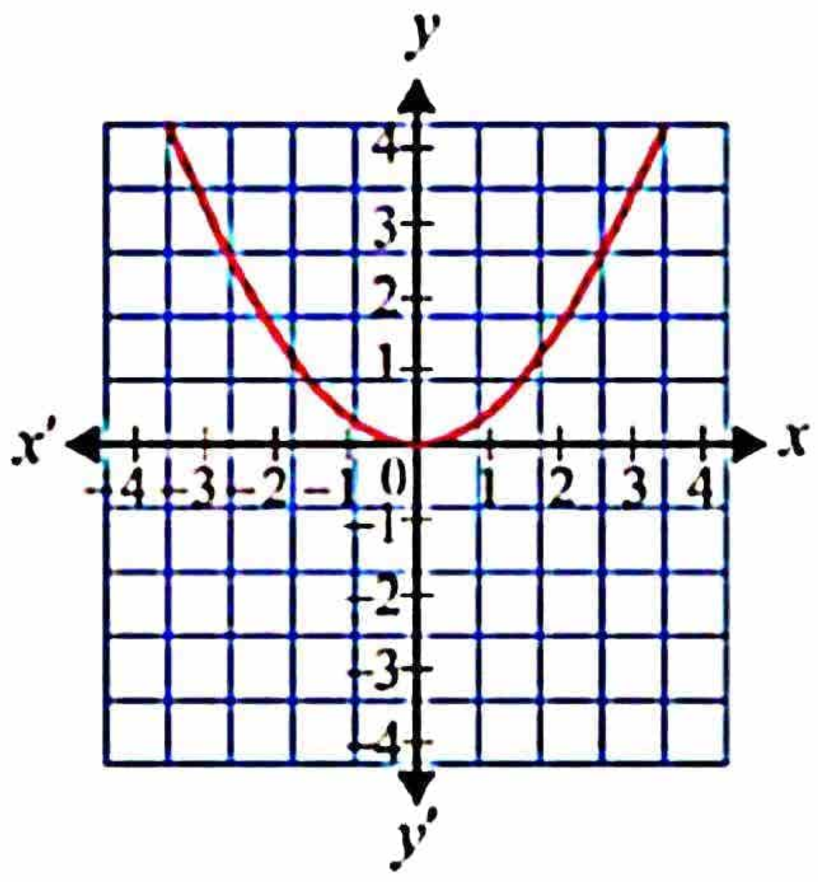
Exercise 2.1

1. Identify the domain and range of the functions through following graph.



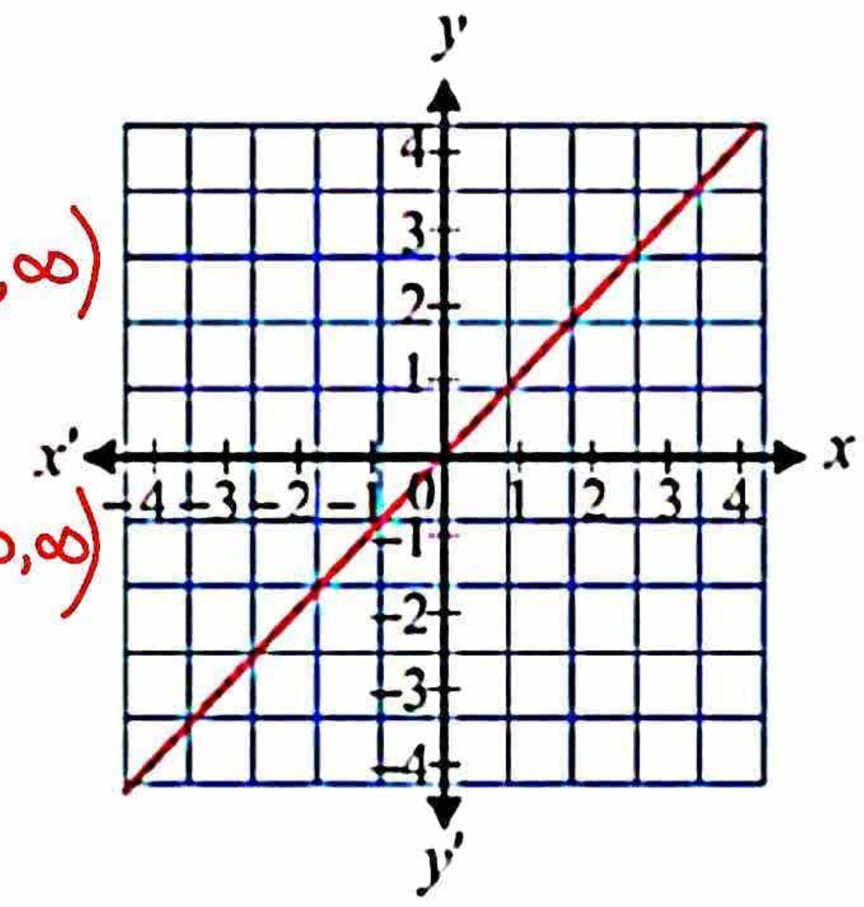
(i)

$$\text{Domain} = \{x \mid x \in \mathbb{R}\} = \mathbb{R} = (-\infty, \infty)$$

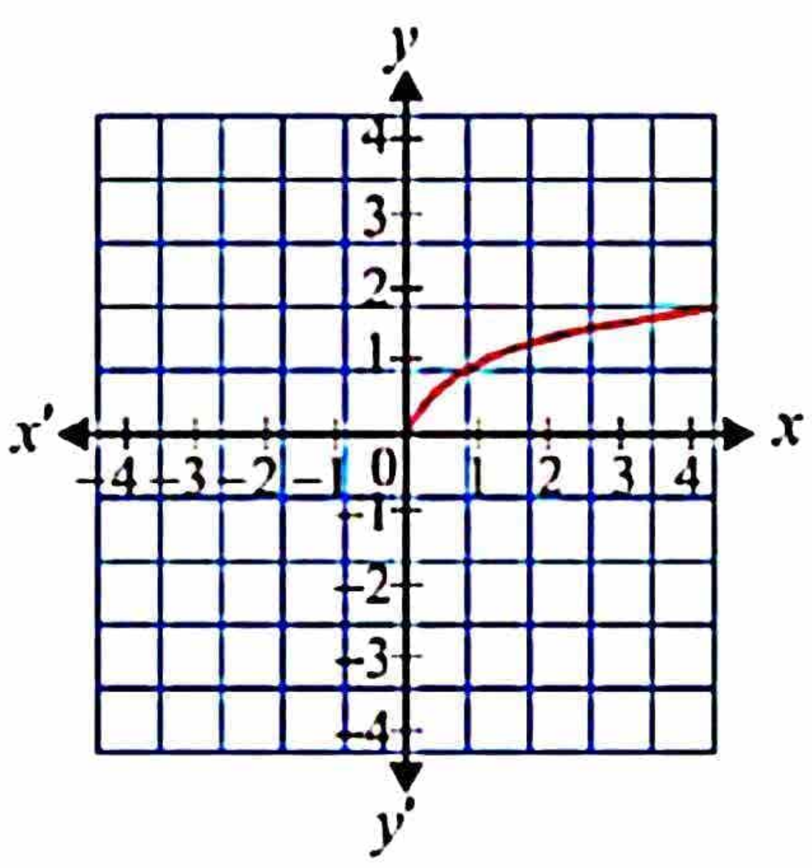
$$\text{Range} = \{y \mid y \geq 0\} = [0, \infty)$$

$$\text{Domain} = \{x \mid x \in \mathbb{R}\} = \mathbb{R} = (-\infty, \infty)$$

$$\text{Range} = \{y \mid y \in \mathbb{R}\} = \mathbb{R} = (-\infty, \infty)$$



(ii)



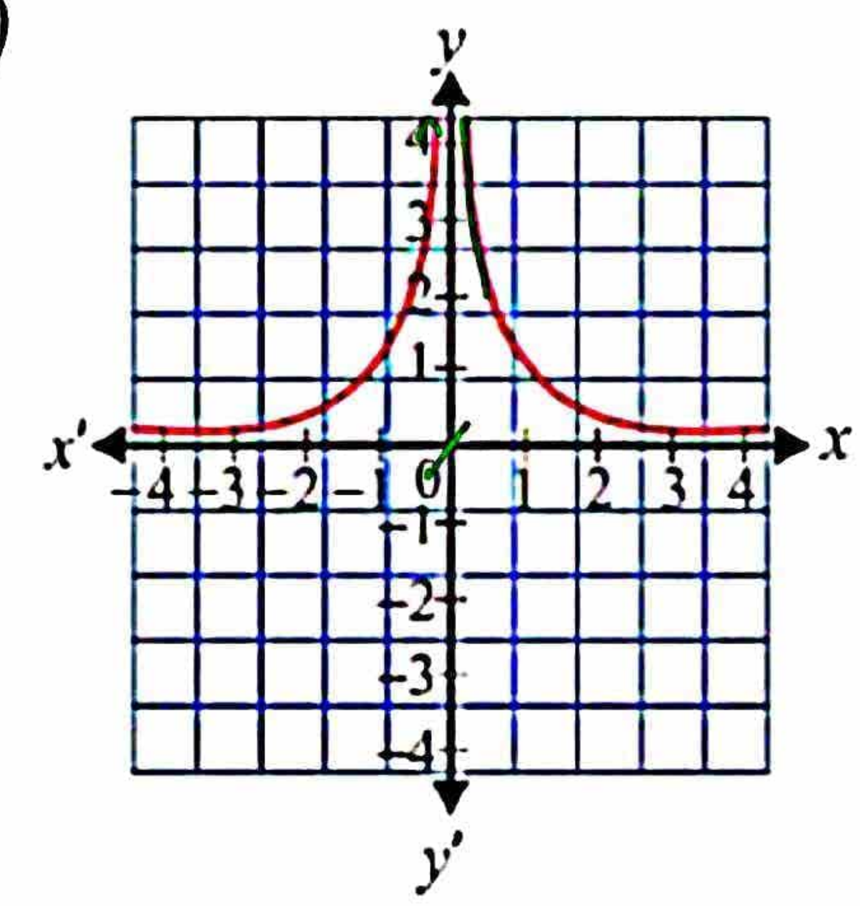
(iii)

$$\text{Domain} = \{x \mid x \in \mathbb{R} \wedge x \geq 0\}$$
$$= \{x \mid x \geq 0\} = [0, \infty)$$

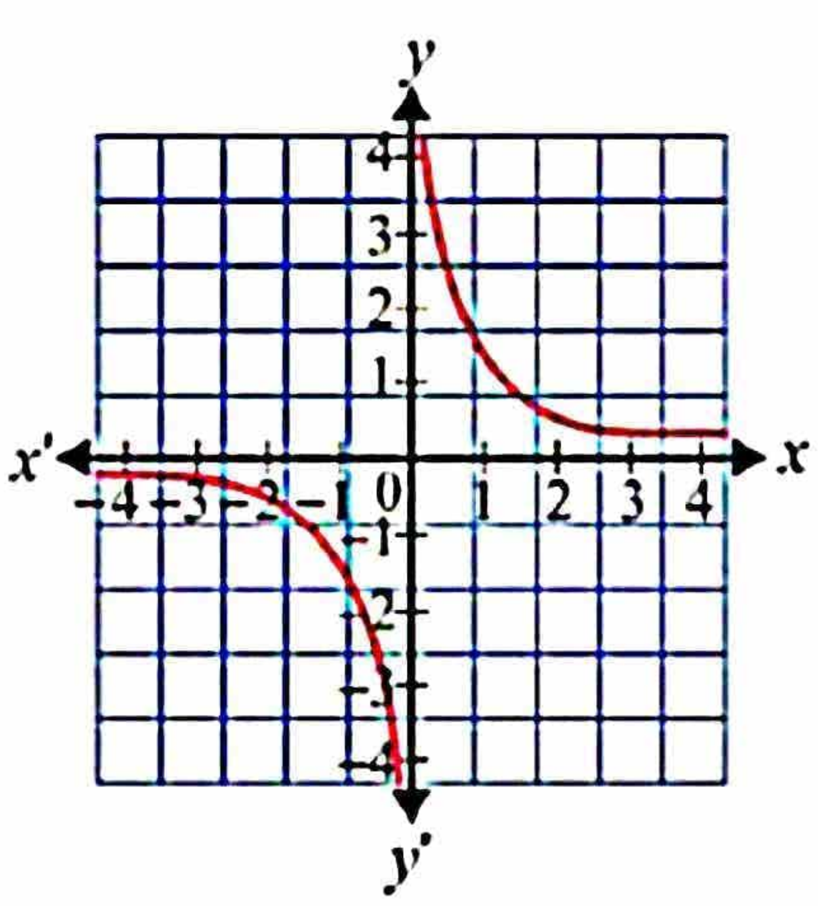
$$\text{Range} = \{y \mid y \in \mathbb{R} \wedge y \geq 0\} = [0, \infty)$$

$$\text{Domain} = \{x \mid x \in \mathbb{R} \wedge x \neq 0\}$$
$$= \mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$$

$$\text{Range} = \{y \mid y \in \mathbb{R} \wedge y > 0\}$$
$$= (0, \infty)$$



(iv)



(v)

$$\text{Domain} = \{x \mid x \in \mathbb{R} \wedge x \neq 0\} = \mathbb{R} - \{0\}$$
$$= (-\infty, 0) \cup (0, \infty).$$

$$\text{Range} = \{y \mid y \in \mathbb{R} \wedge y \neq 0\} = \mathbb{R} - \{0\}$$
$$= (-\infty, 0) \cup (0, \infty).$$

2. If $f(x) = 5x + 2$ and $g(x) = 2x^2 - 3$, then find

(i) fog (ii) gof (iii) fof (iv) gog

(i) $fog(x) = f\left(g(x)\right) = 5g(x) + 2 = 5\left(2x^2 - 3\right) + 2$

$= 10x^2 - 15 + 2$

$= 10x^2 - 13$

(ii) $gof(x) = g\left(f(x)\right) = 2\left(f(x)\right)^2 - 3 = 2\left(5x + 2\right)^2 - 3$

$= 2\left(25x^2 + 4 + 20x\right) - 3$

$= 50x^2 + 8 + 40x - 3$

$= 50x^2 + 40x + 5$

(iii) $fof(x) = f\left(f(x)\right) = 5f(x) + 2$

$= 5\left(5x + 2\right) + 2$

$= 25x + 10 + 2 = 25x + 12$

(iv) $gog(x) = g\left(g(x)\right) = 2g(x)^2 - 3$

$= 2\left(2x^2 - 3\right)^2 - 3$

$= 2\left(4x^4 + 9 - 12x^2\right) - 3$

$= 8x^4 + 18 - 24x^2 - 3$

$= 8x^4 - 24x^2 + 15$

3. If $f(x) = 2x$ and $g(x) = x + 1$, then find $fog(x)$ for $x = -5$.

$$fog(x) = f(g(x)) = 2\,g(x)$$
$$= 2(x+1)$$
$$= 2x + 2$$

$$fog(-5) = 2(-5) + 2$$
$$= -10 + 2$$
$$= -8$$

4. If $f(x) = x + 3$ and $g(x) = x^2$, then find $gof(x)$ for $x = 1$.

$$gof(x) = g(f(x)) = (f(x))^2$$
$$= (x+3)^2$$
$$= x^2 + 9 + 6x$$
$$gof(x) = x^2 + 6x + 9$$

$$gof(1) = (1)^2 + 6(1) + 9$$
$$= 1 + 6 + 9 = 16$$

5. If $c(x) = \cos x$ and $p(x) = x^3 + 1$ then find $poc(x)$.

$$poc(x) = p(c(x)) = (c(x))^3 + 1$$
$$= (\cos x)^3 + 1$$
$$= \cos^3 x + 1.$$

6. Given that $f(x) = x + 2$ and $g(x) = 3x - 2$ are two given functions then find $(fog)^{-1}$ and $(gof)^{-1}$ also show that $(gof)^{-1} = f^{-1}og^{-1}$.

$$fog(x) = f(g(x)) = g(x) + 2 = (3x - 2) + 2 = 3x - \cancel{2} + \cancel{2} = 3x$$

Let $h(x) = fog(x)$.

$h(x) = 3x$

Suppose $y = h(x) \Rightarrow h^{-1}(y) = x$

$y = 3x$

$\frac{y}{3} = x$

$x = \frac{y}{3}$

$h^{-1}(y) = \frac{y}{3}$

Replace y with x

$h^{-1}(x) = \frac{x}{3} \Rightarrow (fog)^{-1} = \frac{x}{3}$.

$$gof(x) = g(f(x)) = 3f(x) - 2 = 3(x+2) - 2 = 3x + 6 - 2 = 3x + 4$$

Let $h_1(x) = (gof)(x)$

$h_1(x) = 3x + 4$

Suppose $y = h_1(x) \Rightarrow h_1^{-1}(y) = x$

$y = 3x + 4$

$3x = y - 4$

$x = \frac{y-4}{3}$

$h_1^{-1}(y) = \frac{y-4}{3}$

Replace y with x

$h_1^{-1}(x) = \frac{x-4}{3} \Rightarrow (gof)^{-1} = \frac{x-4}{3}$.

$f(x) = x+2,$

Let $y = f(x) \Rightarrow f^{-1}(y) = x$

$y = x+2$

$y-2 = x$

$x = y-2$

$f^{-1}(y) = y-2$

Replace y with x

$f^{-1}(x) = x-2$

$g(x) = 3x-2$

Let $y = g(x) \Rightarrow g^{-1}(y) = x$

$y = 3x-2$

$y+2 = 3x$

$x = \frac{y+2}{3}$

$g^{-1}(y) = \frac{y+2}{3}$

Replace y with x

$g^{-1}(x) = \frac{x+2}{3}$

$$(f^{-1} \circ g^{-1})(x) = f^{-1}\left(g^{-1}(x)\right) = g^{-1}(x) - 2$$

$$= \frac{x+2}{3} - 2 = \frac{x+2-6}{3}$$

$$= \frac{x-4}{3}$$

Hence proved that

$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$

7. Given that $h(x) = x - 3$ and $k(x) = 2x + 5$ are two functions then verify that:
(i) $hok \neq koh$ (ii) $(hok)^{-1} = k^{-1}oh^{-1}$ (iii) $(koh)^{-1} = h^{-1}ok^{-1}$

L.H.S

(i) $h \circ k(x) = h(k(x)) = k(x) - 3 = 2x + 5 - 3 = 2x + 2$

R.H.S

$koh(x) = k(h(x)) = 2h(x) + 5 = 2(x-3) + 5 = 2x - 6 + 5 = 2x - 1$

So $hok \neq koh$.

(ii) L.H.S $hok(x) = h(k(x)) = 2x + 2$

Let $f(x) = hok(x)$

$f(x) = 2x + 2$

Suppose $y = f(x) \Rightarrow f^{-1}(y) = x$

$y = 2x + 2$

$y - 2 = 2x \Rightarrow x = \dfrac{y-2}{2} \Rightarrow f^{-1}(y) = \dfrac{y-2}{2}$

$f^{-1}(x) = \dfrac{x-2}{2}$

$\Rightarrow (hok)^{-1} = \dfrac{x-2}{2}$

R.H.S

$k(x) = 2x + 5$ $h(x) = x - 3$

$k^{-1}(x) = \dfrac{x-5}{2}$ $h^{-1}(x) = x + 3$

$k^{-1}oh^{-1}(x) = k^{-1}(h^{-1}(x)) = \dfrac{h^{-1}(x) - 5}{2} = \dfrac{x + 3 - 5}{2}$

$= \dfrac{x - 2}{2}$

So $(hok)^{-1} = k^{-1}oh^{-1}$.

(iii) L.H.S:

$k \circ h(x) = 2x - 1$

$k \circ h^{-1}(x) = \dfrac{x+1}{2}$

R.H.S

$h(x) = x - 3$, $k(x) = 2x + 5$

$h^{-1}(x) = x + 3$ $k^{-1}(x) = \dfrac{x-5}{2}$

$h^{-1} \circ k^{-1}(x) = h^{-1}\left(k^{-1}(x)\right)$

$= k^{-1}(x) + 3 = \dfrac{x-5}{2} + 3$

$= \dfrac{x-5+6}{2} = \dfrac{x+1}{2}$

Hence proved.

Exercise 2.2

1. Which of the following are algebraic, exponential, logarithmic, trigonometric, inverse trigonometric, hyperbolic and inverse hyperbolic functions.

algebraic (i) $y = x^2 + 5x + 6$

(ii) $f(x) = \tan^{-1}x$ inverse trigonometric.

exponential (iii) $y = 2^{x+1}$

(iv) $y = \log_5(x+2)$ logarithmic

trigonometric (v) $f(x) = 3\sin x$

(vi) $y = a^{\sin x}$ exponential.

algebraic (vii) $f(x) = \dfrac{x^2+5x+7}{x+9}$

(viii) $f(x) = \dfrac{\sin x}{\sec x}$ trigonometric.

logarithmic (ix) $y = \log_a \sin x$

(x) $f(x) = \mathrm{cosec}^{-1}\sqrt{x^2-1}$ inverse trigonometric

trigonometric (xi) $f(x) = \tan(\sin x)$

(xii) $y = \dfrac{x}{x+3}$ algebraic.

hyperbolic (xiii) $f(x) = \sinh x$

(xiv) $y = \ln\cosh x$ logarithmic

(xv) $y = \tan h^{-1}x$

(xvi) $y = \cos^{-1}(\ln x)$ inverse trigonometric.

inverse hyperbolic.

2. Identify, whether the y is the explicit or implicit function of independent variable x if:

(i) $xy^2 + 5xy + 7 = 0$

(ii) $y = 3x^2 - 3x + 5$

(iii) $yx^2 + y^2x = 3 - 5y$

(iv) $x^2 + xy^2 = 2 + 3xy$

(v) $y = \dfrac{x+3}{x^2+5}$

(vi) $\dfrac{x}{y} = 3x^3y - 5$

(i) $xy^2 + 5xy + 7 = 0$ implicit.

(ii) $y = 3x^2 - 3x + 5$ explicit

(iii) $yx^2 + y^2x = 3 - 5y$ implicit

(iv) $x^2 + xy^2 = 2 + 3xy$ implicit

(v) $y = \dfrac{x+3}{x^2+5}$ explicit

(vi) $\dfrac{x}{y} = 3x^3y - 5$

$x = 3x^3y^2 - 5y$ implicit.

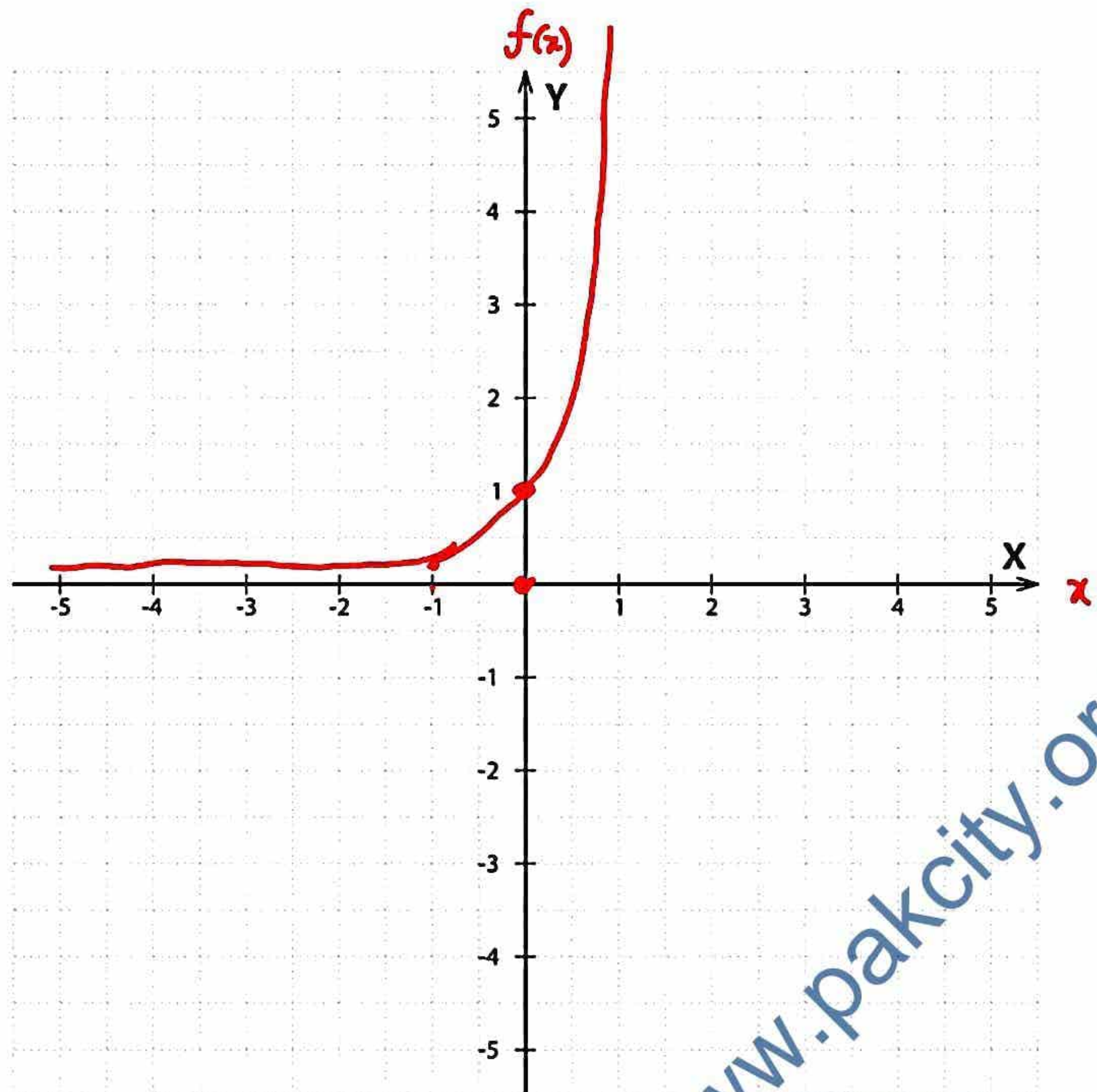
3. Draw the graph of the following functions:

(i) $f(x) = e^{3x}$ (ii) $f(x) = 3log_{10}x$

(iii) $y = \sqrt{36 - x^2}$ (iv) $\frac{x^2}{16} + \frac{y^2}{25} = 1$

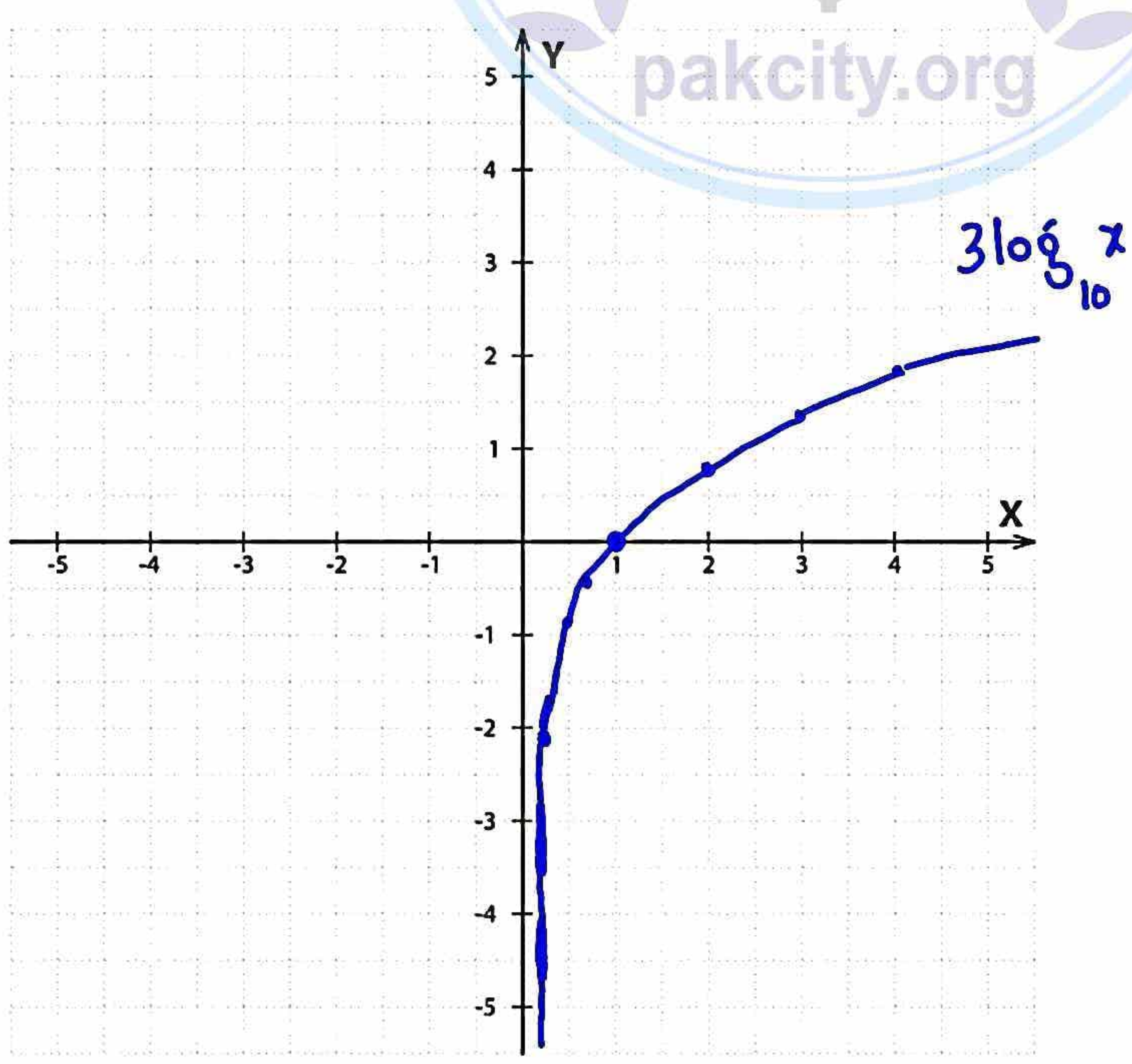
(i) $f(x) = e^{3x}$

x	-4	-3	-2	-1	0	1	2	3	4
f(x)				0.0498	1	20.08	403.43		



(ii) $f(x) = 3\log_{10} x$

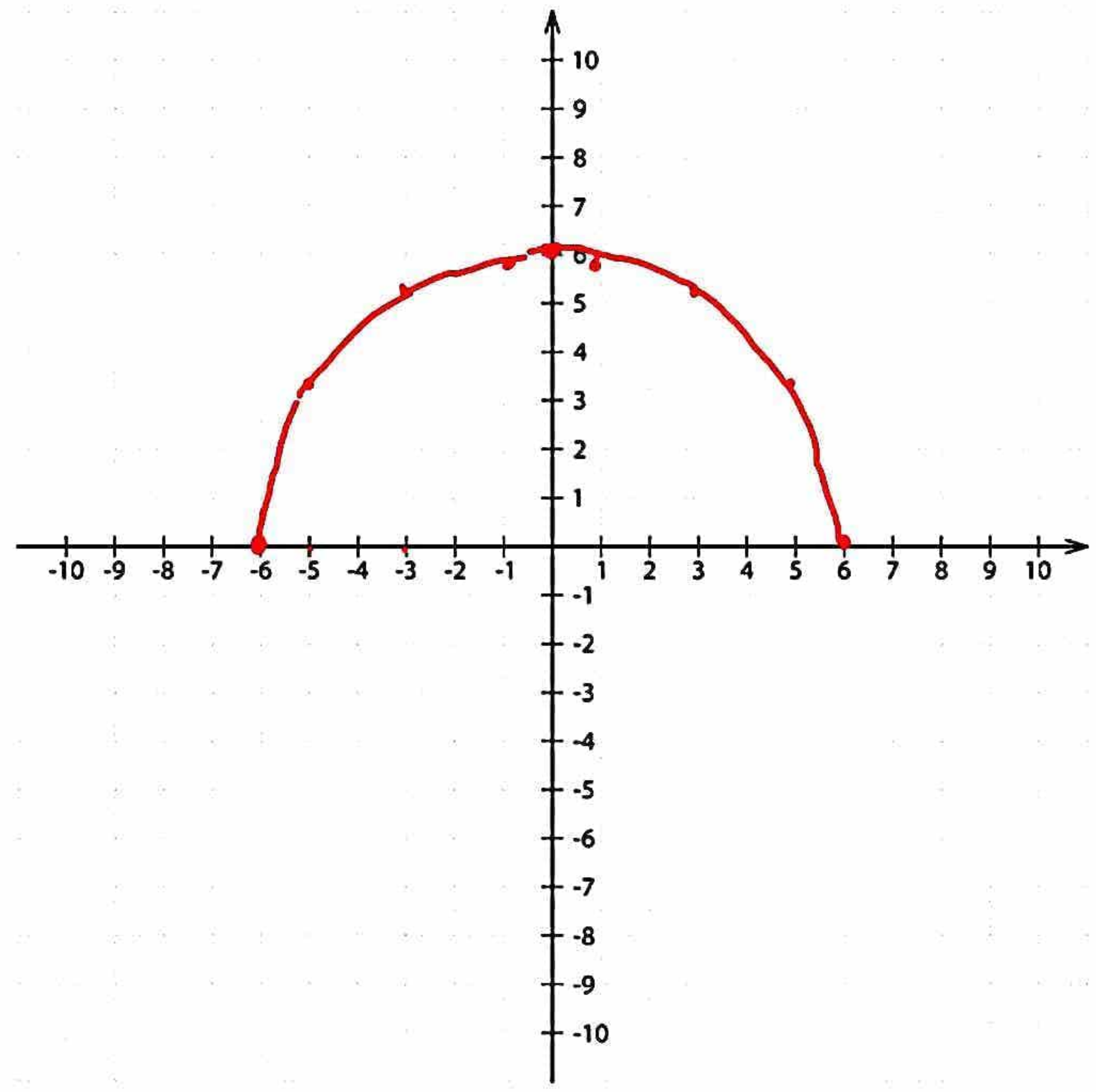
x	0	0.2	0.5	0.7	0.9	1	2	3	4
f(x)	X	-2.1	-0.9	-0.46	-0.137	0	0.9	1.43	1.8



(iii) $y = \sqrt{36-x^2}$

$36 - x^2 = 0$
$x = \pm 6.$

x	−6	−5	−3	−1	0	1	3	5	6
y	0	3.31	5.2	5.9	6	5.9	5.2	3.31	0

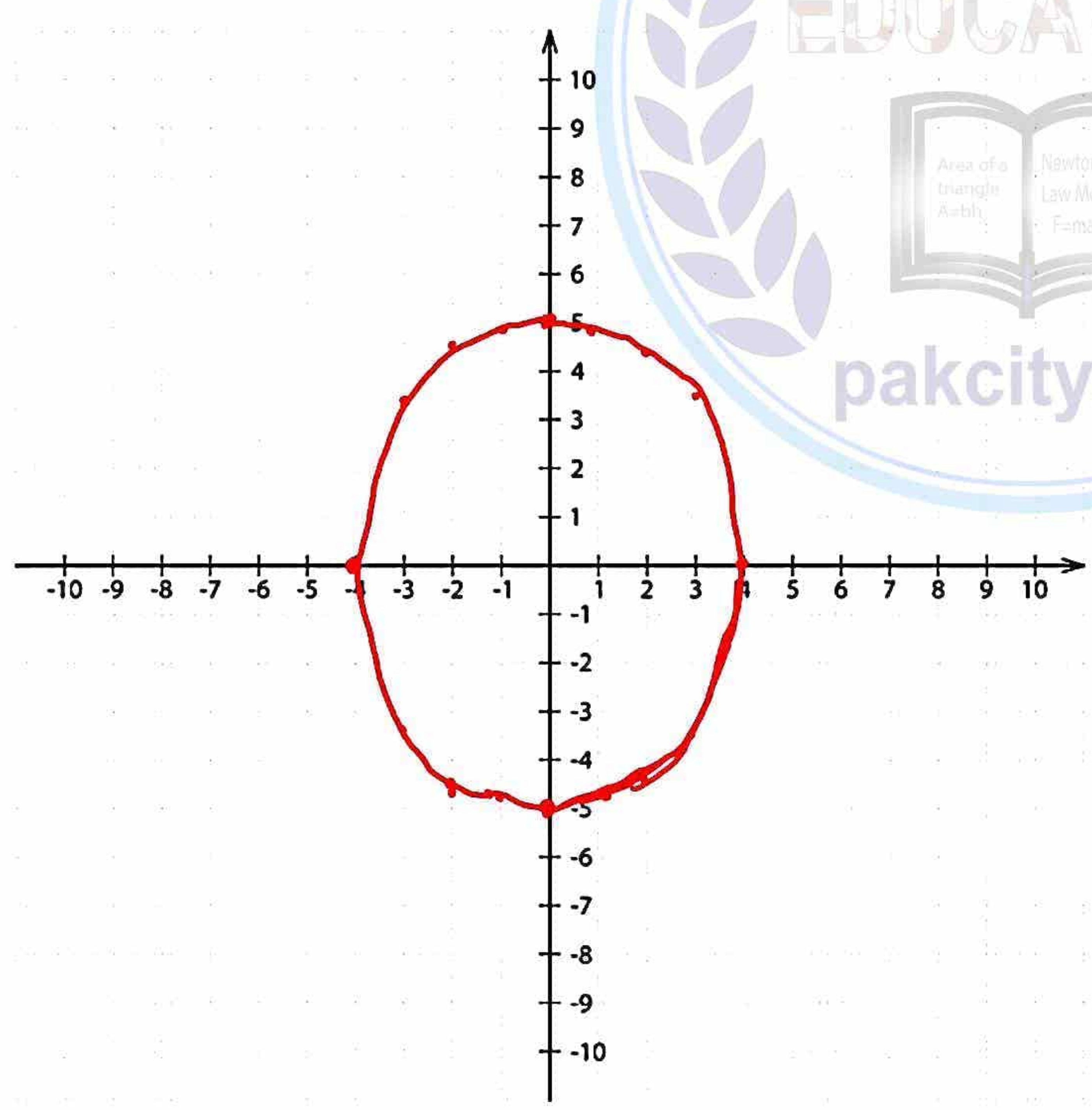


(iv) $\frac{x^2}{16} + \frac{y^2}{25} = 1 \Rightarrow \frac{y^2}{25} = 1 - \frac{x^2}{16} = \frac{16-x^2}{16}$

$y^2 = \frac{25}{16}(16 - x^2)$

$y = \pm \frac{5}{4}\sqrt{16-x^2}$

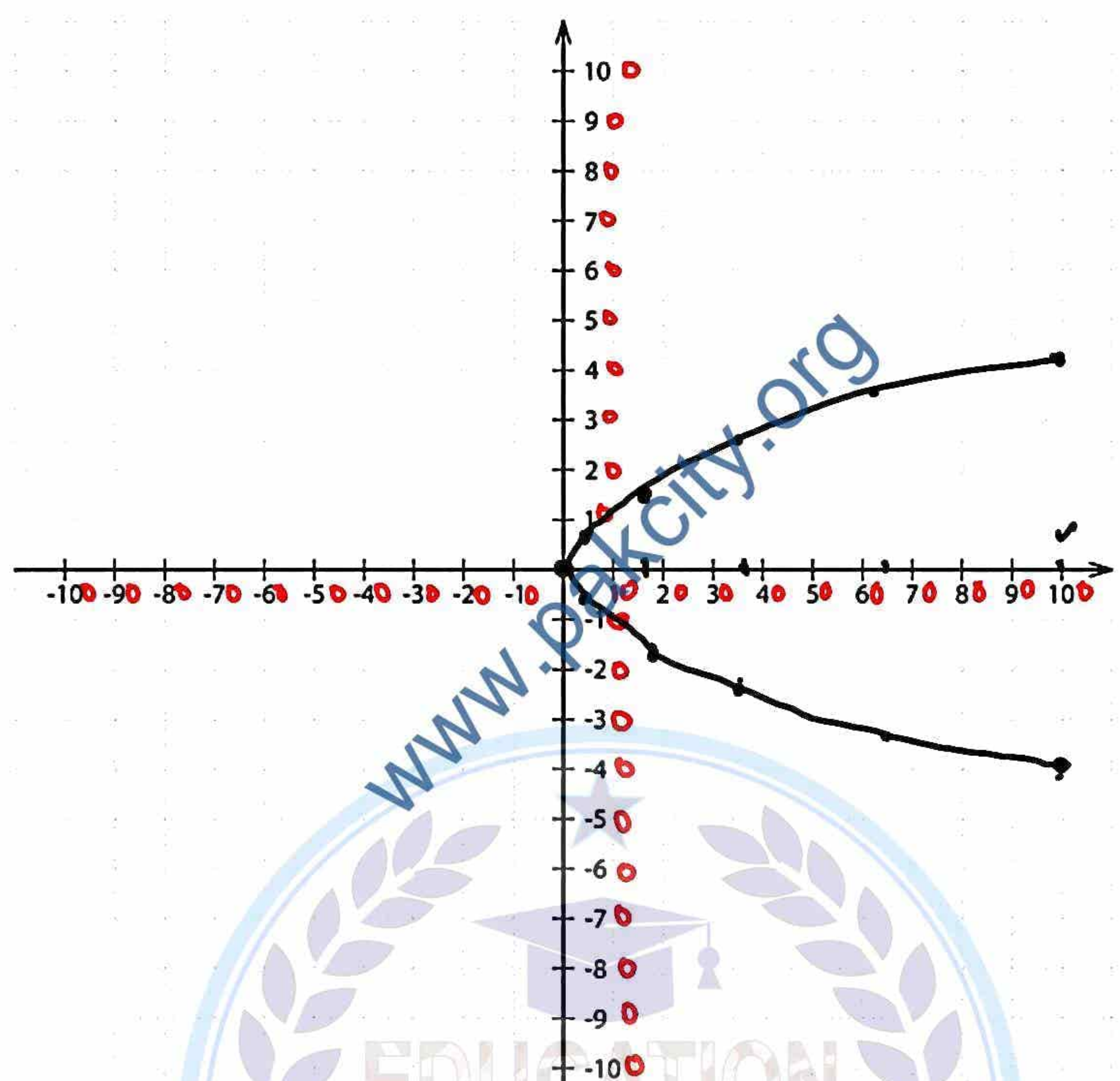
x	−4	−3	−2	−1	0	1	2	3	4
y	±0	±3.3	±4.33	±4.8	±5	±4.8	±4.33	±3.3	±0



4. Draw the graph of parametric equations of function
$x = at^2,\ y = 2at$, when $a = 4$ and $-5 \le t \le 5$

$x = 4t^2, \quad y = 8t \qquad -5 \le t \le 5$

t	-5	-4	-3	-2	-1	0	1	2	3	4	5
x	100	64	36	16	4	0	4	16	36	64	100
y	-40	-32	-24	-16	-8	0	8	16	24	32	40

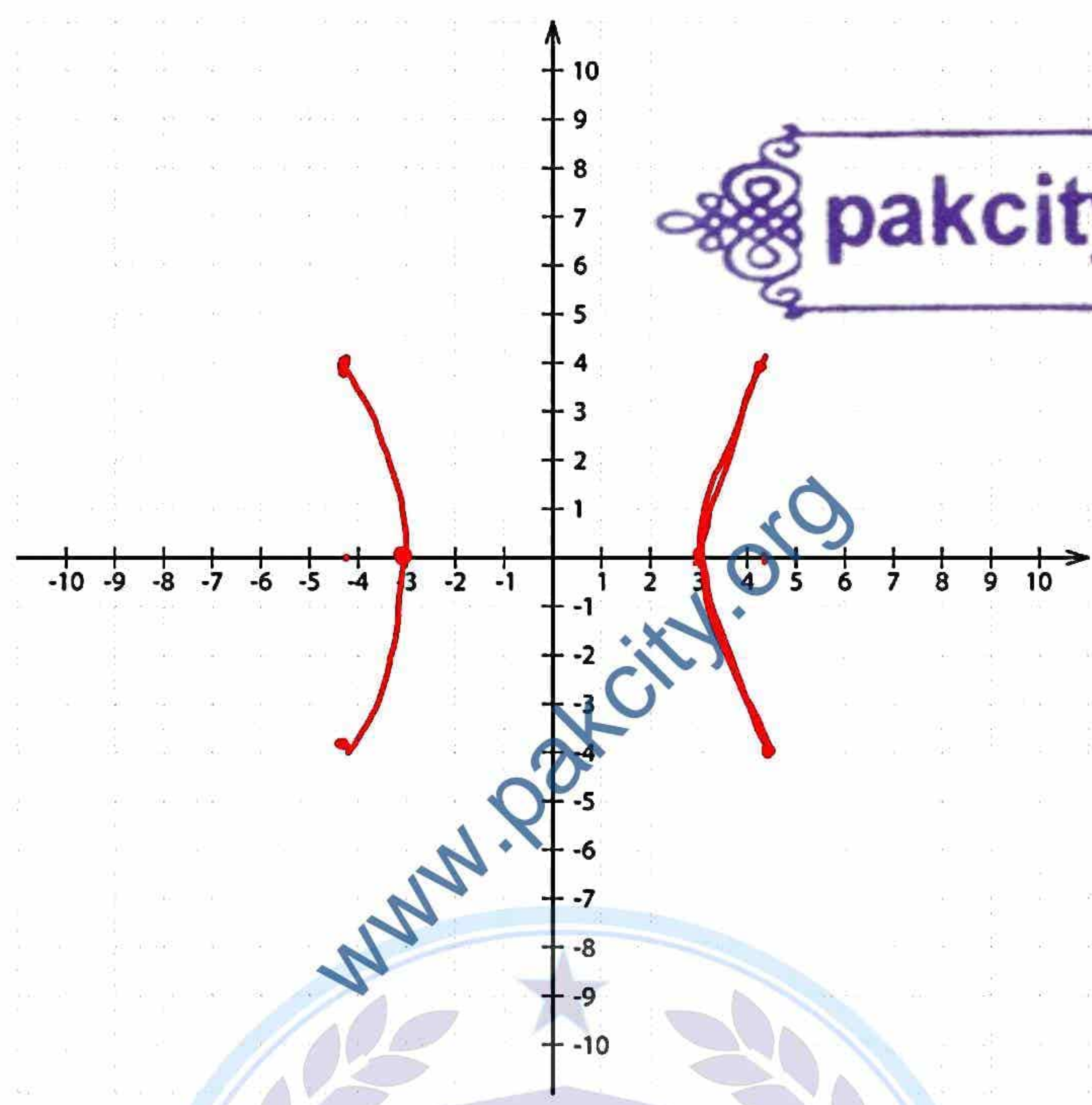


5. Draw the graph of parametric equations of function
$x = a\sec\theta,\ y = b\tan\theta$, when $a = 3, b = 4$ and $-\pi \le \theta \le \pi$

$x = 3\sec\theta$, $y = 4\tan\theta$, $-\pi \le \theta \le \pi$

$x = \frac{3}{\cos\theta}$

θ	-180°		-135	-90°	-45°	0	45°	90°	135°		180°
x	-3		-4.2	undefined.	4.2	3	4.2	undefined	-4.2		-3
y	0		4	undefined.	-4	0	4	undefined	-4		0



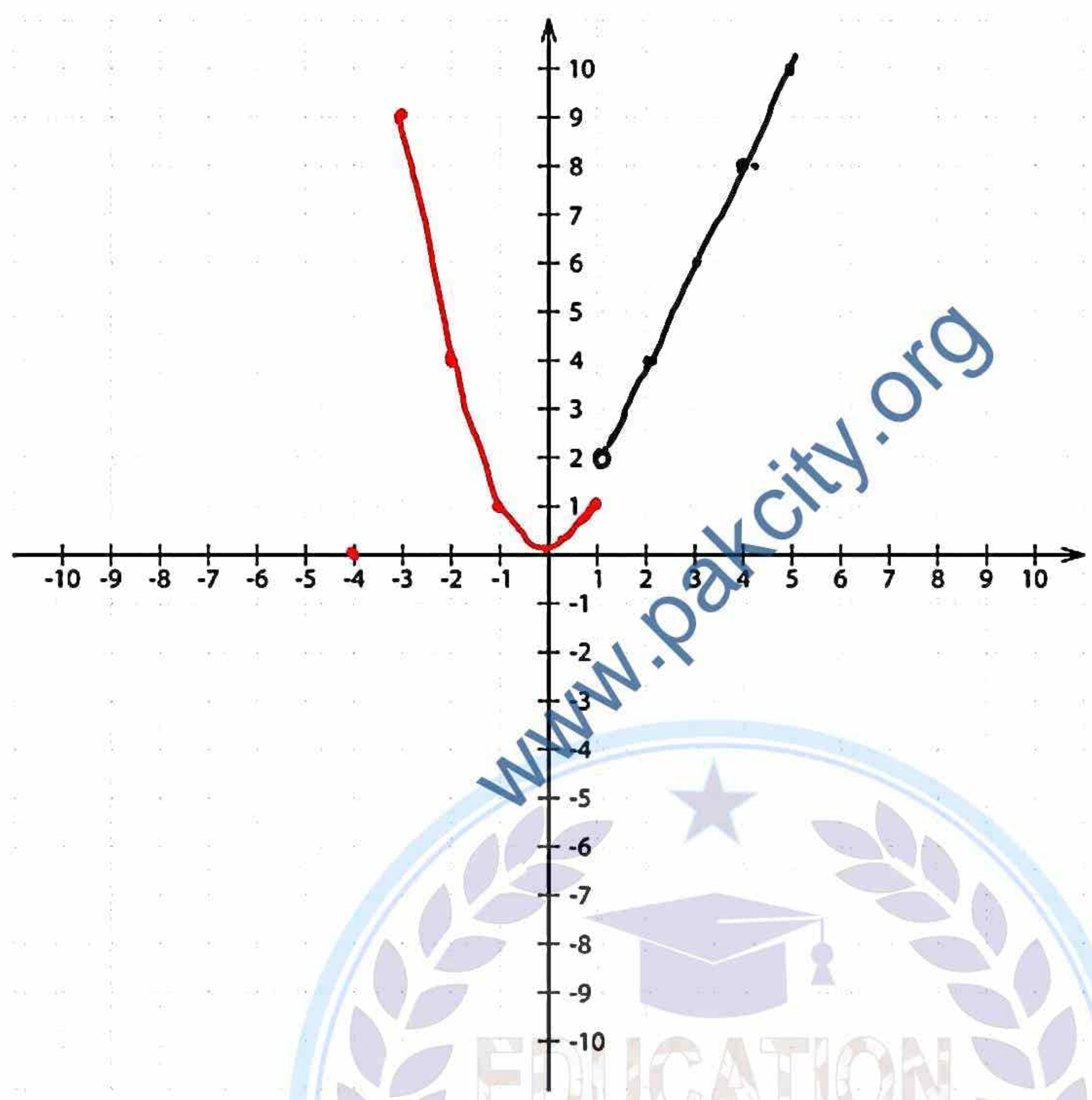
6. Draw the graph of the $f(x) = \begin{cases} x^2 & x \le 1 \\ 2x & x > 1 \end{cases}$

$f(x) = x^2, \quad x \le 1$ Quadratic (parabola)

x	-4	-3	-2	-1	0	1
f(x)	16	9	4	1	0	1

$f(x) = 2x, \quad x > 1$ Linear (Line)

x	1.0001	2	3	4	5	6
f(x)	2.0002	4	6	8	10	12



Important limits in Exercise 2.3

① $\lim_{x\to\infty} \frac{\text{number}}{x} = 0 \Rightarrow \frac{\text{number}}{\infty} = 0$ $\qquad \lim_{x\to a} f(x) = L$

② $\lim_{x\to 0} \frac{\text{number}}{x} = \infty$

③ $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$ $\qquad e \approx 2.718281\ldots$

④ $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

⑤ $\lim_{x\to 0} \frac{a^x-1}{x} = \ln a$, where $a>0$, $a\neq 1$.

⑥ $\lim_{x\to 0} \frac{(1+x)^n-1}{x} = n$, where $n\in Q$.

⑦ $\lim_{x\to 0} \frac{\sin x}{x} = 1$

L-Hospital Rule.

If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is indeterminate form $\left(\frac{0}{0}, \frac{\infty}{\infty}, 0^\infty, \infty^0, 0^0, 1^\infty\right)$ then we use L-Hospital Rule, i.e.,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

① $(k)' = 0$, $\quad (x^n)' = nx^{n-1}$, $\quad (x)' = 1$

$$\left(f(x)^n\right)' = n\, f(x)^{n-1} \cdot f'(x).$$

$a>0, a\neq 1$ $\quad (a^x)' = a^x \ln a$, $\quad (e^x)' = 1.$

$(\sin ax)' = a\cos ax$

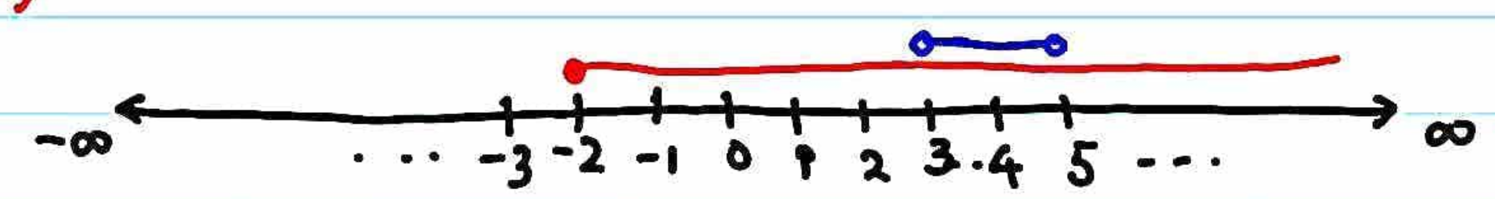
Exercise 2.3

Sindh Board.

Q # 01 Find the following:

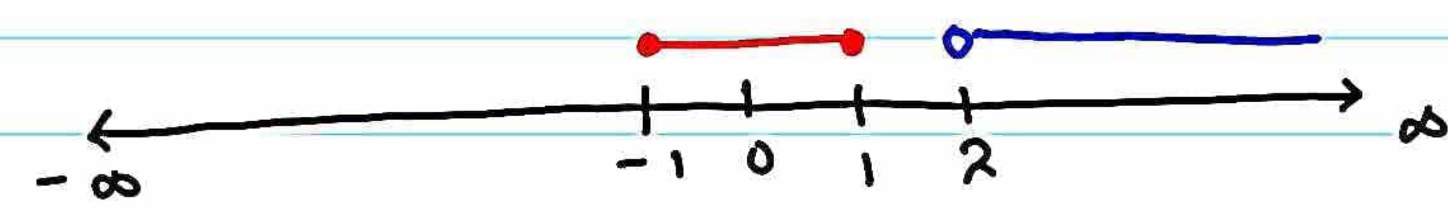
(i) $[2,\infty) \cup (3,5)$

$= [2, \infty)$



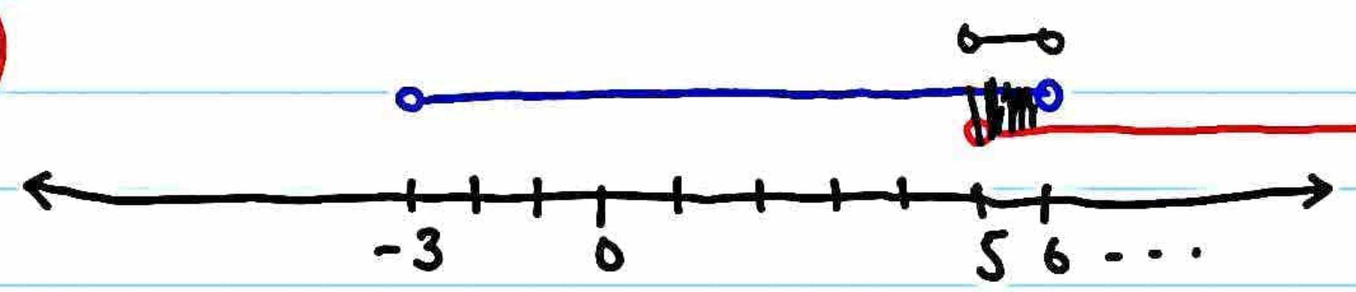
(ii) $[-1,1] - (2,\infty)$

$= [-1, 1]$



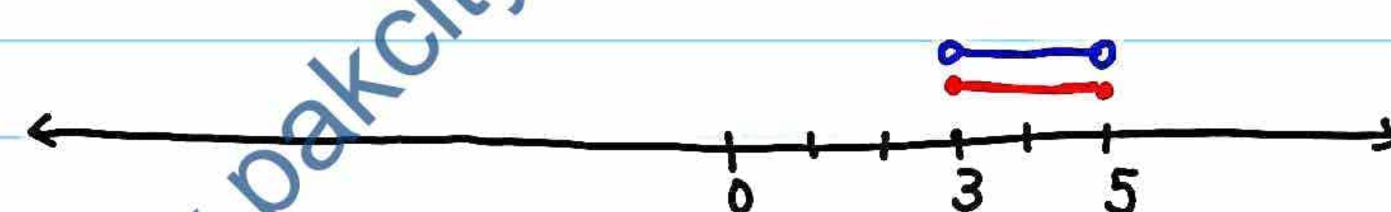
(iii) $(5,\infty) \cap (-3,6)$

$= (5,6)$



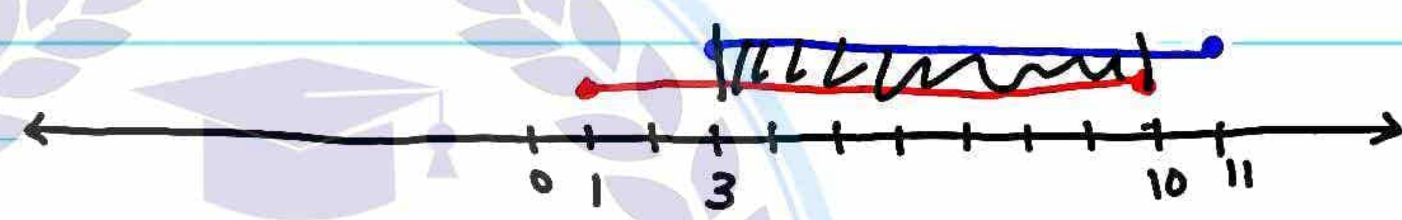
(iv) $[3,5] - (3,5)$

$= \{3, 5\}$



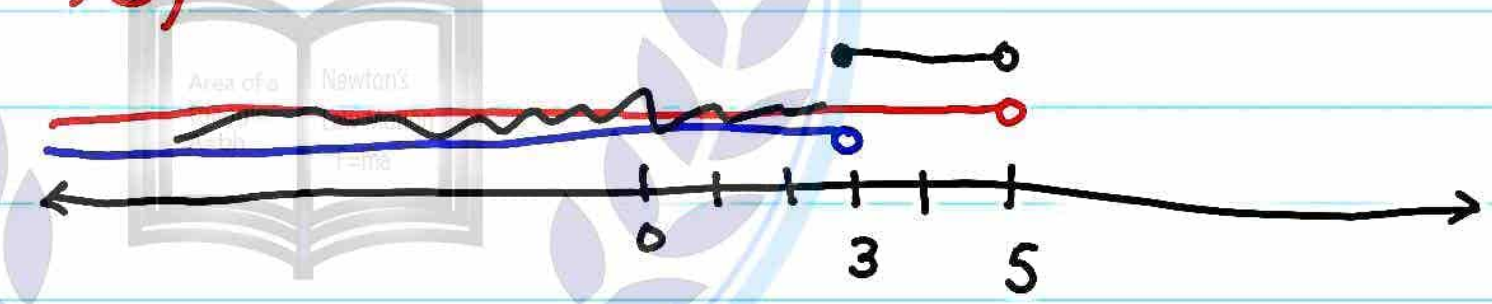
(v) $[1,10] \cap [3,11]$

$= [3, 10]$



(vi) $(-\infty,5) - (-\infty,3)$

$= [3, 5)$



Q # 02 Find the nth term and limit of the following sequences.

(i) $\frac{1}{2^0}, \frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$

$$a_n = \frac{1}{2^{n-1}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^{n-1}} = \frac{1}{2^{\infty-1}} = \frac{1}{2^{\infty}}$$

$$= \frac{1}{\infty} = 0$$ Ans.

(ii) $\frac{1\cdot2}{3\cdot4}, \frac{3\cdot4}{5\cdot6}, \frac{5\cdot6}{7\cdot8}, \ldots$

$1, 3, 5, \ldots$

$b_1 = 1$

$d = 2$

$b_n = b_1 + (n-1)d$

$= 1 + (n-1)2$

$= 1 + 2n - 2$

$= 2n - 1$

$$a_n = \frac{(2n-1)\cdot(2n)}{(2n+1)\cdot(2n+2)}$$

$$a_n = \frac{4n^2 - 2n}{4n^2 + 4n + 2n + 2}$$

$$a_n = \frac{4n^2 - 2n}{4n^2 + 6n + 2}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n^2 - 2n}{4n^2 + 6n + 2}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{4n^2 - 2n}{n^2}\right)}{\left(\frac{4n^2 + 6n + 2}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(4 - \frac{2}{n}\right)}{\left(4 + \frac{6}{n} + \frac{2}{n^2}\right)}$$

$$= \frac{4 - 0}{4 + 0 + 0} = \frac{4}{4} = 1$$

Ans.

Q # 03 Find the limit of the following sequences whose nth terms are

(i) $a_n = \dfrac{1+5n}{7n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1+5n}{7n}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1+5n}{n}\right)}{\left(\frac{7n}{n}\right)} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}+5\right)}{7}$$

$$= \frac{0+5}{7} = \frac{5}{7}$$

(ii) $a_n = \dfrac{(3n-1)(n^4-n)}{(n^2+5)(n^3-7)} = \dfrac{3n^5-3n^2-n^4+n}{n^5-7n^2+5n^3-35}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\left(\frac{3n^5-3n^2-n^4+n}{n^5}\right)}{\left(\frac{n^5-7n^2+5n^3-35}{n^5}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(3-\frac{3}{n^3}-\frac{1}{n}+\frac{1}{n^4}\right)}{\left(1-\frac{7}{n^3}+\frac{5}{n^2}-\frac{35}{n^5}\right)} = \frac{3-0-0+0}{1-0+0-0}$$

$$= \frac{3}{1} = 3 \quad \text{Ans}$$

(iii) $a_n = \dfrac{(n+1)!}{n!-(n+1)!}$

$$a_n = \frac{(n+1)\,n!}{n!-(n+1)\,n!}$$

$7! = 7\cdot 6!$

$(n+1)! = (n+1)\ n!$

$$a_n = \frac{(n+1)\,\cancel{n!}}{\cancel{n!}\,[1-(n+1)]}$$

$$a_n = \frac{n+1}{\cancel{1}-n-\cancel{1}} = \frac{n+1}{n} = 1+\frac{1}{n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1+0 = 1 \quad \text{Ans}$$

Q # 04 Find the limit of the function

$$y = \frac{5x}{x+1} \quad \text{for} \quad x \to \infty.$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{5x}{x+1}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{5\cancel{x}}{\cancel{x}}\right)}{\left(\frac{x+1}{x}\right)} = \lim_{x\to\infty} \frac{5}{\left(1+\frac{1}{x}\right)}$$

$$= \frac{5}{1+0} = \frac{5}{1} = 5$$ Ans

Q # 05 Evaluate

(i) $\lim_{x\to 2}(x^5+x^2+x+1)$ → polynomial.

$= 2^5+2^2+2+1$

$= 32+4+2+1$

$= 39.$

(ii) $\lim_{x\to 5}\left(\frac{1+x}{x^2}\right)$

$= \frac{1+5}{5^2}$

$= \frac{6}{25}.$

(iii) $\lim_{x\to 1}\left[(2x^3+3x^2)(x+1)\right]$

$= \left(2(1)^3+3(1)^2\right)(1+1)$

$= (2+3)(2) = 10.$

(iv) $\lim_{x\to 5}\left\{(x+1)-(x^2+2x+3)\right\}$

$= (5+1)-\left(5^2+2(5)+3\right)$

$= 6-(25+10+3)$

$= 6-38$

$= -32.$

(xi) $\lim_{x\to 0} \frac{17^x - 1}{x}$ $\qquad \lim_{x\to 0}\frac{a^x-1}{x} = \ln a$

$= \ln 17$

Ans.

(xii) $\lim_{h\to 0} \frac{(1+2h)^n - 1}{5h}$ $\qquad \lim_{x\to 0}\frac{(1+x)^n - 1}{x} = n.$

Let $x = 2h \Rightarrow h = \frac{x}{2}$

As $h \to 0$, $x \to 0$

$\lim_{x\to 0} \frac{(1+x)^n - 1}{5\frac{x}{2}} = \lim_{x\to 0} \frac{2[(1+x)^n - 1]}{5x}$

$= \frac{2}{5}\lim_{x\to 0}\frac{(1+x)^n - 1}{x} = \frac{2}{5}n$

Ans.

(xiii) $\lim_{x\to 0} \frac{2^x - 4^x - 8^x - 1}{x+2}$ $\qquad \frac{2^0 - 4^0 - 8^0 - 1}{0+2} = \frac{1-1-1-1}{2}$

$= \frac{2^0 - 4^0 - 8^0 - 1}{0+2}$

$= \frac{\cancel{1} - \cancel{1} - 1 - 1}{2}$

$= \frac{-2}{2} = -1$

(xiv) $\lim_{x\to 0}\left(\frac{1+7x}{1-9x}\right)^{\frac{1}{x}}$

$(\ln f)' = \frac{1}{f}\cdot f'$

Let $y = \left(\frac{1+7x}{1-9x}\right)^{\frac{1}{x}}$

$\ln y = \ln\left(\frac{1+7x}{1-9x}\right)^{\frac{1}{x}}$

$\ln y = \frac{1}{x}\ln\left(\frac{1+7x}{1-9x}\right) = \frac{1}{x}\left[\ln(1+7x) - \ln(1-9x)\right]$

$\lim_{x\to 0}\ln y = \lim_{x\to 0}\left[\frac{\ln(1+7x)}{x} - \frac{\ln(1-9x)}{x}\right]$

$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\left(\frac{1}{1+7x}\cdot 7\right)}{1} - \lim_{x\to 0}\frac{\left(\frac{1}{1-9x}\cdot -9\right)}{1}$

$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{7}{1+7x} - \lim_{x\to 0}\frac{-9}{1-9x}$

$\lim_{x\to 0}\ln y = \frac{7}{1+0} - \frac{-9}{1-0} = 7+9 = 16$

So $\lim_{x\to 0} y = e^{16}$

$\lim_{x\to 0}\left(\frac{1+7x}{1-9x}\right)^{\frac{1}{x}} = e^{16}$

Ans.

$(a^x)' = a^x \ln a$

(xv) $\lim_{x\to 0} \frac{a^x - b^x}{x}$ $\left(\frac{a^0-b^0}{0} = \frac{0}{0}\right)$

Using L-Hospital Rule

$= \lim_{x\to 0} \frac{a^x \ln a - b^x \ln b}{1}$

$= \lim_{x\to 0} a^x \ln a - b^x \ln b$

$= a^0 \ln a - b^0 \ln b = \ln a - \ln b = \ln\left(\frac{a}{b}\right).$

$(e^{kx})' = ke^{kx}$

(xvi) $\lim_{x\to 0} \frac{e^{-2x} - e^{-11x}}{x}$ $\left(\frac{e^0 - e^0}{0} = \frac{0}{0}\right)$

Using L-Hospital Rule

$= \lim_{x\to 0} \frac{-2e^{-2x} - (-11e^{-11}x)}{1}$

$= \lim_{x\to 0} \left(-2e^{-2x} + 11e^{-11}x\right) = -2e^0 + 11e^0$

$= -2 + 11 = 9$ Ans.

(xvii) $\lim_{x\to 0} \frac{3e^{5x} - 5e^{2x} + 2}{x}$ $\left(\frac{0}{0}\right)$

Use L-Hospital Rule

$= \lim_{x\to 0} \frac{3(-5e^{-5x}) - 5(-2e^{-2x}) + 0}{1}$

$= \lim_{x\to 0} -15e^{-5x} + 10e^{-2x}$

$= -15e^0 + 10e^0$

$= -15 + 10 = -5.$

$(\ln f)' = \frac{1}{f}\cdot f'$

(xviii) $\lim_{x\to 0} (1+3\tan x)^{\cot x}$

Let $y = (1+3\tan x)^{\cot x}$

$\ln y = \ln(1+3\tan x)^{\cot x}$

$\ln y = \cot x \ln(1+3\tan x)$

$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1+3\tan x)}{\tan x}$ $\left(\frac{0}{0}\right)$

$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\left(\frac{1}{1+3\tan x}\cdot 3\cancel{\sec^2 x}\right)}{\cancel{\sec^2 x}}$

$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{3}{1+3\tan x} = \frac{3}{1+0} = 3$

$\lim_{x\to 0} y = e^3$

So

$\lim_{x\to 0} (1+3\tan x)^{\cot x} = e^3$

Ans.

Q # 07

(i) $\lim_{x \to 0} \frac{a \sin ax}{x}$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

(x-radian)

$= a \lim_{x \to 0} \frac{\sin ax}{ax} \cdot a$

$= a^2 \left(\lim_{x \to 0} \frac{\sin ax}{ax} \right)$

$= a^2 (1) = a^2$ Ans.

(ii) $\lim_{x \to 0} \frac{\sin \sqrt{a}\, x}{\frac{x}{\sqrt{a}}}$

$= \lim_{x \to 0} \sqrt{a} \frac{\sin \sqrt{a}\, x}{\sqrt{a}\, x} \cdot \sqrt{a}$ $= \sqrt{a}\sqrt{a} \left(\lim_{x \to 0} \frac{\sin \sqrt{a}\, x}{\sqrt{a}\, x} \right)$

$= a \, (1)$

$= a$ Ans

(iii) $\lim_{x \to 0} (3 \cos x + 2 \tan x)^3$

$= (3 \cos 0 + 2 \tan 0)^3$

$= (3 + 0)^3$

$= 3^3 = 27$ Ans

(iv) $\lim_{x \to 0} \frac{3 \sin x - x^3}{2x}$

$= \lim_{x \to 0} \frac{3 \sin x}{2x} - \lim_{x \to 0} \frac{x^3}{2x}$

$= \frac{3}{2} \left(\lim_{x \to 0} \frac{\sin x}{x} \right) - \lim_{x \to 0} \frac{x^2}{2}$

$= \frac{3}{2} (1) - \frac{0^2}{2}$

$= \frac{3}{2}$ Ans

OR

$= \lim_{x \to 0} \frac{3 \cos x - 3x^2}{2}$

$= \frac{3 \cos 0 - 0}{2}$

$= \frac{3}{2}$ Ans

(v) $\lim_{x\to 0} \frac{\sin px}{\sin qx}$

$$= \lim_{x\to 0} \frac{\frac{\sin px}{px}\cdot p\cancel{x}}{\frac{\sin qx}{qx}\cdot q\cancel{x}}$$

$$= \frac{p}{q}\cdot \frac{\left(\lim_{x\to 0}\frac{\sin px}{px}\right)}{\left(\lim_{x\to 0}\frac{\sin qx}{qx}\right)} = \frac{p}{q}\frac{(1)}{(1)} = \frac{p}{q}.$$

OR

$$= \lim_{x\to 0} \frac{p\cos px}{q\cos qx} = \frac{p\cos 0}{q\cos 0}$$

$$= \frac{p}{q}.$$

(vi) $\lim_{x\to 0} \frac{(2\pi - x)\sec(\pi - x)}{\pi/2}$

$$= \frac{(2\pi - 0)\sec(\pi - 0)}{\frac{\pi}{2}} = \frac{2\times 2\pi(-1)}{\pi}$$

$$= -\frac{4\cancel{\pi}}{\cancel{\pi}} = -4.$$

(vii) $\lim_{x\to 0} \frac{\sin x^\circ}{x}$

$1^\circ = \frac{\pi}{180}$ rad

$x^\circ = \frac{\pi}{180}x$ rad

$$= \lim_{x\to 0} \frac{\sin\left(\frac{\pi}{180}x\right)}{\frac{\pi}{180}x}\cdot\frac{\pi}{180}$$

$$= \frac{\pi}{180}\left(\lim_{x\to 0}\frac{\sin\left(\frac{\pi}{180}x\right)}{\frac{\pi}{180}x}\right)$$

$$= \frac{\pi}{180}(1) = \frac{\pi}{180}.$$

(viii) $\lim_{x\to 0} \frac{1-\cos nx}{1-\cos mx}$ $\left(\frac{1-1}{1-1} = \frac{0}{0}\right)$

$(\cos nx)'$

$= -n\sin nx$

$$= \lim_{x\to 0} \frac{0 + n\sin nx}{0 + m\sin mx} = \lim_{x\to 0}\frac{n\sin nx}{m\sin mx} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{n\,n\cos nx}{m\,m\cos mx} = \frac{n^2\cos 0}{m^2\cos 0} = \frac{n^2}{m^2}$$ Ans

(ix) $\lim_{x \to 0} \frac{\sin 3x \sin 5x}{7x^2}$

$= \lim_{x \to 0} \frac{\sin 3x}{7x} \cdot \lim_{x \to 0} \frac{\sin 5x}{x}$

$= \frac{1}{7}\left(\lim_{x \to 0} \frac{\sin 3x}{3x}\right)3 \cdot \left(\lim_{x \to 0} \frac{\sin 5x}{5x}\right) \cdot 5$

$= \frac{1}{7}(1)\,3 \cdot (1) \cdot 5$

$= \frac{15}{7}$ Ans

Q # 08 (i) $\lim_{x\to 0} \frac{\sin^2\left(\frac{x}{2}\right)}{4x^2}$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ (x - radian)

$$= \frac{1}{4}\lim_{x\to 0} \frac{\left(\sin\frac{x}{2}\right)^2}{\frac{x^2}{4}\times 4}$$

$$= \frac{1}{16}\lim_{x\to 0} \frac{\left(\sin\frac{x}{2}\right)^2}{\left(\frac{x}{2}\right)^2} = \frac{1}{16}\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$= \frac{1}{16}\left[\lim_{x\to 0}\frac{\sin x/2}{x/2}\right]^2 = \frac{1}{16}(1)^2 = \frac{1}{16}.$$

(ii) $\lim_{x\to\infty} x\sin\frac{1}{x}$

$$= \lim_{x\to\infty}\frac{\left(\sin\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$$

Let $\frac{1}{x} = \theta$

As $x\to\infty$, $\theta\to 0$

$$= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}$$

$$= 1.$$

(iii) $\lim_{x\to\infty}\left[\sqrt{x^2+x+1} - x\right]$ $(\infty - \infty)$

$$= \lim_{x\to\infty} \sqrt{x^2+x+1} - x \times \frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x} = \lim_{x\to\infty}\frac{\left(\sqrt{x^2+x+1}\right)^{\cancel{2}} - x^2}{\sqrt{x^2+x+1}+x}$$

$$= \lim_{x\to\infty}\frac{\cancel{x^2}+x+1-\cancel{x^2}}{\sqrt{x^2+x+1}+x} = \lim_{x\to\infty}\frac{x+1}{\sqrt{x^2+x+1}+x}$$

$$= \lim_{x\to\infty}\frac{x\left(1+\frac{1}{x}\right)}{x\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+x} = \lim_{x\to\infty}\frac{\cancel{x}\left(1+\frac{1}{x}\right)}{\cancel{x}\left[\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1\right]}$$

$$= \frac{1+0}{\sqrt{1+0+1}+1} = \frac{1}{1+1} = \frac{1}{2} \quad \text{Ans.}$$

$(\cos^3 x)'$

$= 3\cos^2 x\cdot(-\sin x)$

$(\sin^2 x)'$

$= 2\sin x\cdot\cos x$

(iv) $\lim_{x\to 0}\frac{1-\cos^3 x}{\sin^2 x}$ $\left(\frac{1-\cos^3 0}{\sin^2 0} = \frac{0}{0}\right)$

Use L-Hospital Rule

$$= \lim_{x\to 0}\frac{0 - 3\cos^2 x(-\sin x)}{2\sin x\cos x}$$

$$= \lim_{x\to 0}\frac{3\cancel{\sin x}\cos^{\cancel{2}} x}{2\cancel{\sin x}\cancel{\cos x}} = \lim_{x\to 0}\frac{3}{2}\cos x = \frac{3}{2}\cos 0$$

$$= \frac{3}{2} \quad \text{Ans.}$$

(v) $\lim_{x\to 0} \frac{a^x + a^{-x} - 2}{x^2}$ $\left(\frac{a^0+a^0-2}{0} = \frac{0}{0}\right)$

Use L-Hospital Rule

$= \lim_{x\to 0} \frac{a^x \ln a + (-a^{-x}\ln a) - 0}{2x} = \lim_{x\to 0} \frac{a^x \ln a - a^{-x}\ln a}{2x}$ $\left(\frac{0}{0}\right)$

Use L-Hospital Rule

$= \lim_{x\to 0} \frac{a^x \ln a . \ln a - (-a^{-x}\ln a)\ln a}{2} = \lim_{x\to 0} \frac{a^x(\ln a)^2 + a^{-x}(\ln a)^2}{2}$

$= \frac{a^0(\ln 2)^2 + a^0(\ln 2)^2}{2} = \frac{\cancel{2}(\ln 2)^2}{\cancel{2}} = (\ln 2)^2$ Ans.

(vi) $\lim_{x\to 0} \frac{6^x - 3^x - 2^x + 1}{x^2}$

Use L-Hospital Rule $= \lim_{x\to 0} \frac{6^x \ln 6 - 3^x \ln 3 - 2^x \ln 2}{2x}$ $\left(\frac{\ln 6 - \ln 3 - \ln 2}{0} = \frac{\ln 6 - (\ln 3 + \ln 2)}{0} = \frac{\ln 6 - \ln 6}{0} = \frac{0}{0}\right)$

Use L-Hospital Rule $= \lim_{x\to 0} \frac{6^x \ln 6 . \ln 6 - 3^x \ln 3 . \ln 3 - 2^x \ln 2 . \ln 2}{2}$

$= \frac{6^0(\ln 6)^2 - 3^0(\ln 3)^2 - 2^0(\ln 2)^2}{2} = \frac{(\ln 6)^2 - (\ln 3)^2 - (\ln 2)^2}{2}$

(vii) $\lim_{x\to 3} \frac{\frac{x}{x+2} - \frac{3}{5}}{x-3}$ $\left(\frac{\frac{3}{5} - \frac{3}{5}}{3-3} = \frac{0}{0}\right)$

$= \lim_{x\to 3} \frac{\left(\frac{5x - 3x - 6}{5(x+2)}\right)}{x-3} = \lim_{x\to 3} \frac{2x-6}{5(x+2)(x-3)}$

$= \lim_{x\to 3} \frac{2\cancel{(x-3)}}{5(x+2)\cancel{(x-3)}} = \frac{2}{5(3+2)} = \frac{2}{5(5)}$

$= \frac{2}{25}$ Ans.

(viii) $\lim_{y\to 4} \frac{y^{\frac{5}{2}} - 16y^{\frac{1}{2}}}{y-4}$ $\left(\frac{0}{0}\right)$

Use L-Hospital Rule

$= \lim_{y\to 4} \frac{\frac{5}{2}y^{\frac{5}{2}-1} - 16\left(\frac{1}{2}y^{\frac{1}{2}-1}\right)}{1-0} = \lim_{y\to 4}\left(\frac{5}{2}y^{\frac{3}{2}} - 8y^{-\frac{1}{2}}\right)$

$= \frac{5}{2}(4)^{\frac{3}{2}} - 8(4)^{-\frac{1}{2}} = \frac{5}{2}(2^2)^{\frac{3}{2}} - 8(2^2)^{-\frac{1}{2}} = \frac{5}{\cancel{2}}(\cancel{8}^{4}) - \frac{\cancel{8}^{4}}{\cancel{2}}$

$= 20 - 4 = 16$ Ans.

(ix) $\lim_{x\to 1} \dfrac{\frac{1}{\sqrt{x}} - 1}{1-x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 1} \dfrac{x^{-1/2} - 1}{1-x}$

Use L-Hospital Rule

$= \lim_{x\to 1} \dfrac{-\frac{1}{2}x^{-\frac{1}{2}-1} - 0}{0-1} = \lim_{x\to 1} \frac{1}{2} x^{-3/2} = \frac{1}{2}(1)^{-3/2} = \frac{1}{2}$ Ans

(x) $\lim_{x\to \pi} \dfrac{\sqrt{5+\cos x} - 2}{\pi - x}$ $\left(\dfrac{\sqrt{5-1} - 2}{\pi-\pi} = \frac{0}{0}\right)$

$= \lim_{x\to\pi} \dfrac{\frac{1}{2}(5+\cos x)^{\frac{1}{2}-1}(0-\sin x) - 0}{0-1}$

$= \lim_{x\to\pi} -\frac{1}{2}(5+\cos x)^{-\frac{1}{2}}(-\sin x) = -\frac{1}{2}(5-1)^{-\frac{1}{2}}(-\sin\pi)$

$= -\frac{1}{2}(2^2)^{-1/2}(0) = 0$

(xi) $\lim_{x\to 1} x^{\frac{1}{x-1}}$

Let $y = x^{\frac{1}{x-1}}$

$\ln y = \ln x^{\frac{1}{x-1}} = \frac{1}{x-1}\ln x$

$\lim_{x\to 1} \ln y = \lim_{x\to 1} \dfrac{\ln x}{x-1}$ $\left(\frac{0}{0}\right)$

$\lim_{x\to 1} \ln y = \lim_{x\to 1} \dfrac{1/x}{1-0}$

$= \lim_{x\to 1} \frac{1}{x}$

$\lim_{x\to 1} \ln y = \frac{1}{1} = 1$

$\lim_{x\to 1} y = e^1$

So

$\lim_{x\to 1} x^{\frac{1}{x-1}} = e$

(xii) $\lim_{x\to 1} \dfrac{x^2 - \sqrt{x}}{\sqrt{x} - 1}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 1} \dfrac{2x - \frac{1}{2}x^{\frac{1}{2}-1}}{\frac{1}{2}x^{\frac{1}{2}-1} - 0} = \lim_{x\to 1} \dfrac{2x - \frac{1}{2}x^{-\frac{1}{2}}}{2\, x^{-\frac{1}{2}}}$

$= \dfrac{2(1) - \frac{1}{2}(1)^{-1/2}}{2(1)^{-1/2}} = \dfrac{2 - \frac{1}{2}}{2} = \dfrac{\left(\frac{4-1}{2}\right)}{2}$

$= \frac{3}{4}$ Ans.

(xiii) $\lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{(4x-\pi)^2}$ $\left(\frac{0}{0}\right)$

$= \lim_{x \to \frac{\pi}{4}} \frac{0 + \sin x - \cos x}{2(4x-\pi)4} = \lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{8(4x-\pi)}$ $\left(\frac{0}{0}\right)$

$= \lim_{x \to \frac{\pi}{4}} \frac{\cos x + \sin x}{8(4)} = \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{32} = \frac{1}{\cancel{32}_{16}}\left(\frac{\cancel{2}}{\sqrt{2}}\right)$

$= \frac{1}{16\sqrt{2}}$ Ans.

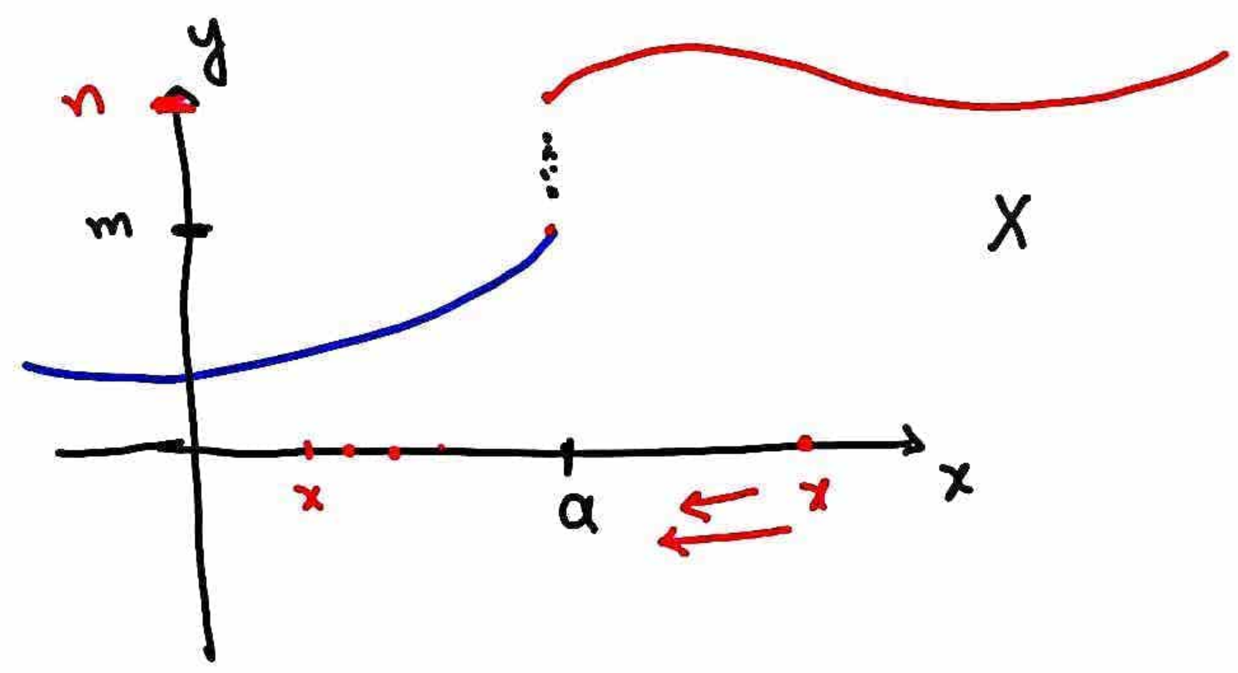
(xiv) $\lim_{x \to e} \frac{\ln x - 1}{x - e}$ $\left(\frac{0}{0}\right)$

$= \lim_{x \to e} \frac{1/x - 0}{1 - 0}$

$= \lim_{x \to e} \frac{1}{x} = \frac{1}{e}$ Ans.

Important Points for Exercise 1.4

One Sided Limits



$$\lim_{x \to a^-} f(x) = m$$

$(x < a)$ (Left-hand limit)

$$\lim_{x \to a^+} f(x) = n$$

$(x > a)$ (Right-hand limit)

$\lim_{x \to a} f(x)$ exists if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

Continuous Function

A function $f(x)$ is continuous at $x = a$ if

(i) $f(a)$ is defined. $(a \in Dom(f))$.

(ii) $\lim_{x \to a} f(x)$ exists. $\left(\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)\right)$.

(iii) $\lim_{x \to a} f(x) = f(a)$.

Exercise 2.4

1. Evaluate the following limits.

(i) $\lim_{x\to 2^+}\frac{x-2}{|x-2|}$ (ii) $\lim_{x\to 1^-}\frac{x^2+2x-3}{|x-1|}$ (iii) $\lim_{x\to 2}\frac{x^2+4x-12}{|x-2|}$

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$|x-2| = \begin{cases} x-2 & \text{if } x-2 \ge 0 \\ -(x-2) & \text{if } x-2 < 0 \end{cases}$$

$$= \begin{cases} x-2 & \text{if } x \ge 2 & \text{R.H.L} \\ -x+2 & \text{if } x < 2 & \text{L.H.L} \end{cases}$$

$$|x-1| = \begin{cases} x-1 & \text{if } x-1 \ge 0 \\ -(x-1) & \text{if } x-1 < 0 \end{cases}$$

$$= \begin{cases} x-1 & \text{if } x \ge 1 \\ -x+1 & \text{if } x < 1 \end{cases}$$

(i) $\lim_{x\to 2^+}\frac{x-2}{|x-2|}$

$x>2$

$$= \lim_{x\to 2}\frac{\cancel{x-2}}{\cancel{x-2}} = \lim_{x\to 2}(1)$$

$$= 1 \quad \text{Ans}$$

(ii) $\lim_{x\to 1^-}\frac{x^2+2x-3}{|x-1|}$

$x<1$

$$= \lim_{x\to 1}\frac{x^2+2x-3}{-x+1}$$

$$= \lim_{x\to 1}\frac{x^2+3x-x-3}{-(x-1)} = \lim_{x\to 1}\frac{x(x+3)-1(x+3)}{-(x-1)}$$

$$= \lim_{x\to 1}\frac{\cancel{(x-1)}(x+3)}{-\cancel{(x-1)}} = -\lim_{x\to 1}(x+3)$$

$$= -(1+3) = -4 \quad \text{Ans}$$

(iii) $\lim_{x\to 2}\frac{x^2+4x-12}{|x-2|}$

L.H.L

$$\lim_{x\to 2^-}\frac{x^2+4x-12}{|x-2|}$$

$$= \lim_{x\to 2}\frac{x^2+4x-12}{-x+2} = \lim_{x\to 2}\frac{x^2+6x-2x-12}{-(x-2)}$$

$$= \lim_{x\to 2}\frac{x(x+6)-2(x+6)}{-(x-2)} = \lim_{x\to 2}\frac{\cancel{(x-2)}(x+6)}{-\cancel{(x-2)}}$$

$$= -(2+6) = -8$$

R.H.L

$$\lim_{x\to 2^+}\frac{x^2+4x-12}{|x-2|}$$

$$= \lim_{x\to 2}\frac{x^2+4x-12}{x-2}$$

$$= \lim_{x\to 2}\frac{\cancel{(x-2)}(x+6)}{\cancel{x-2}}$$

$$= 2+6 = 8$$

Since L.H.L $\ne$ R.H.L so $\lim_{x\to 2}\frac{x^2+4x-12}{|x-2|}$ does not exist.

2. Determine whether $\lim_{x\to1} f(x)$, $\lim_{x\to2} f(x)$, $\lim_{x\to3} f(x)$ and $\lim_{x\to4} f(x)$ exit, when

$$f(x)=\begin{cases}2x+1 & if\ 0\le x\le 2\\ x-7 & if\ 2\le x\le 4\\ x & if\ 4\le x\le 6\end{cases}$$

At $x=1$

L.H.L $\lim_{x\to1^-} f(x) = \lim_{x\to1}(2x+1) = 2(1)+1 = 2+1 = 3$

R.H.L $\lim_{x\to1^+} f(x) = \lim_{x\to1}(2x+1) = 2(1)+1 = 2+1 = 3$

So $\lim_{x\to1} f(x)$ exists.

At $x=2$

L.H.L

$\lim_{x\to2^-} f(x) = \lim_{x\to2}(2x+1) = 2(2)+1 = 5$ ✓

R.H.L

$\lim_{x\to2^+} f(x) = \lim_{x\to2}(x-7) = 2-7 = -5$ ✓

Since L.H.L $\neq$ R.H.L, so

$\lim_{x\to2} f(x)$ does not exist.

At $x=3$

L.H.L

$\lim_{x\to3^-} f(x) = \lim_{x\to3}(x-7) = 3-7 = -4$

R.H.L

$\lim_{x\to3^+} f(x) = \lim_{x\to3}(x-7) = 3-7 = -4$

Since L.H.L = R.H.L, so

$\lim_{x\to3} f(x)$ exists.

At $x=4$

L.H.L

$\lim_{x\to4^-} f(x) = \lim_{x\to4}(x-7) = 4-7 = -3$

R.H.L

$\lim_{x\to4^+} f(x) = \lim_{x\to4}(x) = 4$

Since L.H.L $\neq$ R.H.L, so

$\lim_{x\to4} f(x)$ does not exist.

3. Test the continuity and discontinuity of the following functions.

(i) $f(x) = \sin(x^2 + \pi x) + 7x^2 + x$ at a point $x = 0$

(ii) $f(x) = \frac{2-\cos 3x-\cos 4x}{x}$ at a point $x = 0$

(iii) $f(x) = \begin{cases} 7 + 3x, & \text{when } x < 1 \\ 1 - 5x, & \text{when } x \geq 1 \end{cases}$ at $x = 1$

(i) $f(x) = \sin(x^2 + \pi x) + 7x^2 + x$ at $x = 0.$

Value $f(0) = \sin(0^2 + 0) + 0 + 0 = 0$

L.H.L $\lim_{x\to 0^-} f(x) = \lim_{x\to 0} \sin(x^2+\pi x)+7x^2+x = \sin(0+0)+0+0 = 0$

R.H.L $\lim_{x\to 0^+} f(x) = \lim_{x\to 0} \sin(x^2+\pi x)+7x^2+x = \sin(0+0)+0+0 = 0$

So $\lim_{x\to 0} f(x) = 0$

Since $f(0) = \lim_{x\to 0} f(x)$, so function is continuous at $x=0.$

(ii) $f(x) = \frac{2-\cos 3x-\cos 4x}{x}$ at $x=0$

Value $f(0) = \frac{2-\cos 0-\cos 0}{0} = \frac{2-1-1}{0} = \frac{0}{0}$ (indeterminate form)

$\Rightarrow f$ is not defined at $x=0.$

So $f(x)$ is discontinuous at $x=0.$

(iii) $f(x) = \begin{cases} 7+3x, & x<1 \\ 1-5x, & x\geq 1 \end{cases}$ at $x=1.$

Value $f(1) = 1-5(1) = 1-5 = -4$

L.H.L $\lim_{x\to 1^-} f(x) = \lim_{x\to 1} (7+3x) = 7+3(1) = 7+3 = 10$

R.H.L

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} (1-5x) = 1-5(1) = 1-5 = -4$

Since L.H.L $\neq$ R.H.L, so $\lim_{x\to 1} f(x)$ does not exist.

So $f(x)$ is discontinuous at $x=1.$

4. Determine whether the following function are continuous at $x = 2$

(i) $f(x) = \frac{x^2-4}{x-2}$ (ii) $g(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{when } x \neq 2 \\ 3 & \text{when } x = 2 \end{cases}$

(iii) $h(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{when } x \neq 2 \\ 4 & \text{when } x = 2 \end{cases}$

(i) $f(x) = \frac{x^2-4}{x-2}$

Value $f(2) = \frac{2^2-4}{2-2} = \frac{4-4}{2-2} = \frac{0}{0}$

$\Rightarrow$ $f(2)$ is not defined.

So $f(x)$ is discontinuous at $x = 2$.

(ii) $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{when } x \neq 2 \quad (x<2,\ x>2) \\ 3 & \text{when } x = 2 \end{cases}$

Value $f(2) = 3$

L.H.L $\lim_{x\to 2^-} f(x) = \lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to 2}\left(\frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}\right) = 2+2 = 4.$

R.H.L $\lim_{x\to 2^+} f(x) = \lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right) = 4$

Since L.H.L = R.H.L, so $\lim_{x\to 2} f(x) = 4.$

Since $f(2) \neq \lim_{x\to 2} f(x)$, so $f(x)$ is discontinuous at $x = 2.$

(iii) $f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$

Value $f(2) = 4$

$\lim_{x\to 2} f(x) = \lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to 2}\left(\frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}\right) = 2+2 = 4$

Also $f(2) = \lim_{x\to 2} f(x)$

So $f(x)$ is continuous at $x = 2.$

5. Suppose that $f(x) = \begin{cases} -x^4 + 3 & \text{when } x \le 2 \\ x^2 + 9 & \text{when } x > 2 \end{cases}$

Is continuous everywhere justify your conclusion?

$$f(x) = \begin{cases} -x^4 + 3 & \text{when } x \le 2 \\ x^2 + 9 & \text{when } x > 2 \end{cases}$$

Since for $x < 2$, $f(x) = -x^4 + 3$ (polynomial)

Since polynomials are continuous on their domain, so $f(x)$ is continuous on $x < 2$.

Since for $x > 2$, $f(x) = x^2 + 9$ (polynomial)

So $f(x)$ is continuous on $x > 2$.

At $\underline{x = 2}$

$\underline{L.H.L}$ $\quad \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2} (-x^4 + 3) = -2^4 + 3 = -16 + 3 = -13$

$\underline{R.H.L}$ $\quad \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2} (x^2 + 9) = 2^2 + 9 = 4 + 9 = 13$

Since L.H.L $\neq$ R.H.L, so $\lim\limits_{x \to 2} f(x)$ does not exist.

So $f(x)$ is discontinuous at $x = 2$.

6. Find the value of k if $f(x)=\begin{cases}\frac{\sin kx}{x} & , x\neq 0\\ 2 & , x=0\end{cases}$ is continuous at $x=0$.

If $f(x)$ is continuous at $x=0$

$$\Rightarrow \lim_{x\to 0} f(x) = f(0).$$

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x}=1}$$

$$\lim_{x\to 0}\frac{\sin kx}{x}=2$$

$$\left(\lim_{x\to 0}\frac{\sin kx}{kx}\right)\cdot k = 2$$

$$1\cdot k = 2$$

$$k=2$$

7. Find the value of k if $f(x)=\begin{cases}kx-9 & x<5\\ 9x-k & x>5\\ 36 & x=5\end{cases}$ is continuous at $x=5$.

If $f(x)$ is continuous at $x=5$.

$$\Rightarrow \lim_{x\to 5} f(x) = f(5)$$

$$\Rightarrow \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5)$$

$$\lim_{x\to 5}(kx-9) = \lim_{x\to 5}(9x-k) = 36$$

$$k(5)-9 = 9(5)-k = 36$$

$$5k-9 = 45-k = 36$$

$5k-9=36$

$5k=36+9=45$

$k=\frac{45}{5}$

$k=9$

$45-k=36$

$45-36=k$

$9=k$

8. Find the values of m and n, so that given function f is continuous at $x = 3$.

(i) $f(x) = \begin{cases} mx & if\ x < 3 \\ n & if\ x = 3 \\ -2x+9 & if\ x > 3 \end{cases}$ (ii) $f(x) = \begin{cases} mx & if\ x < 3 \\ x^2 & if\ x \geq 3 \end{cases}$

(i) $f(x) = \begin{cases} mx & \text{if} & x<3 \\ n & \text{if} & x=3 \\ -2x+9 & \text{if} & x>3 \end{cases}$

Since $f(x)$ is continuous at $x=3$, so

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$$

$$\lim_{x\to 3} (mx) = \lim_{x\to 3}(-2x+9) = n$$

$$3m = 3 = n$$

$3m = 3$

$\boxed{m=1}$ $\boxed{3=n}$

(ii) $f(x) = \begin{cases} mx & \text{if} & x<3 \\ x^2 & \text{if} & x\geq 3 \end{cases}$

Since $f(x)$ is continuous at $x=3$,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$$

$$\lim_{x\to 3} (mx) = \lim_{x\to 3}(x^2) = 3^2$$

$$3m = 9 = 9$$

$3m = 9$

$\boxed{m=3}$

9. If $f(x) = \begin{cases} \dfrac{\sqrt{2x+5}-\sqrt{x+7}}{2} & , \ x \neq 2 \\ k & , \ x = 2 \end{cases}$

Find the value of k so that f is continuous at $x = 2$.

Since $f(x)$ is continuous at $x = 2$, so

$$f(2) = \lim_{x \to 2} f(x)$$

$$k = \lim_{x \to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{2}$$

$$k = \lim_{x \to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{2} = \frac{\sqrt{2(2)+5} - \sqrt{2+7}}{2}$$

$$k = \frac{3-3}{2} = 0$$