Important Points for Exercise 3.1

$y = f(x)$

y — dependent variable

x — independent variable

$$y = 3x^3 + 7x^2 - 8$$

x - independent variable
y - dependent variable.

$\delta = \Delta$

If x is changed from x to $x+\delta x$.
Then average rate of y w.r.t. x

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}.$$

If x is changed from a to b.

$$\frac{\delta y}{\delta x} = \frac{f(b) - f(a)}{b - a}$$

Exercise 3.1

pakcity.org

1. Find the average rate of change of the following functions when x varies from a to b.
   (i) $y = f(x) = x^2 + 4;\quad a = 2, b = 2.3$
   (ii) $y = f(x) = x^3 - 4;\quad a = 2, b = 2.3$
   (iii) $y = f(x) = x^3 - 8;\quad a = 3, b = 2.5$

(i) $y = f(x) = x^2 + 4,\quad a = 2,\quad b = 2.3$

$$\frac{\delta y}{\delta x} = \frac{f(b) - f(a)}{b - a}$$

$$\frac{\delta y}{\delta x} = \frac{f(2.3) - f(2)}{2.3 - 2} = \frac{[(2.3)^2 + 4] - [2^2 + 4]}{0.3}$$

$$\frac{\delta y}{\delta x} = \frac{5.29 + \cancel{4} - 4 - \cancel{4}}{0.3} = \frac{1.29}{0.3} = 4.3.$$

(ii) $y = f(x) = x^3 - 4,\quad a = 2,\quad b = 2.3$

$$\frac{\delta y}{\delta x} = \frac{f(b) - f(a)}{b - a}$$

$$\frac{\delta y}{\delta x} = \frac{f(2.3) - f(2)}{2.3 - 2}$$

$$\frac{\delta y}{\delta x} = \frac{[(2.3)^3 - 4] - [2^3 - 4]}{0.3} = \frac{12.167 - \cancel{4} - 8 + \cancel{4}}{0.3}$$

$$\frac{\delta y}{\delta x} = \frac{4.167}{0.3} = 13.89$$

(iii) $y = f(x) = x^3 - 8,\quad a = 3,\quad b = 2.5$

$$\frac{\delta y}{\delta x} = \frac{f(b) - f(a)}{b - a} = \frac{f(2.5) - f(3)}{2.5 - 3}$$

$$= \frac{[(2.5)^3 - 8] - [3^3 - 8]}{-0.5} = \frac{15.625 - \cancel{8} - 27 + \cancel{8}}{-0.5}$$

$$= \frac{\cancel{-}11.375}{\cancel{-}0.5} = 22.75.$$

2. Find out the average rate of change when x changes from a to b.

(i) $A = \pi x^2$, where x is the radius of the ~~sphere~~ circle; $a = 3, b = 3.1$

(ii) $V = \frac{4}{3}\pi x^3$, where x is the radius of the ~~circle~~ sphere. $a = 2, b = 1.9$

(i) $A(x) = \pi x^2$ $\quad a = 3, \quad b = 3.1$

Average rate of change

$$\frac{\delta A}{\delta x} = \frac{A(b) - A(a)}{b - a}$$

$$\frac{\delta A}{\delta x} = \frac{\pi(3.1)^2 - \pi(3)^2}{3.1 - 3}$$

$$\frac{\delta A}{\delta x} = \frac{\pi[9.61 - 9]}{0.1} = \frac{\pi(0.61)\times 10}{1 \times 10}$$

$$= 6.1\pi.$$

(ii) $V(x) = \frac{4}{3}\pi x^3$, $\quad a = 2, \quad b = 1.9$

$$\frac{\delta V}{\delta x} = \frac{V(b) - V(a)}{b - a}$$

$$= \frac{\frac{4}{3}\pi(1.9)^3 - \frac{4}{3}\pi(2)^3}{1.9 - 2}$$

$$= \frac{\frac{4}{3}\pi[6.859 - 8]}{-0.1}$$

$$= \frac{\frac{4}{3}\pi(+1.141)\times 10}{+0.1 \times 100}$$

$$= \frac{4}{3}\pi(11.41)$$

$$\approx 15.213\pi$$

3. The price p in rupees after "t" years is given by $p(t) = 3t^2 + t + 1$. Find the average rate of change of inflation from $t = 3$ to $t = 3.5$ years.

$$p(t) = 3t^2 + t + 1 \qquad a = 3, \quad b = 3.5$$

$$\frac{\delta p(t)}{\delta t} = \frac{p(b) - p(a)}{b - a}$$

$$= \frac{p(3.5) - p(3)}{3.5 - 3}$$

$$= \frac{[3(3.5)^2 + 3.5 + 1] - [3(3)^2 + 3 + 1]}{0.5}$$

$$= \frac{3(12.25) + 3.5 + \cancel{1} - 27 - 3 - \cancel{1}}{0.5}$$

$$= 14.5$$

Ans.

4. A ball is thrown vertically up, its height in metres after t seconds is given by the formula $h(t) = -16t^2 + 80t$. Find the average velocity when t changes from a to b.
(a) $a = 2, b = 2.1$ (b) $a = 2, b = 2.01$

$$h(t) = -16t^2 + 80t$$

(a) $a = 2, \quad b = 2.1$

$$\text{Average velocity} = \frac{\delta h}{\delta t}$$

$$= \frac{h(b) - h(a)}{b - a}$$

$$= \frac{[-16(2.1)^2 + 80(2.1)] - [-16(2)^2 + 80(2)]}{2.1 - 2}$$

$$= \frac{-16(4.41) + 168 + 64 - 160}{0.1}$$

$$= 14.4.$$

(b) $a = 2, \quad b = 2.01$

$$\text{Average velocity} = \frac{\delta h}{\delta t} = \frac{h(b) - h(a)}{b - a} = \frac{[-16(2.01)^2 + 80(2.01)] - [-16(2)^2 + 80(2)]}{2.01 - 2}$$

$$= \frac{-16(4.0401) + 160.8 + 64 - 160}{0.01}$$

$$= 15.84.$$

Derivative of a Function

If $y = f(x)$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$y = f(x)$$

$$y', \ \frac{dy}{dx}, \ f', \ Dy, \ \dot{f}$$

Exercise 3.2

Q# 01 Find by definition (ab-initio) the derivatives w.r.t. x of the following functions defined as:

(i) $f(x) = 2x$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2(x+\delta x) - 2x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\cancel{2x} + 2\delta x - \cancel{2x}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cancel{\delta x}}{\cancel{\delta x}} = 2 \quad \text{Ans}$$

(ii) $f(x) = 1 - \sqrt{x}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[1 - \sqrt{x+\delta x}] - [1 - \sqrt{x}]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{1} - \sqrt{x+\delta x} - \cancel{1} + \sqrt{x}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[(\sqrt{x} - \sqrt{x+\delta x}) \times \frac{(\sqrt{x} + \sqrt{x+\delta x})}{\sqrt{x} + \sqrt{x+\delta x}}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{(\sqrt{x})^2 - (\sqrt{x+\delta x})^2}{\sqrt{x} + \sqrt{x+\delta x}}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{\cancel{x} - \cancel{x} - \delta x}{\sqrt{x} + \sqrt{x+\delta x}}\right]$$

$$= \lim_{\delta x \to 0} \frac{-1}{\sqrt{x} + \sqrt{x+\delta x}} = \frac{-1}{\sqrt{x} + \sqrt{x}} = \frac{-1}{2\sqrt{x}}$$

Ans

(iii) $f(x) = \dfrac{1}{\sqrt{x}}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$f'(x) = \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{1}{\sqrt{x+\delta x}} - \frac{1}{\sqrt{x}}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{(\sqrt{x} - \sqrt{x+\delta x})}{\sqrt{x+\delta x}\,\sqrt{x}} \times \frac{(\sqrt{x} + \sqrt{x+\delta x})}{\sqrt{x} + \sqrt{x+\delta x}}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{\cancel{x} - \cancel{x} - \delta x}{\sqrt{x+\delta x}\,\sqrt{x}\,(\sqrt{x} + \sqrt{x+\delta x})}\right]$$

$$= \lim_{\delta x \to 0} \frac{-1}{\sqrt{x+\delta x}\,\sqrt{x}\,(\sqrt{x} + \sqrt{x+\delta x})}$$

$$= \frac{-1}{\sqrt{x}\,\sqrt{x}\,(\sqrt{x} + \sqrt{x})} = \frac{1}{x\,(2\sqrt{x})} = \frac{1}{2x\sqrt{x}}$$

(iv) $f(x) = 3 - x^2$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\{3 - (x+\delta x)^2\} - \{3 - x^2\}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[3 - (x^2 + \delta x^2 + 2x\,\delta x) - 3 + x^2\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{3} - \cancel{x^2} - \delta x^2 - 2x\,\delta x - \cancel{3} + \cancel{x^2}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[-\delta x - 2x\right]$$

$$= 0 - 2x = 2x \quad \underline{\text{Ans}}$$

(v) $f(x) = x(x+1) = x^2 + x$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left\{(x+\delta x)^2 + (x+\delta x)\right\} - \left\{x^2 + x\right\}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{x^2} + \delta x^2 + 2x\delta x + \cancel{x} + \delta x - \cancel{x^2} - \cancel{x}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[\delta x + 2x + 1\right] = 0 + 2x + 1$$

$$= 2x + 1.$$

(vi) $f(x) = x^2 - 3$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left\{(x+\delta x)^2 - 3\right\} - \left\{x^2 - 3\right\}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{x^2} + \delta x^2 + 2x\delta x - \cancel{3} - \cancel{x^2} + \cancel{3}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[\delta x + 2x\right] = 0 + 2x$$

$$= 2x.$$

(vii) $f(x) = x^3 + 5$ ✓

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left\{(x+\delta x)^3 + 5\right\} - \left\{x^3 + 5\right\}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{x^3} + \delta x^3 + 3x^2\delta x + 3x\delta x^2 + \cancel{5} - \cancel{x^3} - \cancel{5}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[\delta x^2 + 3x^2 + 3x\delta x\right]$$

$$= 0 + 3x^2 + 0 = 3x^2.$$

(viii) $f(x) = 4x^2 - 3x$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left\{4(x+\delta x)^2 - 3(x+\delta x)\right\} - \left\{4x^2 - 3x\right\}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[4(x^2 + \delta x^2 + 2x\delta x) - 3x - 3\delta x - 4x^2 + 3x\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{4x^2} + 4\delta x^2 + 8x\delta x - \cancel{3x} - 3\delta x - \cancel{4x^2} + \cancel{3x}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[4\delta x + 8x - 3\right] = 0 + 8x - 3$$

$$= 8x - 3.$$

(ix) $f(x) = \dfrac{1}{x+2}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{1}{x+\delta x+2} - \frac{1}{x+2}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{\cancel{x}+\cancel{2} - \cancel{x} - \delta x - \cancel{2}}{(x+\delta x+2)(x+2)}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[\frac{-1}{(x+\delta x+2)(x+2)}\right] = \frac{-1}{(x+2)(x+2)}$$

$$= \frac{-1}{(x+2)^2}$$ Ans.

(x) $f(x) = \dfrac{3}{2x+5}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{3}{2(x+\delta x)+5} - \frac{3}{2x+5}\right]$$

$$= \lim_{\delta x \to 0} \frac{3}{\delta x}\left[\frac{1}{2x+2\delta x+5} - \frac{1}{2x+5}\right]$$

$$= \lim_{\delta x \to 0} \frac{3}{\delta x}\left[\frac{\cancel{2x}+\cancel{5} - \cancel{2x} - 2\delta x - \cancel{5}}{(2x+2\delta x+5)(2x+5)}\right]$$

$$= \lim_{\delta x \to 0} 3\frac{\cancel{\delta x}}{\cancel{\delta x}}\left[\frac{-2}{(2x+2\delta x+5)(2x+5)}\right]$$

$$= 3\left[\frac{-2}{(2x+5)(2x+5)}\right]$$

$$= \frac{-6}{(2x+5)^2}$$ Ans

Q #02 Find $f'(x)$ for the following functions using definition:

(i) $f(x) = \sqrt[3]{2x+1} = (2x+1)^{1/3}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[(2(x+\delta x)+1)^{1/3} - (2x+1)^{1/3} \right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[(2x+2\delta x+1)^{1/3} - (2x+1)^{1/3} \right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[(\underline{2x+1} + \underline{2\delta x})^{1/3} - (2x+1)^{1/3} \right]$$

Formula $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots$

$$f'(x) = \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{(2x+1)^{1/3}} + \frac{1}{3}(2x+1)^{\frac{1}{3}-1}(2\delta x) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(2x+1)^{\frac{1}{3}-2}(2\delta x)^2 + \cdots - \cancel{(2x+1)^{1/3}} \right]$$

$$f'(x) = \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{1}{3}(2x+1)^{-\frac{2}{3}} 2\delta x + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(2x+1)^{-\frac{5}{3}} 4\delta x^2 + \cdots \right]$$

$$= \lim_{\delta x \to 0} \cancel{\frac{\delta x}{\delta x}}\left[\frac{1}{3}(2x+1)^{-\frac{2}{3}} 2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(2x+1)^{-\frac{5}{3}} 4\delta x + \cdots \right]$$

$$= \frac{1}{3}(2x+1)^{-\frac{2}{3}} 2 + 0 + 0 + \cdots$$

$$= \frac{2}{3}(2x+1)^{-2/3} = \frac{2}{3(2x+1)^{2/3}} \quad \text{Ans}$$

(ii) $f(x) = (2x-1)^{-\frac{1}{2}}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$f'(x) = \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left(2(x+\delta x)-1\right)^{-1/2} - (2x-1)^{-1/2}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[(2x-1+2\delta x)^{-1/2} - (2x-1)^{-1/2}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{(2x-1)^{-1/2}} + \left(-\frac{1}{2}\right)(2x-1)^{-\frac{1}{2}-1}(2\delta x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(2x-1)^{-\frac{1}{2}-2}(2\delta x)^2 + \cdots - \cancel{(2x-1)^{-1/2}}\right]$$

$$f'(x) = \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[-\frac{1}{2}(2x-1)^{-\frac{3}{2}}2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(2x-1)^{-\frac{5}{2}}4\delta x + \cdots\right]$$

$$= -\frac{1}{\cancel{2}}(2x-1)^{-\frac{3}{2}}\cancel{2} + 0 + 0 + \cdots$$

$$= \frac{-1}{(2x-1)^{3/2}}$$ Ans

(iii) $f(x) = (6x+7)^{\frac{5}{2}}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left(6(x+\delta x)+7\right)^{\frac{5}{2}} - (6x+7)^{\frac{5}{2}}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left(\underline{6x+7} + \underline{6\delta x}\right)^{\frac{5}{2}} - (6x+7)^{\frac{5}{2}}\right]$$

$$f'(x) = \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{(6x+7)^{\frac{5}{2}}} + \frac{5}{2}(6x+7)^{\frac{5}{2}-1} 6\delta x + \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}(6x+7)^{\frac{5}{2}-2}(6\delta x)^2 + \cdots - \cancel{(6x+7)^{\frac{5}{2}}}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[\frac{5}{2}(6x+7)^{\frac{3}{2}} 6 + \frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}(6x+7)^{\frac{1}{2}} 36\,\delta x + \cdots\right]$$

$$= \frac{5}{\cancel{2}}(6x+7)^{\frac{3}{2}} \cancel{6}^{\,3} + 0 + 0 + 0 + \cdots$$

$$= 15(6x+7)^{\frac{3}{2}} \quad \underline{\text{Ans.}}$$

(iv) $f(x) = (3x-5)^{-\frac{3}{2}}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left(3(x+\delta x)-5\right)^{-3/2} - (3x-5)^{-3/2}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\left(\underline{3x-5}+\underline{3\delta x}\right)^{-3/2} - (3x-5)^{-3/2}\right]$$

$$f'(x) = \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cancel{(3x-5)^{-3/2}} + \left(\frac{-3}{2}\right)(3x-5)^{-\frac{3}{2}-1} 3\delta x + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)}{2!}(3x-5)^{-\frac{3}{2}-2}(3\delta x)^2 + \cdots - \cancel{(3x-5)^{-3/2}}\right]$$

$$= \lim_{\delta x \to 0} \frac{\cancel{\delta x}}{\cancel{\delta x}}\left[-\frac{3}{2}(3x-5)^{-\frac{5}{2}} 3 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)}{2!}(3x-5)^{-\frac{7}{2}} 9\delta x + \cdots\right]$$

$$= -\frac{3}{2}(3x-5)^{-\frac{5}{2}} 3 + 0 + 0 + \cdots$$

$$= -\frac{9}{2}(3x-5)^{-5/2} = \frac{-9}{2(3x-5)^{5/2}}$$

Important Formulas for Exercise 3.3

$$f'(x) = \frac{df}{dx}$$

1- $$(cf(x))' = c\,(f(x))'$$

2- $$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

3- $$(f(x)\,g(x))' = f(x)\,g'(x) + g(x)\,f'(x)$$ (Product Rule).

4- $$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}$$ (Quotient Rule).

5- $$(x^n)' = nx^{n-1}, \quad (x)' = 1, \quad (c)' = 0$$

6- $$[f(x)^n]' = n\,f(x)^{n-1} \cdot f'(x)$$

(Power Rule)

Exercise 3.3

Q2 Differentiate the following w.r.t. 'x';

(i) $5x^5$

Let $y = 5x^5$

$y' = (5x^5)'$

$y' = 5(x^5)'$

$= 5(5x^{5-1})$

$= 25x^4$

(ii) $\frac{7}{9}x^9$

Let $y = \frac{7}{9}x^9$

$y' = \left(\frac{7}{9}x^9\right)'$

$y' = \frac{7}{9}(x^9)' = \frac{7}{\cancel{9}}(\cancel{9}x^8)$

$= 7x^8$

(iii) $-25x^{-3/5}$

Let $y = -25x^{-3/5}$

$y' = \left(-25x^{-3/5}\right)'$

$= -25\left(x^{-3/5}\right)'$

$(x^n)' = nx^{n-1}$

$= \overset{5}{\cancel{-25}}\left(\frac{-3}{\cancel{5}}x^{\frac{-3}{5}-1}\right)$

$= 15x^{-8/5}$

(iv) $124\sqrt{x}$

Let $y = 124x^{1/2}$

$y' = (124x^{1/2})' = 124(x^{1/2})'$

$y' = \overset{62}{\cancel{124}}\left(\frac{1}{\cancel{2}}x^{\frac{1}{2}-1}\right) = 62x^{-1/2} = \frac{62}{\sqrt{x}}$

(v) $\frac{1}{22}x^{22}$

Let $y = \frac{1}{22}x^{22}$

$y' = \left(\frac{1}{22}x^{22}\right)' = \frac{1}{22}(x^{22})' = \frac{1}{\cancel{22}}(\cancel{22}x^{21})$

$y = x^{21}$

(vi) x^{-100}

Let $y = x^{-100}$

$y' = (x^{-100})' = -100x^{-101}$

(vii) $15\sqrt[3]{x}$

Let $y = 15x^{1/3}$

$y' = (15x^{1/3})' = 15(x^{1/3})'$

$y' = \overset{5}{\cancel{15}}\left(\frac{1}{\cancel{3}}x^{\frac{1}{3}-1}\right) = 5x^{-2/3}$

(viii) $16\sqrt[4]{x^{3/4}}$

Let $y = 16\,(x^{3/4})^{\frac{1}{4}}$

$y = 16\,x^{\frac{3}{16}}$

$y' = (16\,x^{\frac{3}{16}})'$

$y' = 16\,(x^{\frac{3}{16}})' = \cancel{16}\left(\frac{3}{\cancel{16}}x^{\frac{3}{16}-1}\right)$

$y' = 3\,x^{\frac{-13}{16}} = 3\left(x^{\frac{-13}{4}}\right)^{1/4}$

$y' = 3\sqrt[4]{x^{-\frac{13}{4}}}$

(ix) $\frac{-4}{x^{3/4}}$

$y = -4x^{-3/4}$

$y' = (-4x^{-3/4})' = -4(x^{-3/4})'$

$y' = -\cancel{4}\left(\frac{-3}{\cancel{4}}x^{-3/4-1}\right) = 3x^{\frac{-7}{4}}$

$= \frac{3}{x^{7/4}}$

(x) $\frac{3}{\sqrt[3]{x^2}}$ Let $y = \frac{3}{x^{2/3}} = 3x^{-2/3}$

$y' = (3x^{-2/3})' = \cancel{3}\left(-\frac{2}{\cancel{3}}x^{\frac{-2}{3}-1}\right) = -2x^{-5/3}$

$y' = \frac{-2}{x^{5/3}} = \frac{-2}{\sqrt[3]{x^5}}$

(i) Let $y = \frac{x^5}{a^2+b^2} + \frac{x^2}{a^2-b^2}$

$y' = \left(\frac{x^5}{a^2+b^2} + \frac{x^2}{a^2-b^2}\right)'$

$y' = \left(\frac{x^5}{a^2+b^2}\right)' + \left(\frac{x^2}{a^2-b^2}\right)'$

$y' = \frac{1}{a^2+b^2}(x^5)' + \frac{1}{a^2-b^2}(x^2)'$

$y' = \frac{1}{a^2+b^2}(5x^4) + \frac{1}{a^2-b^2}(2x)$

$y' = \frac{5x^4}{a^2+b^2} + \frac{2x}{a^2-b^2}$.

(ii) Let $y = 2x + \frac{1}{2}x^6$

$y' = \left(2x + \frac{1}{2}x^6\right)'$

$= (2x)' + \left(\frac{1}{2}x^6\right)'$

$= 2(1) + \frac{1}{\cancel{2}}(\cancel{6}^{3}x^5)$

$= 2 + 3x^5$

(iii) Let $y = \sqrt[3]{x^2} + \sqrt{x}$

$y = (x^2)^{1/3} + x^{1/2}$

$y = x^{\frac{2}{3}} + x^{\frac{1}{2}}$

$y' = \left(x^{\frac{2}{3}} + x^{\frac{1}{2}}\right)'$

$y' = \left(x^{\frac{2}{3}}\right)' + \left(x^{\frac{1}{2}}\right)'$

$= \frac{2}{3}x^{\frac{2}{3}-1} + \frac{1}{2}x^{\frac{1}{2}-1}$

$= \frac{2}{3}x^{-\frac{1}{3}} + \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{2}{3x^{1/3}} + \frac{1}{2\sqrt{x}}$

$= \frac{2}{3\sqrt[3]{x}} + \frac{1}{2\sqrt{x}}$ Ans

(iv) Let $y = \frac{1}{21}x^{21} + \frac{1}{22}x^{22}$

$y' = \left(\frac{1}{21}x^{21} + \frac{1}{22}x^{22}\right)'$

$= \left(\frac{1}{21}x^{21}\right)' + \left(\frac{1}{22}x^{22}\right)'$

$= \frac{1}{21}(x^{21})' + \frac{1}{22}(x^{22})'$

$= \frac{1}{\cancel{21}}(\cancel{21}x^{20}) + \frac{1}{\cancel{22}}(\cancel{22}x^{21})$

$= x^{20} + x^{21}$

(v) Let $y = -\frac{5}{4}x^{-\frac{4}{5}} + \frac{2}{3}x^{-\frac{3}{2}}$

$y' = \left(-\frac{5}{4}x^{-4/5} + \frac{2}{3}x^{-3/2}\right)'$

$y' = \left(-\frac{5}{4}x^{-4/5}\right)' + \left(\frac{2}{3}x^{-3/2}\right)'$

$= -\frac{\cancel{5}}{\cancel{4}}\left(-\frac{\cancel{4}}{\cancel{5}}x^{-\frac{4}{5}-1}\right) + \frac{\cancel{2}}{\cancel{3}}\left(-\frac{\cancel{3}}{\cancel{2}}x^{-\frac{3}{2}-1}\right)$

$= x^{-\frac{9}{5}} - x^{-\frac{5}{2}}$

Ans

(3) (i) Let $y = 2ax^3 - \frac{x^2}{b} + 6$

$y' = \left(2ax^3 - \frac{x^2}{b} + 6\right)'$

$y' = (2ax^3)' - \left(\frac{x^2}{b}\right)' + (6)'$

$y' = 2a\,(x^3)' - \frac{1}{b}(x^2)' + 0$

$y' = 2a\,(3x^2) - \frac{1}{b}(2x)$

$y' = 6ax^2 - \frac{2x}{b}.$

(ii) Let $y = x^3 - \frac{3}{7}x^{7/3}$

$y' = \left(x^3 - \frac{3}{7}x^{7/3}\right)'$

$y' = (x^3)' - \left(\frac{3}{7}x^{\frac{7}{3}}\right)'$

$= 3x^2 - \frac{\cancel{3}}{\cancel{7}}\left(\frac{\cancel{7}}{\cancel{3}}x^{\frac{7}{3}-1}\right)$

$= 3x^2 - x^{\frac{4}{3}}$

(iii) Let $y = 5x^{\frac{3}{5}} - \frac{1}{7}x^{\frac{7}{3}}$

$y' = \left(5x^{\frac{3}{5}} - \frac{1}{7}x^{\frac{7}{3}}\right)'$

$y' = (5x^{3/5})' - \left(\frac{1}{7}x^{7/3}\right)'$

$y' = \cancel{5}\left(\frac{3}{\cancel{5}}x^{\frac{3}{5}-1}\right) - \frac{1}{\cancel{7}}\left(\frac{\cancel{7}}{3}x^{\frac{7}{3}-1}\right)$

$y' = 3x^{\frac{-2}{5}} - x^{\frac{4}{3}}$

Ans.

(iv) Let $y = x^{10} - 10x^{15}$

$y' = (x^{10} - 10x^{15})'$

$= (x^{10})' - (10x^{15})'$

$= 10x^9 - 10x^{14}$

Ans.

(v) Let $y = 3\left(\sqrt[3]{x^2}\right) - 4\left(\sqrt[4]{x}\right)$

$y = 3\,(x^2)^{\frac{1}{3}} - 4\,(x)^{\frac{1}{4}}$

$y = 3x^{\frac{2}{3}} - 4x^{\frac{1}{4}}$

$y' = \left(3x^{\frac{2}{3}} - 4x^{\frac{1}{4}}\right)'$

$y' = \left(3x^{\frac{2}{3}}\right)' - \left(4x^{\frac{1}{4}}\right)'$

$y' = \cancel{3}\left(\frac{2}{\cancel{3}}x^{\frac{2}{3}-1}\right) - \cancel{4}\left(\frac{1}{\cancel{4}}x^{\frac{1}{4}-1}\right)$

$y' = 2x^{-1/3} - x^{-3/4}$

Ans.

Q4 (i) $P(x) = x^3 - 3x^2 + 2x + 1$

$$P'(x) = (x^3 - 3x^2 + 2x + 1)'$$
$$= (x^3)' - (3x^2)' + (2x)' + (1)'$$
$$= 3x^2 - 3(2x) + 2(1) + 0$$
$$= 3x^2 - 6x + 2$$

(ii) $P(x) = x^4 - 3x^2 + 2x - 3$

$$P'(x) = (x^4 - 3x^2 + 2x - 3)'$$
$$= (x^4)' - (3x^2)' + (2x)' - (3)'$$
$$= 4x^3 - 3(2x) + 2(1) + 0$$
$$= 4x^3 - 6x + 2$$

(iii) $P(x) = x^6 - x^4 + x^3 + x$

$$P'(x) = (x^6 - x^4 + x^3 + x)'$$
$$= (x^6)' - (x^4)' + (x^3)' + (x)'$$
$$= 6x^5 - 4x^3 + 3x^2 + 1$$

(iv) $P(x) = 9x^9 + 7x^7 + \frac{1}{5}x^5 - \frac{1}{4}x^4 + x + 1$

$$P'(x) = \left(9x^9 + 7x^7 + \frac{1}{5}x^5 - \frac{1}{4}x^4 + x + 1\right)'$$
$$= (9x^9)' + (7x^7)' + \left(\frac{1}{5}x^5\right)' - \left(\frac{1}{4}x^4\right)' + x' + 1'$$
$$= 9(9x^8) + 7(7x^6) + \frac{1}{\cancel{5}}(\cancel{5}x^4) - \frac{1}{\cancel{4}}(\cancel{4}x^3) + 1 + 0$$
$$= 81x^8 + 49x^6 + x^4 - x^3 + 1$$

Ans

(v) $P(x) = x^3 + x^2 + x + 1$

$$P'(x) = (x^3 + x^2 + x + 1)'$$
$$= (x^3)' + (x^2)' + x' + 1'$$
$$= 3x^2 + 2x + 1 + 0$$
$$= 3x^2 + 2x + 1$$

Ans

(5) Find the derivative using product rule.

$(fg)' = fg' + gf'$

(i) $h(x) = (2x-5)\cdot(5x+7)$

$h'(x) = \left[(2x-5)(5x+7)\right]'$

$= (2x-5)(5x+7)' + (5x+7)(2x-5)'$

$= (2x-5)(5+0) + (5x+7)(2-0)$

$= (2x-5)(5) + (5x+7)(2)$

$= 10x - 25 + 10x + 14$

$= 20x - 11$ Ans.

(ii) $h(x) = x\cdot\sqrt{3x^2+4}$

$h'(x) = \left[x\cdot(3x^2+4)^{1/2}\right]'$

$\left(f^n(x)\right)' = n f^{n-1}(x)\cdot f'(x)$

$= x\cdot\left[(3x^2+4)^{1/2}\right]' + (3x^2+4)^{\frac{1}{2}}\cdot(x)'$

$= x\cdot\frac{1}{2}(3x^2+4)^{\frac{1}{2}-1}\cdot(3x^2+4)' + (3x^2+4)^{1/2}\cdot 1$

$= \frac{x}{\cancel{2}}(3x^2+4)^{-\frac{1}{2}}\cdot(\overset{3}{\cancel{6}}x) + (3x^2+4)^{\frac{1}{2}}$

$= \dfrac{3x^2}{\sqrt{3x^2+4}} + \sqrt{3x^2+4}$

$= \dfrac{3x^2 + 3x^2+4}{\sqrt{3x^2+4}}$

$= \dfrac{6x^2+4}{\sqrt{3x^2+4}}$

$= \dfrac{2(3x^2+2)}{\sqrt{3x^2+4}}$ Ans

(iii) $$h(x) = \sqrt[3]{x+1} \cdot \sqrt[5]{x^2+1}$$

$$h'(x) = \left[(x+1)^{\frac{1}{3}} \cdot (x^2+1)^{\frac{1}{5}} \right]'$$

$$= (x+1)^{\frac{1}{3}} \cdot \left[(x^2+1)^{\frac{1}{5}}\right]' + (x^2+1)^{\frac{1}{5}} \left[(x+1)^{\frac{1}{3}}\right]'$$

$$= (x+1)^{\frac{1}{3}} \frac{1}{5}(x^2+1)^{\frac{1}{5}-1}(x^2+1)' + (x^2+1)^{\frac{1}{5}} \cdot \frac{1}{3}(x+1)^{\frac{1}{3}-1}(x+1)'$$

$$= (x+1)^{\frac{1}{3}} \cdot \frac{1}{5}(x^2+1)^{-\frac{4}{5}}(2x) + (x^2+1)^{\frac{1}{5}} \cdot \frac{1}{3}(x+1)^{-\frac{2}{3}}(1)$$

$$= (x+1)^{\frac{1}{3}} \cdot \frac{1}{5} \cdot \frac{2x}{(x^2+1)^{4/5}} + (x^2+1)^{\frac{1}{5}} \cdot \frac{1}{3} \cdot \frac{1}{(x+1)^{2/3}}$$

$$= \frac{2x(x+1)^{1/3}}{5(x^2+1)^{4/5}} + \frac{(x^2+1)^{\frac{1}{5}}}{3(x+1)^{2/3}}$$

$$= \frac{6x(x+1)^{1/3+2/3} + 5(x^2+1)^{\frac{1}{5}+\frac{4}{5}}}{15(x^2+1)^{4/5}(x+1)^{2/3}}$$

$$= \frac{6x(x+1) + 5(x^2+1)}{15(x^2+1)^{4/5}(x+1)^{2/3}} = \frac{6x^2+6x+5x^2+5}{15(x^2+1)^{4/5}(x+1)^{2/3}}$$

$$= \frac{11x^2+6x+5}{15(x^2+1)^{4/5}(x+1)^{2/3}}$$

Ans.

(iv) $$h(x) = x^2(\sqrt{x}+1)$$

$$h'(x) = \left[x^2 \cdot (\sqrt{x}+1)\right]' = \left[x^2 \cdot (x^{\frac{1}{2}}+1)\right]'$$

$$h'(x) = x^2 \cdot (x^{\frac{1}{2}}+1)' + (x^{\frac{1}{2}}+1)(x^2)'$$

$$= x^2\left(\frac{1}{2}x^{\frac{1}{2}-1}+0\right) + (x^{\frac{1}{2}}+1)(2x)$$

$$= x^2\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + (x^{\frac{1}{2}}+1)\,2x$$

$$= \frac{1}{2}x^{2-\frac{1}{2}} + 2x^{\frac{1}{2}+1} + 2x$$

$a^{\frac{3}{2}} = a a^{\frac{1}{2}} = a\sqrt{a}$

$$= \frac{1}{2}x^{\frac{3}{2}} + 2x^{\frac{3}{2}} + 2x$$

$$= x^{\frac{2}{3}}\left(\frac{1}{2}+2\right) + 2x$$

$$= \frac{5}{2}x^{\frac{2}{3}} + 2x = \frac{5}{2}x\sqrt{x} + 2x$$

$$= \frac{x}{2}\left(5\sqrt{x}+4\right)$$

Ans

(v) $h(x) = (x+1)^3 \cdot x^{-3/2}$

$h'(x) = \left[\underline{(x+1)^3} \cdot \underline{x^{-3/2}} \right]'$

$= (x+1)^3 \cdot \left(x^{-3/2}\right)' + x^{-3/2} \cdot \left[(x+1)^3\right]'$

$= (x+1)^3 \cdot \left(-\frac{3}{2} x^{-\frac{3}{2}-1}\right) + x^{-\frac{3}{2}} \left(3(x+1)^2 (x+1)'\right)$

$= -\frac{3}{2}(x+1)^3 x^{-\frac{5}{2}} + 3x^{-\frac{3}{2}}(x+1)^2$

$x^{5/2} = x \cdot x^{3/2}$

$= \dfrac{-3(x+1)^3}{2x \cdot x^{3/2}} + \dfrac{3(x+1)^2}{x^{3/2}}$

$= \dfrac{-3(x+1)^3 + 3(x+1)^2(2x)}{2x^{5/2}}$

$= \dfrac{3(x+1)^2\left[-(x+1) + 2x\right]}{2x^{5/2}}$

$= \dfrac{3(x+1)^2(-x-1+2x)}{2x^{5/2}}$

$= \dfrac{3(x+1)^2(x-1)}{2x^{5/2}}$ Ans.

(6) Find the derivative using quotient rule.

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

(i) $$h(x) = \frac{3x+4}{2x-3}$$

$$h'(x) = \left(\frac{3x+4}{2x-3}\right)'$$

$$= \frac{(2x-3)\cdot(3x+4)' - (3x+4)(2x-3)'}{(2x-3)^2}$$

$$= \frac{(2x-3)(3) - (3x+4)(2)}{(2x-3)^2}$$

$$= \frac{\cancel{6x} - 9 - \cancel{6x} - 8}{(2x-3)^2} = \frac{-17}{(2x-3)^2}$$

(ii) $$h(x) = (x^2-1)(x^2+1)^{-1}$$

$$h'(x) = \left(\frac{x^2-1}{x^2+1}\right)'$$

$$h'(x) = \frac{(x^2+1)(x^2-1)' - (x^2-1)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$$

$$= \frac{\cancel{2x^3} + 2x - \cancel{2x^3} + 2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

(iii)

$$h(x) = \frac{x^2-x+1}{x^2+x+1}$$

$$h'(x) = \left(\frac{x^2-x+1}{x^2+x+1}\right)'$$

$$= \frac{(x^2+x+1)(x^2-x+1)' - (x^2-x+1)(x^2+x+1)'}{(x^2+x+1)^2}$$

$$= \frac{(x^2+x+1)(2x-1) - (x^2-x+1)(2x+1)}{(x^2+x+1)^2}$$

$$= \frac{1}{(x^2+x+1)^2}\left[2x^3-x^2+2x^2-x+2x-1-(2x^3+x^2-2x^2-x+2x+1)\right]$$

$$= \frac{1}{(x^2+x+1)^2}\left[\cancel{2x^3}-\cancel{x^2}+\cancel{2x^2}-\cancel{x}+\cancel{2x}-1-\cancel{2x^3}-\cancel{x^2}+2x^2+\cancel{x}-\cancel{2x}-1\right]$$

$$= \frac{1}{(x^2+x+1)^2}\left(2x^2-2\right) = \frac{2(x^2-1)}{(x^2+x+1)^2}$$

Ans.

(iv)

$$h(x) = \frac{2x^4}{b^2-x^2}$$

$$h'(x) = \left(\frac{2x^4}{b^2-x^2}\right)'$$

$$= \frac{(b^2-x^2)(2x^4)' - 2x^4(b^2-x^2)'}{(b^2-x^2)^2}$$

$$= \frac{(b^2-x^2)(8x^3) - 2x^4(-2x)}{(b^2-x^2)^2}$$

$$= \frac{8b^2x^3 - 8x^5 + 4x^5}{(b^2-x^2)^2}$$

$$= \frac{8b^2x^3 - 4x^5}{(b^2-x^2)^2} = \frac{4x^3(2b^2-x^2)}{(b^2-x^2)^2}$$

Ans.

(v)

$$h(x) = \frac{\sqrt{x}+1}{\sqrt{x}-1}$$

$$h'(x) = \left(\frac{x^{\frac{1}{2}}+1}{x^{\frac{1}{2}}-1}\right)'$$

$$h'(x) = \frac{(x^{\frac{1}{2}}-1)(x^{\frac{1}{2}}+1)' - (x^{\frac{1}{2}}+1)(x^{\frac{1}{2}}-1)'}{(x^{\frac{1}{2}}-1)^2}$$

$$h'(x) = \frac{1}{(\sqrt{x}-1)^2}\left[(x^{\frac{1}{2}}-1)\left(\frac{1}{2}x^{\frac{1}{2}-1}\right) - (x^{\frac{1}{2}}+1)\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)\right]$$

$$= \frac{1}{(\sqrt{x}-1)^2}\left[(\sqrt{x}-1)\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x}+1)\left(\frac{1}{2\sqrt{x}}\right)\right]$$

$$= \frac{1}{2\sqrt{x}(\sqrt{x}-1)^2}\left[\cancel{\sqrt{x}} - 1 - \cancel{\sqrt{x}} - 1\right] = \frac{-\cancel{2}}{\cancel{2}\sqrt{x}(\sqrt{x}-1)^2}$$

$$= \frac{-1}{\sqrt{x}(\sqrt{x}-1)^2}$$ Ans

Important Formulas for Exercise 3.4

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\operatorname{cosec} x)' = -\operatorname{cosec} x \cot x$

$(\sec x)' = \sec x \tan x$

$(\cot x)' = -\operatorname{cosec}^2 x$

$(\sin^{-1} x)' = \dfrac{1}{\sqrt{a^2 - x^2}}$

$(\cos^{-1} x)' = \dfrac{-1}{\sqrt{a^2 - x^2}}$

$(\tan^{-1} x)' = \dfrac{1}{1 + x^2}$

$(\operatorname{cosec}^{-1} x)' = \dfrac{-1}{x\sqrt{x^2 - 1}}, \quad x > 0$

$(\sec^{-1} x)' = \dfrac{1}{x\sqrt{x^2 - 1}}, \quad x > 0$

$(\cot^{-1} x)' = \dfrac{-1}{1 + x^2}$

$(\sin ax)' = (\cos ax)(ax)'$

Chain Rule

$y = f(x)$

$$\begin{tikzcd} y \arrow[d, "\frac{dy}{dx}"] \\ x \end{tikzcd}$$

$y = y(u), \quad u = u(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\begin{tikzcd} y \arrow[d, "\frac{dy}{du}"] \\ u \arrow[d, "\frac{du}{dx}"] \\ x \end{tikzcd}$$

Exercise 3.4

① Find the derivative using chain rule.

(i) $y = (x^4 + 5x^2 + 6)^{\frac{3}{2}}$

Let $u = x^4 + 5x^2 + 6$, $y = u^{3/2}$

$\frac{du}{dx} = 4x^3 + 5(2x) + 0$, $\frac{dy}{du} = \frac{3}{2}u^{\frac{3}{2}-1}$

$= 4x^3 + 10x$, $= \frac{3}{2}u^{\frac{1}{2}}$

$= \frac{3}{2}(x^4+5x^2+6)^{\frac{1}{2}}$

So $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$= \frac{3}{2}(x^4+5x^2+6)^{\frac{1}{2}}\cdot(4x^3+10x)$

$= \frac{3}{\cancel{2}}(x^4+5x^2+6)^{\frac{1}{2}}\,\cancel{2}x(2x^2+5)$

$= 3x\,(x^4+5x^2+6)^{\frac{1}{2}}(2x^2+5)$ Ans.

(ii) $y = \left(\frac{x-1}{x+1}\right)^{\frac{3}{4}}$

Let $u = \frac{x-1}{x+1}$, then $y = u^{\frac{3}{4}}$

$\frac{du}{dx} = \frac{(x+1)(x-1)' - (x-1)(x+1)'}{(x+1)^2}$, $\frac{dy}{du} = \frac{3}{4}u^{\frac{3}{4}-1}$

$= \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2}$, $= \frac{3}{4}u^{-\frac{1}{4}}$

$= \frac{\cancel{x}+1-\cancel{x}+1}{(x+1)^2} = \frac{2}{(x+1)^2}$, $= \frac{3}{4}\left(\frac{x-1}{x+1}\right)^{-\frac{1}{4}}$

$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$-\frac{1}{4}+2 = \frac{-1+8}{4} = \frac{7}{4}$

$= \frac{3}{{}_2\cancel{4}}\left(\frac{x-1}{x+1}\right)^{-\frac{1}{4}}\cdot\frac{\cancel{2}}{(x+1)^2} = \frac{3}{2}\,\frac{(x-1)^{-\frac{1}{4}}}{(x+1)^{-\frac{1}{4}}(x+1)^2}$

$= \frac{3}{2\,(x+1)^{-\frac{1}{4}+2}\,(x-1)^{\frac{1}{4}}} = \frac{3}{2(x+1)^{7/4}(x-1)^{1/4}}$. Ans.

(iii) $y = \sqrt{\dfrac{2+x}{3+x}}$

Let $u = \dfrac{2+x}{3+x}$, then $y = \sqrt{u} = u^{\frac{1}{2}}$

$$\frac{du}{dx} = \frac{(3+x)(1) - (2+x)(1)}{(3+x)^2}$$

$$= \frac{3+\cancel{x}-2-\cancel{x}}{(3+x)^2}$$

$$= \frac{1}{(3+x)^2}$$

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1} = \frac{1}{2}u^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(\frac{2+x}{3+x}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\frac{(2+x)^{-\frac{1}{2}}}{(3+x)^{-1/2}}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{2}\frac{(2+x)^{-\frac{1}{2}}}{(3+x)^{-\frac{1}{2}}}\cdot\frac{1}{(3+x)^2}$$

$$= \frac{1}{2(3+x)^{-\frac{1}{2}+2}(2+x)^{\frac{1}{2}}} = \frac{1}{2(3+x)^{3/2}(2+x)^{1/2}}$$ Ans.

(iv) $y = \left(x + \sqrt{x^2-1}\right)^n$

$(f^n)' = nf^{n-1}\cdot f'$

Let $u = x + \sqrt{x^2-1}$, $y = u^n$

$$\frac{du}{dx} = 1 + \frac{1}{2}(x^2-1)^{\frac{1}{2}-1}\cdot(x^2-1)'$$

$$= 1 + \frac{1}{\cancel{2}}(x^2-1)^{-\frac{1}{2}}(\cancel{2}x)$$

$$= 1 + \frac{x}{\sqrt{x^2-1}}$$

$$= \frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}$$

$$\frac{dy}{du} = nu^{n-1}$$

$$= n\left(x+\sqrt{x^2-1}\right)^{n-1}$$

So
$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= n\left(x+\sqrt{x^2-1}\right)^{n-1}\cdot\left(\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}\right)$$

$$= \frac{n\left(x+\sqrt{x^2-1}\right)^n}{\sqrt{x^2-1}}$$ Ans.

(v) $y = \sqrt[3]{\dfrac{x^3+1}{x^3-1}}$

Let $u = \dfrac{x^3+1}{x^3-1}$, then $y = \sqrt[3]{u} = u^{1/3}$

$$\frac{du}{dx} = \frac{(x^3-1)(3x^2)-(x^3+1)(3x^2)}{(x^3-1)^2}$$

$$= \frac{3x^2(\cancel{x^3}-1-\cancel{x^3}-1)}{(x^3-1)^2}$$

$$= \frac{3x^2(-2)}{(x^3-1)^2}$$

$$= \frac{-6x^2}{(x^3-1)^2}$$

$$\frac{dy}{du} = \frac{1}{3}u^{\frac{1}{3}-1} = \frac{1}{3}u^{-2/3}$$

$$\frac{dy}{du} = \frac{1}{3}\left(\frac{x^3+1}{x^3-1}\right)^{-2/3}$$

$$= \frac{1}{3}\,\frac{(x^3+1)^{-2/3}}{(x^3-1)^{-2/3}}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{\cancel{3}}\,\frac{(x^3+1)^{-2/3}}{(x^3-1)^{-2/3}}\cdot\frac{-\overset{2}{\cancel{6}}x^2}{(x^3-1)^2}$$

$$-\frac{2}{3}+2 = \frac{-2+6}{3} = \frac{4}{3}$$

$$= \frac{-2x^2}{(x^3-1)^{-\frac{2}{3}+2}(x^3+1)^{2/3}}$$

$$= \frac{-2x^2}{(x^3-1)^{4/3}(x^3+1)^{2/3}} \quad \text{Ans.}$$

(2) Differentiate $\dfrac{x^3}{1+x^3}$ w.r.t. x^3.

$\boxed{\dfrac{du}{dv} = ?}$

Let $u = \dfrac{x^3}{1+x^3}$ and $v = x^3$

$$\frac{du}{dx} = \frac{(1+x^3)(3x^2) - x^3(3x^2)}{(1+x^3)^2} \qquad \frac{dv}{dx} = 3x^2$$

$$= \frac{3x^2(1+\cancel{x^3}-\cancel{x^3})}{(1+x^3)^2}$$

$$= \frac{3x^2}{(1+x^3)^2}$$

$$\frac{du}{dv} = \frac{du}{dx} \cdot \frac{dx}{dv}$$

$$= \frac{\cancel{3x^2}}{(1+x^3)^2} \cdot \frac{1}{\cancel{3x^2}} = \frac{1}{(1+x^3)^2} \quad \text{Ans.}$$

Another Method.

Differentiate $\dfrac{x^3}{1+x^3}$ w.r.t. x^3.

Let $u = \dfrac{x^3}{1+x^3}$, $v = x^3$

$$\Rightarrow \quad u = \frac{v}{1+v}$$

$$\frac{du}{dv} = \frac{(1+v)(1) - v(1)}{(1+v)^2} = \frac{1+\cancel{v}-\cancel{v}}{(1+v)^2}$$

$$= \frac{1}{(1+x^3)^2}$$

3) Find $\frac{dy}{dx}$

(i) $y - xy - \sin y = 0$

$$\frac{d}{dx}(y - xy - \sin y) = \frac{d}{dx}(0)$$

$\frac{d}{dx}(\sin x) = \cos x$

$$\frac{dy}{dx} - \frac{d}{dx}(xy) - \frac{d}{dx}(\sin y) = 0$$

$$\frac{dy}{dx} - \left[x\frac{dy}{dx} + y\frac{dx}{dx}\right] - \cos y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} - x\frac{dy}{dx} - y - \cos y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} - x\frac{dy}{dx} - \cos y\frac{dy}{dx} = y$$

$$\frac{dy}{dx}(1 - x - \cos y) = y$$

$$\frac{dy}{dx} = \frac{y}{1 - x - \cos y}$$ Ans.

(ii) $y^3 - 3y + 2x = 0$

$$\frac{d}{dx}(y^3 - 3y + 2x) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(y^3) - \frac{d}{dx}(3y) + \frac{d}{dx}(2x) = 0$$

$$3y^2\frac{dy}{dx} - 3\frac{dy}{dx} + 2 = 0$$

$$3y^2\frac{dy}{dx} - 3\frac{dy}{dx} = -2$$

$$\frac{dy}{dx}(3y^2 - 3) = -2$$

$$\frac{dy}{dx} = \frac{-2}{3y^2 - 3} = \frac{-2}{3(y^2 - 1)}$$

(iii) $x^2 + y^2 + 4x + 6y - 12 = 0$

$$\frac{d}{dx}\left(x^2 + y^2 + 4x + 6y - 12\right) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) + \frac{d}{dx}(4x) + \frac{d}{dx}(6y) - \frac{d}{dx}(12) = 0$$

$$2x + 2y\frac{dy}{dx} + 4 + 6\frac{dy}{dx} - 0 = 0$$

$$2y\frac{dy}{dx} + 6\frac{dy}{dx} = -2x - 4$$

$$\frac{dy}{dx}(2y + 6) = -2x - 4$$

$$\frac{dy}{dx} = \frac{-2x-4}{2y+6} = \frac{-\cancel{2}(x+2)}{\cancel{2}(y+3)} = \frac{-(x+2)}{y+3}$$

(iv) $\sin xy + \sec x = 2$

$$\frac{d}{dx}(\sin xy + \sec x) = \frac{d}{dx}(2)$$

$$\frac{d}{dx}(\sin xy) + \frac{d}{dx}(\sec x) = 0$$

$$(\cos xy)\frac{d}{dx}(xy) + \sec x\tan x = 0$$

$$(\cos xy)\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + \sec x\tan x = 0$$

$$x(\cos xy)\frac{dy}{dx} + y\cos xy + \sec x\tan x = 0$$

$$x\cos xy\,\frac{dy}{dx} = -y\cos xy - \sec x\tan x$$

$$\frac{dy}{dx} = \frac{-y\cos xy - \sec x\tan x}{x\cos xy}$$

Ans.

(v) $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$$\frac{d}{dx}\left(x(1+y)^{\frac{1}{2}} + y(1+x)^{\frac{1}{2}}\right) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}\left[x\cdot(1+y)^{\frac{1}{2}}\right] + \frac{d}{dx}\left[y(1+x)^{\frac{1}{2}}\right] = 0$$

$$x\frac{d}{dx}(1+y)^{\frac{1}{2}} + (1+y)^{\frac{1}{2}}\cdot 1 + y\frac{d}{dx}(1+x)^{\frac{1}{2}} + (1+x)^{\frac{1}{2}}\cdot\frac{dy}{dx} = 0$$

$$x\cdot\frac{1}{2}(1+y)^{\frac{1}{2}-1}\frac{d}{dx}(1+y) + (1+y)^{\frac{1}{2}} + y\cdot\frac{1}{2}(1+x)^{\frac{1}{2}-1}\cdot\frac{d}{dx}(1+x) + (1+x)^{\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\frac{x}{2}(1+y)^{-\frac{1}{2}}\frac{dy}{dx} + (1+y)^{\frac{1}{2}} + \frac{y}{2}(1+x)^{-\frac{1}{2}}\cdot 1 + (1+x)^{\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\frac{x}{2}\cdot\frac{1}{\sqrt{1+y}}\frac{dy}{dx} + \sqrt{1+x}\frac{dy}{dx} = -\sqrt{1+y} - \frac{y}{2}\cdot\frac{1}{\sqrt{1+x}}$$

$$\frac{dy}{dx}\left[\frac{x}{2\sqrt{1+y}} + \sqrt{1+x}\right] = -\left[\sqrt{1+y} + \frac{y}{2\sqrt{1+x}}\right]$$

$$\frac{dy}{dx}\left[\frac{x+2\sqrt{1+x}\sqrt{1+y}}{2\sqrt{1+y}}\right] = -\left[\frac{2\sqrt{1+x}\sqrt{1+y}+y}{2\sqrt{1+x}}\right]$$

$$\frac{dy}{dx} = -\left[\frac{\cancel{2}\sqrt{1+y}}{x+2\sqrt{1+x}\sqrt{1+y}}\right]\left[\frac{y+2\sqrt{1+x}\sqrt{1+y}}{\cancel{2}\sqrt{1+x}}\right]$$

$$\frac{dy}{dx} = -\frac{\sqrt{1+y}\left[y+2\sqrt{1+x}\sqrt{1+y}\right]}{\sqrt{1+x}\left[x+2\sqrt{1+x}\sqrt{1+y}\right]}$$

Ans.

(vi) $$y(x^2+1) = x(y^2+1)$$

$$x^2y + y = xy^2 + x$$

$$\frac{d}{dx}(x^2y + y) = \frac{d}{dx}(xy^2 + x)$$

$$\frac{d}{dx}(x^2y) + \frac{d}{dx}(y) = \frac{d}{dx}(xy^2) + \frac{d}{dx}(x)$$

$$x^2\frac{dy}{dx} + y\frac{d}{dx}(x^2) + \frac{dy}{dx} = x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x) + 1$$

$$x^2\frac{dy}{dx} + y(2x) + \frac{dy}{dx} = x\left(2y\frac{dy}{dx}\right) + y^2 + 1$$

$$x^2\frac{dy}{dx} + 2xy + \frac{dy}{dx} = 2xy\frac{dy}{dx} + y^2 + 1$$

$$x^2\frac{dy}{dx} + \frac{dy}{dx} - 2xy\frac{dy}{dx} = y^2 + 1 - 2xy$$

$$\frac{dy}{dx}(x^2 + 1 - 2xy) = y^2 + 1 - 2xy$$

$$\frac{dy}{dx} = \frac{y^2 + 1 - 2xy}{x^2 + 1 - 2xy}$$ Ans.

(4) If $\frac{u^2}{a^2}+\frac{v^2}{b^2}=1$, where a and b are nonzero constants, find $\frac{du}{dv}$ and $\frac{dv}{du}$.

Given $\frac{u^2}{a^2}+\frac{v^2}{b^2}=1$

$$\frac{d}{dv}\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right)=\frac{d}{dv}(1)$$

$$\frac{d}{dv}\left(\frac{u^2}{a^2}\right)+\frac{d}{dv}\left(\frac{v^2}{b^2}\right)=0$$

$$\frac{1}{a^2}\frac{d}{dv}(u^2)+\frac{1}{b^2}\frac{d}{dv}(v^2)=0$$

$$\frac{1}{a^2}\cdot 2u\frac{du}{dv}+\frac{1}{b^2}\cdot 2v=0$$

$$\frac{2u}{a^2}\frac{du}{dv}=\frac{-2v}{b^2}$$

$$\frac{du}{dv}=-\frac{\cancel{2}v}{b^2}\cdot\frac{a^2}{\cancel{2}u}=-\frac{a^2v}{b^2u}$$

$$\frac{dv}{du}=-\frac{b^2u}{a^2v}$$

⑤ Find the slope of the tangent to the curve

$$3x^2 - 7y^2 + 14y - 27 = 0$$

at the point $(-3,0)$.

$\left.\frac{dy}{dx}\right|_{(a,b)}$ = slope of tangent at (a,b).

$$3x^2 - 7y^2 + 14y - 27 = 0$$

$$\frac{d}{dx}\left(3x^2 - 7y^2 + 14y - 27\right) = \frac{d}{dx}(0)$$

$$6x - 7\left(2y\frac{dy}{dx}\right) + 14\frac{dy}{dx} - 0 = 0$$

$$6x - 14y\frac{dy}{dx} + 14\frac{dy}{dx} = 0$$

$$-14y\frac{dy}{dx} + 14\frac{dy}{dx} = -6x$$

$$\frac{dy}{dx}\left(-14y + 14\right) = -6x$$

$$\frac{dy}{dx} = \frac{-6x}{-14y+14}$$

Slope of tangent at $(-3,0)$ = $\left.\frac{dy}{dx}\right|_{(-3,0)} = \frac{-6(-3)}{-14(0)+14} = \frac{\cancel{18}^{9}}{\cancel{14}_{7}} = \frac{9}{7}$

⑥ Differentiate by using first principle.

(i) $\sin 4x$

Let $f(x) = \sin 4x$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\sin 4(x+\delta x) - \sin 4x\right]$$

$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\sin(4x+4\delta x) - \sin 4x\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\, 2\cos\left(\frac{4x+4\delta x+4x}{2}\right)\sin\left(\frac{\cancel{4x}+4\delta x-\cancel{4x}}{2}\right)$$

$$= \lim_{\delta x \to 0} 2\cos(4x+2\delta x)\,\frac{\sin(2\delta x)}{2\delta x}\cdot 2$$

$$= \left[\lim_{\delta x \to 0} 2\cos(4x+2\delta x)\right]\left[\lim_{\delta x \to 0}\frac{\sin(2\delta x)}{2\delta x}\right]\left[\lim_{\delta x \to 0} 2\right]$$

$$= 2\cos(4x+0)\,(1)\cdot 2$$

$$= 4\cos 4x \qquad \text{Ans.}$$

(ii) Let $f(x) = \cos^2 2x$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cos^2 2(x+\delta x) - \cos^2 2x\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cos^2(2x+2\delta x) - \cos^2 2x\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cos(2x+2\delta x)+\cos 2x\right]\left[\cos(2x+2\delta x) - \cos 2x\right]$$

$\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cos(2x+2\delta x)+\cos 2x\right]\left[-2\sin\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{\cancel{2x}+2\delta x-\cancel{2x}}{2}\right)\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cos(2x+2\delta x)+\cos 2x\right]\left[-2\sin(2x+\delta x)\sin\delta x\right]$$

$$= \lim_{\delta x \to 0} \left[\cos(2x+2\delta x)+\cos 2x\right]\left[-2\sin(2x+\delta x)\,\frac{\sin\delta x}{\delta x}\right]$$

$$= \left[\cos 2x + \cos 2x\right]\left[-2\sin 2x \cdot 1\right]$$

$$= -2\sin 2x\,(2\cos 2x)$$

$$= -4\sin 2x\cos 2x \qquad \text{Ans.}$$

(iii) Let $f(x) = \sec\sqrt{x}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\sec\sqrt{x+\delta x} - \sec\sqrt{x}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{1}{\cos\sqrt{x+\delta x}} - \frac{1}{\cos\sqrt{x}}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{\cos\sqrt{x} - \cos\sqrt{x+\delta x}}{\cos\sqrt{x+\delta x}\,\cos\sqrt{x}}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x\left(\cos\sqrt{x+\delta x}\,\cos\sqrt{x}\right)}\left[\cos\sqrt{x} - \cos\sqrt{x+\delta x}\right]$$

$\boxed{\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}}$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x\left(\cos\sqrt{x+\delta x}\,\cos\sqrt{x}\right)}\left[-2\sin\left(\frac{\sqrt{x}+\sqrt{x+\delta x}}{2}\right)\sin\left(\frac{\sqrt{x}-\sqrt{x+\delta x}}{2}\right)\right]$$

$$= \lim_{\delta x \to 0} \frac{+2\sin\left(\frac{\sqrt{x}+\sqrt{x+\delta x}}{2}\right)}{\cos\sqrt{x+\delta x}\,\cos\sqrt{x}} \cdot \frac{\sin\left(\frac{\sqrt{x}-\sqrt{x+\delta x}}{2}\right)}{x-x-\delta x}$$

$x - x - \delta x$
$= x - (x+\delta x)$
$= (\sqrt{x})^2 - (\sqrt{x+\delta x})^2$
$= (\sqrt{x} + \sqrt{x+\delta x})(\sqrt{x} - \sqrt{x+\delta x})$

$$= \lim_{\delta x \to 0}\left[\frac{\cancel{2}\sin\left(\frac{\sqrt{x}+\sqrt{x+\delta x}}{2}\right)}{\cos\sqrt{x+\delta x}\,\cos\sqrt{x}\left(\sqrt{x}+\sqrt{x+\delta x}\right)\cancel{2}}\right]\left[\frac{\sin\left(\frac{\sqrt{x}-\sqrt{x+\delta x}}{2}\right)}{\left(\frac{\sqrt{x}-\sqrt{x+\delta x}}{2}\right)}\right]$$

$$= \frac{\sin\left(\frac{\sqrt{x}+\sqrt{x}}{2}\right)}{\cos\sqrt{x}\,\cos\sqrt{x}\left(\sqrt{x}+\sqrt{x}\right)} \cdot 1$$

$$= \frac{\sin\sqrt{x}}{2\sqrt{x}\,\cos\sqrt{x}\cdot\cos\sqrt{x}} = \frac{1}{2\sqrt{x}}\tan\sqrt{x}\,.\,\sec\sqrt{x}$$ Ans.

(iv) Let $f(x) = \sqrt{\tan x}$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\sqrt{\tan(x+\delta x)} - \sqrt{\tan x}\right] \times \frac{\left(\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}\right)}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x\left(\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}\right)}\left[\tan(x+\delta x) - \tan x\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x\left(\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}\right)}\left[\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x\left(\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}\right)}\left[\frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{\cos(x+\delta x)\cos x}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\left(\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}\right)} \cdot \frac{1}{\cos(x+\delta x)\cos x} \cdot \frac{\sin(\cancel{x}+\delta x - \cancel{x})}{\delta x}$$

$\boxed{\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)}$

$$= \frac{1}{\sqrt{\tan x} + \sqrt{\tan x}} \cdot \frac{1}{\cos x \cdot \cos x} \cdot 1$$

$$= \frac{1}{2\sqrt{\tan x}} \cdot \frac{1}{\cos^2 x} = \frac{1}{2}\sqrt{\cot x}\ \sec^2 x.$$

Ans.

(v) Let $f(x) = \operatorname{cosec} 3x$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\operatorname{cosec}(3x+3\delta x) - \operatorname{cosec} 3x\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{1}{\sin(3x+3\delta x)} - \frac{1}{\sin 3x}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{\sin 3x - \sin(3x+3\delta x)}{\sin(3x+3\delta x)\sin 3x}\right]$$

$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x\left(\sin(3x+3\delta x)\sin 3x\right)} \cdot 2\cos\left(\frac{3x+3x+3\delta x}{2}\right)\sin\left(\frac{\cancel{3x}-\cancel{3x}-3\delta x}{2}\right)$$

$$= \lim_{\delta x \to 0} \left(\frac{2\cos\left(3x+\frac{3}{2}\delta x\right)}{\sin(3x+3\delta x)\sin 3x}\right) \cdot \left(\frac{\sin\left(-\frac{3}{2}\delta x\right)}{-\frac{3}{2}\delta x}\right) \cdot \left(-\frac{3}{2}\right)$$

$$= \frac{\cancel{2}\cos 3x}{\sin 3x \cdot \sin 3x} \cdot 1 \cdot \left(-\frac{3}{\cancel{2}}\right)$$

$$= -3\frac{\cos 3x}{\sin 3x} \cdot \frac{1}{\sin 3x} = -3\cot 3x \cdot \operatorname{cosec} 3x$$

Ans.

(vi) Let $f(x) = \cot 2x$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\cot(2x+2\delta x) - \cot 2x\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{\cos(2x+2\delta x)}{\sin(2x+2\delta x)} - \frac{\cos 2x}{\sin 2x}\right]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{\sin 2x\cos(2x+2\delta x) - \cos 2x\sin(2x+2\delta x)}{\sin(2x+2\delta x)\sin 2x}\right]$$

$$= \lim_{\delta x \to 0} \left(\frac{1}{\left(\sin(2x+2\delta x)\sin 2x\right)}\right) \cdot \left(\frac{\sin\left(\cancel{2x}-\cancel{2x}-2\delta x\right)}{-2\delta x}\right) \cdot (-2)$$

$$= \frac{1}{\sin 2x \cdot \sin 2x} \cdot 1 \cdot (-2)$$

$$= -2\operatorname{cosec}^2 2x$$

Ans.

Q7 Using differentiation rules, differentiate w.r.t. involved variables:

(i) $f(x) = (x+2)\cdot \sin x$

$$f'(x) = \left((x+2)\cdot \sin x\right)'$$

$$= (x+2)(\sin x)' + \sin x\cdot(x+2)'$$

$$= (x+2)\cos x + \sin x\cdot(1+0)$$

$$= (x+2)\cos x + \sin x \quad \text{Ans.}$$

(ii) $f(\theta) = \tan^2\theta\cdot\sec^3\theta$

$$\frac{d}{d\theta} f(\theta) = \frac{d}{d\theta}\left(\tan^2\theta\cdot\sec^3\theta\right)$$

$$= \tan^2\theta\cdot\frac{d}{d\theta}(\sec^3\theta) + \sec^3\theta\cdot\frac{d}{d\theta}(\tan^2\theta)$$

$$= \tan^2\theta\cdot 3\sec^2\theta\cdot\frac{d}{d\theta}(\sec\theta) + \sec^3\theta\cdot 2\tan\theta\cdot\frac{d}{d\theta}(\tan\theta)$$

$$= 3\tan^2\theta\sec^2\theta\cdot\sec\theta\tan\theta + 2\sec^3\theta\tan\theta\sec^2\theta$$

$$= 3\tan^3\theta\sec^3\theta + 2\tan\theta\sec^5\theta$$

$$= \tan\theta\sec^3\theta\left(3\tan^2\theta + 2\sec^2\theta\right)$$

Ans.

(iii) $f(t) = \sin^3 3t\cdot\cos^3 t$

$$\frac{d}{dt} f(t) = \frac{d}{dt}\left(\sin^3 3t\cdot\cos^3 t\right)$$

$$= \sin^3 3t\cdot\frac{d}{dt}(\cos^3 t) + \cos^3 t\cdot\frac{d}{dt}(\sin^3 3t)$$

$$= \sin^3 3t\cdot 3\cos^2 t\cdot\frac{d}{dt}(\cos t) + \cos^3 t\cdot 3\sin^2 3t\cdot\frac{d}{dt}(\sin 3t)$$

$$= 3\sin^3 3t\cos^2 t(-\sin t) + 3\cos^3 t\sin^2 3t(\cos 3t)\frac{d}{dt}(3t)$$

$$= -3\sin^3 3t\cos^2 t\sin t + 3\cos^3 t\sin^2 3t\cos 3t\,(3)$$

$$= -3\sin^3 3t\cos^2 t\sin t + 9\cos^3 t\sin^2 3t\cos 3t$$

(iv) $f(x) = \sqrt{\dfrac{\sin 2x}{\cos x}}$

$$f(x) = \sqrt{\frac{2\sin x \cancel{\cos x}}{\cancel{\cos x}}}$$

$$f(x) = \sqrt{2}\ \sqrt{\sin x}$$

$$f'(x) = \sqrt{2}\ \left((\sin x)^{\frac{1}{2}}\right)'$$

$$= \sqrt{2} \cdot \frac{1}{2} (\sin x)^{\frac{1}{2}-1} \cdot (\sin x)'$$

$$= \frac{\sqrt{2}}{2} (\sin x)^{-\frac{1}{2}} \cos x = \frac{\sqrt{2}\ \cos x}{2\sqrt{\sin x}} \quad \text{Ans}$$

(v) $f(\theta) = \dfrac{\tan\theta - 1}{\sec\theta}$

$$\frac{d}{d\theta} f(\theta) = \frac{d}{d\theta}\left(\frac{\tan\theta - 1}{\sec\theta}\right)$$

$$= \frac{(\sec\theta)\frac{d}{d\theta}(\tan\theta - 1) - (\tan\theta - 1)\frac{d}{d\theta}(\sec\theta)}{\sec^2\theta}$$

$$= \frac{\sec\theta(\sec^2\theta - 0) - (\tan\theta - 1)\sec\theta\tan\theta}{\sec^2\theta}$$

$$\boxed{\begin{aligned} 1 + \tan^2\theta &= \sec^2\theta \\ 1 &= \sec^2\theta - \tan^2\theta \end{aligned}}$$

$$= \frac{\sec^3\theta - \tan^2\theta\sec\theta + \sec\theta\tan\theta}{\sec^2\theta}$$

$$= \frac{\cancel{\sec\theta}(\sec^2\theta - \tan^2\theta + \tan\theta)}{\sec^{\cancel{2}}\theta}$$

$$= \frac{1 + \tan\theta}{\sec\theta} \quad \text{Ans}$$

(vi) $f(x) = \sin x^2 + \sin^2 x$

$$f'(x) = (\sin x^2)' + (\sin^2 x)'$$

$$= (\cos x^2)(x^2)' + 2\sin x\ (\sin x)'$$

$$= (\cos x^2)(2x) + 2\sin x \cos x$$

$$= 2x\cos x^2 + 2\sin x\cos x .$$

(8) Differentiate $\frac{1+\tan^2 x}{1-\tan^2 x}$ w.r.t. $\tan^2 x$.

Let $u = \frac{1+\tan^2 x}{1-\tan^2 x}$, $v = \tan^2 x$

$$u = \frac{1+v}{1-v}$$

$$\frac{du}{dv} = \frac{d}{dv}\left(\frac{1+v}{1-v}\right)$$

$$= \frac{(1-v)\frac{d}{dv}(1+v) - (1+v)\frac{d}{dv}(1-v)}{(1-v)^2}$$

$$= \frac{(1-v)(0+1) - (1+v)(0-1)}{(1-v)^2} = \frac{1-\cancel{v}+1+\cancel{v}}{(1-v)^2}$$

$$\frac{du}{dv} = \frac{2}{(1-\tan^2 x)^2} \quad \text{Ans}$$

Alternative

$u = \frac{1+\tan^2 x}{1-\tan^2 x}$, $v = \tan^2 x$

$$\frac{du}{dx} = \frac{(1-\tan^2 x)(0+2\tan x\sec^2 x) - (1+\tan^2 x)(0-2\tan x\sec^2 x)}{(1-\tan^2 x)^2}$$

$$\frac{du}{dx} = \frac{1}{(1-\tan^2 x)^2}\left[2\tan x\sec^2 x - \cancel{2\tan^3 x\sec^2 x} + 2\tan x\sec^2 x + \cancel{2\tan^3 x\sec^2 x}\right]$$

$$\frac{du}{dx} = \frac{4\tan x\sec^2 x}{(1-\tan^2 x)^2}$$

$v = \tan^2 x$

$\frac{dv}{dx} = 2\tan x\sec^2 x$

$$\frac{du}{dv} = \frac{du}{dx}\cdot\frac{dx}{dv}$$

$$= \frac{\overset{2}{\cancel{4}}\,\cancel{\tan x\sec^2 x}}{(1-\tan^2 x)^2}\cdot\frac{1}{\cancel{2\tan x\sec^2 x}}$$

$$= \frac{2}{(1-\tan^2 x)^2} \quad \text{Ans.}$$

Q) Find $\frac{dy}{dx}$ of the following:

(i) $y = \sin^{-1}\sqrt{\frac{1-\cos x}{2}}$

$$2\sin^2 x = 1-\cos 2x$$
$$2\sin^2 \frac{x}{2} = 1-\cos x$$
$$\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}$$

$$y = \sin^{-1}\sqrt{\sin^2\frac{x}{2}}$$
$$= \sin^{-1}\left(\sin\frac{x}{2}\right)$$
$$y = \frac{x}{2}$$
$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}\frac{d}{dx}(x) = \frac{1}{2} \quad \text{Ans.}$$

(ii) $y = \cot^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right)$

$$2\cos^2 x = 1+\cos 2x$$
$$2\cos^2 \frac{x}{2} = 1+\cos x$$

$$2\sin^2 x = 1-\cos 2x$$
$$2\sin^2 \frac{x}{2} = 1-\cos x$$

$$y = \cot^{-1}\left(\sqrt{\frac{\cancel{2}\cos^2\frac{x}{2}}{\cancel{2}\sin^2\frac{x}{2}}}\right)$$
$$y = \cot^{-1}\sqrt{\cot^2\frac{x}{2}}$$
$$y = \cot^{-1}\left(\cot\frac{x}{2}\right)$$
$$y = \frac{x}{2}$$
$$\frac{dy}{dx} = \frac{1}{2} \quad \text{Ans.}$$

(iii) $y = \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)$

$$y = \tan^{-1}\left(\frac{\cancel{2}\sin x \cancel{\cos x}}{\cancel{2}\cos^{\cancel{2}} x}\right)$$
$$y = \tan^{-1}(\tan x)$$
$$y = x$$
$$\frac{dy}{dx} = 1 \quad \text{Ans.}$$

(iv) $y = x + (\cos^{-1}x)\cdot(\sqrt{1-x^2})$

$$\frac{dy}{dx} = \frac{d}{dx}\left[x + (\cos^{-1}x)(\sqrt{1-x^2})\right]$$

$$= \frac{d}{dx}(x) + \frac{d}{dx}\left[(\cos^{-1}x)(\sqrt{1-x^2})\right]$$

$$= 1 + \cos^{-1}x \cdot \frac{1}{2}(1-x^2)^{\frac{1}{2}-1}\frac{d}{dx}(1-x^2) + \cancel{\sqrt{1-x^2}} \cdot \frac{-1}{\cancel{\sqrt{1-x^2}}}$$

$$= \cancel{1} + \cos^{-1}x \cdot \frac{1}{\cancel{2}}\frac{1}{\sqrt{1-x^2}}(-\cancel{2}x) \cancel{-1}$$

$$= -\frac{x\cos^{-1}x}{\sqrt{1-x^2}}$$

(v) $y = \frac{x\tan^{-1}x}{1+x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x\tan^{-1}x}{1+x^2}\right)$$

$$= \frac{(1+x^2)\frac{d}{dx}(x\tan^{-1}x) - (x\tan^{-1}x)\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{1}{(1+x^2)^2}\left[(1+x^2)\left(x\cdot\frac{1}{1+x^2} + \tan^{-1}x\right) - (x\tan^{-1}x)(2x)\right]$$

$$= \frac{1}{(1+x^2)^2}\left[\frac{x\,\cancel{(1+x^2)}}{\cancel{1+x^2}} + (1+x^2)\tan^{-1}x - 2x^2\tan^{-1}x\right]$$

$$= \frac{1}{(1+x^2)^2}\left[x + (1+x^2-2x^2)\tan^{-1}x\right]$$

$$= \frac{x + (1-x^2)\tan^{-1}x}{(1+x^2)^2}$$

Ans.

(vi) $$y = \tan^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)$$

$$y = \tan^{-1}\left(\frac{(\sqrt{1+x} - \sqrt{1-x})}{\sqrt{1+x} + \sqrt{1-x}} \times \frac{(\sqrt{1+x} - \sqrt{1-x})}{\sqrt{1+x} - \sqrt{1-x}}\right)$$

$$y = \tan^{-1}\left(\frac{(\sqrt{1+x})^2 + (\sqrt{1-x})^2 - 2\sqrt{1+x}\sqrt{1-x}}{(\sqrt{1+x})^2 - (\sqrt{1-x})^2}\right)$$

$$y = \tan^{-1}\left(\frac{1+\cancel{x}+1-\cancel{x} - 2\sqrt{1-x^2}}{\cancel{1}+x-\cancel{1}+x}\right)$$

$$y = \tan^{-1}\left(\frac{\cancel{2}(1-\sqrt{1-x^2})}{\cancel{2}x}\right)$$

$$\boxed{\left(\tan^{-1} f(x)\right)' = \frac{1}{1+f(x)^2} \cdot f'(x)}$$

$$y = \tan^{-1}\left(\frac{1-(1-x^2)^{\frac{1}{2}}}{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{1-(1-x^2)^{\frac{1}{2}}}{x}\right)^2} \cdot \frac{d}{dx}\left(\frac{1-(1-x^2)^{\frac{1}{2}}}{x}\right)$$

$$= \frac{1}{1+\frac{(1-\sqrt{1-x^2})^2}{x^2}} \cdot \frac{x\left(0-\frac{1}{\cancel{2}}(1-x^2)^{\frac{1}{2}-1}(-\cancel{2}x)\right) - \left(1-(1-x^2)^{\frac{1}{2}}\right) \cdot 1}{x^2}$$

$$= \frac{1}{\frac{x^2+(1-\sqrt{1-x^2})^2}{\cancel{x^2}}} \cdot \frac{\left[\frac{x^2}{\sqrt{1-x^2}} - (1-\sqrt{1-x^2})\right]}{\cancel{x^2}}$$

$$= \frac{1}{x^2+(1-\sqrt{1-x^2})^2}\left[\frac{x^2-\sqrt{1-x^2}(1-\sqrt{1-x^2})}{\sqrt{1-x^2}}\right]$$

$$= \frac{1}{\cancel{x^2}+1+1-\cancel{x^2}-2\sqrt{1-x^2}}\left[\frac{\cancel{x^2}-\sqrt{1-x^2}+1-\cancel{x^2}}{\sqrt{1-x^2}}\right]$$

$$= \frac{1}{2\cancel{(1-\sqrt{1-x^2})}}\left[\frac{\cancel{1-\sqrt{1-x^2}}}{\sqrt{1-x^2}}\right]$$

$$= \frac{1}{2\sqrt{1-x^2}}.$$

(10) If $y = \tan\left(2\tan^{-1}\frac{x}{2}\right)$, then prove that $\frac{dy}{dx} = 4\left(\frac{1+y^2}{4+x^2}\right)$.

$$y = \tan\left(2\tan^{-1}\frac{x}{2}\right)$$

$$\frac{dy}{dx} = \sec^2\left(2\tan^{-1}\frac{x}{2}\right) \cdot \frac{d}{dx}\left(2\tan^{-1}\frac{x}{2}\right)$$

$$= \sec^2\left(2\tan^{-1}\frac{x}{2}\right) \cdot 2\frac{1}{1+\left(\frac{x}{2}\right)^2} \cdot \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$= \sec^2\left(2\tan^{-1}\frac{x}{2}\right) \cdot \frac{\cancel{2}}{\left(1+\frac{x^2}{4}\right)} \cdot \frac{1}{\cancel{2}}$$

$$= \left[1+\tan^2\left(2\tan^{-1}\frac{x}{2}\right)\right] \cdot \frac{1}{\left(\frac{4+x^2}{4}\right)}$$

$$= \frac{4(1+y^2)}{4+x^2} = 4\left(\frac{1+y^2}{4+x^2}\right).$$

(11) If $\frac{y}{x} = \tan^{-1}\left(\frac{x}{y}\right)$, show that $\frac{dy}{dx} = \frac{y}{x}$.

$$\boxed{(\tan^{-1}x)' = \frac{1}{1+x^2}}$$

$$\frac{y}{x} = \tan^{-1}\frac{x}{y}$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{d}{dx}\left(\tan^{-1}\frac{x}{y}\right)$$

$$\frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\quad \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\frac{\left(x\frac{dy}{dx} - y\right)}{x^2} = \frac{1}{\left(1+\frac{x^2}{y^2}\right)} \cdot \frac{y \cdot 1 - x\frac{dy}{dx}}{y^2}$$

$$\frac{\left(x\frac{dy}{dx} - y\right)}{x^2} = \frac{1}{\left(\frac{y^2+x^2}{\cancel{y^2}}\right)} \cdot \frac{\left(y - x\frac{dy}{dx}\right)}{\cancel{y^2}}$$

$$\frac{\left(x\frac{dy}{dx} - y\right)}{x^2} = \frac{\left(y - x\frac{dy}{dx}\right)}{x^2+y^2}$$

$$(x^2+y^2)\left(x\frac{dy}{dx} - y\right) = x^2\left(y - x\frac{dy}{dx}\right)$$

$$x^3\frac{dy}{dx} - x^2y + xy^2\frac{dy}{dx} - y^3 = x^2y - x^3\frac{dy}{dx}$$

$$x^3\frac{dy}{dx} + xy^2\frac{dy}{dx} + x^3\frac{dy}{dx} = x^2y + x^2y + y^3$$

$$\frac{dy}{dx}\left(x^3 + xy^2 + x^3\right) = 2x^2y + y^3$$

$$\frac{dy}{dx} = \frac{2x^2y + y^3}{2x^3 + xy^2} = \frac{y\,\cancel{(2x^2+y^2)}}{x\,\cancel{(2x^2+y^2)}}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

(12) If $y = \tan(a \tan^{-1}x)$, show that $(1+x^2)\frac{dy}{dx} - a(1+y^2) = 0$.

$$y = \tan(a \tan^{-1}x)$$

$$\tan^{-1}y = a \tan^{-1}x$$

$$\frac{d}{dx}(\tan^{-1}y) = a\frac{d}{dx}(\tan^{-1}x)$$

$$\frac{1}{1+y^2} \cdot \frac{dy}{dx} = a \cdot \frac{1}{1+x^2}$$

$$(1+x^2)\frac{dy}{dx} = a(1+y^2)$$

$$(1+x^2)\frac{dy}{dx} - a(1+y^2) = 0$$

as required.

(13) Find $\frac{dy}{dx}$:

(i) $x = a\sin\theta$, $y = a\cos\theta$

$\frac{dx}{d\theta} = a\cos\theta$, $\frac{dy}{d\theta} = -a\sin\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = (-\cancel{a}\sin\theta) \cdot \left(\frac{1}{\cancel{a}\cos\theta}\right)$$

$$= -\frac{\sin\theta}{\cos\theta} = -\tan\theta.$$

(ii) $x = t + \frac{1}{t}$, $y = t+1$

$x = t + t^{-1}$, $y = t+1$

$\frac{dx}{dt} = 1 - t^{-2}$

$= 1 - \frac{1}{t^2}$ $\qquad \frac{dy}{dt} = 1$

$\frac{dx}{dt} = \frac{t^2-1}{t^2}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= 1 \cdot \frac{t^2}{t^2-1} = \frac{t^2}{t^2-1} \text{ Ans.}$$

(iii) $x = \dfrac{a(1-t^2)}{1+t^2}$, $y = \dfrac{2bt}{1+t^2}$

$$\frac{dx}{dt} = a\frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$$

$$= a\left[\frac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2}\right]$$

$$= a\left[\frac{-2t\cancel{-2t^3}-2t+\cancel{2t^3}}{(1+t^2)^2}\right]$$

$$\frac{dx}{dt} = \frac{-4at}{(1+t^2)^2}$$

$$\frac{dy}{dt} = 2b\frac{d}{dt}\left(\frac{t}{1+t^2}\right)$$

$$= 2b\left[\frac{(1+t^2)(1)-t(2t)}{(1+t^2)^2}\right]$$

$$= 2b\left[\frac{1+t^2-2t^2}{(1+t^2)^2}\right]$$

$$\frac{dy}{dt} = \frac{2b(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \frac{\cancel{2}b(1-t^2)}{\cancel{(1+t^2)^2}}\cdot\frac{\cancel{(1+t^2)}^2}{-\cancel{4}_2at}$$

$$= \frac{-b(1-t^2)}{2at}$$ Ans.

(iv) $x = a\theta^2$, $y = 2a\theta$

$$\frac{dx}{d\theta} = 2a\theta, \quad \frac{dy}{d\theta} = 2a$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$\frac{dy}{dx} = \cancel{2a}\cdot\frac{1}{\cancel{2a}\theta} = \frac{1}{\theta}.$$

14/2 If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x}}} + \cdots + \infty$, prove that $(2y-1)\frac{dy}{dx} = \sec^2 x$.

$$y^2 = \tan x + \left(\sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x}}} + \cdots + \infty\right)$$

$$y^2 = \tan x + y$$

$$y^2 - y = \tan x$$

$$\frac{d}{dx}(y^2 - y) = \frac{d}{dx}(\tan x)$$

$$2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

$$(2y-1)\frac{dy}{dx} = \sec^2 x$$

as required.

(15) Find the derivatives of $\cos^{-1}x$, $\csc^{-1}x$ and $\cot^{-1}x$ by using differentiation formulas.

Let $y = \cos^{-1}x$

$$\cos y = x$$

$$\frac{d}{dx}(\cos y) = \frac{d}{dx}(x)$$

$$-\sin y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}.$$

Let $y = \operatorname{cosec}^{-1}x$

$$\operatorname{cosec} y = x$$

$$\frac{d}{dx}(\operatorname{cosec} y) = \frac{d}{dx}(x)$$

$1+\cot^2 y = \operatorname{cosec}^2 y$

$\cot^2 y = \operatorname{cosec}^2 y - 1$

$$-\operatorname{cosec} y \cot y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\operatorname{cosec} y \cot y} = \frac{-1}{\operatorname{cosec} y\sqrt{\operatorname{cosec}^2 y - 1}}$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$$

Let $y = \cot^{-1}x$

$$\cot y = x$$

$$\frac{d}{dx}(\cot y) = \frac{d}{dx}(x)$$

$$-\operatorname{cosec}^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\operatorname{cosec}^2 y} = \frac{-1}{1+\cot^2 y}$$

$$\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}.$$

Important Formulas for Exercise 3.5

$\log_a x$

$a = 10$, common

$a = e$, natural

$\log_e x = \ln x$.

1. $\frac{d}{dx}(a^x) = a^x \ln a$, $a > 0$, $a \neq 1$
2. $\frac{d}{dx}(e^x) = e^x$
3. $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$
4. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

$$\left(a^{f(x)}\right)' = a^{f(x)} \ln a \cdot f'(x).$$

$$\left(e^{f(x)}\right)' = e^{f(x)} \cdot f'(x).$$

$$y = (f(x))^{g(x)}$$

$$\ln y = \ln (f(x))^{g(x)} = g(x) \cdot \ln (f(x)).$$

1) $\ln(ab) = \ln a + \ln b$

2) $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

3) $\ln a^m = m \ln a$

Exercise 3.5

① Differentiate the following w.r.t. 'x'.

(i) Let $y = x^2 + 2^x$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 + 2^x\right)$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(2^x)$$

$$= 2x + 2^x \ln 2.$$

$$\boxed{(a^x)' = a^x \ln a}$$

(ii) Let $y = 4^x + 5^x$

$$\frac{dy}{dx} = \frac{d}{dx}\left(4^x + 5^x\right)$$

$$= \frac{d}{dx}(4^x) + \frac{d}{dx}(5^x)$$

$$= 4^x \ln 4 + 5^x \ln 5.$$

(iii) Let $y = e^{\tan x + \cot x}$

$$\boxed{\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} \cdot f'(x)}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\tan x + \cot x}\right)$$

$$= e^{\tan x + \cot x} \cdot \frac{d}{dx}\left(\tan x + \cot x\right)$$

$$= e^{\tan x + \cot x} \cdot \left(\sec^2 x - \operatorname{cosec}^2 x\right)$$

$$= \left(\sec^2 x - \operatorname{cosec}^2 x\right) e^{\tan x + \cot x}.$$

(iv) Let $y = e^{\tan x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\tan x^2}\right)$$

$$= e^{\tan x^2} \cdot \frac{d}{dx}\left(\tan x^2\right)$$

$$= e^{\tan x^2} \cdot \sec^2 x^2 \cdot \frac{d}{dx}(x^2)$$

$$= e^{\tan x^2} \cdot \sec^2 x^2 \cdot 2x$$

$$= 2x\, e^{\tan x^2} \sec^2 x^2.$$

(v) Let $y = e^{2\ln(2x+1)}$

$\boxed{\ln a^m = m \ln a}$

$$y = e^{\ln(2x+1)^2}$$

$$y = (2x+1)^2$$

$$y = 4x^2 + 1 + 4x$$

$$\frac{dy}{dx} = 8x + 0 + 4$$

$$= 4(2x+1).$$

(vi) Let $y = \log_{10} x$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log_{10} x\right)$$

$\boxed{\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}}$

$$= \frac{1}{x \ln 10}$$

(vii) Let $y = \frac{e^x}{x^2+1}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x}{x^2+1}\right)$$

$$= \frac{(x^2+1)\frac{d}{dx}(e^x) - e^x \frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)e^x - e^x(2x)}{(x^2+1)^2}$$

$$= \frac{e^x(x^2+1-2x)}{(x^2+1)^2}$$

$$= \frac{e^x(x-1)^2}{(x^2+1)^2} \quad \text{Ans.}$$

(viii) Let $y = x^2 + 2^x + a^{2x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 + 2^x + a^{2x}\right)$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(2^x) + \frac{d}{dx}(a^{2x})$$

$\boxed{(a^x)' = a^x \ln a}$

$$= 2x + 2^x \ln 2 + a^{2x} \cdot \ln a \cdot \frac{d}{dx}(2x)$$

$$= 2x + 2^x \ln 2 + 2a^{2x} \ln a.$$

(ix) Let $y = (\ln x)^x$

$$\ln y = \ln (\ln x)^x$$

$$\ln y = x \ln(\ln x)$$

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}\left[x \ln(\ln x)\right]$$

$\boxed{[\ln f]' = \frac{1}{f} \cdot f'}$

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \frac{d}{dx}\left[\ln(\ln x)\right] + \ln(\ln x) \frac{d}{dx}(x)$$

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{\ln x} \cdot \frac{d}{dx}(\ln x) + \ln(\ln x) \cdot 1$$

$$\frac{1}{y}\frac{dy}{dx} = \cancel{x} \cdot \frac{1}{\ln x} \cdot \frac{1}{\cancel{x}} + \ln(\ln x)$$

$$\frac{dy}{dx} = y\left[\frac{1}{\ln x} + \ln(\ln x)\right]$$

$$\frac{dy}{dx} = (\ln x)^x\left[\frac{1}{\ln x} + \ln(\ln x)\right]$$

(x) Let $y = \ln\left(\sqrt{e^{3x} + e^{-3x}}\right)$

$$y = \ln\left(e^{3x} + e^{-3x}\right)^{1/2}$$

$$y = \frac{1}{2}\ln\left(e^{3x} + e^{-3x}\right)$$

$\boxed{(\ln f)' = \frac{1}{f} \cdot f'}$

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\left[\ln\left(e^{3x} + e^{-3x}\right)\right]$$

$$= \frac{1}{2}\frac{1}{e^{3x} + e^{-3x}} \cdot \frac{d}{dx}\left(e^{3x} + e^{-3x}\right)$$

$$= \frac{1}{2} \cdot \frac{1}{e^{3x} + e^{-3x}} \cdot \left(3e^{3x} - 3e^{-3x}\right)$$

$$= \frac{3}{2} \cdot \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$$

$\boxed{\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}}$

$$= \frac{3}{2} \cdot \tanh 3x.$$

(xi) Let $y = \ln\left(\sin(\ln x)\right)$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln\left(\sin(\ln x)\right)\right]$$

$\boxed{(\ln f)' = \frac{1}{f} \cdot f'}$

$\boxed{(\sin f)' = (\cos f) \cdot f'}$

$$= \frac{1}{\sin(\ln x)} \cdot \frac{d}{dx}\left(\sin(\ln x)\right)$$

$$= \frac{1}{\sin(\ln x)} \cdot \cos(\ln x) \cdot \frac{d}{dx}(\ln x)$$

$$= \frac{\cos(\ln x)}{\sin(\ln x)} \cdot \frac{1}{x} = \frac{\cot(\ln x)}{x}$$

(xii) Let $y = \ln\left[\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right]$

$$\frac{dy}{dx} = \frac{d}{dx}\ln\left[\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right]$$

$$\boxed{\frac{d}{dx}(\ln f) = \frac{1}{f}\cdot\frac{d}{dx}f}$$

$$= \frac{1}{\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)} \cdot \frac{d}{dx}\left[\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)} \cdot \sec^2\left(\frac{x}{2}+\frac{\pi}{4}\right) \cdot \frac{d}{dx}\left(\frac{x}{2}+\frac{\pi}{4}\right)$$

$$= \frac{\cancel{\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)}}{\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)} \cdot \frac{1}{\cos^{\cancel{2}}\left(\frac{x}{2}+\frac{\pi}{4}\right)} \cdot \frac{1}{2}$$

$$\boxed{2\sin\theta\cos\theta = \sin 2\theta}$$

$$= \frac{1}{2\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sin 2\left(\frac{x}{2}+\frac{\pi}{4}\right)} = \frac{1}{\sin\left(x+\frac{\pi}{2}\right)}$$

$$= \frac{1}{\cos x} = \sec x$$

Ans

② Use logarithmic differentiation to find $\frac{dy}{dx}$ if

(i) $y = \sqrt{\frac{x^2-1}{x^2+1}}$

$$\ln y = \ln \left(\frac{x^2-1}{x^2+1}\right)^{1/2}$$

$$\ln y = \frac{1}{2} \ln \left(\frac{x^2-1}{x^2+1}\right)$$

$$\ln y = \frac{1}{2}\left[\ln(x^2-1) - \ln(x^2+1)\right]$$

$$\frac{d}{dx} \ln y = \frac{1}{2} \frac{d}{dx}\left[\ln(x^2-1) - \ln(x^2+1)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x^2-1}(2x) - \frac{1}{x^2+1}(2x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\cancel{2}x}{\cancel{2}}\left[\frac{1}{x^2-1} - \frac{1}{x^2+1}\right]$$

$$\frac{dy}{dx} = xy\left[\frac{\cancel{x^2}+1-\cancel{x^2}+1}{(x^2-1)(x^2+1)}\right]$$

$$\frac{dy}{dx} = x\left(\frac{x^2-1}{x^2+1}\right)^{1/2}\left(\frac{2}{(x^2-1)(x^2+1)}\right)$$

$$= x\frac{(x^2-1)^{1/2}}{(x^2+1)^{1/2}} \cdot \frac{2}{(x^2-1)(x^2+1)}$$

$$= \frac{2x}{(x^2+1)^{\frac{1}{2}+1}(x^2-1)^{1-\frac{1}{2}}}$$

$a^{3/2} = a \cdot a^{1/2}$

$$= \frac{2x}{(x^2+1)^{3/2}(x^2-1)^{1/2}}$$

$$= \frac{2x}{(x^2+1)(x^2+1)^{1/2}(x^2-1)^{1/2}} = \frac{2x}{(x^2+1)\sqrt{x^4-1}}$$ Ans

(ii) $y = x^3\sqrt{x}$

$$\ln y = \ln(x^3\sqrt{x})$$

$$\ln y = \ln x^3 + \ln x^{1/2}$$

$$\ln y = \ln x^3 + \frac{1}{2}\ln x$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\ln x^3 + \frac{1}{2}\frac{d}{dx}\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^{\cancel{3}}}\cdot 3x^{\cancel{2}} + \frac{1}{2}\cdot\frac{1}{x}$$

$$\frac{dy}{dx} = y\left[\frac{3}{x} + \frac{1}{2x}\right] = x^{\cancel{3}\,2}\sqrt{x}\left[\frac{6+1}{2\cancel{x}}\right]$$

$$= \frac{7}{2}x^2\sqrt{x} = \frac{7}{2}x^{2+\frac{1}{2}} = \frac{7}{2}x^{5/2}$$

(iii) $y = xe^{\cos x}$

$$\ln y = \ln(xe^{\cos x})$$

$$\ln y = \ln x + \ln e^{\cos x}$$

$$\ln y = \ln x + \cos x(\ln e)$$

$$\ln y = \ln x + \cos x$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}(\ln x) + \frac{d}{dx}(\cos x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} - \sin x$$

$$\frac{dy}{dx} = y\left[\frac{1}{x} - \sin x\right]$$

$$= xe^{\cos x}\left[\frac{1}{x} - \sin x\right]$$

$$= \cancel{x}e^{\cos x}\cdot\frac{1}{\cancel{x}} - xe^{\cos x}\sin x$$

$$= (1 - x\sin x)e^{\cos x}$$

(iv)
$$y = e^{-2x}(x^2+2x+1)$$
$$\ln y = \ln\left[e^{-2x}(x^2+2x+1)\right]$$
$$\ln y = \ln e^{-2x} + \ln(x^2+2x+1)$$
$$\ln y = -2x(\ln e) + \ln(x^2+2x+1)$$
$$\ln y = -2x + \ln(x^2+2x+1)$$
$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(-2x) + \frac{d}{dx}\left(\ln(x^2+2x+1)\right)$$
$$\frac{1}{y}\frac{dy}{dx} = -2 + \frac{1}{x^2+2x+1}(2x+2)$$
$$\frac{dy}{dx} = e^{-2x}(x^2+2x+1)\left[-2 + \frac{2x+2}{x^2+2x+1}\right]$$
$$= e^{-2x}\cancel{(x^2+2x+1)}\left[\frac{-2x^2-4x\cancel{-2}+2x+\cancel{2}}{\cancel{x^2+2x+1}}\right]$$
$$= e^{-2x}(-2x^2-2x)$$
$$= -2x\,e^{-2x}(x+1)$$

(v)
$$y = \ln\left(\frac{e^x}{1+e^x}\right)$$
$$y = \ln e^x - \ln(1+e^x)$$
$$y = x(\ln e) - \ln(1+e^x)$$
$$y = x - \ln(1+e^x)$$
$$\frac{dy}{dx} = \frac{d}{dx}(x) - \frac{d}{dx}\left(\ln(1+e^x)\right)$$
$$= 1 - \frac{1}{1+e^x}\frac{d}{dx}(1+e^x)$$
$$= 1 - \frac{e^x}{1+e^x} = \frac{1+\cancel{e^x}-\cancel{e^x}}{1+e^x}$$
$$= \frac{1}{1+e^x}$$

(vi) $y = \sqrt{\dfrac{1+e^x}{1-e^x}}$

$$\ln y = \ln\left(\frac{1+e^x}{1-e^x}\right)^{1/2}$$

$$\ln y = \frac{1}{2}\ln\left(\frac{1+e^x}{1-e^x}\right)$$

$$\ln y = \frac{1}{2}\left[\ln(1+e^x) - \ln(1-e^x)\right]$$

$$\frac{d}{dx}\ln y = \frac{1}{2}\left[\frac{d}{dx}\ln(1+e^x) - \frac{d}{dx}\ln(1-e^x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{1+e^x}\cdot e^x - \frac{1}{1-e^x}(-e^x)\right]$$

$$\frac{dy}{dx} = y\cdot\frac{e^x}{2}\left[\frac{1}{1+e^x} + \frac{1}{1-e^x}\right]$$

$$\frac{dy}{dx} = \left(\frac{1+e^x}{1-e^x}\right)^{1/2}\cdot\frac{e^x}{2}\left[\frac{1-\cancel{e^x}+1+\cancel{e^x}}{(1+e^x)(1-e^x)}\right]$$

$$\frac{dy}{dx} = \frac{(1+e^x)^{1/2}}{(1-e^x)^{1/2}}\cdot\frac{e^x}{\cancel{2}}\cdot\frac{\cancel{2}}{(1+e^x)(1-e^x)}$$

$$= \frac{e^x}{(1-e^x)^{\frac{1}{2}+1}\cdot(1+e^x)^{1-\frac{1}{2}}}$$

$$= \frac{e^x}{(1-e^x)^{3/2}\cdot(1+e^x)^{1/2}}$$

$$= \frac{e^x}{(1-e^x)(1-e^x)^{1/2}(1+e^x)^{1/2}} = \frac{e^x}{(1-e^x)\sqrt{1-e^{2x}}}$$

③ Find $\frac{dy}{dx}$ if

(i) $y = \frac{1-x^2}{\sqrt{1+x^2}}$

$$\ln y = \ln\left(\frac{1-x^2}{\sqrt{1+x^2}}\right) = \ln(1-x^2) - \ln(1+x^2)^{1/2}$$

$$\ln y = \ln(1-x^2) - \frac{1}{2}\ln(1+x^2)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\ln(1-x^2) - \frac{1}{2}\frac{d}{dx}\ln(1+x^2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{1-x^2}(-2x) - \frac{1}{\cancel{2}}\cdot\frac{1}{1+x^2}(\cancel{2}x)$$

$$\frac{dy}{dx} = y\left[\frac{-2x}{1-x^2} - \frac{x}{1+x^2}\right]$$

$$= \frac{\cancel{1-x^2}}{\sqrt{1+x^2}}\left[\frac{-2x(1+x^2) - x(1-x^2)}{\cancel{(1-x^2)}(1+x^2)}\right]$$

$$= \frac{-2x - 2x^3 - x + x^3}{(1+x^2)^{\frac{1}{2}+1}}$$

$$= \frac{-x^3 - 3x}{(1+x^2)^{3/2}}$$

(ii) $y = \sqrt{\frac{1-x}{1+x}}$

$$\ln y = \ln\left(\frac{1-x}{1+x}\right)^{1/2}$$

$$\ln y = \frac{1}{2}\left[\ln(1-x) - \ln(1+x)\right]$$

$$\frac{d}{dx}\ln y = \frac{1}{2}\left[\frac{d}{dx}\ln(1-x) - \frac{d}{dx}\ln(1+x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{1-x}(-1) - \frac{1}{1+x}(1)\right]$$

$$\frac{dy}{dx} = y\cdot\frac{1}{2}\left[\frac{-1-\cancel{x}-1+\cancel{x}}{(1-x)(1+x)}\right]$$

$$\frac{dy}{dx} = \frac{(1-x)^{1/2}}{(1+x)^{1/2}}\cdot\frac{1}{\cancel{2}}\left[\frac{-\cancel{2}}{(1-x)(1+x)}\right]$$

$$= \frac{-1}{(1+x)^{1/2+1}(1-x)^{1-1/2}} = \frac{-1}{(1+x)^{3/2}(1-x)^{1/2}}.$$

(4) Find $\frac{dy}{dx}$ if

(i) $y = x^{\sin x}$

$$\ln y = \ln x^{\sin x}$$

$$\ln y = \sin x . \ln x$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(\sin x . \ln x\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x . \frac{d}{dx}(\ln x) + \ln x . \frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = y\left[\sin x . \frac{1}{x} + \ln x . \cos x\right]$$

$$= x^{\sin x}\left[\frac{\sin x}{x} + \cos x . \ln x\right]$$

(ii) $y = (\sin^{-1}x)^{\ln x}$

$$\ln y = \ln (\sin^{-1}x)^{\ln x}$$

$$\ln y = \ln x . \ln(\sin^{-1}x)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left[\ln x . \ln(\sin^{-1}x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \ln x . \frac{d}{dx}\left(\ln(\sin^{-1}x)\right) + \ln(\sin^{-1}x) . \frac{d}{dx}(\ln x)$$

$$\frac{dy}{dx} = y\left[\ln x . \frac{1}{\sin^{-1}x} . \frac{1}{\sqrt{1-x^2}} + \ln \sin^{-1}x . \frac{1}{x}\right]$$

$$\frac{dy}{dx} = (\sin^{-1}x)^{\ln x}\left[\frac{\ln x}{\sqrt{1-x^2}\,\sin^{-1}x} + \frac{\ln(\sin^{-1}x)}{x}\right]$$

(iii) $y = (\tan^{-1}x)^{\sin x + \cos x}$

$$\ln y = \ln (\tan^{-1}x)^{\sin x + \cos x}$$

$$\ln y = (\sin x + \cos x) \ln (\tan^{-1}x)$$

$$\frac{d}{dx} \ln y = \frac{d}{dx}\left[\underline{(\sin x + \cos x)}\ \underline{\ln (\tan^{-1}x)} \right]$$

$$\frac{1}{y}\frac{dy}{dx} = (\sin x + \cos x)\frac{d}{dx}\left(\ln(\tan^{-1}x)\right) + \ln(\tan^{-1}x)\cdot\frac{d}{dx}(\sin x + \cos x)$$

$$\frac{dy}{dx} = y\left[(\sin x + \cos x)\frac{1}{\tan^{-1}x}\cdot\frac{1}{1+x^2} + \ln(\tan^{-1}x)(\cos x - \sin x)\right]$$

$$\frac{dy}{dx} = (\tan^{-1}x)^{\sin x + \cos x}\left[\frac{\sin x + \cos x}{(1+x^2)\tan^{-1}x} + (\cos x - \sin x)\ln(\tan^{-1}x)\right]$$

(iv) $y = (\ln x)^{\cos x}$

$$\ln y = \ln (\ln x)^{\cos x}$$

$$\ln y = \cos x \cdot \ln(\ln x)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\underline{\cos x}\cdot \underline{\ln(\ln x)}\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x\cdot\frac{d}{dx}\left(\ln(\ln x)\right) + \ln(\ln x)\cdot\frac{d}{dx}(\cos x)$$

$$\frac{dy}{dx} = y\left[\cos x\cdot\frac{1}{\ln x}\cdot\frac{1}{x} + \ln(\ln x)\cdot(-\sin x)\right]$$

$$= (\ln x)^{\cos x}\left[\frac{\cos x}{x\ln x} - \sin x\cdot\ln(\ln x)\right]$$

(v) $y = x^x$

$$\ln y = \ln x^x = x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}(\ln x) + \ln x \cdot \frac{d}{dx}(x)$$

$$\frac{dy}{dx} = y\left[\cancel{x}\frac{1}{\cancel{x}} + \ln x \cdot 1\right]$$

$$\frac{dy}{dx} = x^x(1 + \ln x).$$

(vi) $y = \ln\left(\dfrac{\sqrt{x^2+1} - x}{\sqrt{x^2+1} + x}\right)$

$$y = \ln\left(\sqrt{x^2+1} - x\right) - \ln\left(\sqrt{x^2+1} + x\right)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln\left(\sqrt{x^2+1} - x\right)\right] - \frac{d}{dx}\left[\ln\left(\sqrt{x^2+1} + x\right)\right]$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1} - x}\cdot\frac{d}{dx}\left((x^2+1)^{\frac{1}{2}} - x\right) - \frac{1}{\sqrt{x^2+1} + x}\cdot\frac{d}{dx}\left((x^2+1)^{\frac{1}{2}} + x\right)$$

$$= \frac{1}{\sqrt{x^2+1} - x}\left[\frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}}(\cancel{2}x) - 1\right] - \frac{1}{\sqrt{x^2+1} + x}\left[\frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}}(\cancel{2}x) + 1\right]$$

$$= \frac{1}{\sqrt{x^2+1} - x}\left[\frac{x}{\sqrt{x^2+1}} - 1\right] - \frac{1}{\sqrt{x^2+1} + x}\left[\frac{x}{\sqrt{x^2+1}} + 1\right]$$

$$= \frac{-1}{\cancel{\sqrt{x^2+1} - x}}\left(\frac{\cancel{-x + \sqrt{x^2+1}}}{\sqrt{x^2+1}}\right) - \frac{1}{\cancel{\sqrt{x^2+1} + x}}\left(\frac{\cancel{x + \sqrt{x^2+1}}}{\sqrt{x^2+1}}\right)$$

$$= \frac{-1}{\sqrt{x^2+1}} - \frac{1}{\sqrt{x^2+1}}$$

$$= \frac{-2}{\sqrt{x^2+1}}$$

(5) Find $\frac{dy}{dx}$, when:

(i) $x^y \cdot y^x = 1$

$$y^x = \frac{1}{x^y} = x^{-y}$$

$$\ln y^x = \ln x^{-y}$$

$$x \ln y = -y \ln x$$

$$\frac{d}{dx}\left(x \ln y\right) = -\frac{d}{dx}\left(y \ln x\right)$$

$$x\frac{d}{dx}(\ln y) + \ln y \frac{d}{dx}(x) = -\left[y\frac{d}{dx}(\ln x) + \ln x \cdot \frac{dy}{dx}\right]$$

$$x \cdot \frac{1}{y}\frac{dy}{dx} + \ln y = -y\frac{1}{x} - \ln x \frac{dy}{dx}$$

$$\frac{x}{y}\frac{dy}{dx} + \ln x \frac{dy}{dx} = -\frac{y}{x} - \ln y$$

$$\frac{dy}{dx}\left(\frac{x}{y} + \ln x\right) = -\left(\frac{y}{x} + \ln y\right)$$

$$\frac{dy}{dx}\left(\frac{x + y\ln x}{y}\right) = -\left(\frac{y + x\ln y}{x}\right)$$

$$\frac{dy}{dx} = -\frac{y}{x}\frac{(y + x\ln y)}{(x + y\ln x)}.$$

(ii) $\ln(xy) = x^2 + y^2$

$$\ln x + \ln y = x^2 + y^2$$

$$\frac{d}{dx}(\ln x + \ln y) = \frac{d}{dx}(x^2 + y^2)$$

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 2x + 2y\frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x - \frac{1}{x}$$

$$\frac{dy}{dx}\left(\frac{1}{y} - 2y\right) = 2x - \frac{1}{x}$$

$$\frac{dy}{dx}\left(\frac{1-2y^2}{y}\right) = \frac{2x^2-1}{x}$$

$$\frac{dy}{dx} = \frac{y}{x}\frac{(2x^2-1)}{(1-2y^2)} = \frac{-y(2x^2-1)}{x(2y^2-1)}$$

(iii) $y = \sin^{-1}(\cos x) + \cos^{-1}(\sin x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\cos^2 x}}\frac{d}{dx}(\cos x) - \frac{1}{\sqrt{1-\sin^2 x}} \cdot \frac{d}{dx}(\sin x)$$

$$= \frac{-\sin x}{\sqrt{\sin^2 x}} - \frac{\cos x}{\sqrt{\cos^2 x}}$$

$$= \frac{-\cancel{\sin x}}{\cancel{\sin x}} - \frac{1\,\cancel{\cos x}}{\cancel{\cos x}}$$

$$= -1 - 1 = -2.$$

(iv) $y = x^y$

$$\ln y = \ln x^y = y \ln x$$

$$\frac{d}{dx} \ln y = \frac{d}{dx}(y \ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = y \cdot \frac{1}{x} + \ln x \cdot \frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} - \ln x \frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{dx}\left(\frac{1}{y} - \ln x\right) = \frac{y}{x}$$

$$\frac{dy}{dx}\left(\frac{1 - y\ln x}{y}\right) = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y^2}{x(1 - y \ln x)}.$$

(v) $y = \cos x \cdot \ln(\sin^{-1}x)$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos x \cdot \ln(\sin^{-1}x)\right)$$

$$= \cos x \cdot \frac{1}{\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}} + \ln(\sin^{-1}x) \cdot (-\sin x)$$

$$= \frac{\cos x}{\sqrt{1-x^2}\,\sin^{-1}x} - \sin x \ln(\sin^{-1}x).$$

(vi) $x^n \cdot y^n = a^n$

$$(xy)^n = a^n$$

$$xy = a$$

$$y = \frac{a}{x}$$

$$\frac{dy}{dx} = a\frac{d}{dx}(x^{-1}) = a(-x^{-2}) = -\frac{a}{x^2}.$$

Exercise 3.6

Sr. No	$y = f(x)$	$\frac{dy}{dx}$
1.	$y = \sinh x$	$\cosh x$
2.	$y = \cosh x$	$\sinh x$
3.	$y = \tanh x$	$\text{sech}^2 x$
4.	$y = \coth x$	$-\text{csch}^2 x$
5.	$y = \text{csch}\, x$	$-\text{csch}\, x \,.\coth x$
6.	$y = \text{sech}\, x$	$-\text{sech}\, x \,.\tanh x$
7.	$y = \sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}, -\infty < x < \infty$
8.	$y = \cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}, \ x > 1$
9.	$y = \tanh^{-1} x$	$\frac{1}{1-x^2}, \lvert x \rvert < 1$
10.	$y = \coth^{-1} x$	$\frac{1}{1-x^2}, \lvert x \rvert > 1$
11.	$y = \text{csch}^{-1} x$	$-\frac{1}{x\sqrt{1+x^2}}, x > 0$
12.	$y = \text{sech}^{-1} x$	$-\frac{1}{x\sqrt{1-x^2}}, 0 < x < 1$

Exercise # 3.6

Q1 Differentiate the following w.r.t. 'x'.

(i) Let $y = \sinh[\ln(x+3)]$

$$\frac{dy}{dx} = \frac{d}{dx}\sinh[\ln(x+3)]$$

$$= \cosh[\ln(x+3)] \cdot \frac{d}{dx}\ln(x+3)$$

$$= \cosh[\ln(x+3)] \cdot \frac{1}{x+3} \cdot \frac{d}{dx}(x+3)$$

$$= \cosh(\ln(x+3)) \cdot \frac{1}{x+3} \cdot 1$$

$$= \frac{\cosh[\ln(x+3)]}{x+3}.$$

(ii) Let $y = \sinh(e^{3x})$

$(\sinh x)' = \cosh x$

$$\frac{dy}{dx} = \frac{d}{dx}\sinh(e^{3x})$$

$$= \cosh(e^{3x}) \cdot \frac{d}{dx}(e^{3x})$$

$(e^x)' = e^x$

$$= \cosh(e^{3x}) \cdot e^{3x}\frac{d}{dx}(3x)$$

$$= \cosh(e^{3x}) \cdot e^{3x} \cdot 3$$

$$= 3e^{3x}\cosh(e^{3x}).$$

(iii) Let $y = \cosh(2x^2+3x)$

$$\frac{dy}{dx} = \frac{d}{dx}\cosh(2x^2+3x)$$

$\frac{d}{dx}(\cosh x) = \sinh x$

$$= \sinh(2x^2+3x) \cdot \frac{d}{dx}(2x^2+3x)$$

$$= \sinh(2x^2+3x) \cdot (4x+3)$$

$$= (4x+3)\sinh(2x^2+3x).$$

(iv) Let $y = \dfrac{\tanh\sqrt{x}}{\sqrt{\cosh x}}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\tanh\sqrt{x}}{\sqrt{\cosh x}}\right)$$

$$= \frac{\sqrt{\cosh x}\cdot\frac{d}{dx}\left(\tanh\sqrt{x}\right) - \tanh\sqrt{x}\,\frac{d}{dx}\left((\cosh x)^{\frac{1}{2}}\right)}{\left(\sqrt{\cosh x}\right)^2}$$

$(\tanh x)' = \operatorname{sech}^2 x$

$$= \frac{1}{\cosh x}\left[\sqrt{\cosh x}\ \operatorname{sech}^2\sqrt{x}\,\frac{d}{dx}\left(x^{\frac{1}{2}}\right) - \tanh\sqrt{x}\cdot\frac{1}{2}(\cosh x)^{\frac{1}{2}-1}\frac{d}{dx}(\cosh x)\right]$$

$$= \frac{1}{\cosh x}\left[\sqrt{\cosh x}\cdot\operatorname{sech}^2\sqrt{x}\cdot\frac{1}{2}\frac{1}{\sqrt{x}} - \tanh\sqrt{x}\cdot\frac{1}{2}\frac{1}{\sqrt{\cosh x}}\cdot\sinh x\right]$$

$$= \frac{1}{\cosh x}\left[\frac{\sqrt{\cosh x}\cdot\operatorname{sech}^2\sqrt{x}}{2\sqrt{x}} - \frac{\tanh\sqrt{x}\cdot\sinh x}{2\sqrt{\cosh x}}\right]$$

$$= \frac{1}{\cosh x}\left[\frac{\sqrt{\cosh x}\ \operatorname{sech}^2\sqrt{x}\,\sqrt{\cosh x} - \tanh\sqrt{x}\cdot\sinh x\,\sqrt{x}}{2\sqrt{x}\,\sqrt{\cosh x}}\right]$$

$$= \frac{\cosh x\cdot\operatorname{sech}^2\sqrt{x} - \sqrt{x}\,\sinh x\,\tanh\sqrt{x}}{2\sqrt{x}\,(\cosh x)^{3/2}}$$

(v) Let $y = \tan\left(e^{\sinh^{-1}x}\right)$

$$\frac{dy}{dx} = \frac{d}{dx}\tan\left(e^{\sinh^{-1}x}\right)$$

$$= \sec^2\left(e^{\sinh^{-1}x}\right)\cdot\frac{d}{dx}e^{\sinh^{-1}x}$$

$(e^x)' = e^x$

$$= \sec^2\left(e^{\sinh^{-1}x}\right)\cdot e^{\sinh^{-1}x}\cdot\frac{d}{dx}\left(\sinh^{-1}x\right)$$

$$= \sec^2\left(e^{\sinh^{-1}x}\right)\cdot e^{\sinh^{-1}x}\cdot\frac{1}{\sqrt{x^2+1}}$$

$(\sinh^{-1}x)' = \dfrac{1}{\sqrt{x^2+1}}$

$$= \frac{e^{\sinh^{-1}x}}{\sqrt{x^2+1}}\cdot\sec^2\left(e^{\sinh^{-1}x}\right).$$

(vi) Let $y = \dfrac{\sinh^{-1}x}{\operatorname{sech}^{-1}x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sinh^{-1}x}{\operatorname{sech}^{-1}x}\right)$$

$$= \frac{(\operatorname{sech}^{-1}x)\frac{d}{dx}(\sinh^{-1}x) - (\sinh^{-1}x)\frac{d}{dx}(\operatorname{sech}^{-1}x)}{(\operatorname{sech}^{-1}x)^2}$$

$(\sinh^{-1}x)' = \dfrac{1}{\sqrt{x^2+1}}$

$$= \frac{1}{(\operatorname{sech}^{-1}x)^2}\left[(\operatorname{sech}^{-1}x)\cdot\frac{1}{\sqrt{x^2+1}} - \sinh^{-1}x\cdot\left(\frac{-1}{x\sqrt{1-x^2}}\right)\right]$$

$(\operatorname{sech}^{-1}x)' = \dfrac{-1}{x\sqrt{1-x^2}}$

$$= \frac{1}{(\operatorname{sech}^{-1}x)^2}\left[\frac{\operatorname{sech}^{-1}x}{\sqrt{1+x^2}} + \frac{\sinh^{-1}x}{x\sqrt{1-x^2}}\right]$$

$$= \frac{1}{(\operatorname{sech}^{-1}x)^2}\left[\frac{\operatorname{sech}^{-1}x\cdot x\sqrt{1-x^2} + \sinh^{-1}x\cdot\sqrt{1+x^2}}{x\sqrt{1+x^2}\sqrt{1-x^2}}\right]$$

$$= \frac{x\sqrt{1-x^2}\operatorname{sech}^{-1}x + \sqrt{1+x^2}\sinh^{-1}x}{x\sqrt{1-x^4}\,(\operatorname{sech}^{-1}x)^2}$$

(vii) Let $y = \cosh x\cdot\coth x^2$

$$\frac{dy}{dx} = \frac{d}{dx}(\cosh x\cdot\coth x^2)$$

$$= (\cosh x)\frac{d}{dx}(\coth x^2) + \coth x^2\cdot\frac{d}{dx}(\cosh x)$$

$(\coth x)' = -\operatorname{cosech}^2 x$

$$= \cosh x\cdot(-\operatorname{cosech}^2 x^2)\cdot\frac{d}{dx}(x^2) + \coth x^2\cdot\sinh x$$

$$= -2x\cosh x\operatorname{cosech}^2 x^2 + \coth x^2\cdot\sinh x.$$

(viii) Let $y = \sinh x \cdot \tanh x^2$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sinh x \cdot \tanh x^2\right)$$

$$= (\sinh x)\frac{d}{dx}(\tanh x^2) + \tanh x^2 \cdot \frac{d}{dx}(\sinh x)$$

$$= (\sinh x)(\text{sech}^2 x^2)\frac{d}{dx}(x^2) + \tanh x^2 \cdot \cosh x$$

$$= 2x \sinh x \,\text{sech}^2 x^2 + \tanh x^2 \cosh x.$$

(ix) Let $y = \ln[\tanh(x^2+2x+1)]$

$$\frac{dy}{dx} = \frac{d}{dx}\ln[\tanh(x^2+2x+1)]$$

$(\ln f)' = \frac{1}{f}\cdot f'$

$$\frac{dy}{dx} = \frac{1}{\tanh(x^2+2x+1)}\cdot\frac{d}{dx}\left(\tanh(x^2+2x+1)\right)$$

$$= \coth(x^2+2x+1)\ \text{sech}^2(x^2+2x+1)\frac{d}{dx}(x^2+2x+1)$$

$$= \frac{\cancel{\cosh(x^2+2x+1)}}{\sinh(x^2+2x+1)}\cdot\frac{1}{\cosh^{\cancel{2}}(x^2+2x+1)}\cdot(2x+2)$$

$$= 2(x+1)\ \text{cosech}(x^2+2x+1)\ \text{sech}(x^2+2x+1).$$

② Find $\frac{dy}{dx}$ for the following functions.

(i) $y = x\cosh^{-1}x - \sqrt{x^2-1}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x\cosh^{-1}x\right) - \frac{d}{dx}\left((x^2-1)^{1/2}\right)$$

$$= x\frac{d}{dx}(\cosh^{-1}x) + \cosh^{-1}x\frac{d}{dx}(x) - \frac{1}{2}(x^2-1)^{\frac{1}{2}-1}\frac{d}{dx}(x^2-1)$$

$$= x\frac{1}{\sqrt{x^2-1}} + \cosh^{-1}x - \frac{1}{\cancel{2}}(x^2-1)^{-1/2}(\cancel{2}x)$$

$$= \cancel{\frac{x}{\sqrt{x^2-1}}} + \cosh^{-1}x - \cancel{\frac{x}{\sqrt{x^2-1}}}$$

$$= \cosh^{-1}x.$$

(ii) $y = x\tanh^{-1}(3x)$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x\tanh^{-1}(3x)\right)$$

$$= x\frac{d}{dx}\left(\tanh^{-1}(3x)\right) + \tanh^{-1}(3x)\cdot\frac{d}{dx}(x)$$

$$= x\cdot\frac{1}{1-(3x)^2}\cdot\frac{d}{dx}(3x) + \tanh^{-1}(3x)\cdot 1$$

$$= \frac{3x}{1-9x^2} + \tanh^{-1}(3x)$$

$$= \frac{3x + (1-9x^2)\tanh^{-1}(3x)}{1-9x^2}$$

(iii) $$\ln(\cosh^{-1}x) + \sinh^{-1}y = C$$

$$\frac{d}{dx}\left[\ln(\cosh^{-1}x)\right] + \frac{d}{dx}(\sinh^{-1}y) = \frac{d}{dx}(C)$$

$$\frac{1}{\cosh^{-1}x}\cdot\frac{d}{dx}(\cosh^{-1}x) + \frac{1}{\sqrt{y^2+1}}\cdot\frac{dy}{dx} = 0$$

$$\frac{1}{\cosh^{-1}x}\cdot\frac{1}{\sqrt{x^2-1}} + \frac{1}{\sqrt{y^2+1}}\,\frac{dy}{dx} = 0$$

$$\frac{1}{\sqrt{y^2+1}}\,\frac{dy}{dx} = -\frac{1}{\sqrt{x^2-1}\,\cosh^{-1}x}$$

$$\frac{dy}{dx} = \frac{-\sqrt{y^2+1}}{\sqrt{x^2-1}\,\cosh^{-1}x}.$$

(iv) $y = \ln(1-x^2) + 2x\tanh^{-1}x$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln(1-x^2)\right] + 2\frac{d}{dx}\left[x\tanh^{-1}x\right]$$

$$= \frac{1}{1-x^2}\cdot\frac{d}{dx}(1-x^2) + 2\left[x\frac{d}{dx}(\tanh^{-1}x) + \tanh^{-1}x\cdot\frac{d}{dx}(x)\right]$$

$$= \frac{1}{1-x^2}(-2x) + 2\left[x\cdot\frac{1}{1-x^2} + \tanh^{-1}x\right]$$

$$= \cancel{\frac{-2x}{1-x^2}} + \cancel{\frac{2x}{1-x^2}} + 2\tanh^{-1}x$$

$$= 2\tanh^{-1}x.$$

$\cancel{\tan^{-1}}\left(\cancel{\tan} -\right)$

(v) $y = \tanh^{-1}(\tan x^3)$

$(\tanh^{-1}x)' = \dfrac{1}{1-x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tanh^{-1}(\tan x^3)\right]$$

$$= \frac{1}{1-(\tan x^3)^2} \cdot \frac{d}{dx}(\tan x^3)$$

$$= \frac{1}{1-\tan^2 x^3} \cdot \sec^2 x^3 \cdot \frac{d}{dx}(x^3)$$

$$= \frac{1}{1-\tan^2 x^3} \cdot \sec^2 x^3 \cdot 3x^2$$

$$= \frac{3x^2 \sec^2 x^3}{1-\tan^2 x^3}.$$

(vi) $y = x \operatorname{sech}^{-1}(\sqrt{x})$

$(\operatorname{sech}^{-1}x)' = \dfrac{-1}{x\sqrt{1-x^2}}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x \operatorname{sech}^{-1}(\sqrt{x})\right)$$

$$= x\frac{d}{dx}\left(\operatorname{sech}^{-1}\sqrt{x}\right) + \operatorname{sech}^{-1}\sqrt{x} \cdot \frac{d}{dx}(x)$$

$$= x \cdot \frac{-1}{\sqrt{x}\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}\left(x^{\frac{1}{2}}\right) + \operatorname{sech}^{-1}\sqrt{x}$$

$$= \frac{-x}{\sqrt{x}\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} + \operatorname{sech}^{-1}\sqrt{x}$$

$$= \frac{-1}{2\sqrt{1-x}} + \operatorname{sech}^{-1}\sqrt{x}$$

$$= \frac{-1+2\sqrt{1-x}\operatorname{sech}^{-1}\sqrt{x}}{2\sqrt{1-x}}.$$