\cancel{2}x}{\cancel{2}\sin^2(\frac{\pi}{4}-\frac{x}{2})}\,dx$$

$$= \int \underset{\text{I}}{x}\,\underset{\text{II}}{\operatorname{cosec}^2}(\frac{\pi}{4}-\frac{x}{2})\,dx$$

$$= x\left[\frac{-\cot(\frac{\pi}{4}-\frac{x}{2})}{(-\frac{1}{2})}\right] - \int \frac{-\cot(\frac{\pi}{4}-\frac{x}{2})}{(-\frac{1}{2})}(1)\,dx$$

$$= x\,(2\cot(\frac{\pi}{4}-\frac{x}{2}) - 2\int \cot(\frac{\pi}{4}-\frac{x}{2})\,dx$$

$$= 2x\cot(\frac{\pi}{4}-\frac{x}{2}) - 2\int \frac{\cos(\frac{\pi}{4}-\frac{x}{2})}{\sin(\frac{\pi}{4}-\frac{x}{2})}\,dx$$

$$= 2x\cot(\frac{\pi}{4}-\frac{x}{2}) - 2(-2)\int \frac{\cos(\frac{\pi}{4}-\frac{x}{2})(-\frac{1}{2})}{\sin(\frac{\pi}{4}-\frac{x}{2})}\,dx$$

$$= 2x\cot(\frac{\pi}{4}-\frac{x}{2}) + 4\ln|\sin(\frac{\pi}{4}-\frac{x}{2})| + c$$

(x) $\int \frac{e^x(1+x)}{(2+x)^2}\,dx$

Solution:- $\int e^x\frac{(2+x-1)}{(2+x)^2}\,dx$

$$= \int e^x\left[\frac{2+x}{(2+x)^2} + \left(\frac{-1}{(2+x)^2}\right)\right]dx$$

$$= \int e^x\left[\frac{1}{2+x} + \frac{-1}{(2+x)^2}\right]dx$$

$$= \int e^x\left[\frac{1}{2+x} + \frac{d}{dx}\left(\frac{1}{2+x}\right)\right]dx$$

$$= e^x\cdot\frac{1}{2+x} + c = \frac{e^x}{2+x} + c$$

$$\left(\begin{aligned}&\because \int e^x[f(x)+f'(x)]\,dx\\ &= e^x f(x)+c\end{aligned}\right)$$

(xi) $\int \left(\frac{1-\sin x}{1-\cos x}\right)e^x\,dx$

Solution:- $\int\left(\frac{1-\sin x}{1-\cos x}\right)e^x\,dx$

$$= \int e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)$$

$$\left(\begin{aligned}&\because \sin 2\theta = 2\sin\theta\cos\theta\\ &\Rightarrow \sin\theta = 2\sin\frac{\theta}{2}\cos\theta\end{aligned}\right)$$

and $\left(1-\cos\theta = 2\sin^2\frac{\theta}{2}\right)$

$$= \int e^x\left(\frac{1}{2\sin^2\frac{x}{2}} - \frac{\cancel{2}\cancel{\sin}\frac{x}{2}\cos\frac{x}{2}}{\cancel{2}\sin^{\cancel{2}}\frac{x}{2}}\right)dx$$

$$= \int e^x\left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right)dx$$

$$= \int e^x\left(-\cot\frac{x}{2} + \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right)dx$$

$$= \int e^x\left(-\cot\frac{x}{2} + \frac{d}{dx}(-\cot\frac{x}{2})\right)dx$$

$$= e^x(-\cot\frac{x}{2}) + c = -e^x\cot\frac{x}{2} + c$$

$$\left(\because \int e^x(f(x)+f'(x))\,dx = e^x f(x)+c\right)$$

Integration involving "Partial Fractions"

If $P(x), Q(x)$ are two polynomial functions and $Q(x)(\neq 0)$ in rational fraction $\frac{P(x)}{Q(x)}$ can be factorized into linear and quadratic (irreducible) factors, then the rational function is written as a sum of simpler rational functions, each of which can be integrated by methods already known.

Here we will discuss examples of the three cases of partial fraction and then apply integration.

Case #1

When $Q(x)$ contains non-repeated linear factors. e.g., $\frac{P(x)}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$

or $\frac{-x+6}{(x-2)(x-3)(x-4)} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{x-4}$

etc

Examples of case I

Example 1. Evaluate $\int \frac{-x+6}{2x^2-7x+6}dx \quad (x>2)$

Solution:- $\int \frac{-x+6}{2x^2-7x+6}dx$

Let $\frac{-x+6}{(x-2)(2x-3)} = \frac{A}{x-2} + \frac{B}{2x-3} \rightarrow (i)$ $\quad \because 2x^2-7x+6 = 2x^2-4x-3x+6 = 2x(x-2)-3(x-2) = (x-2)(2x-3)$

'x' by $(x-2)(2x-3)$, we get

$\rightarrow -x+6 = A(2x-3)+B(x-2) \longrightarrow (ii)$

Put $x-2=0 \rightarrow x=2$ in (ii)

$\rightarrow -2+6 = A(2(2)-3)+B(0)$

$4 = A \rightarrow \boxed{A=4}$

Put $2x-3=0 \rightarrow x=\frac{3}{2}$ in (ii)

$\rightarrow -\frac{3}{2}+6 = A(0)+B(\frac{3}{2}-2)$

$\rightarrow \frac{-3+12}{\cancel{2}} = B(\frac{3-4}{\cancel{2}}) \rightarrow 9 = -B \rightarrow \boxed{B=-9}$

so (i) $\rightarrow \frac{-x+6}{(x-2)(2x-3)} = \frac{4}{x-2} - \frac{9}{2x-3}$

$\rightarrow \int \frac{-x+6}{(x-2)(2x-3)}dx = 4\int \frac{1}{x-2}dx - 9\int \frac{1}{2x-3}dx$

$= 4\int \frac{1}{x-2}dx - \frac{9}{2}\int \frac{2}{2x-3}dx$

$\rightarrow \int \frac{-x+6}{2x^2-7x+6}dx = 4\ln|x-2| - \frac{9}{2}\ln|2x-3| + c$

Example 2. Evaluate $\int \frac{2x^3-9x^2+12x}{2x^2-7x+6}dx, \ (x>2)$

Solution:- $\int \frac{2x^3-9x^2+12x}{2x^2-7x+6}dx$

$$\begin{array}{r|l} & x-1 \\ \hline 2x^2-7x+6 & 2x^3-9x^2+12x \\ & \underset{-}{2x^3}\ \underset{+}{-7x^2}\ \underset{-}{+6x} \\ \hline & -2x^2+6x \\ & \underset{+}{-2x^2}\ \underset{-}{+7x}\ \underset{+}{-6} \\ \hline & 6-x \end{array}$$

Thus

$\int \frac{2x^3-9x^2+12x}{2x^2-7x+6}dx = \int (x-1+\frac{6-x}{2x^2-7x+6})dx$

Now

$\frac{-x+6}{(x-2)(2x-3)} = \frac{A}{x-2} + \frac{B}{2x-3} \rightarrow (i)$ $\quad \because 2x^2-7x+6 = (x-2)(2x-3)$

'x' by $(x-2)(2x-3)$, we get

$\rightarrow -x+6 = A(2x-3)+B(x-2) \longrightarrow (ii)$

Put $x-2=0 \rightarrow x=2$ in (ii)

$\rightarrow -2+6 = A(2(2)-3)+B(0)$

$\rightarrow 4 = A(1) \rightarrow \boxed{A=4}$

Put $2x-3=0 \rightarrow x=\frac{3}{2}$ in (ii)

$\rightarrow -\frac{3}{2}+6 = A(0)+B(\frac{3}{2}-2)$

$\rightarrow \frac{-3+12}{\cancel{2}} = B(\frac{3-4}{\cancel{2}}) \rightarrow 9 = B(-1) \rightarrow \boxed{B=-9}$

so $\frac{-x+6}{(x-2)(2x-3)} = \frac{4}{x-2} - \frac{9}{2x-3}$

Now

$\int \frac{2x^3-9x^2+12x}{2x^2-7x+6}dx = \int (x-1+\frac{4}{x-2}-\frac{9}{2x-3})dx$

$= \int x\,dx - \int 1\,dx + 4\int \frac{1}{x-2}dx - \frac{9}{2}\int \frac{2}{2x-3}dx$

$= \frac{x^2}{2} - x + 4\ln|x-2| - \frac{9}{2}\ln|2x-3| + c$

Example 3. Evaluate (i) $\int \frac{2a}{x^2-a^2}dx, \ (x>a)$

(ii) $\int \frac{2a}{a^2-x^2}dx, \ (x<a)$

Solution:- (i) $\int \frac{2a}{x^2-a^2}dx, \ (x>a)$

Let $\frac{2a}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$ $\quad \because x^2-a^2 = (x-a)(x+a)$

'x' by $(x-a)(x+a)$, we get

$\rightarrow 2a = A(x+a)+B(x-a) \longrightarrow (i)$

Put $x-a=0 \rightarrow x=a$ in (i)

$\rightarrow 2a = A(a+a)+B(0) \rightarrow \cancel{2a} = \cancel{2a}A$

$\rightarrow \boxed{A=1}$

Put $x+a=0 \rightarrow x=-a$ in (i)

$\rightarrow 2a = A(0)+B(-a-a) \rightarrow 2a = -2aB$

$\rightarrow \boxed{B=-1}$

So $\frac{2a}{(x-a)(x+a)} = \frac{1}{x-a} - \frac{1}{x+a}$

$\rightarrow \int \frac{2a}{(x-a)(x+a)}dx = \int \frac{1}{x-a}dx - \int \frac{1}{x+a}dx$

$= \ln|x-a| - \ln|x+a| + c$

$= \ln\left|\frac{x-a}{x+a}\right| + c$

(ii) $\int \frac{2a}{a^2-x^2}dx$

Let $\frac{2a}{(a-x)(a+x)} = \frac{A}{(a-x)} + \frac{B}{a+x}$ $\quad \because a^2-x^2 = (a-x)(a+x)$

'x' by $(a-x)(a+x)$, we get

$\rightarrow 2a = A(a+x)+B(a-x)$ —— (i)

Put $a-x=0 \rightarrow x=a$ in (i)

$\rightarrow 2a = 2aA \rightarrow \boxed{A=1}$

Put $a+x=0 \rightarrow x=-a$ in (i)

$\rightarrow 2a = A(0)+B(a-(-a)) \rightarrow 2a = 2aB$

$\rightarrow \boxed{B=1}$

So $\frac{2a}{(a-x)(a+x)} = \frac{1}{a-x} + \frac{1}{a+x}$

$$\Rightarrow \int \frac{2a}{(a-x)(a+x)}dx = \int \frac{1}{a-x}dx + \int \frac{1}{a+x}dx$$

$$= -\int \frac{-1}{a-x}dx + \int \frac{1}{a+x}dx$$

$$= -\ln|a-x| + \ln|a+x| + c$$

$$= \ln\left|\frac{a+x}{a-x}\right| + c$$

Case # 2

When Q(x) contains non-repeated and repeated linear factors.

e.g., $\frac{p(x)}{(x-a)(x+b)^2} = \frac{A}{x-a} + \frac{B}{x+b} + \frac{C}{(x+b)^2}$

$$\frac{2x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \text{ etc}$$

Examples of Case 2

Example 4. Evaluate $\int \frac{7x-1}{(x-1)^2(x+1)}dx$ $(x>1)$

Solution:- $\int \frac{7x-1}{(x-1)^2(x+1)}dx$

Let $\frac{7x-1}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \rightarrow$ (i)

$\Rightarrow 7x-1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$ —— (ii)

Put $x-1=0 \Rightarrow x=1$ in (ii)

$\Rightarrow 7(1)-1 = A(0)(1+1) + B(1+1) + C(0)^2$

$6 = 2B \Rightarrow \boxed{B=3}$

Put $x+1=0 \Rightarrow x=-1$ in (ii)

$\Rightarrow 7(-1)-1 = A(-1-1)(0) + B(0) + C(-1-1)^2$

$-8 = C(-2)^2 \Rightarrow -8 = 4C \Rightarrow \boxed{C=-2}$

from (ii)

$7x-1 = A(x^2-1) + Bx + B + C(x^2+1-2x)$

$7x-1 = Ax^2 - A + Bx + B + Cx^2 + C - 2Cx$

Equating coefficient of x^2

$\Rightarrow A + C = 0$

$\Rightarrow A - 2 = 0 \Rightarrow \boxed{A=2}$

So $\int \frac{7x-1}{(x-1)^2(x+1)}dx = \int \frac{2}{x-1}dx + 3\int \frac{1}{(x-1)^2}dx - \int \frac{2dx}{x+1}$

$$= 2\ln|x-1| + 3\int (x-1)^{-2}dx - 2\ln|x+1|$$

$$= 2\ln|x-1| + 3\frac{(x-1)^{-2+1}}{-2+1} - 2\ln|x+1|$$

$$= 2\ln\left(\frac{x-1}{x+1}\right) - \frac{3}{x-1} + c$$

Example 5. Evaluate $\int \frac{e^x(x^2+1)}{(x+1)^2}dx$

Solution:- $\int e^x \cdot \frac{x^2+1}{(x+1)^2}dx$

$x^2+1+2x \overline{)\, x^2+1 \,}$ quotient 1

$\underline{\ \ _-x^2 \ \underset{-}{+}1 \ \underset{-}{+}2x\ \ }$

$-2x$

$\because (x+1)^2 = x^2+1+2x$

$$= \int e^x\left(1 - \frac{2x}{(x+1)^2}\right)dx$$

$$= \int e^x dx - 2e^x\int \frac{x}{(x+1)^2}dx \text{ —— (I)}$$

Now

$\frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$

$\Rightarrow x = A(x+1) + B$ —— (i)

Put $x+1=0 \Rightarrow x=-1$ in (i)

$\Rightarrow -1 = A(0) + B \Rightarrow \boxed{B=-1}$

from (i) $\Rightarrow x = Ax + A + B$

Equating coefficient of x, $\boxed{A=1}$

So $\int \frac{x}{(x+1)^2}dx = \int \frac{1}{x+1}dx + \int \frac{-1}{(x+1)^2}dx$

So (I)

$$= \int e^x dx - 2e^x\left[\int \frac{1}{x+1}dx - \int \frac{1}{(x+1)^2}dx\right]$$

$$= e^x - 2\int \frac{e^x}{x+1}dx - 2\int \frac{e^x}{(x+1)^2}dx$$

$$= e^x - 2\int \frac{e^x}{x+1}dx - 2\int \underset{\text{I}}{e^x}\,\underset{\text{II}}{(x+1)^{-2}}dx$$

$$= e^x - 2\int \frac{e^x}{(x+1)}dx - 2\left[e^x\frac{(x+1)^{-2+1}}{-2+1} - \int \frac{(x+1)^{-2+1}}{-2+1}\cdot e^x dx\right]$$

$$= e^x - 2\cancel{\int \frac{e^x}{x+1}dx} + 2\frac{e^x}{x+1} - 2\cancel{\int \frac{e^x}{x+1}dx}$$

$$= e^x - 2\frac{e^x}{x+1} + c$$

$$= \frac{e^x x + e^x - 2e^x}{x+1} + c = \frac{e^x(x-1)}{x+1} + c$$

Case # 3

When Q(x) contains non-repeated irreducible quadratic factors.

e.g., $\frac{p(x)}{(x+b)(x^2+c)} = \frac{A}{x+b} + \frac{Bx+C}{x^2+c}$

$$\frac{1}{(x-1)(x^2+1+x)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1+x} \text{ etc}$$

Examples of Case 3

Example 6. Evaluate $\int \frac{1}{x^3-1}dx$

Solution:– $\int \frac{1}{x^3-1}dx = \int \frac{1}{(x-1)(x^2+1+x)}dx$

$(\because (x)^3-(1)^3 = (x-1)(x^2+1+2x))$

Now

$$\frac{1}{(x-1)(x^2+1+x)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1+x}$$

$$\rightarrow 1 = A(x^2+1+x)+(Bx+C)(x-1) \longrightarrow (i)$$

put $x-1=0 \rightarrow x=1$ in (i)

$$\rightarrow 1 = A((1)^2+1+(1)) + (B(1)+C)(0)$$

$$1 = 3A \rightarrow \boxed{A=\frac{1}{3}}$$

from (i) $\rightarrow 1 = Ax^2+A+Ax+Bx^2-Bx+Cx-C$

Equating coefficients of x^2 and x

For x^2; $0 = A+B \rightarrow A+B=0$

$\rightarrow \frac{1}{3}+B=0 \rightarrow \boxed{B=-\frac{1}{3}}$

For x; $0 = A-B+C \rightarrow 0=\frac{1}{3}-(-\frac{1}{3})+C$

$\rightarrow 0 = \frac{2}{3}+C \rightarrow \boxed{C=-\frac{2}{3}}$

Thus

$$\frac{1}{x^3-1} = \frac{1}{3(x-1)} + \frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+1+x}$$

$$= \frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{3}\left(\frac{x+2}{x^2+x+1}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{6}\left(\frac{2x+4}{x^2+x+1}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{6}\left(\frac{2x+1+3}{x^2+x+1}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{6}\left(\frac{2x+1}{x^2+x+1}\right) - \frac{1}{6}\cdot\frac{3}{x^2+x+1}$$

$$= \frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{6}\cdot\frac{2x+1}{x^2+x+1} - \frac{1}{2}\cdot\frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}$$

$$\frac{1}{x^3-1} = \frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{6}\cdot\frac{2x+1}{x^2+x+1} - \frac{1}{2}\frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$\rightarrow \int\frac{dx}{x^3-1} = \frac{1}{3}\int\frac{1}{x-1}dx - \frac{1}{6}\int\frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int\frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{\cancel{2}}\cdot\frac{1}{\frac{\sqrt{3}}{\cancel{2}}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

$$\left(\because \int\frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right)$$

Example 7. Evaluate $\int \frac{2x}{x^6-1}dx$

Solution:– $\int \frac{2x}{(x^2)^3-1}dx$

put $x^2 = t \rightarrow d(x^2) = dt$

$\rightarrow 2xdx = dt$

$$= \int\frac{dt}{t^3-1} = \int\frac{dt}{(t-1)(t^2+1+t)}$$

Now

$$\frac{1}{(t-1)(t^2+t+1)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+1+t}$$

$$\rightarrow 1 = A(t^2+1+t) + (Bt+C)(t-1) \longrightarrow (i)$$

put $t-1=0 \rightarrow t=1$ in (i)

$$\rightarrow 1 = A[(1)^2+1+1] + (B(1)+C)(1-1)$$

$$1 = 3A \rightarrow \boxed{A=\frac{1}{3}}$$

from (i) $\rightarrow 1 = At^2+A+At+Bt^2-Bt+Ct-C$

Equating coefficients of t^2 and t

For t^2; $0 = A+B \rightarrow \frac{1}{3}+B=0$

$\rightarrow \boxed{B=-\frac{1}{3}}$

For t; $0 = A-B+C \rightarrow 0=\frac{1}{3}-(-\frac{1}{3})+C$

$\rightarrow 0=\frac{2}{3}+C \rightarrow \boxed{C=-\frac{2}{3}}$

Thus

$$\frac{1}{t^3-1} = \frac{1}{3(t-1)} + \frac{(-\frac{1}{3})t-\frac{2}{3}}{t^2+t+1}$$

$$= \frac{1}{3}\cdot\frac{1}{t-1} - \frac{1}{3}\left(\frac{t+2}{t^2+t+1}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{t-1} - \frac{1}{6}\left(\frac{2t+4}{t^2+t+1}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{t-1} - \frac{1}{6}\left(\frac{2t+1+3}{t^2+t+1}\right)$$

$$= \frac{1}{3}\cdot\frac{1}{t-1} - \frac{1}{6}\left(\frac{2t+1}{t^2+t+1}\right) - \frac{1}{2}\cdot\frac{1}{t^2+t+1}$$

$$= \frac{1}{3}\cdot\frac{1}{t-1} - \frac{1}{6}\left(\frac{2t+1}{t^2+t+1}\right) - \frac{1}{2}\frac{1}{t^2+t+\frac{1}{4}+\frac{3}{4}}$$

$$= \frac{1}{3}\cdot\frac{1}{t-1} - \frac{1}{6}\left(\frac{2t+1}{t^2+t+1}\right) - \frac{1}{2}\frac{1}{(t+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

Thus

$$\rightarrow \int\frac{dt}{t^3-1} = \frac{1}{3}\int\frac{1}{t-1}dt - \frac{1}{6}\int\frac{2t+1}{t^2+t+1}dt - \frac{1}{2}\int\frac{1}{(t+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}dt$$

$$= \frac{1}{3}\ln|t-1| - \frac{1}{6}\ln|t^2+t+1| - \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$

$$= \frac{1}{3}\ln|t-1| - \frac{1}{6}\ln|t^2+t+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)+c$$

Replacing t by x^2

$$\rightarrow \int\frac{dx}{x^6-1} = \frac{1}{3}\ln|x^2-1| - \frac{1}{6}\ln|x^4+x^2+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+c$$

Example 8. Evaluate $\int \frac{3}{x(x^3-1)}dx$, $x\neq 0$, $x\neq -1$

Solution:- $\int \frac{3}{x(x-1)(x^2+x+1)}dx$

Now

$$\frac{3}{x(x-1)(x^2+x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1}$$

$$\rightarrow 3 = A(x-1)(x^2+x+1) + B(x)(x^2+x+1) + (Cx+D)(x)(x-1) \longrightarrow (i)$$

Put $x=0$ in (i)

$$3 = A(0-1)((0)^2+0+1) + B(0)(0+0+1) + (C(0)+D)(0)(0-1)$$

$$\rightarrow 3 = A(-1)(1) \rightarrow \boxed{A=-3}$$

Put $x-1=0 \Rightarrow x=1$ in (i)

$$3 = A(0)(1+1+1) + B(1)(1+1+1) + (C(1)+D)(1)(0)$$

$$3 = 3B \rightarrow \boxed{B=1}$$

From (i)

$$3 = A(x^3-1) + Bx^3 + Bx^2 + B + (Cx^2+Dx)(x-1)$$

$$3 = Ax^3 - A + Bx^3 + Bx^2 + B + Cx^3 - Cx^2 + Dx^2 - Dx$$

Equating coefficients of x^3 and x^2

For x^3; $0 = A+B+C$

$\rightarrow 0 = -3+1+C \rightarrow 0 = -2+C$

$\rightarrow \boxed{C=2}$

For x^2; $0 = B - C + D$

$\rightarrow 0 = 1-2+D \rightarrow \boxed{D=1}$

Thus

$$\frac{3}{x(x^3-1)} = \frac{-3}{x} + \frac{1}{x-1} + \frac{2x+1}{x^2+x+1}$$

$$\rightarrow \int \frac{3}{x(x^3-1)}dx = -3\int \frac{1}{x}dx + \int \frac{1}{x-1}dx + \int \frac{2x+1}{x^2+x+1}dx$$

$$= -3\ln|x| + \ln|x-1| + \ln|x^2+x+1| + c$$

$$= -3\ln|x| + \ln(x-1)(x^2+x+1) + c$$

$$\int \frac{3dx}{x(x^3-1)} = -3\ln|x| + \ln(x^3-1) + c$$

Example 9. Evaluate $\int \frac{2x^2+6x}{(x^2+1)(x^2+2x+3)}dx$

Solution:- $\int \frac{2x^2+6x}{(x^2+1)(x^2+2x+3)}dx$

Now

$$\frac{2x^2+6x}{(x^2+1)(x^2+2x+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2x+3}$$

$$\rightarrow 2x^2+6x = (Ax+B)(x^2+2x+3) + (Cx+D)(x^2+1)$$

$$2x^2+6x = Ax^3 + 2Ax^2 + 3Ax + Bx^2 + 2Bx + 3B + Cx^3 + Cx + Dx^2 + D$$

Equating coefficients of x^3, x^2, x and constant term

For x^3; $0 = A + C$ —— (i)

For x^2; $2 = 2A + B + D$ —— (ii)

For x; $6 = 3A + 2B + C$ —— (iii)

For constant term; $0 = 3B + D$ —— (iv)

By (iii)-(ii) $\rightarrow$

$$\begin{array}{l} 6 = 3A + 2B + C \\ \underline{\underset{-}{2} = \underset{-}{2A} \underset{-}{+B} \quad \underset{-}{+D}} \\ 4 = A + B + C - D \end{array}$$

$\rightarrow A + C - D = 4 - B$ —— (v)

By (i)-(iv) $\rightarrow$

$$\begin{array}{l} 0 = A + C \\ \underline{\underset{-}{0} = \underset{-}{3B} \underset{-}{+D}} \end{array}$$

$\rightarrow A + C - 3B - D = 0$

$\rightarrow A + C - D = 3B$ —— (vi)

Comparing (v) and (vi) $3B = 4 - B$

$\rightarrow 4B = 4 \rightarrow \boxed{B=1}$ Put in (iv)

$\rightarrow 0 = 3(1) + D \rightarrow \boxed{D=-3}$

Put $B=1$ and $D=-3$ in (ii)

$\rightarrow 2 = 2A + 1 - 3 \rightarrow 2-1+3 = 2A \rightarrow \boxed{A=2}$

Put $A=2$ in (i) $\rightarrow 0 = 2 + C \rightarrow \boxed{C=-2}$

So

$$\frac{2x^2+6x}{(x^2+1)(x^2+2x+3)} = \frac{2x+1}{x^2+1} + \frac{-2x-3}{x^2+2x+3}$$

$$= \frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2x+3}{x^2+2x+3}$$

$$= \frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2x+2+1}{x^2+2x+3}$$

$$= \frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2x+2}{x^2+2x+3} - \frac{1}{x^2+2x+3}$$

$$\int \frac{2x^2+6x}{(x^2+1)(x^2+2x+3)}dx = \int \frac{2x\,dx}{x^2+1} + \int \frac{dx}{x^2+1} - \int \frac{2x+2}{x^2+2x+3}dx - \int \frac{1}{x^2+2x+1+2}dx$$

$$= \ln|x^2+1| + \tan^{-1}x - \ln|x^2+2x+3| - \int \frac{dx}{(x+1)^2+(\sqrt{2})^2}$$

$$= \ln|x^2+1| + \tan^{-1}x - \ln|x^2+2x+3| - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

$$\left(\because \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right)$$

Exercise 3.5

Evaluate the following integrals.

Q1. $\int \frac{3x+1}{x^2-x-6}dx$

Solution:- $\int \frac{3x+1}{x^2-x-6}dx$

Now

$$\frac{3x+1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$\because x^2-x-6$
$= x^2-3x+2x-6$
$= x(x-3)+2(x-3)$
$= (x-3)(x+2)$

$\Rightarrow 3x+1 = A(x+2)+B(x-3)$ —— (i)

Put $x-3=0 \Rightarrow x=3$ in (i)

$\Rightarrow 3(3)+1 = A(3+2)+B(0) \Rightarrow 5A=10 \Rightarrow \boxed{A=2}$

Put $x+2=0 \Rightarrow x=-2$ in (i)

$3(-2)+1 = A(0) + B(-2-3) \Rightarrow -5B = -6+1$

$\Rightarrow -5B=-5 \Rightarrow \boxed{B=1}$

So $\frac{3x+1}{(x-3)(x+2)} = \frac{2}{x-3} + \frac{1}{x+2}$

$\Rightarrow \int \frac{3x+1}{(x-3)(x+2)}dx = 2\int \frac{1}{x-3}dx + \int \frac{1}{x+2}dx$

$= 2\ln|x-3| + \ln|x+2| + c$

Q2. $\int \frac{5x+8}{(x+3)(2x-1)}dx$

Solution:- $\int \frac{5x+8}{(x+3)(2x-1)}dx$

Now

$$\frac{5x+8}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{2x-1}$$

$\Rightarrow 5x+8 = A(2x-1) + B(x+3) \longrightarrow$ (i)

Put $2x-1=0 \Rightarrow x=\frac{1}{2}$ in (i)

$\Rightarrow 5(\frac{1}{2})+8 = A(0) + B(\frac{1}{2}+3)$

$\Rightarrow \frac{5+16}{\cancel{2}} = B(\frac{1+6}{\cancel{2}}) \Rightarrow 7B=21 \Rightarrow \boxed{B=3}$

Put $x+3=0 \Rightarrow x=-3$ in (i)

$\Rightarrow 5(-3)+8 = A(2(-3)-1) + B(0)$

$-15+8 = -7A \Rightarrow -7=-7A \Rightarrow \boxed{A=1}$

So $\frac{5x+8}{(x+3)(2x-1)} = \frac{1}{x+3} + \frac{3}{2x-1}$

$\Rightarrow \int \frac{5x+8}{(x+3)(2x-1)}dx = \int \frac{1}{x+3}dx + 3\int \frac{1}{2x-1}dx$

$= \ln|x+3| + \frac{3}{2}\int \frac{2}{2x-1}dx$

$\int \frac{5x+8}{(x+3)(2x-1)}dx = \ln|x+3| + \frac{3}{2}\ln|2x-1| + c$

Q3. $\int \frac{x^2+3x-34}{x^2+2x-15}dx$

Solution:- $\int \frac{x^2+3x-34}{x^2+2x-15}dx$

$$\begin{array}{r} 1 \\ x^2+2x-15 \overline{) \; x^2+3x-34} \\ x^2+2x-15 \\ - \quad - \quad + \\ \hline x-19 \end{array}$$

So, $\int \left(1 + \frac{x-19}{x^2+2x-15}\right)dx$

$= \int 1dx + \int \frac{x-19}{x^2+2x-15}dx$

Now

$$\frac{x-19}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5} \rightarrow \text{(i)}$$

$\because x^2+2x-15$
$= x^2+5x-3x-15$
$= x(x+5)-3(x+5)$
$= (x+5)(x-3)$

$\Rightarrow x-19 = A(x+5) + B(x-3) \longrightarrow$ (ii)

Put $x-3=0 \Rightarrow x=3$ in (ii)

$\Rightarrow 3-19 = A(3+5) + B(0) \Rightarrow -16 = 8A \Rightarrow \boxed{A=-2}$

Put $x+5=0 \Rightarrow x=-5$ in (ii)

$\Rightarrow -5-19 = A(0) + B(-5-3) \Rightarrow -24 = -8B$

$\Rightarrow \boxed{B=3}$

(i) $\Rightarrow \frac{x-19}{x^2+2x-15} = \frac{-2}{x-3} + \frac{3}{x+5}$

Thus

$\int \frac{x^2+3x-34}{x^2+2x-15}dx = \int 1dx + \int \frac{-2}{x-3}dx + \int \frac{3}{x+5}dx$

$= x - 2\ln|x-3| + 3\ln|x+5| + c$

Q4. $\int \frac{(a-b)x}{(x-a)(x-b)}dx, \; (a>b)$

Solution:- $\int \frac{(a-b)x}{(x-a)(x-b)}dx$

Now

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

$\Rightarrow (a-b)x = A(x-b) + B(x-a)$ —— (i)

Put $x-a=0 \Rightarrow x=a$ in (i)

$\Rightarrow (a-b)\cdot a = A(a-b) + B(a-a)$

$\cancel{(a-b)}\cdot a = A\cancel{(a-b)} \Rightarrow \boxed{A=a}$

Put $x-b=0 \Rightarrow x=b$ in (i)

$\Rightarrow (a-b)\cdot b = A(0) + B(b-a)$

$\cancel{(a-b)}\cdot b = -B\cancel{(a-b)} \Rightarrow \boxed{B=-b}$

Thus $\frac{(a-b)x}{(x-a)(x-b)} = \frac{a}{x-a} - \frac{b}{x-b}$

$\Rightarrow \int \frac{(a-b)x}{(x-a)(x-b)}dx = \int \frac{a}{x-a}dx - \int \frac{b}{x-b}dx$

$= a\ln|x-a| - b|x-b| + c$

Q5. $\int \frac{3-x}{1-x-6x^2}\,dx$

Solution:- $\int \frac{3-x}{1-x-6x^2}\,dx$

$\because 1-x-6x^2$
$= -6x^2-x+1$
$= -6x^2-3x+2x+1$
$= -3x(2x+1)+1(2x+1)$
$= (2x+1)(1-3x)$

Now $\frac{3-x}{(2x+1)(1-3x)} = \frac{A}{2x+1} + \frac{B}{1-3x}$

$\Rightarrow 3-x = A(1-3x) + B(2x+1) \longrightarrow (i)$

Put $2x+1=0 \Rightarrow x = -\frac{1}{2}$ in (i)

$\Rightarrow 3-(-\frac{1}{2}) = A\left(1-3(-\frac{1}{2})\right) + B(0)$

$\Rightarrow 3+\frac{1}{2} = A\left(1+\frac{3}{2}\right) \Rightarrow \frac{7}{2} = A\left(\frac{5}{2}\right)$

$\Rightarrow \boxed{A = 7/5}$

Put $1-3x=0 \Rightarrow 1=3x \Rightarrow x=\frac{1}{3}$ in (i)

$\Rightarrow 3-\frac{1}{3} = A(0) + B\left(2(\frac{1}{3})+1\right)$

$\frac{9-1}{3} = B\left(\frac{2+3}{3}\right) \Rightarrow 8 = 5B \Rightarrow \boxed{B = 8/5}$

so $\frac{3-x}{(2x+1)(1-3x)} = \frac{7/5}{2x+1} + \frac{8/5}{1-3x}$

$\Rightarrow \int \frac{3-x}{(2x+1)(1-3x)}\,dx = 7/5\int \frac{1}{2x+1}\,dx + \frac{8}{5}\int \frac{1}{1-3x}\,dx$

$= \frac{7}{10}\int \frac{2}{2x+1}\,dx + \frac{8}{5}(-\frac{1}{3})\int \frac{-3}{1-3x}\,dx$

$= \frac{7}{10}\ln|2x+1| - \frac{8}{15}\ln|1-3x| + c$

Q6. $\int \frac{2x}{x^2-a^2}\,dx \qquad (x>a)$

Solution:- $\int \frac{2x}{x^2-a^2}\,dx$

$\because x^2-a^2$
$=(x-a)(x+a)$

Now
$\frac{2x}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$

$\Rightarrow 2x = A(x+a) + B(x-a) \longrightarrow (i)$

Put $x-a=0 \Rightarrow x=a$ in (i)

$\Rightarrow 2a = A(a+a) + B(0) \Rightarrow \cancel{2a} = \cancel{2a}A$

$\Rightarrow \boxed{A=1}$

Put $x+a=0 \Rightarrow x=-a$ in (i)

$\Rightarrow 2(-a) = A(0) + B(-a-a) \Rightarrow -2a = -2aB$

$\Rightarrow \boxed{B=1}$

So $\frac{2x}{x^2-a^2} = \frac{1}{x-a} + \frac{1}{x+a}$

$\Rightarrow \int \frac{2x}{x^2-a^2}\,dx = \int \frac{1}{x-a}\,dx + \int \frac{1}{x+a}\,dx$

$= \ln|x-a| + \ln|x+a| + c$

$= \ln|(x-a)(x+a)| + c$

$= \ln|x^2-a^2| + c$

Q7. $\int \frac{1}{6x^2+5x-4}\,dx$

Solution:- $\int \frac{1}{6x^2+5x-4}\,dx$

$\because 6x^2+5x-4$
$= 6x^2-3x+8x-4$
$= 3x(2x-1)+4(2x-1)$
$= (2x-1)(3x+4)$

Now $\frac{1}{(2x-1)(3x+4)} = \frac{A}{2x-1} + \frac{B}{3x+4}$

$\Rightarrow 1 = A(3x+4) + B(2x-1) \longrightarrow (i)$

Put $2x-1=0 \Rightarrow x=\frac{1}{2}$ in (i)

$\Rightarrow 1 = A\left(3(\frac{1}{2})+4\right) + B(0) \Rightarrow 1 = A\left(\frac{3+8}{2}\right)$

$\Rightarrow 2 = 11A \Rightarrow \boxed{A = \frac{2}{11}}$

Put $3x+4=0 \Rightarrow x = -\frac{4}{3}$ in (i)

$\Rightarrow 1 = A(0) + B\left(2(-\frac{4}{3})-1\right) \Rightarrow 1 = B\left(-\frac{8-3}{3}\right)$

$\Rightarrow 3 = -11B \Rightarrow \boxed{B = \frac{-3}{11}}$

So $\frac{1}{6x^2+5x-4} = \frac{2/11}{2x-1} + \frac{-3/11}{3x+4}$

$\Rightarrow \int \frac{1}{6x^2+5x-4}\,dx = \frac{1}{11}\int \frac{2}{2x-1}\,dx - \frac{1}{11}\int \frac{3}{3x+4}\,dx$

$= \frac{1}{11}\ln|2x-1| - \frac{1}{11}\ln|3x+4| + c$

$= \frac{1}{11}\ln\left|\frac{2x-1}{3x+4}\right| + c$

Q8. $\int \frac{2x^3-3x^2-x-7}{2x^2-3x-2}\,dx$

Solution:- $\int \frac{2x^3-3x^2-x-7}{2x^2-3x-2}\,dx$

$$\begin{array}{r l} & \qquad x \\ 2x^2-3x-2 & \overline{)\,2x^3-3x^2-x-7} \\ & \underline{\underset{-}{2x^3}\ \underset{+}{-}3x^2\ \underset{+}{-}2x} \\ & \qquad x-7 \end{array}$$

$\int \frac{2x^3-3x^2-x-7}{2x^2-3x-2}\,dx = \int\left(x + \frac{x-7}{2x^2-3x-2}\right)dx$

$= \int x\,dx + \int \frac{x-7}{2x^2-3x-2}\,dx$

$\because 2x^2-3x-2$
$= 2x^2-4x+x-2$
$= 2x(x-2)+1(x-2)$
$= (x-2)(2x+1)$

Now $\frac{x-7}{(x-2)(2x+1)} = \frac{A}{x-2} + \frac{B}{2x+1}$

$\Rightarrow x-7 = A(2x+1) + B(x-2) \text{ — } (i)$

Put $x-2=0 \Rightarrow x=2$ in (i)

$\Rightarrow 2-7 = A(2(2)+1) + B(0) \Rightarrow -5 = 5A$

$\Rightarrow \boxed{A=-1}$

Put $2x+1=0 \Rightarrow x = -\frac{1}{2}$ in (i)

$\Rightarrow -\frac{1}{2}-7 = A(0) + B(-\frac{1}{2}-2) \Rightarrow \frac{-1-14}{\cancel{2}} = B\left(\frac{-1-4}{\cancel{2}}\right)$

$\Rightarrow -15 = -5B \Rightarrow \boxed{B=3}$

so $\frac{x-7}{2x^2-3x-2} = \frac{-1}{x-2} + \frac{3}{2x+1}$

Thus $\int \frac{2x^3-3x^2-x-7}{2x^2-3x-2}dx = \int x\,dx - \int \frac{1}{x-2}dx + 3\int \frac{1}{2x+1}dx$

$= \frac{x^2}{2} - \ln|x-2| + \frac{3}{2}\int \frac{2}{2x+1}dx$

$= \frac{x^2}{2} - \ln|x-2| + \frac{3}{2}\ln|2x+1| + c$

Q9. $\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx$

Solution:- $\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx$

Now $\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$

$\rightarrow 3x^2-12x+11 = A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2) \longrightarrow (i)$

Put $x-1=0 \rightarrow x=1$ in (i)

$\rightarrow 3(1)^2-12(1)+11 = A(1-2)(1-3)+B(0)(1-3)+C(0)(1-2)$

$3-12+11 = A(-1)(-2) \rightarrow 2 = 2A \rightarrow \boxed{A=1}$

Put $x-2=0 \rightarrow x=2$ in (i)

$3(2)^2-12(2)+11 = A(0)(2-3)+B(2-1)(2-3)+C(2-1)(2-2)$

$12-24+11 = B(1)(-1) \rightarrow -1 = -B \rightarrow \boxed{B=1}$

Put $x-3=0 \rightarrow x=3$ in (i)

$\rightarrow 3(3)^2-12(3)+11 = A(3-2)(0)+B(3-1)(0)+C(3-1)(3-2)$

$27-36+11 = C(2)(1) \rightarrow 2 = 2C \rightarrow \boxed{C=1}$

So $\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}$

$\rightarrow \int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx = \int \frac{1}{x-1}dx + \int \frac{1}{x-2}dx + \int \frac{1}{x-3}dx$

$= \ln|x-1| + \ln|x-2| + \ln|x-3| + c$

Q10. $\int \frac{2x-1}{x(x-1)(x-3)}dx$

Solution:- $\int \frac{2x-1}{x(x-1)(x-3)}dx$

Now $\frac{2x-1}{x(x-1)(x-3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-3}$

$\rightarrow 2x-1 = A(x-1)(x-3)+B(x)(x-3)+C(x)(x-1) \longrightarrow (i)$

Put $x=0$ in (i)

$\rightarrow 2(0)-1 = A(0-1)(0-3)+B(0)(0-3)+C(0)(0-1)$

$-1 = A(-1)(-3) \rightarrow \boxed{A = \frac{-1}{3}}$

Put $x-1=0 \rightarrow x=1$ in (i)

$\rightarrow 2(1)-1 = A(0)(1-3)+B(1)(1-3)+C(1)(0)$

$1 = B(-2) \rightarrow \boxed{B = -\frac{1}{2}}$

Put $x-3=0 \rightarrow x=3$ in (i)

$\rightarrow 2(3)-1 = A(3-1)(0)+B(3)(0)+C(3)(3-1)$

$5 = C(3)(2) \rightarrow \boxed{C = 5/6}$

So $\frac{2x-1}{x(x-1)(x-3)} = \frac{-1/3}{x} + \frac{-1/2}{x-1} + \frac{5/6}{x-3}$

$\rightarrow \int \frac{2x-1}{x(x-1)(x-3)}dx = -\frac{1}{3}\int \frac{1}{x}dx - \frac{1}{2}\int \frac{1}{x-1}dx + \frac{5}{6}\int \frac{1}{x-3}dx$

$= -\frac{1}{3}\ln|x| - \frac{1}{2}\ln|x-1| + \frac{5}{6}\ln|x-3| + c$

Q11. $\int \frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx$

Solution:- $\int \frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx$

Now $\frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+3}$

$\rightarrow 5x^2+9x+6 = A(x+1)(2x+3)+B(x-1)(2x+3)+C(x-1)(x+1) \longrightarrow (i)$

Put $x-1=0 \rightarrow x=1$ in (i)

$\rightarrow 5(1)^2+9(1)+6 = A(1+1)(2(1)+3)+B(0)(2(1)+3)+C(0)(1+1)$

$\overset{6}{\cancel{10}}\,\cancel{20} = A(\cancel{2})(\cancel{5}) \rightarrow \boxed{A=4}$

Put $x+1=0 \rightarrow x=-1$ in (i)

$5(-1)^2+9(-1)+6 = A(0)(2(-1)+3)+B(-1-1)(2(-1)+3)+C(-1-1)(0)$

$\rightarrow 5-9+6 = B(-2)(1) \rightarrow 2 = -2B$

$\rightarrow \boxed{B=-1}$

Put $2x+3=0 \rightarrow x=\frac{-3}{2}$ in (i)

$5(-\frac{3}{2})^2+9(-\frac{3}{2})+6 = A(-\frac{3}{2}+1)(0)+B(-\frac{3}{2}-1)(0)+C(-\frac{3}{2}-1)(-\frac{3}{2}+1)$

$\frac{45}{4} - \frac{27}{2} + 6 = C(-\frac{5}{2})(-\frac{1}{2})$

$\rightarrow \frac{45-54+24}{\cancel{4}} = C(\frac{5}{\cancel{4}}) \rightarrow C = \frac{15}{5} \rightarrow \boxed{C=3}$

So $\frac{5x^2+9x+6}{(x^2-1)(2x+3)} = \frac{4}{x-1} + \frac{-1}{x+1} + \frac{3}{2x+3}$

$\rightarrow \int \frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx = 4\int \frac{1}{x-1}dx - \int \frac{dx}{x+1} + \frac{3}{2}\int \frac{2}{2x+3}dx$

$= 4\ln|x-1| - \ln|x+1| + \frac{3}{2}\ln|2x+3| + c$

Q12. $\int \frac{4+7x}{(1+x)^2(2+3x)}dx$

Solution:- $\int \frac{4+7x}{(1+x)^2(2+3x)}dx$

Now $\frac{4+7x}{(1+x)^2(2+3x)} = \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{2+3x}$

$\rightarrow 4+7x = A(1+x)(2+3x)+B(2+3x)+C(1+x)^2 \longrightarrow (i)$

Put $1+x=0 \rightarrow x=-1$ in (i)

$\rightarrow 4+7(-1) = A(0)(2+3(-1))+B(2+3(-1))+C(0)^2$

$\rightarrow 4-7 = B(2-3) \rightarrow -B = -3$

$\rightarrow \boxed{B=3}$

Put $2+3x=0 \rightarrow 3x=-2 \rightarrow x=-\frac{2}{3}$ in (i)

$\rightarrow 4+7(-\frac{2}{3}) = A(1-\frac{2}{3})(0) + B(0) + C(1-\frac{2}{3})^2$

$\rightarrow 4-\frac{14}{3} = C(\frac{3-2}{3})^2 \rightarrow \frac{12-14}{3} = C(\frac{1}{3})^2$

$\rightarrow -\frac{2}{3} = C\cdot(\frac{1}{9}) \rightarrow -\frac{2\times 9}{3} = C \rightarrow \boxed{C=-6}$

from (i)

$4+7x = A(2+3x+2x+3x^2)+2B+3Bx+C(1+2x+x^2)$

$\rightarrow 4+7x = 2A+5Ax+3x^2A+2B+3Bx+C+2Cx+Cx^2$

Equating coefficients of x^2,

$0 = 3A+C \rightarrow 3A=-C \rightarrow 3A=-(-6)$

$\rightarrow \boxed{A=2}$ so

$$\frac{4+7x}{(1+x)^2(2+3x)} = \frac{2}{1+x}+\frac{3}{(1+x)^2}-\frac{6}{2+3x}$$

$$\rightarrow \int\frac{4+7x}{(1+x)^2(2+3x)}dx = 2\int\frac{1}{1+x}dx+3\int(1+x)^{-2}dx-\frac{6}{3}\int\frac{3}{2+3x}dx$$

$$= 2\ln|1+x| + 3\frac{(1+x)^{-1}}{-1} - 2\ln|2+3x| + c$$

$$= \ln|1+x|^2 - \frac{3}{1+x} - \ln|2+3x|^2 + c$$

Q13. $\int\frac{2x^2}{(x-1)^2(x+1)}dx$

Solution:- $\int\frac{2x^2}{(x-1)^2(x+1)}dx$

Now

$$= \frac{2x^2}{(x-1)^2(x+1)} = \frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$$

$\rightarrow 2x^2 = A(x-1)(x+1)+B(x+1)+C(x-1)^2 \longrightarrow$ (i)

Put $x-1=0 \rightarrow x=1$ in (i)

$\rightarrow 2(1)^2 = A(0)(1+1)+B(1+1)+C(0)^2$

$2 = 2B \rightarrow \boxed{B=1}$

Put $x+1=0 \rightarrow x=-1$ in (i)

$\rightarrow 2(-1)^2 = A(-1-1)(0)+B(0)+C(-1-1)^2$

$\rightarrow 2 = C(-2)^2 \rightarrow 2=C4 \rightarrow \boxed{C=\frac{1}{2}}$

from (i)

$2x^2 = A(x^2-1)+Bx+B+C(x^2+1-2x)$

$\rightarrow 2x^2 = Ax^2-A+Bx+B+Cx^2+C-2Cx$

Equating coefficients of x^2, we have

$\rightarrow 2 = A+C \rightarrow 2=A+\frac{1}{2} \rightarrow A=2-\frac{1}{2}$

$\rightarrow \boxed{A=\frac{3}{2}}$

$$\text{so } \frac{2x^2}{(x-1)^2(x+1)} = \frac{3/2}{x-1}+\frac{1}{(x-1)^2}+\frac{1/2}{x+1}$$

$$\rightarrow \int\frac{2x^2}{(x-1)^2(x+1)}dx = \frac{3}{2}\int\frac{1}{x-1}dx+\int(x-1)^{-2}dx+\frac{1}{2}\int\frac{dx}{x+1}$$

$$= \frac{3}{2}\ln|x-1| + \frac{(x-1)^{-1}}{-1} + \frac{1}{2}\ln|x+1|$$

$$= \frac{3}{2}\ln|x-1| - \frac{1}{x-1} + \frac{1}{2}\ln|x+1| + c$$

Q14. $\int\frac{1}{(x-1)(x+1)^2}dx$

Solution:- $\int\frac{1}{(x-1)(x+1)^2}dx$

Now,

$$\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$\rightarrow 1 = A(x+1)^2+B(x-1)(x+1)+C(x-1) \longrightarrow$ (i)

Put $x-1=0 \rightarrow x=1$ in (i)

$\rightarrow 1 = A(1+1)^2+B(0)(1+1)+C(0)$

$1 = A4 \rightarrow \boxed{A=\frac{1}{4}}$

Put $x+1=0 \rightarrow x=-1$ in (i)

$\rightarrow 1 = A(-1+1)^2+B(-1-1)(-1+1)+C(-1-1)$

$\rightarrow 1 = C(-2) \rightarrow \boxed{C=-\frac{1}{2}}$

from (i) $\rightarrow 1 = A(x^2+1+2x)+B(x^2-1)+Cx-C$

$\rightarrow 1 = Ax^2+A+2Ax+Bx^2-B+Cx-C$

Equating coefficient of x^2, we have

$0 = A+B \rightarrow 0=\frac{1}{4}+B \rightarrow \boxed{B=-\frac{1}{4}}$

$$\text{so } \frac{1}{(x-1)(x+1)^2} = \frac{1/4}{x-1}-\frac{1/4}{x+1}-\frac{1/2}{(x+1)^2}$$

$$\rightarrow \int\frac{1}{(x-1)(x+1)^2}dx = \frac{1}{4}\int\frac{dx}{x-1}-\frac{1}{4}\int\frac{dx}{x+1}-\frac{1}{2}\int(x+1)^{-2}dx$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\cdot\frac{(x+1)^{-1}}{-1} + c$$

$$= \frac{1}{4}\left\{\ln|x-1| - \ln|x+1|\right\} + \frac{1}{2(x+1)} + c$$

Q15. $\int\frac{x+4}{x^3-3x^2+4}dx$

Solution:- $\int\frac{x+4}{x^3-3x^2+4}dx$

$\because x^3-3x^2+4 = x^3+x^2-4x^2+4$

$= x^2(x+1)-4(x^2-1)$

$= x^2(x+1)-4(x-1)(x+1)$

$= (x+1)(x^2-4x+4)$

$\rightarrow x^3-3x^2+4 = (x+1)(x-2)^2$

Now

$$\frac{x+4}{(x+1)(x-2)^2} = \frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$

$\rightarrow x+4 = A(x-2)^2+B(x+1)(x-2)+C(x+1) \longrightarrow$ (i)

Put $x+1=0 \rightarrow x=-1$ in (i)

$\rightarrow -1+4 = A(-1-2)^2+B(0)(x-2)+C(0)$

$\rightarrow 3 = 9A \rightarrow \boxed{A=\frac{1}{3}}$

Put $x-2=0 \rightarrow x=2$ in (i)

(i) $\Rightarrow 2+4 = A(0)^2 + B(2+1)(0) + C(2+1)$

$\Rightarrow 6 = 3C \Rightarrow \boxed{C=2}$

from (i)

$x+4 = A(x^2-4x+4) + B(x^2-2x+x-2) + Cx + C$

$\Rightarrow x+4 = Ax^2 - 4Ax + 4A + Bx^2 - Bx - 2B + Cx + C$

Equating coefficient of x^2

$\Rightarrow 0 = A+B \Rightarrow 0 = \frac{1}{3} + B \Rightarrow \boxed{B = -\frac{1}{3}}$

so $\dfrac{x+4}{(x+1)(x-2)^2} = \dfrac{1/3}{x+1} + \dfrac{-1/3}{x-2} + \dfrac{2}{(x-2)^2}$

$\Rightarrow \int \dfrac{x+4}{(x+1)(x-2)^2}dx = \frac{1}{3}\int \frac{1}{x+1}dx - \frac{1}{3}\int \frac{dx}{x-2} + 2\int (x-2)^{-2}dx$

$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\ln|x-2| + 2\dfrac{(x-2)^{-1}}{-1} + c$

$= \frac{1}{3}\{\ln|x+1 - \ln|x-2|\} - 2/_{x-2} + c$

Q16. $\int \dfrac{x^3-6x^2+25}{(x+1)^2(x-2)^2}dx$

Solution:- $\int \dfrac{x^3-6x^2+25}{(x+1)^2(x-2)^2}dx$

Now

$\dfrac{x^3-6x^2+25}{(x+1)^2(x-2)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x-2)} + \dfrac{D}{(x-2)^2}$

$\Rightarrow x^3-6x^2+25 = A(x+1)(x-2)^2 + B(x-2)^2 + C(x+1)^2(x-2)$
$+ D(x+1)^2 \longrightarrow$ (i)

Put $x+1=0 \Rightarrow x=-1$ in (i)

$\Rightarrow (-1)^3 - 6(-1)^2 + 25 = A(0)(-1-2)^2 + B(-1-2)^2 + C(0)^2(-1-2)$
$+ D(0)^2$

$\Rightarrow -1-6+25 = 9B \Rightarrow 9B = 18 \Rightarrow \boxed{B=2}$

Put $x-2=0 \Rightarrow x=2$ in (i)

$(2)^3 - 6(2)^2 + 25 = A(2+1)(0)^2 + B(0)^2 + C(2+1)(0) + D(2+1)^2$

$8-24+25 = 9D \Rightarrow 9 = 9D \Rightarrow \boxed{D=1}$

from (i)

$x^3-6x^2+25 = A(x+1)(x^2-4x+4) + B(x^2-4x+4)$
$+ C(x^2+1+2x)(x-2) + D(x^2+1+2x)$

$= A(x^3-4x^2+\cancel{4x}+x^2-\cancel{4x}+4) + Bx^2 - 4Bx + 4B$
$+ C(x^3-\cancel{2x^2}+x-2+\cancel{2x^2}-4x) + Dx^2 + D + 2Dx$

$= Ax^3 - 3Ax^2 + 4A + Bx^2 - 4Bx + 4B + Cx^3 - 3Cx$
$+ Dx^2 + D + 2Dx$

Equating coefficients of x^3 and x^2

For x^3; $1 = A + C$ —— (ii)

For x^2; $-6 = -3A + B + D$ $\quad \because B=2, D=1$

$-6 = -3A + 2 + 1$

$-6-3 = -3A \Rightarrow -9 = -3A$

$\Rightarrow \boxed{A=3}$ put in (ii)

$1 = 3 + C \Rightarrow 1-3 = C$

$\Rightarrow \boxed{C=-2}$

so $\dfrac{x^3-6x^2+25}{(x+1)^2(x-2)^2} = \dfrac{3}{x+1} + \dfrac{2}{(x+1)^2} - \dfrac{2}{x-2} + \dfrac{1}{(x-2)^2}$

$\Rightarrow \int \dfrac{x^3-6x^2+25}{(x+1)^2(x-2)^2}dx = 3\int \frac{1}{x+1}dx + 2\int \frac{1}{(x+1)^2}dx - 2\int \frac{1}{x-2}$
$+ \int \frac{1}{(x-2)^2}dx$

$= 3\ln|x+1| + 2\dfrac{(x+1)^{-1}}{-1} - 2\ln|x-2| + \dfrac{(x-2)^{-1}}{-1} + c$

$= 3\ln|x+1| - \dfrac{2}{x+1} - 2\ln|x-2| - \dfrac{1}{x-2} + c$

Q17. $\int \dfrac{x^3+22x^2+14x-17}{(x-3)(x+2)^3}dx$

Solution:- $\int \dfrac{x^3+22x^2+14x-17}{(x-3)(x+2)^3}dx$

Now

$\dfrac{x^3+22x^2+14x-17}{(x-3)(x+2)^3} = \dfrac{A}{x-3} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2} + \dfrac{D}{(x+2)^3}$

$\Rightarrow x^3+22x^2+14x-17 = A(x+2)^3 + B(x-3)(x+2)^2 + C(x-3)(x+2)$
$+ D(x-3) \longrightarrow$ (i)

Put $x-3=0 \Rightarrow x=3$ in (i)

$(3)^3 + 22(3)^2 + 14(3) - 17 = A(3+2)^3 + B(0)(x+2)^2 + C(0)(3+2)$
$+ D(0)$

$27+198+42-17 = 125A \Rightarrow 250 = 125A$

$\Rightarrow \boxed{A=2}$

Put $x+2=0 \Rightarrow x=-2$ in (i)

$(-2)^3 + 22(-2)^2 + 14(-2) - 17 = A(0)^3 + B(-2-3)(0)^2 + C(-2-3)(0)$
$+ D(-2-3)$

$-8+88-28-17 = -5D \Rightarrow 35 = -5D$

$\Rightarrow \boxed{D=-7}$

from (i)

$x^3+22x^2+14x-17 = A[x^3+6x^2+12x+8] + B(x-3)(x^2+4+4x)$
$+ C(x^2+2x-3x-6) + Dx - 3D$

$= Ax^3 + 6Ax^2 + 12Ax + 8A + B(x^3+4x^2+4x-3x^2-12x$
$-12)$
$+ Cx^2 - Cx - 6C + Dx - 3D$

Equating coefficients of x^3 and x^2

For x^3; $1 = A + B \Rightarrow 1 = 2 + B \Rightarrow \boxed{B=-1}$

For x^2; $22 = 6A + B + C \Rightarrow 22 = 6(2) - 1 + C$

$\Rightarrow C = 22 - 12 + 1 = 11 \Rightarrow \boxed{C=11}$

so $\dfrac{x^3+22x^2+14x-17}{(x-3)(x+2)^3} = \dfrac{2}{x-3} - \dfrac{1}{x+2} + \dfrac{11}{(x+2)^2} - \dfrac{7}{(x+2)^3}$

$\Rightarrow \int \dfrac{x^3+22x^2+14x-17}{(x-3)(x+2)^3}dx = 2\int \frac{dx}{x-3} - \int \frac{dx}{x+2} + 11\int (x+2)^{-2}dx - 7\int (x+2)^{-3}dx$

$$= 2\ln|x-3| - \ln|x+2| + 11\frac{(x+2)^{-1}}{-1} - 7\frac{(x+2)^{-2}}{-2} + c$$

$$= 2\ln|x-3| - \ln|x+2| - \frac{11}{x+2} + \frac{7}{2}\frac{1}{(x+2)^2} + c$$

Q18. $\int \frac{x-2}{(x+1)(x^2+1)}dx$

Solution:- $\int \frac{x-2}{(x+1)(x^2+1)}dx$

Now $\frac{x-2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$\rightarrow x-2 = A(x^2+1) + (Bx+C)(x+1) \longrightarrow (i)$

put $x+1=0 \rightarrow x=-1$ in (i)

$\rightarrow -1-2 = A((-1)^2+1) + (B(-1)+C)(0)$

$-3 = 2A \rightarrow \boxed{A = \frac{-3}{2}}$

from (i)

$\rightarrow x-2 = Ax^2 + A + Bx^2 + Bx + Cx + C$

Equating coefficients of x^2 and x

For x^2; $0 = A+B \rightarrow 0 = \frac{-3}{2} + B$

$\rightarrow \boxed{B = \frac{3}{2}}$

For x; $1 = B+C \rightarrow 1 = \frac{3}{2} + C$

$\rightarrow C = 1 - \frac{3}{2} = \frac{-1}{2} \rightarrow \boxed{C = \frac{-1}{2}}$

So $\frac{x-2}{(x+1)(x^2+1)} = \frac{-3/2}{x+1} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1}$

$\rightarrow \int \frac{x-2}{(x+1)(x^2+2)}dx = \frac{-3}{2}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{3x-1}{x^2+1}dx$

$= \frac{-3}{2}\ln|x+1| + \frac{1}{2}\int \frac{3x}{x^2+1}dx - \frac{1}{2}\int \frac{1}{x^2+1}dx$

$= \frac{-3}{2}\ln|x+1| + \frac{3}{2}\cdot\frac{1}{2}\int \frac{2x}{x^2+1}dx - \frac{1}{2}\tan^{-1}x$

$= -\frac{3}{2}\ln|x+1| + \frac{3}{4}\ln|x^2+1| - \frac{1}{2}\tan^{-1}x + c$

Q19. $\int \frac{x}{(x-1)(x^2+1)}dx$

Solution:- $\int \frac{x}{(x-1)(x^2+1)}dx$

Now $\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$\rightarrow x = A(x^2+1) + (Bx+C)(x-1)$ ——— (i)

put $x-1=0 \rightarrow x=1$ in (i)

$\rightarrow 1 = A((1)^2+1) + (B(1)+C)(0)$

$\rightarrow \boxed{A = \frac{1}{2}}$

from (i)

$\rightarrow x = Ax^2 + A + Bx^2 - Bx + Cx - C$

Equating coefficients of x^2 and x, we have

For x^2;
$\rightarrow 0 = A+B \rightarrow 0 = \frac{1}{2} + B \rightarrow \boxed{B = -\frac{1}{2}}$

For x; $1 = -B+C \rightarrow 1 = -(-\frac{1}{2}) + C$

$\rightarrow 1 = \frac{1}{2} + C \rightarrow 1 - \frac{1}{2} = C \rightarrow \boxed{C = \frac{1}{2}}$

So $\frac{x}{(x-1)(x^2+1)} = \frac{1/2}{x-1} + \frac{-\frac{1}{2}x + 1/2}{x^2+1}$

$\rightarrow \int \frac{x}{(x-1)(x^2+1)}dx = \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{x-1}{x^2+1}dx$

$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\int \frac{2x-2}{x^2+1}dx$

$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$

$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\tan^{-1}x + c$

Q20. $\int \frac{9x-7}{(x+3)(x^2+1)}dx$

Solution:- $\int \frac{9x-7}{(x+3)(x^2+1)}dx$

Now $\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$

$\Rightarrow 9x-7 = A(x^2+1) + (Bx+C)(x+3) \longrightarrow (i)$

Put $x+3=0 \Rightarrow x=-3$ in (i)

$\Rightarrow 9(-3)-7 = A((-3)^2+1) + (B(-3)+C)(0)$

$\Rightarrow -27-7 = 10A \rightarrow A = \frac{-34}{10} \rightarrow \boxed{A = -\frac{17}{5}}$

from (i)

$\Rightarrow 9x-7 = Ax^2 + A + Bx^2 + 3Bx + Cx + 3C$

Equating coefficients of x^2 and x

For x^2;
$\Rightarrow 0 = A+B \rightarrow 0 = -\frac{17}{5} + B \Rightarrow \boxed{B = \frac{17}{5}}$

and For x; $3B + C = 9$

$\rightarrow 3(\frac{17}{5}) + C = 9 \rightarrow \frac{51}{5} + C = 9$

$\rightarrow C = 9 - \frac{51}{5} = \frac{45-51}{5} \rightarrow \boxed{C = -\frac{6}{5}}$

So $\frac{9x-7}{(x+3)(x^2+1)} = \frac{-17/5}{x+3} + \frac{17/5\,x - 6/5}{x^2+1}$

$\rightarrow \int \frac{9x-7}{(x+3)(x^2+1)}dx = -\frac{17}{5}\int \frac{1}{x+3}dx + \frac{17}{5}\int \frac{x}{x^2+1}dx - \frac{6}{5}\int \frac{1}{x^2+1}dx$

$= \frac{-17}{5}\ln|x+3| + \frac{17}{10}\int \frac{2x}{x^2+1}dx - \frac{6}{5}\int \frac{1}{x^2+1}dx$

$= -\frac{17}{5}\ln|x+3| + \frac{17}{10}\ln|x^2+1| - \frac{6}{5}\tan^{-1}x + c$

Q21. $\int \frac{1+4x}{(x-3)(x^2+4)}dx$

Solution:- $\int \frac{1+4x}{(x-3)(x^2+4)}dx$

Now $\frac{1+4x}{(x-3)(x^2+4)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+4}$

$\Rightarrow 1+4x = A(x^2+4) + (Bx+C)(x-3)$ —— (i)

Put $x-3=0 \Rightarrow x=3$ in (i)

$\Rightarrow 1+4(3) = A((3)^2+4) + (B(3)+C)(0)$

$\Rightarrow 13 = A(9+4) \Rightarrow 13 = 13A \Rightarrow \boxed{A=1}$

from (i)

$1+4x = Ax^2+4A+Bx^2-3Bx+Cx-3C$

Equating coefficients of x^2 and x

$\Rightarrow 0 = A+B$ (for x^2)

$0 = 1+B \Rightarrow \boxed{B=-1}$

$\Rightarrow 4 = -3B+C$ (for x)

$\Rightarrow 4 = -3(-1)+C \Rightarrow 4-3 = C \Rightarrow \boxed{C=1}$

So $\frac{1+4x}{(x-3)(x^2+4)} = \frac{1}{x-3} + \frac{(-1)x+1}{x^2+4}$

$\Rightarrow \int \frac{1+4x}{(x-3)(x^2+4)}dx = \int \frac{1}{x-3}dx - \int \frac{x\,dx}{x^2+4} + \int \frac{1}{x^2+4}dx$

$= \ln|x-3| - \frac{1}{2}\int \frac{2x\,dx}{x^2+4} + \int \frac{1}{(x)^2+(2)^2}dx$

$= \ln|x-3| - \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\tan^{-1}\frac{x}{2} + c$

Q22. $\int \frac{12}{x^3+8}dx$

Solution:- $\int \frac{12}{(x)^3+(2)^3}dx$

$\because a^3+b^3 = (a+b)(a^2-ab+b^2)$

Now $\frac{12}{(x+2)(x^2-2x+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4}$

$\Rightarrow 12 = A(x^2-2x+4) + (Bx+C)(x+2) \longrightarrow$ (i)

Put $x+2=0 \Rightarrow x=-2$ in (i)

$\Rightarrow 12 = A((-2)^2-2(-2)+4) + (B(-2)+C)(0)$

$12 = A(4+4+4) \Rightarrow 12 = 12A \Rightarrow \boxed{A=1}$

from (i)

$12 = Ax^2-2Ax+4A+Bx^2+2Bx+Cx+2C$

Equating coefficients of x^2 and x, we have

for x^2; $0 = A+B \Rightarrow 0 = 1+B \Rightarrow \boxed{B=-1}$

For x; $0 = -2A+2B+C$

$0 = -2(1)+2(-1)+C \Rightarrow 0 = -2-2+C$

$\Rightarrow \boxed{C=4}$

So $\frac{12}{(x+2)(x^2-2x+4)} = \frac{1}{x+2} + \frac{-x+4}{x^2-2x+4}$

$\Rightarrow \int \frac{12}{x^3+8}dx = \int \frac{1}{x+2}dx - \int \frac{x-4}{x^2-2x+4}dx$

$= \int \frac{1}{x+2}dx - \frac{1}{2}\int \frac{2x-8}{x^2-2x+4}dx$

$= \ln|x+2| - \frac{1}{2}\int \frac{2x-2-6}{x^2-2x+4}dx$

$= \ln|x+2| - \frac{1}{2}\int \frac{2x-2}{x^2-2x+4}dx + \frac{6}{2}\int \frac{1}{x^2-2x+4}dx$

$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + 3\int \frac{1}{x^2-2x+1+3}dx$

$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + 3\int \frac{1}{(x-1)^2+\sqrt{3}}dx$

$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + c$

$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + \sqrt{3}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + c$

Q23. $\int \frac{9x+6}{x^3-8}dx$

Solution:- $\int \frac{9x+6}{(x)^3-(2)^3}dx$

$\because a^3-b^3 = (a-b)(a^2+ab+b^2)$

Now $\frac{9x+6}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$

$\Rightarrow 9x+6 = A(x^2+2x+4) + (Bx+C)(x-2) \longrightarrow$ (i)

Put $x-2=0 \Rightarrow x=2$ in (i)

$9(2)+6 = A[(2)^2+2(2)+4] + (B(2)+C)(0)$

$\Rightarrow 24 = 12A \Rightarrow \boxed{A=2}$

from (i)

$\Rightarrow 9x+6 = Ax^2+2Ax+4A+Bx^2-2Bx+Cx-2C$

Equating coefficients of x^2 and x

For x^2; $0 = A+B \Rightarrow 0 = 2+B \Rightarrow \boxed{B=-2}$

For x; $9 = 2A-2B+C \Rightarrow 9 = 2(2)-2(-2)+C$

$\Rightarrow 9 = 4+4+C \Rightarrow 9-8 = C \Rightarrow \boxed{C=1}$

So $\frac{9x+6}{x^3-8} = \frac{2}{x-2} + \frac{-2x+1}{x^2+2x+4}$

$\Rightarrow \int \frac{9x+6}{x^3-8}dx = 2\int \frac{1}{x-2}dx - \int \frac{2x-1}{x^2+2x+4}dx$

$= 2\int \frac{1}{x-2}dx - \int \frac{2x+2-2-1}{x^2+2x+4}dx$

$= 2\int \frac{1}{x-2}dx - \int \frac{2x+2}{x^2+2x+4}dx + 3\int \frac{1}{x^2+2x+4}dx$

$= 2\ln|x-2| - \ln|x^2+2x+4| + 3\int \frac{1}{x^2+2x+1+3}dx$

$= 2\ln|x-2| - \ln|x^2+2x+4| + 3\int \frac{1}{(x+1)^2+(\sqrt{3})^2}dx$

$= 2\ln|x-2| - \ln|x^2+2x+4| + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + c$

$= 2\ln|x-2| - \ln|x^2+2x+4| + \sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + c$

Q24. $\int \frac{2x^2+5x+3}{(x-1)^2(x^2+4)}dx$

Solution:- $\int \frac{2x^2+5x+3}{(x-1)^2(x^2+4)}dx$

Now $\frac{2x^2+5x+3}{(x-1)^2(x^2+4)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4}$

$\Rightarrow 2x^2+5x+3 = A(x-1)(x^2+4)+B(x^2+4)+(Cx+D)(x-1)^2 \longrightarrow (i)$

Put $x-1=0 \Rightarrow x=1$ in (i)

$\Rightarrow 2(1)^2+5(1)+3 = A(0)(1+4)+B(1+4)+(C(1)+D)(0)$

$2+5+3 = 5B \Rightarrow 10=5B \Rightarrow \boxed{B=2}$

from (i)

$2x^2+5x+3 = A(x^3+4x-x^2-4)+Bx^2+4B+(Cx+D)(x^2+1-2x)$

$= Ax^3+4Ax-Ax^2-4A+Bx^2+4B+Cx^3+Cx-2Cx^2+Dx^2+D-2Dx$

Equating coefficients of x^3, x^2 and x, we get

For x^3;

$\Rightarrow 0 = A+C \Rightarrow C=-A$ —— (ii)

For x^2; $2=-A+B-2C+D$

Put $B=2$ and $C=-A$

$\Rightarrow 2=-A+2-2(-A)+D$

$\Rightarrow 2-2=-A+2A+D \Rightarrow 0=A+D$

$\Rightarrow D=-A$ —— (iii)

For x; $5=4A+C-2D$ Put $C=-A$ and $D=-A$

$\Rightarrow 5=4A-A-2(-A)$

$5=3A+2A \Rightarrow 5=5A \Rightarrow \boxed{A=1}$

so (ii) $\Rightarrow \boxed{C=-1}$ and (iii) $\Rightarrow \boxed{D=-1}$

Thus $\frac{2x^2+5x+3}{(x-1)^2(x^2+4)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{(-1)x-1}{x^2+4}$

$\Rightarrow \int \frac{2x^2+5x+3}{(x-1)^2(x^2+4)}dx = \int \frac{1}{x-1}dx + 2\int (x-1)^{-2}dx - \int \frac{x+1}{x^2+4}dx$

$= \int \frac{1}{x-1}dx + 2\int (x-1)^{-2}dx - \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx$

$= \ln|x-1| + 2\frac{(x-1)^{-1}}{-1} - \frac{1}{2}\int \frac{2x}{x^2+4}dx - \int \frac{1}{(x)^2+(2)^2}dx$

$= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\frac{x}{2} + c$

Q25. $\int \frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}dx$

Solution:- $\int \frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}dx$

Now $\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2+x+1}$

$\Rightarrow 2x^2-x-7 = A(x+2)(x^2+x+1)+B(x^2+x+1)+(Cx+D)(x+2)^2$

Put $x+2=0 \Rightarrow x=-2$

$\Rightarrow 2(-2)^2-(-2)-7 = A(0)((-2)^2+(-2)+1)+B((-2)^2+(-2)+1)$
$+(C(-2)+D)(0)$

$\Rightarrow 8+2-7 = B(4-2+1)$

$\Rightarrow 3=3B \Rightarrow \boxed{B=1}$

from (i)

$2x^2-x-7 = A(x^3+x^2+x+2x^2+2x+2)+Bx^2+Bx+B$
$+(Cx+D)(x^2+4x+4)$

$= Ax^3+3Ax^2+3Ax+2A+Bx^2+Bx+B$
$+Cx^3+4Cx^2+4Cx+Dx^2+4Dx+4D$

Equating coefficients of x^3, x^2 and x

For x^3; $0=A+C \Rightarrow C=-A$ —— (ii)

For x^2; $2=3A+B+4C+D$

Put $B=1$, $C=-A$

$\Rightarrow 2=3A+1-4A+D \Rightarrow 2-1=-A+D$

$\Rightarrow D=A+1$ —— (iii)

For x; $-1=3A+B+4C+4D$

Put $B=1$, $C=-A$, $D=A+1$

$\Rightarrow -1=3A+1-\cancel{4A}+\cancel{4A}+4$

$-1-1-4=3A \Rightarrow -6=3A \Rightarrow \boxed{A=-2}$

so (ii) $\Rightarrow \boxed{C=2}$ and (iii) $\Rightarrow \boxed{D=-1}$

Thus

$\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)} = \frac{-2}{x+2} + \frac{1}{(x+2)^2} + \frac{2x-1}{x^2+x+1}$

$\int \frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}dx = -2\int \frac{1}{x+2}dx + \int (x+2)^{-2}dx + \int \frac{2x+1-2}{x^2+x+1}dx$

$= -2\ln|x+2| + \frac{(x+2)^{-1}}{-1} + \int \frac{2x+1}{x^2+x+1}dx - 2\int \frac{1}{x^2+x+1}dx$

$= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - 2\int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx$

$= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - 2\int \frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$

$= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - \frac{2}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{(x+\frac{1}{2})}{\frac{\sqrt{3}}{2}} + c$

$= -2\ln|x+2| - \frac{1}{x+2} + \ln|x^2+x+1| - \frac{4}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + c$

Q26. $\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}dx$

Solution:- $\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}dx$

$\because \frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{x^2-x+1}$

$\Rightarrow 3x+1 = (Ax+B)(x^2-x+1)+(Cx+D)(4x^2+1)$

$3x+1 = Ax^3-Ax^2+Ax+Bx^2-Bx+B+4Cx^3+Cx+4Dx^2+D$

Equating coefficients of x^3, x^2, x and constant term,

For x^3; $0=A+4C$ —— (i)

For x^2; $0=-A+B+4D$ —— (ii)

For x; $3=A+B+C$ —— (iii)

For constant term; $1=B+D$ —— (iv)

from (i) $\Rightarrow A=-4C$ and (iv) $\Rightarrow B=1-D$ Put in (ii) and (iii)

$\rightarrow 0 = -(-4C) + (1-D) + 4D$ and $3 = -4C - (1-D) + C$

$0 = 4C + 1 - D + 4D$

$0 = 4C + 3D + 1$ —— (v)

$3 = -4C - 1 + D + C$

$0 = -3C + D - 4$

$\rightarrow D = 3C + 4$ put in (v)

$\rightarrow 0 = 4C + 3(3C+4) + 1$

$0 = 4C + 9C + 12 + 1 \rightarrow 0 = 13C + 13$

$\rightarrow -13C = 13 \rightarrow \boxed{C=-1}$

As $A = -4C \rightarrow A = -4(-1) \rightarrow \boxed{A=4}$ $\because C=-1$

As $D = 3C+4 \rightarrow D = 3(-1)+4 = -3+4 \rightarrow \boxed{D=1}$

As $B = 1-D = 1-1 = 0 \rightarrow \boxed{B=0}$

Thus

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{4x+0}{4x^2+1} + \frac{(-1)x+1}{x^2-x+1}$$

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{1}{2}\frac{8x}{4x^2+1} + \frac{(-1)(x-1)}{x^2-x+1}$$

$$\rightarrow \int \frac{3x+1}{(4x^2+1)(x^2-x+1)}dx = \frac{1}{2}\int \frac{8x}{4x^2+1}dx - \frac{1}{2}\int \frac{2x-2}{x^2-x+1}dx$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\int \frac{2x-1-1}{x^2-x+1}dx$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{1}{x^2-x+1}dx$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\ln|x^2-x+1| + \frac{1}{2}\int \frac{1}{x^2-x+\frac{1}{4}+\frac{3}{4}}dx$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\ln|x^2-x+1| + \frac{1}{2}\int \frac{1}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}dx$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\ln|x^2-x+1| + \frac{1}{\cancel{2}} \cdot \frac{1}{\frac{\sqrt{3}}{\cancel{2}}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\ln|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c$$

Q27. $\int \frac{4x+1}{(x^2+4)(x^2+4x+5)}dx$

Solution:- $\int \frac{4x+1}{(x^2+4)(x^2+4x+5)}dx$

$$\because \frac{4x+1}{(x^2+4)(x^2+4x+5)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+4x+5}$$

$\rightarrow 4x+1 = (Ax+B)(x^2+4x+5) + (Cx+D)(x^2+4)$

$\rightarrow 4x+1 = Ax^3 + 4Ax^2 + 5Ax + Bx^2 + 4Bx + 5B$
$+ Cx^3 + 4Cx + Dx^2 + 4D$

Equating coefficients of x^3, x^2, x and const. term

For x^3; $0 = A + C$ —— (i)

For x^2; $0 = 4A + B + D$ —— (ii)

For x; $4 = 5A + 4B + 4C$ —— (iii)

For constant term; $1 = 5B + 4D$ —— (iv)

from (i) $\rightarrow A = -C$ and (iv) $\rightarrow 5B = 1-4D$
$\rightarrow B = \frac{1-4D}{5}$ Put in (ii) and (iii)

so

(ii) $\rightarrow 0 = 4(-C) + \frac{1-4D}{5} + D$ and (iii) $\rightarrow 4 = 5(-C) + 4(\frac{1-4D}{5}) + 4C$

$\rightarrow 0 = -4C + \frac{1-4D}{5} + D$

$0 = -20C + 1 - 4D + 5D$

$\rightarrow 0 = -20C + D + 1$

$\rightarrow D = 20C - 1$ —— (v)

$\rightarrow 20 = -25C + 4 - 16D + 20C$

$\rightarrow 16D = -5C + 4 - 20$

$\rightarrow D = \frac{-5C-16}{16}$ —— (vi)

By (v) and (vi) $\rightarrow 20C - 1 = \frac{-5C-16}{16}$

$\rightarrow 320C - \cancel{16} = -5C - \cancel{16} \rightarrow 320C + 5C = 0$

$\rightarrow 325C = 0 \rightarrow \boxed{C=0}$

As $A = -C \rightarrow \boxed{A=0}$

As $D = 20C - 1 \rightarrow D = 20(0) - 1 \rightarrow \boxed{D=-1}$

As $B = \frac{1-4D}{5} \rightarrow B = \frac{1-4(-1)}{5} = \frac{5}{5} = 1$

$\rightarrow \boxed{B=1}$

so $$\frac{4x+1}{(x^2+4)(x^2+4x+5)} = \frac{0x+1}{x^2+4} + \frac{0x+(-1)}{x^2+4x+5}$$

$$\int \frac{4x+1}{(x^2+4)(x^2+4x+5)}dx = \int \frac{1}{x^2+4}dx - \int \frac{1}{x^2+4x+4+1}dx$$

$$= \frac{1}{2}\tan^{-1}\frac{x}{2} - \int \frac{1}{(x+2)^2+(1)^2}dx$$

$$= \frac{1}{2}\tan^{-1}\frac{x}{2} - \tan^{-1}(x-2) + c$$

Q28. $\int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)}dx$

Solution:- $\int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)}dx$

$$\because \frac{6a^2}{(x^2+a^2)(x^2+4a^2)} = \frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{x^2+4a^2}$$

$\rightarrow 6a^2 = (Ax+B)(x^2+4a^2) + (Cx+D)(x^2+a^2)$

$6a^2 = Ax^3 + 4a^2Ax + Bx^2 + 4Ba^2 + Cx^3 + Ca^2x + Dx^2 + Da^2$

Equating coefficients of x^3, x^2, x and constant term,

For x^3; $0 = A + C$ —— (i)

For x^2; $0 = B + D$ —— (ii)

For x; $0 = 4a^2A + a^2C \rightarrow 0 = (4A+C)a^2$

$\rightarrow 4A + C = 0$ —— (iii)

For constant term; $6a^2 = 4a^2B + Da^2 \rightarrow 6\cancel{a^2} = (4B+D)\cancel{a^2}$

$\rightarrow 4B + D = 6$ —— (iv)

From (i) $\rightarrow A = -C$ and from (ii) $B = -D$

Put in (iii) and (iv) so

(iii) $4(-C) + C = 0 \rightarrow -4C + C = 0 \rightarrow -3C = 0$

$\rightarrow \boxed{C=0}$

(iv) $\rightarrow 4(-D) + D = 6 \rightarrow -4D + D = 6 \rightarrow -3D = 6$

$\rightarrow \boxed{D=-2}$

As $A=-C \rightarrow \boxed{A=0}$ $\because C=0$

As $B=-D \rightarrow B=-(-2) \rightarrow \boxed{B=2}$ $\because D=-2$

so
$$\frac{6a^2}{(x^2+a^2)(x^2+4a^2)} = \frac{0x+2}{x^2+a^2} + \frac{0x+(-2)}{x^2+4a^2}$$

$$\rightarrow \int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)}dx = 2\int\frac{1}{x^2+a^2}dx - 2\int\frac{1}{x^2+(2a)^2}dx$$

$$= \frac{2}{a}\tan^{-1}\frac{x}{a} - \frac{2}{2a}\tan^{-1}\frac{x}{2a}+c$$

$$= \frac{2}{a}\tan^{-1}\frac{x}{a} - \frac{1}{a}\tan^{-1}\frac{x}{2a}+c$$

Q29. $\int \frac{2x^2-2}{x^4+x^2+1}dx$

Solution:- $\int \frac{2x^2-2}{x^4+x^2+1}dx$ $\quad \because x^4+x^2+1 = (x^2+x+1)(x^2-x+1)$

$$\because \frac{2x^2-2}{(x^2+x+1)(x^2-x+1)} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1}$$

$$\rightarrow 2x^2-2 = (Ax+B)(x^2-x+1)+(Cx+D)(x^2+x+1)$$
$$= Ax^3-Ax^2+Ax+Bx^2-Bx+B+Cx^3+Cx^2+Cx+Dx^2+Dx+D$$

Equating coefficients of x^3, x^2, x and constant term,

For x^3; $0=A+C$ —— (i)

For x^2; $0=A-B+C+D$ —— (ii)

For x; $2=-A+B+C+D$ —— (iii)

For constant term; $-2=B+D$ —— (iv)

Put $B+D=-2$ in (iii) $\rightarrow 2=-A+C-2$

$\rightarrow 2+2=-A+C \rightarrow -A+C=4$ —— (v)

Put $A+C=0$ in (ii) $\rightarrow 0=-B+D$ —— (vi)

Now by (i) + (v) $\rightarrow 2C=4 \rightarrow \boxed{C=2}$

As $A+C=0 \rightarrow A+2=0 \rightarrow \boxed{A=-2}$

Now by (iv) + (vi) $\rightarrow 2D=-2 \rightarrow \boxed{D=-1}$

As $B+D=-2 \rightarrow B-1=-2 \rightarrow \boxed{B=-1}$

so
$$\frac{2x^2-2}{(x^2+x+1)(x^2-x+1)} = \frac{-2x-1}{x^2+x+1} + \frac{2x-1}{x^2-x+1}$$

$$\rightarrow \int\frac{2x^2-2}{(x^2+x+1)(x^2-x+1)}dx = -\int\frac{2x+1}{x^2+x+1}dx + \int\frac{2x-1}{x^2-x+1}dx$$
$$= -\ln|x^2+x+1| + \ln|x^2-x+1| + c$$
$$= \ln\left|\frac{x^2-x+1}{x^2+x+1}\right| + c$$

Q30. $\int \frac{3x-8}{(x^2-x+2)(x^2+x+2)}dx$

Solution:- $\int \frac{3x-8}{(x^2-x+2)(x^2+x+2)}dx$

$$\because \frac{3x-8}{(x^2-x+2)(x^2+x+2)} = \frac{Ax+B}{x^2-x+2} + \frac{Cx+D}{x^2+x+2}$$

$$\rightarrow 3x-8 = (Ax+B)(x^2+x+2) + (Cx+D)(x^2-x+2)$$
$$= Ax^3+Ax^2+2Ax+Bx^2+Bx+2B+Cx^3-Cx^2+2Cx+Dx^2-Dx+2D$$

Equating coefficients of x^3, x^2, x and constant term.

For x^3; $0=A+C$ —— (i)

For x^2; $0=A+B-C+D$ —— (ii)

For x; $3=2A+B+2C-D$ —— (iii)

For constant term; $-8=2B+2D \rightarrow B+D=-4$ —— (iv)

from (i) $\rightarrow A=-C$ and from (iv) $\rightarrow B=-4-D$

Put in (ii) and (iii) so

(ii) $\rightarrow 0=-C+(-4-D)-C+D$

$0=-C-4-\cancel{D}-C+\cancel{D}$

$0=-2C-4$

$\rightarrow 2C=-4$

$\rightarrow \boxed{C=-2}$ $\quad$ As $A=-C \rightarrow \boxed{A=2}$

(iii) $\rightarrow 3=2(-C)-4-D+2C-D$

$3=-\cancel{2C}-4+\cancel{2C}-2D$

$\rightarrow 3+4=-2D \rightarrow \boxed{D=\frac{-7}{2}}$

As $B=-4-D = -4(-\frac{7}{2}) = -4+\frac{7}{2} = \frac{-8+7}{2} = \frac{-1}{2}$

$\rightarrow \boxed{B=\frac{-1}{2}}$ so

$$\frac{3x-8}{(x^2-x+2)(x^2+x+2)} = \frac{2x-\frac{1}{2}}{x^2-x+2} + \frac{-2x+(-\frac{7}{2})}{x^2+x+2}$$

$$\int\frac{3x-8}{(x^2-x+2)(x^2+x+2)}dx = \int\frac{2x-1+1-\frac{1}{2}}{x^2-x+2}dx - \int\frac{2x+1-1+\frac{7}{2}}{x^2+x+2}dx$$

$$= \int\frac{2x-1+\frac{1}{2}}{x^2-x+2}dx - \int\frac{2x+1+\frac{5}{2}}{x^2+x+2}dx$$

$$= \int\frac{2x-1}{x^2-x+2}dx + \frac{1}{2}\int\frac{1}{x^2-x+2}dx - \int\frac{2x+1}{x^2+x+2}dx - \int\frac{\frac{5}{2}}{x^2+x+2}dx$$

$$= \ln|x^2-x+2| + \frac{1}{2}\int\frac{dx}{x^2-x+\frac{1}{4}-\frac{1}{4}+2} - \ln|x^2+x+2| - \frac{5}{2}\int\frac{1}{x^2+x+\frac{1}{4}-\frac{1}{4}+2}dx$$

$$= \ln|x^2-x+2| + \frac{1}{2}\int\frac{1}{(x-\frac{1}{2})^2+\frac{7}{4}}dx - \ln|x^2+x+2| - \frac{5}{2}\int\frac{1}{(x+\frac{1}{2})^2+\frac{7}{4}}dx$$

$$= \ln|x^2-x+1| + \frac{1}{2}\int \frac{1}{(x-\frac{1}{2})^2+(\frac{\sqrt{7}}{2})^2}dx - \ln|x^2+x+1|$$

$$-\frac{5}{2}\int \frac{1}{(x+\frac{1}{2})^2+(\sqrt{7}/2)^2}dx$$

$$= \ln|x^2-x+1| + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{7}}{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right) - \ln|x^2+x+1|$$

$$-\frac{5}{2}\cdot\frac{1}{\frac{\sqrt{7}}{2}}\tan^{-1}\left(\frac{x+1/2}{\sqrt{7}/2}\right) + c$$

$$= \ln|x^2-x+1| + \frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right) - \ln|x^2+x+1|$$

$$-\frac{5}{\sqrt{7}}\tan^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + c$$

Q31. $\int \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}dx$

Solution:- $\int \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}dx$

$$\because \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2+2x+3}$$

$$\Rightarrow 3x^3+4x^2+9x+5 = (Ax+B)(x^2+2x+3) + ((x+D)(x^2+x+1)$$

$$= Ax^3+2Ax^2+3Ax+Bx^2+2Bx+3B+Cx^3+Cx^2+Cx$$

$$+Dx^2+Dx+D$$

Equation coefficients of x^3, x^2, x and constant term

For x^3; $3 = A+C$ —— (i)

For x^2; $4 = 2A+B+C+D$ —— (ii)

For x; $9 = 3A+2B+C+D$ —— (iii)

For constant term; $5 = 3B+D$ —— (iv)

From (i) $\Rightarrow A = 3-C$ and from (iv) $\Rightarrow D = 5-3B$

Put in (ii) and (iii)

(ii) $\Rightarrow$ $4 = 2(3-C)+B+C+5-3B$

$4 = 6-2C+B+C+5-3B$

$4-6-5 = -C-2B \Rightarrow -7 = -(C+2B)$

$\Rightarrow C+2B = 7 \longrightarrow$ (v)

(iii) $\Rightarrow$ $9 = 3(3-C)+2B+C+5-3B$

$9 = 9-3C+2B+C+5-3B$

$\Rightarrow \cancel{9}-\cancel{9}-5 = -2C-B \Rightarrow B = -2C+5$ put in (v)

$\Rightarrow C+2(-2C+5) = 7 \Rightarrow C-4C+10 = 7$

$\Rightarrow -3C+10 = 7 \Rightarrow -3C = 7-10$

$\Rightarrow -3C = -3 \Rightarrow \boxed{C=1}$

As $B = 5-2C = 5-2(1) = 3 \Rightarrow \boxed{B=3}$

As $D = 5-3B = 5-3(3) = 5-9 = -4 \Rightarrow \boxed{D=-4}$

As $A = 3-C = 3-1 = 2 \Rightarrow \boxed{A=2}$

So

$$\frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2-x+1)} = \frac{2x+3}{x^2+x+1} + \frac{(1)x-4}{x^2+2x+3}$$

$$\Rightarrow \int \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2-x+1)}dx = \int \frac{2x+1+2}{x^2+x+1}dx + \frac{1}{2}\int \frac{2x-8}{x^2+2x+3}dx$$

$$= \int \frac{2x+1}{x^2+x+1}dx + 2\int \frac{1}{x^2+x+1}dx + \frac{1}{2}\int \frac{2x+2-10}{x^2+2x+3}dx$$

$$= \ln|x^2+x+1| + 2\int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx + \frac{1}{2}\int \frac{2x+2}{x^2+2x+3}dx$$

$$-5\int \frac{1}{x^2+2x+3}dx$$

$$= \ln|x^2+x+1| + 2\int \frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}dx + \frac{1}{2}\ln|x^2+2x+3)$$

$$-5\int \frac{1}{x^2+2x+1+2}dx$$

$$= \ln|x^2+x+1| + 2\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right) + \frac{1}{2}\ln|x^2+2x+3|$$

$$-5\int \frac{1}{(x+1)^2+(\sqrt{2})^2}dx$$

$$= \ln|x^2+x+1| + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{2}\ln|x^2+2x+3)$$

$$-\frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

$$= \ln|x^2+x+1| + \ln|x^2+2x+3|^{1/2} + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$-\frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

$$= \ln|x^2+x+1|\sqrt{x^2+2x+3} + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$-\frac{5}{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

The Definit Integrals

If $\int f(x)dx = \phi(x)+c$, then the integral of $f(x)$ from a to b is denoted by $\int_a^b f(x)dx$ and read as "definit integral of $f(x)$. Here a is called Lower limit and b is called upper limit.

★ The interval $[a, b]$ is called range of integration.

We evaluat $\int_a^b f(x)\,dx$ as;

Consider $\int f(x)dx = \phi(x)+c$

$$\rightarrow \int_a^b f(x)dx = \left|\phi(x)+c\right|_a^b$$

$$= [\phi(b)+c]-[\phi(a)+c]$$

$$= \phi(b)+c-\phi(a)-c$$

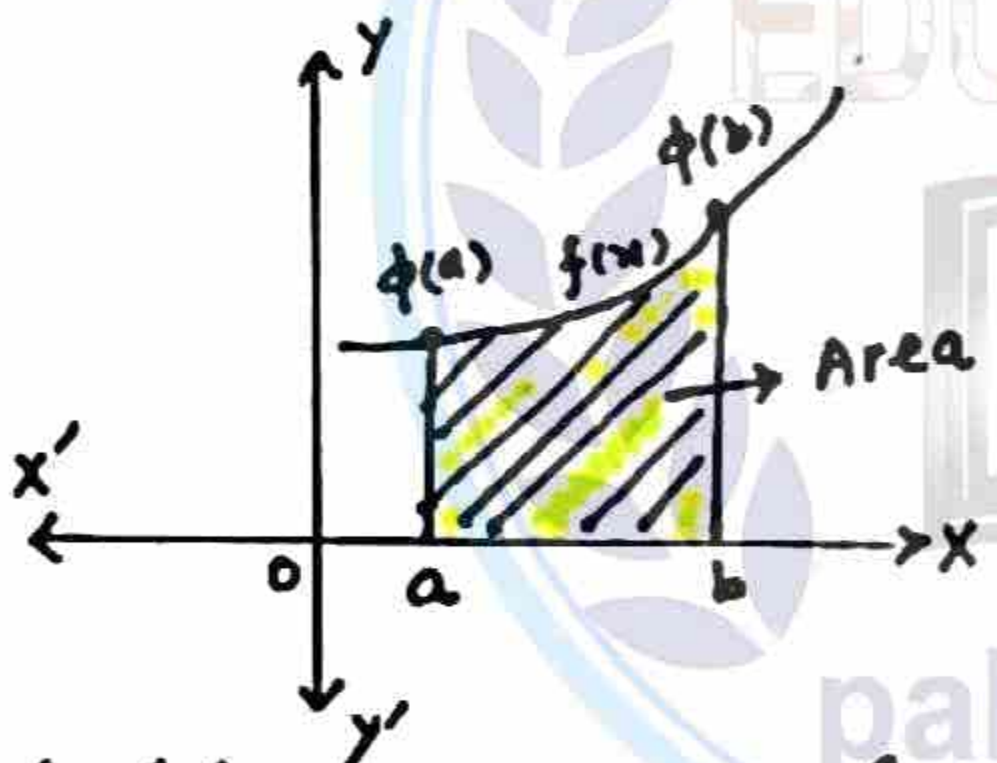
$$\int_a^b f(x)dx = \phi(b)-\phi(a)$$

Note:- If the lower limit is a constant and upper limit is a variable, then the integral is a function of the upper limit i-e., $\int_a^x f(t)dt = |\phi(t)|_a^x = \phi(x)-\phi(a)$

The Area Under the Curve

$$\int_a^b f(x)dx = \phi(b)-\phi(a)$$

represents the "area of region" bounded under the curve of function $f(x)$ the x-axis and between two ordinates $x=a$, $x=b$ as shown in fig.



Fundamental theorem of Calculus

If $f(x)$ is continuous $\forall x \in [a, b]$ and $\phi'(x) = f(x)$ then $\int_a^b f(x)dx = \phi(b)-\phi(a)$ is called Fundamental theorem of integral calculus.

Properties of Definit integrals

(a) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Proof:- $\because \int_a^b f(x)dx = \phi(b)-\phi(a)$

$$= -[\phi(a)-\phi(b)]$$

$$= -\int_b^a f(x)dx$$

$$\rightarrow \int_a^b f(x)dx = -\int_b^a f(x)dx$$

(b) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\ a<c<b$

Proof:- $\because \int_a^c f(x)dx = \phi(c)-\phi(a)$

and $\int_c^b f(x)dx = \phi(b)-\phi(c)$

Thus $\int_a^b f(x)dx + \int_c^b f(x)\,dx = \cancel{\phi(c)}-\phi(a)+\phi(b)-\cancel{\phi(c)}$

$$= \phi(b)-\phi(a)$$

$$= \int_a^b f(x)\,dx$$

$$\rightarrow \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\ a<c<b$$

(c) $\int_a^a f(x)dx = 0$

Proof:- $\because \int_a^a f(x)dx = \phi(a)-\phi(a)$

$$= 0$$

$$\rightarrow \int_a^a f(x)\,dx = 0$$

Also remember, $\int_a^b c f(x)dx = c\int_a^b f(x)dx$

and $\int_a^b [f(x)\pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Example 1. Evaluate (i) $\int_{-1}^3 (x^3+3x^2)dx$

(ii) $\int_1^2 \frac{x^2+1}{x+1}dx$

Solution:- (i) $\int_1^3 (x^3+3x^2dx$

$$= \int_{-1}^3 x^3dx + 3\int_{-1}^3 x^2dx = \left|\frac{x^4}{4}\right|_{-1}^3 + 3\left|\frac{x^3}{3}\right|_{-1}^3$$

$$= \frac{1}{4}|x^4|_{-1}^3 + \frac{\cancel{3}}{\cancel{3}}|x^3|_{-1}^3 = \frac{1}{4}[(3)^4-(-1)^4]+[(3)^3-(-1)^3]$$

$$= \frac{1}{4}(81-1)+(27+1) = \frac{1}{4}(80)+28 = 20+28$$

$$= 48$$

(ii) $\int_1^2 \frac{x^2+1}{x+1}dx$

$$= \int_1^2 \left(x-1+\frac{2}{x-1}\right)dx$$

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2+1 \\ & -x^2 \pm x \\ \hline & -x+1 \\ & \mp x \mp 1 \\ \hline & 2 \end{array}$$

$$= \int_1^2 x\,dx - \int_1^2 dx + 2\int_1^2 \frac{1}{x+1}\,dx$$

$$= \left|\frac{x^2}{2}\right|_1^2 - |x|_1^2 + 2\,\Big|\ln|x+1|\Big|_1^2$$

$$= \frac{1}{2}\left((2)^2-(1)^2\right) - (2-1) + 2\left[\ln|2+1| - \ln|1+1|\right]$$

$$= \frac{1}{2}(3) - 1 + 2(\ln 3 - \ln 2)$$

$$= \frac{3}{2} - 1 + 2\ln\frac{3}{2} = \frac{1}{2} + 2\ln\frac{3}{2}$$

Example 2. Evaluate (i) $\int_0^{\sqrt{3}} \frac{x^3+9x+1}{x^2+9}\,dx$

(ii) $\int_0^{\pi/4} \sec x(\sec x + \tan x)\,dx$

Solution:- (i) $\int_0^{\sqrt{3}} \frac{x^3+9x+1}{x^2+9}\,dx$

$$= \int_0^{\sqrt{3}} \left(\frac{x^3+9x}{x^2+9} + \frac{1}{x^2+9}\right) dx$$

$$= \int_0^{\sqrt{3}} \left(\frac{x\cancel{(x^2+9)}}{\cancel{x^2+9}} + \frac{1}{x^2+9}\right) dx = \int_0^{\sqrt{3}} x\,dx + \int_0^{\sqrt{3}} \frac{1}{x^2+(3)^2}\,dx$$

$$= \left|\frac{x^2}{2}\right|_0^{\sqrt{3}} + \left|\frac{1}{3}\tan^{-1}\frac{x}{3}\right|_0^{\sqrt{3}}$$

$$= \frac{1}{2}\left[(\sqrt{3})^2 - (0)^2\right] + \frac{1}{3}\left[\tan^{-1}\frac{\sqrt{3}}{3} - \tan^{-1}\frac{0}{3}\right]$$

$$= \frac{1}{2}(3) + \frac{1}{3}\left(\tan^{-1}\frac{1}{\sqrt{3}} - 0\right)$$

$$= \frac{3}{2} + \frac{1}{3}\left(\frac{\pi}{6}\right) \qquad \because \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

$$= \frac{3}{2} + \frac{\pi}{18}$$

(ii) $\int_0^{\pi/4} \sec x(\sec x + \tan x)\,dx$

$$= \int_0^{\pi/4} \sec^2 x\,dx + \int_0^{\pi/4} \sec x \tan x\,dx$$

$$= |\tan x|_0^{\pi/4} + |\sec x|_0^{\pi/4}$$

$$= \left(\tan\frac{\pi}{4} - \tan 0\right) + \left(\sec\frac{\pi}{4} - \sec 0\right)$$

$$= (1-0) + (\sqrt{2}-1) = 1 + \sqrt{2} - 1 = \sqrt{2}$$

Example 3. Evaluate $\int_0^{\pi/4} \frac{1}{1-\sin x}\,dx$

Solution:- $\int_0^{\pi/4} \frac{1}{1-\sin x}\,dx$

$$= \int_0^{\pi/4} \frac{1+\sin x}{(1-\sin x)(1+\sin x)}\,dx$$

$$= \int_0^{\pi/4} \frac{1+\sin x}{1-\sin^2 x}\,dx$$

$$= \int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\,dx$$

$$= \int_0^{\pi/4} \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right) dx$$

$$= \int_0^{\pi/4} \sec^2 x\,dx + \int_0^{\pi/4} \sec x \tan x\,dx$$

$$= |\tan x|_0^{\pi/4} + |\sec x|_0^{\pi/4}$$

$$= \left(\tan\frac{\pi}{4} - \tan 0\right) + \left(\sec\frac{\pi}{4} - \sec 0\right)$$

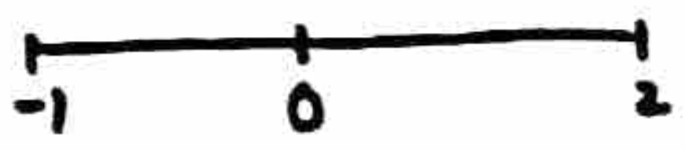
$$= (1-0) + (\sqrt{2}-1) = 1 + \sqrt{2} - 1 = \sqrt{2}$$

Example 4. Evaluate $\int_{-1}^{2} (x + |x|)\,dx$

Solution:-

$$\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a<c<b$$

$$\Rightarrow \int_{-1}^{2} (x+|x|)\,dx = \int_{-1}^{0} (x+|x|)\,dx + \int_0^2 (x+|x|)\,dx$$



$$= \int_{-1}^{0} (x+(-x))\,dx + \int_0^2 (x+(x))\,dx \qquad \because \text{If } x>0,\ |x| = x \quad \text{If } x<0,\ |x| = -x$$

$$= \int_{-1}^{0} 0\,dx + \int_0^2 2x\,dx$$

$$= 0 + 2\int_0^2 x\,dx = \cancel{2}\left|\frac{x^2}{\cancel{2}}\right|_0^2 = (2)^2 - (0)^2$$

$$= 4 - 0 = 4$$

Example 5. Evaluate $\int_0^{\sqrt{7}} \frac{3x}{\sqrt{x^2+9}}\,dx$

Solution:- $\int_0^{\sqrt{7}} \frac{3x}{\sqrt{x^2+9}}\,dx$

$$= 3\int_0^{\sqrt{7}} (x^2+9)^{-1/2} \cdot x\,dx$$

$$= \frac{3}{2}\int_0^{\sqrt{7}} (x^2+9)^{-1/2} \cdot 2x\,dx \qquad \because \int f^n(x) f'(x)\,dx = \frac{f^{n+1}(x)}{n+1} + c$$

$$= \frac{3}{2} \cdot \left|\frac{(x^2+9)^{1/2}}{1/2}\right|_0^{\sqrt{7}} = \frac{3}{2}\cdot\frac{2}{1}\left|\sqrt{x^2+9}\right|_0^{\sqrt{7}}$$

$$= 3\left(\sqrt{(\sqrt{7})^2+9} - \sqrt{(0)^2+9}\right) = 3\left(\sqrt{7+9} - \sqrt{9}\right)$$

$$= 3\left(\sqrt{16} - \sqrt{9}\right) = 3(4-3) = 3(1) = 3$$

Example 6. Evaluate $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx, \ x \neq -1, 1$

Solution:- $\int_{\frac{1}{2}}^{\sqrt{3}/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} (\sin^{-1}x)^1 \frac{1}{\sqrt{1-x^2}}\,dx \qquad \because \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$= \left|\frac{(\sin^{-1}x)^2}{2}\right|_{1/2}^{\sqrt{3}/2}$$

$$= \frac{1}{2}\left[\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)^2 - \left(\sin^{-1}\frac{1}{2}\right)^2\right] \qquad \because \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3} \quad \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{3}\right)^2 - \left(\frac{\pi}{6}\right)^2\right]$$

$$= \frac{1}{2}\left[\frac{\pi^2}{9} - \frac{\pi^2}{36}\right] = \frac{1}{2}\left[\frac{4\pi^2-\pi^2}{36}\right] = \frac{1}{2}\left(\frac{3\pi^2}{36}\right) = \frac{1}{2}\left(\frac{\pi^2}{12}\right)$$

$$= \frac{\pi^2}{24}$$

Example 7. Evaluate $\int_0^{\pi/6} x\cos x\,dx$

Solution:- $\int_0^{\pi/6} \underset{\text{I}}{x}\,\underset{\text{II}}{\cos x}\,dx$

$$= \Big|x\sin x\Big|_0^{\pi/6} - \int_0^{\pi/6}\sin x\,(1)\,dx$$

$$= \Big|x\sin x\Big|_0^{\pi/6} - \Big|(-\cos x)\Big|_0^{\pi/6}$$

$$= \left[\frac{\pi}{6}\sin\frac{\pi}{6} - 0\cdot\sin 0\right] + \left[\cos\frac{\pi}{6} - \cos 0\right]$$

$$= \left[\frac{\pi}{6}\cdot\frac{1}{2} - 0\right] + \left(\frac{\sqrt{3}}{2} - 1\right) = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

Example 8. Evaluate $\int_1^e x\ln x\,dx$

Solution:- $\int_1^e \underset{\text{II}}{x}\,\underset{\text{I}}{\ln x}\,dx$

$$= \Big|\ln x\cdot\frac{x^2}{2}\Big|_1^e - \int_1^e \frac{x^2}{2}\cdot\frac{1}{x}\,dx$$

$$= \frac{1}{2}\Big|x^2\ln x\Big|_1^e - \frac{1}{2}\int_1^e x\,dx$$

$$= \frac{1}{2}\Big|x^2\ln x\Big|_1^e - \frac{1}{2}\cdot\Big|\frac{x^2}{2}\Big|_1^e$$

$$= \frac{1}{2}\left[e^2\ln e - (1)^2\ln 1\right] - \frac{1}{4}\left[e^2 - (1)^2\right]$$

$$= \frac{1}{2}\left[e^2 - 0\right] - \frac{1}{4}(e^2 - 1) \qquad \because \ln e = 1,\ \ln 1 = 0$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{2e^2 - e^2}{4} + \frac{1}{4} = \frac{e^2}{4} + \frac{1}{4}$$

Example 9. If $\int_{-2}^{1} f(x)\,dx = 5$, $\int_1^3 f(x)\,dx = 3$, $\int_{-2}^{1} g(x)\,dx = 4$ evaluate the following definit integrals: (i) $\int_{-2}^{3} f(x)\,dx$ (ii) $\int_{-2}^{1}[2f(x)+3g(x)]\,dx$ (iii) $\int_{-2}^{1} 3f(x)\,dx - \int_{-2}^{1} 2g(x)\,dx$

Solution:- (i) $\int_{-2}^{3} f(x)\,dx$

$$\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a<c<b$$

$$\Rightarrow \int_{-2}^{3} f(x)\,dx = \int_{-2}^{1} f(x)\,dx + \int_1^3 f(x) = 5 + 3 = 8$$

(ii) $$\int_{-2}^{1}\left[2f(x)+3g(x)\right]dx = 2\int_{-2}^{1} f(x)\,dx + 3\int_{-2}^{1} g(x)\,dx$$

$$= 2(5) + 3(4) = 10 + 12 = 22$$

(iii) $$\int_{-2}^{1} 3f(x)\,dx - \int_{-2}^{1} 2g(x)\,dx$$

$$= 3\int_{-2}^{1} f(x)\,dx - 2\int_{-2}^{1} g(x)\,dx$$

$$= 3(5) - 2(4) = 15 - 8 = 7$$

Exercise 3.6

Evaluate the following definite integrals

Q1. $\int_1^2 (x^2+1)\,dx$

Solution:- $\int_1^2 (x^2+1)\,dx = \int_1^2 x^2\,dx + \int_1^2 1\,dx$

$$= \Big|\frac{x^3}{3}\Big|_1^2 + \Big|x\Big|_1^2 = \frac{1}{3}\left[(2)^3 - (1)^3\right] + [2-1]$$

$$= \frac{1}{3}[8-1] + 1 = \frac{7}{3} + 1 = \frac{10}{3}$$

Q2. $\int_{-1}^{1} (x^{1/3}+1)\,dx$

Solution:- $\int_{-1}^{1}(x^{1/3}+1)\,dx = \int_{-1}^{1} x^{1/3}\,dx + \int_{-1}^{1} 1\,dx$

$$= \Big|\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\Big|_{-1}^{1} + \Big|x\Big|_{-1}^{1} = \Big|\frac{x^{4/3}}{4/3}\Big|_{-1}^{1} + \Big|x\Big|_{-1}^{1}$$

$$= \frac{3}{4}\left((1)^{4/3} - (-1)^{4/3}\right) + \left(1 - (-1)\right)$$

$$= \frac{3}{4}(1 - 1) + (1+1) = \frac{3}{4}(0) + 2 = 2$$

Q3. $\int_{-2}^{0} \frac{1}{(2x-1)^2}\,dx$

Solution:- $\int_{-2}^{0}\frac{1}{(2x-1)^2}\,dx = \int_{-2}^{0}(2x-1)^{-2}\,dx$

$$= \frac{1}{2}\int_{-2}^{0}(2x-1)^{-2}\cdot 2\,dx = \frac{1}{2}\Big|\frac{(2x-1)^{-2+1}}{-2+1}\Big|_{-2}^{0}$$

$$= -\frac{1}{2}\Big|\frac{1}{2x-1}\Big|_{-2}^{0} = -\frac{1}{2}\left[\frac{1}{2(0)-1} - \frac{1}{2(-2)-1}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{-1} - \frac{1}{-5}\right] = -\frac{1}{2}\left(-1 + \frac{1}{5}\right) = -\frac{1}{2}\left(\frac{-5+1}{5}\right)$$

$$= -\frac{1}{2}\left(-\frac{4}{5}\right) = \frac{2}{5}$$

Q4. $\int_{-6}^{2}\sqrt{3-x}\,dx$

Solution:- $\int_{-6}^{2}\sqrt{3-x}\,dx = \int_{-6}^{2}(3-x)^{\frac{1}{2}}\,dx$

$$= -\int_{-6}^{2}(3-x)^{1/2}(-1)\,dx = -\Big|\frac{(3-x)^{3/2}}{3/2}\Big|_{-6}^{2}$$

$$= -\frac{2}{3}\left[(3-2)^{3/2} - (3-(-6))^{3/2}\right]$$

$$= -\frac{2}{3}\left[(1)^{3/2} - (9)^{3/2}\right] = -\frac{2}{3}\left[1 - (3^2)^{3/2}\right]$$

$$= -\frac{2}{3}\left[1 - 3^3\right] = -\frac{2}{3}\left[1 - 27\right] = -\frac{2}{3}(-26) = \frac{52}{3}$$

Q5. $\int_1^{\sqrt{5}}\sqrt{(2t-1)^3}\,dt$

Solution:- $\int_1^{\sqrt{5}}\sqrt{(2t-1)^3}\,dt = \int_1^{\sqrt{5}}(2t-1)^{3/2}\,dt$

$$= \frac{1}{2}\int_1^{\sqrt{5}}(2t-1)^{3/2}(2)\,dt = \frac{1}{2}\Big|\frac{(2t-1)^{5/2}}{5/2}\Big|_1^{\sqrt{5}}$$

$$= \frac{1}{2}\cdot\frac{2}{5}\left[(2\sqrt{5}-1)^{5/2} - (2(1)-1)^{5/2}\right]$$

$$= \frac{1}{5}\left[(2\sqrt{5}-1)^{2/5} - (1)^{5/2}\right] = \frac{1}{5}\left[(2\sqrt{5}-1)^{2/5} - 1\right]$$

Q6. $\int_2^{\sqrt{5}} x\sqrt{x^2-1}\,dx$

Solution:- $\int_2^{\sqrt{5}} x\sqrt{x^2-1}\,dx = \int_2^{\sqrt{5}} (x^2-1)^{1/2}\cdot x\,dx$

$$= \frac{1}{2}\int_2^{\sqrt{5}} (x^2-1)^{1/2}\cdot 2x\,dx = \frac{1}{2}\left|\frac{(x^2-1)^{3/2}}{3/2}\right|_2^{\sqrt{5}}$$

$$= \frac{1}{2}\cdot\frac{2}{3}\left[((\sqrt{5})^2-1)^{3/2} - ((2)^2-1)^{3/2}\right]$$

$$= \frac{1}{3}\left[(5-1)^{3/2} - (4-1)^{3/2}\right] = \frac{1}{3}\left[(4)^{3/2} - (3)^{3/2}\right]$$

$$= \frac{1}{3}\left((2^2)^{3/2} - 3^1\cdot 3^{1/2}\right) = \frac{1}{3}\left(2^3 - 3\sqrt{3}\right) = \frac{1}{3}(8-3\sqrt{3})$$

Q7. $\int_1^2 \frac{x}{x^2+2}\,dx$

Solution:- $\int_1^2 \frac{x}{x^2+2}\,dx = \frac{1}{2}\int_1^2 \frac{2x}{x^2+2}\,dx$

$$= \frac{1}{2}\left|\ln|x^2+2|\right|_1^2 \qquad \because \int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+c$$

$$= \frac{1}{2}\left[\ln|(2)^2+2| - \ln|(1)^2+2|\right]$$

$$= \frac{1}{2}\left[\ln|4+2| - \ln|1+2|\right] = \frac{1}{2}\left[\ln 6 - \ln 3\right]$$

$$= \frac{1}{2}\ln\frac{6}{3} = \frac{1}{2}\ln 2$$

Q8. $\int_2^3 \left(x-\frac{1}{x}\right)^2 dx$

Solution:- $\int_2^3 \left(x-\frac{1}{x}\right)^2 dx = \int_2^3 \left(x^2+\frac{1}{x^2}-2\right)dx$

$$= \int_2^3 x^2\,dx + \int_2^3 x^{-2}\,dx - 2\int_2^3 dx$$

$$= \left|\frac{x^3}{3}\right|_2^3 + \left|\frac{x^{-1}}{-1}\right|_2^3 - 2|x|_2^3 = \frac{1}{3}|x^3|_2^3 - \left|\frac{1}{x}\right|_2^3 - 2|x|_2^3$$

$$= \frac{1}{3}\left[(3)^3-(2)^3\right] - \left[\frac{1}{3}-\frac{1}{2}\right] - 2\left[3-2\right]$$

$$= \frac{1}{3}(27-8) - \left(\frac{2-3}{6}\right) - 2(1)$$

$$= \frac{19}{3} + \frac{1}{6} - 2 = \frac{38+1-12}{6} = \frac{27}{6} = \frac{9}{2}$$

Q9. $\int_{-1}^1 \left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}\,dx$

Solution:- $\int_{-1}^1 \left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}\,dx$

$$= \int_{-1}^1 (x^2+x+1)^{1/2}\left(\frac{2x+1}{2}\right)dx$$

$$= \frac{1}{2}\int_{-1}^1 (x^2+x+1)^{1/2}(2x+1)\,dx$$

$$= \frac{1}{2}\cdot\left|\frac{(x^2+x+1)^{3/2}}{3/2}\right|_{-1}^1 = \frac{1}{2}\cdot\frac{2}{3}\left|(x^2+x+1)^{3/2}\right|_{-1}^1$$

$$= \frac{1}{3}\left[((1)^2+1+1)^{3/2} - ((-1)^2-1+1)^{3/2}\right]$$

$$= \frac{1}{3}\left[(3)^{3/2} - (1)^{3/2}\right] = \frac{1}{3}\left[3\sqrt{3}-1\right] = \sqrt{3} - \frac{1}{\sqrt{3}}$$

Q10. $\int_0^3 \frac{dx}{x^2+9}$

Solution:- $\int_0^3 \frac{dx}{x^2+9} = \int_0^3 \frac{dx}{(x)^2+(3)^2}$

$$= \frac{1}{3}\left|\tan^{-1}\frac{x}{3}\right|_0^3 \qquad \because \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$$

$$= \frac{1}{3}\left[\tan^{-1}\frac{3}{3} - \tan^{-1}\frac{0}{3}\right]$$

$$= \frac{1}{3}\left(\tan^{-1}1 - 0\right) = \frac{1}{3}\left(\frac{\pi}{4}-0\right) = \frac{\pi}{12}$$

Q11. $\int_{\pi/6}^{\pi/3} \cos t\,dt$

Solution:- $\int_{\pi/6}^{\pi/3} \cos t\,dt$

$$= |\sin t|_{\pi/6}^{\pi/3} = \sin\frac{\pi}{3} - \sin\frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$$

Q12. $\int_1^2 \left(x+\frac{1}{x}\right)^{1/2}\left(1-\frac{1}{x^2}\right)dx$

Solution:- $\int_1^2 \left(x+\frac{1}{x}\right)^{1/2}\left(1-\frac{1}{x^2}\right)dx$

$$= \left|\frac{\left(x+\frac{1}{x}\right)^{3/2}}{3/2}\right|_1^2 \qquad \because \frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$= 1+\frac{d}{dx}(x^{-1}) = 1+(-1)x^{-1-1}$$

$$= 1 - \frac{1}{x^2}$$

$$= \frac{2}{3}\left[\left(2+\frac{1}{2}\right)^{3/2} - \left(1+\frac{1}{1}\right)^{3/2}\right]$$

$$= \frac{2}{3}\left[\left(\frac{5}{2}\right)^{3/2} - (2)^{3/2}\right]$$

$$= \frac{2}{3}\left[\frac{(5)^{3/2}}{(2)^{3/2}} - (2)^{3/2}\right] = \frac{2}{3}\left(\frac{5\sqrt{5} - \left[(2)^{3/2}\right]^2}{(2)^{3/2}}\right)$$

$$= \frac{\cancel{2}}{3}\left[\frac{5\sqrt{5} - 2^3}{\cancel{2}\sqrt{2}}\right] = \frac{5\sqrt{5}-8}{3\sqrt{2}}$$

Q13. $\int_1^2 \ln x\,dx$

Solution:- $\int_1^2 \ln x\,dx = \int_1^2 \underset{\text{I}}{\ln x}\cdot\underset{\text{II}}{1}\,dx$

$$= |x\ln x|_1^2 - \int_1^2 x\cdot\frac{1}{x}\,dx$$

$$= |x\ln x|_1^2 - \int_1^2 1\,dx = |x\ln x|_1^2 - |x|_1^2$$

$$= (2\ln 2 - 1\ln 1) - (2-1) \qquad (\because \ln 1 = 0)$$

$$= (2\ln 2 - 0) - 1 = 2\ln 2 - 1$$

Q14. $\int_0^2 (e^{x/2} - e^{-x/2})\,dx$

Solution:- $\int_0^2 (e^{x/2} - e^{-x/2})\,dx$

$= \int_0^2 e^{x/2}dx - \int_0^2 e^{-x/2}dx = \left|\frac{e^{x/2}}{1/2}\right|_0^2 - \left|\frac{e^{-x/2}}{-1/2}\right|_0^2$

$= 2\left|e^{x/2}\right|_0^2 + 2\left|e^{-x/2}\right|_0^2$

$= 2[e^{2/2} - e^{0/2}] + 2[e^{-2/2} - e^{-0/2}]$

$= 2[e^1 - e^0] + 2[e^{-1} - e^0]$

$= 2(e^1 - 1) + 2(\frac{1}{e} - 1)$

$= 2\left[e - 1 + \frac{1}{e} - 1\right] = 2\left(e + \frac{1}{e} - 2\right)$

$= 2\left(\frac{e^2+1-2e}{e}\right) = \frac{2}{e}(e-1)^2$

Q15. $\int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{\cos 2\theta + 1}d\theta$

Solution:- $\int_0^{\pi/4} \frac{\cos\theta + \sin\theta}{\cos 2\theta + 1}d\theta$

$\left(\because \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} \rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2} \rightarrow 2\cos^2\theta = 1+\cos 2\theta\right)$

$= \int_0^{\pi/4} \frac{\cos\theta+\sin\theta}{2\cos^2\theta}d\theta = \frac{1}{2}\int_0^{\pi/4}\left(\frac{\cos\theta}{\cos^2\theta} + \frac{\sin\theta}{\cos^2\theta}\right)d\theta$

$= \frac{1}{2}\int_0^{\pi/4}\left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}\right)d\theta$

$= \frac{1}{2}\int_0^{\pi/4}(\sec\theta + \sec\theta\tan\theta)d\theta$

$= \frac{1}{2}\left[\int_0^{\pi/4}\sec\theta d\theta + \int_0^{\pi/4}\sec\theta\tan\theta d\theta\right]$

$= \frac{1}{2}\left\{\left|\ln|\sec\theta+\tan\theta|\right|_0^{\pi/4} + \left|\sec\theta\right|_0^{\pi/4}\right]$

$= \frac{1}{2}\left\{\ln|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}| - \ln|\sec 0 + \tan 0| + \sec\frac{\pi}{4} - \sec 0\right]$

$= \frac{1}{2}\left\{\ln|\sqrt{2}+1| - \ln|1+0| + \sqrt{2} - 1\right]$

$= \frac{1}{2}\left[\ln|\sqrt{2}+1| - 0 + \sqrt{2} - 1\right]$

$= \frac{1}{2}\left[\ln|\sqrt{2}+1| + \sqrt{2} - 1\right]$

Q16. $\int_0^{\pi/6}\cos^3\theta\,d\theta$

Solution:- $\int^{\pi/2}\cos^3\theta d\theta$

$= \int_0^{\pi/2}\cos^2\theta\cdot\cos\theta d\theta$

$= \int_0^{\pi/2}(1-\sin^2\theta)\cos\theta d\theta$

$= \int_0^{\pi/2}\cos\theta d\theta - \int_0^{\pi/2}\sin^2\theta\cos\theta d\theta$

$= \left|\sin\theta\right|_0^{\pi/6} - \left|\frac{\sin^3\theta}{3}\right|_0^{\pi/6}$

$= (\sin\frac{\pi}{6} - \sin 0) - \frac{1}{3}(\sin^3\frac{\pi}{6} - \sin^3 0)$

$= (\frac{1}{2} - 0) - \frac{1}{3}((\frac{1}{2})^3 - 0)$

$= (\frac{1}{2}) - \frac{1}{3}(\frac{1}{8}) = \frac{1}{2} - \frac{1}{24} = \frac{12-1}{24}$

$= \frac{11}{24}$

Q17. $\int_{\pi/6}^{\pi/4}\cos^2\theta\cot^2\theta\,d\theta$

Solution:- $\int_{\pi/6}^{\pi/4}\cos^2\theta\cot^2\theta d\theta$

$= \int_{\pi/6}^{\pi/4}\cos^2\theta(\operatorname{cosec}^2\theta - 1)d\theta$

$= \int_{\pi/6}^{\pi/4}\cos^2\theta\operatorname{cosec}^2\theta d\theta - \int_{\pi/6}^{\pi/4}\cos^2\theta d\theta$

$= \int_{\pi/6}^{\pi/4}\frac{\cos^2\theta}{\sin^2\theta}d\theta - \int_{\pi/6}^{\pi/4}\cos^2\theta d\theta$

$= \int_{\pi/6}^{\pi/4}\cot^2\theta d\theta - \int_{\pi/6}^{\pi/4}\cos^2\theta d\theta$

$\because 2\cos^2\theta = 1+\cos 2\theta \rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$= \int_{\pi/6}^{\pi/4}(\operatorname{cosec}^2\theta - 1)d\theta - \int_{\pi/6}^{\pi/4}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$

$= \int_{\pi/6}^{\pi/4}\operatorname{cosec}^2\theta d\theta - \int_{\pi/6}^{\pi/4}1\,d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/4}(1+\cos 2\theta)d\theta$

$= \int_{\pi/6}^{\pi/4}\operatorname{cosec}^2\theta d\theta - \int_{\pi/6}^{\pi/4}1\,d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/4}d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/4}\cos 2\theta d\theta$

$= \int_{\pi/6}^{\pi/4}\operatorname{cosec}^2\theta d\theta - \int_{\pi/6}^{\pi/4}(1+\frac{1}{2})d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/4}\cos 2\theta d\theta$

$= \left|-\cot\theta\right|_{\pi/6}^{\pi/4} - \frac{3}{2}\left|\theta\right|_{\pi/6}^{\pi/4} - \frac{1}{2}\left|\frac{\sin 2\theta}{2}\right|_{\pi/6}^{\pi/4}$

$= -\left[\cot\frac{\pi}{4} - \cot\frac{\pi}{6}\right] - \frac{3}{2}\left[\frac{\pi}{4} - \frac{\pi}{6}\right] - \frac{1}{4}\left[\sin\cancel{2}\cdot\frac{\pi}{\cancel{4}_2} - \sin\cancel{2}\cdot\frac{\pi}{\cancel{6}_3}\right]$

$= -(1-\sqrt{3}) - \frac{3}{2}\left(\frac{3\pi-2\pi}{\cancel{4\times 2}}\right) - \frac{1}{4}\left(1 - \frac{\sqrt{3}}{2}\right)$

$= -1 + \sqrt{3} - \frac{\pi}{8} - \frac{1}{4} + \frac{\sqrt{3}}{8} = \frac{-8+8\sqrt{3}-\pi-2+\sqrt{3}}{8}$

$= \frac{9\sqrt{3}-10-\pi}{8}$

$\left(\because \cot\theta = \frac{1}{\tan\theta} \quad \tan\frac{\pi}{4} = 1 \rightarrow \cot\frac{\pi}{4} = 1 \quad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \rightarrow \cot\pi/6 = \sqrt{3}\right)$

Q18. $\int_0^{\pi/4}\cos^4 t\,dt$

Solution:- $\int_0^{\pi/4}\cos^4 t\,dt$

$\because \cos^2\theta = \frac{1+\cos 2\theta}{2} \rightarrow \cos^2 2\theta = \frac{1+\cos 4\theta}{2}$

$= \int_0^{\pi/4}(\cos^2 t)^2 dt$

$= \int_0^{\pi/4}\left(\frac{1+\cos 2t}{2}\right)^2 dt$

$= \int_0^{\pi/4}\left(\frac{1+\cos^2 2t + 2\cos 2t}{4}\right)dt$

$$= \int_0^{\pi/4} \left(\frac{1 + 2\cos 2t + \cos^2 2t}{4} \right) dt$$

$$= \frac{1}{4} \int_0^{\pi/4} (1 + 2\cos 2t + \cos^2 2t)\, dt$$

$$= \frac{1}{4} \int_0^{\pi/4} \left(1 + 2\cos 2t + \frac{1 + \cos 4t}{2} \right) dt$$

$$= \frac{1}{4} \left[\int_0^{\pi/4} 1\, dt + 2 \int_0^{\pi/4} \cos 2t\, dt + \frac{1}{2} \int_0^{\pi/4} (1 + \cos 4t)\, dt \right]$$

$$= \frac{1}{4} \left[|t|_0^{\pi/4} + \cancel{2} \left| \frac{\sin 2t}{\cancel{2}} \right|_0^{\pi/4} + \frac{1}{2} |t|_0^{\pi/4} + \frac{1}{2} \left| \frac{\sin 4t}{4} \right|_0^{\pi/4} \right]$$

$$= \frac{1}{4} \left[\left(\frac{\pi}{4} - 0\right) + \left(\sin 2\left(\frac{\pi}{4}\right) - \sin 2(0)\right) + \frac{1}{2}\left(\frac{\pi}{4} - 0\right) \cdot + \frac{1}{8}\left(\sin 4\left(\frac{\pi}{4}\right) - \sin 4(0) \right) \right]$$

$$= \frac{1}{4} \left[\frac{\pi}{4} + \sin \frac{\pi}{2} - 0 + \frac{\pi}{8} + \frac{1}{8}(\sin \pi - 0) \right]$$

$$= \frac{1}{4} \left[\frac{\pi}{4} + 1 + \frac{\pi}{8} + \frac{1}{8}(0) \right]$$

$$= \frac{1}{4} \left(\frac{2\pi + 8 + \pi}{8} \right) = \frac{3\pi + 8}{32}$$

Q19. $\int_0^{\pi/3} \cos^2\theta \sin\theta\, d\theta$

Solution:- $\int_0^{\pi/3} \cos^2\theta \sin\theta\, d\theta$

$$= -\int_0^{\pi/3} \cos^2\theta(-\sin\theta)\, d\theta$$

$$= -\left| \frac{\cos^3\theta}{3} \right|_0^{\pi/3} = -\frac{1}{3} \left| (\cos\theta)^3 \right|_0^{\pi/3}$$

$$= -\frac{1}{3} \left[\left(\cos\frac{\pi}{3}\right)^3 - (\cos 0)^3 \right] = -\frac{1}{3} \left[\left(\frac{1}{2}\right)^3 - (1)^3 \right]$$

$$= -\frac{1}{3} \left[\frac{1}{8} - 1 \right] = -\frac{1}{3} \left[\frac{1 - 8}{8} \right] = -\frac{1}{3} \left[-\frac{7}{8} \right]$$

$$= \frac{7}{24}$$

Q20. $\int_0^{\pi/4} (1 + \cos^2\theta) \tan^2\theta\, d\theta$

Solution:- $\int_0^{\pi/4} (1 + \cos^2\theta) \tan^2\theta\, d\theta$

$$= \int_0^{\pi/4} (\tan^2\theta + \cos^2\theta \tan^2\theta)\, d\theta$$

$$= \int_0^{\pi/4} \tan^2\theta\, d\theta + \int_0^{\pi/4} \cos^2\theta \tan^2\theta\, d\theta$$

$$= \int_0^{\pi/4} (\sec^2\theta - 1)\, d\theta + \int_0^{\pi/4} \cancel{\cos^2\theta} \cdot \frac{\sin^2\theta}{\cancel{\cos^2\theta}}\, d\theta$$

$$= \int_0^{\pi/4} \sec^2\theta\, d\theta - \int_0^{\pi/4} 1\, d\theta + \int_0^{\pi/4} \sin^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \sec^2\theta\, d\theta - \int_0^{\pi/4} 1\, d\theta + \int_0^{\pi/4} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta$$

$$= |\tan\theta|_0^{\pi/4} - |\theta|_0^{\pi/4} + \frac{1}{2} \int_0^{\pi/4} 1\, d\theta - \frac{1}{2} \int_0^{\pi/4} \cos 2\theta\, d\theta$$

$$= |\tan\theta|_0^{\pi/4} - |\theta|_0^{\pi/4} + \frac{1}{2} |\theta|_0^{\pi/4} - \frac{1}{2} \left| \frac{\sin 2\theta}{2} \right|_0^{\pi/4}$$

$$= \left(\tan\frac{\pi}{4} - \tan 0\right) - \left(\frac{\pi}{4} - 0\right) + \frac{1}{2}\left(\frac{\pi}{4} - 0\right) \cdot \frac{1}{4}\left(\sin 2\left(\frac{\pi}{4}\right) - \sin 2(0)\right)$$

$$= (1 - 0) - \left(\frac{\pi}{4} - 0\right) - \frac{\pi}{8} - \frac{1}{4}(1 - 0)$$

$$= 1 - \frac{\pi}{4} + \frac{\pi}{8} - \frac{1}{4} = \frac{8 - 2\pi + \pi - 2}{8} = \frac{6 - \pi}{8}$$

Q21. $\int_0^{\pi/4} \frac{\sec\theta}{\sin\theta + \cos\theta}\, d\theta$

Solution:- $\int_0^{\pi/4} \frac{\sec\theta}{\sin\theta + \cos\theta}\, d\theta$

$$= \int_0^{\pi/4} \frac{\sec\theta}{\cos\theta\left(\frac{\sin\theta}{\cos\theta} + 1\right)}\, d\theta = \int_0^{\pi/4} \frac{\sec\theta \cdot \sec\theta}{\tan\theta + 1}\, d\theta$$

$$= \int_0^{\pi/4} \frac{\sec^2\theta}{\tan\theta + 1}\, d\theta \qquad \because \int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$$

$$= \left| \ln|\tan\theta + 1| \right|_0^{\pi/4}$$

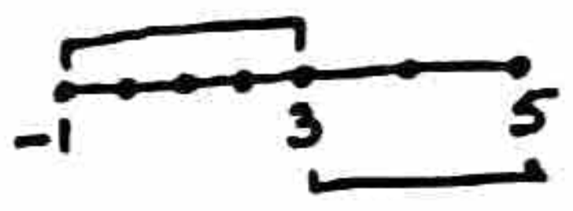
$$= \ln\left|\tan\frac{\pi}{4} + 1\right| - \ln|\tan 0 + 1|$$

$$= \ln|1 + 1| - \ln|0 + 1| \qquad \because \ln 1 = 0$$

$$= \ln 2 - \ln 1 = \ln 2$$

Q22. $\int_{-1}^{5} |x - 3|\, dx$

Solution:- $\int_{-1}^{5} |x - 3|\, dx$



$\because |x - 3| = -(x - 3)$ if $-1 < x < 3$

$|x - 3| = (x - 3)$ if $3 < x < 5$ so

$$= \int_{-1}^{3} -(x - 3)\, dx + \int_{3}^{5} (x - 3)\, dx$$

$$= -\left| \frac{(x - 3)^2}{2} \right|_{-1}^{3} + \left| \frac{(x - 3)^2}{2} \right|_{3}^{5}$$

$$= -\frac{1}{2} \left\{ (3 - 3)^2 - (-1 - 3)^2 \right\} + \frac{1}{2} \left\{ (5 - 3)^2 - (3 - 3)^2 \right\}$$

$$= -\frac{1}{2} \left\{ 0 - (-4)^2 \right\} + \frac{1}{2} \left\{ (2)^2 - 0 \right\}$$

$$= -\frac{1}{2}(-16) + \frac{1}{2}(4) = \frac{16}{2} + \frac{4}{2} = 10$$

Q23. $\int_{1/8}^{1} \left(\frac{x^{1/3} + 2)^2}{x^{2/3}} \right) dx$

Solution:- $\int_{1/8}^{1} \frac{(x^{1/3} + 2)^2}{x^{2/3}}\, dx = \int_{1/8}^{1} (x^{1/3} + 2)^2 \cdot x^{-2/3}\, dx$

$$\because \frac{d}{dx}(x^{1/3} + 2) = \frac{1}{3} x^{\frac{1}{3} - 1} + 0 = \frac{1}{3} x^{-2/3}$$

$$= 3 \int_{1/8}^{1} (x^{1/3} + 2)^2 \cdot \frac{1}{3} x^{-2/3}\, dx$$

$$= \cancel{3} \left| \frac{(x^{1/3} + 2)^3}{\cancel{3}} \right|_{1/8}^{1}$$

$$= \left[((1)^{1/3} + 2) \right]^3 - \left[\left(\frac{1}{8}\right)^{1/3} + 2 \right]^3$$

$$= (1 + 2)^3 - \left(\left(\frac{1}{2^3}\right)^{1/3} + 2 \right)^3$$

$$= 27 - \left(\frac{1}{2} + 2 \right)^3 = 27 - \left(\frac{1 + 4}{2} \right)^3$$

$$= 27 - \left(\frac{5}{2}\right)^3 = 27 - \frac{125}{8} = \frac{216 - 125}{8} = \frac{91}{8}$$

Q24. $\int_1^3 \frac{x^2-2}{x+1}dx$

Solution:- $\int_1^3 \frac{x^2-2}{x+1}dx$

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2-2 \\ & \underline{x^2 \pm x} \\ & -x-2 \\ & \underline{\mp x \mp 1} \\ & -1 \end{array}$$

$= \int_1^3 \left(x-1+\frac{-1}{x+1}\right)dx$

$= \int_1^3 x\,dx - \int_1^3 1\,dx - \int_1^3 \frac{1}{x+1}dx$

$= \Big|\frac{x^2}{2}\Big|_1^3 - |x|_1^3 - \big|\ln|x+1|\big|_1^3$

$= \frac{1}{2}((3)^2-(1)^2)-(3-1)-\left[\ln|3+1|-\ln|1+1|\right]$

$= \frac{1}{2}(9-1)-2-(\ln 4-\ln 2)$

$= \frac{1}{2}(8)-2-\ln\frac{4}{2} = 4-2-\ln 2 = 2-\ln 2$

Q25. $\int_2^3 \frac{3x^2-2x+1}{(x-1)(x^2+1)}dx$

Solution:- $\int_2^3 \frac{3x^2-2x+1}{(x-1)(x^2+1)}dx$

$\because (x-1)(x^2+1) = x^3+x-x^2-1$

$= x^3-x^2+x-1$

$= \int_2^3 \frac{3x^2-2x+1}{x^3-x^2+x-1}dx = \Big|\ln|x^3-x^2+x-1|\Big|_2^3$

$= \ln|(3)^3-(3)^2+3-1| - \ln|(2)^3-(2)^2+2-1|$

$= \ln|27-9+2| - \ln|8-4+1|$

$= \ln 20 - \ln 5 = \ln\frac{20}{5} = \ln 4$

Q26. $\int_0^{\pi/4} \frac{\sin x-1}{\cos^2 x}dx$

Solution:- $\int_0^{\pi/4}\left(\frac{\sin x}{\cos^2 x}-\frac{1}{\cos^2 x}\right)dx$

$= \int_0^{\pi/4}\sec x\tan x\,dx - \int_0^{\pi/4}\sec^2 x\,dx$

$= |\sec x|_0^{\pi/4} - |\tan x|_0^{\pi/4}$

$= (\sec\frac{\pi}{4}-\sec 0)-(\tan\frac{\pi}{4}-\tan 0)$

$= (\sqrt{2}-1)-(1-0) = \sqrt{2}-1-1 = \sqrt{2}-2$

Q27. $\int_0^{\pi/4}\frac{1}{1+\sin x}dx$

Solution:- $\int_0^{\pi/4}\frac{1}{1+\sin x}dx$

$= \int_0^{\pi/4}\frac{1-\sin x}{(1+\sin x)(1-\sin x)}dx$

$= \int_0^{\pi/4}\frac{1-\sin x}{1-\sin^2 x}dx$

$= \int_0^{\pi/4}\frac{1-\sin x}{\cos^2 x}dx$

$= \int_0^{\pi/4}\left(\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}\right)dx$

$= \int_0^{\pi/4}\sec^2 x\,dx - \int_0^{\pi/4}\sec x\tan x\,dx$

$= |\tan x|_0^{\pi/4} - |\sec x|_0^{\pi/4}$

$= \tan\frac{\pi}{4}-\tan 0-(\sec\frac{\pi}{4}-\sec 0)$

$= 1-0-(\sqrt{2}-1) = 1-\sqrt{2}+1$

$= 2-\sqrt{2}$

Q28. $\int_0^1 \frac{3x}{\sqrt{4-3x}}dx$

Solution:- $\int_0^1 \frac{3x}{\sqrt{4-3x}}dx$

$= -\int_0^1 \frac{-3x}{\sqrt{4-3x}}dx = -\int_0^1 \frac{4-3x-4}{\sqrt{4-3x}}dx$

$= -\int_0^1\left(\frac{4-3x}{\sqrt{4-3x}}-\frac{4}{\sqrt{4-3x}}\right)dx$

$= -\int_0^1\left(\sqrt{4-3x}-4(4-3x)^{-\frac{1}{2}}\right)dx$

$= -\int_0^1(4-3x)^{\frac{1}{2}}dx + 4\int_0^1(4-3x)^{-\frac{1}{2}}dx$

$= \frac{1}{3}\int_0^1(4-3x)^{\frac{1}{2}}(-3)dx - \frac{4}{3}\int_0^1(4-3x)^{-\frac{1}{2}}(-3)dx$

$= \frac{1}{3}\cdot\Big|\frac{(4-3x)^{3/2}}{3/2}\Big|_0^1 - \frac{4}{3}\Big|\frac{(4-3x)^{\frac{1}{2}}}{\frac{1}{2}}\Big|_0^1$

$= \frac{1}{3}\cdot\frac{2}{3}\Big|(4-3x)^{3/2}\Big|_0^1 - \frac{4}{3}\cdot\frac{2}{1}\Big|(4-3x)^{\frac{1}{2}}\Big|_0^1$

$= \frac{2}{9}\left[(4-3(1))^{3/2}-(4-3(0))^{3/2}\right]-\frac{8}{3}\left[(4-3(1))^{\frac{1}{2}}-(4-3(0))^{\frac{1}{2}}\right]$

$= \frac{2}{9}\left((1)^{3/2}-(4)^{3/2}\right)-\frac{8}{3}\left((1)^{\frac{1}{2}}-(4)^{\frac{1}{2}}\right)$

$= \frac{2}{9}\left(1-(2^2)^{3/2}\right)-\frac{8}{3}\left(1-(2^2)^{\frac{1}{2}}\right)$

$= \frac{2}{9}(1-2^3)-\frac{8}{3}(1-2^1)$

$= \frac{2}{9}(1-8)-\frac{8}{3}(-1) = \frac{-14}{9}+\frac{8}{3} = \frac{-14+24}{9}$

$= \frac{10}{9}$

Q29. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin(2+\sin x)} dx$

Solution:- $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x.\sin x(\frac{2}{\sin x}+1)} dx$

$= \int_{\frac{\pi}{6}}^{\pi/2} \frac{1}{\sin x}.\frac{\cos x}{\sin x}.\frac{1}{(2\operatorname{cosec} x+1)} dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\operatorname{cosec} x.\cot x}{2\operatorname{cosec} x+1} dx$

$= -\frac{1}{2} \int_{\frac{\pi}{6}}^{\pi/2} \frac{-2\operatorname{cosec} x \cot x}{2\operatorname{cosec} x+1} dx$

$\because \int \frac{f'(x)}{f(x)} = \ln|f(x)| + c$

$= -\frac{1}{2} \left|\ln|2\operatorname{cosec} x+1|\right|_{\pi/6}^{\pi/2}$

$= -\frac{1}{2}\left[\ln|2\operatorname{cosec}\frac{\pi}{2}+1| - \ln|2\operatorname{cosec}\frac{\pi}{6}+1|\right]$

$\because \sin\frac{\pi}{2} = 1 \rightarrow \operatorname{cosec}\frac{\pi}{2} = 1$

$\sin\frac{\pi}{6} = \frac{1}{2} \rightarrow \operatorname{cosec}\frac{\pi}{6} = 2$

$= -\frac{1}{2}\left[\ln|2(1)+1| - \ln|2(2)+1|\right]$

$= -\frac{1}{2}[\ln 3 - \ln 5] = \frac{1}{2}[\ln 5 - \ln 3]$

$= \frac{1}{2}\ln 5/3$

Q30. $\int_0^{\pi/2} \frac{\sin x}{(1+\cos x)(2+\cos x)} dx$

Solution:- $\int_0^{\pi/2} \frac{\sin x}{(1+\cos x)(2+\cos x)} dx$

Put $\cos x = t$

$\rightarrow d(\cos x) = dt \rightarrow -\sin x dx = dt$

$\rightarrow \sin x dx = -dt$

As $x \rightarrow 0$ then $t \rightarrow 1$ $\quad \because \cos 0 = 1$

and $x \rightarrow \frac{\pi}{2}$ then $t \rightarrow 0$ $\quad \because \cos\frac{\pi}{2} = 0$

$= -\int_1^0 \frac{dt}{(1+t)(2+t)} = \int_0^1 \frac{1}{(1+t)(2+t)} dt$

$\because \frac{1}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t}$

$\rightarrow 1 = A(2+t) + B(1+t) \longrightarrow (i)$

Put $1+t = 0 \rightarrow t = -1$ in (i)

$\rightarrow 1 = A(2-1) + B(0) \rightarrow \boxed{A = 1}$

Put $2+t = 0 \rightarrow t = -2$ in (i)

$\rightarrow 1 = A(0) + B(1-2)$

$1 = -B \rightarrow \boxed{B = -1}$

so $\frac{1}{(1+t)(2+t)} = \frac{1}{1+t} + \frac{-1}{2+t}$

Now $\int_0^1 \frac{1}{(1+t)(2+t)} dt = \int_0^1 \frac{1}{1+t} dt - \int_0^1 \frac{1}{2+t} dt$

$= \left|\ln|1+t|\right|_0^1 - \left|\ln|2+t|\right|_0^1$

$= [\ln|1+1| - \ln|1+0|] - [\ln|2+1| - \ln|2+0|]$

$= [\ln 2 - \ln 1] - [\ln 3 - \ln 2)$

$= \ln 2 - 0 - \ln 3 + \ln 2$

$= 2\ln 2 - \ln 3$

$= \ln 2^2 - \ln 3$

$= \ln 4 - \ln 3$

$= \ln\frac{4}{3}$

$\left(\begin{array}{l} \because a\ln b = \ln b^a \\ \because \ln a - \ln b \\ = \ln a/b \end{array}\right)$

Application of Definite integrals

Area under the curve:-

Case I:- If $f(x) \geq 0 \quad \forall x \in [a,b]$ then curve lies above x-axis. So

$$\boxed{A = \int_a^b f(x) dx \quad \text{where } a < b}$$

A is area of region above x-axis, under the curve of function $y = f(x)$ from a to b.

Case II:- If $f(x) \leq 0 \quad \forall x \in [a,b]$ then curve lies below x-axis. so

$$\boxed{A = -\int_a^b f(x) dx \quad \text{where } a < b}$$

A is area of region below x-axis, under the curve of function $y = f(x)$ from a to b.

Example 1:- Find the area bounded by the curve $y = 4 - x^2$ and the x-axis.

Solution:- $y = 4 - x^2$

For x-intercept; Put $y = 0$

$\rightarrow 0 = 4 - x^2 \rightarrow x^2 = 4 \rightarrow x = \pm 2$

$\rightarrow -2 \leq x \leq 2$ or $x \in [-2,2]$

$\because y \geq 0 \quad \forall x \in [-2,2]$

x	$y = 4 - x^2$
−2	0
−1	3
0	4
1	3
2	0

so $A = \int_{-2}^{2} (4 - x^2) dx$

$= 4\int_{-2}^{2} dx - \int_{-2}^{2} x^2 dx = 4|x|_{-2}^{2} - \left|\frac{x^3}{3}\right|_{-2}^{2}$

$= 4(2+2) - \frac{1}{3}((2)^3 - (-2)^3)$

$= 16 - \frac{1}{3}(8+8) = 16 - \frac{16}{3} = \frac{48-16}{3}$

$= 32/3$ sq. units

Example 2. Find the area bounded by the curve $y = x^3 + 3x^2$ and the x-axis.

Solution:- $y = x^3 + 3x^2$

For x-intercept; put $y = 0$

$\rightarrow x^3 + 3x^2 = 0 \rightarrow x^2(x+3) = 0$

$\rightarrow x^2 = 0, \quad x + 3 = 0$

$\rightarrow x = 0, \quad x = -3$

$\rightarrow -3 \le x \le 0$ or $x \in [-3, 0]$

$\because y \geq 0 \quad \forall\, x \in [-3, 0]$ so

$\rightarrow A = \int_{-3}^{0} (x^3 + 3x^2)\, dx$

$= \int_{-3}^{0} x^3 dx + 3 \int_{-3}^{0} x^2 dx$

x	$y = x^3 + 3x^2$
-3	0
-2	4
-1	2
0	0

$= \Big| \frac{x^4}{4} \Big|_{-3}^{0} + \cancel{3} \Big| \frac{x^3}{\cancel{3}} \Big|_{-3}^{0}$

$= \frac{1}{4}\left[(0)^4 - (-3)^4\right] + \left((0)^3 - (-3)^3\right)$

$= \frac{1}{4}(-(81)) + (-(-27))$

$= -\frac{81}{4} + 27 = \frac{-81+108}{4} = \frac{27}{4}$ sq. units

Example 3. Find the area bounded by $y = x(x^2 - 4)$ and the x-axis.

Solution:- $y = x(x^2 - 4)$

$\rightarrow y = x^3 - 4x$

For x-intercept; Put $y = 0$

$\rightarrow x(x^2 - 4) = 0 \rightarrow x(x-2)(x+2) = 0$

$\rightarrow x = 0, \quad x - 2 = 0, \quad x + 2 = 0$

$\rightarrow x = 0, \quad x = 2, \quad x = -2$

so $0 \le x \le 2$ and $-2 \le x \le 0$

or $x \in [0, 2]$ and $x \in [-2, 0]$

$\because y \leq 0 \quad \forall\, x \in [0, 2]$

$\therefore A = -\int_{0}^{2} (x^3 - 4x)\, dx$

x	$y = x^3 - 4x$
0	0
1	-3
2	8

$= -\int_{0}^{2} x^3 dx + 4 \int_{0}^{2} x\, dx$

$= -\Big| \frac{x^4}{4} \Big|_{0}^{2} + 4 \Big| \frac{x^2}{2} \Big|_{0}^{2}$

$= -\frac{1}{4}\left((2)^4 - (0)^4\right) + 2\left((2)^2 - (0)^2\right)$

$= -\frac{1}{4}(16) + 2(4) = -4 + 8 = 4$

$\because y \geq 0 \quad \forall\, x \in [-2, 0]$

$\rightarrow A = \int_{-2}^{0} (x^3 - 4x)\, dx$

x	$y = x^3 - 4x$
-2	0
-1	3
0	0

$= \int_{-2}^{0} x^3 dx - 4 \int_{-2}^{0} x\, dx$

$= \Big| \frac{x^4}{4} \Big|_{-2}^{0} - 4 \Big| \frac{x^2}{2} \Big|_{-2}^{0}$

$= \frac{1}{4}\left((0)^4 - (-2)^4\right) - 2\left((0)^2 - (-2)^2\right)$

$= \frac{1}{4}(-16) - 2(-4) = -4 + 8 = 4$

Thus total area $= 4 + 4 = 8$ sq. units.

Example 4. Find the area bounded by the curve $f(x) = x^3 - 2x^2 + 1$ and the x-axis in the I^{st} quadrant.

Solution:- $f(x) = x^3 - 2x^2 + 1$

$\rightarrow y = x^3 - 2x^2 + 1 \qquad \because f(x) = y$

For x-intercept; $y = 0$

$\rightarrow x^3 - 2x^2 + 1 = 0$

$\rightarrow (x-1)(x^2 - x - 1) = 0$

$\Rightarrow x - 1 = 0, \qquad x^2 - x - 1 = 0$

$x = 1 \qquad x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2}$

$x = \frac{1 \pm \sqrt{5}}{2}$

For $x = 1$

$\therefore (1)^3 - 2(1)^2 + 1$

$1 - 2 + 1 = 0$

$$\begin{array}{c|cccc} 1 & 1 & -2 & 0 & 1 \\ & & 1 & -1 & -1 \\ \hline & 1 & -1 & -1 & \boxed{0} \end{array}$$

$\therefore (x-1)$ and $x^2 - x - 1$ are factors of $x^3 - 2x^2 + 1 = 0$

For I^{st} quadrant;

$y \geq 0 \quad \forall\, x \in [0, 1]$ so

Area in I^{st} quad $= A = \int_{0}^{1} (x^3 - 2x^2 + 1)\, dx$

$\rightarrow A = \int_{0}^{1} x^3 dx - 2\int_{0}^{1} x^2 dx + \int_{0}^{1} 1\, dx$

$= \Big| \frac{x^4}{4} \Big|_{0}^{1} - 2 \Big| \frac{x^3}{3} \Big|_{0}^{1} + |x|_{0}^{1}$

$= \frac{1}{4}\left[(1)^4 - (0)^4\right] - \frac{2}{3}\left[(1)^3 - (0)^3\right] + [1 - 0]$

$= \frac{1}{4}(1) - \frac{2}{3}(1) + 1 = \frac{1}{4} - \frac{2}{3} + 1$

$= \frac{3 - 8 + 12}{12} = \frac{7}{12}$ sq. units.

Example 5. Find the area between the x-axis and the curve $y^2 = 4 - x$ in the I^{st} quadrant from $x = 0$ to $x = 3$

Solution:- $y^2 = 4 - x \quad , x \in [0, 3]$

$\rightarrow y = \pm\sqrt{4-x}$

$y = \sqrt{4-x}$ and $y = -\sqrt{4-x}$

$\because y > 0$ in I^{st} quadrant so we reject $y = -\sqrt{4-x}$. Thus area

$$A = \int_0^3 \sqrt{4-x}\, dx$$

$$= -\int_0^3 (4-x)^{\frac{1}{2}}(-1)dx$$

$$= -\left|\frac{(4-x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^3 = -\frac{2}{3}\left[(4-3)^{\frac{3}{2}} - (4-0)^{\frac{3}{2}}\right]$$

$$= -\frac{2}{3}\left[(1)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right] = -\frac{2}{3}\left[1 - (2^2)^{\frac{3}{2}}\right]$$

$$= -\frac{2}{3}\left[1 - 2^3\right] = -\frac{2}{3}\left[1-8\right] = -\frac{2}{3}\left[-7\right]$$

$\rightarrow A = \frac{14}{3}$ sq. units

Exercise 3.7

Q1. Find the area between the x-axis and the curve $y = x^2 + 1$ from $x = 1$ to $x = 2$

Solution:- $y = x^2 + 1 \quad , x \in [1, 2]$

$\because y > 0 \quad \forall x \in [1, 2]$ so

x	$y = x^2 + 1$
1	2
2	5

$$A = \int_1^2 (x^2+1)dx$$

$$= \int_1^2 x^2 dx + \int_1^2 1 dx$$

$$= \left|\frac{x^3}{3}\right|_1^2 + |x|_1^2 = \frac{1}{3}\left[(2)^3 - (1)^3\right] + [2-1]$$

$$\rightarrow A = \frac{1}{3}(8-1) + 1 = \frac{7}{3} + 1 = \frac{10}{3} \text{ sq. units}$$

Q2. Find the area, above the x-axis and under the curve $y = 5 - x^2$ from $x = -1$ to $x = 2$

Solution:- $y = 5 - x^2 \quad , x \in [-1, 2]$

Area above x-axis, so

$$A = \int_{-1}^2 (5 - x^2)dx$$

$$= 5\int_{-1}^2 dx - \int_{-1}^2 x^2 dx = 5|x|_{-1}^2 - \left|\frac{x^3}{3}\right|_{-1}^2$$

$$= 5(2-(-1)) - \frac{1}{3}((2)^3 - (-1)^3)$$

$$= 5(3) - \frac{1}{3}(8-(-1))$$

$$= 15 - \frac{1}{3}(9) = 15 - 3 = 12 \text{ sq. units.}$$

Q3. Find the area below the curve $y = 3\sqrt{x}$ and above x-axis between $x = 1$ and $x = 4$

Solution:- $y = 3\sqrt{x} \quad , x \in [1, 4]$

Area above x-axis, so

$$A = \int_1^4 3\sqrt{x}\,dx = 3\int_1^4 x^{\frac{1}{2}} dx$$

$$= 3 \cdot \left|\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^4 = 3 \cdot \frac{2}{3}\left[(4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right]$$

$$= 2((2^2)^{\frac{3}{2}} - 1) = 2(8-1) = 14 \text{ sq. units}$$

Q4. Find the area bounded by cos function from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$

Solution:- $y = \cos x \quad , x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\because y \geq 0 \quad \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ so

x	$y = Cosx$
$-\frac{\pi}{2}$	0
0	1
$\frac{\pi}{2}$	0

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx$$

$$= |\sin x|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \sin\frac{\pi}{2} - \sin(-\frac{\pi}{2}) = \sin\frac{\pi}{2} + \sin\frac{\pi}{2}$$

$= 1 + 1 = 2$ sq. units $\quad \because \sin(-\theta) = -\sin\theta$

Q5. Find the area between the x-axis and curve $y = 4x - x^2$

Solution:- $y = 4x - x^2$

For x-intercept, $y = 0$

$\rightarrow 4x - x^2 = 0$

$\rightarrow x(4-x) = 0 \quad \rightarrow x = 0, \; 4 - x = 0$

$\rightarrow x = 0, \; x = 4 \quad \rightarrow 0 \leq x \leq 4$ or $x \in [0, 4]$

$\because y \geq 0 \quad \forall x \in [0, 4]$ so

x	$y = 4x - x^2$
0	0
1	3
2	4
3	3
4	0

$$A = \int_0^4 (4x - x^2)dx$$

$$= 4\int_0^4 x\,dx - \int_0^4 x^2 dx$$

$$= 4\left|\frac{x^2}{2}\right|_0^4 - \left|\frac{x^3}{3}\right|_0^4$$

$$= \frac{4}{2}((4)^2 - (0)^2) - \frac{1}{3}((4)^3 - (0)^3)$$

$$= 2(16) - \frac{1}{3}(64) = 32 - \frac{64}{3} = \frac{96-64}{3}$$

$$= \frac{32}{3} \text{ sq. units}$$

Q6. Determine the area bounded by parabola $y = x^2 + 2x - 3$ and the x-axis.

Solution:- $y = x^2 + 2x - 3$

For x-intercept; $y = 0$

$\rightarrow x^2 + 2x - 3 = 0 \rightarrow x^2 - x + 3x - 3 = 0$

$\rightarrow x(x-1) + 3(x-1) = 0 \rightarrow (x-1)(x+3) = 0$

$\rightarrow x = 1, \; x = -3 \rightarrow -3 \leq x \leq 1$

$\rightarrow x \in [-3, 1]$

$\because y \leq 0 \;\; \forall \; x \in [-3, 1]$ so

x	$y = x^2 + 2x - 3$
−3	0
−2	−3
−1	−4
0	−3
1	0

$$A = -\int_{-3}^{1} (x^2 + 2x - 3)\,dx$$

$$= \int_{1}^{-3} (x^2 + 2x - 3)\,dx$$

$$= \int_{1}^{-3} x^2 dx + 2\int_{1}^{-3} x\,dx - 3\int_{1}^{-3} dx \qquad \because \int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$= \left|\frac{x^3}{3}\right|_1^{-3} + \cancel{2}\cdot\left|\frac{x^2}{\cancel{2}}\right|_1^{-3} - 3|x|_1^{-3}$$

$$= \frac{1}{3}\left((-3)^3 - (-1)^3\right) + \left((-3)^2 - (1)^2\right) - 3(-3-1)$$

$$= \frac{1}{3}(-27+1) + (9-1) - 3(-4)$$

$$= -\frac{28}{3} + 8 + 12 = \frac{-28}{3} + 20 = \frac{-28+60}{3}$$

$$\rightarrow A = \frac{32}{3} \text{ sq. units}$$

Q7. Find the area bounded by the curve $y = x^3 + 1$, the x-axis and the line $x = 2$.

Solution:- $y = x^3 + 1 \qquad x = 2$

For x-intercept; put $y = 0$

$\rightarrow x^3 + 1 = 0 \rightarrow (x)^3 + (1)^3 = 0$

$\rightarrow (x+1)(x^2 - x + 1) = 0$

$\rightarrow x + 1 = 0, \qquad x^2 - x + 1 = 0$ (rejected being complex roots)

so $-1 \leq x \leq 2$ or $x \in [-1, 2]$

$\because y \geq 0 \;\; \forall \; x \in [-1, 2]$. so

x	$y = x^3 + 1$
−1	0
0	1
2	9

$$A = \int_{-1}^{2} (x^3 + 1)\,dx$$

$$= \int_{-1}^{2} x^3 dx + \int_{-1}^{2} 1\,dx$$

$$= \left|\frac{x^4}{4}\right|_{-1}^{2} + |x|_{-1}^{2}$$

$$= \frac{1}{4}\left[(2)^4 - (-1)^4\right] + \left[2 - (-1)\right]$$

$$= \frac{1}{4}\left[16 - 1\right] + (2+1) = \frac{15}{4} + 3 = \frac{15+12}{4}$$

$$\rightarrow A = \frac{27}{4} \text{ sq. units}$$

Q8. Find the area bounded by the curve $y = x^3 - 4x$ and the x-axis.

Solution:- $y = x^3 - 4x$

$\rightarrow y = x(x^2 - 4)$

Q8 is same as Example # 3 (Page #48)

Q9. Find the area between the curve $y = x(x-1)(x+1)$ and the x-axis.

Solution:- $y = x(x-1)(x+1)$ $\rightarrow y = x(x^2 - 1)$ $\rightarrow y = x^3 - x$

For x-intercept; $y = 0$

$\rightarrow x(x-1)(x+1) = 0$

$\rightarrow x = 0, \; x - 1 = 0, \; x + 1 = 0$

$\rightarrow x = 0, \; x = 1, \; x = -1$

$\rightarrow 0 \leq x \leq 1$ and $-1 \leq x \leq 0$

or $x \in [0, 1]$ and $x \in [-1, 0]$

$\because y \leq 0 \;\; \forall \; x \in [0, 1]$

x	$y = x^3 - x$
0	0
$\frac{1}{2}$	$-\frac{3}{8}$
1	0

$$A = -\int_0^1 (x^3 - x)\,dx$$

$$= -\left[\int_0^1 x^3 dx - \int_0^1 x\,dx\right]$$

$$= -\left[\left|\frac{x^4}{4}\right|_0^1 - \left|\frac{x^2}{2}\right|_0^1\right]$$

$$= -\left[\frac{1}{4}\left((1)^4 - (0)^4\right) - \frac{1}{2}\left((1)^2 - (0)^2\right)\right]$$

$$A = -\left[\frac{1}{4}(1-0) - \frac{1}{2}(1-0)\right] = \frac{-1}{4} + \frac{1}{2} = \frac{-1+2}{4}$$

$$\rightarrow A = \frac{1}{2}$$

$\because y \geq 0 \;\; \forall \; x \in [-1, 0]$ so

x	$y = x^3 - x$
−1	0
$-\frac{1}{2}$	$\frac{3}{8}$
0	0

$$A = \int_{-1}^{0} (x^3 - x)\,dx$$

$$= \int_{-1}^{0} x^3 dx - \int_{-1}^{0} x\,dx$$

$$= \left|\frac{x^4}{4}\right|_{-1}^{0} - \left|\frac{x^2}{2}\right|_{-1}^{0}$$

$$= \frac{1}{4}\left[(0)^4 - (-1)^4\right] - \frac{1}{2}\left[(0)^2 - (-1)^2\right]$$

$$= \frac{1}{4}(0-1) - \frac{1}{2}(0-1)$$

$$\rightarrow A = \frac{-1}{4} + \frac{1}{2} = \frac{-1+2}{4} = \frac{1}{4}$$

Thus area $= \frac{1}{2} + \frac{1}{4} = \frac{1+1}{4} = \frac{2}{4} = \frac{1}{2}$ sq. units

Q10. Find the area above the x-axis, bounded by the curve $y^2 = 3-x$ from $x=-1$ to $x=2$

Solution:- $y^2 = 3-x$

$\rightarrow y = \pm\sqrt{3-x}$

$\rightarrow y = \sqrt{3-x}$ or $y = -\sqrt{3-x}$

$\because$ we are to find area above x-axis so we reject $y = -\sqrt{3-x}$

($\because$ above x-axis $y > 0$) so

$$\text{Area} = A = \int_{-1}^{2} \sqrt{3-x}\,dx$$

$$\rightarrow A = -\int_{-1}^{2} (3-x)^{\frac{1}{2}}(-1)\,dx = -\left|\frac{(3-x)^{3/2}}{3/2}\right|_{-1}^{2}$$

$$= -\frac{2}{3}\left[(3-2)^{3/2} - (3-(-1))^{3/2}\right]$$

$$= -\frac{2}{3}\left[(1)^{3/2} - (4)^{3/2}\right] = -\frac{2}{3}\left[1 - (2^{\cancel{2}})^{3/\cancel{2}}\right]$$

$$\rightarrow A = -\frac{2}{3}\left[1-8\right] = -\frac{2}{3}(-7) = \frac{14}{3}\text{ sq.units}$$

Q11. Find the area between the x-axis and the curve $y = \cos\frac{1}{2}x$ from $x = -\pi$ to $x = \pi$.

Solution:- $y = \cos\frac{1}{2}x$ $\quad \forall x \in [-\pi, \pi]$

$\because y \geqslant 0 \ \forall x \in (-\pi, \pi)$

So

x	$y = Cos\frac{1}{2}x$
$-\pi$	0
0	1
π	0

$$A = \int_{-\pi}^{\pi} \cos\frac{1}{2}x\,dx$$

$$= \left|\frac{\sin\frac{1}{2}x}{1/2}\right|_{-\pi}^{\pi}$$

$$= 2\left[\sin\frac{1}{2}(\pi) - \sin\frac{1}{2}(-\pi)\right]$$

$$= 2\left[\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right] = 2\left[1+1\right]$$

$$= 2(2) = 4\text{ sq.units}$$ $\quad \because \sin(-\theta) = -\sin\theta$

Q12. Find the area between the x-axis and the curve $y = \sin 2x$ from $x = 0$ to $x = \frac{\pi}{3}$

Solution:- $y = \sin 2x$ $\quad x \in [0, \frac{\pi}{3}]$

$\because y \geqslant 0 \ \forall x \in [0, \frac{\pi}{3}]$ so

x	$y = Sin2\,x$
0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	0

$$A = \int_{0}^{\pi/3} \sin 2x\,dx$$

$$= \left|-\frac{\cos 2x}{2}\right|_{0}^{\pi/3}$$

$$= -\frac{1}{2}\left[\cos 2(\tfrac{\pi}{3}) - \cos 2(0)\right]$$

$$= -\frac{1}{2}\left[\cos\frac{2\pi}{3} - \cos 0\right]$$

$$A = -\frac{1}{2}\left(-\frac{1}{2} - 1\right) = -\frac{1}{2}\left(-\frac{3}{2}\right) = \frac{3}{4}\text{ sq.units}$$

Q13. Find the area between the x-axis and the curve $y = \sqrt{2ax - x^2}$ when $a > 0$

Solution:- $y = \sqrt{2ax - x^2}$

For x-intercept; $y = 0$

$\rightarrow 0 = \sqrt{2ax - x^2} \rightarrow 2ax - x^2 = 0$

$\rightarrow x(2a - x) = 0 \rightarrow x = 0,\ 2a - x = 0$

$\rightarrow x = 0,\ x = 2a$

$\rightarrow 0 \leqslant x \leq 2a$

or $x \in [0, 2a]$

$\because y \geqslant 0 \ \forall x \in [0, 2a]$ so

x	$y = \sqrt{2ax - x^2}$
0	0
a	a
$2a$	0

$$A = \int_{0}^{2a} \sqrt{2ax - x^2}\,dx$$

$$= \int_{0}^{2a} \sqrt{-x^2 + 2ax}\,dx$$

$$= \int_{0}^{2a} \sqrt{-(x^2 - 2ax)}\,dx$$

$$= \int_{0}^{2a} \sqrt{-(x^2 - 2ax + a^2 - a^2)}\,dx$$

$$= \int_{0}^{2a} \sqrt{a^2 - (x-a)^2}\,dx$$

Put $x - a = a\sin\theta$

$dx = a\cos\theta\,d\theta$

when $x = 0$

$0 - \cancel{a} = \cancel{a}\sin\theta;$

$\rightarrow \theta = \sin^{-1}(-1) = -\frac{\pi}{2}$

when $x = 2a$

$2a - a = a\sin\theta$

$\rightarrow \cancel{a} = \cancel{a}\sin\theta \rightarrow 1 = \sin\theta$

$\theta = \sin^{-1}(1) = \frac{\pi}{2}$

$$\rightarrow A = \int_{-\frac{\pi}{2}}^{\pi/2} \sqrt{a^2 - (a\sin\theta)^2}\cdot a\cos\theta\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \sqrt{a^2(1 - \sin^2\theta)}\cdot a\cos\theta\,d\theta$$

$$= a^2 \int_{-\pi/2}^{\pi/2} \sqrt{\cos^2\theta}\cdot\cos\theta\,d\theta$$

$$= a^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta = a^2 \int_{-\frac{\pi}{2}}^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{2}a^2 \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\,d\theta$$

$$= \frac{1}{2}a^2\left[\left|\theta\right|_{-\pi/2}^{\pi/2} + \left|\frac{\sin 2\theta}{2}\right|_{-\pi/2}^{\pi/2}\right]$$

$$= \frac{1}{2}a^2\left[\left(\frac{\pi}{2} - (-\frac{\pi}{2})\right) + \frac{1}{2}\left(\sin 2(\tfrac{\pi}{2}) - \sin 2(-\tfrac{\pi}{2})\right)\right]$$

$$= \frac{1}{2}a^2\left[\frac{\pi}{2} + \frac{\pi}{2} + \frac{1}{2}(\sin\pi + \sin\pi)\right]$$

$$= \frac{1}{2}a^2\left[\pi + \frac{1}{2}(0+0)\right] = \frac{a^2\pi}{2}$$

$$\rightarrow A = \frac{\pi a^2}{2}\text{ sq.units}$$

Differential Equations

An equation containing at least on derivative of a dependent variable with respect to an independent variable is called differential equation.

e.g., $y\frac{dy}{dx}+2x=0$, $x\frac{d^2y}{dx^2}+\frac{dy}{dx}-2x=0$

Order of differential Equation:-

The order of a differential equation is the order of the highest derivative in the equation. e.g.,

order of $\frac{dy}{dx}=3y$ is 1.

order of $\frac{d^2y}{dx^2}+\frac{dy}{dx}=12y$ is 2. etc

Degree of differential Equation:-

The highest power of the highest ordered derivative is called degree of differential equation. e.g.,

Degree of $\frac{d^2y}{dx^2}-\sin y=0$ is 1

Degree of $\left(\frac{d^2y}{dx^2}\right)^2+4x=0$ is 2.

Degree of $\frac{d^2y}{dx^2}+\frac{dy}{dx}-12y=0$ is 1. etc

★ A solution of differential equation is any relation between the variable that is free of derivatives or differentials.

Separable Equation:-

A differentiable equation which is solved by separable variables is called separable equation. e.g $\frac{dy}{dx}=\frac{\sin x}{y}$

$\rightarrow y\,dy=\sin x\,dx$ By separating variables we solve differential equation by integrating it.

Example 1. Solve the differential equation $(x-1)dx+y\,dy=0$

Solution:- $(x-1)dx+y\,dy=0$

$\rightarrow y\,dy=-(x-1)dx$

$\rightarrow y\,dy=(1-x)dx$

Integrating on both sides

$\rightarrow \int y\,dy=\int 1dx-\int x\,dx$

$\rightarrow \frac{y^2}{2}=x-\frac{x^2}{2}+c_1$

$\rightarrow y^2=2x-x^2+2c_1$

$\rightarrow y^2=2x-x^2+c$

$\rightarrow x^2+y^2=2x+c$ $\rightarrow x^2+y^2-2x=c$

Thus required general solution is
$x^2+y^2-2x=c$ where $c=2c_1$

Example 2. Solve the differential equation $x^2(2y+1)\frac{dy}{dx}-1=0$

Solution:- $x^2(2y+1)\frac{dy}{dx}=1$

$\rightarrow (2y+1)\frac{dy}{dx}=\frac{1}{x^2}$ $\rightarrow (2y+1)dy=\frac{1}{x^2}dx$

$\rightarrow \int(2y+1)dy=\int\frac{1}{x^2}dx$ Integrating both sides

$\rightarrow 2\int y\,dy+\int 1dy=\int x^{-2}dx$

$\rightarrow \cancel{2}\cdot\frac{y^2}{\cancel{2}}+y=\frac{x^{-1}}{-1}+c$

$\rightarrow y^2+y=-\frac{1}{x}+c$ $\rightarrow y^2+y+\frac{1}{x}=c$

Thus required general solution is $y^2+y+\frac{1}{x}=c$

Example 3. Solve the differential equation $\frac{1}{x}\frac{dy}{dx}-2y=0$, $x\neq0,\ y>0$

Solution:- $\frac{1}{x}\frac{dy}{dx}-2y=0$

$\rightarrow \frac{dy}{x\,dx}=2y$

$\rightarrow \frac{dy}{y}=2x\,dx$

$\rightarrow \int\frac{dy}{y}=2\int x\,dx$ Integrating both sides

$\rightarrow \ln y=\cancel{2}\cdot\frac{x^2}{\cancel{2}}+c_1$

$\rightarrow \ln y=x^2+c_1$

$\rightarrow y=e^{x^2+c_1}$

$\rightarrow y=e^{x^2}\cdot e^{c_1}=ce^{x^2}$ required general solution

$\rightarrow y=ce^{x^2}$

Example 4. Solve $\frac{dy}{dx}=\frac{y^2+1}{e^{-x}}$

Solution:- $\frac{dy}{dx}=\frac{y^2+1}{e^{-x}}$

$\rightarrow e^{-x}dy=(y^2+1)dx$

$\rightarrow \frac{dy}{y^2+1}=\frac{1}{e^{-x}}dx$

$\rightarrow \int\frac{dy}{y^2+1}=\int e^{x}dx$

$\rightarrow \tan^{-1}y=e^{x}+c$

$\rightarrow y=\tan(e^{x}+c)$ required general solution

Example 5:- Solve

$2e^{x}\tan y\,dx+(1-e^{x})\sec^2y\,dy=0$ $\left(\begin{matrix}0<y<\frac{\pi}{2}\\ \text{or }\pi<y<\frac{3\pi}{2}\end{matrix}\right)$

Solution:-

$2e^{x}\tan y\,dx+(1-e^{x})\sec^2y\,dy=0$

$\rightarrow (1-e^{x})(\sec^2y)\,dy=-2e^{x}\tan y\,dx$

$\rightarrow \frac{\sec^2y}{\tan y}dy=\frac{-2e^{x}}{1-e^{x}}dx$

$\rightarrow \int \frac{\sec^2 y dy}{\tan y} dy = 2\int \frac{e^{-x}}{1-e^{x}} dx$

$\rightarrow \ln|\tan y| = 2\ln|1-e^{x}| + \ln|c|$

$\ln|\tan y| = \ln|1-e^{x}|^2 + \ln|c|$

$\ln|\tan y| = \ln|c(1-e^{x})^2|$

$\rightarrow \tan y = c(1-e^{x})^2$

Example 6. Solve

$(\sin y + y\cos y)dy = [x(2\ln x+1)]dx$

Solution:- $(\sin y + y\cos y)dy = [x(2\ln x + 1)]dx$

$\rightarrow \int(\sin y + y\cos y)dy = \int(2\ln x+1).x dx$

$\rightarrow \int \sin y dy + \int \underset{I}{y}\,\underset{II}{\cos y} dy = \int \underset{I}{(2\ln x+1)}\cdot \underset{II}{x}\, dx$

$\rightarrow -\cos y + y\sin y - \int \sin y(1)dy = (2\ln x+1).\frac{x^2}{2} - \int \frac{x^{x}}{\cancel{x}}\cdot\cancel{x}\cdot\frac{1}{\cancel{x}}dx$

$-\cancel{\cos y} + y\sin y + \cancel{\cos y} = \frac{x^2}{\cancel{x}}(\cancel{2}\ln x) + \cancel{\frac{x^2}{2}} - \cancel{\frac{x^2}{2}} + c$

$\rightarrow y\sin y = x^2\ln x + c$

Initial Conditions

The arbitrary constants involving in the solution of differential equations can be determined by the given conditions. Such conditions are called initial value conditions.

★ The general solution of differential equation of order n contains n arbitrary constants which can be determined by n initial value conditions.

★ The general solution of differential equation in variable separable form containts only one variable.

Example 1. The slope of the tangent at any point of the curve is given by $\frac{dy}{dx} = 2x-2$, find the equation of the curve if $y=0$ when $x=-1$

Solution:- $\because \frac{dy}{dx} = 2x-2$

$\rightarrow dy = (2x-2)dx$

$\rightarrow \int dy = 2\int x dx - 2\int dx$

$\rightarrow y = 2\cdot\frac{x^2}{2} - 2x + c$

$\rightarrow y = x^2 - 2x + c \longrightarrow (i)$

Put $y=0$, $x=-1$ so

$(i) \rightarrow 0 = (-1)^2 - 2(-1) + c \quad \rightarrow 0 = 1+2+c$

$\rightarrow \boxed{c=-3}$

Thus $(i) \rightarrow y = x^2 - 2x - 3$

Example 2. Solve $\frac{dy}{dx} = \frac{3}{4}x^2 + x - 3$, if $y=0$ when $x=2$

Solution:- $\frac{dy}{dx} = \frac{3}{4}x^2 + x - 3$

$\rightarrow dy = (\frac{3}{4}x^2 + x - 3)dx$

$\rightarrow dy = \frac{3}{4}x^2dx + xdx - 3dx$

$\rightarrow \int dy = \frac{3}{4}\int x^2dx + \int xdx - 3\int dx$

$\rightarrow y = \frac{3}{4}\frac{x^3}{3} + \frac{x^2}{2} - 3x + c$

$\rightarrow y = \frac{x^3}{4} + \frac{x^2}{2} - 3x + c \longrightarrow (i)$

Put $y=0$, $x=2$ in (i) so

$(i) \rightarrow 0 = \frac{(2)^3}{4} + \frac{(2)^2}{2} - 3(2) + c$

$0 = 8/4 + 4/2 - 6 + c$

$0 = 2 + 2 - 6 + c$

$0 = 4 - 6 + c \quad \rightarrow 0 = -2 + c$

$\rightarrow \boxed{c=2}$

So $(i) \rightarrow y = \frac{x^3}{4} + \frac{x^2}{2} - 3x + 2$

$\rightarrow 4y = x^3 + 2x^2 - 12x + 8$

Example 3. A particle is moving in a straight line and its acceleration is given by $a = 2t-7$

(i) find v(velocity) in terms of t if $v = 10 m/sec$, $t=0$

(ii) find s(distance) in terms of t if $s=0$, when $t=0$

Solution:- (i) $a = 2t-7$

$\rightarrow \frac{dv}{dt} = 2t-7 \qquad \because a = \frac{dv}{dt}$

$\rightarrow dv = 2tdt - 7dt$

$\rightarrow \int dv = 2\int tdt - 7\int dt$

$v = 2\cdot\frac{t^2}{2} - 7t + c_1 \text{——} (i)$

Put $v = 10$, $t = 0$ in (i)

$\rightarrow 10 = 2\cdot\frac{(0)^2}{2} - 7(0) + c_1 \quad \rightarrow c_1 = 10$

$\rightarrow \boxed{c_1 = 10}$ so (i)

$v = t^2 - 7t + 10$

(ii) $v = t^2 - 7t + 10 \qquad \because v = \frac{ds}{dt}$

$\rightarrow \frac{ds}{dt} = t^2 - 7t + 10$

$\rightarrow ds = t^2dt - 7tdt + 10dt$

$$\rightarrow \int ds = \int t^2 dt - 7\int t\,dt + 10\int dt$$

$$s = \frac{t^3}{3} - 7\cdot\frac{t^2}{2} + 10t + c_2 \longrightarrow (ii)$$

Put $s = 0,\ t = 0$ in (ii)

$$(ii) \rightarrow 0 = \frac{(0)^3}{3} - 7\frac{(0)^2}{2} + 10(0) + c_2$$

$\rightarrow c_2 = 0$ so (ii) becomes

$$s = \frac{t^3}{3} - \frac{7}{2}t^2 + 10t + c_2$$

Example 4. In a culture, bacteria increases at the rate proportional to the number of bacteria present. If bacteria are 100 initially and are doubled in 2 hours, find the number of bacteria present four hours later.

Solution:- Let p be number of bacteria present at time t, then

$$\frac{dp}{dt} \propto p$$

$$\rightarrow \frac{dp}{dt} = kp\ , \quad (k>0)$$

$$\rightarrow dp = kp\,dt \rightarrow \frac{1}{p}dp = k\,dt$$

$$\rightarrow \int \frac{1}{p}dp = k\int dt$$

$$\rightarrow \ln p = kt + c_1$$

$$\rightarrow p = e^{kt+c_1} = e^{kt}\cdot e^{c_1} \quad \because e^{c_1} = c$$

$$P = ce^{kt} \longrightarrow (i)$$

Put $t = 0,\ P = 100$ in (i)

$$100 = c\,e^{k(0)} \rightarrow 100 = c \quad \because e^0 = 1$$

$$\text{so } (i) \rightarrow P = 100e^{kt} \longrightarrow (ii)$$

Put $t = 2, t = 2(100) = 200$

$$(ii) \rightarrow 200 = 100\,e^{k(2)}$$

$$\rightarrow 2 = e^{2k}$$

$$\rightarrow \ln 2 = \ln e^{2k}$$

$$\ln 2 = 2k \ln e$$

$$\rightarrow k = \frac{\ln 2}{2} \quad \because \ln e = 1$$

$$(ii) \rightarrow P = 100\,e^{(\ln 2/2)t}$$

For $t = 4$

$$P = 100\left(e^{(\frac{\ln 2}{\cancel{2}})\cancel{4}^{2}}\right)$$

$$= 100\,e^{\ln 4}$$

$$P = 100(4)$$

$\rightarrow P = 400$ required numbers of bacteria present 4 hours later.

Example 5. A ball is thrown vertically upward with a velocity of 1470 cm/sec, Neglecting air resistance, find

(i) velocity of ball at any time 't'

(ii) distance traveled in any time 't'

(iii) maximum height attained by the ball

Solution:- (i) Let v be velocity of ball at any time t so $\frac{dv}{dt} = -g$ (for upward motion)

$$\rightarrow dv = -g\,dt$$

$$\rightarrow \int dv = -g\int dt \quad \text{(integrating both sides)}$$

$$\rightarrow v = -gt + c_1 \longrightarrow (i)$$

Put $t = 0$ then $v = 1470$ cm/sec so (i)

$$1470 = -g(0) + c_1 \rightarrow c_1 = 1470$$

Thus (i) $v = -gt + 1470$

$v = -980t + 1470$ —(ii) $\quad$ ($\because$ In c.g.s system $g = 980\ \text{cm/sec}^2$)

(ii) Let h be the hight of ball then

$$\because v = \frac{dh}{dt} \rightarrow \frac{dh}{dt} = -980t + 1470$$

$$\rightarrow \int dh = \int(-980t + 1470)dt = -980\int t\,dt + 1470\int dt$$

$$\rightarrow h = -980\cdot\frac{t^2}{2} + 1470t + c_2 \longrightarrow (iii)$$

when $t = 0$ then $h = 0$ so (iii)

$$0 = -980\cdot\frac{(0)^2}{2} + 1470(0) + c_2$$

$\rightarrow c_2 = 0$ so (iii) becomes

$$\rightarrow h = -490t^2 + 1470t \text{ —(iv)}$$

(iii) When ball will be at maximum height then $v = 0$ so put in (ii)

$$0 = -980t + 1470$$

$$\rightarrow 980t = 1470 \rightarrow t = \frac{1470}{980}$$

$t = 3/2$ Put in (iv)

$$h = -490\left(\frac{3}{2}\right)^2 + 1470\left(\frac{3}{2}\right)$$

$$\rightarrow h = -490(9/4) + 735(3)$$

$$\rightarrow h = -1102.5 + 2205$$

$$\rightarrow h = 1102.5\ \text{cm}$$

Exercise 3.8

Q1. Check that each of the following equations written against the differential equation is its solution.

(i) $x\frac{dy}{dx} = 1+y$, $y = cx-1$

Solution:- $x.\frac{dy}{dx} = 1+y$

→ $xdy = (1+y)dx$ → $\frac{dy}{1+y} = \frac{dx}{x}$

→ $\int\frac{dy}{1+y} = \int\frac{dx}{x}$ integrating both sides

→ $\ln|1+y| = \ln x+\ln c$

→ $\ln|1+y| = \ln cx$

→ $1+y = cx$ → $y = cx-1$

→ which is solution of given differential equation.

(ii) $x^2(2y+1)\frac{dy}{dx}-1=0$, $y^2+y = c-\frac{1}{x}$

Solution:- $x^2(2y+1)\frac{dy}{dx}-1=0$

→ $x^2(2y+1)\frac{dy}{dx} = 1$

→ $(2y+1)\frac{dy}{dx} = \frac{1}{x^2}$ → $(2y+1)dy = \frac{1}{x^2}dx$

→ $2\int ydy+\int dy = \int x^{-2}dx$ Taking integration

→ $2\cdot\frac{y^2}{2}+y = \frac{x^{-1}}{-1}+c$

→ $y^2+y = c-\frac{1}{x}$ → which is solution of given differential equation.

(iii) $y\frac{dy}{dx}-e^{2x} = 1$, $y^2 = e^{2x}+2x+c$

Solution:- $y\frac{dy}{dx}-e^{2x} = 1$

→ $y\frac{dy}{dx} = 1+e^{2x}$

→ $y\,dy = (1+e^{2x})dx$

→ $\int ydy = \int dx+\int e^{2x}dx$

→ $\frac{y^2}{2} = x+\frac{e^{2x}}{2}+c_1$

→ $y^2 = e^{2x}+2x+2c_1$

→ $y^2 = e^{2x}+2x+c$ ∵ $2c_1 = c$

→ which is solution of given diff. eq.

(iv) $\frac{1}{x}\frac{dy}{dx}-2y=0$, $y = ce^{x^2}$

Solution:- $\frac{1}{x}\frac{dy}{dx}-2y=0$

→ $\frac{1}{x}\frac{dy}{dx} = 2y$

→ $\frac{dy}{y} = 2x\,dx$

→ $\int\frac{dy}{y} = 2\int x\,dx$ Taking integration

→ $\ln y = 2\cdot\frac{x^2}{2}+c$

→ $\ln y = x^2+c$

→ $y = e^{x^2+c_1} = e^{x^2}\cdot e^{c_1}$ ∵ $e^{c_1} = c$

→ $y = ce^{x^2}$ which is solution of diff. eq.

(v) $\frac{dy}{dx} = \frac{y^2+1}{e^{-x}}$, $y = \tan(e^x+c)$

Solution:- $\frac{dy}{dx} = \frac{y^2+1}{e^{-x}}$

→ $\frac{dy}{y^2+1} = \frac{dx}{e^{-x}}$

→ $\frac{dy}{y^2+1} = e^x dx$

→ $\int\frac{dy}{y^2+1} = \int e^x dx$ Taking integration

→ $\tan^{-1}y = e^x+c$

→ $y = \tan(e^x+c)$ which is solution of diff. equation.

Solve the following differential equations:

Q2. $\frac{dy}{dx} = -y$

Solution:- $\frac{dy}{dx} = -y$

→ $\frac{dy}{y} = -dx$

→ $\int\frac{dy}{y} = -\int dx$ → $\ln y = -x+c_1$

→ $y = e^{-x+c_1} = e^{-x}\cdot e^{c_1} = ce^{-x}$ ∵ $e^{c_1} = c$

→ $y = ce^{-x}$

Q3. $ydx+xdy = 0$

Solution:- $ydx+xdy = 0$

→ $xdy = -ydx$

→ $\frac{dy}{y} = -\frac{dx}{x}$

→ $\int\frac{dy}{y} = -\int\frac{dx}{x}$ Take integration

→ $\ln y = -\ln|x|+\ln c$

→ $\ln y+\ln x = \ln c$

→ $\ln xy = \ln c$ → $xy = c$

Q4. $\frac{dy}{dx} = \frac{1-x}{y}$

Solution:- $\frac{dy}{dx} = \frac{1-x}{y}$

→ $y\,dy = (1-x)dx$

→ $\int ydy = \int dx-\int xdx$ Take integration

→ $\frac{y^2}{2} = x-\frac{x^2}{2}+c_1$

→ $y^2 = 2x-x^2+2c_1$

→ $y^2 = x(2-x)+c$ ∵ $2c_1 = c$

Q5. $\frac{dy}{dx} = \frac{y}{x^2} \quad , (y>0)$

Solution:- $\frac{dy}{dx} = \frac{y}{x^2}$

$\rightarrow \frac{dy}{y} = \frac{dx}{x^2}$

$\rightarrow \int \frac{dy}{y} = \int \frac{dx}{x^2}$ Take integration

$\rightarrow \ln y = \int x^{-2} dx = \frac{x^{-1}}{-1} + c_1$

$\rightarrow \ln y = -\frac{1}{x} + c_1$

$\rightarrow y = e^{-1/x + c_1} = e^{-1/x} \cdot e^{c_1} \quad \because e^{c_1} = c$

$\rightarrow y = ce^{-1/x}$

Q6. $\sin y \operatorname{cosec} x \frac{dy}{dx} = 1$

Solution:- $\sin y \operatorname{cosec} x \frac{dy}{dx} = 1$

$\rightarrow \sin y\, dy = \frac{1}{\operatorname{cosec} x} dx$

$\rightarrow \sin y\, dy = \sin x\, dx$

$\rightarrow \int \sin y\, dy = \int \sin x\, dx$ Take integration

$\rightarrow -\cos y = -\cos x - c_1$

$\rightarrow \cos y = \cos x + c \quad \because c = -c_1$

Q7. $x\,dy + y(x-1)dx = 0$

Solution:- $x\,dy + y(x-1)\,dx = 0$

$\rightarrow x\,dy = -y(x-1)\,dx$

$\rightarrow \frac{1}{y}dy = \frac{1-x}{x}dx$

$\rightarrow \frac{dy}{y} = (\frac{1}{x} - 1)dx$

$\rightarrow \int \frac{dy}{y} = \int \frac{1}{x}dx - \int 1 dx$ Take integration

$\rightarrow \ln y = \ln x - x + \ln c$

$\rightarrow \ln y = \ln cx - x$

$\rightarrow \ln y - \ln cx = -x$

$\rightarrow \ln y/cx = -x$

$\rightarrow \frac{y}{cx} = e^{-x} \rightarrow y = cxe^{-x}$

Q8. $\frac{x^2+1}{y+1} = \frac{x}{y}\frac{dy}{dx}$

Solution:- $\frac{x^2+1}{y+1} = \frac{x}{y}\frac{dy}{dx}$

$\rightarrow \frac{x^2+1}{x} = \frac{y+1}{y} \cdot \frac{dy}{dx}$

$\rightarrow \frac{x^2+1}{x}dx = \frac{y+1}{y}dy$

$\rightarrow (x + \frac{1}{x})dx = (1 + \frac{1}{y})dy$

$\rightarrow (1+\frac{1}{y})dy = (x + \frac{1}{x})dx$

Take integration

$\rightarrow \int dy + \int \frac{1}{y}dy = \int x\,dx + \int \frac{1}{x}dx$

$\rightarrow y + \ln y = \frac{x^2}{2} + \ln x + \ln c$

$\ln e^y + \ln y = \frac{x^2}{2} + \ln cx \quad \because \ln e^y = y$

$\rightarrow \ln ye^y = \ln cx + \frac{x^2}{2}$

$\rightarrow \ln ye^y - \ln cx = \frac{x^2}{2}$

$\rightarrow \ln \frac{ye^y}{cx} = \frac{x^2}{2} \rightarrow \frac{ye^y}{cx} = e^{x^2/2}$

$\rightarrow ye^y = cxe^{x^2/2}$

Q9. $\frac{1}{x}\frac{dy}{dx} = \frac{1}{2}(1+y^2)$

Solution:- $\frac{1}{x}\frac{dy}{dx} = \frac{1}{2}(1+y^2)$

$\rightarrow \frac{dy}{1+y^2} = \frac{x}{2}dx$

$\rightarrow \int \frac{dy}{1+y^2} = \frac{1}{2}\int x\,dx$ Take integration

$\rightarrow \tan^{-1} y = \frac{1}{2} \cdot \frac{x^2}{2} + c \quad$ or $\tan^{-1} y = \frac{x^2}{4} + c$

$\rightarrow y = \tan(\frac{x^2}{4} + c)$

Q10. $2x^2y\frac{dy}{dx} = x^2 - 1$

Solution:- $2x^2y\frac{dy}{dx} = x^2 - 1$

$\rightarrow 2y\frac{dy}{dx} = \frac{x^2-1}{x^2}$

$\rightarrow 2y\frac{dy}{dx} = 1 - \frac{1}{x^2}$

$\rightarrow 2y\,dy = (1 - x^{-2})dx$ Take integration

$\rightarrow 2\int y\,dy = \int 1 dx - \int x^{-2}dx$

$\rightarrow 2\frac{y^2}{2} = x - \frac{x^{-1}}{-1} + c$

$\rightarrow y^2 = x + \frac{1}{x} + c$

Q11. $\frac{dy}{dx} + \frac{2xy}{2y+1} = x$

Solution:- $\frac{dy}{dx} + \frac{2xy}{2y+1} = x$

$\rightarrow \frac{dy}{dx} = x - \frac{2xy}{2y+1}$

$\rightarrow \frac{dy}{dx} = \frac{\cancel{2xy} + x - \cancel{2xy}}{2y+1}$

$\rightarrow (2y+1)\,dy = x\,dx$

Take integration

$\rightarrow 2\int y\,dy + \int 1 dy = \int x\,dx$

$\rightarrow \cancel{2} \cdot \frac{y^2}{\cancel{2}} + y = \frac{x^2}{2} + c$

$\rightarrow y(y+1) = \frac{x^2}{2} + c$

Q12. $(x^2-yx^2)\frac{dy}{dx}+y^2+xy^2=0$

Solution:- $(x^2-yx^2)\frac{dy}{dx}+y^2+xy^2=0$

$\rightarrow x^2(1-y)\frac{dy}{dx}+y^2(1+x)=0$

$\rightarrow x^2(1-y)\frac{dy}{dx}=-y^2(1+x)$

$\rightarrow \frac{1-y}{-y^2}\frac{dy}{dx}=\frac{1+x}{x^2}$

$\rightarrow \left(\frac{-1}{y^2}+\frac{1}{y}\right)dy=\left(\frac{1}{x^2}+\frac{1}{x}\right)dx$

Take integration

$\rightarrow -\int y^{-2}dy+\int\frac{1}{y}dy=\int x^{-2}dx+\int\frac{1}{x}dx$

$\rightarrow -\frac{y^{-1}}{-1}+\ln y=\frac{x^{-1}}{-1}+\ln x+\ln c$

$\rightarrow \frac{1}{y}+\ln y \quad =-\frac{1}{x}+\ln cx$

$\because \frac{1}{y}=\ln e^{1/y}$

$\rightarrow \ln e^{1/y}+\ln y=-\frac{1}{x}+\ln cx$

$\rightarrow \ln|ye^{1/y}|-\ln cx=-\frac{1}{x}$

$\rightarrow \ln\frac{ye^{1/y}}{cx}=-1/x$

$\rightarrow \frac{ye^{1/y}}{cx}=e^{-1/x} \rightarrow ye^{1/y}=cxe^{-1/x}$

Q13. $\sec^2x\tan y\,dx+\sec^2y\tan x\,dy=0$

Solution:- $\sec^2x\tan y\,dx+\sec^2y\tan x\,dy=0$

$\rightarrow \sec^2y\tan x\,dy=-\sec^2x\tan y\,dx$

$\rightarrow \frac{\sec^2y}{\tan y}dy=-\frac{\sec^2x}{\tan x}dx$

$\rightarrow \int\frac{\sec^2y}{\tan y}dy=-\int\frac{\sec^2x}{\tan x}dx$

$\rightarrow \ln\tan y=-\ln\tan x+\ln c$

$\rightarrow \ln\tan y+\ln\tan x=\ln c$

$\rightarrow \ln(\tan y\tan x)=\ln c$

$\rightarrow \tan y\tan x=c$

Q14. $y-x\frac{dy}{dx}=2\left(y^2+\frac{dy}{dx}\right)$

Solution:- $y-x\frac{dy}{dx}=2\left(y^2+\frac{dy}{dx}\right)$

$\rightarrow y-x\frac{dy}{dx}=2y^2+2\frac{dy}{dx}$

$\rightarrow y-2y^2=x\frac{dy}{dx}+2\frac{dy}{dx}$

$\rightarrow y(1-2y)=(x+2)\,dy/dx$

$\rightarrow y(1-2y)\,dx=(2+x)\,dy$

$\rightarrow \frac{dx}{2+x}=\frac{dy}{y(1-2y)}$

$\rightarrow \frac{dy}{y(1-2y)}=\frac{dx}{2+x} \longrightarrow$ Ⓘ

Using partial fraction to solve $\frac{1}{y(1-2y)}$

so $\frac{1}{y(1-2y)}=\frac{A}{y}+\frac{B}{1-2y}$

$\rightarrow 1=A(1-2y)+By$ —— (ii)

Put $y=0$ in (ii)

$1=A(1-2(0))+B(0)$

$\rightarrow 1=A \rightarrow \boxed{A=1}$

Put $1-2y=0 \rightarrow 1=2y \rightarrow y=\frac{1}{2}$ in (ii)

$\rightarrow 1=A(0)+B\cdot\frac{1}{2} \rightarrow \boxed{B=2}$

so $\frac{1}{y(1-2y)}=\frac{1}{y}+\frac{2}{1-2y}$

$=\frac{1}{y}-\frac{2}{2y-1}$

Thus (I) becomes as

$\left(\frac{1}{y}-\frac{2}{2y-1}\right)dy=\frac{dx}{2+x}$

Take integration

$\int\frac{1}{y}dy-\int\frac{2}{2y-1}dy=\int\frac{dx}{2+x}$

$\rightarrow \ln y-\ln|2y-1|=\ln|2+x|+\ln c$

$\rightarrow \ln\frac{y}{2y-1}=\ln|c(2+x)|$

$\rightarrow \frac{y}{2y-1}=c(x+2)$

Q15. $1+\cos x\tan y\frac{dy}{dx}=0$

Solution:- $1+\cos x\tan y\frac{dy}{dx}=0$

$\rightarrow \cos x\tan y\frac{dy}{dx}=-1$

$\rightarrow \tan y\frac{dy}{dx}=-\frac{1}{\cos x}$

$\rightarrow -\tan y\,dy=\sec x\,dx$

Take integration

$\rightarrow \int\tan y\,dy=\int\sec x\,dx$

$\rightarrow \int\frac{-\sin y}{\cos y}dy=\int\sec x\,dx$

$\rightarrow \ln|\cos y|=\ln|\sec x+\tan x|+\ln|c|$

$\ln|\cos y|=\ln|c(\sec x+\tan x|$

$\rightarrow \cos y=c(\sec x+\tan x)$

Q16. $y - x\frac{dy}{dx} = 3(1 + x\frac{dy}{dx})$

Solution:- $y - x\frac{dy}{dx} = 3(1 + x\frac{dy}{dx})$

$$\rightarrow y - x\frac{dy}{dx} = 3 + 3x\frac{dy}{dx}$$

$$\rightarrow y - 3 = x\frac{dy}{dx} + 3x\frac{dy}{dx}$$

$$\rightarrow y - 3 = (x + 3x)\frac{dy}{dx}$$

$$\rightarrow y - 3 = 4x\frac{dy}{dx}$$

$$\rightarrow (y-3)dx = 4x dy$$

$$\rightarrow \frac{1}{4}\frac{dx}{x} = \frac{dy}{y-3}$$

$$\rightarrow \frac{dy}{y-3} = \frac{1}{4}\frac{dx}{x}$$

Take integration

$$\rightarrow \int\frac{dy}{y-3} = \frac{1}{4}\int\frac{dx}{x}$$

$$\rightarrow \ln|y-3| = \frac{1}{4}\ln|x| + \ln c$$

$$\rightarrow \ln|y-3| = \ln x^{1/4} + \ln c$$

$$\rightarrow \ln|y-3| = \ln cx^{1/4}$$

$$\rightarrow y - 3 = cx^{1/4} \rightarrow y = 3 + cx^{1/4}$$

Q17. $\sec x + \tan y\frac{dy}{dx} = 0$

Solution:- $\sec x + \tan y\frac{dy}{dx} = 0$

$$\rightarrow \tan y\frac{dy}{dx} = -\sec x$$

$$\rightarrow \tan y\, dy = -\sec x\, dx$$

Take integration

$$\rightarrow \int\tan y\, dy = \int -\sec x\, dx$$

$$\rightarrow \int -\tan y\, dy = \int\sec x\, dx$$

$$\rightarrow \int -\frac{\sin y}{\cos y}dy = \int\sec x\, dx$$

$$\rightarrow \ln|\cos y| = \ln|\sec x + \tan x| + \ln|c|$$

$$\rightarrow \ln|\cos y| = \ln|c(\sec x + \tan x)|$$

$$\rightarrow \cos y = c(\sec x + \tan x)$$

Q18. $(e^x + e^{-x})\frac{dy}{dx} = e^x - e^{-x}$

Solution:- $(e^x + e^{-x})\frac{dy}{dx} = e^x - e^{-x}$

$$\rightarrow \frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\rightarrow dy = \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$$

Take integration

$$\int dy = \int\frac{e^x - e^{-x}}{e^x + e^{-x}}dx$$

$$\rightarrow y = \ln|e^x + e^{-x}| + c$$

Q19. Find the general solution of the equation $\frac{dy}{dx} - x = xy^2$

Also find the particular solution if $y = 1$ when $x = 0$

Solution:- $\frac{dy}{dx} - x = xy^2$

$$\rightarrow \frac{dy}{dx} = xy^2 + x$$

$$\rightarrow \frac{dy}{dx} = x(y^2 + 1)$$

$$\rightarrow \frac{dy}{dx} = x(1 + y^2)$$

$$\rightarrow \frac{dy}{1+y^2} = x dx$$

Take integration

$$\rightarrow \int\frac{dy}{1+y^2} = \int x dx$$

$$\rightarrow \tan^{-1} y = \frac{x^2}{2} + c \longrightarrow ①$$

which is general solution.

using $y = 1,\ x = 0$ ① becomes

$$① \rightarrow \tan^{-1}(1) = \frac{(0)^2}{2} + c \quad \rightarrow c = \frac{\pi}{4}$$

Now ① $\rightarrow \tan^{-1} y = \frac{x^2}{2} + \frac{\pi}{4}$ (req. particular sol.)

Q20. Solve the differential equation $\frac{dx}{dt} = 2x$ given that $x = 4$ when $t = 0$

Solution:- $\frac{dx}{dt} = 2x$

$$\rightarrow \frac{dx}{x} = 2dt$$

$$\rightarrow \int\frac{dx}{x} = 2\int dt$$

$$\rightarrow \ln|x| = 2\cdot t + c_1$$

$$\rightarrow x = e^{2t + c_1} = e^{2t}\cdot e^{c_1} \quad \because e^{c_1} = c$$

$$x = ce^{2t} \longrightarrow (i) \quad \text{(general solution)}$$

Put $x = 4$ when $t = 0$

$$\rightarrow 4 = ce^{2(0)} = ce^{0} = c \quad \because e^{0} = 1$$

$$\rightarrow c = 4$$

so (i) $\rightarrow x = 4e^{2t}$ (required particular solution)

Q21. Solve the differential equation $\frac{ds}{dt} + 2st = 0$. Also find the particular solution if $s = 4e$, when $t = 0$

Solution:- $\frac{ds}{dt} + 2st = 0$

$\rightarrow \frac{ds}{dt} = -2st \rightarrow ds = -2stdt$

$\rightarrow \frac{ds}{s} = -2tdt$

Take integration

$\rightarrow \int \frac{ds}{s} = -2\int tdt$

$\rightarrow \ln|s| = -2\frac{t^2}{2} + \ln|c|$

$\rightarrow \ln|s| \ \ln|c| = -t^2$

$\rightarrow \ln\left|\frac{s}{c}\right| = -t^2$

$\rightarrow \frac{s}{c} = -\frac{t^2}{e^2} \rightarrow s = ce^{-t^2} \longrightarrow ①$

which is general solution

put $s = 4e$ when $t = 0$ in ①

so $4e = ce^{-(0)^2} \rightarrow 4e = ce^{0}$

$\rightarrow c = 4e^{1} \quad \because e^{0} = 1$

so ① $\rightarrow s = 4e^{1} \cdot e^{-t^2} = 4e^{1-t^2}$

$\rightarrow s = 4e^{1-t^2}$ (required particular solution)

Q22. In a culture, bacteria increases at rate proportional to the number of bacteria present. If bacteria are 200 initially and are doubled in 2 hours, find the number of bacteria present four hours later.

Solution:- Let p be numbers of bacteria then

$\frac{dp}{dt} \propto p$ (according to condition)

$\rightarrow \frac{dp}{dt} = kp$

$\rightarrow \frac{1}{p}dp = k\,dt$

Take integral

$\rightarrow \int \frac{1}{p}dp = k\int dt$

$\rightarrow \ln p = kt + \ln c$

$\rightarrow \ln p - \ln c = kt$

$\rightarrow \ln p/c = kt$

$\rightarrow p/c = e^{kt}$

$\rightarrow p = ce^{kt}$ —— (i)

Put $p = 200$, $t = 0$ (condition I)

$200 = ce^{k(0)} = ce^{0}$

$\rightarrow c = 200 \quad \because e^{0} = 1$

so (i) $p = 200e^{kt}$ —— (ii)

Put $p = 400$ when $t = 2$ (condition II)

so (i) $\rightarrow 400 = 200e^{2k}$

$\rightarrow 2 = e^{2k} \rightarrow \ln 2 = \ln e^{2k}$

$\rightarrow 2k = \ln 2$

$\rightarrow k = \frac{1}{2}\ln 2$

so (ii) $\rightarrow p = 200e^{\frac{1}{2}\ln 2}$

$\rightarrow p = 200e^{(\frac{\ln 2}{2})(4)}$ For $t = 4$

$p = 200e^{2\ln 2} = 200e^{\ln 2^2} = 200e^{\ln(4)}$

$\rightarrow p = 200(4) \rightarrow p = 800$

which is req. number of bacteria present four hours later.

Q23. A ball is thrown vertically upward with a velocity of 2450cm/sec. Neglecting air resistance, find

(i) velocity of ball at any time t

(ii) distance traveled in any time t

(iii) maximum height attained by the ball.

Solution:- Let v is velocity and g is acceleration, so

(i) $\frac{dv}{dt} = -g$ (For upward motion)

$\rightarrow dv = -g\,dt$

$\rightarrow \int dv = -g\int dt$

$\rightarrow v = -gt + c_1$

Put $v = 2450$, $t = 0$ so

$2450 = -g(0) + c_1 \rightarrow c_1 = 2450$

Thus $v = -gt + 2450$

$\because g = 9.8\,m/sec$

$\rightarrow g = 980\,cm/sec$

$\rightarrow v = -980t + 2450$

(ii) Let h be height so

$v = \frac{dh}{dt}$

$\rightarrow \frac{dh}{dt} = v$

$\rightarrow \frac{dh}{dt} = -980t+2450$

$\rightarrow dh = -980tdt+2450dt$

$\rightarrow \int dh = -980\int tdt + 2450\int dt$

$\rightarrow h = -980\frac{t^2}{2} + 2450t + c_2$

Put $h=0,\ t=0$

$\rightarrow 0 = -490(0)^2 + 2450(0) + c_2$

$\rightarrow c_2 = 0$

so $h = -490t^2+2450$

(iii) For max. hieght, $v=0$

so $0 = -980t + 2450$ from (i)

$\rightarrow 980t = 2450$

$\rightarrow t = \frac{2450}{980}$

$\rightarrow t = \frac{5}{2}$

so $h = 2450\left(\frac{5}{2}\right) - 490\left(\frac{5}{2}\right)^2$

$= 6125 - 3062.5$

$\rightarrow h = 3062.5$

so max hieght = 3062.5 cm

$\rightarrow$ max hieght = 30.6m (÷ by 100)

Remaining QNo 18 « ——— »
Exercise 3.3

Q18. $\int \frac{x}{x^4+2x^2+5}\,dx$

Solution:- $\int \frac{xdx}{x^4+2x^2+5}$

$= \int \frac{xdx}{(x^2)^2+2x^2+1+4}$

$= \int \frac{xdx}{(x^2+1)^2+(2)^2}$

put $x^2+1 = 2\tan\theta$

$2xdx = 2\sec^2\theta d\theta$

$\rightarrow xdx = \sec^2\theta\, d\theta$

$= \int \frac{\sec^2\theta d\theta}{(2\tan\theta)^2+(2)^2}$

$= \int \frac{\sec^2\theta d\theta}{4\tan^2\theta+4}$

$= \int \frac{\sec^2\theta d\theta}{4(\tan^2\theta+1)}$

$= \int \frac{\sec^2\theta d\theta}{4\sec^2\theta}$

$= \frac{1}{4}\int 1 d\theta = \frac{1}{4}\theta + c$

$= \frac{1}{4}\tan^{-1}\left(\frac{x^2+1}{2}\right) + c$

$\because 2\tan\theta = x^2+1$

$\tan\theta = \frac{x^2+1}{2}$

$\rightarrow \theta = \tan^{-1}\left(\frac{x^2+1}{2}\right)$