

MATHEMATICS 2nd YEAR

UNIT #

02

DIFFERENTIATION

EDUCATION



Muhammad Salman Sherazi

M.Phil (Math)

Contents	
Exercise	Page #
Exercise 2.1	3
Exercise 2.2	7
Exercise 2.3	10
Exercise 2.4	15
Exercise 2.5	22
Exercise 2.6	33
Exercise 2.7	39
Exercise 2.8	45
Exercise 2.9	50
Exercise 2.10	55

Sherazi Mathematics

اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Average rate of change

Let f be a real valued function the (difference quotient) $\frac{f(x_1) - f(x)}{x_1 - x}$ is called average rate of change.

Derivative:- Let $f(x)$ be a function, then its derivative is denoted by $f'(x)$ or $\frac{d}{dx}f(x)$ and defined as; $f'(x) = \lim_{\delta x \rightarrow 0} \frac{f(x+\delta x) - f(x)}{\delta x}$.

* The process of finding f' is called differentiation.

Notation for Derivative

For the function $y = f(x)$

$$\rightarrow y + \delta y = f(x + \delta x) \rightarrow \delta y = f(x + \delta x) - f(x) \rightarrow (i)$$

where δy is the increment of y (change in the value of y corresponding to δx (increment of x)).

Dividing (i) by δx on both sides, we get

$$\rightarrow \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Taking limit on both sides as $\delta x \rightarrow 0$, we get

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} \rightarrow (ii)$$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}$ is denoted by $\frac{dy}{dx}$, so (ii) is written

$$\text{as } \frac{dy}{dx} = f'(x)$$

Note:- The symbol $\frac{dy}{dx}$ is used for the derivative of y with respect to x and here it is not a quotient of dy and dx . $\frac{dy}{dx}$ is also denoted by f' .

* Different mathematicians used symbols given below.

Leibniz $\frac{dy}{dx}$ or $\frac{df}{dx}$

Newton $f'(x)$

Lagrange $f'(x)$

Cauchy $Df(x)$

Finding $f'(x)$ from definition of derivative

derivative:- Given a function f , $f'(x)$ if it exists, can be found by the following four steps

Step I:- Find $f(x + \delta x)$ **Step II:-** Simplify $f(x + \delta x) - f(x)$

Step III:- Divide $f(x + \delta x) - f(x)$ by δx to get $\frac{f(x + \delta x) - f(x)}{\delta x}$ and simplify it.

Step IV:- Find $\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$ The method of finding derivatives by this process is called differentiation by definition or by ab-initio or from first principle.

Example 1. Find the derivative of the following functions by definition (a) $f(x) = c$ (b) $f(x) = x^2$

Solution:- (a) $f(x) = c \rightarrow (i)$

$$\rightarrow f(x + \delta x) = c \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow f(x + \delta x) - f(x) = c - c$$

$$\rightarrow f(x + \delta x) - f(x) = 0$$

Dividing both sides by δx

$$\rightarrow \frac{f(x + \delta x) - f(x)}{\delta x} = \frac{0}{\delta x} \rightarrow \frac{f(x + \delta x) - f(x)}{\delta x} = 0$$

Applying limit as $\delta x \rightarrow 0$, so

$$\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{\delta x \rightarrow 0} (0)$$

$\rightarrow f'(x) = 0$ i.e., Derivative of a constant function is zero.

(b) $f(x) = x^2 \rightarrow (i)$

$$\rightarrow f(x + \delta x) = (x + \delta x)^2 \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow f(x + \delta x) - f(x) = (x + \delta x)^2 - x^2$$

$$\rightarrow f(x + \delta x) - f(x) = x^2 + (\delta x)^2 + 2x(\delta x) - x^2$$

$$= (\delta x)^2 + 2x(\delta x) = \delta x(\delta x + 2x)$$

$$\rightarrow f(x + \delta x) - f(x) = \delta x(\delta x + 2x)$$

Dividing both sides by δx

$$\rightarrow \frac{f(x + \delta x) - f(x)}{\delta x} = \frac{\delta x(\delta x + 2x)}{\delta x}$$

Applying limit as $\delta x \rightarrow 0$, so

$$\rightarrow \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{\delta x \rightarrow 0} (\delta x + 2x)$$

$$\rightarrow f'(x) = 2x$$

Example 2. Find the derivative of $\sqrt{x}$ at $x=a$ from first principles.

Solution:- Let $y = \sqrt{x} \rightarrow (i)$

$$\rightarrow y + \delta y = \sqrt{x + \delta x} \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = \sqrt{x + \delta x} - \sqrt{x}$$

$$\rightarrow \delta y = \frac{(\sqrt{x + \delta x} - \sqrt{x})(\sqrt{x + \delta x} + \sqrt{x})}{\sqrt{x + \delta x} + \sqrt{x}}$$

$$\delta y = \frac{x + \delta x - x}{\sqrt{x + \delta x} + \sqrt{x}} = \frac{\delta x}{\sqrt{x + \delta x} + \sqrt{x}}$$

Dividing both sides by δx

$$\rightarrow \frac{\delta y}{\delta x} = \frac{1}{\sqrt{x + \delta x} + \sqrt{x}}$$

Applying limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{\sqrt{x+\delta x} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \quad \text{Now } \left. \frac{dy}{dx} \right|_{x=a} = \frac{1}{2\sqrt{a}}$$

Example 3. If $y = \frac{1}{x^2}$, then find $\frac{dy}{dx}$ at $x = -1$ by ab-initio method.

Solution:- $y = \frac{1}{x^2} \rightarrow (i)$

$$\rightarrow y + \delta y = \frac{1}{(x+\delta x)^2} \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = \frac{1}{(x+\delta x)^2} - \frac{1}{x^2} = \frac{x^2 - (x+\delta x)^2}{(x+\delta x)^2 \cdot x^2}$$

$$\rightarrow \delta y = \frac{x^2 - (x^2 + 2x(\delta x) + (\delta x)^2)}{(x+\delta x)^2 \cdot x^2} = \frac{-2x(\delta x) - (\delta x)^2}{(x+\delta x)^2 \cdot x^2}$$

$$\delta y = -\frac{\delta x(x+2x)}{(x+\delta x)^2 \cdot x^2}$$

$$\rightarrow \frac{\delta y}{\delta x} = -\frac{\delta x(x+2x)}{\delta x(x+\delta x)^2 \cdot x^2} \quad (\text{Dividing both sides by } \delta x)$$

Applying limit as $\delta x \rightarrow 0$

$$\rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} -\frac{(x+2x)}{(x+\delta x)^2 \cdot x^2} = -\frac{2x}{x^2 \cdot x^2}$$

$$\rightarrow \frac{dy}{dx} = -\frac{2}{x^3} \quad \text{Now } \left. \frac{dy}{dx} \right|_{x=-1} = \frac{-2}{(-1)^3} = \frac{-2}{-1} = 2$$

Example 4. Find the derivative of $x^{2/3}$ and also calculate the value of derivative at $x = 8$.

Solution:- Let $y = x^{2/3} \rightarrow (i)$

$$\rightarrow y + \delta y = (x+\delta x)^{2/3} \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = (x+\delta x)^{2/3} - x^{2/3}$$

$$\rightarrow \delta y = [x(1 + \frac{\delta x}{x})]^{2/3} - x^{2/3} = x^{2/3} (1 + \frac{\delta x}{x})^{2/3} - x^{2/3}$$

$$\rightarrow \delta y = x^{2/3} \left\{ (1 + \frac{\delta x}{x})^{2/3} - 1 \right\}$$

$$\because (1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$$

$$= x^{2/3} \left\{ 1 + \frac{2}{3} (\frac{\delta x}{x}) + \frac{2(\frac{2}{3}-1)}{2!} (\frac{\delta x}{x})^2 + \dots \right\}$$

$$\delta y = x^{2/3} \frac{2}{3} (\frac{\delta x}{x}) \left\{ 1 + \frac{(\frac{2}{3}-1)}{2!} (\frac{\delta x}{x}) + \dots \right\}$$

Dividing both sides by δx

$$\rightarrow \frac{\delta y}{\delta x} = \frac{2}{3} \frac{x^{2/3}}{x} \left\{ 1 + \frac{(\frac{2}{3}-1)}{2!} (\frac{\delta x}{x}) + \dots \right\}$$

Applying limit as $\delta x \rightarrow 0$

$$\rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{2}{3x^{1/3}} \left\{ 1 + \frac{(\frac{2}{3}-1)}{2!} (\frac{\delta x}{x}) + \dots \right\}$$

$$\rightarrow \frac{dy}{dx} = \frac{2}{3x^{1/3}} \left\{ 1 + 0 + \dots \right\}$$

$$\frac{dy}{dx} = \frac{2}{3x^{1/3}} \quad \text{Now } \left. \frac{dy}{dx} \right|_{x=8} = \frac{2}{3(8)^{1/3}} = \frac{2}{3(2)^3} = \frac{2}{3(2)} = \frac{1}{3}$$

Example 5. Find the derivative of $x^3 + 2x + 3$

Solution:- Let $y = x^3 + 2x + 3 \rightarrow (i)$

$$\rightarrow y + \delta y = (x+\delta x)^3 + 2(x+\delta x) + 3 \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = (x+\delta x)^3 + 2(x+\delta x) + 3 - x^3 - 2x - 3$$

$$\rightarrow \delta y = x^3 + (\delta x)^3 + 3x(\delta x)^2 + 3x^2(\delta x) + 2x + 2(\delta x) + 3 - x^3 - 2x - 3$$

$$\rightarrow \delta y = \delta x((\delta x)^2 + 3x(\delta x) + 3x^2 + 2)$$

Dividing both sides by δx $\frac{(a+b)^3}{= a^3 + b^3 + 3a^2b + 3ab^2}$

$$\frac{\delta y}{\delta x} = \frac{(\delta x)^2 + 3x(\delta x) + 3x^2 + 2}{\delta x}$$

Applying limit as $\delta x \rightarrow 0$

$$\rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} ((\delta x)^2 + 3x(\delta x) + 3x^2 + 2)$$

$$\rightarrow \frac{dy}{dx} = 3x^2 + 2$$

Derivation of x^n when $n \in \mathbb{Z}$

(a) Let $y = x^n$ when n is +iv integer.

$$\rightarrow y = x^n \rightarrow (i), \quad y + \delta y = (x+\delta x)^n \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = (x+\delta x)^n - x^n$$

$$\rightarrow \delta y = (x+\delta x)^n - x^n$$

using binomial theorem, we have

$$\rightarrow \delta y = [x^n + nx^{n-1}\delta x + \frac{n(n-1)}{2!} x^{n-2}(\delta x)^2 + \dots + (\delta x)^n] - x^n$$

$$= nx^{n-1}\delta x + \frac{n(n-1)}{2!} x^{n-2}(\delta x)^2 + \dots + (\delta x)^n - x^n$$

$$= \delta x \left[nx^{n-1} + \frac{n-1}{2!} x^{n-2}(\delta x) + \dots + (\delta x)^{n-1} \right]$$

Dividing both sides by δx

$$\rightarrow \frac{\delta y}{\delta x} = \frac{\delta x}{\delta x} \left[nx^{n-1} + \frac{n-1}{2!} x^{n-2}(\delta x) + \dots + (\delta x)^{n-1} \right]$$

Applying limit as $\delta x \rightarrow 0$

$$\rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[nx^{n-1} + \frac{n-1}{2!} x^{n-2}(\delta x) + \dots + (\delta x)^{n-1} \right]$$

$$\rightarrow \frac{dy}{dx} = nx^{n-1} + \frac{n-1}{2!} x^{n-2}(0) + \dots + (0)^{n-1}$$

$$\rightarrow \frac{dy}{dx} = nx^{n-1}$$

(b) Let $y = x^n$ where n is -ive integer.

Let $n = -m$ (m is +iv integer). Then

$$y = x^{-m} \rightarrow y = \frac{1}{x^m} \rightarrow (i)$$

$$\rightarrow y + \delta y = \frac{1}{(x + \delta x)^m} \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = \frac{1}{(x + \delta x)^m} - \frac{1}{x^m}$$

$$\begin{aligned} \delta y &= \frac{x^m - (x + \delta x)^m}{x^m (x + \delta x)^m} \quad \text{using binomial theorem} \\ &= \frac{x^m - (x^m + m x^{m-1} \delta x + \frac{m(m-1)}{2!} x^{m-2} (\delta x)^2 + \dots + (\delta x)^m)}{x^m (x + \delta x)^m} \\ &= \frac{-m x^{m-1} \delta x - \frac{m(m-1)}{2!} x^{m-2} (\delta x)^2 - \dots - (\delta x)^m}{x^m (x + \delta x)^m} \end{aligned}$$

$$\delta y = \frac{-\delta x}{x (x + \delta x)^m} \left(m x^{m-1} + \frac{m(m-1)}{2!} x^{m-2} (\delta x) + \dots + (\delta x)^{m-1} \right)$$

Dividing both sides by δx

$$\rightarrow \frac{\delta y}{\delta x} = \frac{-1}{x (x + \delta x)^m} \left(m x^{m-1} + \frac{m(m-1)}{2!} x^{m-2} (\delta x) + \dots + (\delta x)^{m-1} \right)$$

Applying limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[\frac{-1}{x (x + \delta x)^m} \left(m x^{m-1} + \frac{m(m-1)}{2!} x^{m-2} (\delta x) + \dots + (\delta x)^{m-1} \right) \right]$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{x \cdot x^m} \left(m x^{m-1} + \frac{m(m-1)}{2!} x^{m-2} (0) + \dots + (0)^{m-1} \right)$$

$$\frac{dy}{dx} = -\frac{m x^{m-1}}{x^{m+1}} = -m x^{m-1-2m} = -m x^{-m-1}$$

$$\rightarrow \frac{dy}{dx} = -m x^{-m-1} \quad \text{or} \quad \frac{dy}{dx} = n x^{n-1}$$

so for we have proved that $\frac{d}{dx}(x^n) = n x^{n-1}$, $n \in \mathbb{Z}$

Note that $\frac{d}{dx}(x^n) = n x^{n-1}$ is called power rule and holds if $n \in \mathbb{Q} - \mathbb{Z}$.

Exercise 2.1

Q1. Find by definition, the derivative w.r.t 'x' of the following functions defined as

(i) $2x^2 + 1$

Solution:- Let $y = 2x^2 + 1 \rightarrow (i)$

$$\rightarrow y + \delta y = 2(x + \delta x)^2 + 1 \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = 2(x + \delta x)^2 + 1 - (2x^2 + 1)$$

$$\begin{aligned} \delta y &= 2(x^2 + (\delta x)^2 + 2x(\delta x)) + 1 - 2x^2 - 1 \\ &= 2x^2 + 2(\delta x)^2 + 4x(\delta x) - 2x^2 \end{aligned}$$

$$\delta y = \delta x (2(\delta x) + 4x)$$

Dividing both sides by δx

$$\rightarrow \frac{\delta y}{\delta x} = \frac{\delta x (2(\delta x) + 4x)}{\delta x} = 2(\delta x) + 4x$$

Applying limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} [2(\delta x) + 4x] \rightarrow \frac{dy}{dx} = 4x$$

(ii) $2 - \sqrt{x}$

Solution:- Let $y = 2 - \sqrt{x} \rightarrow (i)$

$$\rightarrow y + \delta y = 2 - \sqrt{x + \delta x} \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = 2 - \sqrt{x + \delta x} - (2 - \sqrt{x})$$

$$\rightarrow \delta y = 2 - \sqrt{x + \delta x} - 2 + \sqrt{x} = \sqrt{x} - \sqrt{x + \delta x}$$

$$\rightarrow \delta y = \sqrt{x} - \sqrt{x + \delta x} \times \frac{\sqrt{x} + \sqrt{x + \delta x}}{\sqrt{x} + \sqrt{x + \delta x}}$$

$$\delta y = \frac{(\sqrt{x})^2 - (\sqrt{x + \delta x})^2}{\sqrt{x} + \sqrt{x + \delta x}} = \frac{x - (x + \delta x)}{\sqrt{x} + \sqrt{x + \delta x}}$$

$$\delta y = \frac{-\delta x}{\sqrt{x} + \sqrt{x + \delta x}}$$

Dividing both sides by δx

$$\rightarrow \frac{\delta y}{\delta x} = \frac{-1}{\delta x (\sqrt{x} + \sqrt{x + \delta x})} = \frac{-1}{\sqrt{x} + \sqrt{x + \delta x}}$$

Applying limit as $\delta x \rightarrow 0$,

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[\frac{-1}{\sqrt{x} + \sqrt{x + \delta x}} \right]$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{2\sqrt{x}}$$

(iii) $\frac{1}{\sqrt{x}}$

Solution:- Let $y = \frac{1}{\sqrt{x}} \rightarrow (i)$

$$\rightarrow y + \delta y = \frac{1}{\sqrt{x + \delta x}} \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = \frac{1}{\sqrt{x + \delta x}} - \frac{1}{\sqrt{x}}$$

$$\rightarrow \delta y = \frac{\sqrt{x} - \sqrt{x + \delta x}}{(\sqrt{x + \delta x})\sqrt{x}}$$

$$\delta y = \frac{\sqrt{x} - \sqrt{x + \delta x}}{(\sqrt{x + \delta x})(\sqrt{x})} \times \frac{\sqrt{x} + \sqrt{x + \delta x}}{\sqrt{x} + \sqrt{x + \delta x}}$$

$$\rightarrow \delta y = \frac{(\sqrt{x})^2 - (\sqrt{x + \delta x})^2}{(\sqrt{x + \delta x})\sqrt{x}(\sqrt{x} + \sqrt{x + \delta x})} = \frac{x - x - \delta x}{(\sqrt{x + \delta x})\sqrt{x}(\sqrt{x} + \sqrt{x + \delta x})}$$

$$\delta y = \frac{-\delta x}{(\sqrt{x + \delta x})\sqrt{x}(\sqrt{x} + \sqrt{x + \delta x})}$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{-1}{(\sqrt{x + \delta x})\sqrt{x}(\sqrt{x} + \sqrt{x + \delta x})}$$



$$\rightarrow \frac{dy}{dx} = \frac{-1}{(\sqrt{x+0})\sqrt{x}(\sqrt{x}+\sqrt{x+0})} = \frac{-1}{\sqrt{x}\sqrt{x}(2\sqrt{x})}$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{2x^{3/2}}$$

$$(iv) \frac{1}{x^3}$$

Solution:- Let $y = \frac{1}{x^3} \rightarrow (i)$

then $y + \delta y = \frac{1}{(x + \delta x)^3} \rightarrow (ii)$

By (ii) - (i) $\Rightarrow y + \delta y - y = \frac{1}{(x + \delta x)^3} - \frac{1}{x^3}$

$$\rightarrow \delta y = \frac{x^3 - (x + \delta x)^3}{(x + \delta x)^3 \cdot x^3} = \frac{x^3 - (x^3 + 3x^2\delta x + 3x\delta x^2 + \delta x^3)}{(x + \delta x)^3 \cdot x^3}$$

$$\delta y = \frac{-3x^2\delta x - 3x\delta x^2 - \delta x^3}{(x + \delta x)^3 \cdot x^3}$$

$$\rightarrow \delta y = \frac{-\delta x (3x^2 + 3x\delta x + \delta x^2)}{(x + \delta x)^3 \cdot x^3}$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[\frac{-(3x^2 + 3x\delta x + \delta x^2)}{(x + \delta x)^3 \cdot x^3} \right]$$

$$\rightarrow \frac{dy}{dx} = \frac{-(3x^2 + 3x(0) + 0)}{(x + 0)^3 \cdot x^3} = \frac{-3x^2}{x^6}$$

$$\rightarrow \frac{dy}{dx} = \frac{-3}{x^4}$$

$$(v) \frac{1}{x-a}$$

Solution:- Let $y = \frac{1}{x-a} \rightarrow (i)$

$\rightarrow y + \delta y = \frac{1}{(x + \delta x) - a} \rightarrow (ii)$

By (ii) - (i) $\Rightarrow y + \delta y - y = \frac{1}{x + \delta x - a} - \frac{1}{x - a}$

$$\rightarrow \delta y = \frac{(x-a) - (x + \delta x - a)}{(x + \delta x - a)(x - a)} = \frac{-\delta x}{(x + \delta x - a)(x - a)}$$

$$\delta y = \frac{-\delta x}{(x + \delta x - a)(x - a)}$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{-1}{(x + \delta x - a)(x - a)}$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{(x + 0 - a)(x - a)} = \frac{-1}{(x - a)^2}$$

$$(vi) x(x-3)$$

Solution:- Let $y = x(x-3) = x^2 - 3x \rightarrow (i)$

$\rightarrow y + \delta y = (x + \delta x)^2 - 3(x + \delta x) \rightarrow (ii)$

By (ii) - (i) $\Rightarrow y + \delta y - y = (x + \delta x)^2 - 3(x + \delta x) - (x^2 - 3x)$

$$\rightarrow \delta y = x^2 + 2x\delta x + \delta x^2 - 3x - 3\delta x - x^2 + 3x$$

$$\delta y = \delta x (2x + \delta x - 3)$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (2x + \delta x - 3) = 2x + 0 - 3$$

$$\rightarrow \frac{dy}{dx} = 2x - 3$$

$$(vii) \frac{2}{x^4}$$

Solution:- Let $y = \frac{2}{x^4} = 2x^{-4} \rightarrow (i)$

$\rightarrow y + \delta y = 2(x + \delta x)^{-4} \rightarrow (ii)$

By (ii) - (i) $\Rightarrow y + \delta y - y = 2(x + \delta x)^{-4} - 2x^{-4}$

$$\rightarrow \delta y = 2x^{-4} \left(1 + \frac{\delta x}{x}\right)^{-4} - 2x^{-4} = 2x^{-4} \left[\left(1 + \frac{\delta x}{x}\right)^{-4} - 1\right]$$

$$= 2x^{-4} \left[1 + (-4) \cdot \frac{\delta x}{x} + \frac{-4(-4-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots - 1\right]$$

$$= 2x^{-4} \left(\frac{\delta x}{x}\right) \left[-4 + \frac{-4(-4-1)}{2!} \left(\frac{\delta x}{x}\right) + \dots\right]$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{2x^{-4} \cancel{(\delta x)} \left[-4 + \frac{-4(-4-1)}{2!} \left(\frac{\delta x}{x}\right) + \dots\right]}{\delta x}$$

$$\rightarrow \frac{dy}{dx} = 2x^{-5} (-4 + 0) = -8x^{-5} = \frac{-8}{x^5}$$

$$(viii) (x+4)^{1/3}$$

Solution:- Let $y = (x+4)^{1/3} \rightarrow (i)$

$\rightarrow y + \delta y = [x + \delta x + 4]^{1/3} = [x + 4 + \delta x]^{1/3} \rightarrow (ii)$

By (ii) - (i) $\Rightarrow y + \delta y - y = [x + 4 + \delta x]^{1/3} - (x + 4)^{1/3}$

$$\rightarrow \delta y = (x+4)^{1/3} \left[1 + \frac{\delta x}{x+4}\right]^{1/3} - (x+4)^{1/3}$$

$$= (x+4)^{1/3} \left[\left(1 + \frac{\delta x}{x+4}\right)^{1/3} - 1\right]$$

$$= (x+4)^{1/3} \left[1 + \frac{1}{3} \left(\frac{\delta x}{x+4}\right) + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!} \left(\frac{\delta x}{x+4}\right)^2 + \dots - 1\right]$$

$$= (x+4)^{1/3} \left(\frac{\delta x}{x+4}\right) \left[\frac{1}{3} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!} \left(\frac{\delta x}{x+4}\right) + \dots\right]$$

Dividing by δx and take limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{(x+4)^{1/3} \cancel{(\delta x)} \left[\frac{1}{3} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!} \left(\frac{\delta x}{x+4}\right) + \dots\right]}{\delta x}$$

$$\rightarrow \frac{dy}{dx} = (x+4)^{-2/3} \left(\frac{1}{3} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!} (0) + \dots\right)$$

$$\frac{dy}{dx} = \frac{1}{3(x+4)^{2/3}}$$



(ix) $x^{\frac{3}{2}}$

Solution:- Let $y = x^{\frac{3}{2}} \rightarrow (i)$

$\rightarrow y + \delta y = (x + \delta x)^{\frac{3}{2}} \rightarrow (ii)$

By (ii) - (i) $\rightarrow y + \delta y - y = (x + \delta x)^{\frac{3}{2}} - x^{\frac{3}{2}}$

$\rightarrow \delta y = x^{\frac{3}{2}} \left(1 + \frac{\delta x}{x}\right)^{\frac{3}{2}} - x^{\frac{3}{2}} = x^{\frac{3}{2}} \left[\left(1 + \frac{\delta x}{x}\right)^{\frac{3}{2}} - 1\right]$

$\rightarrow \delta y = x^{\frac{3}{2}} \left[1 + \frac{3}{2} \cdot \frac{\delta x}{x} + \frac{3}{2} \cdot \frac{(3-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots - 1\right]$

Dividing by δx and take limit $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{x^{\frac{3}{2}} \cdot \frac{\delta x}{x} \left[\frac{3}{2} + \frac{3}{2} \cdot \frac{(3-1)}{2!} \left(\frac{\delta x}{x}\right) + \dots \right]}{\delta x}$

$\rightarrow \frac{dy}{dx} = x^{\frac{3}{2}-1} \left(\frac{3}{2} + \frac{3}{2} \cdot \frac{(3-1)}{2!} (0) + \dots \right)$

$\rightarrow \frac{dy}{dx} = \frac{3}{2} x^{\frac{1}{2}}$

(x) $x^{\frac{5}{2}}$

Solution:- Let $y = x^{\frac{5}{2}} \rightarrow (i)$

$\rightarrow y + \delta y = (x + \delta x)^{\frac{5}{2}} \rightarrow (ii)$

By (ii) - (i) $\rightarrow y + \delta y - y = (x + \delta x)^{\frac{5}{2}} - x^{\frac{5}{2}}$

$\rightarrow \delta y = x^{\frac{5}{2}} \left(1 + \frac{\delta x}{x}\right)^{\frac{5}{2}} - x^{\frac{5}{2}} = x^{\frac{5}{2}} \left[\left(1 + \frac{\delta x}{x}\right)^{\frac{5}{2}} - 1\right]$

$\delta y = x^{\frac{5}{2}} \left[1 + \left(\frac{5}{2}\right) \left(\frac{\delta x}{x}\right) + \frac{(5)(5-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots - 1\right]$

Dividing by δx and take limit $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{x^{\frac{5}{2}} \cdot \frac{\delta x}{x} \left[\frac{5}{2} + \frac{(5)(5-1)}{2!} \left(\frac{\delta x}{x}\right) + \dots \right]}{\delta x}$

$\rightarrow \frac{dy}{dx} = x^{\frac{5}{2}-1} \left[\frac{5}{2} + \frac{(5)(5-1)}{2!} (0) + \dots \right]$

$\rightarrow \frac{dy}{dx} = \frac{5}{2} x^{\frac{3}{2}}$

(xi) $x^m, m \in \mathbb{N}$

Solution:- Let $y = x^m \rightarrow (i)$

$y + \delta y = (x + \delta x)^m \rightarrow (ii)$

By (ii) - (i) $\rightarrow y + \delta y - y = (x + \delta x)^m - x^m$

$\rightarrow \delta y = x^m + m x^{m-1} (\delta x) + \frac{m(m-1)}{2!} (\delta x)^2 + \dots + (\delta x)^m - x^m$

$\delta y = m x^{m-1} (\delta x) + \frac{m(m-1)}{2!} (\delta x)^2 + \dots + (\delta x)^m$

$\delta y = m (\delta x) \left[x^{m-1} + \frac{m-1}{2!} (\delta x) + \dots + (\delta x)^{m-1} \right]$

Dividing by δx and take limit $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} m (\delta x) \left[x^{m-1} + \frac{m-1}{2!} (\delta x) + \dots + (\delta x)^{m-1} \right]$

$\rightarrow \frac{dy}{dx} = m \left[x^{m-1} + \frac{m-1}{2!} (0) + \dots + (0)^{m-1} \right]$

$\rightarrow \frac{dy}{dx} = m x^{m-1}$

(xii) $\frac{1}{x^m}, m \in \mathbb{N}$

Solution:- Let $y = \frac{1}{x^m} = x^{-m} \rightarrow (i)$

$\rightarrow y + \delta y = (x + \delta x)^{-m}$

By (ii) - (i) $\rightarrow y + \delta y - y = (x + \delta x)^{-m} - x^{-m} = x^{-m} \left(1 + \frac{\delta x}{x}\right)^{-m} - x^{-m}$

$\delta y = x^{-m} \left[\left(1 + \frac{\delta x}{x}\right)^{-m} - 1\right]$

$= x^{-m} \left[1 + (-m) \frac{\delta x}{x} + \frac{(-m)(-m-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots - 1\right]$

$= x^{-m} \left[-m \left(\frac{\delta x}{x}\right) + \frac{(-m)(-m-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots\right]$

Dividing by δx and take limit $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{x^{-m} \cdot \frac{\delta x}{x} \left[-m + \frac{(-m)(-m-1)}{2!} \left(\frac{\delta x}{x}\right) + \dots\right]}{\delta x}$

$\rightarrow \frac{dy}{dx} = x^{-m-1} \left[-m + \frac{(-m)(-m-1)}{2!} (0) + \dots\right]$

$\rightarrow \frac{dy}{dx} = -m x^{-m-1}$

(xiii) x^{40}

Solution:- Let $y = x^{40} \rightarrow (i)$

$y + \delta y = (x + \delta x)^{40} \rightarrow (ii)$

By (ii) - (i) $\rightarrow y + \delta y - y = (x + \delta x)^{40} - x^{40}$

$\rightarrow \delta y = (x + \delta x)^{40} - x^{40}$

$= x^{40} + 40 x^{39} (\delta x) + \frac{40(39)}{2!} (\delta x)^2 x^{38} + \dots - x^{40}$

$\delta y = 40 (\delta x) \left[x^{39} + \frac{39}{2!} (\delta x) x^{38} + \dots + (\delta x)^{39} \right]$

Dividing by δx and take limit $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} 40 (\delta x) \left[x^{39} + \frac{39}{2!} (\delta x) x^{38} + \dots + (\delta x)^{39} \right]$

$\rightarrow \frac{dy}{dx} = 40 \left[x^{39} + \frac{39}{2!} x (0) + \dots + (0)^{39} \right]$

$\rightarrow \frac{dy}{dx} = 40 x^{39}$

(xiv) x^{-100}

Solution:- Let $y = x^{-100} \rightarrow (i)$

$\rightarrow y + \delta y = (x + \delta x)^{-100} = x^{-100} \left(1 + \frac{\delta x}{x}\right)^{-100} \rightarrow (ii)$

By (ii) - (i) $\rightarrow y + \delta y - y = x^{-100} \left(1 + \frac{\delta x}{x}\right)^{-100} - x^{-100}$

$\rightarrow \delta y = x^{-100} \left[\left(1 + \frac{\delta x}{x}\right)^{-100} - 1\right]$

$= x^{-100} \left[1 + (-100) \frac{\delta x}{x} + \frac{(-100)(-100-1)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots - 1\right]$

$\delta y = x^{-100} \left[-100 \frac{\delta x}{x} + \frac{(-100)(-101)}{2!} \left(\frac{\delta x}{x}\right)^2 + \dots\right]$

Dividing by δx and take limit $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{x^{-100} \cdot \frac{\delta x}{x} \left[-100 + \frac{(-100)(-101)}{2!} \left(\frac{\delta x}{x}\right) + \dots\right]}{\delta x}$



$$\Rightarrow \frac{dy}{dx} = x^{-100} \left(-100 + \frac{(-100)(-101)}{2!} (0) + \dots \right)$$

$$\Rightarrow \frac{dy}{dx} = -100 x^{-101}$$

Q2. Find $\frac{dy}{dx}$ from first principles if

(i) $\sqrt{x+2}$ (ii) $\frac{1}{\sqrt{x+a}}$

(i) $\sqrt{x+2}$

Solution:- Let $y = \sqrt{x+2} = (x+2)^{\frac{1}{2}} \rightarrow$ (i)

$$\Rightarrow y + \delta y = (x + \delta x + 2)^{\frac{1}{2}} = [(x+2) + \delta x]^{\frac{1}{2}} \rightarrow$$
 (ii)

$$\text{By (ii) - (i)} \Rightarrow y + \delta y - y = [(x+2) + \delta x]^{\frac{1}{2}} - (x+2)^{\frac{1}{2}}$$

$$\Rightarrow \delta y = (x+2)^{\frac{1}{2}} \left[1 + \frac{\delta x}{x+2} \right]^{\frac{1}{2}} - (x+2)^{\frac{1}{2}}$$

$$= (x+2)^{\frac{1}{2}} \left[\left(1 + \frac{\delta x}{x+2} \right)^{\frac{1}{2}} - 1 \right]$$

$$\delta y = (x+2)^{\frac{1}{2}} \left[1 + \frac{1}{2} \left(\frac{\delta x}{x+2} \right) + \frac{1}{2} \left(\frac{1}{2} - 1 \right) \left(\frac{\delta x}{x+2} \right)^2 + \dots - 1 \right]$$

$$= (x+2)^{\frac{1}{2}} \left[\frac{1}{2} \left(\frac{\delta x}{x+2} \right) + \frac{1}{2} \left(-\frac{1}{2} \right) \left(\frac{\delta x}{x+2} \right)^2 + \dots \right]$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\Rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (x+2)^{\frac{1}{2}} \cdot \frac{\delta x}{\delta x} \left[\frac{1}{2} + \frac{1}{2} \left(-\frac{1}{2} \right) \left(\frac{\delta x}{x+2} \right) + \dots \right]$$

$$\Rightarrow \frac{dy}{dx} = (x+2)^{\frac{1}{2}-1} \left[\frac{1}{2} + \frac{1}{2} \left(-\frac{1}{2} \right) (0) + \dots \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} (x+2)^{-\frac{1}{2}} = \frac{1}{2(x+2)^{\frac{1}{2}}} = \frac{1}{2\sqrt{x+2}}$$

(ii) $\frac{1}{\sqrt{x+a}}$

Solution:- Let $y = \frac{1}{\sqrt{x+a}} = (x+a)^{-\frac{1}{2}} \rightarrow$ (i)

$$\Rightarrow y + \delta y = (x + \delta x + a)^{-\frac{1}{2}} = [(x+a) + \delta x]^{-\frac{1}{2}} \rightarrow$$
 (ii)

$$\text{By (ii) - (i)} \Rightarrow y + \delta y - y = [(x+a) + \delta x]^{-\frac{1}{2}} - (x+a)^{-\frac{1}{2}}$$

$$\Rightarrow \delta y = (x+a)^{-\frac{1}{2}} \left[\left(1 + \frac{\delta x}{x+a} \right)^{-\frac{1}{2}} - 1 \right]$$

$$= (x+a)^{-\frac{1}{2}} \left[1 + \left(-\frac{1}{2} \right) \frac{\delta x}{x+a} + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!} \left(\frac{\delta x}{x+a} \right)^2 + \dots - 1 \right]$$

$$= (x+a)^{-\frac{1}{2}} \left[-\frac{1}{2} \left(\frac{\delta x}{x+a} \right) + \frac{1}{2} \left(-\frac{3}{2} \right) \left(\frac{\delta x}{x+a} \right)^2 + \dots \right]$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (x+a)^{-\frac{1}{2}} \cdot \frac{\delta x}{\delta x} \left[-\frac{1}{2} + \frac{1}{2} \left(-\frac{3}{2} \right) \left(\frac{\delta x}{x+a} \right) + \dots \right]$$

$$\Rightarrow \frac{dy}{dx} = (x+a)^{-\frac{1}{2}-1} \left[-\frac{1}{2} + \frac{1}{2} \left(-\frac{3}{2} \right) (0) + \dots \right]$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2} (x+a)^{-\frac{3}{2}}$$



Differentiation of expression of the types:

$$(ax+b)^n \text{ and } \frac{1}{(ax+b)^n}, n = 1, 3, \dots$$

Example 1. Find from definition the differential coefficient of $(ax+b)^n$ w.r.t 'x' when n is +ve integer

Solution:- Let $y = (ax+b)^n \rightarrow$ (i)

$$y + \delta y = [a(x + \delta x) + b]^n \rightarrow$$
 (ii)

$$\text{By (ii) - (i)} \Rightarrow y + \delta y - y = [ax + a\delta x + b]^n - (ax+b)^n$$

$$\Rightarrow \delta y = [(ax+b) + a\delta x]^n - (ax+b)^n$$

$$= (ax+b)^n + n(ax+b)^{n-1}(a\delta x) + \frac{n(n-1)}{2!}(ax+b)^{n-2}(a\delta x)^2$$

$$+ \dots + (a\delta x)^n - (ax+b)^n$$

$$\delta y = na\delta x [(ax+b)^{n-1} + \frac{n(n-1)}{2!}(ax+b)^{n-2}(a\delta x)$$

$$+ \dots + (a\delta x)^{n-1}]$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} na\delta x [(ax+b)^{n-1} + \frac{n(n-1)}{2!}(ax+b)^{n-2}(a\delta x)$$

$$+ \dots + (a\delta x)^{n-1}]$$

$$\Rightarrow \frac{dy}{dx} = na[(ax+b)^{n-1} + \frac{n(n-1)}{2!}(ax+b)^{n-2}(a \cdot 0)$$

$$+ \dots + (a \cdot 0)^{n-1}]$$

$$\frac{dy}{dx} = na(ax+b)^{n-1}$$

Example 2. Find from first principles, the derivative of $\frac{1}{(ax+b)^n}$ w.r.t 'x'

Solution:- Let $y = \frac{1}{(ax+b)^n} = (ax+b)^{-n} \rightarrow$ (i)

$$\Rightarrow y + \delta y = [a(x + \delta x) + b]^{-n} = [ax + a\delta x + b]^{-n}$$

$$\Rightarrow y + \delta y = [(ax+b) + a\delta x]^{-n} \rightarrow$$
 (ii)

$$\text{By (ii) - (i)} \Rightarrow y + \delta y - y = [(ax+b) + a\delta x]^{-n} - (ax+b)^{-n}$$

$$\Rightarrow \delta y = (ax+b)^{-n} \left[\left(1 + \frac{a\delta x}{ax+b} \right)^{-n} - 1 \right]$$

$$= (ax+b)^{-n} \left[\left(1 + \frac{a\delta x}{ax+b} \right)^{-n} - 1 \right]$$

$$= (ax+b)^{-n} \left[1 + (-n) \left(\frac{a\delta x}{ax+b} \right) + \frac{(-n)(-n-1)}{2!} \left(\frac{a\delta x}{ax+b} \right)^2 + \dots - 1 \right]$$

$$= (ax+b)^{-n} \left[-n \left(\frac{a\delta x}{ax+b} \right) + \frac{(-n)(-n-1)}{2!} \left(\frac{a\delta x}{ax+b} \right)^2 + \dots \right]$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (ax+b)^{-n} \cdot \frac{a\delta x}{\delta x} \left[-n + \frac{(-n)(-n-1)}{2!} \left(\frac{a\delta x}{ax+b} \right) + \dots \right]$$

$$\frac{dy}{dx} = (ax+b)^{-n-1} \left[-n + \frac{(-n)(-n-1)}{2!} (0) + \dots \right]$$

$$\Rightarrow \frac{dy}{dx} = -na(ax+b)^{-n-1} = -na(ax+b)^{-(n+1)}$$

Exercise 2.2

Q1. Find from first principles, the derivatives of the following expressions w.r.t their respective independent variables:

(i) $(ax+b)^3$

Solution:- Let $y = (ax+b)^3 \rightarrow (i)$

$y + \delta y = [a(x+\delta x) + b]^3 \rightarrow (ii)$

By (ii) - (i) $\rightarrow y + \delta y - y = (ax + a\delta x + b)^3 - (ax+b)^3$

$$\rightarrow \delta y = [(ax+b) + a\delta x]^3 - (ax+b)^3$$

$$= (ax+b)^3 + 3(ax+b)^2(a\delta x) + 3(ax+b)(a\delta x)^2 + (a\delta x)^3 - (ax+b)^3$$

$$\delta y = 3a(\delta x) [(ax+b)^2 + (ax+b)(a\delta x) + (a\delta x)^2]$$

Dividing by δx and take Limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{3a(\delta x) [(ax+b)^2 + (ax+b)(a\delta x) + (a\delta x)^2]}{\delta x}$$

$$\rightarrow \frac{dy}{dx} = 3a [(ax+b)^2 + (ax+b)(a \cdot 0) + (a \cdot 0)^2]$$

$$\rightarrow \frac{dy}{dx} = 3a(ax+b)^2$$

(ii) $(2x+3)^5$

Solution:- Let $y = (2x+3)^5 \rightarrow (i)$

$y + \delta y = [2(x+\delta x) + 3]^5 \rightarrow (ii)$

By (ii) - (i) $\rightarrow$

$$y + \delta y - y = [2x + 2\delta x + 3]^5 - (2x+3)^5$$

$$\rightarrow \delta y = [(2x+3) + 2\delta x]^5 - (2x+3)^5$$

$$= (2x+3)^5 + 5(2x+3)^4(2\delta x) + \frac{5 \cdot 4}{2!} (2x+3)^3 (2\delta x)^2 + \dots + (2\delta x)^5 - (2x+3)^5$$

$$\delta y = 5(2\delta x) [(2x+3)^4 + \frac{4}{2!} (2x+3)^3 (2\delta x) + \dots + (2\delta x)^4]$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{10\delta x [(2x+3)^4 + \frac{4}{2!} (2x+3)^3 (2\delta x) + \dots + (2\delta x)^4]}{\delta x}$$

$$\frac{dy}{dx} = 10 [(2x+3)^4 + \frac{4}{2!} (2x+3)^3 (2 \cdot 0) + \dots + (2 \cdot 0)^4]$$

$$\rightarrow \frac{dy}{dx} = 10(2x+3)^4$$

(iii) $(3t+2)^{-2}$

Solution:- Let $y = (3t+2)^{-2} \rightarrow (i)$

$$\rightarrow y + \delta y = [3(t+\delta t) + 2]^{-2} = [3t + 3\delta t + 2]^{-2}$$

$$= (3t+2 + 3\delta t)^{-2} \rightarrow (ii)$$

By (ii) - (i) $\rightarrow y + \delta y - y = (3t+2)^{-2} \left(1 + \frac{3\delta t}{3t+2}\right)^{-2} - (3t+2)^{-2}$

$$\rightarrow \delta y = (3t+2)^{-2} \left[\left(1 + \frac{3\delta t}{3t+2}\right)^{-2} - 1 \right]$$

$$\rightarrow \delta y = (3t+2)^{-2} \left[1 + (-2) \left(\frac{3\delta t}{3t+2}\right) + \frac{(-2)(-2-1)}{2!} \left(\frac{3\delta t}{3t+2}\right)^2 + \dots \right]$$

$$= (3t+2)^{-2} \left[-2 \left(\frac{3\delta t}{3t+2}\right) + \frac{(-2)(-3)}{2!} \left(\frac{3\delta t}{3t+2}\right)^2 + \dots \right]$$

Dividing by δt and take limit $\delta t \rightarrow 0$

$$\lim_{\delta t \rightarrow 0} \frac{\delta y}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{(3t+2)^{-2} \left(\frac{3\delta t}{3t+2}\right) \left[-2 + \frac{(-2)(-3)}{2!} \left(\frac{3\delta t}{3t+2}\right) + \dots\right]}{\delta t}$$

$$\rightarrow \frac{dy}{dt} = (3t+2)^{-2} (3) \left[-2 + \frac{(-2)(-3)}{2!} (0) + \dots\right]$$

$$\rightarrow \frac{dy}{dt} = 3(3t+2)^{-3}$$

(iv) $\frac{1}{(ax+b)^5}$

Solution:- Let $y = \frac{1}{(ax+b)^5} = (ax+b)^{-5} \rightarrow (i)$

$$\rightarrow y + \delta y = [a(x+\delta x) + b]^{-5} = [ax + a\delta x + b]^{-5}$$

$$y + \delta y = [(ax+b) + a\delta x]^{-5} \rightarrow (ii)$$

By (ii) - (i) $\rightarrow y + \delta y - y = (ax+b)^{-5} \left(1 + \frac{a\delta x}{ax+b}\right)^{-5} - (ax+b)^{-5}$

$$\rightarrow \delta y = (ax+b)^{-5} \left[\left(1 + \frac{a\delta x}{ax+b}\right)^{-5} - 1 \right]$$

$$= (ax+b)^{-5} \left[1 + (-5) \left(\frac{a\delta x}{ax+b}\right) + \frac{(-5)(-5-1)}{2!} \left(\frac{a\delta x}{ax+b}\right)^2 + \dots \right]$$

$$= (ax+b)^{-5} \left[-5 \left(\frac{a\delta x}{ax+b}\right) + \frac{(-5)(-6)}{2!} \left(\frac{a\delta x}{ax+b}\right)^2 + \dots \right]$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{(ax+b)^{-5} \left(\frac{a\delta x}{ax+b}\right) \left[-5 + \frac{(-5)(-6)}{2!} \left(\frac{a\delta x}{ax+b}\right) + \dots\right]}{\delta x}$$

$$\rightarrow \frac{dy}{dx} = (ax+b)^{-5-1} (a) \left[-5 + \frac{(-5)(-6)}{2!} (0) + \dots\right]$$

$$\rightarrow \frac{dy}{dx} = -5a(ax+b)^{-6}$$

(v) $\frac{1}{(az-b)^7}$

Solution:- Let $y = \frac{1}{(az-b)^7} = (az-b)^{-7} \rightarrow (i)$

$$\rightarrow y + \delta y = [a(z+\delta z) - b]^{-7} = [az + a\delta z - b]^{-7}$$

$$\rightarrow y + \delta y = [(az-b) + a\delta z]^{-7} \rightarrow (ii)$$

By (ii) - (i) $\rightarrow y + \delta y - y = (az-b)^{-7} \left(1 + \frac{a\delta z}{az-b}\right)^{-7} - (az-b)^{-7}$

$$\rightarrow \delta y = (az-b)^{-7} \left[\left(1 + \frac{a\delta z}{az-b}\right)^{-7} - 1 \right]$$

$$= (az-b)^{-7} \left[1 + (-7) \left(\frac{a\delta z}{az-b}\right) + \frac{(-7)(-7-1)}{2!} \left(\frac{a\delta z}{az-b}\right)^2 + \dots \right]$$

$$\delta y = (az-b)^{-7} \left[-7 \left(\frac{a\delta z}{az-b}\right) + \frac{(-7)(-8)}{2!} \left(\frac{a\delta z}{az-b}\right)^2 + \dots \right]$$

Dividing by δz and take limit $\delta z \rightarrow 0$

$$\lim_{\delta z \rightarrow 0} \frac{\delta y}{\delta z} = \lim_{\delta z \rightarrow 0} \frac{(az-b)^{-7} \left(\frac{a\delta z}{az-b}\right) \left[-7 + \frac{(-7)(-8)}{2!} \left(\frac{a\delta z}{az-b}\right) + \dots\right]}{\delta z}$$

$$\rightarrow \frac{dy}{dz} = (az-b)^{-7-1} (a) \left[-7 + \frac{(-7)(-8)}{2!} (0) + \dots\right]$$

$$\rightarrow \frac{dy}{dz} = -7a(az-b)^{-8}$$

Theorems on differentiation

We have, so far proved the following two formulae;

1. $\frac{d}{dx}(c) = 0$ i.e., derivative of constant function is zero.

2. $\frac{d}{dx}(x^n) = nx^{n-1}$ power rule when n is rational no.

3. Derivative of $y = cf(x)$ i.e., $\frac{dy}{dx} = cf'(x)$

Proof:- $y = cf(x) \rightarrow (i)$, $y + \delta y = cf(x + \delta x) \rightarrow (ii)$

By (ii) - (i) $\rightarrow y + \delta y - y = cf(x + \delta x) - cf(x)$

$\rightarrow \delta y = c(f(x + \delta x) - f(x))$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[c \left(\frac{f(x + \delta x) - f(x)}{\delta x} \right) \right]$$

$\rightarrow \frac{dy}{dx} = cf'(x)$ Thus proved.

Example 1:- Calculate $\frac{d}{dx}(3x^{\frac{4}{3}})$

$$\begin{aligned} \text{Solution:- } \frac{d}{dx}(3x^{\frac{4}{3}}) &= 3 \frac{d}{dx}(x^{\frac{4}{3}}) \\ &= 3 \times \frac{4}{3} x^{\frac{4}{3}-1} = 4x^{\frac{1}{3}} \quad (\text{using power rule}) \end{aligned}$$

4. Derivative of a sum of functions:
 $[f(x) + g(x)]' = f'(x) + g'(x)$

Proof:- Let $y = f(x) + g(x) \rightarrow (i)$

$y + \delta y = f(x + \delta x) + g(x + \delta x) \rightarrow (ii)$

By (ii) - (i) $\rightarrow y + \delta y - y = f(x + \delta x) + g(x + \delta x) - f(x) - g(x)$

$\rightarrow \delta y = f(x + \delta x) - f(x) + g(x + \delta x) - g(x)$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} + \lim_{\delta x \rightarrow 0} \frac{g(x + \delta x) - g(x)}{\delta x}$$

$\rightarrow \frac{dy}{dx} = f'(x) + g'(x)$ Thus proved

5. Derivative of difference of functions:

$$[f(x) - g(x)]' = f'(x) - g'(x)$$

Proof:- Let $y = f(x) - g(x) \rightarrow (i)$

$y + \delta y = f(x + \delta x) - g(x + \delta x) \rightarrow (ii)$

By (ii) - (i) $\rightarrow y + \delta y - y = f(x + \delta x) - g(x + \delta x) - f(x) + g(x)$

$\rightarrow \delta y = f(x + \delta x) - f(x) - (g(x + \delta x) - g(x))$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} - \lim_{\delta x \rightarrow 0} \frac{g(x + \delta x) - g(x)}{\delta x}$$

$\rightarrow \frac{dy}{dx} = f'(x) - g'(x)$

Thus proved.

Example 1. Find the derivative of

$$y = \frac{3}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + 2x + 5 \text{ w.r.t } x.$$

Solution:- $y = \frac{3}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + 2x + 5$

Diff. w.r.t 'x' we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{3}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + 2x + 5 \right) \\ &= \frac{3}{4}(4)x^{4-1} + \frac{2}{3}(3)x^{3-1} + \frac{1}{2}(2x) + 2 + 0 \\ &= 3x^3 + 2x^2 + x + 2 \end{aligned}$$

Example 2. Find the derivative of $y = (x^2 + 5)(x^3 + 7)$ w.r.t 'x'.

Solution:- $y = (x^2 + 5)(x^3 + 7)$

$$\rightarrow y = x^5 + 7x^2 + 5x^3 + 35$$

$$\rightarrow y = x^5 + 5x^3 + 7x^2 + 35$$

Diff w.r.t 'x' we have

$$\begin{aligned} \frac{dy}{dx} &= 5x^4 + 5(3x^2) + 7(2x) + 0 \\ &= 5x^4 + 15x^2 + 14x \end{aligned}$$

Example 3. Find the derivative of $y = (2\sqrt{x} + 2)(x - \sqrt{x})$

Solution:- $y = (2\sqrt{x} + 2)(x - \sqrt{x})$

$$\rightarrow y = 2(\sqrt{x} + 1) \cdot (\sqrt{x}\sqrt{x} - \sqrt{x}) \quad \because \sqrt{x}\sqrt{x} = \sqrt{x^2} = x$$

$$\rightarrow y = 2(1 + \sqrt{x}) \cdot \sqrt{x}(\sqrt{x} - 1)$$

$$\begin{aligned} \rightarrow y &= 2\sqrt{x}(1 + \sqrt{x})(\sqrt{x} - 1) \\ &= 2\sqrt{x}(x - 1) = 2x^{\frac{1}{2}}(x - 1) \end{aligned}$$

$$\rightarrow y = 2x^{\frac{3}{2}} - 2x^{\frac{1}{2}}$$

Diff w.r.t 'x' we have

$$\begin{aligned} \frac{dy}{dx} &= 2 \frac{d}{dx}(x^{\frac{3}{2}}) - 2 \frac{d}{dx}(x^{\frac{1}{2}}) \\ &= 2 \left(\frac{3}{2} \right) x^{\frac{3}{2}-1} - 2 \left(\frac{1}{2} \right) x^{\frac{1}{2}-1} \\ &= 3x^{\frac{1}{2}} - x^{-\frac{1}{2}} = 3\sqrt{x} - \frac{1}{\sqrt{x}} = \frac{3x - 1}{\sqrt{x}} \end{aligned}$$

6. Derivative of a product (product rule)

If f and g are differentiable at x then $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$

Proof:- Let $y = f(x)g(x) \rightarrow (i)$

$\rightarrow y + \delta y = f(x + \delta x) \cdot g(x + \delta x) \rightarrow (ii)$

By (ii) - (i) $\rightarrow y + \delta y - y = f(x + \delta x)g(x + \delta x) - f(x)g(x)$
 subtracting and adding $f(x + \delta x)g(x)$

$$\rightarrow \delta y = f(x + \delta x)g(x + \delta x) - f(x + \delta x)g(x) + f(x + \delta x)g(x) - f(x)g(x)$$

$$\delta y = f(x + \delta x) \{g(x + \delta x) - g(x)\} + g(x) \{f(x + \delta x) - f(x)\}$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[f(x+\delta x) \left\{ \frac{g(x+\delta x) - g(x)}{\delta x} \right\} \right] + \lim_{\delta x \rightarrow 0} \left[g(x) \left\{ \frac{f(x+\delta x) - f(x)}{\delta x} \right\} \right]$$

$$\rightarrow \frac{dy}{dx} = f(x) \cdot g'(x) + g(x) f'(x) \text{ Thus proved}$$

Example: Find derivative of $y = (2\sqrt{x} + 2)(x - \sqrt{x})$ w.r.t x .

$$\text{Solution:- } y = (2\sqrt{x} + 2)(x - \sqrt{x})$$

$$\rightarrow y = 2(1 + \sqrt{x})(x - \sqrt{x})$$

Diff. w.r.t 'x' we have

$$\frac{dy}{dx} = 2 \frac{d}{dx} [(1 + \sqrt{x})(x - \sqrt{x})]$$

$$= 2 \left[\frac{d}{dx} (1 + \sqrt{x})(x - \sqrt{x}) + (1 + \sqrt{x}) \frac{d}{dx} (x - \sqrt{x}) \right]$$

$$= 2 \left[\left(0 + \frac{1}{2}x^{-\frac{1}{2}}\right)(x - \sqrt{x}) + (1 + \sqrt{x}) \left(1 - \frac{1}{2}x^{-\frac{1}{2}}\right) \right]$$

$$= 2 \left[\frac{1}{2\sqrt{x}}(x - \sqrt{x}) + (\sqrt{x} + 1) \left(1 - \frac{1}{2\sqrt{x}}\right) \right]$$

$$= 2 \left[\frac{x - \sqrt{x}}{2\sqrt{x}} + (\sqrt{x} + 1) \left(\frac{2\sqrt{x} - 1}{2\sqrt{x}}\right) \right]$$

$$= \frac{2(x - \sqrt{x} + 2x + 2\sqrt{x} - \sqrt{x} - 1)}{2\sqrt{x}} = \frac{3x - 1}{\sqrt{x}}$$

7. Derivative of a Quotient (Quotient rule)

For two differentiable functions $f(x)$ and $g(x)$,

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Proof:- Let $y = \frac{f(x)}{g(x)} \rightarrow (i)$

$$y + \delta y = \frac{f(x + \delta x)}{g(x + \delta x)} \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = \frac{f(x + \delta x)}{g(x + \delta x)} - \frac{f(x)}{g(x)}$$

$$\rightarrow \delta y = \frac{f(x + \delta x) \cdot g(x) - f(x)g(x + \delta x)}{g(x + \delta x) \cdot g(x)}$$

By subtracting and adding $g(x)f(x)$

$$\delta y = \frac{1}{g(x + \delta x) \cdot g(x)} \{ f(x + \delta x)g(x) - g(x)f(x) + g(x)f(x) - f(x)g(x + \delta x) \}$$

$$= \frac{1}{g(x + \delta x) \cdot g(x)} [g(x)(f(x + \delta x) - f(x)) - f(x)(g(x + \delta x) - g(x))]$$

Dividing by δx and applying limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{g(x + \delta x)g(x)} \left[g(x) \cdot \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} - f(x) \cdot \lim_{\delta x \rightarrow 0} \frac{g(x + \delta x) - g(x)}{\delta x} \right]$$

$$\rightarrow \frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \text{ Thus proved.}$$

Example 2. Find $\frac{dy}{dx}$ if $y = \frac{(\sqrt{x} + 1)(x^{\frac{3}{2}} - 1)}{x^{\frac{1}{2}} - 1}$, $x \neq 1$

$$\text{Solution:- } y = \frac{(\sqrt{x} + 1)(x^{\frac{3}{2}} - 1)}{\sqrt{x} - 1}$$

$$\rightarrow y = \frac{(\sqrt{x} + 1)((\sqrt{x})^3 - (1)^3)}{\sqrt{x} - 1}$$

$$= \frac{(\sqrt{x} + 1)(\sqrt{x} - 1)(x + 1 + \sqrt{x})}{\sqrt{x} - 1}$$

$$\rightarrow y = (\sqrt{x} + 1)(x + 1 + \sqrt{x}) = (\sqrt{x} + 1)(\sqrt{x} + 1 + x)$$

$$\rightarrow y = (\sqrt{x} + 1)^2 + (\sqrt{x} + 1)x = x + 1 + 2\sqrt{x} + x^{\frac{3}{2}} + x$$

$$\rightarrow y = x^{\frac{3}{2}} + 2x + 2x^{\frac{1}{2}} + 1$$

Diff. w.r.t 'x' we have

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + 2 + 2\left(\frac{1}{2}\right)x^{-\frac{1}{2}} + 0$$

$$= \frac{3}{2}x^{\frac{1}{2}} + 2 + x^{-\frac{1}{2}} = \frac{3}{2}\sqrt{x} + 2 + \frac{1}{\sqrt{x}}$$

Example 3. Differentiate $\frac{(\sqrt{x} + 1)(x^{\frac{3}{2}} - 1)}{x^{\frac{1}{2}} - x^{\frac{1}{2}}}$ w.r.t x

$$\text{Solution:- } y = \frac{(\sqrt{x} + 1)(x^{\frac{3}{2}} - 1)}{x^{\frac{3}{2}} - x^{\frac{1}{2}}}$$

$$\rightarrow y = \frac{(\sqrt{x} + 1)((\sqrt{x})^3 - (1)^3)}{x\sqrt{x} - \sqrt{x}} = \frac{(\sqrt{x} + 1)(\sqrt{x} - 1)(x + 1 + \sqrt{x})}{\sqrt{x}(x - 1)}$$

$$\rightarrow y = \frac{(x + 1 + \sqrt{x})}{\sqrt{x}(x - 1)} = \frac{x + \sqrt{x} + 1}{\sqrt{x}}$$

Diff. w.r.t 'x' we have

$$\frac{dy}{dx} = \frac{\sqrt{x} \left(1 + \frac{1}{2}x^{-\frac{1}{2}} + 0\right) - (x + \sqrt{x} + 1) \left(\frac{1}{2}x^{-\frac{1}{2}}\right)}{(\sqrt{x})^2}$$

$$= \frac{\sqrt{x} \left(1 + \frac{1}{2\sqrt{x}}\right) - (x + \sqrt{x} + 1) \cdot \frac{1}{2\sqrt{x}}}{x}$$

$$= \frac{\sqrt{x} \left(\frac{2\sqrt{x} + 1}{2\sqrt{x}}\right) - \frac{x + \sqrt{x} + 1}{2\sqrt{x}}}{x}$$

$$\frac{dy}{dx} = \frac{2\sqrt{x} + \sqrt{x} - x - \sqrt{x} - 1}{x \cdot 2\sqrt{x}} = \frac{x - 1}{2x^{\frac{3}{2}}}$$

Example 4. Differentiate $\frac{2x^3 - 3x^2 + 5}{x^2 + 1}$ w.r.t 'x'.

$$\text{Solution:- Let } y = \frac{2x^3 - 3x^2 + 5}{x^2 + 1}$$

Diff w.r.t 'x' we have

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{2x^3 - 3x^2 + 5}{x^2 + 1} \right)$$

$$= \frac{(x^2 + 1) \frac{d}{dx} (2x^3 - 3x^2 + 5) - (2x^3 - 3x^2 + 5) \frac{d}{dx} (x^2 + 1)}{(x^2 + 1)^2}$$

$$= \frac{(x^2 + 1)(6x^2 - 6x) - (2x^3 - 3x^2 + 5)(2x)}{(x^2 + 1)^2}$$

$$\frac{dy}{dx} = \frac{6x^4 - 6x^3 + 6x^2 - 6x - 4x^4 + 6x^3 - 10x}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{2x^4 + 6x^2 - 16x}{(x^2+1)^2}$$

Reciprocal rule:- If $g(x)$ is differentiable at x and $g(x) \neq 0$ then

$\frac{1}{g(x)}$ is differentiable at x and

$$\frac{d}{dx} \left(\frac{1}{g(x)} \right) = \frac{-g'(x)}{[g(x)]^2} \text{ is called reciprocal rule.}$$

Exercise 2.3

Differentiate w.r.t 'x'.

Q1. $x^4 + 2x^3 + x^2$

Solution:- Let $y = x^4 + 2x^3 + x^2$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} (x^4 + 2x^3 + x^2) = 4x^3 + 2(3)x^2 + 2x$$

$$\rightarrow \frac{dy}{dx} = 4x^3 + 6x^2 + 2x$$

Q2. $x^{-3} + 2x^{-\frac{3}{2}} + 3$

Solution:- Let $y = x^{-3} + 2x^{-\frac{3}{2}} + 3$

$$\frac{dy}{dx} = \frac{d}{dx} (x^{-3} + 2x^{-\frac{3}{2}} + 3) = -3x^{-4} + 2(-\frac{3}{2})x^{-\frac{3}{2}-1} + 0$$

$$\rightarrow \frac{dy}{dx} = -3x^{-4} - 3x^{-\frac{5}{2}}$$

Q3. $\frac{a+x}{a-x}$

Solution:- Let $y = \frac{a+x}{a-x}$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\frac{a+x}{a-x} \right) = \frac{(a-x)(0+1) - (a+x)(0-1)}{(a-x)^2}$$

$$= \frac{a-x - (a+x)}{(a-x)^2} = \frac{a-x-a-x}{(a-x)^2} = \frac{-2x}{(a-x)^2}$$

Q4. $\frac{2x-3}{2x+1}$

Solution:- Let $y = \frac{2x-3}{2x+1}$

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{2x-3}{2x+1} \right) = \frac{(2x+1)\frac{d}{dx}(2x-3) - (2x-3)\frac{d}{dx}(2x+1)}{(2x+1)^2}$$

$$= \frac{(2x+1)(2) - (2x-3)(2)}{(2x+1)^2} = \frac{4x+2-4x+6}{(2x+1)^2}$$

$$= \frac{8}{(2x+1)^2}$$

Q5. $(x-5)(3-x)$

Solution:- Let $y = (x-5)(3-x)$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} [(x-5)(3-x)] = (x-5)\frac{d}{dx}(3-x) + (3-x)\frac{d}{dx}(x-5)$$

$$\rightarrow \frac{dy}{dx} = (x-5)(0-1) + (3-x)(1-0) = -x+5+3-x$$

$$\rightarrow \frac{dy}{dx} = -2x+8$$



Q6. $(\sqrt{x} - \frac{1}{\sqrt{x}})^2$

Solution:- Let $y = (\sqrt{x} - \frac{1}{\sqrt{x}})^2$

$$\rightarrow y = (\sqrt{x})^2 + (\frac{1}{\sqrt{x}})^2 - 2(\sqrt{x})(\frac{1}{\sqrt{x}}) \quad \because (a-b)^2 = a^2 + b^2 - 2ab$$

$$\rightarrow y = x + \frac{1}{x} - 2 = x + x^{-1} - 2$$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} (x + x^{-1} - 2) = 1 + (-1)x^{-2} - 0 = 1 - x^{-2}$$

$$\rightarrow \frac{dy}{dx} = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$$

Q7. $\frac{(1+\sqrt{x})(x-x^{\frac{3}{2}})}{\sqrt{x}}$

Solution:- Let $y = \frac{(1+\sqrt{x})(x-x^{\frac{3}{2}})}{\sqrt{x}}$

$$\rightarrow y = \frac{(1+\sqrt{x})(x-x\sqrt{x})}{\sqrt{x}} = \frac{(1+\sqrt{x}) \cdot x(1-\sqrt{x})}{\sqrt{x}}$$

$$y = \frac{(1+\sqrt{x}) \cdot \sqrt{x} \sqrt{x} (1-\sqrt{x})}{\sqrt{x}} = \sqrt{x}(1-x)$$

$$\rightarrow y = x^{\frac{1}{2}} - x^{\frac{3}{2}} \rightarrow \frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} - \frac{3}{2}x^{\frac{3}{2}-1}$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} = \frac{1}{2\sqrt{x}} - \frac{3}{2}\sqrt{x}$$

$$\rightarrow \frac{dy}{dx} = \frac{1-3x}{2\sqrt{x}}$$

Q8. $\frac{(x^2+1)^2}{x^2-1}$

Solution:- Let $y = \frac{(x^2+1)^2}{x^2-1}$

$$\rightarrow y = \frac{x^4+1+2x^2}{x^2-1} \rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^4+1+2x^2}{x^2-1} \right)$$

$$\rightarrow \frac{dy}{dx} = \frac{(x^2-1)\frac{d}{dx}(x^4+1+2x^2) - (x^4+1+2x^2)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)(4x^3+4x) - (x^4+1+2x^2)(2x)}{(x^2-1)^2}$$

$$= \frac{4x^5+4x^3-4x^5-4x-2x^5-2x-4x^3}{(x^2-1)^2}$$

$$= \frac{2x^5-4x^3-6x}{(x^2-1)^2} = \frac{2x(x^4-2x^2-3)}{(x^2-1)^2}$$

$$= \frac{2x[x^4+x^2-3x^2-3]}{(x^2-1)^2} = \frac{2x[x^2(x^2+1)-3(x^2+1)]}{(x^2-1)^2}$$

$$\rightarrow \frac{dy}{dx} = \frac{2x(x^2+1)(x^2-3)}{(x^2-1)^2}$$

Q9. $\frac{x^2+1}{x^2-3}$

Solution:- Let $y = \frac{x^2+1}{x^2-3}$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^2+1}{x^2-3} \right) = \frac{(x^2-3) \frac{d}{dx}(x^2+1) - (x^2+1) \frac{d}{dx}(x^2-3)}{(x^2-3)^2}$$

$$= \frac{(x^2-3)(2x) - (x^2+1)(2x)}{(x^2-3)^2} = \frac{2x\{x^2-3-x^2-1\}}{(x^2-3)^2}$$

$$\frac{dy}{dx} = \frac{-8x}{(x^2-3)^2}$$

Q10. $\frac{\sqrt{1+x}}{\sqrt{1-x}}$

Solution:- Let $y = \frac{\sqrt{1+x}}{\sqrt{1-x}} = \sqrt{\frac{1+x}{1-x}} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\frac{1+x}{1-x} \right)^{\frac{1}{2}} = \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\frac{1+x}{1-x} \right)$$

$$= \frac{1}{2} \left(\frac{1-x}{1+x} \right)^{\frac{1}{2}} \cdot \frac{(1-x) \frac{d}{dx}(1+x) - (1+x) \frac{d}{dx}(1-x)}{(1-x)^2}$$

$$= \frac{1}{2} \left(\frac{1-x}{1+x} \right)^{\frac{1}{2}} \cdot \frac{(1-x)(1) - (1+x)(-1)}{(1-x)^2}$$

$$= \frac{1}{2} \cdot \frac{1}{(1+x)^{\frac{1}{2}}} \cdot \frac{1-x+1-x}{(1-x)^{2-\frac{1}{2}}}$$

$$= \frac{2}{2(1+x)^{\frac{1}{2}}(1-x)^{\frac{3}{2}}} = \frac{1}{(1+x)^{\frac{1}{2}}(1-x)^{\frac{3}{2}}}$$

$$= \frac{1}{(1+x)^{\frac{1}{2}}(1-x)\sqrt{1-x}} = \frac{1}{\sqrt{1+x}\sqrt{1-x}(1-x)}$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2} \cdot (1-x)}$$

Q11. $\frac{2x-1}{\sqrt{x^2+1}}$

Solution:- Let $y = \frac{2x-1}{\sqrt{x^2+1}} = \frac{2x-1}{(x^2+1)^{\frac{1}{2}}}$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\frac{2x-1}{(x^2+1)^{\frac{1}{2}}} \right) = \frac{(x^2+1)^{\frac{1}{2}} \frac{d}{dx}(2x-1) - (2x-1) \frac{d}{dx}(x^2+1)^{\frac{1}{2}}}{[(x^2+1)^{\frac{1}{2}}]^2}$$

$$= \frac{1}{x^2+1} \left[(x^2+1)^{\frac{1}{2}}(2) - (2x-1) \cdot \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \frac{d}{dx}(x^2+1) \right]$$

$$= \frac{1}{x^2+1} \left[2\sqrt{x^2+1} - \frac{2x-1}{2\sqrt{x^2+1}} \cdot (2x) \right]$$

$$= \frac{1}{x^2+1} \left[\frac{2(1+x^2) - (2x^2-x)}{\sqrt{x^2+1}} \right]$$

$$= \frac{2+2x^2-2x^2+x}{(x^2+1)\sqrt{x^2+1}} = \frac{x+2}{(x^2+1)^{\frac{3}{2}}}$$

Q12. $\sqrt{\frac{a-x}{a+x}}$

Solution:- Let $y = \sqrt{\frac{a-x}{a+x}} = \left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{a-x}{a+x} \right)^{\frac{1}{2}}$$

$$= \frac{1}{2} \left(\frac{a-x}{a+x} \right)^{-\frac{1}{2}} \cdot \frac{d}{dx} \left(\frac{a-x}{a+x} \right)$$

$$= \frac{1}{2} \frac{(a-x)^{-\frac{1}{2}}}{(a+x)^{\frac{1}{2}}} \cdot \frac{(a+x) \frac{d}{dx}(a-x) - (a-x) \frac{d}{dx}(a+x)}{(a+x)^2}$$

$$= \frac{1}{2} \frac{(a-x)^{-\frac{1}{2}}}{(a+x)^{\frac{1}{2}}} \cdot \frac{(a+x)(-1) - (a-x)(1)}{(a+x)^2}$$

$$= \frac{(a-x)^{-\frac{1}{2}}}{2(a+x)^{\frac{3}{2}}} \cdot \frac{-a-x-a+x}{1} = \frac{(a-x)^{-\frac{1}{2}}}{2(a+x)^{\frac{3}{2}}} \cdot (-2a)$$

$$\frac{dy}{dx} = \frac{-a}{\sqrt{a-x}(a+x)^{\frac{3}{2}}}$$

Q13. $\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$

Solution:- Let $y = \sqrt{\frac{x^2+1}{x^2-1}} = \left(\frac{x^2+1}{x^2-1}\right)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^2+1}{x^2-1} \right)^{\frac{1}{2}} = \frac{1}{2} \left(\frac{x^2+1}{x^2-1} \right)^{-\frac{1}{2}} \cdot \frac{d}{dx} \left(\frac{x^2+1}{x^2-1} \right)$$

$$= \frac{1}{2} \left(\frac{x^2-1}{x^2+1} \right)^{\frac{1}{2}} \cdot \frac{(x^2-1) \frac{d}{dx}(x^2+1) - (x^2+1) \frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)^{\frac{1}{2}} [(x^2-1)(2x) - (x^2+1)(2x)]}{2(x^2-1)^{\frac{3}{2}}}$$

$$= \frac{(x^2-1)^{\frac{1}{2}} [2x^2 - 2x - 2x^2 - 2x]}{2(x^2-1)^{\frac{3}{2}}}$$

$$\rightarrow \frac{dy}{dx} = \frac{-4x(x^2-1)^{\frac{1}{2}}}{2(x^2-1)^{\frac{3}{2}}} = \frac{-2x}{(x^2+1)^{\frac{1}{2}}(x^2-1)^{\frac{3}{2}}}$$

Q14. $\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$

Solution:- Let $y = \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$

$$= \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \times \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$$

$$= \frac{(\sqrt{1+x})^2 - (\sqrt{1-x})^2}{(\sqrt{1+x} + \sqrt{1-x})^2} = \frac{1+x - (1-x)}{1+x+1-x+2\sqrt{(1+x)(1-x)}}$$

$$= \frac{1+x-1+x}{2+2\sqrt{1-x^2}} = \frac{2x}{2(1+\sqrt{1-x^2})} = \frac{x}{1+\sqrt{1-x^2}}$$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\frac{x}{1+\sqrt{1-x^2}} \right)$$

$$= \frac{1+\sqrt{1-x^2} \frac{d}{dx}(x) - x \frac{d}{dx}(1+\sqrt{1-x^2})}{(1+\sqrt{1-x^2})^2}$$

$$= \frac{1}{(1+\sqrt{1-x^2})^2} \left[1+\sqrt{1-x^2} - x \left(0 + \frac{d}{dx}(1-x^2)^{\frac{1}{2}} \right) \right]$$

$$= \frac{1}{(1+\sqrt{1-x^2})^2} \left[1+\sqrt{1-x^2} - x \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \frac{d}{dx}(1-x^2) \right]$$

$$= \frac{1}{(1+\sqrt{1-x^2})^2} \left[1+\sqrt{1-x^2} - \frac{x}{\sqrt{1-x^2}} (-2x) \right]$$

$$\begin{aligned}
&= \frac{1}{(1+\sqrt{1-x^2})^2} \left[1 + \sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}} \right] \\
&= \frac{1}{(1+\sqrt{1-x^2})^2} \left[\frac{\sqrt{1-x^2}(1+\sqrt{1-x^2}) + x^2}{\sqrt{1-x^2}} \right] \\
&= \frac{1}{(1+\sqrt{1-x^2})^2} \left[\frac{\sqrt{1-x^2} + (\sqrt{1-x^2})^2 + x^2}{\sqrt{1-x^2}} \right] \\
&= \frac{1}{(1+\sqrt{1-x^2})^2} \left[\frac{\sqrt{1-x^2} + (1-x^2) + x^2}{\sqrt{1-x^2}} \right] \\
&= \frac{1}{(1+\sqrt{1-x^2})^2} \left[\frac{\sqrt{1-x^2} + 1 - x^2 + x^2}{\sqrt{1-x^2}} \right] \\
&= \frac{1}{(1+\sqrt{1-x^2})^2} \cdot \frac{1+\sqrt{1-x^2}}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}(1+\sqrt{1-x^2})}
\end{aligned}$$

Q15. $\frac{x\sqrt{a+x}}{\sqrt{a-x}}$

Solution:- Let $y = \frac{x\sqrt{a+x}}{\sqrt{a-x}}$

$$\begin{aligned}
\rightarrow \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{x\sqrt{a+x}}{\sqrt{a-x}} \right) = x \frac{d}{dx} \left(\frac{\sqrt{a+x}}{\sqrt{a-x}} \right) + \frac{\sqrt{a+x}}{\sqrt{a-x}} \frac{d}{dx} (x) \\
&= x \left[\frac{\sqrt{a-x} \frac{d}{dx} (a+x) - \sqrt{a+x} \frac{d}{dx} (a-x)}{(\sqrt{a-x})^2} \right] + \frac{\sqrt{a+x}}{\sqrt{a-x}}
\end{aligned}$$

$$= \frac{x}{a-x} \left[\sqrt{a-x} \cdot \frac{1}{2} (a+x)^{-\frac{1}{2}} (1) - \sqrt{a+x} \cdot \frac{1}{2} (a-x)^{-\frac{1}{2}} (-1) \right] + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$= \frac{x}{a-x} \left[\frac{\sqrt{a-x}}{2\sqrt{a+x}} + \frac{\sqrt{a+x}}{2\sqrt{a-x}} \right] + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$= \frac{x}{a-x} \left[\frac{a-x + a+x}{2\sqrt{a+x}\sqrt{a-x}} \right] + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$= \frac{x}{a-x} \left[\frac{2a}{2\sqrt{a+x}\sqrt{a-x}} \right] + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$= \frac{ax}{(a-x)\sqrt{a+x}\sqrt{a-x}} + \frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$= \frac{ax + (a-x)(a+x)}{(a-x)\sqrt{a+x}\sqrt{a-x}} = \frac{ax + a^2 - x^2}{\sqrt{a+x}(a-x)^{\frac{3}{2}}}$$

Q16. If $y = \sqrt{x} - \frac{1}{\sqrt{x}}$, show that $2x \frac{dy}{dx} + y = 2\sqrt{x}$

Solution:- $y = \sqrt{x} - \frac{1}{\sqrt{x}} = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$

$$\begin{aligned}
\rightarrow \frac{dy}{dx} &= \frac{d}{dx} \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}} \right) = \frac{1}{2} x^{\frac{1}{2}-1} - \left(-\frac{1}{2} \right) x^{-\frac{1}{2}-1} \\
&= \frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} x^{-\frac{3}{2}} = \frac{1}{2\sqrt{x}} + \frac{1}{2x^{\frac{3}{2}}}
\end{aligned}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{2x\sqrt{x}}$$

$$\rightarrow 2x \frac{dy}{dx} = \frac{2x}{2\sqrt{x}} + \frac{2x}{2x\sqrt{x}} \quad ('x' \text{ by } 2x)$$

$$\rightarrow 2x \frac{dy}{dx} = \frac{\sqrt{x}\sqrt{x}}{\sqrt{x}} + \frac{1}{\sqrt{x}} = \sqrt{x} + \frac{1}{\sqrt{x}}$$

$$\text{Now } 2x \frac{dy}{dx} + y = \sqrt{x} + \frac{1}{\sqrt{x}} + \sqrt{x} - \frac{1}{\sqrt{x}} = 2\sqrt{x}$$

$$\rightarrow 2x \frac{dy}{dx} + y = 2\sqrt{x} \quad \text{Hence proved.}$$

Q17. If $y = x^4 + 2x^2 + 2$, prove that $\frac{dy}{dx} = 4x\sqrt{y-1}$

Proof:- $y = x^4 + 2x^2 + 2$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} (x^4 + 2x^2 + 2) = 4x^3 + 4x$$

$$\frac{dy}{dx} = 4x(x^2 + 1)$$

$$\begin{aligned}
\rightarrow \frac{dy}{dx} &= 4x\sqrt{(x^2+1)^2} = 4x\sqrt{x^4+1+2x^2} \\
&= 4x\sqrt{x^4+2x^2+1+1-1} = 4x\sqrt{x^4+2x^2+2-1}
\end{aligned}$$

$$\rightarrow \frac{dy}{dx} = 4x\sqrt{y-1} \quad \because y = x^4 + 2x^2 + 2$$

Hence proved.

The chain Rule

If $y = (f \circ g)(x)$ or $y = f(g(x))$

Let $u = g(x)$ then $y = f(u)$ and $u = g(x)$

so we find $\frac{dy}{du}$ and $\frac{du}{dx}$ then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{is called chain rule.}$$

Example 1. Find the derivative of $(x^3+1)^9$ w.r.t 'x'.

Solution:- Let $y = (x^3+1)^9$

and $u = x^3+1$ then $y = u^9$

$$\text{Now } \frac{dy}{du} = \frac{d}{du} (u^9) = 9u^8 \quad \left| \quad \frac{du}{dx} = \frac{d}{dx} (x^3+1) = 3x^2 \right.$$

By chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 9u^8 \cdot 3x^2 = 27x^2(x^3+1)^8 \quad (\because u = x^3+1)$$

Example 2. Differentiate $\sqrt{\frac{a-x}{a+x}}$, ($x \neq -a$) w.r.t 'x'.

Solution:- Let $y = \sqrt{\frac{a-x}{a+x}}$ and $u = \frac{a-x}{a+x}$

then $y = \sqrt{u} = u^{\frac{1}{2}}$

$$\rightarrow \frac{dy}{du} = \frac{d}{du} (u^{\frac{1}{2}}) = \frac{1}{2} u^{-\frac{1}{2}}$$

$$\text{Now } \frac{du}{dx} = \frac{d}{dx} \left(\frac{a-x}{a+x} \right) = \frac{(a+x) \frac{d}{dx}(a-x) - (a-x) \frac{d}{dx}(a+x)}{(a+x)^2}$$

$$= \frac{(a+x)(0-1) - (a-x)(0+1)}{(a+x)^2} = \frac{-a-x-a+x}{(a+x)^2}$$

$$\rightarrow \frac{du}{dx} = \frac{-2a}{(a+x)^2} \quad \text{By chain rule, } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{2} u^{-\frac{1}{2}} \cdot \left(\frac{-2a}{(a+x)^2} \right) = \left(\frac{a-x}{a+x} \right)^{-\frac{1}{2}} \cdot \frac{-a}{(a+x)^2}$$

$$= \frac{(a-x)^{-\frac{1}{2}} (-a)}{(a+x)^{\frac{1}{2}} (a+x)^2} = \frac{-a}{(a+x)^{\frac{5}{2}} (a-x)^{\frac{1}{2}}}$$

$$\rightarrow \frac{dy}{dx} = \frac{-a}{(a+x)^{\frac{5}{2}} (a-x)^{\frac{1}{2}}}$$

Example 3. Find $\frac{dy}{dx}$ if $y = \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}}$ ($x \neq 0$)

Solution:- $y = \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}}$

$$\rightarrow y = \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} \times \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} + \sqrt{a-x}}$$

$$= \frac{(\sqrt{a+x})^2 - (\sqrt{a-x})^2}{(\sqrt{a+x} - \sqrt{a-x})(\sqrt{a+x} + \sqrt{a-x})} = \frac{a+x - (a-x)}{(\sqrt{a+x})^2 - (\sqrt{a-x})^2}$$

$$= \frac{a+x-a+x}{a+x+a-x-2\sqrt{(a+x)(a-x)}} = \frac{2x}{2a-2\sqrt{a^2-x^2}}$$

$$= \frac{2x}{2(a-\sqrt{a^2-x^2})} = \frac{x}{a-\sqrt{a^2-x^2}}$$

$$\rightarrow y = \frac{x}{a-\sqrt{a^2-x^2}} \rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\frac{x}{a-\sqrt{a^2-x^2}} \right)$$

$$= \frac{(a-\sqrt{a^2-x^2}) \frac{d}{dx}(x) - x \frac{d}{dx}(a-\sqrt{a^2-x^2})}{(a-\sqrt{a^2-x^2})^2}$$

$$= \frac{a-\sqrt{a^2-x^2}(1) - x(0 - \frac{d}{dx}(a^2-x^2)^{\frac{1}{2}})}{(a-\sqrt{a^2-x^2})^2}$$

$$= \frac{a-\sqrt{a^2-x^2} + x \cdot \frac{1}{2} (a^2-x^2)^{-\frac{1}{2}} \frac{d}{dx}(a^2-x^2)}{(a-\sqrt{a^2-x^2})^2}$$

$$= \frac{1}{(a-\sqrt{a^2-x^2})^2} \left[a-\sqrt{a^2-x^2} + \frac{x}{\sqrt{a^2-x^2}} (-2x) \right]$$

$$= \frac{1}{(a-\sqrt{a^2-x^2})^2} \left[\frac{a\sqrt{a^2-x^2} - (\sqrt{a^2-x^2})^2 - x^2}{\sqrt{a^2-x^2}} \right]$$

$$= \frac{1}{(a-\sqrt{a^2-x^2})^2} \left[\frac{a\sqrt{a^2-x^2} - (a^2-x^2) - x^2}{\sqrt{a^2-x^2}} \right]$$

$$= \frac{1}{(a-\sqrt{a^2-x^2})^2} \left[\frac{a\sqrt{a^2-x^2} - a^2 + x^2 - x^2}{\sqrt{a^2-x^2}} \right]$$

$$= \frac{1}{(a-\sqrt{a^2-x^2})^2} \left[\frac{-a(a-\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} \right] = \frac{-a}{(a-\sqrt{a^2-x^2})\sqrt{a^2-x^2}}$$

Example 4. Find $\frac{dy}{dx}$ if $y = (1+2\sqrt{x})^3 \cdot x^{\frac{3}{2}}$

Solution:- $y = (1+2\sqrt{x})^3 \cdot x^{\frac{3}{2}}$

$$\rightarrow y = (1+2\sqrt{x})^3 \cdot (x^{\frac{1}{2}})^3 = (1+2\sqrt{x})^3 (\sqrt{x})^3$$

$$= [(1+2\sqrt{x})\sqrt{x}]^3, \text{ Let } u = (1+2\sqrt{x})\sqrt{x}$$

$$\rightarrow y = u^3 \quad \frac{du}{dx} = \frac{d}{dx} \{ (1+2\sqrt{x})\sqrt{x} \}$$

$$\frac{dy}{du} = 3u^2 = (1+2\sqrt{x}) \frac{d}{dx} x^{\frac{1}{2}} + \sqrt{x} \frac{d}{dx} (1+2\sqrt{x})$$

$$\rightarrow \frac{du}{dx} = (1+2\sqrt{x}) \cdot \frac{1}{2} x^{-\frac{1}{2}} + \sqrt{x} (0+2 \cdot \frac{1}{2} x^{-\frac{1}{2}})$$

$$= (1+2\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} + 2\sqrt{x} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1+2\sqrt{x}}{2\sqrt{x}} + \frac{2\sqrt{x}}{2\sqrt{x}} = \frac{1+2\sqrt{x}+2\sqrt{x}}{2\sqrt{x}}$$

$$\rightarrow \frac{du}{dx} = \frac{1+4\sqrt{x}}{2\sqrt{x}}$$

By chain Rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\rightarrow \frac{dy}{dx} = 3u^2 \cdot \left(\frac{1+4\sqrt{x}}{2\sqrt{x}} \right)$$

$$= 3[(1+2\sqrt{x})\sqrt{x}]^2 \left(\frac{1+4\sqrt{x}}{2\sqrt{x}} \right)$$

$$= 3(1+2\sqrt{x})^2 (\sqrt{x})^2 \left(\frac{1+4\sqrt{x}}{2\sqrt{x}} \right)$$

$$\frac{dy}{dx} = \frac{3}{2} (1+2\sqrt{x})^2 (\sqrt{x} + 4x)$$

Example 5. If $y = (ax+b)^n$ where n is a negative integer. Find $\frac{dy}{dx}$ using quotient theorem.

Solution:- $y = (ax+b)^n$ Let $n = -m$

$$\rightarrow y = (ax+b)^{-m} = \frac{1}{(ax+b)^m}$$

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{(ax+b)^m} \right) = \frac{(ax+b)^m \frac{d}{dx}(1) - (1) \frac{d}{dx}(ax+b)^m}{[(ax+b)^m]^2}$$

$$= \frac{(ax+b)^m (0) - 1 \cdot m(ax+b)^{m-1} \cdot \frac{d}{dx}(ax+b)}{(ax+b)^{2m}}$$

$$= \frac{-m(ax+b)^{m-1}(a)}{(ax+b)^{2m}} = -ma(ax+b)^{m-1-2m}$$

$$= -ma(ax+b)^{-m-1}$$

$$\rightarrow \frac{dy}{dx} = na(ax+b)^{n-1} \quad \because n = -m$$

Example 6. Find $\frac{dy}{dx}$ if $y = x^n$ where $n = \frac{p}{q}$

$q \neq 0$ **Solution:-**

$$y = x^n \rightarrow y = x^{p/q} \quad \because n = p/q$$

$$\rightarrow y^q = (x^{p/q})^q \quad (\text{Taking } q\text{th power on both sides})$$

$$\rightarrow y^q = x^p \rightarrow \frac{d}{dx}(y^q) = \frac{d}{dx}(x^p)$$

$$\rightarrow q y^{q-1} \cdot \frac{dy}{dx} = p x^{p-1} \cdot \frac{d}{dx}(x)$$

$$\rightarrow q y^{q-1} \cdot \frac{dy}{dx} = p x^{p-1} \rightarrow y \frac{dy}{dx} = \frac{p}{q} x^{p/q-1}$$

Multiplying on both sides by y

$$\rightarrow y \cdot y^{q-1} \frac{dy}{dx} = \frac{p}{q} x^{p/q-1} \cdot y$$

$$\rightarrow y^q \frac{dy}{dx} = \frac{p}{q} x^{p/q-1} \cdot x^{p/q} \rightarrow x^p \frac{dy}{dx} = \frac{p}{q} x^{p-1} \cdot x^{p/q}$$

$$\rightarrow \frac{dy}{dx} = \frac{p}{q} \frac{x^{p-1} \cdot x^{p/q}}{x^p} = \frac{p}{q} x^{p-1-p/q} = \frac{p}{q} x^{p/q-1}$$

$$\rightarrow \frac{dy}{dx} = n x^{n-1} \quad \because n = \frac{p}{q}$$

Derivative of a function given in the form of parametric equations

Example 1. Find $\frac{dy}{dx}$ if $x = at^2$ and $y = 2at$

Solution:- $x = at^2$, $y = 2at$

$$\rightarrow \frac{dx}{dt} = \frac{d}{dt}(at^2)$$

$$= a \frac{d}{dt}(t^2) = a(2t)$$

$$\rightarrow \frac{dx}{dt} = 2at$$

$$\frac{dy}{dt} = \frac{d}{dt}(2at)$$

$$= 2a \frac{d}{dt}(t) = 2a(1)$$

$$\frac{dy}{dt} = 2a$$

By chain Rule, we have

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 2a \times \frac{1}{2at} = \frac{1}{t}$$

Example 2:- Find $\frac{dy}{dx}$ if $x = 1-t^2$ and $y = 3t^2 - 2t^3$

Solution:- $x = 1-t^2$, $y = 3t^2 - 2t^3$

$$\rightarrow \frac{dx}{dt} = \frac{d}{dt}(1-t^2)$$

$$= 0 - \frac{d}{dt}(t^2)$$

$$= -2t$$

$$\frac{dy}{dt} = \frac{d}{dt}(3t^2 - 2t^3)$$

$$= 3 \frac{d}{dt}(t^2) - 2 \frac{d}{dt}(t^3)$$

$$= 3(2t) - 2(3t^2)$$

$$= 6t - 6t^2$$

$$\frac{dy}{dx} = 6t(1-t)$$

By chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 6t(1-t) \times \frac{1}{-2t} = -3(1-t)$$

$$\rightarrow \frac{dy}{dx} = 3(t-1)$$

Example 3. Find $\frac{dy}{dx}$ if $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$

Solution:- $x = \frac{1-t^2}{1+t^2}$

$$\rightarrow \frac{dx}{dt} = \frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right) = \frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2} = \frac{-2t-2t^3-2t+2t^3}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{-4t}{(1+t^2)^2}$$

Now $y = \frac{2t}{1+t^2} \rightarrow \frac{dy}{dt} = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$

$$= \frac{(1+t^2)\frac{d}{dt}(2t) - (2t)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)(2) - (2t)(2t)}{(1+t^2)^2} = \frac{2+2t^2-4t^2}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2-2t^2}{(1+t^2)^2} = \frac{2(1-t^2)}{(1+t^2)^2}$$

using chain rule, we have

$$\rightarrow \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4t}$$

$$\frac{dy}{dx} = \frac{1-t^2}{-2t} \rightarrow \frac{dy}{dx} = \frac{t^2-1}{2t}$$

Differentiation of Implicit Relations

Example 1. Find $\frac{dy}{dx}$ if $x^2 + y^2 = 4$

Solution:- $x^2 + y^2 = 4$

$$\rightarrow \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(4)$$

$$\rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0 \rightarrow 2x + 2y \frac{dy}{dx} = 0$$

$$\rightarrow x y \frac{dy}{dx} = -x \rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Example 2. Find $\frac{dy}{dx}$ if $y^2 + x^2 - 4x = 5$

Solution:- $y^2 + x^2 - 4x = 5$

$$\rightarrow \frac{d}{dx}(y^2 + x^2 - 4x) = \frac{d}{dx}(5)$$

$$\rightarrow \frac{d}{dx}(y^2) + \frac{d}{dx}(x^2) - \frac{d}{dx}(4x) = 0$$

$$\rightarrow 2y \frac{dy}{dx} + 2x - 4 = 0$$

$$\rightarrow 2y \frac{dy}{dx} = 4 - 2x \rightarrow \frac{dy}{dx} = \frac{2-x}{y}$$

Example 3. Find $\frac{dy}{dx}$ if $y^2 - xy - x^2 + 4 = 0$

Solution:- $y^2 - xy - x^2 + 4 = 0$

$$\rightarrow \frac{d}{dx}(y^2 - xy - x^2 + 4) = \frac{d}{dx}(0)$$

$$\rightarrow \frac{d}{dx}(y^2) - \frac{d}{dx}(xy) - \frac{d}{dx}(x^2) + \frac{d}{dx}(4) = 0$$

$$\rightarrow 2y \frac{dy}{dx} - \{x \frac{dy}{dx} + y \cdot 1\} - 2x + 0 = 0$$

$$\rightarrow 2y \frac{dy}{dx} - x \frac{dy}{dx} - y - 2x = 0$$

$$\rightarrow \frac{dy}{dx}(2y - x) = 2x + y \Rightarrow \frac{dy}{dx} = \frac{2x + y}{2y - x}$$

Example 4:- Find $\frac{dy}{dx}$ if $y^3 - 2xy^2 + x^2y + 3x = 0$

Solution:- $y^3 - 2xy^2 + x^2y + 3x = 0$

$$\rightarrow \frac{d}{dx}(y^3 - 2xy^2 + x^2y + 3x) = \frac{d}{dx}(0)$$

$$\rightarrow \frac{d}{dx}(y^3) - 2 \frac{d}{dx}(xy^2) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(3x) = 0$$

$$\rightarrow 3y^2 \frac{dy}{dx} - 2 \{x \cdot \frac{d}{dx}y^2 + y^2 \frac{d}{dx}(x)\} + (x^2 \frac{dy}{dx} + y(2x)) + 3 = 0$$

$$\rightarrow 3y^2 \frac{dy}{dx} - 2 \{x(2y \frac{dy}{dx}) + y^2(1)\} + (x^2 \frac{dy}{dx} + 2xy) + 3 = 0$$

$$\rightarrow 3y^2 \frac{dy}{dx} - 4xy \frac{dy}{dx} - 2y^2 + x^2 \frac{dy}{dx} + 2xy + 3 = 0$$

$$\rightarrow (3y^2 - 4xy + x^2) \frac{dy}{dx} = 2y^2 - 2xy - 3$$

$$\rightarrow \frac{dy}{dx} = \frac{2y^2 - 2xy - 3}{3y^2 - 4xy + x^2}$$

Example 5. Differentiate $x^2 + \frac{1}{x^2}$ w.r.t $x - \frac{1}{x}$.

Solution:- Let $y = x^2 + \frac{1}{x^2}$ and $u = x - \frac{1}{x}$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2 + x^{-2})$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(x^{-2})$$

$$= 2x + (-2)x^{-2-1}$$

$$\rightarrow \frac{dy}{dx} = 2x - \frac{2}{x^3}$$

$$= \frac{2x^4 - 2}{x^3}$$

$$= \frac{2(x^4 - 1)}{x^3}$$

$$\frac{dy}{dx} = \frac{2(x^2 - 1)(x^2 + 1)}{x^3}$$

Now we find $\frac{dy}{du}$ by chain rule

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du} = \frac{2(x^2 - 1)(x^2 + 1)}{x^3} \times \frac{x^2}{x^2 + 1}$$

$$\rightarrow \frac{dy}{du} = \frac{2(x^2 - 1)}{x} = 2 \left(x - \frac{1}{x}\right)$$

Exercise 2.4

Q1. Find $\frac{dy}{dx}$ by making suitable substitutions in the following functions defined as:

(i) $y = \sqrt{\frac{1-x}{1+x}}$

Solution:- $y = \sqrt{\frac{1-x}{1+x}} = \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$

Let $u = \frac{1-x}{1+x}$ then $y = u^{\frac{1}{2}} \Rightarrow \frac{dy}{du} = \frac{d}{du}(u^{\frac{1}{2}})$

$$\rightarrow \frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1} = \frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{1-x}{1+x}\right) = \frac{(1+x)\frac{d}{dx}(1-x) - (1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2} = \frac{-1-x-1+x}{(1+x)^2}$$

$$\rightarrow \frac{du}{dx} = \frac{-2}{(1+x)^2}$$

By chain rule, we have $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\rightarrow \frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \cdot \frac{-2}{(1+x)^2} = \left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}} \cdot \frac{-1}{(1+x)^2}$$

$$= \frac{(1-x)^{-\frac{1}{2}}}{(1+x)^{\frac{1}{2}}} \cdot \frac{-1}{(1+x)^2} = -\frac{(1-x)^{-\frac{1}{2}}}{(1+x)^{\frac{5}{2}}}$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x}(1+x)^{\frac{5}{2}}}$$

(ii) $y = \sqrt{x + \sqrt{x}}$

Solution:- $y = \sqrt{x + \sqrt{x}}$ Let $u = x + \sqrt{x}$

$$\rightarrow y = u^{\frac{1}{2}} \Rightarrow \frac{dy}{du} = \frac{d}{du}(u^{\frac{1}{2}}) = \frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{d}{dx}(x + \sqrt{x}) = \frac{d}{dx}(x) + \frac{d}{dx}(x^{\frac{1}{2}})$$

$$= 1 + \frac{1}{2}x^{-\frac{1}{2}} = 1 + \frac{1}{2\sqrt{x}} = \frac{2\sqrt{x} + 1}{2\sqrt{x}}$$

By chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \rightarrow \frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \cdot \frac{2\sqrt{x} + 1}{2\sqrt{x}}$$

$$= \frac{1}{2}(x + \sqrt{x})^{-\frac{1}{2}} \cdot \frac{2\sqrt{x} + 1}{2\sqrt{x}} = \frac{2\sqrt{x} + 1}{4\sqrt{x}(\sqrt{x} + \sqrt{x})}$$

(iii) $y = x \sqrt{\frac{a+x}{a-x}}$

Solution:- $y = x \sqrt{\frac{a+x}{a-x}} = \sqrt{\frac{x^2(a+x)}{a-x}}$

$$\rightarrow y = \left(\frac{x^2(a+x)}{a-x}\right)^{\frac{1}{2}} \quad \text{Let } u = \frac{ax^2 + x^3}{a-x}$$

$$\rightarrow y = \left(\frac{ax^2 + x^3}{a-x}\right)^{\frac{1}{2}}$$

$$\rightarrow y = u^{\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{d}{du}(u^{\frac{1}{2}}) = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$$

$$\text{Now } \frac{du}{dx} = \frac{d}{dx} \left(\frac{ax^2 + x^3}{a-x} \right) = \frac{(a-x) \frac{d}{dx} (ax^2 + x^3) - (ax^2 + x^3) \frac{d}{dx} (a-x)}{(a-x)^2}$$

$$= \frac{(a-x)(2ax + 3x^2) - (ax^2 + x^3)(-1)}{(a-x)^2}$$

$$= \frac{2a^2x + 3ax^2 - 2ax^2 - 3x^3 + ax^2 + x^3}{(a-x)^2}$$

$$\frac{du}{dx} = \frac{-2x^3 + 2ax^2 + 2a^2x}{(a-x)^2} = \frac{2x(-x^2 + ax + a^2)}{(a-x)^2}$$

By chain Rule, we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \times \frac{2x(a^2 + ax - x^2)}{(a-x)^2}$$

$$= \frac{1}{2\sqrt{ax^2 + x^3}} \times \frac{2x(a^2 + ax - x^2)}{(a-x)^2}$$

$$= \frac{\sqrt{a-x}}{\sqrt{ax^2 + x^3}} \times \frac{x(a^2 + ax - x^2)}{(a-x)^2}$$

$$= \frac{x(a^2 + ax - x^2)}{\sqrt{ax^2 + x^3} (a-x)^{3/2}} = \frac{x(a^2 + ax - x^2)}{\sqrt{x^2(ax+x)} (a-x)^{3/2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x(a^2 + ax - x^2)}{x\sqrt{a+x} (a-x)^{3/2}} = \frac{a^2 + ax - x^2}{\sqrt{a+x} (a-x)^{3/2}}$$

$$(iv) y = (3x^2 - 2x + 7)^6$$

$$\text{Solution:- } y = (3x^2 - 2x + 7)^6, \text{ Let } u = 3x^2 - 2x + 7$$

$$\rightarrow y = u^6 \rightarrow \frac{dy}{du} = 6u^5, \quad \frac{du}{dx} = \frac{d}{dx} (3x^2 - 2x + 7) = 6x - 2$$

By chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 6u^5 \times (6x - 2) = 6(3x^2 - 2x + 7)^5 \times (6x - 2)$$

$$\Rightarrow \frac{dy}{dx} = 6(6x - 2)(3x^2 - 2x + 7)^5$$

$$(v) y = \sqrt{\frac{a^2 + x^2}{a^2 - x^2}}$$

$$\text{Solution:- } y = \sqrt{\frac{a^2 + x^2}{a^2 - x^2}}, \text{ Let } u = \frac{a^2 + x^2}{a^2 - x^2}$$

$$\rightarrow y = \sqrt{u} \rightarrow \frac{dy}{du} = \frac{d}{du} (u^{1/2}) = \frac{1}{2} u^{-1/2}$$

$$\text{Now, } \frac{du}{dx} = \frac{d}{dx} \left(\frac{a^2 + x^2}{a^2 - x^2} \right) = \frac{(a^2 - x^2) \frac{d}{dx} (a^2 + x^2) - (a^2 + x^2) \frac{d}{dx} (a^2 - x^2)}{(a^2 - x^2)^2}$$

$$= \frac{(a^2 - x^2)(2x) - (a^2 + x^2)(-2x)}{(a^2 - x^2)^2}$$

$$= \frac{2x(a^2 - x^2 + a^2 + x^2)}{(a^2 - x^2)^2} = \frac{2x(2a^2)}{(a^2 - x^2)^2} = \frac{4xa^2}{(a^2 - x^2)^2}$$

By chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2} u^{-1/2} \cdot \frac{4xa^2}{(a^2 - x^2)^2}$$

$$= \left(\frac{a^2 + x^2}{a^2 - x^2} \right)^{-1/2} \cdot \frac{2a^2x}{(a^2 - x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(a^2 + x^2)^{-1/2} \cdot 2a^2x}{(a^2 - x^2)^{1/2} (a^2 - x^2)^2} = \frac{2a^2x}{\sqrt{a^2 + x^2} (a^2 - x^2)^{3/2}}$$

Q2. Find $\frac{dy}{dx}$ if

$$(i) 3x + 4y + 7 = 0$$

$$\text{Solution:- } \frac{d}{dx} (3x + 4y + 7) = \frac{d}{dx} (0)$$

$$\rightarrow \frac{d}{dx} (3x) + \frac{d}{dx} (4y) + \frac{d}{dx} (7) = 0$$

$$\rightarrow 3 + 4 \frac{dy}{dx} + 0 = 0 \Rightarrow \frac{dy}{dx} = -3/4$$

$$(ii) xy + y^2 = 2$$

$$\text{Solution:- } \frac{d}{dx} (xy + y^2) = \frac{d}{dx} (2)$$

$$\rightarrow \frac{d}{dx} (xy) + \frac{d}{dx} (y^2) = 0$$

$$\rightarrow x \frac{dy}{dx} + y \frac{d}{dx} (x) + 2y \frac{dy}{dx} = 0$$

$$\rightarrow x \frac{dy}{dx} + y + 2y \frac{dy}{dx} = 0$$

$$\rightarrow \frac{dy}{dx} (x + 2y) = -y \Rightarrow \frac{dy}{dx} = \frac{-y}{x + 2y}$$

$$(iii) x^2 - 4xy - 5y = 0$$

$$\text{Solution:- } \frac{d}{dx} (x^2 - 4xy - 5y) = \frac{d}{dx} (0)$$

$$\rightarrow \frac{d}{dx} (x^2) - 4 \frac{d}{dx} (xy) - 5 \frac{d}{dx} (y) = 0$$

$$\rightarrow 2x - 4 \left\{ x \frac{dy}{dx} + y \frac{d}{dx} (x) \right\} - 5 \frac{dy}{dx} = 0$$

$$\rightarrow 2x - 4x \frac{dy}{dx} - 4y - 5 \frac{dy}{dx} = 0$$

$$\rightarrow 2x - 4y = 4x \frac{dy}{dx} + 5 \frac{dy}{dx}$$

$$\rightarrow 2(x - 2y) = (4x + 5) \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(x - 2y)}{4x + 5}$$

$$(iv) 4x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$\text{Solution:- } \frac{d}{dx} (4x^2 + 2hxy + by^2 + 2gx + 2fy + c) = \frac{d}{dx} (0)$$

$$4 \frac{d}{dx} (x^2) + 2h \frac{d}{dx} (xy) + b \frac{d}{dx} (y^2) + 2g \frac{d}{dx} (x) + 2f \frac{d}{dx} (y) + \frac{d}{dx} (c) = 0$$

$$\rightarrow 4(2x) + 2h \left(x \frac{dy}{dx} + y \right) + 2by \frac{dy}{dx} + 2g + 2f \frac{dy}{dx} + 0 = 0$$

$$8x + 2hx \frac{dy}{dx} + 2hy + 2by \frac{dy}{dx} + 2g + 2f \frac{dy}{dx} = 0$$

$$(2hx + 2by + 2f) \frac{dy}{dx} = -(8x + 2hy + 2g)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2(4x + hy + g)}{2(hx + by + f)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(4x + hy + g)}{hx + by + f}$$

$$(v) x\sqrt{1+y} + y\sqrt{1+x} = 0$$

$$\text{Solution:- } x(1+y)^{\frac{1}{2}} + y(1+x)^{\frac{1}{2}} = 0$$

$$\Rightarrow \frac{d}{dx}(x(1+y)^{\frac{1}{2}} + y(1+x)^{\frac{1}{2}}) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(x(1+y)^{\frac{1}{2}}) + \frac{d}{dx}(y(1+x)^{\frac{1}{2}}) = 0$$

$$\Rightarrow x \cdot \frac{1}{2}(1+y)^{-\frac{1}{2}} \cdot \frac{dy}{dx} + (1+y)^{\frac{1}{2}} \cdot (1) + y \cdot \frac{1}{2}(1+x)^{-\frac{1}{2}} \cdot (1) + (1+x)^{\frac{1}{2}} \cdot \frac{dy}{dx} = 0$$

$$\frac{x}{2\sqrt{1+y}} \frac{dy}{dx} + \sqrt{1+y} + \frac{y}{2\sqrt{1+x}} + \sqrt{1+x} \frac{dy}{dx} = 0$$

$$\left(\frac{x}{2\sqrt{1+y}} + \sqrt{1+x}\right) \frac{dy}{dx} = -\left(\sqrt{1+y} + \frac{y}{2\sqrt{1+x}}\right)$$

$$\Rightarrow \frac{x+2\sqrt{1+x}\sqrt{1+y}}{2\sqrt{1+y}} \cdot \frac{dy}{dx} = -\left(\frac{2\sqrt{1+x}\sqrt{1+y} + y}{x\sqrt{1+x}}\right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{1+y}(y+2\sqrt{1+x}\sqrt{1+y})}{\sqrt{1+x}(x+2\sqrt{1+x}\sqrt{1+y})}$$

$$(vi) y(x^2-1) = x\sqrt{x^2+4}$$

$$\text{Solution:- } y(x^2-1) = x(x^2+4)^{\frac{1}{2}}$$

$$\Rightarrow \frac{d}{dx}\{y(x^2-1)\} = \frac{d}{dx}\{x(x^2+4)^{\frac{1}{2}}\}$$

$$\Rightarrow y \frac{d}{dx}(x^2-1) + (x^2-1) \frac{dy}{dx} = x \frac{d}{dx}(x^2+4)^{\frac{1}{2}} + (x^2+4)^{\frac{1}{2}} \cdot \frac{d}{dx}(x)$$

$$\Rightarrow 2xy + (x^2-1) \frac{dy}{dx} = x \cdot \frac{1}{2}(x^2+4)^{-\frac{1}{2}} \cdot \frac{d}{dx}(x^2+4) + (x^2+4)^{\frac{1}{2}} \cdot (1)$$

$$\Rightarrow 2xy + (x^2-1) \frac{dy}{dx} = \frac{x}{2\sqrt{x^2+4}} (2x) + \sqrt{x^2+4}$$

$$\Rightarrow 2xy + (x^2-1) \frac{dy}{dx} = \frac{x^2}{\sqrt{x^2+4}} + \sqrt{x^2+4}$$

$$2xy + (x^2-1) \frac{dy}{dx} = \frac{x^2 + x^2 + 4}{\sqrt{x^2+4}}$$

$$\Rightarrow (x^2-1) \frac{dy}{dx} = \frac{2x^2+4}{\sqrt{x^2+4}} - 2xy$$

$$\because y(x^2-1) = x\sqrt{x^2+4}$$

$$\Rightarrow y = \frac{x\sqrt{x^2+4}}{x^2-1} \quad \text{Thus}$$

$$(x^2-1) \frac{dy}{dx} = \frac{2x^2+4}{\sqrt{x^2+4}} - 2x \left(\frac{x\sqrt{x^2+4}}{x^2-1}\right)$$

$$= \frac{2x^2+4}{\sqrt{x^2+4}} - \frac{2x^2\sqrt{x^2+4}}{x^2-1}$$

$$= \frac{(2x^2+4)(x^2-1) - 2x^2(\sqrt{x^2+4})^2}{\sqrt{x^2+4}(x^2-1)}$$

$$= \frac{2x^4 - 2x^2 + 4x^2 - 4 - 2x^2(x^2+4)}{\sqrt{x^2+4}(x^2-1)}$$

$$(x^2-1) \frac{dy}{dx} = \frac{2x^4 - 2x^2 + 4x^2 - 4 - 2x^4 - 8x^2}{\sqrt{x^2+4}(x^2-1)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-4-6x^2}{\sqrt{x^2+4}(x^2-1)}$$

Q3. Find $\frac{dy}{dx}$ of the following parametric functions.

$$(i) x = \theta + \frac{1}{\theta} \text{ and } y = \theta + 1$$

$$\text{Solution:- } x = \theta + \frac{1}{\theta}, \quad y = \theta + 1$$

$$\Rightarrow \frac{dx}{d\theta} = \frac{d}{d\theta}(\theta + \theta^{-1}) \quad ; \quad \frac{dy}{d\theta} = \frac{d}{d\theta}(\theta + 1)$$

$$= \frac{d}{d\theta}(\theta) + \frac{d}{d\theta}(\theta^{-1}) \quad , \quad \frac{dy}{d\theta} = 1$$

$$= 1 + (-1)\theta^{-1-1}$$

$$= 1 - \frac{1}{\theta^2} \quad \text{By chain rule, we have}$$

$$\frac{dx}{d\theta} = \frac{\theta^2-1}{\theta^2} \quad \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = (1) \cdot \frac{\theta^2}{\theta^2-1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\theta^2}{\theta^2-1}$$

$$(ii) x = \frac{a(1-t^2)}{1+t^2}, \quad y = \frac{2bt}{1+t^2}$$

$$\text{Solution:- } x = \frac{a(1-t^2)}{1+t^2}$$

$$\frac{dx}{dt} = a \cdot \frac{d}{dt} \left(\frac{1-t^2}{1+t^2} \right) = a \left[\frac{(1+t^2) \frac{d}{dt}(1-t^2) - (1-t^2) \frac{d}{dt}(1+t^2)}{(1+t^2)^2} \right]$$

$$\frac{dx}{dt} = \frac{-4at}{(1+t^2)^2}$$

$$\text{Now } y = \frac{2bt}{1+t^2}$$

$$\Rightarrow \frac{dy}{dt} = \frac{d}{dt} \left(\frac{2bt}{1+t^2} \right)$$

$$= 2b \left[\frac{(1+t^2) \frac{d}{dt}(t) - t \frac{d}{dt}(1+t^2)}{(1+t^2)^2} \right]$$

$$= 2b \left[\frac{(1+t^2)(1) - (t)(2t)}{(1+t^2)^2} \right] = 2b \left[\frac{1+t^2-2t^2}{(1+t^2)^2} \right]$$

$$\frac{dy}{dt} = \frac{2b(1-t^2)}{(1+t^2)^2}$$

By chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{2b(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4at} = \frac{b(1-t^2)}{-2at}$$

$$\frac{dy}{dx} = \frac{b(t^2-1)}{2at}$$

Q4. Prove that $y \frac{dy}{dx} + x = 0$ if $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$

Solution:- $x = \frac{1-t^2}{1+t^2}$

$$\rightarrow \frac{dx}{dt} = \frac{d}{dt} \left(\frac{1-t^2}{1+t^2} \right) = \frac{(1+t^2) \frac{d}{dt}(1-t^2) - (1-t^2) \frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2} = \frac{-2t-2t^3-2t+2t^3}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{-4t}{(1+t^2)^2}$$

Now $y = \frac{2t}{1+t^2}$

$$\rightarrow \frac{dy}{dt} = \frac{d}{dt} \left(\frac{2t}{1+t^2} \right)$$

$$= \frac{(1+t^2) \frac{d}{dt}(2t) - (2t) \frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)(2) - (2t)(2t)}{(1+t^2)^2} = \frac{2+2t^2-4t^2}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2-2t^2}{(1+t^2)^2} \Rightarrow \frac{dy}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}$$

By chain rule, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$$\rightarrow \frac{dy}{dx} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4t} = \frac{1-t^2}{-2t}$$

Now $y \frac{dy}{dx} = \left(\frac{2t}{1+t^2} \right) \left(\frac{1-t^2}{-2t} \right) = - \left(\frac{1-t^2}{1+t^2} \right)$

$$\Rightarrow y \frac{dy}{dx} + x = - \left(\frac{1-t^2}{1+t^2} \right) + \left(\frac{1-t^2}{1+t^2} \right) = 0$$

Thus $y \frac{dy}{dx} + x = 0$ Hence proved.

Q5. Differentiate

(i) $x^2 - \frac{1}{x^2}$ w.r.t x^4

Solution:- Let $y = x^2 - \frac{1}{x^2}$ and $u = x^4$
we have to find $\frac{dy}{du}$.

$$\frac{dy}{dx} = \frac{d}{dx} (x^2 - x^{-2}) = \frac{d}{dx} (x^2) - \frac{d}{dx} (x^{-2})$$

$$= 2x - (-2)x^{-3} = 2x + 2x^{-3}$$

$$\frac{dy}{dx} = 2x + \frac{2}{x^3} = \frac{2x^4 + 2}{x^3}$$

and $\frac{du}{dx} = \frac{d}{dx} (x^4) = 4x^3$

By chain rule, we have

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du} = \frac{2x^4 + 2}{x^3} \times \frac{1}{4x^3}$$

$$\Rightarrow \frac{dy}{du} = \frac{2(x^4 + 1)}{4x^6} = \frac{x^4 + 1}{2x^6}$$

(ii) $(1+x^2)^n$ w.r.t x^2

Solution:- Let $y = (1+x^2)^n$ and $u = x^2$
we have to find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx} ((1+x^2)^n) = n(1+x^2)^{n-1} \cdot \frac{d}{dx} (1+x^2)$$

$$= n(1+x^2)^{n-1} \cdot (2x) = 2xn(1+x^2)^{n-1}$$

and $\frac{du}{dx} = \frac{d}{dx} (x^2) = 2x$

By chain rule, we have

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du} = 2xn(1+x^2)^{n-1} \times \frac{1}{2x}$$

$$\rightarrow \frac{dy}{du} = n(1+x^2)^{n-1}$$

(iii) $\frac{x^2+1}{x^2-1}$ w.r.t $\frac{x-1}{x+1}$

Solution:- Let $y = \frac{x^2+1}{x^2-1}$ and $u = \frac{x-1}{x+1}$

we are to find $\frac{dy}{du}$.

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^2+1}{x^2-1} \right) = \frac{(x^2-1) \frac{d}{dx}(x^2+1) - (x^2+1) \frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} = \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2-1)^2}$$

$$\rightarrow \frac{dy}{dx} = \frac{-4x}{(x^2-1)^2}$$

and $\frac{du}{dx} = \frac{d}{dx} \left(\frac{x-1}{x+1} \right) = \frac{(x+1) \frac{d}{dx}(x-1) - (x-1) \frac{d}{dx}(x+1)}{(x+1)^2}$

$$= \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \frac{x+1-x+1}{(x+1)^2}$$

$$\rightarrow \frac{du}{dx} = \frac{2}{(x+1)^2}$$

By chain rule, we have $\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$

$$\rightarrow \frac{dy}{du} = \frac{-4x}{(x^2-1)^2} \times \frac{(x+1)^2}{2} = \frac{-2x(x+1)^2}{[(x-1)(x+1)]^2}$$

$$\rightarrow \frac{dy}{du} = \frac{-2x(x+1)^2}{(x-1)^2(x+1)^2} = \frac{-2x}{(x-1)^2}$$

(iv) $\frac{ax+b}{cx+d}$ w.r.t $\frac{ax^2+b}{cx^2+d}$

Solution:- Let $y = \frac{ax+b}{cx+d}$ and $u = \frac{ax^2+b}{cx^2+d}$

we have to find $\frac{dy}{du}$.

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{ax+b}{cx+d} \right) = \frac{(cx+d) \frac{d}{dx}(ax+b) - (ax+b) \frac{d}{dx}(cx+d)}{(cx+d)^2}$$

$$\rightarrow \frac{dy}{dx} = \frac{(cx+d)(a) - (ax+b)(c)}{(cx+d)^2} = \frac{acx+ad-acx-bc}{(cx+d)^2}$$

$$\rightarrow \frac{dy}{dx} = \frac{ad-bc}{(cx+d)^2}$$

$$\frac{du}{dx} = \frac{d}{dx} \left(\frac{ax^2+b}{cx^2+d} \right) = \frac{(cx^2+d) \frac{d}{dx}(ax^2+b) - (ax^2+b) \frac{d}{dx}(cx^2+d)}{(cx^2+d)^2}$$

$$= \frac{(cx^2+d)(2ax) - (ax^2+b)(2cx)}{(cx^2+d)^2}$$

$$= \frac{2x\{ax^2+ad-bx^2-bc\}}{(cx^2+d)^2} = \frac{2x(ad-bc)}{(cx^2+d)^2}$$

By chain rule, $\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$

$$\rightarrow \frac{dy}{du} = \frac{ad-bc}{(cx^2+d)^2} \cdot \frac{(cx^2+d)^2}{2x(ad-bc)} = \frac{(cx^2+d)^2}{2x(ad-bc)} \cdot \frac{2x(ad-bc)}{(cx^2+d)^2}$$

(v) $\frac{x^2+1}{x^2-1}$ w.r.t x^3

Solution:- Let $y = \frac{x^2+1}{x^2-1}$ and $u = x^3$

we are to find $\frac{dy}{du}$.

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^2+1}{x^2-1} \right) = \frac{(x^2-1) \frac{d}{dx}(x^2+1) - (x^2+1) \frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} = \frac{2x^3-2x-2x^3-2x}{(x^2-1)^2}$$

$$\rightarrow \frac{dy}{dx} = \frac{-4x}{(x^2-1)^2}$$

$$\frac{du}{dx} = \frac{d}{dx}(x^3) = 3x^2$$

By chain rule, $\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$

$$\rightarrow \frac{dy}{du} = \frac{-4x}{(x^2-1)^2} \times \frac{1}{3x^2} = \frac{-4}{3x(x^2-1)^2}$$

Derivatives of Trigonometric functions:-

1. $\frac{d}{dx}(\sin x) = \cos x$

Proof:- Let $y = \sin x \rightarrow (i)$

$$\rightarrow y + \delta y = \sin(x + \delta x) \rightarrow (ii)$$

$$\text{By (ii) - (i)} \Rightarrow y + \delta y - y = \sin(x + \delta x) - \sin x$$

$$(\because \sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2})$$

$$\rightarrow \delta y = 2 \cos \left(\frac{x + \delta x + x}{2} \right) \sin \left(\frac{x + \delta x - x}{2} \right)$$

$$\delta y = 2 \cos \left(\frac{2x + \delta x}{2} \right) \sin \frac{\delta x}{2}$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[2 \cos \left(x + \frac{\delta x}{2} \right) \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \right]$$

$$\rightarrow \frac{dy}{dx} = 2 \lim_{\delta x \rightarrow 0} \left[\cos \left(x + \frac{\delta x}{2} \right) \cdot \frac{\sin \frac{\delta x}{2}}{2 \times \frac{\delta x}{2}} \right]$$

$$= 2 \lim_{\delta x \rightarrow 0} \cos \left(x + \frac{\delta x}{2} \right) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \frac{\delta x}{2}}{2 \times \frac{\delta x}{2}}$$

$$\frac{dy}{dx} = \cos x \quad \left(\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right)$$

Thus $\frac{d}{dx}(\sin x) = \cos x$

2. $\frac{d}{dx}(\cos x) = -\sin x$

Proof:- Let $y = \cos x \rightarrow (i)$

$$\rightarrow y + \delta y = \cos(x + \delta x) \rightarrow (ii)$$

$$\text{By (ii) - (i)} \Rightarrow y + \delta y - y = \cos(x + \delta x) - \cos x$$

$$\rightarrow \delta y = \cos(x + \delta x) - \cos x$$

$$(\because \cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2})$$

$$= -2 \sin \left(\frac{x + \delta x + x}{2} \right) \sin \left(\frac{x + \delta x - x}{2} \right)$$

$$\delta y = -2 \sin \left(\frac{2x + \delta x}{2} \right) \sin \frac{\delta x}{2}$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[-2 \sin \left(x + \frac{\delta x}{2} \right) \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \right]$$

$$\rightarrow \frac{dy}{dx} = -2 \lim_{\delta x \rightarrow 0} \sin \left(x + \frac{\delta x}{2} \right) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \frac{\delta x}{2}}{2 \times \frac{\delta x}{2}}$$

$$\rightarrow \frac{dy}{dx} = -\sin x \quad \because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

Hence $\frac{d}{dx}(\cos x) = -\sin x$

3. $\frac{d}{dx}(\tan x) = \sec^2 x$

Proof:- Let $y = \tan x \rightarrow (i)$

$$\rightarrow y + \delta y = \tan(x + \delta x) \rightarrow (ii)$$

$$\text{By (ii) - (i)} \Rightarrow y + \delta y - y = \tan(x + \delta x) - \tan x$$

$$\rightarrow \delta y = \frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x + \delta x) \cos x - \cos(x + \delta x) \sin x}{\cos(x + \delta x) \cdot \cos x}$$

$$(\because \sin \alpha \cos \beta - \cos \alpha \sin \beta = \sin(\alpha - \beta))$$

$$\delta y = \frac{\sin(x + \delta x - x)}{\cos(x + \delta x) \cdot \cos x} = \frac{\sin \delta x}{\cos(x + \delta x) \cdot \cos x}$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{\cos(x + \delta x) \cdot \cos x} \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x}$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{\cos x \cdot \cos x} (1) = \frac{1}{\cos^2 x} = \sec^2 x$$

Hence $\frac{d}{dx}(\tan x) = \sec^2 x$



$$4. \frac{d}{dx}(\sec x) = \sec x \tan x$$

Proof:- Let $y = \sec x \rightarrow (i)$

$$\rightarrow y + \delta y = \sec(x + \delta x) \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = \sec(x + \delta x) - \sec x$$

$$\rightarrow \delta y = \frac{1}{\cos(x + \delta x)} - \frac{1}{\cos x} = \frac{\cos x - \cos(x + \delta x)}{\cos(x + \delta x) \cdot \cos x}$$

$$\begin{aligned} (\because \cos A - \cos B &= -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}) \\ &= \frac{-2 \sin \left(\frac{x + x + \delta x}{2} \right) \sin \left(\frac{x - x - \delta x}{2} \right)}{\cos(x + \delta x) \cdot \cos x} \end{aligned}$$

$$= \frac{-2}{\cos(x + \delta x) \cdot \cos x} \left[\sin \left(x + \frac{\delta x}{2} \right) \sin \left(-\frac{\delta x}{2} \right) \right]$$

$$\delta y = \frac{2}{\cos(x + \delta x) \cdot \cos x} \left[\sin \left(x + \frac{\delta x}{2} \right) \sin \frac{\delta x}{2} \right] \quad \because \sin(-\theta) = -\sin \theta$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{2}{\cos(x + \delta x) \cdot \cos x} \left[\lim_{\delta x \rightarrow 0} \sin \left(x + \frac{\delta x}{2} \right) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \right]$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{\cos x \cdot \cos x} (\sin x \cdot 1) = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$\frac{dy}{dx} = \sec x \cdot \tan x \quad \text{Hence } \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$5. \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

Proof:- Let $y = \operatorname{cosec} x \rightarrow (i)$

$$\rightarrow y + \delta y = \operatorname{cosec}(x + \delta x) \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = \operatorname{cosec}(x + \delta x) - \operatorname{cosec} x$$

$$\rightarrow \delta y = \frac{1}{\sin(x + \delta x)} - \frac{1}{\sin x} = \frac{\sin x - \sin(x + \delta x)}{\sin(x + \delta x) \cdot \sin x}$$

$$\begin{aligned} (\because \sin A - \sin B &= 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}) \\ &= \frac{1}{\sin(x + \delta x) \cdot \sin x} \left[2 \cos \left(\frac{x + x + \delta x}{2} \right) \sin \left(\frac{x - x - \delta x}{2} \right) \right] \end{aligned}$$

$$\delta y = \frac{1}{\sin(x + \delta x) \cdot \sin x} \left[2 \cos \left(x + \frac{\delta x}{2} \right) \sin \left(-\frac{\delta x}{2} \right) \right]$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{-1}{\sin(x + \delta x) \sin x} \left[\lim_{\delta x \rightarrow 0} 2 \cos \left(x + \frac{\delta x}{2} \right) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \right]$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{\sin x \cdot \sin x} (\cos x \cdot 1) = \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = -\operatorname{cosec} x \cot x$$

$$\text{Hence } \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$6. \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

Proof:- Let $y = \cot x \rightarrow (i)$

$$y + \delta y = \cot(x + \delta x) \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = \cot(x + \delta x) - \cot x$$

$$\begin{aligned} \rightarrow \delta y &= \frac{\cos(x + \delta x)}{\sin(x + \delta x)} - \frac{\cos x}{\sin x} \\ &= \frac{\cos(x + \delta x) \sin x - \cos x \sin(x + \delta x)}{\sin(x + \delta x) \cdot \sin x} \end{aligned}$$

$$(\because \sin \alpha \cos \beta - \cos \alpha \sin \beta = \sin(\alpha - \beta))$$

$$= \frac{\sin(x - x - \delta x)}{\sin x \sin(x + \delta x)} = \frac{\sin(-\delta x)}{\sin x \cdot \sin(x + \delta x)}$$

$$\rightarrow \delta y = -\frac{\sin \delta x}{\sin x \cdot \sin(x + \delta x)}$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left(\frac{-1}{\sin x \cdot \sin(x + \delta x)} \right) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x}$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{\sin x \cdot \sin x} (1) = \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

$$\text{Hence } \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

Example 1. Find the derivative of $\tan x$ from first principles.

Solution:- see proof # 3 $\frac{d}{dx}(\tan x) = \sec^2 x$ at page # 19.

Example 2. Differentiate ab-initio w.r.t 'x'

- (i) $\cos 2x$ (ii) $\sin \sqrt{x}$ (iii) $\cot^2 x$

Solution:- (i) Let $y = \cos 2x \rightarrow (i)$

$$\rightarrow y + \delta y = \cos 2(x + \delta x) \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = \cos(2x + 2\delta x) - \cos 2x$$

$$(\because \cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2})$$

$$\rightarrow \delta y = -2 \sin \left(\frac{2x + 2\delta x + 2x}{2} \right) \sin \left(\frac{2x + 2\delta x - 2x}{2} \right)$$

$$\delta y = -2 \sin \left(\frac{4x + 2\delta x}{2} \right) \sin \delta x$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (-2 \sin(2x + \delta x)) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x}$$

$$\rightarrow \frac{dy}{dx} = -2 \sin 2x (1) = -2 \sin 2x$$

(ii) $\sin \sqrt{x}$ Let $y = \sin \sqrt{x} \rightarrow (i)$

$$\rightarrow y + \delta y = \sin \sqrt{x + \delta x} \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = \sin \sqrt{x + \delta x} - \sin \sqrt{x}$$

$$(\because \sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2})$$

$$\rightarrow \delta y = 2 \cos \frac{\sqrt{x + \delta x} + \sqrt{x}}{2} \sin \frac{\sqrt{x + \delta x} - \sqrt{x}}{2}$$

Dividing by δx

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{2 \cos \frac{\sqrt{x+\delta x} + \sqrt{x}}{2} \sin \frac{\sqrt{x+\delta x} - \sqrt{x}}{2}}{\delta x}$$

As $(\sqrt{x+\delta x} + \sqrt{x})(\sqrt{x+\delta x} - \sqrt{x}) = \delta x$, so

$$\begin{aligned} \frac{\delta y}{\delta x} &= \frac{2 \cos \frac{\sqrt{x+\delta x} + \sqrt{x}}{2} \sin \frac{\sqrt{x+\delta x} - \sqrt{x}}{2}}{(\sqrt{x+\delta x} + \sqrt{x})(\sqrt{x+\delta x} - \sqrt{x})} \\ &= \frac{2 \cos \frac{\sqrt{x+\delta x} + \sqrt{x}}{2}}{\sqrt{x+\delta x} + \sqrt{x}} \cdot \frac{\sin \frac{\sqrt{x+\delta x} - \sqrt{x}}{2}}{\frac{\sqrt{x+\delta x} - \sqrt{x}}{2}} \end{aligned}$$

Take limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \cos \frac{\sqrt{x+\delta x} + \sqrt{x}}{2} \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \frac{\sqrt{x+\delta x} - \sqrt{x}}{2}}{\frac{\sqrt{x+\delta x} - \sqrt{x}}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos \frac{\sqrt{x} + \sqrt{x}}{2}}{\sqrt{x} + \sqrt{x}} (1) = \frac{\cos \frac{2\sqrt{x}}{2}}{2\sqrt{x}} = \frac{\cos \sqrt{x}}{2\sqrt{x}}$$

(iii) $\cot^2 x$

Solution:- Let $y = \cot^2 x \rightarrow$ (i)

$$\rightarrow y + \delta y = \cot^2(x + \delta x) \rightarrow$$
 (ii)

$$\text{By (ii) - (i)} \Rightarrow y + \delta y - y = \cot^2(x + \delta x) - \cot^2 x$$

$$\Rightarrow \delta y = \cot^2(x + \delta x) - \cot^2 x = [\cot(x + \delta x) + \cot x][\cot(x + \delta x) - \cot x]$$

$$= [\cot(x + \delta x) + \cot x] \left[\frac{\cos(x + \delta x)}{\sin(x + \delta x)} - \frac{\cos x}{\sin x} \right]$$

$$= [\cot(x + \delta x) + \cot x] \left[\frac{\sin x \cdot \cos(x + \delta x) - \cos x \cdot \sin(x + \delta x)}{\sin x \cdot \sin(x + \delta x)} \right]$$

$$(\because \sin \alpha \cos \beta - \cos \alpha \sin \beta = \sin(\alpha - \beta))$$

$$= [\cot(x + \delta x) + \cot x] \left[\frac{\sin(x - x - \delta x)}{\sin x \cdot \sin(x + \delta x)} \right]$$

$$\delta y = (\cot(x + \delta x) + \cot x) \cdot \frac{\sin(-\delta x)}{\sin x \cdot \sin(x + \delta x)}$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\Rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (\cot(x + \delta x) + \cot x) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x} \cdot \lim_{\delta x \rightarrow 0} \frac{-1}{\sin x \cdot \sin(x + \delta x)}$$

$$\frac{dy}{dx} = (\cot x + \cot x) (1) \left(\frac{-1}{\sin x \cdot \sin x} \right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2 \cot x}{\sin^2 x} = -2 \cot x \sec^2 x$$

Example 3. Differentiate $\sin^3 x$ w.r.t $\cos^2 x$

Solution:- Let $y = \sin^3 x$ and $u = \cos^2 x$

we have to find $\frac{dy}{du}$. so

$$\frac{dy}{dx} = \frac{d}{dx} (\sin^3 x) = 3 \sin^2 x \frac{d}{dx} (\sin x)$$

$$= 3 \sin^2 x \cos x$$

$$\frac{du}{dx} = \frac{d}{dx} (\cos^2 x) = 2 \cos x \frac{d}{dx} (\cos x)$$

$$= 2 \cos x (-\sin x) = -2 \cos x \sin x$$

By chain rule, we have

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$$

$$\Rightarrow \frac{dy}{du} = (3 \sin^2 x \cos x) \left(\frac{-1}{2 \cos x \sin x} \right)$$

$$\Rightarrow \frac{dy}{du} = -\frac{3}{2} \sin x$$

Derivatives of inverse trigonometric functions:-

$$1. \frac{d}{dx} [\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}} \quad x \in (-1, 1) \text{ or } -1 < x < 1$$

Proof:- Let $y = \sin^{-1} x$

$$\rightarrow \sin y = x \rightarrow \frac{d}{dx} (\sin y) = \frac{d}{dx} (x)$$

$$\rightarrow \cos y \cdot \frac{dy}{dx} = 1 \rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} \quad \because \cos^2 \theta = 1 - \sin^2 \theta$$

$$\rightarrow \cos \theta = \sqrt{1-\sin^2 \theta}$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \because \sin y = x$$

$$\text{Hence } \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$2. \frac{d}{dx} (\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, \quad x \in (-1, 1)$$

Proof:- Let $y = \cos^{-1} x \rightarrow \cos y = x$

$$\rightarrow \frac{d}{dx} (\cos y) = \frac{d}{dx} (x) \rightarrow -\sin y \frac{dy}{dx} = 1$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}}$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} \quad \because \sin^2 \theta = 1 - \cos^2 \theta$$

$$\rightarrow \sin \theta = \sqrt{1-\cos^2 \theta}$$

$$\because \cos y = x$$

$$\text{Hence } \frac{d}{dx} (\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$3. \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

Proof:- Let $y = \tan^{-1} x$

$$\rightarrow \tan y = x \rightarrow \frac{d}{dx} (\tan y) = \frac{d}{dx} (x)$$

$$\rightarrow \sec^2 y \frac{dy}{dx} = 1 \rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{1+\tan^2 y} \quad \because 1+\tan^2 \theta = \sec^2 \theta$$

$$\because \tan y = x$$

$$\text{Hence } \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$$

$$4. \frac{d}{dx} (\cot^{-1} x) = \frac{-1}{1+x^2}, \quad x \in (-\infty, \infty) \text{ or } (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

Proof:- Let $y = \cot^{-1} x$

$$\rightarrow \cot y = x \rightarrow \frac{d}{dx} (\cot y) = \frac{d}{dx} (x)$$

$$\rightarrow -\operatorname{cosec}^2 y \frac{dy}{dx} = 1 \rightarrow \frac{dy}{dx} = \frac{-1}{\operatorname{cosec}^2 y}$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{1+\cot^2 y} \quad \because \operatorname{cosec}^2 \theta = 1 + \cot^2 \theta$$

$$\because \cot y = x$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{1+x^2} \quad \text{Hence } \frac{d}{dx} (\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$5. \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}, \quad x \in (-1, 1)$$

Proof:- Let $y = \sec^{-1}x \Rightarrow \sec y = x$

$$\Rightarrow \frac{d}{dx}(\sec y) = \frac{d}{dx}(x) \Rightarrow \sec y \tan y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y \sqrt{\sec^2 y - 1}} \quad \because 1 + \tan^2 \theta = \sec^2 \theta$$

$$\Rightarrow \tan \theta = \sqrt{\sec^2 \theta - 1}$$

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}} \quad \because \sec y = x$$

$$\text{Hence } \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$6. \frac{d}{dx}(\csc^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}, \quad x \in \mathbb{R}$$

Proof:- Let $y = \csc^{-1}x$

$$\Rightarrow \csc y = x \Rightarrow \frac{d}{dx}(\csc y) = \frac{d}{dx}(x)$$

$$\Rightarrow -\csc y \cot y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{-1}{\csc y \cot y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\csc y \sqrt{\csc^2 y - 1}} \quad \because 1 + \cot^2 \theta = \csc^2 \theta$$

$$\Rightarrow \cot^2 \theta = \csc^2 \theta - 1$$

$$\Rightarrow \cot \theta = \sqrt{\csc^2 \theta - 1}$$

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}} \quad \because \csc y = x$$

$$\text{Hence } \frac{d}{dx}(\csc^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$$

Example 1. Find $\frac{dy}{dx}$ if $y = x \sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2}$

Solution:- $y = x \sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2}$

$$\frac{dy}{dx} = \frac{d}{dx} \left\{ x \sin^{-1}\left(\frac{x}{a}\right) + (a^2 - x^2)^{\frac{1}{2}} \right\}$$

$$= x \frac{d}{dx} \sin^{-1} \frac{x}{a} + \sin^{-1} \left(\frac{x}{a} \right) \cdot \frac{d}{dx}(x) + \frac{d}{dx} (a^2 - x^2)^{\frac{1}{2}}$$

$$= x \cdot \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} \cdot \frac{d}{dx} \left(\frac{x}{a} \right) + \sin^{-1} \left(\frac{x}{a} \right) (1) + \frac{1}{2} (a^2 - x^2)^{\frac{1}{2}-1} \cdot \frac{d}{dx} (a^2 - x^2)$$

$$= \frac{x}{\sqrt{a^2 - x^2}} \cdot \frac{1}{a} \cdot \frac{d}{dx}(x) + \sin^{-1} \left(\frac{x}{a} \right) + \frac{1}{2} (a^2 - x^2)^{-\frac{1}{2}} (-2x)$$

$$= \frac{x}{\sqrt{a^2 - x^2}} \cdot \frac{1}{a} (1) + \sin^{-1} \left(\frac{x}{a} \right) - \frac{x}{\sqrt{a^2 - x^2}}$$

$$= \frac{x}{\sqrt{a^2 - x^2}} + \sin^{-1} \left(\frac{x}{a} \right) - \frac{x}{\sqrt{a^2 - x^2}}$$

$$= \sin^{-1} \left(\frac{x}{a} \right) \Rightarrow \frac{dy}{dx} = \sin^{-1} \left(\frac{x}{a} \right)$$

Example 2. If $y = \tan(2 \tan^{-1} \frac{x}{2})$, show that $\frac{dy}{dx} = \frac{4(1+y^2)}{4+x^2}$

Solution:- $y = \tan(2 \tan^{-1} \frac{x}{2})$

$$\frac{dy}{dx} = \frac{d}{dx} \left\{ \tan(2 \tan^{-1} \frac{x}{2}) \right\}$$

$$= \sec^2(2 \tan^{-1} \frac{x}{2}) \cdot \frac{d}{dx} (2 \tan^{-1} \frac{x}{2})$$

$$= 1 + \tan^2(2 \tan^{-1} \frac{x}{2}) \cdot 2 \cdot \frac{1}{1 + (\frac{x}{2})^2} \cdot \frac{d}{dx} \left(\frac{x}{2} \right)$$

$$(\because 1 + \tan^2 \theta = \sec^2 \theta)$$

$$= (1+y^2) \cdot \cancel{x} \cdot \frac{1}{1 + \frac{x^2}{4}} \cdot \frac{1}{2} \quad \because \tan(2 \tan^{-1} x) = y$$

$$= (1+y^2) \cdot \frac{1}{\frac{4+x^2}{4}} = \frac{4(1+y^2)}{4+x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4(1+y^2)}{4+x^2}$$



Exercise 2.5

Q1. Differentiate the following trigonometric functions from the first principles.

(i) $\sin 2x$

Solution:- Let $y = \sin 2x \rightarrow$ (i)

$$\Rightarrow y + \delta y = \sin 2(x + \delta x) = \sin(2x + 2\delta x) \rightarrow$$

$$\text{By (i) - (i)} \Rightarrow y + \delta y - y = \sin(2x + 2\delta x) - \sin 2x$$

$$\Rightarrow \delta y = \sin(2x + 2\delta x) - \sin 2x$$

$$(\because \sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2})$$

$$= 2 \cos \left(\frac{2x + 2\delta x + 2x}{2} \right) \sin \left(\frac{2x + 2\delta x - 2x}{2} \right)$$

$$\delta y = 2 \cos \left(\frac{4x + 2\delta x}{2} \right) \sin \delta x$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = 2 \lim_{\delta x \rightarrow 0} \cos(2x + \delta x) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = 2 \cos(2x) (1) = 2 \cos 2x$$

(ii) $\tan 3x$

Solution:- $y = \tan 3x \rightarrow$ (i)

$$\Rightarrow y + \delta y = \tan 3(x + \delta x) = \tan(3x + 3\delta x) \rightarrow$$

$$\text{By (i) - (i)} \Rightarrow y + \delta y - y = \tan(3x + 3\delta x) - \tan 3x$$

$$\Rightarrow \delta y = \frac{\sin(3x + 3\delta x)}{\cos(3x + 3\delta x)} - \frac{\sin 3x}{\cos 3x}$$

$$= \frac{\sin(3x + 3\delta x) \cdot \cos 3x - \cos(3x + 3\delta x) \sin 3x}{\cos 3x \cdot \cos(3x + 3\delta x)}$$

$$(\because \sin \alpha \cos \beta - \cos \alpha \sin \beta = \sin(\alpha - \beta))$$

$$\delta y = \frac{\sin(3x + 3\delta x - 3x)}{\cos 3x \cdot \cos(3x + 3\delta x)}$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\Rightarrow \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{\cos 3x \cdot \cos(3x + 3\delta x)} \cdot \lim_{\delta x \rightarrow 0} \frac{\sin 3\delta x}{3\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos 3x \cdot \cos 3x} 3(1) = \frac{3}{\cos^2 3x}$$

$$\frac{dy}{dx} = 3 \sec^2 3x$$

(iii) $\sin 2x + \cos 2x$

Solution:- $y = \sin 2x + \cos 2x \rightarrow (i)$

$\rightarrow y + \delta y = \sin(2x + 2\delta x) + \cos(2x + 2\delta x) \rightarrow (ii)$

By (ii) - (i) $\rightarrow y + \delta y - y = \sin(2x + 2\delta x) + \cos(2x + 2\delta x) - \sin 2x - \cos 2x$

$\rightarrow \delta y = \sin(2x + 2\delta x) - \sin 2x + \cos(2x + 2\delta x) - \cos 2x$

$(\because \sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2})$
 $(\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2})$

$= 2 \cos \left(\frac{2x + 2\delta x + 2x}{2} \right) \sin \left(\frac{2x + 2\delta x - 2x}{2} \right)$
 $+ (-2 \sin \left(\frac{2x + 2\delta x + 2x}{2} \right) \sin \left(\frac{2x + 2\delta x - 2x}{2} \right))$

$\delta y = 2 \cos \left(\frac{4x + 2\delta x}{2} \right) \sin \delta x - 2 \sin \left(\frac{4x + 2\delta x}{2} \right) \sin \delta x$

Dividing by δx and take limit $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = 2 \cdot \lim_{\delta x \rightarrow 0} \cos(2x + \delta x) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x} - 2 \lim_{\delta x \rightarrow 0} \sin(2x + \delta x) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x}$

$\rightarrow \frac{dy}{dx} = 2 \cos 2x (1) - 2 \sin 2x (1) \quad \because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$

$\rightarrow \frac{dy}{dx} = 2 \cos 2x - 2 \sin 2x$

(iv) $\cos x^2$

Solution:- Let $y = \cos x^2 \rightarrow (i)$

$\rightarrow y + \delta y = \cos(x + \delta x)^2 \rightarrow (ii)$

By (ii) - (i) $\rightarrow y + \delta y - y = \cos(x + \delta x)^2 - \cos x^2$

$(\because \cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2})$

$\rightarrow \delta y = -2 \sin \frac{(x + \delta x)^2 + x^2}{2} \cdot \sin \frac{(x + \delta x)^2 - x^2}{2}$

$= -2 \sin \frac{(x + \delta x)^2 + x^2}{2} \cdot \sin \frac{x^2 + (\delta x)^2 + 2x(\delta x) - x^2}{2}$

$= -2 \sin \frac{(x + \delta x)^2 + x^2}{2} \cdot \sin \frac{(\delta x)^2 + 2x(\delta x)}{2}$

$\delta y = -2 \sin \frac{(x + \delta x)^2 + x^2}{2} \cdot \frac{\sin \left(\frac{(\delta x)^2 + 2x(\delta x)}{2} \right) \delta x}{\left(\frac{(\delta x)^2 + 2x(\delta x)}{2} \right)}$

Dividing by δx and take limit $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left(-2 \sin \frac{(x + \delta x)^2 + x^2}{2} \right) \cdot \lim_{\delta x \rightarrow 0} \left(\frac{\sin \left(\frac{(\delta x)^2 + 2x(\delta x)}{2} \right) \cdot \delta x}{\left(\frac{(\delta x)^2 + 2x(\delta x)}{2} \right)} \right)$

$\cdot \lim_{\delta x \rightarrow 0} \left(\frac{(\delta x)^2 + 2x(\delta x)}{2} \right)$

$\rightarrow \frac{dy}{dx} = -2 \sin \frac{x^2 + x^2}{2} (1) \cdot x = -2 \sin \frac{2x^2}{2} (x)$

$\frac{dy}{dx} = -2x \sin x^2$

(v) $\tan^2 x$

Solution:- Let $y = \tan^2 x \rightarrow (i)$

$\rightarrow y + \delta y = \tan^2(x + \delta x) \rightarrow (ii)$

By (ii) - (i) $\rightarrow y + \delta y - y = \tan^2(x + \delta x) - \tan^2 x$

$(\because a^2 - b^2 = (a - b)(a + b))$

$\rightarrow \delta y = (\tan(x + \delta x) + \tan x)(\tan(x + \delta x) - \tan x)$

$= [\tan(x + \delta x) + \tan x] \left[\frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x} \right]$

$= [\tan(x + \delta x) + \tan x] \left[\frac{\cos x \cdot \sin(x + \delta x) - \sin x \cdot \cos(x + \delta x)}{\cos x \cdot \cos(x + \delta x)} \right]$

$(\because \sin \alpha \cos \beta - \cos \alpha \sin \beta = \sin(\alpha - \beta))$

$= [\tan(x + \delta x) + \tan x] \left[\frac{\sin(x + \delta x - x)}{\cos x \cdot \cos(x + \delta x)} \right]$

Dividing by δx and take limit $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (\tan(x + \delta x) + \tan x) \cdot \lim_{\delta x \rightarrow 0} \frac{1}{\cos x \cdot \cos(x + \delta x)}$
 $\cdot \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x}$

$\rightarrow \frac{dy}{dx} = (\tan x + \tan x) \cdot \frac{1}{\cos x \cdot \cos x} (1)$

$\frac{dy}{dx} = \frac{2 \tan x}{\cos^2 x} = 2 \tan x \sec^2 x$

(vi) $\sqrt{\tan x}$

Solution:- Let $y = \sqrt{\tan x} \rightarrow (i)$

$\rightarrow y + \delta y = \sqrt{\tan(x + \delta x)} \rightarrow (ii)$

By (ii) - (i) $\rightarrow y + \delta y - y = \sqrt{\tan(x + \delta x)} - \sqrt{\tan x}$

$\rightarrow \delta y = \sqrt{\tan(x + \delta x)} - \sqrt{\tan x} \times \frac{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}}$

$= \frac{(\sqrt{\tan(x + \delta x)})^2 - (\sqrt{\tan x})^2}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}}$

$= \frac{\tan(x + \delta x) - \tan x}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}}$

$= \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \left[\frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x} \right]$

$= \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \left[\frac{\sin(x + \delta x) \cdot \cos x - \cos(x + \delta x) \cdot \sin x}{\cos x \cdot \cos(x + \delta x)} \right]$

$(\because \sin \alpha \cos \beta - \cos \alpha \sin \beta = \sin(\alpha - \beta))$

$= \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \left[\frac{\sin(x + \delta x - x)}{\cos x \cdot \cos(x + \delta x)} \right]$

$= \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \left[\frac{\sin(x + \delta x - x)}{\cos x \cdot \cos(x + \delta x)} \right]$

Dividing by δx and take limit $\delta x \rightarrow 0$

$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \cdot \lim_{\delta x \rightarrow 0} \frac{1}{\cos x \cdot \cos(x + \delta x)}$
 $\cdot \lim_{\delta x \rightarrow 0} \frac{\sin \delta x}{\delta x}$

$\rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{\tan x} + \sqrt{\tan x}} \cdot \frac{1}{\cos x \cdot \cos x} (1)$

$\rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\tan x} \cos^2 x} = \frac{\sec^2 x}{2\sqrt{\tan x}}$

(vii) $\cos \sqrt{x}$

Solution:- Let $y = \cos \sqrt{x} \longrightarrow$ (i)

$$\rightarrow y + \delta y = \cos \sqrt{x + \delta x} \longrightarrow$$
 (ii)

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = \cos \sqrt{x + \delta x} - \cos \sqrt{x}$$

$$\rightarrow \delta y = \cos \sqrt{x + \delta x} - \cos \sqrt{x}$$

$$\left(\because \cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \right)$$

$$= -2 \sin \frac{\sqrt{x + \delta x} + \sqrt{x}}{2} \sin \frac{\sqrt{x + \delta x} - \sqrt{x}}{2}$$

Dividing by δx

$$\rightarrow \frac{\delta y}{\delta x} = \frac{-2 \sin \frac{\sqrt{x + \delta x} + \sqrt{x}}{2} \sin \frac{\sqrt{x + \delta x} - \sqrt{x}}{2}}{\delta x}$$

$$\therefore (\sqrt{x + \delta x} + \sqrt{x})(\sqrt{x + \delta x} - \sqrt{x}) = \delta x$$

$$\rightarrow \frac{\delta y}{\delta x} = \frac{-2 \sin \frac{\sqrt{x + \delta x} + \sqrt{x}}{2} \cdot \frac{\sin \frac{\sqrt{x + \delta x} - \sqrt{x}}{2}}{\frac{\sqrt{x + \delta x} + \sqrt{x}}{2}}}{\frac{\delta x}{\sqrt{x + \delta x} + \sqrt{x}}}$$

Take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{-2 \sin \frac{\sqrt{x + \delta x} + \sqrt{x}}{2}}{\sqrt{x + \delta x} + \sqrt{x}} \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \frac{\sqrt{x + \delta x} - \sqrt{x}}{2}}{\frac{\sqrt{x + \delta x} - \sqrt{x}}{2}}$$

$$\rightarrow \frac{dy}{dx} = \frac{-2 \sin \frac{\sqrt{x} + \sqrt{x}}{2}}{\sqrt{x} + \sqrt{x}} (1) = \frac{-2 \sin \frac{2\sqrt{x}}{2}}{2\sqrt{x}}$$

$$\rightarrow \frac{dy}{dx} = \frac{-\sin \sqrt{x}}{\sqrt{x}}$$

Q2. Differentiate the following w.r.t the variable involved

(i) $x^2 \sec 4x$

Solution:- Let $y = x^2 \sec 4x$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} (x^2 \sec 4x) = x^2 \frac{d}{dx} (\sec 4x) + \sec 4x \frac{d}{dx} (x^2)$$

$$= x^2 (\sec 4x \tan 4x) \frac{d}{dx} (4x) + \sec 4x (2x)$$

$$\rightarrow \frac{dy}{dx} = 4x^2 (\sec 4x \tan 4x) + 2x \sec 4x$$

(ii) $\tan^3 \theta \sec^2 \theta$

Solution:- Let $y = \tan^3 \theta \sec^2 \theta$

$$\frac{dy}{d\theta} = \frac{d}{d\theta} (\tan^3 \theta \sec^2 \theta) = \tan^3 \theta \frac{d}{d\theta} (\sec^2 \theta) + \sec^2 \theta \frac{d}{d\theta} (\tan^3 \theta)$$

$$= \tan^3 \theta (2 \sec \theta) \frac{d}{d\theta} (\sec \theta) + \sec^2 \theta (3 \tan^2 \theta) \frac{d}{d\theta} (\tan \theta)$$

$$= 2 \tan^3 \theta \sec \theta (\sec \theta \tan \theta) + 3 \sec^2 \theta \tan^2 \theta (\sec^2 \theta)$$

$$\rightarrow \frac{dy}{d\theta} = 2 \tan^4 \theta \sec^2 \theta + 3 \sec^4 \theta \tan^2 \theta$$

(iii) $(\sin 2\theta - \cos 3\theta)^2$

Solution:- Let $y = (\sin 2\theta - \cos 3\theta)^2$

$$\frac{dy}{d\theta} = \frac{d}{d\theta} (\sin 2\theta - \cos 3\theta)^2$$

$$= 2 (\sin 2\theta - \cos 3\theta) \frac{d}{d\theta} (\sin 2\theta - \cos 3\theta)$$

$$\rightarrow \frac{dy}{d\theta} = 2 (\sin 2\theta - \cos 3\theta) \left[\cos 2\theta \cdot \frac{d}{d\theta} (2\theta) - (-\sin 3\theta) \frac{d}{d\theta} (3\theta) \right]$$

$$\rightarrow \frac{dy}{d\theta} = 2 (\sin 2\theta - \cos 3\theta) (2 \cos 2\theta + 3 \sin 3\theta)$$

(iv) $\cos \sqrt{x} + \sqrt{\sin x}$

Solution:- Let $y = \cos x^{\frac{1}{2}} + (\sin x)^{\frac{1}{2}}$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} (\cos x^{\frac{1}{2}} + (\sin x)^{\frac{1}{2}})$$

$$= \frac{d}{dx} (\cos x^{\frac{1}{2}}) + \frac{d}{dx} (\sin x)^{\frac{1}{2}}$$

$$= -\sin x^{\frac{1}{2}} \frac{d}{dx} (x^{\frac{1}{2}}) + \frac{1}{2} (\sin x)^{-\frac{1}{2}} \cdot \frac{d}{dx} (\sin x)$$

$$= -\sin \sqrt{x} \cdot \frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} (\sin x)^{-\frac{1}{2}} \cdot (\cos x)$$

$$\frac{dy}{dx} = \frac{-\sin \sqrt{x}}{2\sqrt{x}} + \frac{\cos x}{2\sqrt{\sin x}}$$

Q3. Find $\frac{dy}{dx}$ if (i) $y = x \cos y$

(ii) $x = y \sin y$

(i) $y = x \cos y$

Solution:- $y = x \cos y \rightarrow \frac{dy}{dx} = \frac{d}{dx} (x \cos y)$

$$= x \frac{d}{dx} (\cos y) + \cos y \frac{d}{dx} (x)$$

$$= x (-\sin y) \frac{dy}{dx} + \cos y (1)$$

$$\rightarrow \frac{dy}{dx} = -x \sin y \frac{dy}{dx} + \cos y$$

$$\rightarrow \frac{dy}{dx} + x \sin y \frac{dy}{dx} = \cos y$$

$$\rightarrow \frac{dy}{dx} (1 + x \sin y) = \cos y \rightarrow \frac{dy}{dx} = \frac{\cos y}{1 + x \sin y}$$

(ii) $x = y \sin y$

$$\rightarrow \frac{d}{dx} (x) = \frac{d}{dx} (y \sin y) \Rightarrow 1 = y \frac{d}{dx} (\sin y) + \sin y \frac{d}{dx} (y)$$

$$\rightarrow 1 = y (\cos y) \frac{dy}{dx} + \sin y \frac{dy}{dx}$$

$$1 = y \cos y \frac{dy}{dx} + \sin y \frac{dy}{dx} = (y \cos y + \sin y) \frac{dy}{dx}$$

$$\rightarrow \frac{1}{y \cos y + \sin y} = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{y \cos y + \sin y}$$

Q4. Find the derivative w.r.t x

(i) $\cos \sqrt{\frac{1+x}{1+2x}}$

(ii) $\sin \sqrt{\frac{1+2x}{1+x}}$

(i) $\cos \sqrt{\frac{1+x}{1+2x}}$

Solution:- Let $y = \cos \sqrt{\frac{1+x}{1+2x}}$

$$\frac{dy}{dx} = \frac{d}{dx} \left(\cos \sqrt{\frac{1+x}{1+2x}} \right) = -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{d}{dx} \left(\frac{1+x}{1+2x} \right)^{\frac{1}{2}}$$

$$= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{1}{2} \left(\frac{1+x}{1+2x} \right)^{-\frac{1}{2}} \cdot \frac{d}{dx} \left(\frac{1+x}{1+2x} \right)$$

$$\begin{aligned}
 &= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{1}{2} \left(\frac{1+x}{1+2x} \right)^{-\frac{1}{2}} \cdot \frac{\frac{d}{dx}(1+x) - (1+x) \frac{d}{dx}(1+2x)}{(1+2x)^2} \\
 &= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{1}{2} \left(\frac{1+x}{1+2x} \right)^{-\frac{1}{2}} \cdot \frac{(1+2x)(1) - (1+x)(2)}{(1+2x)^2} \\
 &= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{(1+x)^{-\frac{1}{2}}}{2(1+2x)^{\frac{3}{2}}} \cdot \frac{1+2x-2-2x}{(1+2x)^2} \\
 &= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{(1+x)^{-\frac{1}{2}}}{2(1+2x)^{\frac{3}{2}}} \cdot \frac{-1}{(1+2x)^2} \\
 &= \sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{1}{2(1+2x)^{\frac{3}{2}}(1+x)^{\frac{1}{2}}} \\
 \Rightarrow \frac{dy}{dx} &= \frac{\sin \sqrt{\frac{1+x}{1+2x}}}{2(1+2x)^{\frac{3}{2}} \sqrt{1+x}}
 \end{aligned}$$

(ii) $\sin \sqrt{\frac{1+2x}{1+x}}$

Solution:- Let $y = \sin \sqrt{\frac{1+2x}{1+x}}$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \left(\sin \sqrt{\frac{1+2x}{1+x}} \right) = \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{d}{dx} \left(\frac{1+2x}{1+x} \right)^{\frac{1}{2}} \\
 &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{1}{2} \left(\frac{1+2x}{1+x} \right)^{-\frac{1}{2}} \cdot \frac{d}{dx} \left(\frac{1+2x}{1+x} \right) \\
 &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{1}{2} \left(\frac{1+2x}{1+x} \right)^{-\frac{1}{2}} \cdot \frac{\frac{d}{dx}(1+2x) - (1+2x) \frac{d}{dx}(1+x)}{(1+x)^2} \\
 &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{(1+2x)^{-\frac{1}{2}}}{2(1+x)^{\frac{3}{2}}} \cdot \frac{(1+x)(2) - (1+2x)(1)}{(1+x)^2} \\
 &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{(1+2x)^{-\frac{1}{2}}}{2(1+x)^{\frac{3}{2}}} \cdot \frac{2+2x-1-2x}{(1+x)^2} \\
 &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{1}{2(1+x)^{\frac{3}{2}}(1+2x)^{\frac{1}{2}}} \\
 \frac{dy}{dx} &= \cos \sqrt{\frac{1+2x}{1+x}} \cdot \frac{1}{2(1+x)^{\frac{3}{2}} \sqrt{1+2x}}
 \end{aligned}$$

Q5. Differentiate

(i) $\sin x$ w.r.t $\cot x$

(ii) $\sin^2 x$ w.r.t $\cos^4 x$

Solution:- (i) $\sin x$ w.r.t $\cot x$

Let $y = \sin x$ and $u = \cot x$

we have to find $\frac{dy}{du}$. so

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x) = \cos x$$

$$\frac{du}{dx} = \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

By chain rule, we have

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du} = \cos x \cdot \frac{1}{-\operatorname{cosec}^2 x}$$

$$\frac{dy}{du} = \cos x (-\sin^2 x) = -\cos x \sin^2 x$$

(ii) Let $y = \sin^2 x$, $u = \cos^4 x$

we have to find $\frac{dy}{du}$, so

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^2 x) = 2 \sin x \frac{d}{dx}(\sin x) = 2 \sin x \cos x$$

$$\begin{aligned} \frac{du}{dx} &= \frac{d}{dx}(\cos^4 x) = 4 \cos^3 x \frac{d}{dx}(\cos x) = 4 \cos^3 x (-\sin x) \\ &= -4 \cos^3 x \sin x \end{aligned}$$

By chain rule, we have

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du} = \frac{2 \sin x \cos x}{-4 \cos^3 x \sin x} = \frac{1}{-2 \cos^2 x}$$

$$\Rightarrow \frac{dy}{du} = \frac{-1}{2 \cos^2 x} = -\frac{1}{2} \sec^2 x$$

Q6. If $\tan y(1+\tan x) = 1 - \tan x$, show that $\frac{dy}{dx} = -1$

Solution:- $\tan y(1+\tan x) = 1 - \tan x$

$$\Rightarrow \tan y = \frac{1 - \tan x}{1 + \tan x} \quad \text{put } 1 = \tan \frac{\pi}{4}$$

$$\Rightarrow \tan y = \frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \tan x}$$

$$\left(\because \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \right)$$

$$\Rightarrow \tan y = \tan \left(\frac{\pi}{4} - x \right)$$

$$\Rightarrow y = \frac{\pi}{4} - x \Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\frac{\pi}{4} - x \right)$$

$$\Rightarrow \frac{dy}{dx} = -1 \quad \text{Hence proved.}$$

Q7. If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots + \infty}}}$ prove that $(2y-1) \frac{dy}{dx} = \sec^2 x$

Solution:- $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots + \infty}}}$

we can observe that

$$\Rightarrow y = \sqrt{\tan x + y}$$

squaring on both sides

$$\Rightarrow y^2 = \tan x + y$$

$$\Rightarrow \frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x + y) \Rightarrow 2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow 2y \frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

$$\Rightarrow (2y-1) \frac{dy}{dx} = \sec^2 x \quad \text{Hence proved.}$$

Q8. If $x = a \cos^3 \theta$, $y = b \sin^3 \theta$, show that $a \frac{dy}{dx} + b \tan \theta = 0$

Solution:- $x = a \cos^3 \theta$

$$\Rightarrow \frac{d}{d\theta}(x) = \frac{d}{d\theta}(a \cos^3 \theta) = a \frac{d}{d\theta}(\cos^3 \theta)$$

$$= a \cdot 3 \cos^2 \theta \frac{d}{d\theta}(\cos \theta) = a \cdot 3 \cos^2 \theta (-\sin \theta) = -3a \cos^2 \theta \sin \theta$$

$$\text{Now } y = b \sin^3 \theta \rightarrow \frac{dy}{d\theta} = \frac{d}{d\theta} (b \sin^3 \theta)$$

$$\rightarrow \frac{dy}{d\theta} = b \cdot \frac{d}{d\theta} (\sin^3 \theta) = b \cdot 3 \sin^2 \theta \cdot \frac{d}{d\theta} (\sin \theta)$$

$$= 3b \sin^2 \theta \cos \theta$$

By chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = 3b \sin^2 \theta \cos \theta \left(\frac{1}{-3a \cos^3 \theta \sin \theta} \right)$$

$$\frac{dy}{dx} = -\frac{b}{a} \frac{\sin \theta}{\cos \theta} = -\frac{b}{a} \tan \theta$$

$$\rightarrow a \frac{dy}{dx} = -b \tan \theta \rightarrow a \frac{dy}{dx} + b \tan \theta = 0$$

Hence proved.

Q9. Find $\frac{dy}{dx}$ if $x = a(\cos t + \sin t)$,
 $y = a(\sin t - t \cos t)$

Solution:- $x = a(\cos t + \sin t)$

$$\rightarrow \frac{d}{dt}(x) = \frac{d}{dt} \{a(\cos t + \sin t)\}$$

$$= a \left\{ \frac{d}{dt}(\cos t) + \frac{d}{dt}(\sin t) \right\}$$

$$\rightarrow \frac{dx}{dt} = a(-\sin t + \cos t)$$

$$y = a(\sin t - t \cos t)$$

$$\rightarrow \frac{dy}{dt} = \frac{d}{dt} \{a(\sin t - t \cos t)\}$$

$$= a \left\{ \frac{d}{dt}(\sin t) - \frac{d}{dt}(t \cos t) \right\}$$

$$= a \left(\cos t - \left(t \frac{d}{dt}(\cos t) + \cos t \frac{d}{dt}(t) \right) \right)$$

$$= a \left(\cos t - (t(-\sin t) + \cos t(1)) \right)$$

$$\frac{dy}{dt} = a(\cos t + t \sin t - \cos t) = at \sin t$$

By chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = at \sin t \left(\frac{1}{a(\cos t - \sin t)} \right)$$

$$\rightarrow \frac{dy}{dx} = \frac{t \sin t}{\cos t - \sin t}$$

Q10. Differentiate w.r.t x

(i) $\cos^{-1} \frac{x}{a}$

Solution:- Let $y = \cos^{-1} \frac{x}{a}$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\cos^{-1} \frac{x}{a} \right) \quad \because \frac{d}{dx} (\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$= \frac{-1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \cdot \frac{d}{dx} \left(\frac{x}{a} \right)$$

$$= \frac{-1}{\sqrt{1 - \frac{x^2}{a^2}}} \cdot \frac{1}{a} \frac{d}{dx}(x)$$

$$= \frac{-1}{\sqrt{\frac{a^2 - x^2}{a^2}}} \cdot \frac{1}{a} (1) = \frac{-1}{\sqrt{a^2 - x^2}} \cdot \frac{1}{a}$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{a^2 - x^2}} \cdot \frac{1}{a} = \frac{-1}{a\sqrt{a^2 - x^2}}$$

(ii) $\cot^{-1} \frac{x}{a}$

Solution:- Let $y = \cot^{-1} \frac{x}{a}$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\cot^{-1} \frac{x}{a} \right) = \frac{-1}{1 + \left(\frac{x}{a}\right)^2} \cdot \frac{d}{dx} \left(\frac{x}{a} \right)$$

$$\left(\because \frac{d}{dx} (\cot^{-1} x) = \frac{-1}{1+x^2} \right)$$

$$= \frac{-1}{1 + \frac{x^2}{a^2}} \cdot \frac{1}{a} \frac{d}{dx}(x) = \frac{-1}{\frac{a^2 + x^2}{a^2}} \cdot \frac{1}{a} (1)$$

$$\frac{dy}{dx} = \frac{-a^2}{a^2 + x^2} \cdot \frac{1}{a} = \frac{-a}{a^2 + x^2}$$

(iii) $\frac{1}{a} \sin^{-1} \frac{a}{x}$

Solution:- Let $y = \frac{1}{a} \sin^{-1} \frac{a}{x}$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{a} \sin^{-1} \frac{a}{x} \right) = \frac{1}{a} \frac{d}{dx} \left(\sin^{-1} \frac{a}{x} \right)$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{1 - \left(\frac{a}{x}\right)^2}} \cdot \frac{d}{dx} \left(\frac{a}{x} \right) \quad \because \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{a\sqrt{1 - \frac{a^2}{x^2}}} \cdot \frac{d}{dx} \left(\frac{1}{x} \right) = \frac{1}{\sqrt{\frac{x^2 - a^2}{x^2}}} \cdot \frac{d}{dx} (x^{-1})$$

$$= \frac{1}{\sqrt{x^2 - a^2}} (-1 \cdot x^{-2}) = \frac{-x}{\sqrt{x^2 - a^2}} \left(\frac{-1}{x^2} \right)$$

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2 - a^2}}$$

(iv) $\sin^{-1} \sqrt{1-x^2}$

Solution:- Let $y = \sin^{-1} \sqrt{1-x^2}$

$$\frac{dy}{dx} = \frac{d}{dx} \left(\sin^{-1} \sqrt{1-x^2} \right) = \frac{1}{\sqrt{1 - (\sqrt{1-x^2})^2}} \cdot \frac{d}{dx} \sqrt{1-x^2}$$

$$= \frac{1}{\sqrt{1 - (1-x^2)}} \cdot \frac{d}{dx} (1-x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - 1 + x^2}} \cdot \frac{1}{2} (1-x^2)^{\frac{1}{2}-1} \cdot \frac{d}{dx} (1-x^2)$$

$$= \frac{1}{\sqrt{x^2}} \cdot \frac{1}{2} (1-x^2)^{-\frac{1}{2}} (-2x)$$

$$\frac{dy}{dx} = \frac{1}{x} (1-x^2)^{-\frac{1}{2}} (-x) = \frac{-1}{\sqrt{1-x^2}}$$

(v) $\sec^{-1} \left(\frac{x^2+1}{x^2-1} \right)$

Solution:- $y = \sec^{-1} \left(\frac{x^2+1}{x^2-1} \right)$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\sec^{-1} \left(\frac{x^2+1}{x^2-1} \right) \right)$$

$$= \frac{1}{\left(\frac{x^2+1}{x^2-1} \right) \sqrt{\left(\frac{x^2+1}{x^2-1} \right)^2 - 1}} \cdot \frac{d}{dx} \left(\frac{x^2+1}{x^2-1} \right)$$

$$\begin{aligned}\therefore \frac{d}{dx}(\sec^{-1}x) &= \frac{1}{x\sqrt{x^2-1}} \\ \frac{dy}{dx} &= \frac{x^2-1}{(x^2+1)\sqrt{\frac{(x^2+1)^2}{(x^2-1)^2}-1}} \cdot \frac{\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2} \\ &= \frac{x^2-1}{(x^2+1)\sqrt{\frac{(x^2+1)^2-(x^2-1)^2}{(x^2-1)^2}}} \cdot \frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2} \\ &= \frac{x^2-1}{(x^2+1)\sqrt{\frac{(x^2+1)^2-(x^2-1)^2}{(x^2-1)^2}}} \cdot \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2} \\ &= \frac{x^2-1}{(x^2+1)\sqrt{\frac{(x^2+1)^2-(x^2-1)^2}{(x^2-1)^2}}} \cdot \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2} \\ &= \frac{(x^2-1)^2}{(x^2+1)\sqrt{x^4+2x^2+1-(x^4-2x^2+1)}} \cdot \frac{2x(-2)}{(x^2-1)^2} \\ &= \frac{-4x}{(x^2+1)\sqrt{4x^2}} = \frac{-4x}{(x^2+1)2x} = \frac{-2}{(x^2+1)}\end{aligned}$$

$$\frac{dy}{dx} = \frac{-2}{(x^2+1)}$$

(vi) $\cot^{-1}\left(\frac{2x}{1-x^2}\right)$
Solution:- Let $y = \cot^{-1}\left(\frac{2x}{1-x^2}\right)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\cot^{-1}\left(\frac{2x}{1-x^2}\right)\right) \\ &= \frac{-1}{1+\left(\frac{2x}{1-x^2}\right)^2} \cdot \frac{d}{dx}\left(\frac{2x}{1-x^2}\right) \\ &= \frac{-1}{1+\frac{(2x)^2}{(1-x^2)^2}} \cdot \frac{(1-x^2)\frac{d}{dx}(2x)-(2x)\frac{d}{dx}(1-x^2)}{(1-x^2)^2} \\ &= \frac{-1}{1+\frac{(2x)^2}{(1-x^2)^2}} \cdot \frac{(1-x^2)(2)-(2x)(-2x)}{(1-x^2)^2} \\ &= \frac{-1}{\frac{(1-x^2)^2+4x^2}{(1-x^2)^2}} \cdot \frac{(1-x^2)(2)+4x^2}{(1-x^2)^2} \\ &= \frac{-(1-x^2)^2}{(1-x^2)^2+4x^2} \cdot \frac{2-2x^2+4x^2}{(1-x^2)^2} \\ &= \frac{-1}{x^4+1+2x^2} \cdot \frac{2+2x^2}{1} \\ \frac{dy}{dx} &= \frac{-1}{(x^2+1)^2} \cdot 2(1+x^2) = \frac{-2(1+x^2)}{(x^2+1)^2}\end{aligned}$$

$$\frac{dy}{dx} = \frac{-2}{x^2+1}$$

(vii) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$
Solution:- Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right) \\ &= \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \cdot \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)\end{aligned}$$

$$\begin{aligned}&= \frac{-1}{\sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}}} \cdot \frac{(1+x^2)\frac{d}{dx}(1-x^2)-(1-x^2)\frac{d}{dx}(1+x^2)}{(1+x^2)^2} \\ &= \frac{-1}{\sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}}} \cdot \frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2} \\ &= \frac{-(1+x^2)}{\sqrt{(1+x^2)^2-(1-x^2)^2}} \cdot \frac{-2x-2x-2x+2x}{(1+x^2)^2} \\ &= \frac{-1}{\sqrt{4x^2}} \cdot \frac{-4x}{(1+x^2)}\end{aligned}$$

$$\frac{dy}{dx} = \frac{-1}{2x} \cdot \frac{-4x}{(1+x^2)} = \frac{2}{1+x^2}$$

Q11. Show that $\frac{dy}{dx} = \frac{y}{x}$ if $\frac{y}{x} = \tan^{-1}\frac{x}{y}$

Solution:- $\frac{y}{x} = \tan^{-1}\frac{x}{y}$
 $\rightarrow y = x \tan^{-1}\frac{x}{y} \rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(x \tan^{-1}\frac{x}{y}\right)$

$$\begin{aligned}&= x \cdot \frac{d}{dx}\left(\tan^{-1}\frac{x}{y}\right) + \tan^{-1}\frac{x}{y} \cdot \frac{d}{dx}(x) \\ &= x \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{d}{dx}\left(\frac{x}{y}\right) + \tan^{-1}\frac{x}{y} (1) \\ &= \frac{x}{1+\frac{x^2}{y^2}} \cdot \frac{y\frac{d}{dx}(x)-x\frac{dy}{dx}}{y^2} + \tan^{-1}\frac{x}{y}\end{aligned}$$

$$\begin{aligned}&= \frac{x}{\frac{y^2+x^2}{y^2}} \cdot \frac{y(1)-x\frac{dy}{dx}}{y^2} + \frac{y}{x} \left[\begin{aligned} \because \frac{d}{dx}\left(\tan^{-1}x\right) \\ &= \frac{1}{1+x^2} \\ \tan^{-1}\frac{x}{y} &= \frac{y}{x} \end{aligned} \right] \\ &= \frac{xy^2}{x^2+y^2} \left(\frac{y-x\frac{dy}{dx}}{y^2} \right) + \frac{y}{x}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{x}{x^2+y^2} \left(y - x \frac{dy}{dx} \right) + \frac{y}{x} \\ &= \frac{xy}{x^2+y^2} - \frac{x^2}{x^2+y^2} \frac{dy}{dx} + \frac{y}{x}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} + \frac{x^2}{x^2+y^2} \frac{dy}{dx} &= \frac{xy}{x^2+y^2} + \frac{y}{x} \\ \left[1 + \frac{x^2}{x^2+y^2}\right] \frac{dy}{dx} &= \frac{x^2y + y(x^2+y^2)}{x(x^2+y^2)}\end{aligned}$$

$$\begin{aligned}\left(\frac{x^2+y^2+x^2}{x^2+y^2}\right) \frac{dy}{dx} &= \frac{y(x^2+x^2+y^2)}{x(x^2+y^2)} \\ \rightarrow \left(\frac{2x^2+y^2}{x^2+y^2}\right) \frac{dy}{dx} &= \frac{y(2x^2+y^2)}{x(x^2+y^2)}\end{aligned}$$

$$\begin{aligned}\rightarrow \frac{dy}{dx} &= \frac{y}{x} \text{ Hence proved}\end{aligned}$$

Q12. If $y = \tan(p \tan^{-1} x)$, show that $(1+x^2)y_1 - p(1+y^2) = 0$

Solution:- $y = \tan(p \tan^{-1} x)$

$$\rightarrow \tan^{-1} y = p \tan^{-1} x$$

$$\rightarrow \frac{d}{dx} (\tan^{-1} y) = \frac{d}{dx} (p \tan^{-1} x)$$

$$\frac{1}{1+y^2} \cdot \frac{dy}{dx} = p \cdot \frac{1}{1+x^2} \quad \because \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\rightarrow \frac{1}{1+y^2} y_1 = \frac{p}{1+x^2} \quad \because \frac{dy}{dx} = y_1$$

$$\rightarrow (1+x^2) y_1 = p(1+y^2)$$

$$\rightarrow (1+x^2) y_1 - p(1+y^2) = 0 \text{ Hence proved.}$$

Derivative of exponential functions

If $f(x) = a^x$ where $a > 0$ then $f(x)$ is called general exponential function. Here the base a is constant and exponent x is variable.

If $f(x) = e^x$ where $e \approx 2.71$ then $f(x)$ is called natural exponential function. Here base e is constant and exponent x is variable.

1. $\frac{d}{dx} (e^x) = e^x$

Proof:- Let $y = e^x \rightarrow (i)$

$$\rightarrow y + \delta y = e^{x+\delta x} \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = e^{x+\delta x} - e^x$$

$$\rightarrow \delta y = e^x \cdot e^{\delta x} - e^x = e^x (e^{\delta x} - 1)$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[e^x \left(\frac{e^{\delta x} - 1}{\delta x} \right) \right]$$

$$\rightarrow \frac{dy}{dx} = e^x \cdot \lim_{\delta x \rightarrow 0} \frac{e^{\delta x} - 1}{\delta x}$$

$$\frac{dy}{dx} = e^x \cdot \ln e$$

$$\because \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$$

$$\text{and } \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \ln e$$

$$\because \ln e = 1$$

$$\text{Hence } \frac{d}{dx} (e^x) = e^x$$

2. $\frac{d}{dx} (a^x) = a^x \ln a$

Proof:- Let $y = a^x \rightarrow (i)$

$$\rightarrow y + \delta y = a^{x+\delta x} \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = a^{x+\delta x} - a^x$$

$$\rightarrow \delta y = a^x \cdot a^{\delta x} - a^x = a^x (a^{\delta x} - 1)$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[a^x \left(\frac{a^{\delta x} - 1}{\delta x} \right) \right]$$

$$\rightarrow \frac{dy}{dx} = a^x \lim_{\delta x \rightarrow 0} \frac{a^{\delta x} - 1}{\delta x} = a^x \ln a$$

$$\text{Hence } \frac{d}{dx} (a^x) = a^x \ln a \quad \left(\because \lim_{x \rightarrow a} \frac{a^x - 1}{x} = \ln a \right)$$

Example 1. Find $\frac{dy}{dx}$ if: (i) $y = e^{x^2+1}$ (ii) $y = a^{\sqrt{x}}$

Solution:- (i) $y = e^{x^2+1}$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} (e^{x^2+1}) = e^{x^2+1} \cdot \frac{d}{dx} (x^2+1)$$

$$= e^{x^2+1} (2x) = 2x e^{x^2+1}$$

$$(ii) y = a^{\sqrt{x}} \rightarrow \frac{dy}{dx} = \frac{d}{dx} (a^{\sqrt{x}})$$

$$= a^{\sqrt{x}} \ln a \cdot \frac{d}{dx} (\sqrt{x}) = a^{\sqrt{x}} \ln a \cdot \frac{d}{dx} (x^{\frac{1}{2}})$$

$$\rightarrow \frac{dy}{dx} = a^{\sqrt{x}} \ln a \cdot \frac{1}{2} x^{\frac{1}{2}-1} = \frac{a^{\sqrt{x}} \ln a}{2\sqrt{x}}$$

Example 2. Differentiate $y = a^x$ w.r.t x

Solution:- $y = a^x = e^{x \ln a} \quad \because a^x = e^{x \ln a}$

$$\rightarrow y = e^{x \ln a} \rightarrow \frac{dy}{dx} = \frac{d}{dx} (e^{x \ln a})$$

$$\rightarrow \frac{dy}{dx} = e^{x \ln a} \cdot \frac{d}{dx} (x \ln a) = e^{x \ln a} \cdot \ln a \cdot \frac{d}{dx} (x)$$

$$= e^{x \ln a} \cdot \ln a (1) = a^x \ln a$$

Note:- As $a^x = a^x \rightarrow \ln a^x = \ln a^x$

$$\rightarrow \ln a^x = x \ln a$$

$$= x \ln a \cdot 1$$

$$\because \ln a^b = b \ln a$$

$$= x \ln a \cdot \ln e$$

$$\because \ln e = 1$$

$$\ln a^x = \ln e^{x \ln a}$$

$$\rightarrow a^x = e^{x \ln a}$$

Derivative of the logarithmic function:-

Logarithmic function:- If $a > 0$, $a \neq 1$ and $x = a^y$, then the function defined by

$$y = \log_a x, (x > 0) \text{ is called the}$$

logarithm of x to the base a .

The logarithmic functions $\log_e x$ and $\log_{10} x$

are called natural and common logarithms

resp. $y = \log_e x$ is written as $y = \ln x$



$$1. \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Proof:- Let $y = \ln x \rightarrow (i)$

$$\rightarrow y + \delta y = \ln(x + \delta x) \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = \ln(x + \delta x) - \ln x$$

$$\rightarrow \delta y = \ln\left(\frac{x + \delta x}{x}\right) \quad (\because \ln x - \ln y = \ln \frac{x}{y})$$

$$\delta y = \ln\left(\frac{x}{x} + \frac{\delta x}{x}\right) = \ln\left(1 + \frac{\delta x}{x}\right)$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[\frac{1}{\delta x} \ln\left(1 + \frac{\delta x}{x}\right) \right]$$

$$= \lim_{\delta x \rightarrow 0} \left[\frac{1}{x} \cdot \frac{x}{\delta x} \ln\left(1 + \frac{\delta x}{x}\right) \right]$$

$$\frac{dy}{dx} = \frac{1}{x} \lim_{\delta x \rightarrow 0} \ln\left(1 + \frac{\delta x}{x}\right) \quad (\because n \ln m = \ln m^n)$$

$$\frac{dy}{dx} = \frac{1}{x} \ln e = \frac{1}{x} \quad (\because \lim_{x \rightarrow 0} \left(1 + \frac{1}{x}\right)^x = e) \quad \because \ln e = 1$$

$$\text{Hence } \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$2. \frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}$$

Proof:- Let $y = \log_a x \rightarrow (i)$

$$\rightarrow y + \delta y = \log_a(x + \delta x) \rightarrow (ii)$$

$$\text{By (ii) - (i)} \rightarrow y + \delta y - y = \log_a(x + \delta x) - \log_a x$$

$$\rightarrow \delta y = \log_a\left(\frac{x + \delta x}{x}\right) \quad (\because \log_a m - \log_a n = \log_a \frac{m}{n})$$

$$\delta y = \log_a\left(\frac{x}{x} + \frac{\delta x}{x}\right) = \log_a\left(1 + \frac{\delta x}{x}\right)$$

Dividing by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[\frac{1}{\delta x} \log_a\left(1 + \frac{\delta x}{x}\right) \right]$$

$$= \lim_{\delta x \rightarrow 0} \left[\frac{1}{x} \cdot \frac{x}{\delta x} \log_a\left(1 + \frac{\delta x}{x}\right) \right]$$

$$= \frac{1}{x} \lim_{\delta x \rightarrow 0} \log_a\left(1 + \frac{\delta x}{x}\right) \quad (\because n \log m = \log m^n)$$

$$= \frac{1}{x} \log_a e \quad \because \lim_{x \rightarrow 0} \left(1 + \frac{1}{x}\right)^x = e$$

$$\frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{\log_a e} \quad \because \log_a b = \frac{1}{\log_b a}$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{\ln a} \quad \because \log_e a = \ln a$$

$$\text{Hence } \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Example 1. Find $\frac{dy}{dx}$ if $y = \log_{10}(ax^2 + bx + c)$

Solution:- $y = \log_{10}(ax^2 + bx + c)$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx}(\log_{10}(ax^2 + bx + c))$$

$$= \frac{1}{(ax^2 + bx + c) \ln 10} \frac{d}{dx}(ax^2 + bx + c)$$

$$= \frac{1}{(ax^2 + bx + c) \ln 10} (2ax + b) \quad \left(\because \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \right)$$

$$\rightarrow \frac{dy}{dx} = \frac{2ax + b}{(ax^2 + bx + c) \ln 10}$$

Example 2. Differentiate $\ln(x^2 + 2x)$ w.r.t 'x'.

Solution:- Let $y = \ln(x^2 + 2x)$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx}(\ln(x^2 + 2x)) = \frac{1}{x^2 + 2x} \frac{d}{dx}(x^2 + 2x)$$

$$= \frac{1}{x^2 + 2x} (2x + 2) \quad \because \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\rightarrow \frac{dy}{dx} = \frac{2x + 2}{x^2 + 2x}$$

Logarithmic differentiation

Example 1. Differentiate $y = e^{f(x)}$ w.r.t 'x'

Solution:- $y = e^{f(x)}$

Taking log on both sides

$$\rightarrow \ln y = \ln e^{f(x)} = f(x) \ln e$$

$$\ln y = f(x) (1) = f(x) \quad \because n \log m = \log m^n \quad \because \ln e = 1$$

$$\rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}(f(x))$$

$$\rightarrow \frac{1}{y} \frac{dy}{dx} = f'(x) \rightarrow \frac{dy}{dx} = y f'(x) = e^{f(x)} \cdot f'(x)$$

Example 2. Find derivative of $\frac{x\sqrt{x^2+3}}{x^2+1}$

Solution:- Let $y = \frac{x\sqrt{x^2+3}}{x^2+1}$

$$\rightarrow \ln y = \ln \frac{x\sqrt{x^2+3}}{x^2+1}$$

$$= \ln(x\sqrt{x^2+3}) - \ln(x^2+1)$$

$$= \ln x + \ln \sqrt{x^2+3} - \ln(x^2+1) \quad (\because \ln mn = \ln m + \ln n)$$

$$= \ln x + \ln(x^2+3)^{\frac{1}{2}} - \ln(x^2+1)$$

$$\ln y = \ln x + \frac{1}{2} \ln(x^2+3) - \ln(x^2+1) \quad (\because \ln m^n = n \ln m)$$

Differentiating w.r.t x, we have

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\ln x) + \frac{d}{dx}\left(\frac{1}{2} \ln(x^2+3)\right) - \frac{d}{dx}(\ln(x^2+1))$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + \frac{1}{2} \frac{d}{dx}(\ln(x^2+3)) - \frac{d}{dx}(\ln(x^2+1))$$

$$= \frac{1}{x} + \frac{1}{2} \frac{1}{x^2+3} \frac{d}{dx}(x^2+3) - \frac{1}{x^2+1} \frac{d}{dx}(x^2+1)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + \frac{1}{2(x^2+3)} (2x) - \frac{1}{x^2+1} (2x)$$

$$= \frac{1}{x} + \frac{x}{x^2+3} - \frac{2x}{x^2+1}$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{(1)(x^2+3)(x^2+1) + x \cdot x(x^2+1) - 2x \cdot x(x^2+3)}{x(x^2+3)(x^2+1)}$$

$$\rightarrow \frac{dy}{dx} = y \frac{x^4 + x^2 + 3x^2 + 3 + x^4 + x^2 - 2x^4 - 6x^2}{x(x^2+3)(x^2+1)}$$

$$= \frac{\sqrt{x^2+3}}{x^2+1} \frac{(3-x^2)}{\sqrt{x^2+3}(x^2+1)}$$

$$\frac{dy}{dx} = \frac{3-x^2}{(x^2+3)^{1/2} (x^2+1)^2} = \frac{3-x^2}{\sqrt{x^2+3} (x^2+1)^2}$$

Example 3. Differentiate $(\ln x)^x$ w.r.t 'x'.

Solution:- Let $y = (\ln x)^x$

$$\rightarrow \ln y = \ln[(\ln x)^x] = x \ln(\ln x)$$

$$\rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln(\ln x))$$

$$\frac{1}{y} \frac{dy}{dx} = x \cdot \frac{d}{dx}(\ln(\ln x)) + \ln(\ln x) \cdot \frac{d}{dx}(x)$$

$$\rightarrow \frac{dy}{dx} = y \left[x \cdot \frac{1}{\ln x} \frac{d}{dx}(\ln x) + \ln(\ln x) (1) \right]$$

$$= (\ln x)^x \left[\frac{x}{\ln x} \cdot \frac{1}{x} + \ln(\ln x) \right]$$

$$\rightarrow \frac{dy}{dx} = (\ln x)^x \left[\frac{1}{\ln x} + \ln(\ln x) \right]$$

Derivative of hyperbolic functions:-

1. $\frac{d}{dx}(\sinh x) = \cosh x$

Proof:- we know that $\sinh x = \frac{e^x - e^{-x}}{2}$

$$\text{so } \frac{d}{dx}(\sinh x) = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right)$$

$$= \frac{1}{2} \frac{d}{dx}(e^x - e^{-x}) = \frac{1}{2} \left\{ \frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x}) \right\}$$

$$= \frac{1}{2} (e^x - e^{-x} \frac{d}{dx}(-x)) = \frac{1}{2} (e^x - e^{-x}(-1))$$

$$= \frac{1}{2} (e^x + e^{-x}) = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\rightarrow \frac{d}{dx}(\sinh x) = \cosh x \quad (\because \cosh x = \frac{e^x + e^{-x}}{2})$$

2. $\frac{d}{dx}(\cosh x) = \sinh x$

Proof:- we know that $\cosh x = \frac{e^x + e^{-x}}{2}$

$$\text{so } \frac{d}{dx}(\cosh x) = \frac{d}{dx} \left(\frac{e^x + e^{-x}}{2} \right)$$

$$= \frac{1}{2} \frac{d}{dx}(e^x + e^{-x})$$

$$= \frac{1}{2} \left\{ \frac{d}{dx} e^x + \frac{d}{dx} e^{-x} \right\} = \frac{1}{2} (e^x + e^{-x} \frac{d}{dx}(-x))$$

$$= \frac{1}{2} (e^x + e^{-x}(-1)) = \frac{1}{2} (e^x - e^{-x})$$

$$= \frac{e^x - e^{-x}}{2} = \sinh x \quad \therefore \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\rightarrow \frac{d}{dx}(\cosh x) = \sinh x$$

3. $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$

Proof:- We know that $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$$\text{so } \frac{d}{dx}(\tanh x) = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)$$

$$= \frac{(e^x + e^{-x}) \frac{d}{dx}(e^x - e^{-x}) - (e^x - e^{-x}) \frac{d}{dx}(e^x + e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})(e^x - e^{-x}(-1)) - (e^x - e^{-x})(e^x + e^{-x}(-1))}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2e^x e^{-x} - (e^{2x} - e^{-2x} - 2e^x e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} + e^{-2x} + 2}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}$$

$$\frac{d}{dx}(\tanh x) = \left(\frac{2}{e^x + e^{-x}} \right)^2 = (\text{sech } x)^2 = \text{sech}^2 x$$

$$\rightarrow \frac{d}{dx}(\tanh x) = \text{sech}^2 x \quad \therefore \text{sech } x = \frac{2}{e^x + e^{-x}}$$

4. $\frac{d}{dx}(\text{cosech } x) = -\text{cosech } x \coth x$

Proof:- we know that $\text{cosech } x = \frac{1}{\sinh x}$

$$\text{so } \frac{d}{dx}(\text{cosech } x) = \frac{d}{dx} \left(\frac{1}{\sinh x} \right)$$

$$= \frac{\sinh x \frac{d}{dx}(1) - (1) \frac{d}{dx}(\sinh x)}{\sinh^2 x}$$

$$= \frac{\sinh x (0) - \cosh x}{\sinh^2 x}$$

$$\frac{d}{dx}(\text{cosech } x) = -\frac{\cosh x}{\sinh x \cdot \sinh x} = -\coth x \text{cosech } x$$

$$\rightarrow \frac{d}{dx}(\text{cosech } x) = -\text{cosech } x \coth x$$

$$5. \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

Proof:- We know that

$$\begin{aligned} \operatorname{sech} x &= \frac{1}{\cosh x} \\ \rightarrow \frac{d}{dx}(\operatorname{sech} x) &= \frac{d}{dx}\left(\frac{1}{\cosh x}\right) \\ &= \frac{\cosh x \frac{d}{dx}(1) - (1) \frac{d}{dx}(\cosh x)}{\cosh^2 x} \\ &= \frac{\cosh x(0) - (1)(\sinh x)}{\cosh^2 x} \\ &= \frac{-\sinh x}{\cosh x \cdot \cosh x} = -\tanh x \cdot \operatorname{sech} x \end{aligned}$$

$$\rightarrow \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$6. \frac{d}{dx}(\operatorname{coth} x) = -\operatorname{cosech}^2 x$$

Proof:- We know that $\operatorname{coth} x = \frac{1}{\tanh x}$

$$\begin{aligned} \rightarrow \frac{d}{dx}(\operatorname{coth} x) &= \frac{d}{dx}\left(\frac{1}{\tanh x}\right) \\ &= \frac{\tanh x \frac{d}{dx}(1) - (1) \frac{d}{dx}(\tanh x)}{\tanh^2 x} \\ &= \frac{\tanh x(0) - (1)(\operatorname{sech}^2 x)}{\tanh^2 x} \\ &= -\frac{\operatorname{sech}^2 x}{\frac{\sinh^2 x}{\cosh^2 x}} = -\frac{\operatorname{sech}^2 x}{\sinh^2 x} \times \cosh^2 x \\ &= -\frac{\operatorname{sech}^2 x}{\sinh^2 x} \times \frac{1}{\operatorname{sech}^2 x} = \frac{-1}{\sinh^2 x} \end{aligned}$$

$$\frac{d}{dx}(\operatorname{coth} x) = -\operatorname{cosech}^2 x$$

Example 1. Find $\frac{dy}{dx}$ if $y = \sinh 2x$

Solution:- $y = \sinh 2x$

$$\begin{aligned} \rightarrow \frac{dy}{dx} &= \frac{d}{dx}(\sinh 2x) \\ &= \cosh 2x \cdot \frac{d}{dx}(2x) \end{aligned}$$

$$\rightarrow \frac{dy}{dx} = 2 \cosh 2x$$

Example 2. Find $\frac{dy}{dx}$ if $y = \tanh(x^2)$

Solution:- $y = \tanh(x^2)$

$$\begin{aligned} \rightarrow \frac{dy}{dx} &= \frac{d}{dx}(\tanh(x^2)) \\ &= \operatorname{sech}^2(x^2) \cdot \frac{d}{dx}(x^2) = \operatorname{sech}^2(x^2) \cdot 2x \end{aligned}$$

$$\rightarrow \frac{dy}{dx} = 2x \operatorname{sech}^2(x^2)$$

Derivatives of inverse hyperbolic functions:-

The inverse hyperbolic functions are

1. $y = \sinh^{-1} x$ iff $x = \sinh y$; $x, y \in \mathbb{R}$
2. $y = \cosh^{-1} x$ iff $x = \cosh y$; $x \in [1, \infty)$, $y \in [0, \infty)$
3. $y = \tanh^{-1} x$ iff $x = \tanh y$; $x \in (-1, 1)$, $y \in \mathbb{R}$
4. $y = \operatorname{coth}^{-1} x$ iff $x = \operatorname{coth} y$; $x \in (-1, 1)$, $y \in \mathbb{R} - \{0\}$
5. $y = \operatorname{sech}^{-1} x$ iff $x = \operatorname{sech} y$; $x \in (0, 1]$, $y \in [0, \infty)$
6. $y = \operatorname{cosech}^{-1} x$ iff $x = \operatorname{cosech} y$; $x \in \mathbb{R} - \{0\}$, $y \in \mathbb{R} - \{0\}$

Prove that $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

Proof:- Let $y = \sinh^{-1} x \Rightarrow x = \sinh y$

$$\rightarrow x = \frac{e^y - e^{-y}}{2} \quad \therefore \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\rightarrow 2x = \frac{e^y - e^{-y}}{1} = e^y - \frac{1}{e^y} = \frac{(e^y)^2 - 1}{e^y}$$

$$\rightarrow 2x e^y = e^{2y} - 1 \rightarrow e^{2y} - 2x e^y - 1 = 0$$

$$\rightarrow (e^y)^2 - 2x e^y - 1 = 0$$

$$\rightarrow e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)} \quad \text{using } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2x \pm \sqrt{4x^2 + 4}}{2} = \frac{2x \pm \sqrt{4(x^2 + 1)}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2 + 1}}{2} = x \pm \sqrt{x^2 + 1}$$

$$\frac{y}{e} = x \pm \sqrt{x^2 + 1}$$

$$\rightarrow \frac{y}{e} = x + \sqrt{x^2 + 1}, \quad \frac{y}{e} = x - \sqrt{x^2 + 1} \quad \text{(rejected being -ive for } x \in \mathbb{R})$$

As e is +ive for $y \in \mathbb{R}$, so

$$\rightarrow \frac{y}{e} = x + \sqrt{x^2 + 1}$$

$$\rightarrow y = \ln(x + \sqrt{x^2 + 1})$$

$$\text{Hence } \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

Prove that $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$

Proof:- Let $y = \cosh^{-1} x \Rightarrow x = \cosh y$

$$\rightarrow x = \frac{e^y + e^{-y}}{2} \quad \therefore \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\rightarrow 2x = \frac{e^y + e^{-y}}{1} = e^y + \frac{1}{e^y} = \frac{(e^y)^2 + 1}{e^y}$$

$$\rightarrow 2x e^y = e^{2y} + 1 \rightarrow e^{2y} - 2x e^y + 1 = 0$$

$$\rightarrow (e^y)^2 - 2x e^y + 1 = 0$$

$$\rightarrow e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)} \quad \text{using } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note:-

$$\begin{aligned} \therefore \text{if } e^y &= x \\ \rightarrow \ln e^y &= \ln x \\ \rightarrow y \ln e &= \ln x \\ \rightarrow y(1) &= \ln x \\ \rightarrow y &= \ln x \end{aligned}$$

$$\rightarrow e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = \frac{2x \pm \sqrt{4(x^2 - 1)}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2 - 1}}{2} = 2 \left(\frac{x \pm \sqrt{x^2 - 1}}{2} \right)$$

$$\rightarrow e^y = x \pm \sqrt{x^2 - 1}$$

$$\rightarrow e^y = x + \sqrt{x^2 - 1}, \quad e^y = x - \sqrt{x^2 - 1}$$

(rejected being
-ive for all $x > 1$)

As e^y is +ive for $y \in \mathbb{R}$

$$\text{so } e^y = x + \sqrt{x^2 - 1}$$

$$\rightarrow y = \ln(x + \sqrt{x^2 - 1})$$

$$\text{Thus } \cosh^{-1}x = \ln(x + \sqrt{x^2 - 1})$$

$$1. \frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}, \quad x, y \in \mathbb{R}$$

Proof:- Let $y = \sinh^{-1}x$

$$\rightarrow x = \sinh y$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sinh y) = \cosh y \frac{dy}{dx}$$

$$\rightarrow \cosh y \frac{dy}{dx} = 1 \rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2 y}} = \frac{1}{\sqrt{1 + \sinh^2 y}}$$

$$(\because \cosh^2 x - \sinh^2 x = 1)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} \quad (\because \sinh y = x)$$

$$2. \frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2 - 1}}$$

Proof:- Let $y = \cosh^{-1}x$; $x \in [1, \infty)$, $y \in [0, \infty)$

$$\rightarrow \cosh y = x$$

$$\rightarrow \frac{d}{dx}(\cosh y) = \frac{d}{dx}(x) \rightarrow \sinh y \frac{dy}{dx} = 1$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} \quad (\because \cosh^2 y - \sinh^2 y = 1)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

$$3. \frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$$

Proof:- Let $y = \tanh^{-1}x$; $x \in (-1, 1)$, $y \in \mathbb{R}$

$$\rightarrow \tanh y = x$$

$$\rightarrow \frac{d}{dx}(\tanh y) = \frac{d}{dx}(x)$$

$$\rightarrow \text{sech}^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\text{sech}^2 y} = \frac{1}{1 - \tanh^2 y}$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{1-x^2} \quad (\because \text{sech}^2 x + \tanh^2 x = 1)$$

$$4. \frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2}, \quad |x| > 1$$

Proof:- Let $y = \coth^{-1}x$

$$\rightarrow \coth y = x \rightarrow \frac{d}{dx}(\coth y) = \frac{d}{dx}(x)$$

$$\rightarrow -\text{cosech}^2 y \frac{dy}{dx} = 1 \rightarrow \frac{dy}{dx} = \frac{1}{-\text{cosech}^2 y}$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{1 - \coth^2 x} \quad \because \coth^2 x - \text{cosech}^2 x = 1$$

$$\rightarrow -\text{cosech}^2 x = 1 - \coth^2 x$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{1-x^2}$$

$$5. \frac{d}{dx}(\text{sech}^{-1}x) = \frac{-1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$$

Proof:- Let $y = \text{sech}^{-1}x$

$$\rightarrow \text{sech} y = x \rightarrow \frac{d}{dx}(\text{sech} y) = \frac{d}{dx}(x)$$

$$\rightarrow -\text{sech} y \tanh y \frac{dy}{dx} = 1$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{\text{sech} y \tanh y} = \frac{-1}{\text{sech} y \sqrt{\tanh^2 y}}$$

$$\frac{dy}{dx} = \frac{-1}{\text{sech} y \sqrt{1 - \text{sech}^2 y}} \quad (\because \text{sech}^2 y + \tanh^2 y = 1)$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}}$$

$$6. \frac{d}{dx}(\text{cosec}^{-1}x) = -\frac{1}{x\sqrt{1+x^2}}; \quad x \in \mathbb{R} - \{0\}$$

Proof:- Let $y = \text{cosec}^{-1}x$

$$\rightarrow \text{cosec} y = x \rightarrow \frac{d}{dx}(\text{cosec} y) = \frac{d}{dx}(x)$$

$$\rightarrow -\text{cosec} y \coth y \frac{dy}{dx} = 1$$

$$\rightarrow \frac{dy}{dx} = \frac{-1}{\text{cosec} y \coth y} = \frac{-1}{\text{cosec} y \sqrt{\coth^2 y}}$$

$$(\because \coth^2 x - \text{cosech}^2 x = 1)$$

$$\coth^2 x = 1 + \text{cosech}^2 x$$

$$\frac{dy}{dx} = \frac{-1}{\text{cosec} y \sqrt{1 + \text{cosech}^2 x}} = \frac{-1}{x\sqrt{1+x^2}}$$

Example 1. Find $\frac{dy}{dx}$ if $y = \sinh^{-1}(ax+b)$

Solution:- $y = \sinh^{-1}(ax+b)$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sinh^{-1}(ax+b))$$

$$= \frac{1}{\sqrt{1+(ax+b)^2}} \cdot \frac{d}{dx}(ax+b) \quad \because \frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$$

$$\rightarrow \frac{dy}{dx} = \frac{a}{\sqrt{1+(ax+b)^2}}$$

Example 2. Find $\frac{dy}{dx}$ if $y = \cosh^{-1}(\sec x)$

Solution:- $y = \cosh^{-1}(\sec x)$

$$\begin{aligned} \rightarrow \frac{dy}{dx} &= \frac{d}{dx} [\cosh^{-1}(\sec x)] \\ &= \frac{1}{\sqrt{\sec^2 x - 1}} \cdot \frac{d}{dx}(\sec x) \\ &= \frac{\sec x \tan x}{\sqrt{\tan^2 x}} = \frac{\sec x \tan x}{\tan x} \end{aligned}$$

$$\frac{dy}{dx} = \sec x \quad \left(\because \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}} \right)$$

$$1 + \tan^2 x = \sec^2 x$$

Exercise 2.6

Q1. Find $f'(x)$ if

(i) $f(x) = e^{\sqrt{x}-1}$

Solution:- $f(x) = e^{\sqrt{x}-1}$

$$\begin{aligned} \frac{d}{dx}(f(x)) &= \frac{d}{dx}(e^{\sqrt{x}-1}) = e^{\sqrt{x}-1} \cdot \frac{d}{dx}(\sqrt{x}-1) \\ &= e^{\sqrt{x}-1} \left(\frac{d}{dx} x^{\frac{1}{2}} - \frac{d}{dx}(1) \right) = e^{\sqrt{x}-1} \left(\frac{1}{2} x^{-\frac{1}{2}} - 0 \right) \\ \rightarrow f'(x) &= e^{\sqrt{x}-1} \cdot \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}-1}}{2\sqrt{x}} \end{aligned}$$

(ii) $f(x) = x^3 e^{\frac{1}{x}}$

Solution:- $f(x) = x^3 e^{\frac{1}{x}}$

$$\begin{aligned} \frac{d}{dx}(f(x)) &= \frac{d}{dx}(x^3 e^{\frac{1}{x}}) = x^3 \frac{d}{dx}(e^{\frac{1}{x}}) + e^{\frac{1}{x}} \frac{d}{dx}(x^3) \\ &= x^3 \cdot e^{\frac{1}{x}} \frac{d}{dx}\left(\frac{1}{x}\right) + e^{\frac{1}{x}} \cdot 3x^2 \\ &= x^3 \cdot e^{\frac{1}{x}} \frac{d}{dx}(x^{-1}) + 3x^2 e^{\frac{1}{x}} \\ &= x^3 e^{\frac{1}{x}} (-1 \cdot x^{-2}) + 3x^2 e^{\frac{1}{x}} \\ f'(x) &= x^3 e^{\frac{1}{x}} \left(-\frac{1}{x^2}\right) + 3x^2 e^{\frac{1}{x}} = -x e^{\frac{1}{x}} + 3x^2 e^{\frac{1}{x}} \\ \rightarrow f'(x) &= x e^{\frac{1}{x}} (3x - 1) \end{aligned}$$

(iii) $f(x) = e^x (1 + \ln x)$

Solution:- $f(x) = e^x (1 + \ln x)$

$$\begin{aligned} \frac{d}{dx}(f(x)) &= \frac{d}{dx} [e^x (1 + \ln x)] \\ &= e^x \frac{d}{dx}(1 + \ln x) + (1 + \ln x) \frac{d}{dx}(e^x) \\ &= e^x \left(0 + \frac{1}{x}\right) + (1 + \ln x) e^x \\ &= e^x \cdot \frac{1}{x} + (1 + \ln x) e^x \\ f'(x) &= e^x \left(\frac{1}{x} + 1 + \ln x\right) \end{aligned}$$

(iv) $f(x) = \frac{e^x}{e^x + 1}$

Solution:- $f(x) = \frac{e^x}{e^x + 1}$

$$\begin{aligned} \frac{d}{dx}(f(x)) &= \frac{d}{dx} \left(\frac{e^x}{e^x + 1} \right) = \frac{(e^x + 1) \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(e^x + 1)}{(e^x + 1)^2} \\ &= \frac{(e^x + 1) e^x - e^x (e^x \frac{d}{dx}(-x) + 0)}{(e^x + 1)^2} \\ &= \frac{(e^x + 1) e^x - e^x (e^x (-1))}{(e^x + 1)^2} = \frac{e^x \cdot e^x + e^x + e^x \cdot e^x}{(e^x + 1)^2} \\ f'(x) &= \frac{e^{-x+x} + e^x + e^{x-x}}{(e^x + 1)^2} = \frac{e^0 + e^x + e^0}{(e^x + 1)^2} = \frac{1 + e^x + 1}{(e^x + 1)^2} \end{aligned}$$

$\rightarrow f'(x) = \frac{e^x + 2}{(e^x + 1)^2}$

(v) $\ln(e^x + e^{-x})$

Solution:- $f(x) = \ln(e^x + e^{-x})$

$$\begin{aligned} \frac{d}{dx}(f(x)) &= \frac{d}{dx} [\ln(e^x + e^{-x})] = \frac{1}{e^x + e^{-x}} \frac{d}{dx}(e^x + e^{-x}) \\ &= \frac{1}{e^x + e^{-x}} (e^x (1) + e^{-x} (-1)) = \frac{1}{e^x + e^{-x}} (e^x - e^{-x}) \\ f'(x) &= \frac{1}{e^x + \frac{1}{e^x}} \left(e^x - \frac{1}{e^x} \right) = \frac{1}{\frac{(e^x)^2 + 1}{e^x}} \left(\frac{(e^x)^2 - 1}{e^x} \right) \\ \rightarrow f'(x) &= \frac{e^x}{e^x + 1} \cdot \frac{e^x - 1}{e^x} = \frac{e^x - 1}{e^x + 1} \end{aligned}$$

(vi) $f(x) = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$

Solution:- $f(x) = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$

$$\begin{aligned} \frac{d}{dx}(f(x)) &= \frac{d}{dx} \left(\frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}} \right) \\ &= \frac{(e^{ax} + e^{-ax}) \frac{d}{dx}(e^{ax} - e^{-ax}) - (e^{ax} - e^{-ax}) \frac{d}{dx}(e^{ax} + e^{-ax})}{(e^{ax} + e^{-ax})^2} \\ &= \frac{(e^{ax} + e^{-ax}) (e^{ax} \cdot a - e^{-ax} (-a)) - (e^{ax} - e^{-ax}) (e^{ax} \cdot a + e^{-ax} (-a))}{(e^{ax} + e^{-ax})^2} \\ &= \frac{(e^{ax} + e^{-ax}) (a e^{ax} + a e^{-ax}) - (e^{ax} - e^{-ax}) (a e^{ax} - a e^{-ax})}{(e^{ax} + e^{-ax})^2} \\ &= \frac{a (e^{ax} + e^{-ax}) (e^{ax} + e^{-ax}) - a (e^{ax} - e^{-ax}) (e^{ax} - e^{-ax})}{(e^{ax} + e^{-ax})^2} \\ &= \frac{a [e^{2ax} + e^{-2ax} + 2e^{ax} \cdot e^{-ax}] - a [e^{2ax} - 2e^{ax} \cdot e^{-ax} + e^{-2ax}]}{(e^{ax} + e^{-ax})^2} \\ &= \frac{a [e^{2ax} + e^{-2ax} + 2 - e^{2ax} + 2e^{ax} \cdot e^{-ax} - e^{-2ax} + 2]}{(e^{ax} + e^{-ax})^2} \end{aligned}$$

$$f'(x) = \frac{a(2+2)}{(e^x + e^{-x})^2} \rightarrow f'(x) = \frac{4a}{(e^x + e^{-x})^2}$$

$$(vii) f(x) = \sqrt{\ln(e^{2x} + e^{-2x})}$$

$$\text{Solution:- } f(x) = \sqrt{\ln(e^{2x} + e^{-2x})}$$

$$\begin{aligned} \frac{d}{dx}(f(x)) &= \frac{d}{dx} [\ln(e^{2x} + e^{-2x})]^{\frac{1}{2}} \\ &= \frac{1}{2} [\ln(e^{2x} + e^{-2x})]^{-\frac{1}{2}} \cdot \frac{d}{dx} [\ln(e^{2x} + e^{-2x})] \\ &= \frac{1}{2 \sqrt{\ln(e^{2x} + e^{-2x})}} \cdot \frac{1}{(e^{2x} + e^{-2x})} \cdot \frac{d}{dx} (e^{2x} + e^{-2x}) \end{aligned}$$

$$f'(x) = \frac{1}{2(e^{2x} + e^{-2x}) \sqrt{\ln(e^{2x} + e^{-2x})}} \cdot (e^{2x} \cdot 2 + e^{-2x} \cdot (-2))$$

$$\rightarrow f'(x) = \frac{2(e^{2x} - e^{-2x})}{2(e^{2x} + e^{-2x}) \sqrt{\ln(e^{2x} + e^{-2x})}} = \frac{e^{2x} - e^{-2x}}{(e^{2x} + e^{-2x}) \sqrt{\ln(e^{2x} + e^{-2x})}}$$

$$(viii) f(x) = \ln(\sqrt{e^{2x} + e^{-2x}})$$

$$\text{Solution:- } f(x) = \ln(e^{2x} + e^{-2x})^{\frac{1}{2}}$$

$$f(x) = \frac{1}{2} \ln(e^{2x} + e^{-2x})$$

$$\begin{aligned} \rightarrow \frac{d}{dx}(f(x)) &= \frac{d}{dx} \left[\frac{1}{2} \ln(e^{2x} + e^{-2x}) \right] = \frac{1}{2} \frac{d}{dx} (\ln(e^{2x} + e^{-2x})) \\ &= \frac{1}{2} \cdot \frac{1}{e^{2x} + e^{-2x}} \cdot \frac{d}{dx} (e^{2x} + e^{-2x}) \\ &= \frac{1}{2(e^{2x} + e^{-2x})} (e^{2x} \cdot 2 + e^{-2x} \cdot (-2)) \end{aligned}$$

$$f'(x) = \frac{x(e^{2x} - e^{-2x})}{x(e^{2x} + e^{-2x})} = \tanh 2x$$

$$\left(\because \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \tanh 2x = \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \right)$$

Q2. Find $\frac{dy}{dx}$ if

$$(i) y = x^2 \ln \sqrt{x}$$

$$\text{Solution:- } y = x^2 \ln x^{\frac{1}{2}}$$

$$\begin{aligned} \rightarrow \frac{dy}{dx} &= \frac{d}{dx} (x^2 \ln x^{\frac{1}{2}}) = \frac{d}{dx} (x^2 \cdot \frac{1}{2} \ln x) \\ &= \frac{1}{2} \frac{d}{dx} (x^2 \ln x) \\ &= \frac{1}{2} \left[x^2 \frac{d}{dx} (\ln x) + \ln x \frac{d}{dx} (x^2) \right] \\ &= \frac{1}{2} \left[x^2 \cdot \frac{1}{x} + \ln x \cdot 2x \right] = \frac{1}{2} (x + 2x \ln x) \end{aligned}$$

$$\frac{dy}{dx} = \frac{x}{2} + x \ln x$$

$$(ii) y = x \sqrt{\ln x}$$

$$\text{Solution:- } y = x (\ln x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{d}{dx} \{ x (\ln x)^{\frac{1}{2}} \} = x \frac{d}{dx} (\ln x)^{\frac{1}{2}} + (\ln x)^{\frac{1}{2}} \frac{d}{dx} (x)$$

$$= x \cdot \frac{1}{2} (\ln x)^{-\frac{1}{2}} \frac{d}{dx} (\ln x) + (\ln x)^{\frac{1}{2}} (1)$$

$$= x \cdot \frac{1}{2 \sqrt{\ln x}} \cdot \frac{1}{x} + \sqrt{\ln x}$$

$$= \frac{1}{2 \sqrt{\ln x}} + \sqrt{\ln x} = \frac{1 + 2 \sqrt{\ln x} \sqrt{\ln x}}{2 \sqrt{\ln x}}$$

$$\rightarrow f'(x) = \frac{1 + 2 \ln x}{2 \sqrt{\ln x}}$$

$$(iii) y = \frac{x}{\ln x}$$

$$\text{Solution:- } y = \frac{x}{\ln x}$$

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x}{\ln x} \right) = \frac{\ln x \frac{d}{dx} (x) - x \frac{d}{dx} (\ln x)}{(\ln x)^2}$$

$$\rightarrow \frac{dy}{dx} = \frac{\ln x (1) - x \cdot \frac{1}{x}}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$$

$$(iv) y = x^2 \ln \frac{1}{x}$$

$$\text{Solution:- } y = x^2 \ln \frac{1}{x} = x^2 (\ln 1 - \ln x) = x^2 (0 - \ln x)$$

$$\frac{dy}{dx} = \frac{d}{dx} (-x^2 \ln x)$$

$$= - \left\{ x^2 \frac{d}{dx} (\ln x) + \ln x \frac{d}{dx} (x^2) \right\}$$

$$= - \left\{ x^2 \cdot \frac{1}{x} + \ln x (2x) \right\} = - \{ x + 2x \ln x \}$$

$$\frac{dy}{dx} = - \{ x + 2x \ln x \} = -x(1 + 2 \ln x)$$

$$(v) y = \ln \sqrt{\frac{x^2 - 1}{x^2 + 1}}$$

$$\text{Solution:- } y = \ln \left(\frac{x^2 - 1}{x^2 + 1} \right)^{\frac{1}{2}}$$

$$\rightarrow y = \frac{1}{2} \ln \left(\frac{x^2 - 1}{x^2 + 1} \right) = \frac{1}{2} [\ln(x^2 - 1) - \ln(x^2 + 1)]$$

$$\frac{dy}{dx} = \frac{1}{2} \frac{d}{dx} [\ln(x^2 - 1) - \ln(x^2 + 1)]$$

$$= \frac{1}{2} \left[\frac{d}{dx} \ln(x^2 - 1) - \frac{d}{dx} \ln(x^2 + 1) \right]$$

$$= \frac{1}{2} \left[\frac{1}{x^2 - 1} \frac{d}{dx} (x^2 - 1) - \frac{1}{x^2 + 1} \frac{d}{dx} (x^2 + 1) \right]$$

$$= \frac{1}{2} \left[\frac{2x}{x^2 - 1} - \frac{2x}{x^2 + 1} \right]$$

$$= \frac{2x}{2} \left[\frac{1}{x^2 - 1} - \frac{1}{x^2 + 1} \right]$$

$$= x \left[\frac{x^2 + 1 - (x^2 - 1)}{(x^2 - 1)(x^2 + 1)} \right] = x \left[\frac{x^2 + 1 - x^2 + 1}{x^4 - 1} \right]$$

$$\rightarrow \frac{dy}{dx} = \frac{2x}{x^4 - 1}$$

$$(vi) y = \ln(x + \sqrt{x^2 + 1})$$

$$\text{Solution:- } y = \ln(x + \sqrt{x^2 + 1})$$

$$\begin{aligned} \rightarrow \frac{dy}{dx} &= \frac{d}{dx} (\ln(x + \sqrt{x^2 + 1})) \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \frac{d}{dx} (x + \sqrt{x^2 + 1}) \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \left[\frac{d}{dx} (x) + \frac{d}{dx} (x^2 + 1)^{\frac{1}{2}} \right] \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \left[1 + \frac{1}{2} (x^2 + 1)^{-\frac{1}{2}} \frac{d}{dx} (x^2 + 1) \right] \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \left[1 + \frac{1}{\sqrt{x^2 + 1}} \cdot 2x \right] \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \left[1 + \frac{2x}{\sqrt{x^2 + 1}} \right] \\ \rightarrow \frac{dy}{dx} &= \frac{1}{x + \sqrt{x^2 + 1}} \left[\frac{\sqrt{x^2 + 1} + 2x}{\sqrt{x^2 + 1}} \right] = \frac{1}{\sqrt{x^2 + 1}} \end{aligned}$$

$$(vii) y = \ln(9 - x^2)$$

$$\text{Solution:- } y = \ln(9 - x^2)$$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} [\ln(9 - x^2)] = \frac{1}{9 - x^2} \frac{d}{dx} (9 - x^2)$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{9 - x^2} (0 - 2x) = -\frac{2x}{9 - x^2}$$

$$(viii) y = e^{-2x} \sin 2x$$

$$\text{Solution:- } y = e^{-2x} \sin 2x$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} (e^{-2x} \sin 2x) = e^{-2x} \frac{d}{dx} (\sin 2x) + \sin 2x \frac{d}{dx} (e^{-2x}) \\ &= e^{-2x} \cos 2x \frac{d}{dx} (2x) + \sin 2x \cdot e^{-2x} \frac{d}{dx} (-2x) \\ &= 2e^{-2x} \cos 2x - 2e^{-2x} \sin 2x \end{aligned}$$

$$\rightarrow \frac{dy}{dx} = 2e^{-2x} (\cos 2x - \sin 2x)$$

$$(ix) y = e^{-x} (x^3 + 2x^2 + 1)$$

$$\text{Solution:- } y = e^{-x} (x^3 + 2x^2 + 1)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \{ e^{-x} (x^3 + 2x^2 + 1) \} \\ &= e^{-x} \frac{d}{dx} (x^3 + 2x^2 + 1) + (x^3 + 2x^2 + 1) \frac{d}{dx} (e^{-x}) \\ &= e^{-x} (3x^2 + 4x) + (x^3 + 2x^2 + 1) (e^{-x}) (-1) \\ &= e^{-x} (3x^2 + 4x - x^3 - 2x^2 - 1) \\ &= e^{-x} (-x^3 + x^2 + 4x - 1) \end{aligned}$$

$$\frac{dy}{dx} = -e^{-x} (x^3 - x^2 - 4x + 1)$$

$$(x) y = x e^{\sin x}$$

$$\text{Solution:- } y = x e^{\sin x}$$

$$\begin{aligned} \rightarrow \frac{dy}{dx} &= \frac{d}{dx} (x e^{\sin x}) \\ &= x \frac{d}{dx} (e^{\sin x}) + e^{\sin x} \frac{d}{dx} (x) \\ &= x e^{\sin x} \frac{d}{dx} (\sin x) + e^{\sin x} (1) \\ &= x e^{\sin x} \cdot \cos x + e^{\sin x} \end{aligned}$$

$$\rightarrow \frac{dy}{dx} = e^{\sin x} (x \cos x + 1)$$

$$(xi) y = 5 e^{3x-4}$$

$$\text{Solution:- } y = 5 e^{3x-4}$$

$$\begin{aligned} \rightarrow \frac{dy}{dx} &= \frac{d}{dx} (5 e^{3x-4}) = 5 \frac{d}{dx} (e^{3x-4}) \\ &= 5 \cdot e^{3x-4} \frac{d}{dx} (3x-4) = 5 \cdot e^{3x-4} (3 - 0) \end{aligned}$$

$$\rightarrow \frac{dy}{dx} = 15 e^{3x-4}$$

$$(xii) y = (x+1)^x$$

$$\text{Solution:- } y = (x+1)^x$$

Taking ln on both sides

$$\rightarrow \ln y = \ln (x+1)^x$$

$$\rightarrow \ln y = x \ln (x+1)$$

$$\frac{d}{dx} (\ln y) = \frac{d}{dx} (x \ln (x+1))$$

$$\rightarrow \frac{1}{y} \frac{dy}{dx} = x \frac{d}{dx} (\ln (x+1)) + \ln (x+1) \frac{d}{dx} (x)$$

$$= x \cdot \frac{1}{x+1} + \ln (x+1) (1)$$

$$\rightarrow \frac{dy}{dx} = y \left[\frac{x}{x+1} + \ln (x+1) \right]$$

$$\frac{dy}{dx} = (x+1)^x \left[\frac{x}{x+1} + \ln (x+1) \right] \because y = (x+1)^x$$

$$(xiii) y = (\ln x)^{\ln x}$$

$$\text{Solution:- } y = (\ln x)^{\ln x}$$

Taking ln on both sides

$$\ln y = \ln (\ln x)^{\ln x} = \ln x \ln (\ln x)$$

$$\frac{d}{dx} (\ln y) = \frac{d}{dx} (\ln x \ln (\ln x))$$

$$\rightarrow \frac{1}{y} \frac{dy}{dx} = \ln x \frac{d}{dx} \ln (\ln x) + \ln (\ln x) \frac{d}{dx} (\ln x)$$

$$= \cancel{\ln x} \cdot \frac{1}{\cancel{\ln x}} \frac{d}{dx} (\ln x) + \ln (\ln x) \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y \left[\frac{1}{x} + \frac{\ln (\ln x)}{x} \right]$$

$$\rightarrow \frac{dy}{dx} = (\ln x)^{\ln x} \left[\frac{1 + \ln (\ln x)}{x} \right] \because y = (\ln x)^{\ln x}$$

$$(xiv) y = \frac{\sqrt{x^2-1}(x+1)}{(x^2+1)^{3/2}}$$

$$\text{Solution:- } y = \frac{(x^2-1)^{1/2}(x+1)}{[(x+1)(x^2-x+1)]^{3/2}}$$

$$y = \frac{(x-1)^{1/2}(x+1)^{1/2}(x+1)}{(x+1)^{3/2}(x^2-x+1)^{3/2}} = \frac{(x-1)^{1/2}(x+1)^{1/2}}{(x+1)^{3/2}(x^2-x+1)^{3/2}}$$

$$\Rightarrow y = \frac{(x-1)^{1/2}}{(x^2-x+1)^{3/2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} \left[\frac{(x-1)^{1/2}}{(x^2-x+1)^{3/2}} \right]$$

$$= \frac{(x^2-x+1)^{3/2} \cdot \frac{d}{dx}(x-1)^{1/2} - (x-1)^{1/2} \cdot \frac{d}{dx}(x^2-x+1)^{3/2}}{[(x^2-x+1)^3]^2}$$

$$= \frac{(x^2-x+1)(x^2-x+1)^{1/2} \cdot \frac{1}{2}(x-1)^{-1/2} - (x-1)^{1/2} \cdot \frac{3}{2}(x^2-x+1)^{1/2}(2x-1)}{(x^2-x+1)^3}$$

$$= \frac{(x^2-x+1)^{1/2} \cdot \frac{1}{2} \left[(x^2-x+1) \frac{1}{\sqrt{x-1}} - \sqrt{x-1} \cdot 3(2x-1) \right]}{(x^2-x+1)^3}$$

$$= \frac{(x^2-x+1) - (x-1)(6x-3)}{2(x^2-x+1)^{3-1/2}}$$

$$= \frac{x^2-x+1 - (6x^2-3x-6x+3)}{2\sqrt{x-1}(x^2-x+1)^{5/2}}$$

$$\frac{dy}{dx} = \frac{x^2-x+1-6x^2+9x-3}{2\sqrt{x-1}(x^2-x+1)^{5/2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-5x^2+8x-2}{2\sqrt{x-1}(x^2-x+1)^{5/2}}$$

Q3. Find $\frac{dy}{dx}$ if

(i) $y = \cosh 2x$

$$\text{Solution:- } y = \cosh 2x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\cosh 2x)$$

$$\Rightarrow \frac{dy}{dx} = \sinh 2x \cdot \frac{d}{dx}(2x) = \sinh 2x \cdot 2$$

$$\Rightarrow \frac{dy}{dx} = 2 \sinh 2x$$

(ii) $y = \sinh 3x$

$$\text{Solution:- } y = \sinh 3x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sinh 3x)$$

$$\Rightarrow \frac{dy}{dx} = \cosh 3x \cdot \frac{d}{dx}(3x) = \cosh 3x \cdot 3$$

$$= 3 \cosh 3x$$

(iii) $y = \tanh^{-1}(\sin x) \quad , \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$\text{Solution:- } y = \tanh^{-1}(\sin x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\tanh^{-1}(\sin x))$$

$$= \frac{1}{1-(\sin x)^2} \cdot \frac{d}{dx}(\sin x) \quad \because \frac{d}{dx} \tanh^{-1}x = \frac{1}{1-x^2}$$

$$= \frac{1}{1-\sin^2 x} \cdot \cos x \Rightarrow \frac{dy}{dx} = \frac{\cos x}{\cos^2 x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos x} = \sec x$$

(iv) $y = \sinh^{-1}(x^3)$

$$\text{Solution:- } y = \sinh^{-1}(x^3)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sinh^{-1}(x^3))$$

$$= \frac{1}{\sqrt{1+(x^3)^2}} \cdot \frac{d}{dx}(x^3) \quad \because \frac{d}{dx} \sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^6}} (3x^2) = \frac{3x^2}{\sqrt{1+x^6}}$$

(v) $y = \ln(\tanh x)$

$$\text{Solution:- } y = \ln(\tanh x)$$

$$\frac{dy}{dx} = \frac{d}{dx}(\ln(\tanh x)) = \frac{1}{\tanh x} \cdot \frac{d}{dx}(\tanh x)$$

$$= \coth x \cdot \operatorname{sech}^2 x = \frac{\cosh x}{\sinh x} \cdot \frac{1}{\cosh^2 x}$$

$$= \frac{1}{\sinh x \cosh x} = \frac{2}{2 \sinh x \cosh x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sinh 2x} = 2 \operatorname{cosech} 2x$$

(vi) $y = \sinh^{-1}\left(\frac{x}{2}\right)$

$$\text{Solution:- } y = \sinh^{-1}\left(\frac{x}{2}\right)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{2}\right)\right)$$

$$= \frac{1}{\sqrt{1+\left(\frac{x}{2}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$= \frac{1}{\sqrt{1+\frac{x^2}{4}}} \cdot \frac{1}{2} = \frac{1}{2\sqrt{4+x^2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+4}}$$

Successive differentiation (or higher derivatives)

Let $y = f(x)$ be a function, then its successive or higher derivatives are given below

1 st derivative	2 nd derivative	3 rd derivative
y'	y''	y'''
$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$
y_1	y_2	y_3
D_y	D_y^2	D_y^3
$f'(x)$	$f''(x)$	$f'''(x)$
$\frac{df}{dx}$	$\frac{d^2f}{dx^2}$	$\frac{d^3f}{dx^3}$

nth derivative $y^{(n)}$ $\frac{d^ny}{dx^n}$ y_n D_y^n $f^{(n)}(x)$ $\frac{d^nf}{dx^n}$

Example 1. Find higher derivatives of the polynomial

$$f(x) = \frac{1}{12}x^4 - \frac{1}{6}x^3 + \frac{1}{4}x^2 + 2x + 7$$

Solution:- $f(x) = \frac{1}{12}x^4 - \frac{1}{6}x^3 + \frac{1}{4}x^2 + 2x + 7$

$$f'(x) = \frac{1}{12}(4x^3) - \frac{1}{6}(3x^2) + \frac{1}{4}(2x) + 2(1) + 0$$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x + 2$$

$$f''(x) = \frac{1}{3}(3x^2) - \frac{1}{2}(2x) + \frac{1}{2}(1) + 0 = x^2 - x + \frac{1}{2}$$

$$f'''(x) = 2x - 1 + 0 \quad , \quad f^{(iv)}(x) = 2$$

$$f^{(v)}(x) = 0 \quad \text{All other derivatives are zero.}$$

Example 2. Find $\frac{d^3y}{dx^3}$ if $y = \ln(x + \sqrt{x^2 + a^2})$.

Solution:- $y = \ln(x + \sqrt{x^2 + a^2})$

$$\frac{dy}{dx} = \frac{d}{dx} \ln(x + \sqrt{x^2 + a^2})$$

$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \frac{d}{dx} (x + (x^2 + a^2)^{\frac{1}{2}})$$

$$= \frac{1}{x + \sqrt{x^2 + a^2}} \left(1 + \frac{1}{2}(x^2 + a^2)^{-\frac{1}{2}} \frac{d}{dx} (x^2 + a^2) \right)$$

$$= \frac{1}{x + \sqrt{x^2 + a^2}} \left(1 + \frac{1}{x\sqrt{x^2 + a^2}} (2x) \right)$$

$$= \frac{1}{x + \sqrt{x^2 + a^2}} \left(\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}} \right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + a^2}} \Rightarrow \frac{dy}{dx} = (x^2 + a^2)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} (x^2 + a^2)^{-\frac{1}{2}} = -\frac{1}{2}(x^2 + a^2)^{-\frac{3}{2}} (2x)$$

$$\frac{d^2y}{dx^2} = -x(x^2 + a^2)^{-\frac{3}{2}} = \frac{-x}{(x^2 + a^2)^{\frac{3}{2}}}$$

$$\begin{aligned} \rightarrow \frac{d^3y}{dx^3} &= -\frac{d}{dx} \left(\frac{x}{(x^2 + a^2)^{\frac{3}{2}}} \right) \\ &= -\left[\frac{(x^2 + a^2)^{\frac{3}{2}} \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(x^2 + a^2)^{\frac{3}{2}}}{[(x^2 + a^2)^{\frac{3}{2}}]^2} \right] \\ &= -\left[\frac{(x^2 + a^2)(x^2 + a^2)^{\frac{1}{2}}(1) - x \cdot \frac{3}{2}(x^2 + a^2)^{-\frac{1}{2}}(2x)}{(x^2 + a^2)^3} \right] \\ &= -\frac{(x^2 + a^2)^{\frac{1}{2}} [x^2 + a^2 - 3x^2]}{(x^2 + a^2)^3} = -\frac{(-2x^2 + a^2)}{(x^2 + a^2)^{\frac{5}{2}}} \end{aligned}$$

$$\rightarrow \frac{d^3y}{dx^3} = \frac{2x^2 - a^2}{(x^2 + a^2)^{\frac{5}{2}}}$$

Example 3. Find $\frac{d^2y}{dx^2}$ if $y^3 + 3ax^2 + x^3 = 0$

Solution:- $y^3 + 3ax^2 + x^3 = 0$

$$\frac{d}{dx} (y^3 + 3ax^2 + x^3) = \frac{d}{dx} (0)$$

$$\rightarrow 3y^2 \frac{dy}{dx} + 3a \cdot (2x) + 3x^2 = 0$$

$$\rightarrow y^2 \frac{dy}{dx} + 2ax + x^2 = 0$$

$$\rightarrow y^2 \frac{dy}{dx} = -(2ax + x^2) \Rightarrow \frac{dy}{dx} = -\frac{(2ax + x^2)}{y^2} \quad \text{By (i)}$$

$$+ \frac{d^2y}{dx^2} = -\frac{d}{dx} \left[\frac{x^2 + 2ax}{y^2} \right]$$

$$= -\left[\frac{y^2 \frac{d}{dx}(x^2 + 2ax) - (x^2 + 2ax) \frac{d}{dx}(y^2)}{(y^2)^2} \right]$$

$$= -\left[\frac{y^2(2x + 2a) - (x^2 + 2ax) \cdot 2y \frac{dy}{dx}}{y^4} \right]$$

$$= -\frac{2y}{y^4} \left[y(x + a) + (x^2 + 2ax) \cdot \left(\frac{x^2 + 2ax}{y^2} \right) \right] \quad \text{By (i)}$$

$$= -\frac{2}{y^3} \left(\frac{y^3(x + a) + (x^2 + 2ax)^2}{y^2} \right)$$

$$= -\frac{2}{y^5} \{ y^3(x + a) + x^2(x + 2a)^2 \}$$

$$= -\frac{2}{y^5} \left(-x^2(x + 3a)(x + a) + x^2(x + 2a)^2 \right)$$

$$= -\frac{2x^2}{y^5} \left((x + 2a)^2 - (x + 3a)(x + a) \right) \quad \because y^3 + 3ax^2 + x^3 = 0 \Rightarrow y^3 = -x^2(x + 3a)$$

$$= -\frac{2x^2}{y^5} \left(x^2 + 4ax + 4a^2 - (x^2 + ax + 3ax + 3a^2) \right)$$

$$\frac{d^2y}{dx^2} = -\frac{2x^2}{y^5} (x^2 + 4ax + 4a^2 - x^2 - 4ax - 3a^2)$$

$$\rightarrow \frac{d^2y}{dx^2} = -\frac{2x^2}{y^5} (a^2) = -\frac{2a^2x^2}{y^5}$$

Example 4:- If $x = a(\theta - \sin \theta)$, $y = a(1 + \cos \theta)$
then show that $y^2 \frac{d^2 y}{dx^2} + a = 0$

Solution:- $x = a(\theta - \sin \theta)$

$$\rightarrow \frac{dx}{d\theta} = a \frac{d}{d\theta} (\theta - \sin \theta) = a \left(\frac{d}{d\theta} (\theta) - \frac{d}{d\theta} (\sin \theta) \right)$$

$$\rightarrow \frac{dx}{d\theta} = a(1 - \cos \theta)$$

$$\text{Now, } y = a(1 + \cos \theta) \rightarrow \frac{dy}{d\theta} = a \frac{d}{d\theta} (1 + \cos \theta)$$

$$= a \left(\frac{d}{d\theta} (1) + \frac{d}{d\theta} (\cos \theta) \right) = a(0 - \sin \theta)$$

$$\frac{dy}{d\theta} = -a \sin \theta$$

By chain rule, $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$

$$\rightarrow \frac{dy}{dx} = -a \sin \theta \cdot \frac{1}{a(1 + \cos \theta)} = \frac{-\sin \theta}{1 + \cos \theta}$$

$$\rightarrow \frac{d^2 y}{dx^2} = -\frac{d}{dx} \left(\frac{\sin \theta}{1 + \cos \theta} \right)$$

$$= - \left[\frac{(1 + \cos \theta) \frac{d}{dx} (\sin \theta) - (\sin \theta) \frac{d}{dx} (1 + \cos \theta)}{(1 + \cos \theta)^2} \right]$$

$$= \frac{-1}{(1 + \cos \theta)^2} \left[(1 + \cos \theta) \cdot \cos \theta \frac{d\theta}{dx} - \sin \theta (-\sin \theta) \frac{d\theta}{dx} \right]$$

$$= \frac{-1}{(1 + \cos \theta)^2} \frac{d\theta}{dx} (\cos \theta + \cos^2 \theta + \sin^2 \theta)$$

$$= \frac{-1}{(1 + \cos \theta)^2} \frac{d\theta}{dx} \cdot \frac{1}{a(1 + \cos \theta)}$$

$$\frac{d^2 y}{dx^2} = \frac{-1}{a(1 + \cos \theta)^2}$$

$$\frac{d^2 y}{dx^2} = \frac{-1}{a \left(\frac{y}{a} \right)^2}$$

$$\frac{d^2 y}{dx^2} = \frac{-1}{a \cdot \frac{y^2}{a^2}} \rightarrow \frac{d^2 y}{dx^2} = -\frac{a}{y^2}$$

$$\rightarrow y^2 \frac{d^2 y}{dx^2} + a = 0 \text{ Hence proved.}$$

Example 6. Find the first four derivatives of $\cos(ax+b)$.

Solution:- Let $y = \cos(ax+b)$

$$\frac{dy}{dx} = y_1 = \frac{d}{dx} (\cos(ax+b))$$

$$= -\sin(ax+b) \cdot \frac{d}{dx} (ax+b)$$

$$\frac{dy}{dx} = y_1 = -\sin(ax+b) \cdot (a) = -a \sin(ax+b)$$

$$\rightarrow \frac{d^2 y}{dx^2} = y_2 = \frac{d}{dx} (-a \sin(ax+b))$$

$$\rightarrow \frac{d^2 y}{dx^2} = y_2 = -a \cdot \cos(ax+b) \frac{d}{dx} (ax+b)$$

$$\rightarrow \frac{d^2 y}{dx^2} = y_2 = -a^2 \cos(ax+b)$$

$$\rightarrow \frac{d^3 y}{dx^3} = y_3 = -a^2 \frac{d}{dx} (\cos(ax+b))$$

$$= -a^2 (-\sin(ax+b)) \frac{d}{dx} (ax+b)$$

$$\frac{d^3 y}{dx^3} = y_3 = a^3 \sin(ax+b)$$

$$\rightarrow \frac{d^4 y}{dx^4} = y_4 = a^3 \frac{d}{dx} (\sin(ax+b))$$

$$= a^3 (\cos(ax+b)) \frac{d}{dx} (ax+b)$$

$$\rightarrow \frac{d^4 y}{dx^4} = y_4 = a^3 \cos(ax+b) (a) = a^4 \cos(ax+b)$$

Example 6. If $y = e^{-ax}$, then show that

$$\frac{d^3 y}{dx^3} + a^3 y = 0$$

Solution:- $y = e^{-ax}$

$$\frac{dy}{dx} = \frac{d}{dx} (e^{-ax}) = e^{-ax} \frac{d}{dx} (-ax) = -a e^{-ax}$$

$$\rightarrow \frac{d^2 y}{dx^2} = -a \frac{d}{dx} (e^{-ax}) = -a \cdot e^{-ax} (-a) = a^2 e^{-ax}$$

$$\rightarrow \frac{d^3 y}{dx^3} = a^2 \frac{d}{dx} (e^{-ax}) = a^2 (e^{-ax}) \frac{d}{dx} (-ax)$$

$$\frac{d^3 y}{dx^3} = -a^3 e^{-ax} \rightarrow \frac{d^3 y}{dx^3} = -a^3 y \because y = e^{-ax}$$

$$\rightarrow \frac{d^3 y}{dx^3} + a^3 y = 0 \text{ Hence proved.}$$

Example 7. If $y = \sin^{-1} \frac{x}{a}$, then show that $y_2 = x(a^2 - x^2)^{-3/2}$

Solution:- $y = \sin^{-1} \frac{x}{a}$

$$y_1 = \frac{dy}{dx} = \frac{d}{dx} \left(\sin^{-1} \frac{x}{a} \right)$$

$$= \frac{1}{\sqrt{1 - \left(\frac{x}{a} \right)^2}} \cdot \frac{d}{dx} \left(\frac{x}{a} \right)$$

$$= \frac{1}{\sqrt{a^2 - x^2}} \cdot \frac{1}{a} (1) \because \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

$$= \frac{1}{\sqrt{a^2 - x^2}} \cdot \frac{1}{a} = \frac{a}{\sqrt{a^2 - x^2}} \cdot \frac{1}{a}$$

$$y_1 = \frac{1}{(a^2 - x^2)^{1/2}} = (a^2 - x^2)^{-1/2}$$

$$\rightarrow y_2 = \frac{d^2 y}{dx^2} = \frac{d}{dx} (a^2 - x^2)^{-1/2} = -\frac{1}{2} (a^2 - x^2)^{-3/2} \cdot \frac{d}{dx} (a^2 - x^2)$$

$$y_2 = -\frac{1}{2} (a^2 - x^2)^{-3/2} (-2x) = x(a^2 - x^2)^{-3/2}$$

Hence proved.

Exercise 2.7

Q1. Find y_2 if
 (i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$ (ii) $y = (2x+5)^{\frac{3}{2}}$
 (iii) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

Solution:- (i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$

$$\frac{dy}{dx} = y_1 = \frac{d}{dx}(2x^5 - 3x^4 + 4x^3 + x - 2)$$

$$= 2(5x^4) - 3(4x^3) + 4(3x^2) + 1 - 0$$

$$y_1 = 10x^4 - 12x^3 + 12x^2 + 1$$

$$\rightarrow \frac{d^2y}{dx^2} = y_2 = \frac{d}{dx}(10x^4 - 12x^3 + 12x^2 + 1)$$

$$= 10(4x^3) - 12(3x^2) + 12(2x) + 0$$

$$y_2 = 40x^3 - 36x^2 + 24x$$

$$(ii) y = (2x+5)^{\frac{3}{2}}$$

$$\rightarrow y_1 = \frac{d}{dx}(2x+5)^{\frac{3}{2}} = \frac{3}{2}(2x+5)^{\frac{3}{2}-1} \cdot \frac{d}{dx}(2x+5)$$

$$y_1 = \frac{3}{2}(2x+5)^{\frac{1}{2}} \cdot 2 = 3(2x+5)^{\frac{1}{2}}$$

$$\rightarrow y_2 = 3 \frac{d}{dx}(2x+5)^{\frac{1}{2}} = 3 \left(\frac{1}{2}(2x+5)^{-\frac{1}{2}} \right) \cdot \frac{d}{dx}(2x+5)$$

$$y_2 = \frac{3}{2}(2x+5)^{-\frac{1}{2}} \cdot 2 = \frac{3}{\sqrt{2x+5}}$$

$$(iii) y = \sqrt{x} + \frac{1}{\sqrt{x}} \rightarrow y = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

$$\rightarrow y_1 = \frac{d}{dx}(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) = \frac{d}{dx}(x^{\frac{1}{2}}) + \frac{d}{dx}(x^{-\frac{1}{2}})$$

$$y_1 = \frac{1}{2}x^{-\frac{1}{2}} + (-\frac{1}{2})x^{-\frac{3}{2}} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$$

$$\rightarrow y_2 = \frac{d}{dx}\left(\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}\right) = \frac{1}{2}\left(\frac{d}{dx}x^{-\frac{1}{2}} - \frac{d}{dx}x^{-\frac{3}{2}}\right)$$

$$= \frac{1}{2}\left[-\frac{1}{2}x^{-\frac{3}{2}} - (-\frac{3}{2})x^{-\frac{5}{2}}\right] = \frac{1}{2}\left[-\frac{1}{2}x^{-\frac{3}{2}} + \frac{3}{2}x^{-\frac{5}{2}}\right]$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)\left[\frac{-1}{x^{\frac{3}{2}}} + \frac{3}{x^{\frac{5}{2}}}\right] = \frac{1}{4}\left[\frac{-1}{x^{\frac{3}{2}}} + \frac{3}{x^{\frac{5}{2}}}\right]$$

$$\rightarrow y_2 = \frac{1}{4}\left(\frac{-x+3}{x^{\frac{3}{2}} \cdot x}\right) = \frac{-x+3}{4x^{\frac{5}{2}}}$$

Q2. Find y_2 if

$$(i) y = x^2 \cdot e^x \quad (ii) y = \ln\left(\frac{2x+3}{3x+2}\right)$$

Solution:- (i) $y = x^2 \cdot e^x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \cdot e^x) = x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^2)$$

$$y_1 = x^2 e^x \frac{d}{dx}(-x) + e^x \cdot 2x$$

$$y_1 = -x^2 e^x + 2x e^x$$

$$\rightarrow y_2 = \frac{d}{dx}(-x^2 e^x + 2x e^x)$$

$$= \frac{d}{dx}(-x^2 e^x) + 2 \frac{d}{dx}(x e^x)$$

$$= -x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(-x^2) + 2\left\{x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x)\right\}$$

$$= -x^2 \cdot e^x(-1) + e^x(-2x) + 2\{x e^x(-1) + e^x(1)\}$$

$$= x^2 e^x - 2x e^x - 2x e^x + 2 e^x$$

$$y_2 = x^2 e^x - 4x e^x + 2 e^x$$

$$\rightarrow y_2 = e^x(x^2 - 4x + 2)$$

$$(ii) y = \ln\left(\frac{2x+3}{3x+2}\right)$$

$$\rightarrow \frac{dy}{dx} = y_1 = \frac{d}{dx} \cdot \ln\left(\frac{2x+3}{3x+2}\right) = \frac{d}{dx}[\ln(2x+3) - \ln(3x+2)]$$

$$= \frac{d}{dx} \ln(2x+3) - \frac{d}{dx} \ln(3x+2) = \frac{2}{2x+3} - \frac{3}{3x+2}$$

$$y_1 = 2(2x+3)^{-1} - 3(3x+2)^{-1}$$

$$\rightarrow \frac{d^2y}{dx^2} = y_2 = 2 \frac{d}{dx}(2x+3)^{-1} - 3 \frac{d}{dx}(3x+2)^{-1}$$

$$= 2(-1)(2x+3)^{-2} \frac{d}{dx}(2x+3) - 3(-1)(3x+2)^{-2} \frac{d}{dx}(3x+2)$$

$$= -2(2x+3)^{-2}(2) + 3(3x+2)^{-2}(3)$$

$$y_2 = \frac{-4}{(2x+3)^2} + \frac{9}{(3x+2)^2}$$

$$= \frac{-4(3x+2)^2 + 9(2x+3)^2}{(2x+3)^2(3x+2)^2}$$

$$y_2 = \frac{-4(9x^2+4+12x) + 9(4x^2+9+12x)}{(2x+3)^2(3x+2)^2}$$

$$y_2 = \frac{-36x^2 - 16 - 48x + 36x^2 + 81 + 108x}{(2x+3)^2(3x+2)^2}$$

$$\rightarrow y_2 = \frac{60x+65}{(2x+3)^2(3x+2)^2}$$

Q3. Find y_2 if

$$(i) x^2 + y^2 = a^2$$

Solution:- $x^2 + y^2 = a^2 \rightarrow y^2 = a^2 - x^2$

$$\rightarrow \frac{d}{dx}(y^2) = \frac{d}{dx}(a^2 - x^2)$$

$$\rightarrow 2y \frac{dy}{dx} = 0 - 2x \rightarrow x y \frac{dy}{dx} = -x^2$$

$$\rightarrow \frac{dy}{dx} = -\frac{x}{y} \rightarrow y_1 = -\frac{x}{y}$$

$$\rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{x}{y}\right) = \frac{y \frac{d}{dx}(-x) - (-x) \frac{dy}{dx}}{y^2}$$

$$= \frac{y(-1) + x \frac{dy}{dx}}{y^2}$$

$$= \frac{-y + x \frac{dy}{dx}}{y^2} = \frac{-1}{y^2} \left\{ y - x \frac{dy}{dx} \right\}$$

$$= -\frac{1}{y^2} \left\{ y - x \left(-\frac{x}{y} \right) \right\} \therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$= -\frac{1}{y^2} \left\{ y + \frac{x^2}{y} \right\} = -\frac{1}{y^2} \left\{ \frac{y^2 + x^2}{y} \right\}$$

$$\frac{d^2y}{dx^2} = -\frac{(x^2 + y^2)}{y^3} = -\frac{a^2}{y^3} \therefore x^2 + y^2 = a^2$$

$$\rightarrow y_2 = -\frac{a^2}{y^3}$$

(ii) $x^3 - y^3 = a^3$

Solution:- $x^3 - y^3 = a^3 \rightarrow y^3 = x^3 - a^3$

$$\rightarrow \frac{d}{dx}(y^3) = \frac{d}{dx}(x^3 - a^3) = \frac{d}{dx}(x^3) - \frac{d}{dx}(a^3)$$

$$3y^2 \frac{dy}{dx} = 3x^2 \rightarrow \frac{dy}{dx} = \frac{x^2}{y^2} \rightarrow y_1 = \frac{x^2}{y^2}$$

$$\rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} \left\{ \frac{x^2}{y^2} \right\} = \frac{y^2 \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(y^2)}{y^4}$$

$$= \frac{1}{y^4} \left\{ y^2(2x) - x^2 \cdot 2y \frac{dy}{dx} \right\}$$

$$= \frac{2xy}{y^4} \left\{ y - x \frac{dy}{dx} \right\} = \frac{2x}{y^3} \left\{ y - x \left(\frac{x^2}{y^2} \right) \right\}$$

$$= \frac{2x}{y^3} \left\{ \frac{y^3 - x^3}{y^2} \right\} = -\frac{2x}{y^5} (x^3 - y^3)$$

$$\therefore \frac{dy}{dx} = \frac{x^2}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{2x}{y^5} (a^3)$$

$$\rightarrow y_2 = -\frac{2xa^3}{y^5}$$

(iii) $x = a \cos \theta$, $y = a \sin \theta$

Solution:- $x = a \cos \theta$, $y = a \sin \theta$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a \cos \theta) = a(-\sin \theta) = -a \sin \theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(a \sin \theta) = a(\cos \theta)$$

By chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = (a \cos \theta) \left(\frac{1}{-a \sin \theta} \right)$$

$$\frac{dy}{dx} = -\frac{\cos \theta}{\sin \theta} = -\cot \theta \rightarrow y_1 = -\cot \theta$$

$$\rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(-\cot \theta) = -(-\operatorname{cosec}^2 \theta) \frac{d\theta}{dx}$$

$$= \operatorname{cosec}^2 \theta \left(\frac{1}{-a \sin \theta} \right) = \frac{1}{\sin^3 \theta} \left(\frac{1}{-a \sin \theta} \right)$$

$$\frac{d^2y}{dx^2} = \frac{-1}{a \sin^3 \theta} \rightarrow y_2 = \frac{-1}{a \sin^3 \theta}$$

(iv) $x = at^2$, $y = bt^4$

Solution:- $x = at^2$, $y = bt^4$

$$\frac{dx}{dt} = \frac{d}{dt}(at^2) = a(2t) = 2at$$

$$\frac{dy}{dt} = \frac{d}{dt}(bt^4) = b(4t^3) = 4bt^3$$

By chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 4bt^3 \cdot \frac{1}{2at} = \frac{2bt^2}{a}$$

$$\rightarrow \frac{dy}{dx} = \frac{2bt^2}{a}, \quad y_1 = \frac{2bt^2}{a}$$

$$\rightarrow \frac{d^2y}{dx^2} = \frac{2b}{a} \frac{d}{dx}(t^2) = \frac{2b}{a} \left(2t \frac{dt}{dx} \right)$$

$$= \frac{2b}{a} \left(2t \cdot \frac{1}{2at} \right) \therefore \frac{dt}{dx} = \frac{1}{2at}$$

$$\frac{d^2y}{dx^2} = \frac{2b}{a^2} \rightarrow y_2 = \frac{2b}{a^2}$$

(v) $x^2 + y^2 + 2gx + 2fy + c = 0$

Solution:- $x^2 + y^2 + 2gx + 2fy + c = 0$

$$\frac{d}{dx}(x^2 + y^2 + 2gx + 2fy + c) = \frac{d}{dx}(0)$$

$$\rightarrow 2x + 2y \frac{dy}{dx} + 2g(1) + 2f \frac{dy}{dx} + 0 = 0$$

$$\rightarrow x + y \frac{dy}{dx} + g + f \frac{dy}{dx} = 0$$

$$\rightarrow (x + g) + (y + f) \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\left(\frac{x + g}{y + f} \right) \rightarrow y_1 = -\left(\frac{x + g}{y + f} \right)$$

$$\frac{d^2y}{dx^2} = -\frac{d}{dx} \left(\frac{x + g}{y + f} \right) = -\left\{ \frac{(y + f) \frac{d}{dx}(x + g) - (x + g) \frac{d}{dx}(y + f)}{(y + f)^2} \right\}$$

$$= \frac{-1}{(y + f)^2} \left\{ (y + f)(1) - (x + g) \cdot \frac{dy}{dx} \right\}$$

$$= \frac{-1}{(y + f)^2} \left\{ (y + f) - (x + g) \left(-\left(\frac{x + g}{y + f} \right) \right) \right\}$$

$$= \frac{-1}{(y + f)^2} \left\{ (y + f) + \frac{(x + g)^2}{y + f} \right\}$$

$$= \frac{-1}{(y + f)^2} \left\{ \frac{(y + f)^2 + (x + g)^2}{y + f} \right\}$$

$$= \frac{-1}{(y + f)^3} \left\{ y^2 + f^2 + 2fy + x^2 + g^2 + 2gx \right\}$$

$$= \frac{-1}{(y + f)^3} \left\{ f^2 + g^2 + (x^2 + y^2 + 2gx + 2fy) \right\}$$

$$\therefore x^2 + y^2 + 2gx + 2fy + c = 0$$

$$\rightarrow x^2 + y^2 + 2gx + 2fy = -c$$

$$\frac{d^2y}{dx^2} = \frac{-1}{(y + f)^3} (f^2 + g^2 - c) = \frac{c - f^2 - g^2}{(y + f)^3}$$

Q4. Find y_4 if

- (i) $y = \sin 3x$ (ii) $y = \cos^3 x$
 (iii) $y = \ln(x^2 - 9)$

Solution:- (i) $y = \sin 3x$

$$\rightarrow \frac{dy}{dx} = y_1 = \frac{d}{dx}(\sin 3x) = \cos 3x \frac{d}{dx}(3x)$$

$$y_1 = 3 \cos 3x$$

$$\rightarrow \frac{d^2y}{dx^2} = y_2 = \frac{d}{dx}(3 \cos 3x) = 3(-\sin 3x) \frac{d}{dx}(3x)$$

$$y_2 = -9 \sin 3x$$

$$\rightarrow \frac{d^3y}{dx^3} = y_3 = \frac{d}{dx}(-9 \sin 3x) = -9(\cos 3x) \frac{d}{dx}(3x)$$

$$y_3 = -27 \cos 3x$$

$$\rightarrow \frac{d^4y}{dx^4} = y_4 = \frac{d}{dx}(-27 \cos 3x) = -27(-\sin 3x) \frac{d}{dx}(3x)$$

$$y_4 = 81 \sin 3x$$

(ii) $y = \cos^3 x$

$$\begin{aligned} (\because \cos 3x &= 4\cos^3 x - 3\cos x) \\ \rightarrow \cos 3x + 3\cos x &= 4\cos^3 x \\ \rightarrow \frac{1}{4}(\cos 3x + 3\cos x) &= \cos^3 x \end{aligned}$$

Thus $y = \frac{1}{4}(\cos 3x + 3\cos x)$

$$\frac{dy}{dx} = y_1 = \frac{1}{4} \left[\frac{d}{dx}(\cos 3x) + \frac{d}{dx}(3\cos x) \right]$$

$$= \frac{1}{4}(-\sin 3x(3) + 3(-\sin x))$$

$$y_1 = -\frac{3}{4}(\sin 3x + \sin x)$$

$$\rightarrow \frac{d^2y}{dx^2} = y_2 = -\frac{3}{4} \left(\frac{d}{dx}(\sin 3x) + \frac{d}{dx}(\sin x) \right)$$

$$= -\frac{3}{4}(\cos 3x(3) + \cos x)$$

$$y_2 = -\frac{3}{4}(3\cos 3x + \cos x)$$

$$\rightarrow \frac{d^3y}{dx^3} = y_3 = -\frac{3}{4} \left[3 \frac{d}{dx}(\cos 3x) + \frac{d}{dx}(\cos x) \right]$$

$$= -\frac{3}{4} \{ 3(-\sin 3x)(3) + (-\sin x) \}$$

$$= -\frac{3}{4}(-9\sin 3x - \sin x)$$

$$y_3 = \frac{3}{4}(9\sin 3x + \sin x)$$

$$\rightarrow \frac{d^4y}{dx^4} = y_4 = \frac{3}{4} \left[9 \frac{d}{dx}(\sin 3x) + \frac{d}{dx}(\sin x) \right]$$

$$= \frac{3}{4} \{ 9(\cos 3x)(3) + \cos x \}$$

$$= \frac{3}{4} \{ 27\cos 3x + \cos x \}$$

$$= \frac{3}{4} \{ 27(4\cos^3 x - 3\cos x) + \cos x \}$$

$$= \frac{3 \times 27 \times 4}{4} \cos^3 x - \frac{3 \times 27 \times 3}{4} \cos x + \frac{3}{4} \cos x$$

$$= 81\cos^3 x - \frac{243}{4} \cos x + \frac{3}{4} \cos x$$

$$y_4 = 81\cos^3 x + \left(\frac{3-243}{4} \right) \cos x$$

$$\rightarrow y_4 = 81\cos^3 x - \frac{240}{4} \cos x = 81\cos^3 x - 60\cos x$$

(iii) $y = \ln(x^2 - 9) = \ln(x-3)(x+3)$

$$y = \ln(x-3) + \ln(x+3)$$

$$\rightarrow \frac{dy}{dx} = y_1 = \frac{d}{dx}[\ln(x-3) + \ln(x+3)]$$

$$y_1 = \frac{1}{x-3} + \frac{1}{x+3} = (x-3)^{-1} + (x+3)^{-1}$$

$$\rightarrow \frac{d^2y}{dx^2} = y_2 = \frac{d}{dx}(x-3)^{-1} + \frac{d}{dx}(x+3)^{-1}$$

$$y_2 = (-1)(x-3)^{-2} + (-1)(x+3)^{-2}$$

$$\rightarrow \frac{d^3y}{dx^3} = y_3 = \frac{d}{dx}((-1)(x-3)^{-2} + (-1)(x+3)^{-2})$$

$$y_3 = (-1)(-2)(x-3)^{-3} + (-1)(-2)(x+3)^{-3}$$

$$\rightarrow \frac{d^4y}{dx^4} = y_4 = \frac{d}{dx}((-1)(-2)(x-3)^{-3} + (-1)(-2)(x+3)^{-3})$$

$$= (-1)(-2)(-3)(x-3)^{-4} + (-1)(-2)(-3)(x+3)^{-4}$$

$$y_4 = -\frac{6}{(x-3)^4} - \frac{6}{(x+3)^4} = -6 \left[\frac{1}{(x-3)^4} + \frac{1}{(x+3)^4} \right]$$

Q5. If $x = \sin \theta$, $y = \sin m\theta$, show that $(1-x^2)y_2 - xy_1 + m^2y = 0$

Solution:- $\because x = \sin \theta$

$$\rightarrow \sin^{-1} x = \theta \rightarrow y = \sin(m\sin^{-1} x)$$

Now $y = \sin(m\sin^{-1} x)$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin(m\sin^{-1} x))$$

$$y_1 = \cos(m\sin^{-1} x) \cdot \frac{d}{dx}(m\sin^{-1} x)$$

$$y_1 = \cos(m\sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}$$

$$\rightarrow \sqrt{1-x^2} y_1 = m \cos(m\sin^{-1} x)$$

$$\rightarrow \frac{d}{dx}(\sqrt{1-x^2} y_1) = m \frac{d}{dx}(\cos(m\sin^{-1} x))$$

$$\sqrt{1-x^2} \frac{d}{dx}(y_1) + y_1 \frac{d}{dx}(\sqrt{1-x^2}) = m(-\sin(m\sin^{-1} x)) \cdot \frac{d}{dx}(m\sin^{-1} x)$$

$$\sqrt{1-x^2} y_2 + y_1 \left(\frac{1}{2} \right) (1-x^2)^{-\frac{1}{2}} \cdot \frac{d}{dx}(1-x^2) = -m \sin(m\sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2} y_2 + \frac{y_1}{2\sqrt{1-x^2}} (-2x) = -\frac{m^2 \sin(m\sin^{-1} x)}{\sqrt{1-x^2}}$$

Multiplying b/s by $\sqrt{1-x^2}$

$$\rightarrow (1-x^2) y_2 - xy_1 = -m^2 \sin(m\sin^{-1} x)$$

$$\rightarrow (1-x^2) y_2 - xy_1 = -m^2 y \quad \because y = \sin(m\sin^{-1} x)$$

$$\rightarrow (1-x^2) y_2 - xy_1 + m^2 y = 0$$

Hence proved.

Q6. If $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

Solution:- $y = e^x \sin x$

$$\frac{dy}{dx} = \frac{d}{dx} (e^x \sin x)$$

$$= e^x \frac{d}{dx} (\sin x) + \sin x \frac{d}{dx} (e^x)$$

$$\frac{dy}{dx} = e^x \cos x + e^x \sin x \longrightarrow (I)$$

$$\rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} (e^x \cos x) + \frac{d}{dx} (e^x \sin x)$$

$$= e^x \frac{d}{dx} (\cos x) + \cos x \frac{d}{dx} (e^x) + e^x \frac{d}{dx} (\sin x)$$

$$+ \sin x \frac{d}{dx} (e^x)$$

$$= e^x (-\sin x) + \cos x \cdot e^x + e^x \cos x + \sin x \cdot e^x$$

$$= -e^x \sin x + e^x \cos x + e^x \cos x + e^x \sin x$$

$$\frac{d^2y}{dx^2} = 2e^x \cos x$$

$$\text{Now L.H.S} = \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y$$

$$= 2e^x \cos x - 2(e^x \cos x + e^x \sin x) + 2e^x \sin x$$

$$= 2e^x \cos x - 2e^x \cos x - 2e^x \sin x + 2e^x \sin x$$

$$= 0 = R.H.S$$

$$\text{Hence } \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

Q7. If $y = e^{ax} \sin bx$ show that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$

Solution:- $y = e^{ax} \sin bx$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} (e^{ax} \sin bx)$$

$$= e^{ax} \frac{d}{dx} (\sin bx) + \sin bx \frac{d}{dx} (e^{ax})$$

$$= e^{ax} \cdot \cos bx \frac{d}{dx} (bx) + \sin bx \cdot e^{ax} \frac{d}{dx} (ax)$$

$$= be^{ax} \cos bx + ae^{ax} \sin bx$$

$$\frac{dy}{dx} = e^{ax} [a \sin bx + b \cos bx]$$

$$\rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} \{ e^{ax} (a \sin bx + b \cos bx) \}$$

$$= e^{ax} \left(a \frac{d}{dx} \sin bx + b \frac{d}{dx} \cos bx \right) + (a \sin bx + b \cos bx) \frac{d}{dx} (e^{ax})$$

$$= e^{ax} [a \cos bx \cdot b + b(-\sin bx) \cdot a] + (a \sin bx + b \cos bx) e^{ax} \cdot a$$

$$= e^{ax} (abc \cos bx - b^2 \sin bx + a^2 \sin bx + abc \cos bx)$$

$$\frac{d^2y}{dx^2} = e^{ax} (2abc \cos bx - b^2 \sin bx + a^2 \sin bx)$$

$$\text{Now L.H.S} = \frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y$$

$$= e^{ax} (2abc \cos bx - b^2 \sin bx + a^2 \sin bx) - 2a (e^{ax} (a \sin bx + b \cos bx))$$

$$+ (a^2 + b^2) e^{ax} \sin bx$$

$$= e^{ax} 2abc \cos bx - b^2 e^{ax} \sin bx + e^{ax} a^2 \sin bx - 2a^2 e^{ax} \sin bx$$

$$- 2abc \cos bx e^{ax} + a^2 e^{ax} \sin bx + b^2 e^{ax} \sin bx$$

$$= 0 = R.H.S$$

$$\text{Hence } \frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0$$

Q8. If $y = (\cos^{-1} x)^2$, prove that $(1-x^2)y_2 - xy_1 - 2 = 0$

Solution:- $y = (\cos^{-1} x)^2$

$$y_1 = \frac{d}{dx} (\cos^{-1} x)^2 = 2 \cos^{-1} x \cdot \frac{d}{dx} (\cos^{-1} x)$$

$$y_1 = 2 \cos^{-1} x \cdot \frac{-1}{\sqrt{1-x^2}}$$

$$\rightarrow \sqrt{1-x^2} y_1 = -2 \cos^{-1} x$$

$$\rightarrow \frac{d}{dx} (\sqrt{1-x^2} y_1) = -2 \frac{d}{dx} (\cos^{-1} x)$$

$$\rightarrow \sqrt{1-x^2} y_2 + y_1 \frac{d}{dx} \sqrt{1-x^2} = -2 \cdot \left(\frac{-1}{\sqrt{1-x^2}} \right)$$

$$\rightarrow \sqrt{1-x^2} y_2 + y_1 \cdot \frac{1}{2} (1-x^2)^{-\frac{1}{2}} \frac{d}{dx} (1-x^2) = \frac{2}{\sqrt{1-x^2}}$$

$$\rightarrow \sqrt{1-x^2} y_2 + \frac{y_1}{2\sqrt{1-x^2}} (-2x) = \frac{2}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2} y_2 - \frac{y_1 x}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

Multiplying both sides by $\sqrt{1-x^2}$

$$\rightarrow (1-x^2) y_2 - x y_1 = 2$$

$$\rightarrow (1-x^2) y_2 - x y_1 - 2 = 0 \text{ Hence proved.}$$

Q9. If $y = a \cos(\ln x) + b \sin(\ln x)$ prove that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$

Solution:- $y = a \cos(\ln x) + b \sin(\ln x)$

$$\frac{dy}{dx} = a \frac{d}{dx} \cos(\ln x) + b \frac{d}{dx} \sin(\ln x)$$

$$= a (-\sin(\ln x)) \frac{d}{dx} (\ln x) + b (\cos(\ln x)) \frac{d}{dx} (\ln x)$$

$$\frac{dy}{dx} = -a \sin(\ln x) \cdot \frac{1}{x} + b \cos(\ln x) \cdot \frac{1}{x}$$

$$\rightarrow x \frac{dy}{dx} = -a \sin(\ln x) + b \cos(\ln x)$$

$$\rightarrow \frac{d}{dx} \left(x \frac{dy}{dx} \right) = -a \frac{d}{dx} \sin(\ln x) + b \frac{d}{dx} \cos(\ln x)$$

$$x \frac{d^2 y}{dx^2} + \frac{dy}{dx} (1) = -a \cos(\ln x) \cdot \frac{1}{x} + b(-\sin(\ln x)) \cdot \frac{1}{x}$$

$$\rightarrow x \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = -a \cos(\ln x) - b \sin(\ln x) \\ = -(a \cos(\ln x) + b \sin(\ln x))$$

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = -y \quad \because y = a \cos(\ln x) + b \sin(\ln x)$$

$$\rightarrow x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0 \quad \text{Hence proved.}$$

Power series expansion of a function:-

A series of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots + a_n x^n$ is called a power series expansion of a function $f(x)$, where $a_0, a_1, a_2, \dots, a_n$ are constants and x is a variable.

Machlaurin series:-

The series $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^{(n)}(x)$ is known as "Machlaurin's series expansion".

Proof:- We know that power series of $f(x)$ in x

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots + a_n x^n \rightarrow (I)$$

$$\rightarrow f(0) = a_0 \rightarrow a_0 = f(0)$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots + n a_n x^{n-1}$$

$$\rightarrow f'(0) = a_1 \rightarrow a_1 = f'(0)$$

$$f''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + \dots + n(n-1)a_n x^{n-2}$$

$$\rightarrow f''(0) = 2a_2 \rightarrow a_2 = \frac{f''(0)}{2} = \frac{f''(0)}{2!}$$

$$f'''(x) = 6a_3 + 24a_4 x + \dots + n(n-1)(n-2)a_n x^{n-3}$$

$$\rightarrow f'''(0) = 6a_3 \rightarrow a_3 = \frac{f'''(0)}{6} = \frac{f'''(0)}{3!}$$

$$\text{Similarly, } \rightarrow a_n = \frac{f^{(n)}(0)}{n!}$$

Putting values of $a_1, a_2, a_3, \dots, a_n$ in (I)

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0)$$

Note:- A function f can be expanded in Machlaurin series if the function is defined in the interval containing 0 and its derivatives exist at $x=0$. The expansion is only valid if it is convergent.

*The above expansion is also named as "Machlaurin's theorem"

Example 1. Expand $f(x) = \frac{1}{1+x}$ in the Machlaurin series.

$$\text{Solution:- } f(x) = \frac{1}{1+x} \rightarrow f(0) = \frac{1}{1+0} = 1$$

$$f'(x) = \frac{d}{dx} \left(\frac{1}{1+x} \right) \rightarrow f'(x) = \frac{d}{dx} (1+x)^{-1}$$

$$f'(x) = (-1)(1+x)^{-2} \rightarrow f'(0) = (-1)(1+0)^{-2} = (-1)(1) = -1$$

$$f''(x) = (-1) \frac{d}{dx} (1+x)^{-2} = (-1)(-2)(1+x)^{-3} = 2(1+x)^{-3}$$

$$\rightarrow f''(0) = 2(1+0)^{-3} = 2 \rightarrow f''(0) = (1)^3 2!$$

$$f'''(x) = 2 \frac{d}{dx} (1+x)^{-3} = 2(-3)(1+x)^{-4} = -6(1+x)^{-4}$$

$$\rightarrow f'''(0) = -6(1+0)^{-4} = -6 = (-1)^3 3!$$

$$f^{(iv)}(x) = -6 \frac{d}{dx} (1+x)^{-4} = -6(-4)(1+x)^{-5} = 24(1+x)^{-5}$$

$$\rightarrow f^{(iv)}(0) = 24(1+0)^{-5} = 24 = (-1)^4 4!$$

Now by Machlaurin series expansion

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \dots$$

$$\frac{1}{1+x} = 1 + x(-1) + \frac{x^2}{2!} (-1)^2 2! + \frac{x^3}{3!} (-1)^3 3! + \frac{x^4}{4!} (-1)^4 4! + \dots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots$$

Machlaurin's series for $\frac{1}{1+x}$ is the geometric series with first term 1 and common ratio $-x$.
 $\therefore S = \frac{a}{1-r}$ we have
 $1 - x + x^2 - x^3 + \dots = \frac{1}{1-(-x)} = \frac{1}{1+x}$

Example 2. Find the Machlaurin series for $\sin x$.

Solution:- Let $f(x) = \sin x$

$$\rightarrow f(0) = \sin 0 = 0$$

$$f'(x) = \frac{d}{dx} (\sin x) = \cos x$$

$$\rightarrow f'(0) = \cos 0 = 1$$

$$f''(x) = \frac{d}{dx} (\cos x) = -\sin x$$

$$\rightarrow f''(0) = -\sin 0 = 0, \quad f'''(x) = \frac{d}{dx} (-\sin x) = -\cos x$$

$$\rightarrow f'''(0) = -\cos 0 = -1$$

$$f^{(iv)}(x) = \frac{d}{dx} (-\cos x) = -(-\sin x) = \sin x$$

$$f^{(iv)}(0) = \sin 0 = 0, \quad f^{(v)}(x) = \frac{d}{dx} (\sin x) = \cos x$$

$$\rightarrow f^{(v)}(0) = \cos 0 = 1$$

Now by Machlaurin series expansion

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \frac{x^5}{5!} f^{(v)}(0) + \frac{x^6}{6!} f^{(vi)}(0) + \dots$$

$$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Example 3. Expand a^x in the Machlaurin series.

Solution:- Let $f(x) = a^x \rightarrow f(0) = a^0 = 1$
 $f'(x) = \frac{d}{dx}(a^x) = a^x \ln a \rightarrow f'(0) = a^0 \ln a = \ln a$

$$f''(x) = \ln a \frac{d}{dx}(a^x) = \ln a (a^x \ln a) = a^x (\ln a)^2$$

$$\rightarrow f''(0) = a^0 (\ln a)^2 = (\ln a)^2$$

$$f'''(x) = (\ln a)^2 \frac{d}{dx}(a^x) = (\ln a)^2 (a^x \ln a)$$

$$f'''(x) = a^x (\ln a)^3 \rightarrow f'''(0) = a^0 (\ln a)^3 = (\ln a)^3$$

$$f^{(iv)}(x) = (\ln a)^3 \frac{d}{dx}(a^x) = (\ln a)^3 (a^x \ln a) = a^x (\ln a)^4$$

$$\rightarrow f^{(iv)}(0) = a^0 (\ln a)^4 = (\ln a)^4$$

Now by Machlaurin series expansion,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \dots$$

$$a^x = 1 + x \ln a + \frac{x^2}{2!} (\ln a)^2 + \frac{x^3}{3!} (\ln a)^3 + \frac{x^4}{4!} (\ln a)^4 + \dots$$

Note:- If we put $a=e$ in the above expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots \quad (\because \ln e = 1)$$

Replace x by 1, we have

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$$

Example 4. Expand $(1+x)^n$ in the Machlaurin series.

Solution:- Let $f(x) = (1+x)^n$

$$\rightarrow f(0) = (1+0)^n = 1$$

$$f'(x) = n(1+x)^{n-1} \rightarrow f'(0) = n(1+0)^{n-1} = n$$

$$f''(x) = n \frac{d}{dx}(1+x)^{n-1} = n(n-1)(1+x)^{n-2}$$

$$f''(0) = n(n-1)(1+0)^{n-2} = n(n-1)$$

$$f'''(x) = n(n-1) \frac{d}{dx}(1+x)^{n-2} = n(n-1)(n-2)(1+x)^{n-3}$$

$$f'''(0) = n(n-1)(n-2)(1+0)^{n-3} = n(n-1)(n-2)$$

$$f^{(iv)}(x) = n(n-1)(n-2) \frac{d}{dx}(1+x)^{n-3}$$

$$= n(n-1)(n-2)(n-3)(1+x)^{n-4}$$

$$f^{(iv)}(0) = n(n-1)(n-2)(n-3)(1+0)^{n-4} = n(n-1)(n-2)(n-3)$$

Now by Machlaurin series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \dots$$

$$(1+x)^n = 1 + x \cdot n + \frac{x^2}{2!} n(n-1) + \frac{x^3}{3!} n(n-1)(n-2) + \frac{x^4}{4!} n(n-1)(n-2)(n-3) + \dots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \frac{n(n-1)(n-2)(n-3)}{4!} x^4 + \dots$$

Taylor series:- The series

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \dots + \frac{f^{(n)}(x)}{n!}h^n$$

is known as "Taylor series expansion".

Proof:- Let the power series of $f(x)$ in $(x-a)$

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4$$

$$+ \dots + a_n(x-a)^n + \dots$$

$$\rightarrow f(a) = a_0 + a_1(a-a) + a_2(a-a)^2 + a_3(a-a)^3 + \dots + a_n(a-a)^n$$

$$\rightarrow f(a) = a_0 \rightarrow a_0 = f(a)$$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + \dots + na_n(x-a)^{n-1}$$

$$\rightarrow f'(a) = a_1 \rightarrow a_1 = f'(a)$$

$$f''(x) = 2a_2 + 6a_3(x-a) + 12a_4(x-a)^2 + \dots + n(n-1)a_n(x-a)^{n-2}$$

$$\rightarrow f''(a) = 2a_2 \rightarrow a_2 = \frac{f''(a)}{2!}$$

$$f'''(x) = 6a_3 + 24a_4(x-a) + \dots + n(n-1)(n-2)a_n(x-a)^{n-3}$$

$$\rightarrow f'''(a) = 6a_3 \rightarrow a_3 = \frac{f'''(a)}{3!}$$

$$\text{Similarly } a_n = \frac{f^{(n)}(a)}{n!}$$

Putting values of $a_1, a_2, a_3, \dots, a_n$ in (I)

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

This expansion is the Taylor series for f at $x=a$.

Replace x by $x+h$ and a by x , we get

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \dots + \frac{f^{(n)}(x)}{n!}h^n + \dots$$

Note:- If we put $a=0$ in (II) then

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

which is Machlaurin's series

* The above expansion is also named as "Taylor theorem".

Example 1:- Find the Taylor series expansion of $\ln(1+x)$ at $x=2$

Solution:- Let $f(x) = \ln(1+x)$

$$\rightarrow f(2) = \ln(1+2) = \ln 3$$

$$f'(x) = \frac{1}{1+x} \rightarrow f'(2) = \frac{1}{1+2} = \frac{1}{3}$$

$$f''(x) = \frac{d}{dx} (1+x)^{-1} = (-1)(1+x)^{-2}$$

$$\rightarrow f''(2) = -(1+2)^{-2} = \frac{-1}{(3)^2} = -\frac{1}{9}$$

$$f'''(x) = -\frac{d}{dx} (1+x)^{-2} = -(-2)(1+x)^{-3} = \frac{2}{(1+x)^3}$$

$$\rightarrow f'''(2) = \frac{2}{(1+2)^3} = \frac{2}{27}$$

Now by Taylor series of f at $x=a=2$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

$$\ln(1+x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \dots$$

$$\rightarrow \ln(1+x) = \ln 3 + \frac{1}{3}(x-2) + \frac{-1/9}{2!}(x-2)^2 + \frac{2/27}{3!}(x-2)^3 + \dots$$

$$\ln(1+x) = \ln 3 + \frac{1}{3}(x-2) - \frac{(x-2)^2}{2! \cdot 9} + \frac{(x-2)^3}{3! \cdot 27} - \dots$$

Example 2. Use Taylor series expansion to find the value of $\sin 31^\circ$.

Solution:- Let $f(x) = \sin x$

$$\rightarrow f(x+h) = \sin(x+h)$$

$$f'(x) = \cos x, \quad f''(x) = -\sin x$$

$$f'''(x) = -\cos x, \quad f^{(iv)}(x) = \sin x$$

Now by Taylor series expansion,

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \dots$$

$$\sin(x+h) = \sin x + h \cos x - \frac{h^2}{2!} \sin x - \frac{h^3}{3!} \cos x + \dots$$

$$\text{Take } x = 30^\circ, \quad h = 1^\circ \approx 0.01745$$

$$\rightarrow \sin(30^\circ + 1^\circ) \approx \sin 30^\circ + (0.01745) \cos 30^\circ - \frac{(0.01745)^2}{2!} \sin 30^\circ - \frac{(0.01745)^3}{3!} \cos 30^\circ + \dots$$

$$\sin 31^\circ \approx 0.5 + 0.015112 - 0.000077 - 0.000001$$

$$\approx 0.515112 - 0.000077 + \dots$$

$$\sin 31^\circ \approx 0.515035$$

Example 3. Prove that

$$e^{x+h} = e^x \left\{ 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \dots \right\}$$

Solution:- Let $f(x+h) = e^{x+h}$

$$\rightarrow f(x) = e^x, \quad f'(x) = e^x$$

$$f''(x) = e^x, \quad f'''(x) = e^x$$

Now by Taylor theorem,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

$$\rightarrow e^{x+h} = e^x + he^x + \frac{h^2}{2!}e^x + \frac{h^3}{3!}e^x + \dots$$

$$e^{x+h} = e^x \left\{ 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \dots \right\} \text{ Hence proved.}$$

Exercise 2.8

Q1. Apply the Maclaurin series expansion to prove that:

$$(i) \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Solution:- Let $f(x) = \ln(1+x)$

$$\rightarrow f(0) = \ln(1+0) = \ln 1 = 0$$

$$f'(x) = \frac{1}{1+x}, \quad f'(0) = \frac{1}{1+0} = 1$$

$$f''(x) = \frac{d}{dx} (1+x)^{-1} = -(1+x)^{-2} \rightarrow f''(0) = \frac{-1}{(1+0)^2} = -1$$

$$f'''(x) = -\frac{d}{dx} (1+x)^{-2} = -(-2)(1+x)^{-3} = \frac{2}{(1+x)^3}$$

$$\rightarrow f'''(0) = \frac{2}{(1+0)^3} = 2$$

$$f^{(iv)}(x) = 2 \frac{d}{dx} (1+x)^{-3} = 2(-3)(1+x)^{-4} = \frac{-6}{(1+x)^4}$$

$$\rightarrow f^{(iv)}(0) = \frac{-6}{(1+0)^4} = -6$$

Now by Maclaurin series, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(iv)}(0) + \dots$$

$$\ln(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \begin{matrix} \because 2! = 2 \cdot 1 = 2 \\ 3! = 3 \cdot 2 \cdot 1 = 6 \\ 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \end{matrix}$$

$$(ii) \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$$

Solution:- Let $f(x) = \cos x$

$$\rightarrow f(0) = \cos 0 = 1$$

$$f'(x) = -\sin x, \quad f'(0) = -\sin 0 = 0$$

$$f''(x) = -\cos x, \quad f''(0) = -\cos 0 = -1$$

$$f'''(x) = \sin x, \quad f'''(0) = \sin 0 = 0$$

$$f^{(iv)}(x) = \cos x, \quad f^{(iv)}(0) = \cos 0 = 1$$

$$f^{(v)}(x) = -\sin x, \quad f^{(v)}(0) = -\sin 0 = 0$$

$$f^{(vi)}(x) = -\cos x, \quad f^{(vi)}(0) = -\cos 0 = -1$$

Now by Maclaurin series, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \frac{x^5}{5!} f^{(v)}(0) + \frac{x^6}{6!} f^{(vi)}(0) + \dots$$

$$\cos x = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(-1) + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \text{Hence proved}$$

$$(iii) \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots$$

Solution:- Let $f(x) = (1+x)^{\frac{1}{2}}$

$$\rightarrow f(0) = (1+0)^{\frac{1}{2}} = 1$$

$$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} = \frac{1}{2\sqrt{1+x}}, \quad f'(0) = \frac{1}{2\sqrt{1+0}} = \frac{1}{2}$$

$$f''(x) = \frac{1}{2} \frac{d}{dx} (1+x)^{-\frac{1}{2}} = \frac{1}{2} \left(-\frac{1}{2}\right) (1+x)^{-\frac{3}{2}} = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$$

$$f''(0) = -\frac{1}{4}(1+0)^{-\frac{3}{2}} = -\frac{1}{4}$$

$$f'''(x) = -\frac{1}{4} \frac{d}{dx} (1+x)^{-\frac{3}{2}} = -\frac{1}{4} \left(-\frac{3}{2}\right) (1+x)^{-\frac{5}{2}} = \frac{3}{8}(1+x)^{-\frac{5}{2}}$$

$$\rightarrow f'''(0) = \frac{3}{8(1+0)^{\frac{5}{2}}} = \frac{3}{8}$$

$$f^{(iv)}(x) = \frac{3}{8} \frac{d}{dx} (1+x)^{-\frac{5}{2}} = \frac{3}{8} \left(-\frac{5}{2}\right) (1+x)^{-\frac{7}{2}} = -\frac{15}{16}(1+x)^{-\frac{7}{2}}$$

$$f^{(iv)}(0) = -\frac{15}{16(1+0)^{\frac{7}{2}}} = -\frac{15}{16}$$

Now by Maclaurin series, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \dots$$

$$\sqrt{1+x} = 1 + x \cdot \frac{1}{2} + \frac{x^2}{2!} \left(-\frac{1}{4}\right) + \frac{x^3}{3!} \left(\frac{3}{8}\right) + \frac{x^4}{4!} \left(-\frac{15}{16}\right) + \dots$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} + \dots \text{Hence proved}$$

$$(iv) e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Solution:- Let $f(x) = e^x$

$$\rightarrow f(0) = e^0 = 1$$

$$f'(x) = e^x, \quad f'(0) = e^0 = 1$$

$$f''(x) = e^x, \quad f''(0) = e^0 = 1$$

$$f'''(x) = e^x, \quad f'''(0) = e^0 = 1$$

$$f^{(iv)}(x) = e^x, \quad f^{(iv)}(0) = e^0 = 1$$

Now by Maclaurin series, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \dots$$

$$\rightarrow e^x = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(1) + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \text{Hence proved}$$

$$(v) e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \dots$$

Solution:- Let $f(x) = e^{2x}$

$$\rightarrow f(0) = e^{2(0)} = e^0 = 1$$

$$f'(x) = e^{2x} \frac{d}{dx} (2x) = 2e^{2x}, \quad f'(0) = 2e^{2(0)} = 2$$

$$f''(x) = 4e^{2x}, \quad f''(0) = 4e^{2(0)} = 4$$

$$f'''(x) = 8e^{2x}, \quad f'''(0) = 8e^{2(0)} = 8$$

Now by Maclaurin series, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \dots \text{Hence proved.}$$

Q2. Show that

$$\cos(x+h) = \cos x - h \sin x - \frac{h^2}{2!} \cos x + \frac{h^3}{3!} \sin x + \dots$$

and evaluate $\cos 61^\circ$

Solution:- Let $f(x+h) = \cos(x+h)$

$$\rightarrow f(x) = \cos x, \quad f'(x) = -\sin x$$

$$f''(x) = -\cos x, \quad f'''(x) = \sin x$$

Now by Taylor theorem, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

$$\cos(x+h) = \cos x - h \sin x - \frac{h^2}{2!} \cos x + \frac{h^3}{3!} \sin x + \dots$$

$$\text{Take } x = 60^\circ, h = 1^\circ \approx 0.01745$$

$$\cos(60^\circ + 1^\circ) \approx \cos 60^\circ - (0.01745) \sin 60^\circ - \frac{(0.01745)^2}{2} \cos 60^\circ + \frac{(0.01745)^3}{6} \sin 60^\circ + \dots$$

$$\rightarrow \cos 61^\circ \approx 0.5 - 0.01512 - 0.000076 + 0.000001 + \dots$$

$$\cos 61^\circ \approx 0.484813$$

Q3. Show that

$$2^{x+h} = 2^x \left\{ 1 + (\ln 2)h + \frac{(\ln 2)^2 h^2}{2!} + \frac{(\ln 2)^3 h^3}{3!} + \dots \right\}$$

Solution:- Let $f(x+h) = 2^{x+h}$

$$\rightarrow f(x) = 2^x$$

$$f'(x) = 2^x \ln 2, \quad f''(x) = 2^x (\ln 2)^2$$

$$f'''(x) = 2^x (\ln 2)^3$$

Now by Taylor theorem, we have

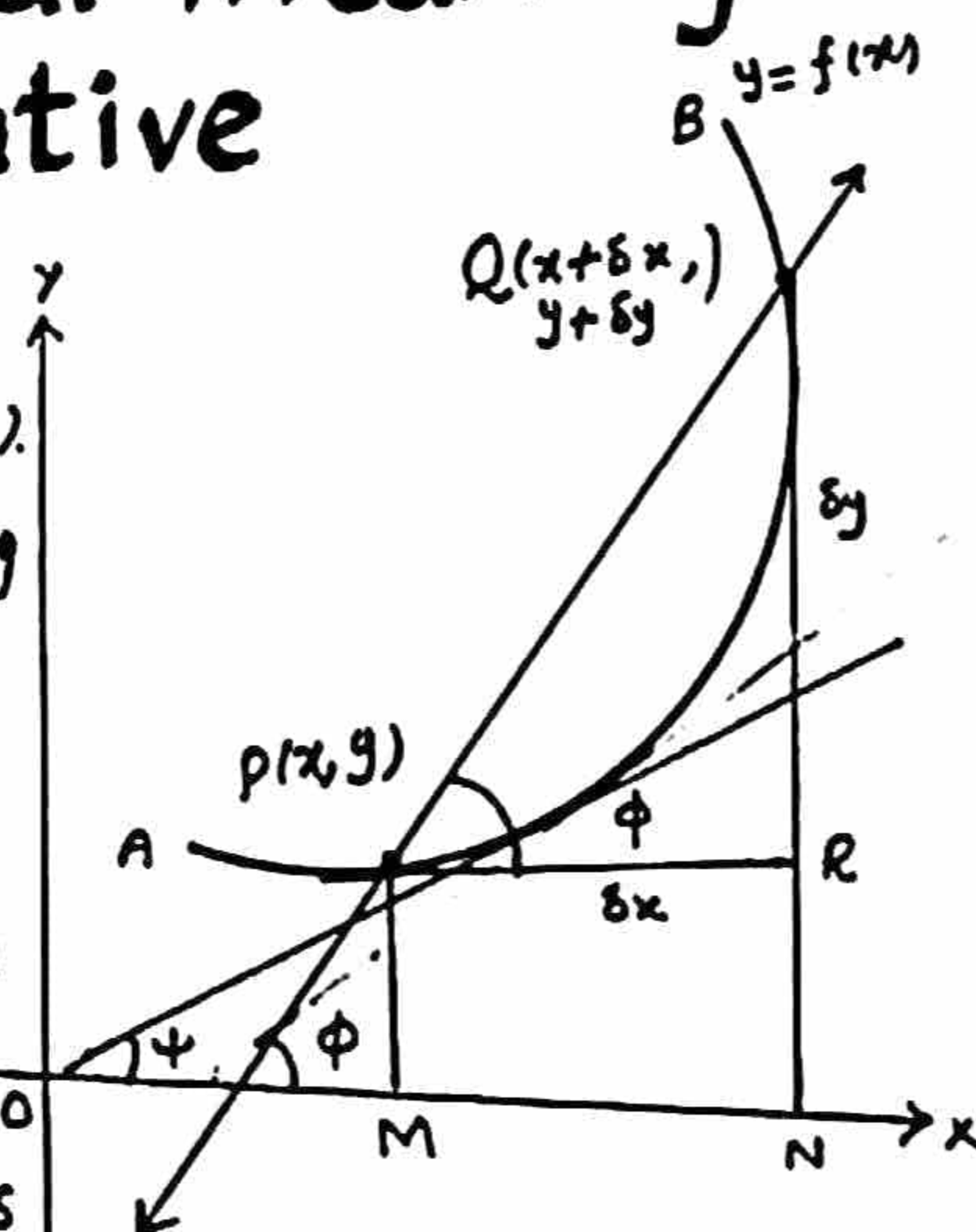
$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

$$2^{x+h} = 2^x + h(2^x(\ln 2)) + \frac{h^2}{2!}(2^x(\ln 2)^2) + \frac{h^3}{3!}(2^x(\ln 2)^3) + \dots$$

$$2^{x+h} = 2^x \left\{ 1 + \ln 2 + \frac{h^2}{2!}(\ln 2)^2 + \frac{h^3}{3!}(\ln 2)^3 + \dots \right\}$$

Geometrical meaning of a derivative

Let us draw a curve $y=f(x)$ of the function $y=f(x)$. Take two neighbouring points $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ on the curve. Draw a tangent line to the curve at pt $P(x, y)$. such that it makes



an angle ψ with x -axis. Also draw a secant line to the curve passing through the pts. $P(x, y)$ and $Q(x+\delta x, y+\delta y)$. The secant line makes angle ϕ with x -axis. Draw $\perp$ rs PM and QN from P and Q on x -axis. Also draw $\perp$ r PR on QN .

$$\text{In } \triangle PQR, \tan \phi = \frac{|QR|}{|PR|}, \quad |QR| = \delta y, \quad |PR| = \delta x$$

$$\Rightarrow \tan \phi = \frac{\delta y}{\delta x}$$

when $Q \rightarrow P$, then $\delta x \rightarrow 0$, $\delta y \rightarrow 0$. $\phi \rightarrow \psi$. It means secant line becomes tangent line. Now

$$\lim_{\phi \rightarrow \psi} \tan \phi = \lim_{\phi \rightarrow \psi} \frac{\delta y}{\delta x} \Rightarrow \tan \psi = \frac{dy}{dx}$$

But $\tan \psi$ is slope of tangent line. Thus Geometrically derivative of y at x represents the slope of tangent line at pt $P(x, y)$.

Example 1. Discuss the tangent line to the graph of the function $|x|$ at $x=0$.

Solution:- Let $f(x) = |x|$

$$\Rightarrow f(x+\delta x) = |x+\delta x|$$

We know that

$$f'(x) = \lim_{\delta x \rightarrow 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$f'(x) = \lim_{\delta x \rightarrow 0} \frac{|x+\delta x| - |x|}{\delta x}$$

$$\text{at } x=0, f'(0) = \lim_{\delta x \rightarrow 0} \frac{|0+\delta x| - |0|}{\delta x}$$

$$\Rightarrow f'(0) = \lim_{\delta x \rightarrow 0} \frac{|\delta x|}{\delta x}$$

$$\text{L.H.L.:- } \lim_{\delta x \rightarrow 0^-} \frac{|\delta x|}{\delta x} = \lim_{\delta x \rightarrow 0^-} \frac{-\delta x}{\delta x} = -1$$

$$\text{R.H.L.:- } \lim_{\delta x \rightarrow 0^+} \frac{|\delta x|}{\delta x} = \lim_{\delta x \rightarrow 0^+} \frac{\delta x}{\delta x} = 1$$

$$\therefore \text{L.H.L.} \neq \text{R.H.L.}$$

$$\Rightarrow \lim_{\delta x \rightarrow 0} \frac{|\delta x|}{\delta x} \text{ does not exist.}$$

Thus there is no tangent line to the graph of f at $x=0$

Example 2. Find the equations of the tangents to the curve $x^2 - y^2 - 6y = 0$ at the pt. whose abscissa is 4.

Solution:- $\because x^2 - y^2 - 6y = 0 \rightarrow (i)$ and abscissa $= x=4$ put in (i)

$$(ii) \rightarrow (4)^2 - y^2 - 6y = 0 \rightarrow 16 - y^2 - 6y = 0$$

$$\rightarrow y^2 + 6y - 16 = 0$$

$$\rightarrow y^2 + 8y - 2y - 16 = 0$$

$$\rightarrow y(y+8) - 2(y+8) = 0 \rightarrow (y+8)(y-2) = 0$$

$$y+8=0, y-2=0 \rightarrow y=-8, y=2$$

Thus the points are $(4, 2)$ and $(4, -8)$

Now diff (i) w.r.t 'x'

$$\frac{d}{dx}(x^2 - y^2 - 6y) = 0$$

$$\rightarrow 2x - 2y \frac{dy}{dx} - 6 \frac{dy}{dx} = 0$$

$$\rightarrow x - y \frac{dy}{dx} - 3 \frac{dy}{dx} = 0 \rightarrow x = y \frac{dy}{dx} + 3 \frac{dy}{dx}$$

$$\rightarrow x = (y+3) \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{y+3}$$

$$\text{Now } \left. \frac{dy}{dx} \right|_{(4,2)} = \frac{4}{2+3} = \frac{4}{5} = \text{slope of tangent line}$$

$$\text{Also } \left. \frac{dy}{dx} \right|_{(4,-8)} = \frac{4}{-8+3} = \frac{-4}{5} = \text{slope of tangent line}$$

Now Eq. of tangent line having slope $\frac{4}{5}$ and passing through pt $(4, 2)$

$$y - y_1 = m(x - x_1) \quad \because m = \frac{4}{5}$$

$$\rightarrow y - 2 = \frac{4}{5}(x - 4) \rightarrow 5y - 10 = 4x - 16$$

$$\rightarrow 4x - 5y - 16 + 10 = 0 \quad 4x - 5y - 6 = 0 \quad (\text{req. tangent line})$$

Also Eq. of tangent line having slope $-\frac{4}{5}$ and passing through pt $(4, -8)$

$$y - y_1 = m(x - x_1) \quad \because m = -\frac{4}{5}$$

$$\rightarrow y + 8 = -\frac{4}{5}(x - 4)$$

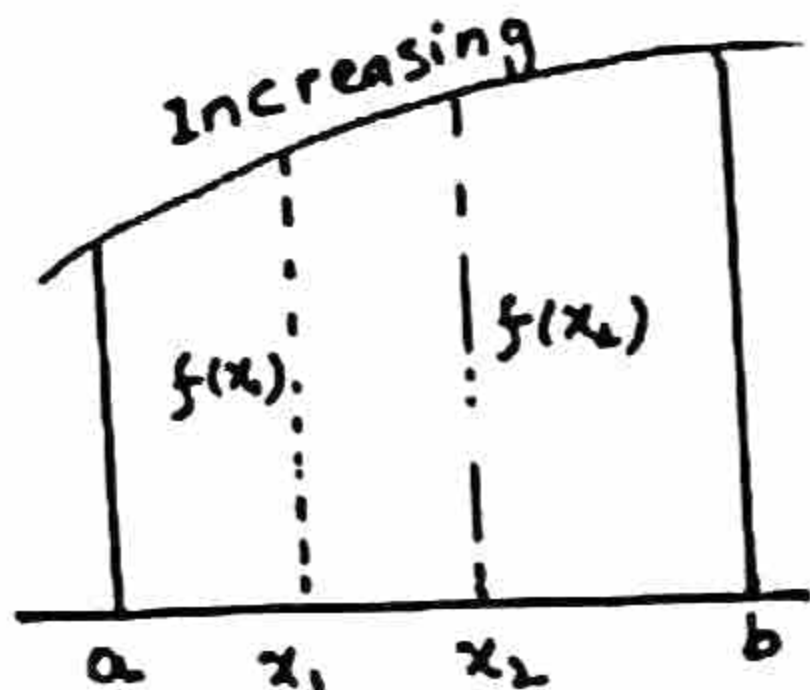
$$\rightarrow 5y + 40 = -4x + 16 \rightarrow 4x + 5y + 40 - 16 = 0$$

$$\rightarrow 4x + 5y + 24 = 0 \quad (\text{req. tangent line})$$

Increasing and decreasing functions:-

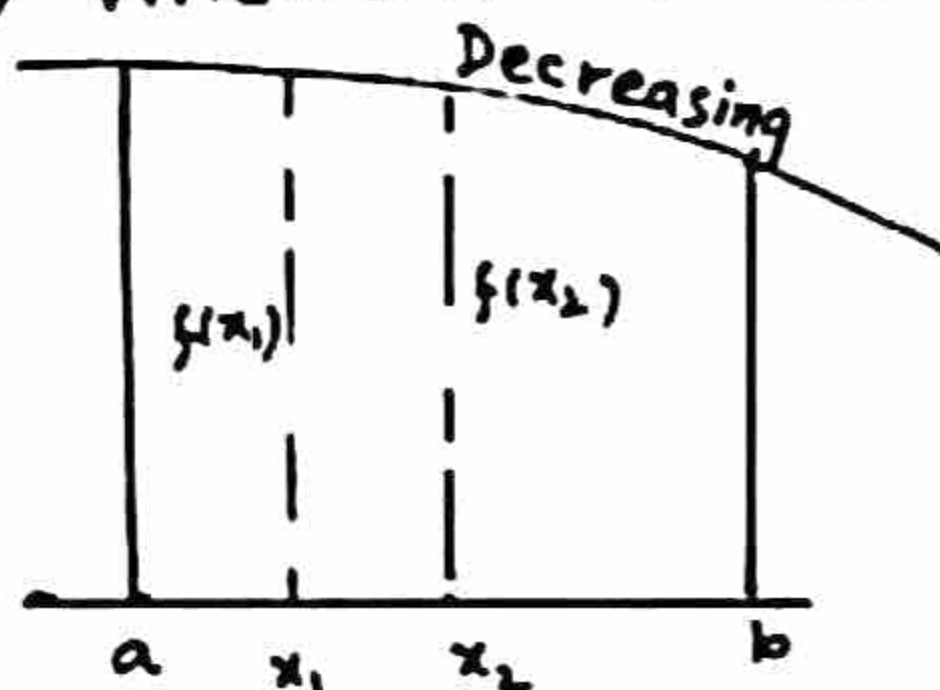
Increasing function:- A function f is said to be increasing function on an interval (a, b) if for every $x_1, x_2 \in (a, b)$ then

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2$$



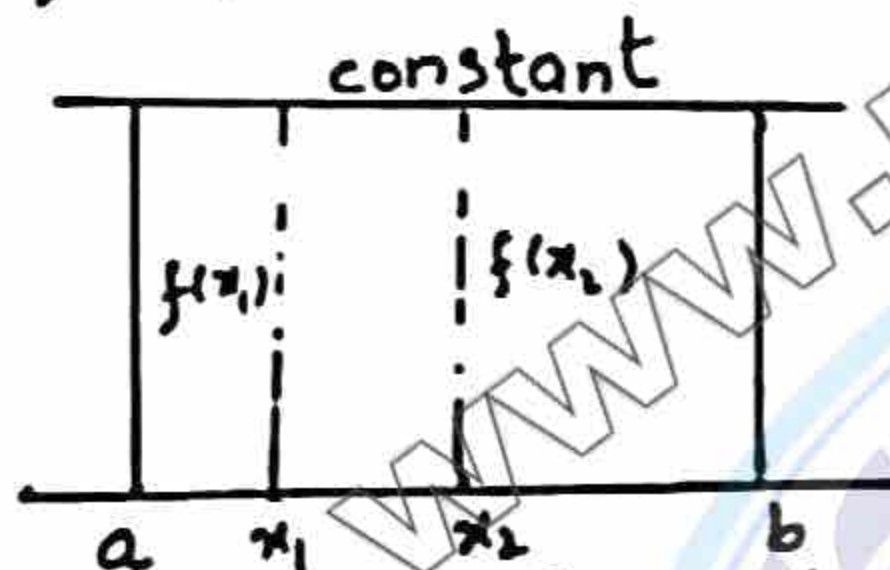
Decreasing function:- A function f is said to be decreasing function on an interval (a, b) if for every $x_1, x_2 \in (a, b)$ then

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2$$



Constant function:- A function f is said to be constant function on an interval (a, b) if for every $x_1, x_2 \in (a, b)$ then

$$f(x_1) = f(x_2) \text{ whenever } x_1 < x_2$$



Alternative definitions

Increasing function:-

A differentiable function f is said to be increasing function if $f'(x) > 0 \forall x \in (a, b)$

Decreasing function:-

A differentiable function f is said to be decreasing function if $f'(x) < 0 \forall x \in (a, b)$

Constant function:-

A differentiable function f is said to be constant function if $f'(x) = 0 \forall x \in (a, b)$

Example 1. Determine the values of x for which $f(x) = x^2 + 2x - 3$ is (i) increasing (ii) decreasing (iii) find the point where the function is neither increasing nor decreasing.

Solution:- $f(x) = x^2 + 2x - 3$

$$\Rightarrow f'(x) = 2x + 2 = 2(x+1)$$

(i) For increasing function;

$$f'(x) > 0$$

$$\text{So } 2(x+1) > 0$$

$$\Rightarrow x+1 > 0 \Rightarrow x > -1$$

Hence f is increasing function in the interval $(-1, +\infty)$

(ii) For decreasing function;

$$f'(x) < 0$$

$$\Rightarrow 2(x+1) < 0 \Rightarrow x+1 < 0$$

$$\Rightarrow x < -1$$

Hence f is decreasing function in the interval $(-\infty, -1)$

(iii) For constant function;
(Neither increasing nor decreasing)

$$f'(x) = 0$$

$$\Rightarrow 2(x+1) = 0 \Rightarrow 2 \neq 0, x+1 = 0$$

$$\Rightarrow x = -1$$

Put $x = -1$ in $f(x) = x^2 + 2x - 3$

$$\Rightarrow f(-1) = (-1)^2 + 2(-1) - 3 = 1 - 2 - 3 = -4$$

So $(-1, -4)$ is the pt. where $f(x)$ is neither increasing nor decreasing.

Stationary point:- Any point where f is neither increasing nor decreasing is called a stationary point, provided that $f'(x) = 0$ at that point.

Example 2. Determine the intervals in which f is increasing or it is decreasing it. $f(x) = x^3 - 6x^2 + 9x$

Solution:- $f(x) = x^3 - 6x^2 + 9x$

$$\Rightarrow f'(x) = 3x^2 - 12x + 9$$

$$= 3(x^2 - 4x + 3)$$

$$= 3(x^2 - 3x - x + 3)$$

$$= 3[x(x-3) - 1(x-3)]$$

$$f'(x) = 3(x-3)(x-1)$$

For increasing function;

$$f'(x) > 0$$

$$\Rightarrow 3(x-3)(x-1) > 0$$

$$\Rightarrow x-3 > 0, x-1 > 0$$

$$\Rightarrow x > 3, x > 1$$

Hence f is increasing function in the interval $(1, +\infty)$ and $(3, +\infty)$

For decreasing functions;

$$f'(x) < 0$$

$$\Rightarrow 3(x-3)(x-1) < 0$$

$$\Rightarrow x-3 < 0, x-1 > 0$$

$$\Rightarrow x < 3, x > 1$$

Hence f is decreasing function in the interval $(1, \infty)$ and $(-\infty, 3)$

Relative Maxima:- A function $f(x)$ has relative maxima $f(c)$ at $x=c \in (a,b)$ if
 (i) $f'(c-\delta x) > 0$ (ii) $f'(c) = 0$ (iii) $f'(c+\delta x) < 0$
 where δx is small +ve number.

Relative Minima:- A function $f(x)$ has relative minima $f(c)$ at $x=c \in (a,b)$ if
 (i) $f'(c-\delta x) < 0$ (ii) $f'(c) = 0$ (iii) $f'(c+\delta x) > 0$ where δx is small +ve no.

Alternative definitions

Relative Maxima:- A function $f(x)$ has relative maxima $f(c)$ if $f'(x)$ changes sign from +ve to -ve as x increases through c .

Relative minima:- A function $f(x)$ has relative minima $f(c)$ if $f'(x)$ changes sign from -ve to +ve as x changes through c .

Relative Extrema:- Both relative maxima and relative minima are called in general "relative extrema".

Critical values and critical points:-

If $c \in D_f$ and $f'(c) = 0$ or $f'(c)$ does not exist, then the number c is called a critical value for f while the point $(c, f(c))$ on the graph of f is named as a critical point.

Important note:- There are functions which have extrema (maxima or minima) at the points where their derivatives do not exist. For example,
 $f(x) = |x|$ and $\phi(x) = \begin{cases} 2-x & x > 0 \\ x+2 & x \leq 0 \end{cases}$
 do not exist at $(0,0)$ and $(0,2)$ resp. But f has minima at $(0,0)$ and ϕ has relative maxima at $(0,2)$.

"First derivative rule"

Let f be differentiable in neighbourhood of c where $f'(c) = 0$

(1) A function $f(x)$ has relative maxima at $x = c \in (a,b)$ if $f'(c-\delta x) > 0$, $f'(c) = 0$ and $f'(c+\delta x) < 0$

(2) A function $f(x)$ has relative minima at $x = c \in (a,b)$ if $f'(c-\delta x) < 0$, $f'(c) = 0$ and $f'(c+\delta x) > 0$

"Second derivative rule"

Let f be differential function in neighbourhood of c where $f'(c) = 0$. then

- (1) f has relative maxima at c if $f''(c) < 0$
- (2) f has relative minima at c if $f''(c) > 0$

Turning point:- A stationary point is called a turning point if it is either a maximum point or a minimum point.

Point of inflexion:- A point at which the function has neither maximum nor minimum value is called point of inflexion.

Procedure for finding Maxima and Minima of a function

Let $y = f(x)$ be given function.

Step I:- Find $\frac{dy}{dx}$ or $f'(x)$

Step II:- Put $\frac{dy}{dx} = 0$ or $f'(x) = 0$

Step III:- Find $\frac{d^2y}{dx^2}$ or $f''(x)$

Step IV:- check the sign of $f''(x)$ at the roots.

* If $f''(x) > 0$ then $f(x)$ has minimum value

* If $f''(x) < 0$ then $f(x)$ has maximum value.

* If $f''(x) = 0$ then no information about the function then use "first derivative rule."

Example 1. Examine the function $f(x) = x^3 - 6x^2 + 9x$ for extreme values.

Solution:- $f(x) = x^3 - 6x^2 + 9x$

$$\rightarrow f'(x) = 3x^2 - 12x + 9$$

$$\text{Put } f'(x) = 0 \rightarrow 3x^2 - 12x + 9 = 0$$

$$\rightarrow 3(x^2 - 4x + 3) = 0$$

$$\rightarrow x^2 - 4x + 3 = 0 \rightarrow x^2 - 3x - x + 3 = 0$$

$$\rightarrow x(x-3) - 1(x-3) = 0 \rightarrow (x-3)(x-1) = 0$$

$$\rightarrow x-3 = 0, \quad x-1 = 0$$

$$\rightarrow x=3, \quad x=1$$

$$\text{Now } f''(x) = 6x - 12 = 6(x-2)$$

$$\text{At } x=1:- f''(1) = 6(1-2) = -6 < 0$$

$\rightarrow f$ has relative maxima at $x=1$

$$\text{and } f(1) = (1)^3 - 6(1)^2 + 9(1) = 1 - 6 + 9 = 4$$

$$\text{At } x=3:- f''(3) = 6(3-2) = 6 > 0$$

→ f has relative minima at $x=3$
and $f(3) = (3)^3 - 6(3)^2 + 9(3) = 27 - 54 + 27 = 0$
→ $f(3) = 0$

Example 2. Examine the function defined as $f(x) = 1 + x^3$ for extreme values.

Solution:- $f(x) = 1 + x^3$

$$\rightarrow f'(x) = 3x^2 \quad (I) \rightarrow f''(x) = 6x \quad (II)$$

$$\text{Put } f'(x) = 0 \rightarrow 3x^2 = 0 \rightarrow x = 0$$

$$\text{Put } x = 0 \text{ in (II)} \rightarrow f''(0) = 6(0) = 0$$

The "second derivative" does not help in finding the extreme values. so we use "first derivative"

$$\text{Put } x = 0 - \delta x \text{ and } x = 0 + \delta x \text{ in (I)}$$

$$f'(0 - \delta x) = 3(0 - \delta x)^2 = 3(\delta x)^2 > 0$$

$$f'(0 + \delta x) = 3(0 + \delta x)^2 = 3(\delta x)^2 > 0$$

→ first derivative does not change sign at $x > 0$ or $x < 0$ so put $x = 0$ in $f(x) = 1 + x^3$

$$\rightarrow f(0) = 1 + (0)^3 = 1$$

Thus $(0, 1)$ is the point of inflexion.

Example 3. Discuss the function defined as $f(x) = \sin x + \frac{1}{2\sqrt{2}} \cos 2x$ for extreme values in the interval $(0, 2\pi)$.

Solution:- $f(x) = \sin x + \frac{1}{2\sqrt{2}} \cos 2x$

$$\rightarrow f'(x) = \cos x + \frac{1}{2\sqrt{2}} (-\sin 2x) \quad (2)$$

$$f'(x) = \cos x - \frac{1}{\sqrt{2}} \sin 2x$$

$$f'(x) = \cos x - \frac{1}{\sqrt{2}} (2 \sin x \cos x)$$

$$f'(x) = \cos x - \frac{\sqrt{2} \cancel{\sqrt{2}}}{\sqrt{2}} \sin x \cos x \quad (\because \sin 2\theta = 2 \sin \theta \cos \theta)$$

$$f'(x) = \cos x - \sqrt{2} \sin x \cos x = \cos x (1 - \sqrt{2} \sin x)$$

$$\text{Put } f'(x) = 0 \Rightarrow \cos x (1 - \sqrt{2} \sin x) = 0$$

$$\rightarrow \cos x = 0, \quad \sin x = \frac{1}{\sqrt{2}}$$

$$\rightarrow x = \cos^{-1}(0) = \frac{\pi}{2}, \frac{3\pi}{2}$$

($\because \sin x$ is +ve in I and II quadrant with reference angle $x = \frac{\pi}{4}$)

$$\text{so } x = \frac{\pi}{4} \text{ (For I quadrant)}$$

$$x = \pi - \frac{\pi}{4} = \frac{4\pi - \pi}{4} = \frac{3\pi}{4}$$

$$\text{so } x = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\text{Now } f''(x) = -\sin x - \frac{1}{\sqrt{2}} (\cos 2x) \quad (2)$$

$$f''(x) = -\sin x - \sqrt{2} \cos 2x$$

$$\text{Now } f''\left(\frac{\pi}{2}\right) = -\sin \frac{\pi}{2} - \sqrt{2} \cos 2\left(\frac{\pi}{2}\right)$$

$$= -\sin \frac{\pi}{2} - \sqrt{2} \cos \pi = -1 - \sqrt{2}(-1)$$

$$f''\left(\frac{\pi}{2}\right) = \sqrt{2} - 1 > 0$$

so $f(x)$ has minimum value at $x = \frac{\pi}{2}$

$$f''\left(\frac{3\pi}{2}\right) = -\sin \frac{3\pi}{2} - \sqrt{2} \cos 2\left(\frac{3\pi}{2}\right)$$

$$= -\sin \frac{3\pi}{2} - \sqrt{2} \cos (3\pi)$$

$$= -(-1) - \sqrt{2}(-1) = \sqrt{2} + 1 > 0$$

so $f(x)$ has minimum value at $x = \frac{3\pi}{2}$

$$f''\left(\frac{\pi}{4}\right) = -\sin \frac{\pi}{4} - \sqrt{2} \cos 2\left(\frac{\pi}{4}\right)$$

$$f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \sqrt{2} \cos \left(\frac{\pi}{2}\right) = -\frac{1}{\sqrt{2}} < 0$$

so $f(x)$ has maximum value at $x = \frac{\pi}{4}$

$$f''\left(\frac{3\pi}{4}\right) = -\sin \frac{3\pi}{4} - \sqrt{2} \cos 2\left(\frac{3\pi}{4}\right)$$

$$f''\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \sqrt{2} \cos \frac{3\pi}{2} = -\frac{1}{\sqrt{2}} < 0 \quad \because \cos \frac{3\pi}{2} = 0$$

so $f(x)$ has maximum value at $x = \frac{3\pi}{4}$

Exercise 2.9

Q1. Determine the intervals in which f is increasing or decreasing for the domain mentioned in each case:

(i) $f(x) = \sin x$; $x \in (-\pi, \pi)$

Solution:- $f(x) = \sin x$

$$\rightarrow f'(x) = \cos x$$

For Increasing;

$$f'(x) > 0$$

$$\rightarrow \cos x > 0$$

so required intervals

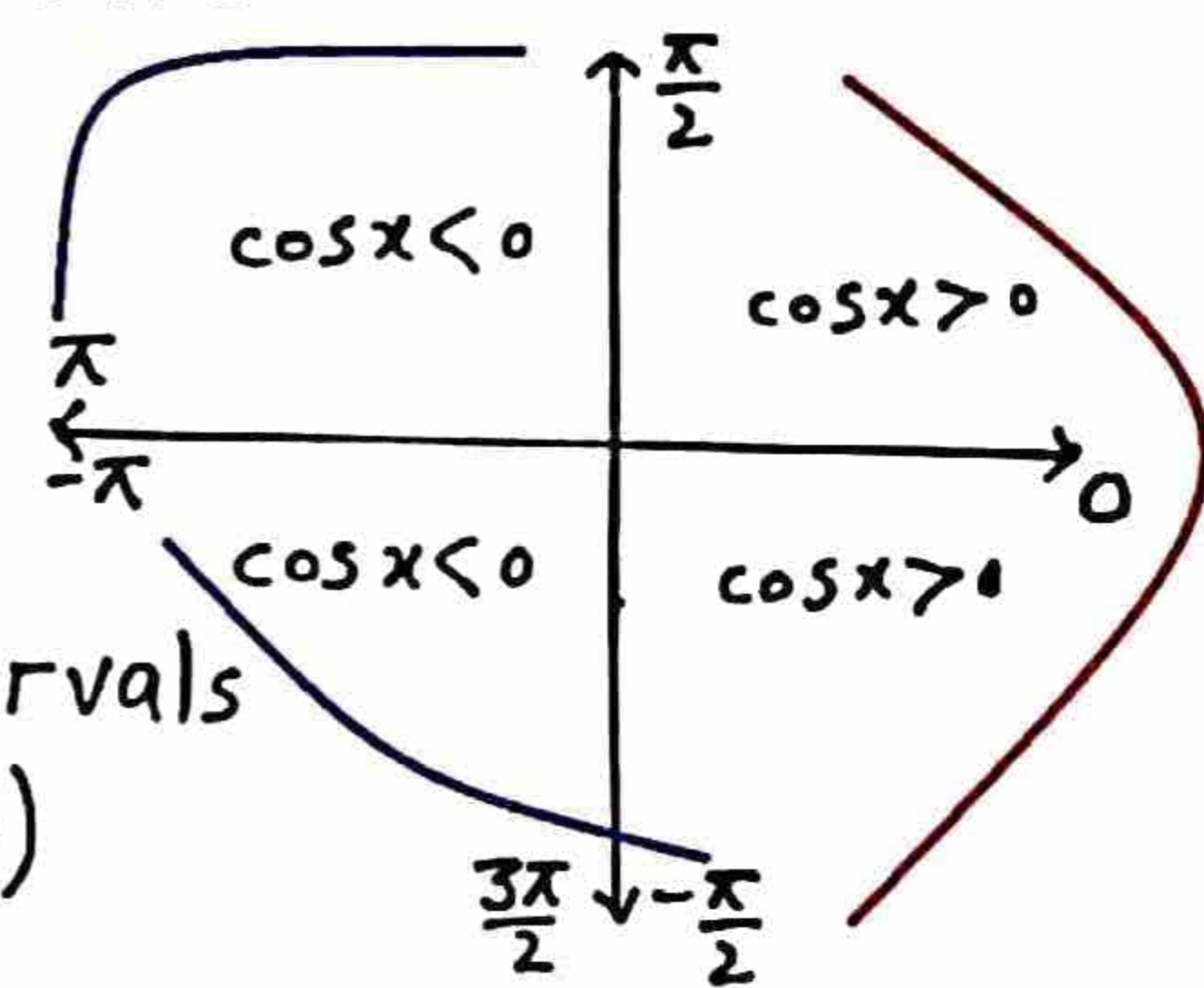
$$\text{are } \left(0, \frac{\pi}{2}\right) \& \left(-\frac{\pi}{2}, 0\right)$$

$$\text{or } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

For Decreasing;

$$f'(x) < 0 \quad \text{so required intervals}$$

$$\rightarrow \cos x < 0 \quad \text{are } \left(\frac{\pi}{2}, \pi\right) \& \left(-\pi, -\frac{\pi}{2}\right).$$



(ii) $f(x) = \cos x$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Solution:- $f(x) = \cos x$

$$f'(x) = -\sin x$$

For Increasing;

$$f'(x) > 0 \Rightarrow -\sin x > 0$$

$$\rightarrow \sin x < 0$$

so req. Interval

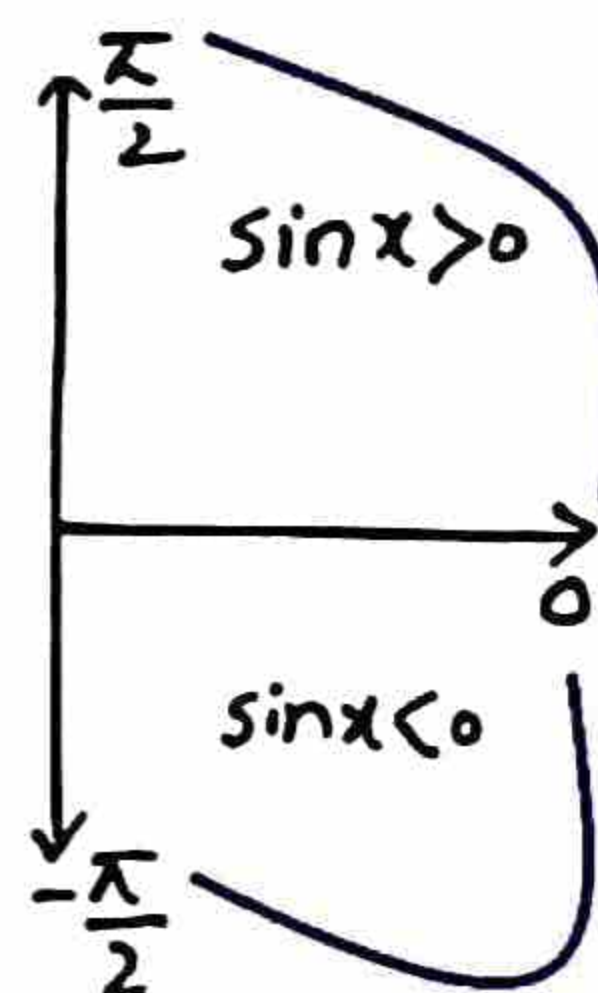
$$\text{is } \left(-\frac{\pi}{2}, 0\right)$$

For Decreasing; $f'(x) < 0$

$$\rightarrow -\sin x < 0 \Rightarrow \sin x > 0$$

so req. Interval is

$$\left(0, \frac{\pi}{2}\right)$$



(iii) $f(x) = 4 - x^2$; $x \in (-2, 2)$

Solution:- $f(x) = 4 - x^2 \rightarrow f'(x) = -2x$

For Increasing;

$f'(x) > 0$
 $\rightarrow -2x > 0 \rightarrow 2x < 0$

$\rightarrow x < 0$

so req. Interval is $(-2, 0)$

For Decreasing;

$f'(x) < 0 \rightarrow -2x < 0 \rightarrow 2x > 0$

$\rightarrow x > 0$ so req. Interval is $(0, 2)$

(iv) $f(x) = x^2 + 3x + 2$; $x \in (-4, 1)$

Solution:- $f(x) = x^2 + 3x + 2$

$f'(x) = 2x + 3$

For Increasing;

$f'(x) > 0 \rightarrow 2x + 3 > 0 \rightarrow x > -\frac{3}{2}$

so req. Interval is $(-\frac{3}{2}, 1)$

For Decreasing;

$f'(x) < 0 \rightarrow 2x + 3 < 0 \rightarrow x < -\frac{3}{2}$

so req. Interval is $(-4, -\frac{3}{2})$.

Q2. Find the extreme values for the following functions defined as:

(i) $f(x) = 1 - x^3$

Solution:- $f(x) = 1 - x^3$

$\rightarrow f'(x) = -3x^2 \rightarrow f''(x) = -6x$ (I)

Put $f'(x) = 0 \rightarrow -3x^2 = 0 \rightarrow x = 0$

Put $x = 0$ in (I) $\rightarrow f''(0) = -6(0) = 0$

The 2nd derivative does not help in finding the extreme values. so we use "first derivative".

Put $x = 0 - \delta x$ and $x = 0 + \delta x$ in (I)

$f'(0 - \delta x) = -3(0 - \delta x)^2 = -3(\delta x)^2 < 0$

$f'(0 + \delta x) = -3(0 + \delta x)^2 = -3(\delta x)^2 < 0$

$\rightarrow$ 1st derivative does not change sign at $x < 0$ or $x > 0$. so put $x = 0$ in $f(x) = 1 - x^3$

$\rightarrow f(0) = 1 - (0)^3 = 1$

Thus $(0, 1)$ is the point of inflexion.

(ii) $f(x) = x^2 - x - 2$

Solution:- $f(x) = x^2 - x - 2$ (I)

$\rightarrow f'(x) = 2x - 1 \rightarrow f''(x) = 2$ (II)

Put $f'(x) = 0 \rightarrow 2x - 1 = 0$

$\rightarrow x = \frac{1}{2}$ put in (II)

$\rightarrow f''(\frac{1}{2}) = 2 > 0$, so $f(x)$ has minimum value at $x = \frac{1}{2}$

and $f(\frac{1}{2}) = (\frac{1}{2})^2 - \frac{1}{2} - 2 = \frac{1}{4} - \frac{1}{2} - 2 = -\frac{9}{4}$

(iii) $f(x) = 5x^2 - 6x + 2$

Solution:- $f(x) = 5x^2 - 6x + 2$

$\rightarrow f'(x) = 10x - 6 \rightarrow f''(x) = 10$ (I)

Put $f'(x) = 0 \rightarrow 10x - 6 = 0 \rightarrow x = \frac{6}{10} = \frac{3}{5}$

Put $x = \frac{3}{5}$ in (I), so $f''(\frac{3}{5}) = 10 > 0$

so $f(x)$ has minimum value at $x = \frac{3}{5}$.

and $f(\frac{3}{5}) = 5(\frac{3}{5})^2 - 6(\frac{3}{5}) + 2 = 5(\frac{9}{25}) - \frac{18}{5} + 2$
 $= \frac{9}{5} - \frac{18}{5} + 2 = \frac{9 - 18 + 10}{5} = \frac{1}{5}$

(iv) $f(x) = 3x^2$

Solution:- $f(x) = 3x^2$

$\rightarrow f'(x) = 6x \rightarrow f''(x) = 6$ (I)

Put $f'(x) = 0 \rightarrow 6x = 0 \rightarrow x = 0$ put in (I)

$\rightarrow f''(0) = 6 > 0$ so $f(x)$ has minimum value at $x = 0$. and

$f(0) = 3(0)^2 = 0$

(v) $f(x) = 3x^2 - 4x + 5$

Solution:- $f(x) = 3x^2 - 4x + 5$

$\rightarrow f'(x) = 6x - 4 \rightarrow f''(x) = 6$ (I)

Put $f'(x) = 0 \rightarrow 6x - 4 = 0 \rightarrow x = \frac{4}{6} = \frac{2}{3}$

Put $x = \frac{2}{3}$ in (I), $f''(\frac{2}{3}) = 6 > 0$

$\rightarrow f(x)$ has minimum value at $x = \frac{2}{3}$

and $f(\frac{2}{3}) = 3(\frac{2}{3})^2 - 4(\frac{2}{3}) + 5 = 3(\frac{4}{9}) - \frac{8}{3} + 5$

$= \frac{4}{3} - \frac{8}{3} + 5 = \frac{4 - 8 + 15}{3} = \frac{11}{3}$

(vi) $f(x) = 2x^3 - 2x^2 - 36x + 3$

Solution:- $f(x) = 2x^3 - 2x^2 - 36x + 3$

$\rightarrow f'(x) = 6x^2 - 4x - 36 \rightarrow f''(x) = 12x - 4$ (I)

Put $f'(x) = 0$ so

$6x^2 - 4x - 36 = 0$

$\rightarrow 3x^2 - 2x - 18 = 0$

$\rightarrow x = \frac{2 \pm \sqrt{(-2)^2 - 4(3)(-18)}}{2(3)}$ using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{2 \pm \sqrt{4 + 4(54)}}{6} = \frac{2 \pm 2\sqrt{1+54}}{6} = \frac{1 \pm \sqrt{55}}{3}$

$\rightarrow x = \frac{1 \pm \sqrt{55}}{3}$

$\rightarrow x = \frac{1 + \sqrt{55}}{3}$; $x = \frac{1 - \sqrt{55}}{3}$

Put $x = \frac{1 + \sqrt{55}}{3}$ in (I)

$f''(\frac{1 + \sqrt{55}}{3}) = 12(\frac{1 + \sqrt{55}}{3}) - 4 = 4 + 4\sqrt{55} - 4 = 4\sqrt{55}$

$\rightarrow f''(\frac{1 + \sqrt{55}}{3}) = 4\sqrt{55} > 0$

so $f(x)$ has minimum value at $x = \frac{1+\sqrt{55}}{3}$

and

$$f\left(\frac{1+\sqrt{55}}{3}\right) = 2\left(\frac{1+\sqrt{55}}{3}\right)^3 - 2\left(\frac{1+\sqrt{55}}{3}\right)^2 - \frac{11}{3}\left(\frac{1+\sqrt{55}}{3}\right) + 3$$

$$= 2\left(\frac{1 + (\sqrt{55})^3 + 3\sqrt{55} + 3(\sqrt{55})^2}{27}\right) - 2\left(\frac{1 + 55 + 2\sqrt{55}}{9}\right) - \frac{11(1+\sqrt{55})}{9} + 3$$

$$= \frac{2}{27}(1 + 3\sqrt{55} + 55\sqrt{55} + 165) - \frac{2}{9}(56 + 2\sqrt{55}) - \frac{11}{9} - \frac{11\sqrt{55}}{9} + 3$$

$$= \frac{2}{27}(166 + 58\sqrt{55}) - \frac{2}{9}(56 + 2\sqrt{55}) - \frac{11}{9} - \frac{11\sqrt{55}}{9} + 3$$

$$= \frac{332}{27} + \frac{116\sqrt{55}}{27} - \frac{112}{9} - \frac{4\sqrt{55}}{9} - \frac{11}{9} - \frac{11\sqrt{55}}{9} + 3$$

$$= \frac{332}{27} - \frac{112}{9} - \frac{11}{9} + \left(\frac{116}{27} - \frac{4}{9} - \frac{11}{9}\right)\sqrt{55} + 3$$

$$= \frac{332 - 336 - 243}{27} + \left(\frac{116 - 12 - 324}{27}\right)\sqrt{55} + 3$$

$$= -\frac{247}{27} + \frac{220}{27}\sqrt{55} = -\frac{1}{27}(247 + 220\sqrt{55})$$

Now for $x = \frac{1-\sqrt{55}}{3}$ put in (I)

$$f''\left(\frac{1-\sqrt{55}}{3}\right) = \frac{4}{3}\left(\frac{1-\sqrt{55}}{3}\right) - 4 = \frac{4}{9} - \frac{4\sqrt{55}}{3} < 0$$

$$\rightarrow f''\left(\frac{1-\sqrt{55}}{3}\right) = -\frac{4\sqrt{55}}{3} < 0$$

so $f(x)$ has maximum value at $x = \frac{1-\sqrt{55}}{3}$

$$\text{and since } f\left(\frac{1+\sqrt{55}}{3}\right) = -\frac{1}{27}(247 + 220\sqrt{55})$$

therefore by replacing $\sqrt{55}$ by $-\sqrt{55}$ we have

$$f\left(\frac{1-\sqrt{55}}{3}\right) = -\frac{1}{27}(247 - 220\sqrt{55})$$

$$(vii) f(x) = x^4 - 4x^2$$

$$\text{Solution: } f(x) = x^4 - 4x^2$$

$$\rightarrow f'(x) = 4x^3 - 8x \rightarrow f''(x) = 12x^2 - 8 \quad (I)$$

$$\text{Put } f'(x) = 0 \text{ so } 4x^3 - 8x = 0$$

$$\rightarrow 4x(x^2 - 2) = 0$$

$$\rightarrow 4x = 0, \quad x^2 - 2 = 0$$

$$\rightarrow x = 0, \quad x = \pm\sqrt{2}$$

Thus $x = 0, x = \sqrt{2}, x = -\sqrt{2}$

Put $x = 0$ in (I)

$$\rightarrow f''(0) = 12(0)^2 - 8 = -8 < 0$$

so $f(x)$ has maximum value at $x = 0$

$$\text{and } f(0) = (0)^4 - 4(0)^2 = 0$$

Put $x = \sqrt{2}$ in (I)

$$\rightarrow f''(\sqrt{2}) = 12(\sqrt{2})^2 - 8 = 12(2) - 8 = 24 - 8 = 16 > 0$$

so $f(x)$ has minimum value at $x = \sqrt{2}$

$$\text{and } f(\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2$$

$$= (2)^2 - 4(2) = 4 - 8 = -4 < 0$$

Put $x = -\sqrt{2}$ in (I)

$$f''(-\sqrt{2}) = 12(-\sqrt{2})^2 - 8 = 12(2) - 8 = 24 - 8 = 16 > 0$$

so $f(x)$ has minimum value at $x = -\sqrt{2}$

$$\text{and } f(-\sqrt{2}) = (-\sqrt{2})^4 - 4(-\sqrt{2})^2 = (2)^2 - 4(2) = 4 - 8 = -4$$

$$(viii) f(x) = (x-2)^2(x-1)$$

$$\text{Solution: } f(x) = (x-2)^2(x-1)$$

$$\rightarrow f'(x) = \frac{d}{dx} \{(x-2)^2(x-1)\}$$

$$= (x-2)^2 \frac{d}{dx}(x-1) + (x-1) \frac{d}{dx}(x-2)^2$$

$$= (x-2)^2(1) + (x-1) \cdot 2(x-2) \frac{d}{dx}(x-2)$$

$$= (x-2)^2 + 2(x-1)(x-2)(1)$$

$$f'(x) = (x-2)^2 + 2(x-1)(x-2)$$

$$= (x-2) \{x-2 + 2x-2\}$$

$$f'(x) = (x-2)(3x-4)$$

$$\rightarrow f''(x) = \frac{d}{dx} \{(x-2)(3x-4)\}$$

$$= (x-2) \frac{d}{dx}(3x-4) + (3x-4) \frac{d}{dx}(x-2)$$

$$= (x-2)(3) + (3x-4)(1)$$

$$f''(x) = 3x-6 + 3x-4 = 6x-10 \quad (I)$$

$$\text{Put } f'(x) = 0, \text{ so } (x-2)(3x-4) = 0$$

$$\rightarrow x-2 = 0, \quad 3x-4 = 0$$

$$x = 2, \quad x = \frac{4}{3}$$

Put $x = 2$ in (I), $f''(2) = 6(2) - 10 = 12 - 10 = 2 > 0$

so $f(x)$ has minimum value at $x = 2$. and

$$f(2) = (2-2)^2(2-1) = 0$$

Put $x = \frac{4}{3}$ in (I), $f''\left(\frac{4}{3}\right) = 6\left(\frac{4}{3}\right) - 10 = 8 - 10 = -2 < 0$

so $f(x)$ has maximum value at $x = \frac{4}{3}$ and

$$f\left(\frac{4}{3}\right) = \left(\frac{4}{3}-2\right)^2\left(\frac{4}{3}-1\right) = \left(\frac{4-6}{3}\right)^2\left(\frac{4-3}{3}\right) = \left(-\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{4}{27}$$

$$(ix) f(x) = 5 + 3x - x^3$$

$$\text{Solution: } f(x) = 5 + 3x - x^3$$

$$\rightarrow f'(x) = 3 - 3x^2 \rightarrow f''(x) = -6x \quad (I)$$

$$\text{Put } f'(x) = 0, \quad 3 - 3x^2 = 0$$

$$\rightarrow 3(1-x^2) = 0, \quad 3 \neq 0 \text{ so } 1-x^2 = 0$$

$$\rightarrow 1 = x^2 \rightarrow x = \pm 1$$

Put $x = 1$ in (I), $f''(1) = -6 < 0$

$\rightarrow f(x)$ has maximum value at $x = 1$.

$$\text{and } f(1) = 5 + 3(1) - (1)^3 = 5 + 3 - 1 = 7$$

$$\text{Put } x = -1 \text{ in (I), } f''(-1) = -6(-1) = 6 > 0$$

so $f(x)$ has minimum value at $x = -1$
 and $f(-1) = 5 + 3(-1) - (-1)^3 = 5 - 3 - (-1)$
 $\rightarrow f(-1) = 2 + 1 = 3$

Q3. Find the maximum and minimum values of the function defined by the following equation occurring in the interval $[0, 2\pi]$. $f(x) = \sin x + \cos x$

Solution:- $f(x) = \sin x + \cos x$

$$\rightarrow f'(x) = \cos x - \sin x, \quad f''(x) = -\sin x - \cos x \quad \text{--- (I)}$$

$$\text{put } f'(x) = 0 \rightarrow \cos x - \sin x = 0$$

$$\rightarrow \cos x = \sin x \rightarrow 1 = \frac{\sin x}{\cos x}$$

$$\rightarrow \tan x = 1 \rightarrow x = \tan^{-1}(1)$$

$\therefore \tan x$ is +ve in I and III quadrants with reference angle $x = \frac{\pi}{4}$ so

$$x = \frac{\pi}{4} \quad (\text{For I quad.})$$

$$x = \pi + \frac{\pi}{4} = \frac{4\pi + \pi}{4} = \frac{5\pi}{4} \quad (\text{For III quad.})$$

$$\text{so put } x = \frac{\pi}{4} \text{ in (I)}$$

$$\rightarrow f''\left(\frac{\pi}{4}\right) = -\sin \frac{\pi}{4} - \cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\rightarrow f''\left(\frac{\pi}{4}\right) = -\sqrt{2} < 0$$

so $f(x)$ has maximum value at $x = \frac{\pi}{4}$

$$\text{and } f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\text{put } x = \frac{5\pi}{4} \text{ in (I), } f''\left(\frac{5\pi}{4}\right) = -\sin \frac{5\pi}{4} - \cos \frac{5\pi}{4}$$

$$\rightarrow f''\left(\frac{5\pi}{4}\right) = -\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

$$\rightarrow f''\left(\frac{5\pi}{4}\right) = \sqrt{2} > 0$$

$\rightarrow f(x)$ has minimum value at $x = \frac{5\pi}{4}$

$$\text{and } f\left(\frac{5\pi}{4}\right) = \sin \frac{5\pi}{4} + \cos \frac{5\pi}{4} = -\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)$$

$$f\left(\frac{5\pi}{4}\right) = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

Q4. Show that $y = \frac{\ln x}{x}$ has maximum value at $x = e$.

Solution:- $y = \frac{\ln x}{x}$

$$\rightarrow \frac{dy}{dx} = \frac{d}{dx} \left(\frac{\ln x}{x} \right) = \frac{x \frac{d}{dx}(\ln x) - (\ln x) \frac{d}{dx}(x)}{x^2}$$

$$\rightarrow \frac{dy}{dx} = \frac{x \cdot \frac{1}{x} - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

$$\rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{1 - \ln x}{x^2} \right) = \frac{x^2 \frac{d}{dx}(1 - \ln x) - (1 - \ln x) \frac{d}{dx}(x^2)}{x^4}$$

$$= \frac{x^2 \left(-\frac{1}{x}\right) - (1 - \ln x)(2x)}{x^4} = \frac{-x - 2x + 2x \ln x}{x^4}$$

$$= \frac{-3x - 2x \ln x}{x^4} = \frac{-x(3 + 2 \ln x)}{x^4}$$

$$\frac{d^2y}{dx^2} = -\frac{(3 + 2 \ln x)}{x^3} \quad \text{--- (I)}$$

$$\text{put } \frac{dy}{dx} = 0$$

$$\rightarrow \frac{1 - \ln x}{x} = 0 \rightarrow 1 - \ln x = 0$$

$$\rightarrow 1 = \ln x \rightarrow \ln e = \ln x$$

$$\rightarrow x = e \text{ put in (I)} \quad \therefore \ln e = 1$$

$$\frac{d^2y}{dx^2} \Big|_{x=e} = \frac{-3 + 2 \ln e}{e^3} = \frac{-3 + 2}{e^3} = \frac{-1}{e^3} < 0$$

Thus y has maximum value at $x = e$.

Q5. show that $y = x^x$ has minimum value at $x = \frac{1}{e}$

Solution:- $y = x^x$

Taking $\ln$ on both sides

$$\rightarrow \ln y = \ln x^x$$

$$\rightarrow \ln y = x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$$

$$\frac{1}{y} \frac{dy}{dx} = x \frac{d}{dx}(\ln x) + (\ln x) \frac{d}{dx}(x)$$

$$= x \cdot \frac{1}{x} + (\ln x)(1)$$

$$\frac{1}{y} \frac{dy}{dx} = 1 + \ln x$$

$$\rightarrow \frac{dy}{dx} = y(1 + \ln x) = x^x(1 + \ln x) \quad \therefore y = x^x$$

$$\rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} \{ x^x(1 + \ln x) \}$$

$$= x^x \frac{d}{dx}(1 + \ln x) + (1 + \ln x) \frac{d}{dx}(x^x)$$

$$= x^x \cdot \frac{1}{x} + (1 + \ln x) \cdot \{ x^x(1 + \ln x) \}$$

$$\frac{d^2y}{dx^2} = x^x \left\{ \frac{1}{x} + (1 + \ln x)^2 \right\} \quad \text{--- (II)}$$

$$\text{put } \frac{dy}{dx} = 0, \text{ so}$$

$$x^x(1 + \ln x) = 0$$

$$\therefore x^x \neq 0, \quad 1 + \ln x = 0 \rightarrow \ln e + \ln x = 0$$

$$\rightarrow \ln ex = \ln 1 \quad (\because \ln 1 = 0)$$

$$\rightarrow ex = 1 \rightarrow x = \frac{1}{e}$$

$$\text{put } x = \frac{1}{e} \text{ in (II)}$$

$$\frac{d^2y}{dx^2} \Big|_{x=\frac{1}{e}} = \left(\frac{1}{e}\right)^{\frac{1}{e}} \left\{ \frac{1}{\frac{1}{e}} + (\ln e + \ln \frac{1}{e})^2 \right\}$$

$$= \left(\frac{1}{e}\right)^{\frac{1}{e}} \{ e + (\ln e - \ln e)^2 \}$$

$$= \left(\frac{1}{e}\right)^{\frac{1}{e}} (e) > 0 \quad \therefore \ln 1 = 0$$

Hence y has minimum value at $x = \frac{1}{e}$

Application of Maxima and Minima

- 1) We first form the relation of the form $y = f(x)$ from the given information
- 2) Find the maximum or minimum value of f as required by using 1st or 2nd derivative rule.

* Mostly we will use 2nd derivative rule.

Example 1. Find two positive integers whose sum is 9 and the product of one with the square of the other will be maximum.

Solution:- Let x and y be req. integers.
so $x + y = 9$ (according to given condition)

$$\rightarrow y = 9 - x \rightarrow \text{(i)}$$

$$\text{Product} = f(x) = x^2 y = x^2 (9 - x)$$

$$\rightarrow f(x) = 9x^2 - x^3 \rightarrow f'(x) = 18x - 3x^2$$

$$\text{and } f''(x) = 18 - 6x \rightarrow \text{(ii)}$$

$$\text{Put } f'(x) = 0 \rightarrow 18x - 3x^2 = 0$$

$$\rightarrow 6x - x^2 = 0 \rightarrow x(6 - x) = 0$$

$$\rightarrow x = 0, x = 6$$

$$\text{Put } x = 0 \text{ in (ii)} \rightarrow f''(0) = 18 - 6(0) = 18 > 0$$

so $f(x)$ has minimum value at $x = 0$

This is not required.

$$\text{Put } x = 6 \text{ in (ii)} \rightarrow f''(6) = 18 - 6(6) = 18 - 36 = -18 < 0$$

$\rightarrow f(x)$ has maximum at $x = 6$

$$\text{and put in (i)}: y = 9 - 6 = 3$$

Hence required integers are 6 and 3.

Example 2. What are the dimensions of a box of a square base having largest volume if the sum of one side of the base and its height is 12cm.

Solution:- Let length, width and height of box with square base are x, x and y

$$\therefore \text{Volume} = f(x) = x \times x \times y$$

$$\rightarrow f(x) = x^2 y$$

$$\text{By given condition, } x + y = 12$$

$$\rightarrow y = 12 - x \rightarrow \text{(i)}$$

$$\text{so } f(x) = x^2 (12 - x) = 12x^2 - x^3$$

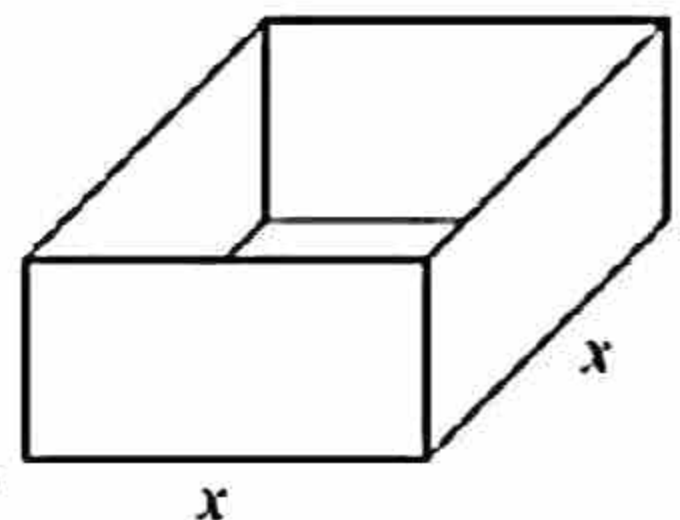
$$\rightarrow f'(x) = 24x - 3x^2, f''(x) = 24 - 6x \rightarrow \text{(ii)}$$

$$\text{Put } f'(x) = 0 \rightarrow 24x - 3x^2 = 0$$

$$\rightarrow 8x - x^2 = 0 \rightarrow x(8 - x) = 0$$

$$\rightarrow x = 0, x = 8$$

$x = 0$ (Not possible because x being length)



$$\text{so put } x = 8 \text{ in (ii)} \rightarrow f''(8) = 24 - 6(8)$$

$$\rightarrow f''(8) = 24 - 48 = -24 < 0$$

$\rightarrow f(x)$ has minimum value at $x = 8$ and put in (i)

$$\rightarrow y = 12 - 8 = 4$$

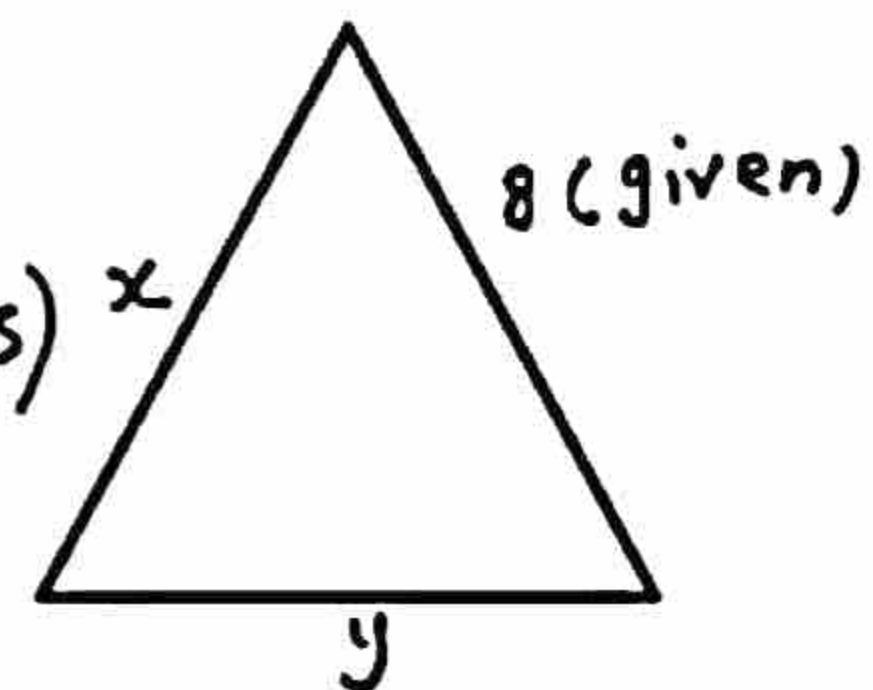
so required dimensions are 8cm and 4cm

Example 3. The perimeter of triangle is 20cm. If one side is of length 8cm. What are the length of the other sides for maximum area of the triangle?

Solution:- Let x and y be required lengths of triangle.

$$\therefore \text{perimeter} = x + y + 8$$

(perimeter = sum of all sides)



$$\rightarrow 20 = x + y + 8$$

$$\rightarrow x + y = 20 - 8$$

$$\rightarrow x + y = 12 \rightarrow y = 12 - x \rightarrow \text{(i)}$$

$$\text{Also } s = \frac{x + y + 8}{2} \rightarrow s = \frac{20}{2} = 10$$

We know that

$$\text{Area} = f(x) = \sqrt{s(s-x)(s-y)(s-8)}$$

$$\rightarrow A^2 = f(x) = s(s-x)(s-y)(s-8)$$

$$\rightarrow f(x) = s(s-x)(s-y)(s-8)$$

$$= (10)(10-x)(10-y)(10-8)$$

$$= 10(10-x)(10-y)(2)$$

$$f(x) = 20(10-x)(10-(12-x)) \quad \because y = 12 - x$$

$$\rightarrow f'(x) = 20 \frac{d}{dx} (10-x)(x-2)$$

$$= 20 \{ (10-x)(1) + (x-2)(-1) \}$$

$$= 20(10 - x - x + 2) = 20(12 - 2x)$$

$$f'(x) = 40(6 - x) \rightarrow f''(x) = 40(-1)$$

$$f''(x) = -40 \rightarrow \text{(ii)}$$

$$\text{Put } f'(x) = 0 \rightarrow 40(6 - x) = 0$$

$$\rightarrow 6 - x = 0 \rightarrow x = 6 \text{ put in (i)}$$

$f''(6) = -40 < 0$ Thus $f(x)$ has maximum value at $x = 6$ and put in (i)

$$y = 12 - 6 = 6.$$

Hence lengths of req. sides are 6cm and 6cm.



Example 4. An open box of rectangular base is to be made from 24cm by 45cm cardboard by cutting out square sheets of equal size from each corner and bending the sides. Find the dimensions of corner squares to obtain a box having largest possible volume.

Solution:-

∴ Length of cardboard = 45cm
width of cardboard = 24cm

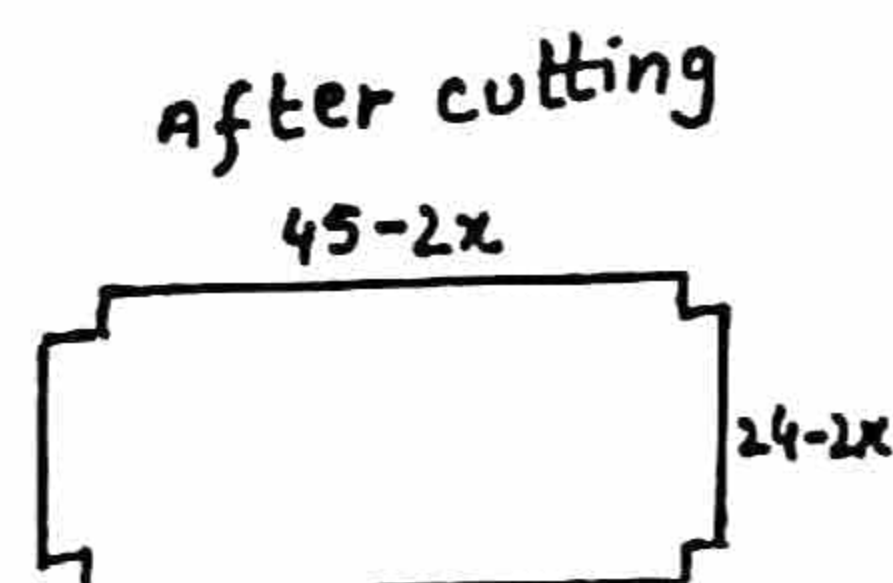
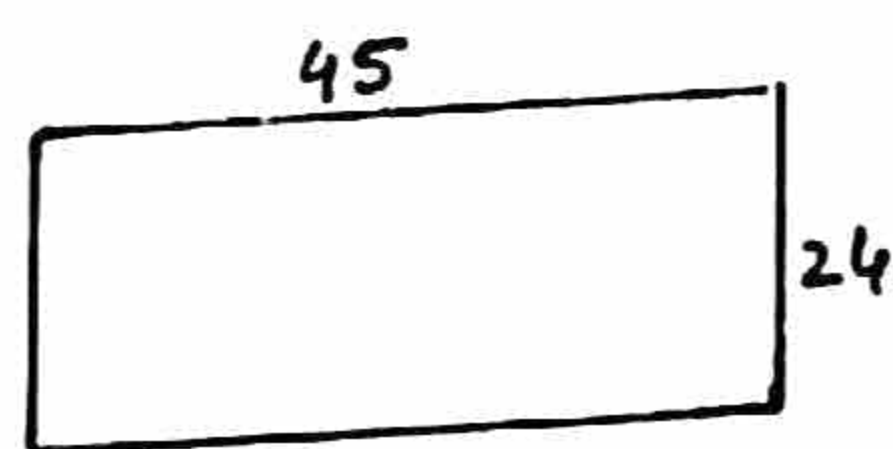
Let length of each side of square be x cm.

After cutting x cm from each corner we get

length of box = $45 - 2x$

width of box = $24 - 2x$

height of box = x



∴ Volume = $f(x) = (45-2x)(24-2x)(x)$

→ $f(x) = (45-2x)(24x-2x^2)$

= $1080x - 90x^2 - 48x^2 + 4x^3$

$f(x) = 4x^3 - 138x^2 + 1080x$

→ $f'(x) = 12x^2 - 276x + 1080$

→ $f''(x) = 24x - 276$ — (i)

Put $f'(x) = 0$, $12x^2 - 276x + 1080 = 0$

→ $12(x^2 - 23x + 90) = 0$

→ $x^2 - 23x + 90 = 0$

→ $x(x-5) - 18(x-5) = 0$

→ $(x-5)(x-18) = 0$

→ $x = 5$, $x = 18$

Put $x = 5$ in (i) → $f''(5) = 24(5) - 276$

$f''(5) = 120 - 276 = -156 < 0$

Thus $f(x)$ will be maximum (largest).
Hence 5cm is required length of each side of square corner to be cut off.

Example 5. Find the point on the graph of the curve $y = 4 - x^2$ which is closest to the point $(3, 4)$.

Solution:- Let $P(x, y)$ be the req. point.

∴ $y = 4 - x^2$ — (i)

and $d = \text{distance b/w } (x, y) \text{ and } (3, 4)$

→ $d = \sqrt{(x-3)^2 + (y-4)^2}$

→ $d^2 = f(x) = (x-3)^2 + (y-4)^2$

→ $f(x) = (x-3)^2 + (4-x^2-4)^2$

= $(x-3)^2 + (-x^2)^2$

$f(x) = (x-3)^2 + x^4 = x^2 + 9 - 6x + x^4$

→ $f'(x) = 2x - 6 + 4x^3$

→ $f''(x) = 2 + 12x^2$ — (ii)

Now put $f'(x) = 0$ so

$4x^3 + 2x - 6 = 0$

→ $2x^3 + x - 3 = 0$

→ $(x-1)(2x^2+2x+3) = 0$

$x = 1$, $2x^2+2x+3 = 0$
(rejected being complex roots)

For $x = 1$
∴ $2(1)^3 + 1 - 3$
= $2 + 1 - 3 = 3 - 3 = 0$

1	2	0	1	-3
	2	2	3	0

→ $2x^2+2x+3$
and $x-1$ are factors
of $2x^3+x-3=0$

so put $x = 1$ in (ii)

→ $f''(1) = 2 + 12(1)^2 = 14 > 0$

Thus $f(x)$ has minimum value at

$x = 1$ and put in (i) → $y = 4 - (1)^2 = 3$

Hence $P(1, 3)$ is req. pt. closest to $(3, 4)$.

Exercise 2.10

Q1. Find two integers whose sum is 30 and their product will be maximum.

Solution:- Let x and y be req. integers.

product = $f(x) = xy$

By given condition, $x + y = 30$

→ $y = 30 - x$ — (i)

so $f(x) = x(30-x) = 30x - x^2$

→ $f'(x) = 30 - 2x$

→ $f''(x) = -2$ — (ii)

Put $f'(x) = 0$, $30 - 2x = 0$

→ $2x = 30$ → $x = 15$ put in (i)

→ $f''(15) = -2 < 0$ Thus $f(x)$ has maximum value at $x = 15$ and put in (i)

→ $y = 30 - 15 = 15$

so req. integers are 15 and 15.

Q2. Divide 20 into two parts so that the sum of their squares will be minimum.

Solution:- Let x and y be req. two parts of 20. so $x + y = 20$

→ $y = 20 - x$ — (i)

By given condition, $f(x) = x^2 + y^2$

→ $f(x) = x^2 + (20-x)^2 = x^2 + 400 + x^2 - 40x$

→ $f'(x) = 4x - 40$ → $f''(x) = 4$ — (ii)

Put $f'(x) = 0$, $4x - 40 = 0$

→ $x - 10 = 0$ → $x = 10$ put in (i)

$f''(10) = 4 > 0$, so $f(x)$ has minimum value at $x = 10$ and put in (i) $y = 20 - 10 = 10$

so req. two parts of 20 are 10 and 10.

Q3. Find two positive integers whose sum is 12 and the product of one with the square of the other will be maximum.

Solution:- Let x and y be req. integers
By given condition; $x + y = 12 \Rightarrow y = 12 - x \rightarrow (i)$

$$\text{Product} = f(x) = x^2 y = x^2 (12 - x)$$

$$\Rightarrow f(x) = 12x^2 - x^3$$

$$\Rightarrow f'(x) = 24x - 3x^2 \quad \Rightarrow f''(x) = 24 - 6x \quad (ii)$$

$$\text{Put } f'(x) = 0 \Rightarrow 24x - 3x^2 = 0 \Rightarrow x(8 - x) = 0$$

$$\Rightarrow x = 0 \text{ and } x = 8$$

$$\text{Put } x = 0 \text{ in (ii)} \Rightarrow f''(0) = 24 > 0$$

so $f(x)$ has minimum value at $x = 0$
which is not required.

$$\text{so put } x = 8 \text{ in (ii)} \Rightarrow f''(8) = 24 - 6(8) = 24 - 48$$

$$f''(8) = -24 < 0$$

so $f(x)$ has maximum value at $x = 8$

$$\text{and put in (i)} \Rightarrow y = 12 - 8 = 4$$

so req. integers are 8 and 4.

Q4. The perimeter of a triangle is 16cm. If one side is of length 6cm, what are length of the other sides for maximum area of the triangle?

Solution:- Let x and y be other sides.

$$\therefore \text{perimeter} = x + y + 6 \text{ (sum of all sides)}$$

$$\Rightarrow 16 = x + y + 6$$

$$\Rightarrow 16 - 6 = x + y$$

$$\Rightarrow x + y = 10$$

$$\text{and } y = 10 - x \rightarrow (i)$$

$$\therefore s = \frac{x + y + 6}{2} = \frac{16}{2} \text{ or } s = 8$$

$$\text{Also Area} = A^2 = f(x) = s(s-x)(s-y)(s-6)$$

$$\Rightarrow f(x) = 8(8-x)(8-y)(8-6)$$

$$f(x) = 16(8-x)(8-y)$$

$$= 16(8-x)(8-(10-x))$$

$$f(x) = 16(8-x)(x-2)$$

$$\Rightarrow f'(x) = 16 \frac{d}{dx} \{(8-x)(x-2)\}$$

$$= 16 \{(8-x)(1) + (x-2)(-1)\}$$

$$= 16(8 - x - x + 2) = 16(10 - 2x)$$

$$f'(x) = 160 - 32x \Rightarrow f''(x) = -32 \quad (ii)$$

$$\text{Put } f'(x) = 0 \Rightarrow 160 - 32x = 0$$

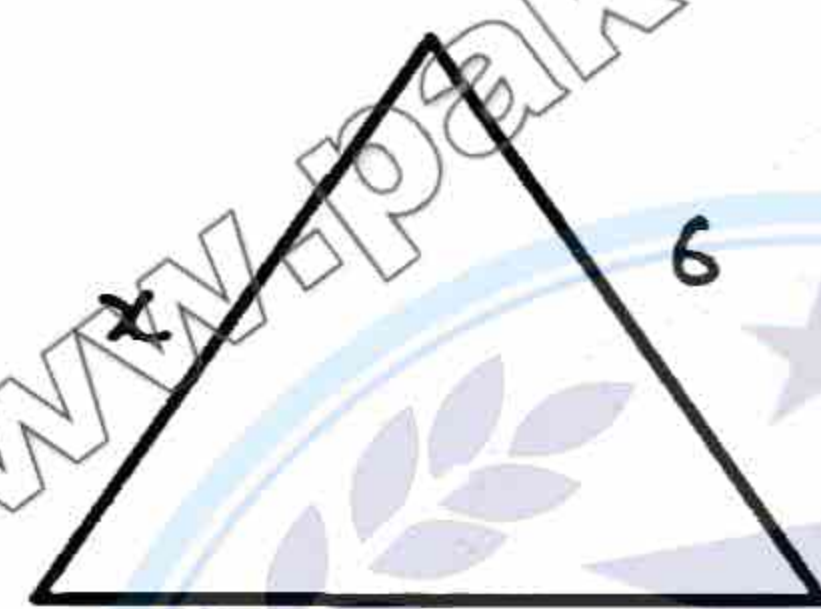
$$\Rightarrow 32x = 160 \Rightarrow x = 5$$

$$\text{Put } x = 5 \text{ in (ii)} \Rightarrow f''(5) = -32 < 0$$

Thus $f(x)$ has maximum value at $x = 5$

$$\text{and put in (i)} \Rightarrow y = 10 - 5 = 5$$

Hence req. sides are 5cm and 5cm.



Q5. Find the dimensions of a rectangle of largest area having perimeter 120cm.

Solution:- Let x and y be required dimensions.

$$\text{Perimeter} = x + y + x + y \text{ (sum of all sides)}$$

$$\Rightarrow 120 = 2x + 2y$$

$$\Rightarrow x + y = 60$$

$$\text{and } y = 60 - x \rightarrow (i)$$

$$\therefore \text{Area of rectangle} = f(x) = xy$$

$$\Rightarrow f(x) = x(60 - x)$$

$$f(x) = 60x - x^2 \Rightarrow f'(x) = 60 - 2x$$

$$\Rightarrow f''(x) = -2 \quad (ii)$$

$$\text{Put } f'(x) = 0, \text{ so } 60 - 2x = 0 \Rightarrow 2x = 60$$

$$\Rightarrow x = 30 \text{ put in (i)} \Rightarrow f''(30) = -2 < 0$$

so $f(x)$ has maximum (largest) value

$$\text{at } x = 30 \text{ and put in (i)} \Rightarrow y = 60 - 30 = 30$$

Thus required dimensions are 30cm and 30cm.

Q6. Find the lengths of sides of a variable rectangle having area 36cm² where its perimeter is minimum.

Solution:- Let x and y be required lengths.

$$\text{perimeter} = x + y + x + y$$

$$\Rightarrow f(x) = 2x + 2y$$

$$\Rightarrow f(x) = 2(x + y) \quad (i)$$

$$\text{Area of rectangle} = xy$$

$$\Rightarrow 36 = xy \Rightarrow y = \frac{36}{x} \quad (ii)$$

$$\text{so (i)} \Rightarrow f(x) = 2\left(x + \frac{36}{x}\right)$$

$$f(x) = 2x + 72x^{-1}$$

$$f'(x) = 2 - \frac{72}{x^2}, \quad f''(x) = -72(-2)x^{-3}$$

$$f''(x) = \frac{144}{x^3} \rightarrow (iii)$$

$$\text{Put } f'(x) = 0 \Rightarrow 2 - \frac{72}{x^2} = 0$$

$$\Rightarrow 2 = \frac{72}{x^2} \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$$

$$\Rightarrow x = 6, \quad x = -6 \text{ (rejected } \because \text{length always +ive)}$$

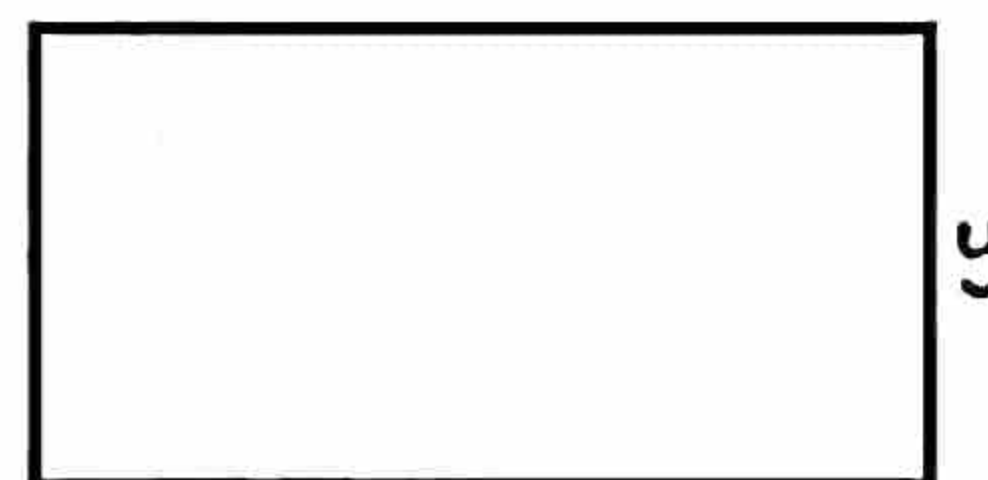
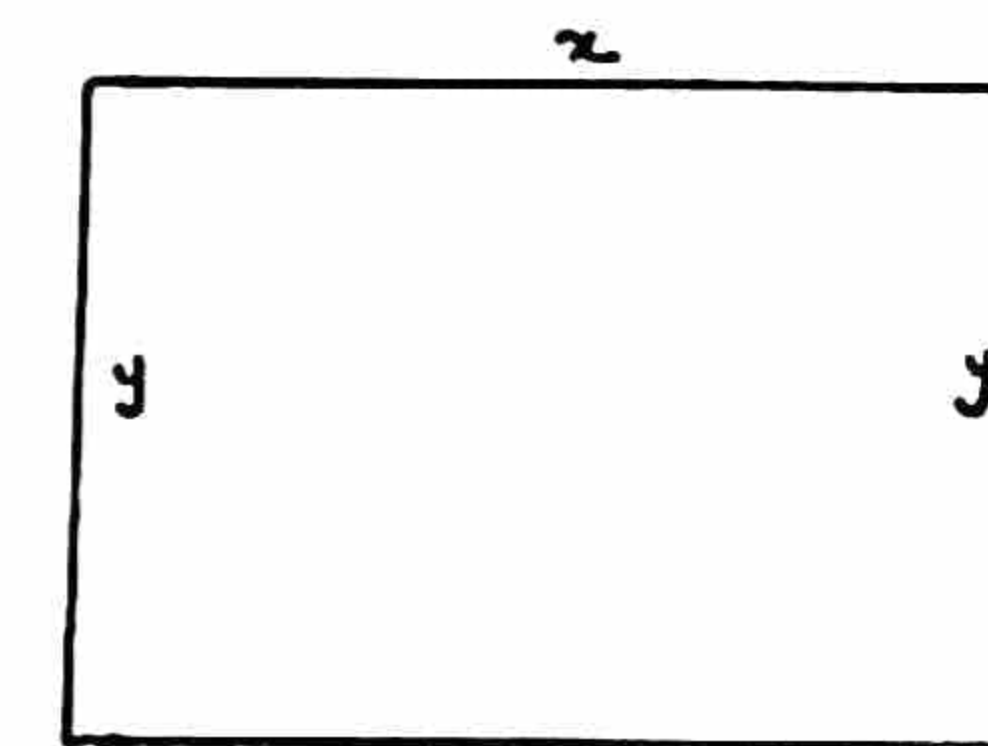
$$\text{Put } x = 6 \text{ in (iii)}$$

$$\Rightarrow f''(6) = \frac{144}{(6)^3} = \frac{144}{216} > 0$$

so $f(x)$ has minimum value at $x = 6$

$$\text{and put in (ii)} \Rightarrow y = \frac{36}{6} = 6$$

so sides of rectangle are 6cm and 6cm



$\therefore$ radius = 8 cm (given) so $|OC| = 8$ cm
 In $\triangle OAC$ $x^2 + y^2 = (8)^2$ (using Pythagoras's theorem)
 $\rightarrow y^2 = 64 - x^2 \rightarrow y = \pm \sqrt{64 - x^2}$
 $\rightarrow y = \sqrt{64 - x^2}$; $y = -\sqrt{64 - x^2}$
 (rejected, being length)

$\therefore$ Area of rectangle = $2xy \rightarrow A = f(x) = 2xy$
 $\rightarrow A^2 = f(x) = 4x^2(64 - x^2) = 256x^2 - 4x^4$
 $\rightarrow f'(x) = 512x - 16x^3$
 $\rightarrow f''(x) = 512 - 48x^2 \rightarrow$ (ii)
 Put $f'(x) = 0$, $\rightarrow 512x - 16x^3 = 0$
 $\rightarrow 16x(32 - x^2) = 0$
 $\rightarrow 16x = 0$, $x^2 = 32 \rightarrow x = \pm \sqrt{32}$
 $\rightarrow x = 0$, $x = \sqrt{32}$, $x = -\sqrt{32}$
 (rejected, being length)
 so put $x = \sqrt{32}$ in (i) $\rightarrow f'(\sqrt{32}) = 512 - 48(\sqrt{32})^2$
 $\rightarrow f'(\sqrt{32}) = 512 - 48(32) = -1024 < 0$
 $\rightarrow f(x)$ has maximum value at $x = \sqrt{32}$
 and put in (i) $y = \sqrt{64 - (\sqrt{32})^2} = \sqrt{64 - 32}$
 $y = \sqrt{32}$

Thus length of rectangle = $2x = 2(4\sqrt{2}) = 8\sqrt{2}$
 width of rectangle = $y = 4\sqrt{2}$

Hence dimensions are $8\sqrt{2}$ and $4\sqrt{2}$.

Q11. Find the point on the curve $y = x^2 - 1$ that is closest to the point $(3, -1)$.

Solution:- $\therefore y = x^2 - 1 \rightarrow$ (i)
 Let $P(x, y)$ be required point.

Let d = distance b/w (x, y) and $(3, -1)$

$\rightarrow d = \sqrt{(x-3)^2 + (y+1)^2}$
 $\rightarrow d^2 = (x-3)^2 + (y+1)^2$
 $\rightarrow f(x) = (x-3)^2 + (y+1)^2$
 $\rightarrow f(x) = x^2 + 9 - 6x + y^2 + 1 + 2y$
 $= x^2 + 9 - 6x + (x^2 - 1)^2 + 1 + 2(x^2 - 1)$

$= x^2 + 9 - 6x + x^4 + 1 - 2x^2 + 1 + 2x^2 - 2$
 $\therefore y = x^2 - 1$
 $f(x) = x^4 + x^2 - 6x + 9$

$\rightarrow f'(x) = 4x^3 + 2x - 6$

$\rightarrow f''(x) = 12x^2 + 2 \rightarrow$ (ii)

Put $f'(x) = 0 \rightarrow 4x^3 + 2x - 6 = 0$

$\rightarrow 2x^3 + x - 3 = 0$

For $x = 1$
 $2(1)^3 + 1 - 3 = 0$ so

$\rightarrow (x-1)(2x^2 + 2x + 3) = 0$

$\rightarrow x = 1$, $2x^2 + 2x + 3 = 0$
 (rejected being complex)

so put $x = 1$ in (ii)

1	2	0	1	-3
	2	2	3	0
	2x ² + 2x + 3			
	x-1 are factors of 2x ³ + x - 3 = 0			

$f''(1) = 12(1)^2 + 2 = 14 > 0$

$\rightarrow f(x)$ has minimum value at $x = 1$

and put in (i) $\rightarrow y = (1)^2 - 1 = 0$
 so $(1, 0)$ is the required point closest to $(3, -1)$.

Q12. Find the point on the curve $y = x^2 + 1$ that is closest to the point $(18, 1)$.

Solution:- $\therefore y = x^2 + 1 \rightarrow$ (i)
 Let $P(x, y)$ be the required point.

$\therefore d$ = distance b/w $P(x, y)$ and $(18, 1)$

$\rightarrow d = \sqrt{(x-18)^2 + (y-1)^2}$

$\rightarrow d^2 = (x-18)^2 + (y-1)^2$ Take $d^2 = f(x)$

$\rightarrow f(x) = (x-18)^2 + (x^2 + 1 - 1)^2 \therefore y = x^2 + 1$
 $= x^2 + 324 - 36x + x^4$

$f(x) = x^4 + x^2 - 36x + 324$

$\rightarrow f'(x) = 4x^3 + 2x - 36$

$\rightarrow f''(x) = 12x^2 + 2 \rightarrow$ (ii)

Put $f'(x) = 0 \rightarrow 4x^3 + 2x - 36 = 0$

$\rightarrow 2x^3 + x - 18 = 0$

For $x = 2$
 $2(2)^3 + 2 - 18 = 0$

$\rightarrow (x-2)(2x^2 + 4x + 9) = 0$

$\rightarrow x = 2$, $2x^2 + 4x + 9 = 0$
 (rejected being complex)

Put $x = 2$ in (ii)

$f''(2) = 12(2)^2 + 2 > 0$

$\rightarrow f(x)$ has minimum value at $x = 2$

and put in (i) $\rightarrow y = (2)^2 + 1 = 5$

so $(2, 5)$ is the required point which is closest to $(18, 1)$

2	0	1	-18
	2	4	9
	2x ² + 4x + 9		
	x-2 are factors of 2x ³ + x - 18 = 0		