MATHEMATICS 1st YEAR

UNIT # 06

SEQUENCE & SERIES

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Contents

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اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Sequence:- A sequence is a function whose domain is subset of natural numbers.

e.g., 2, 4, 6, 8, ...

1, -1, 1, -1, 1, -1, ... and

a, (a+d), (a+2d), ... etc

Real sequence:- If all members of a sequence are real numbers, then it is called a real sequence.

Finite sequence:- If the domain of a sequence is a finite set, then the sequence is called finite sequence.

Infinite sequence:- If the domain of a sequence is an infinite set, then the sequence is called infinite sequence.

Note:- *The numbers $a_1, a_2, a_3, \dots, a_n$ are called terms of a sequence and here $a_1 =$ first term

$a_2 = 2^{nd}$ term, ... , $a_n =$ nth term

* The last term of the sequence is called its nth term or general term denoted by a_n.
* Sequences are also called progressions.
* An infinite sequence has no last term.
* We can find a sequence by putting $n = 1, 2, 3, \dots$ in the nth or general term.

Example 1. Write first two, 21st and 26th terms of the sequences whose general term is $(-1)^{n+1}$.

Solution:- Here

$a_n = (-1)^{n+1}$

Put $n = 1, 2, 21$ and 26

$a_1 = (-1)^{1+1} = (-1)^2 = 1$

$a_2 = (-1)^{2+1} = (-1)^3 = -1$

$a_{21} = (-1)^{21+1} = (-1)^{22} = 1$

$a_{26} = (-1)^{26+1} = (-1)^{27} = -1$

Example 2. Find the sequence if $a_n - a_{n-1} = n+1$ and $a_4 = 14$

Solution:- Given that

$a_n - a_{n-1} = n+1 \longrightarrow (i)$

Put $n = 2$ in (i)

$a_2 - a_1 = 2+1$

$\rightarrow a_2 - a_1 = 3 \longrightarrow (ii)$

Put $n = 3$ in (i)

$a_3 - a_2 = 3+1$

$a_3 - a_2 = 4 \longrightarrow (iii)$

Put $n = 4$ in (i)

$a_4 - a_3 = 4+1$

$a_4 - a_3 = 5 \longrightarrow (iv)$

From (iv) $\rightarrow a_3 = a_4 - 5$ $\quad \because a_4 = 14$

$a_3 = 14 - 5 = 9$

From (iii) $\rightarrow a_2 = a_3 - 4$

$a_2 = 9 - 4 = 5$ $\quad \because a_3 = 9$

From (ii) $a_1 = a_2 - 3$

$a_1 = 5 - 3 = 2$ $\quad \because a_2 = 5$

Thus 2, 5, 9, 14, . . . req. sequence

Exercise 6.1

Q1. Write the first four terms of the following sequences, if

(i) $a_n = 2n - 3$

Solution:- $a_n = 2n - 3 \longrightarrow (i)$

Put $n = 1, 2, 3, 4$ in (i)

$a_1 = 2(1) - 3 = 2 - 3 = -1$

$a_2 = 2(2) - 3 = 4 - 3 = 1$

$a_3 = 2(3) - 3 = 6 - 3 = 3$

$a_4 = 2(4) - 3 = 8 - 3 = 5$

First four terms are -1, 1, 3, 5

(ii) $a_n = (-1)^n n^2$

Solution:- $a_n = (-1)^n n^2 \longrightarrow (i)$

Put $n = 1, 2, 3, 4$ in (i)

$a_1 = (-1)^1 (1)^2 = (-1)(1) = -1$

$a_2 = (-1)^2 (2)^2 = (1)(4) = 4$

$a_3 = (-1)^3 (3)^2 = (-1)(9) = -9$

$a_4 = (-1)^4 (4)^2 = (1)(16) = 16$

First four terms are -1, 4, -9, 16

(iii) $a_n = (-1)^n (2n-3)$

Solution:- $a_n = (-1)^n (2n-3) \longrightarrow (i)$

Put $n = 1, 2, 3, 4$ in (i)

$a_1 = (-1)^{(1)} (2(1)-3) = (-1)(-1) = 1$

$a_2 = (-1)^2 (2(2)-3) = (1)(4-3) = (1)(1) = 1$

$a_3 = (-1)^3 (2(3)-3) = (-1)(6-3)$

$= (-1)(3) = -3$

$a_4 = (-1)^4 (2(4)-3) = (1)(8-3)$

$= (1)(5) = 5$

First four terms are 1, 1, -3, 5

(iv) $a_n = 3n-5$

Solution:- $a_n = 3n-5 \longrightarrow (i)$

Put $n = 1, 2, 3, 4$ in (i)

$a_1 = 3(1)-5 = 3-5 = -2$

$a_2 = 3(2)-5 = 6-5 = 1$

$a_3 = 3(3)-5 = 9-5 = 4$

$a_4 = 3(4)-5 = 12-5 = 7$

First four terms are -2, 1, 4, 7

(v) $a_n = \frac{n}{2n+1}$

Solution:- $a_n = \frac{n}{2n+1} \longrightarrow (i)$

Put $n = 1, 2, 3, 4$ in (i)

$a_1 = \frac{1}{2(1)+1} = \frac{1}{2+1} = \frac{1}{3}$

$a_2 = \frac{2}{2(2)+1} = \frac{2}{4+1} = \frac{2}{5}$

$a_3 = \frac{3}{2(3)+1} = \frac{3}{6+1} = \frac{3}{7}$

$a_4 = \frac{4}{2(4)+1} = \frac{4}{8+1} = \frac{4}{9}$

First four terms are $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$

vi) $a_n = \frac{1}{2^n}$

Solution:- $a_n = \frac{1}{2^n} \longrightarrow (i)$

Put $n = 1, 2, 3, 4$ in (i)

$a_1 = \frac{1}{2^1} = \frac{1}{2}$

$a_2 = \frac{1}{2^2} = \frac{1}{4}$

$a_3 = \frac{1}{2^3} = \frac{1}{8}$, $a_4 = \frac{1}{2^4} = \frac{1}{16}$

First four terms are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

vii) $a_n - a_{n-1} = n+2$, $a_1 = 2$

Solution:- $a_n - a_{n-1} = n+2 \longrightarrow (i)$

Put $n = 2, 3, 4$ in (i)

$a_2 - a_1 = 2+2$

$\rightarrow a_2 - a_1 = 4$ $\quad \because a_1 = 2$

$a_2 - 2 = 4$

$\rightarrow a_2 = 6$

$a_3 - a_2 = 3+2$

$a_3 - a_2 = 5$

$\rightarrow a_3 - 6 = 5$ $\quad \because a_2 = 6$

$a_3 = 11$

$a_4 - a_3 = 4+2$

$a_4 - a_3 = 6$ $\quad \because a_3 = 11$

$\rightarrow a_4 - 11 = 6$

$a_4 = 17$

First four terms are 2, 6, 11, 17

viii) $a_n = n a_{n-1}$, $a_1 = 1$

Solution:- $a_n = n a_{n-1} \longrightarrow (i)$

Put $n = 2, 3, 4$

$a_2 = 2a_1 \rightarrow a_2 = 2(1) = 2$ $\quad \because a_1 = 1$

$a_3 = 3a_2 \rightarrow a_3 = 3(2) = 6$ $\quad \because a_2 = 2$

$a_4 = 4a_3 \rightarrow a_4 = 4(6) = 24$ $\quad \because a_3 = 6$

First four terms are 1, 2, 6, 24

ix) $a_n = (n+1)a_{n-1}$, $a_1 = 1$

Solution:- $a_n = (n+1)a_{n-1} \longrightarrow (i)$

Put $n = 2, 3, 4$

$a_2 = (2+1)a_1 = 3a_1 = 3(1) = 3 \quad \because a_1 = 1$

$a_3 = (3+1)a_2 = 4a_2 = 4(3) = 12 \quad \because a_2 = 3$

$a_4 = (4+1)a_3 = 5a_3 = 5(12) = 60 \quad \because a_3 = 12$

First four terms are 1, 3, 12, 60

(x) $a_n = \frac{1}{a+(n-1)d}$

Solution:- $a_n = \frac{1}{a+(n-1)d} \longrightarrow (i)$

Put $n = 1, 2, 3, 4$ in (i)

$a_1 = \frac{1}{a+(1-1)d} = \frac{1}{a}$

$a_2 = \frac{1}{a+(2-1)d} = \frac{1}{a+d}$

$a_3 = \frac{1}{a+(3-1)d} = \frac{1}{a+2d}$

$a_4 = \frac{1}{a+(4-1)d} = \frac{1}{a+3d}$

First four terms are $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}$

Q2. Find the indicated terms of the following sequences:

i) 2, 6, 11, 17, ..., a_7

Solution:-

$a_1 = 2$

$a_2 = 2+4 = 6$

$a_3 = 6+5 = 11$

$a_4 = 11+6 = 17$

$a_5 = 17+7 = 24$

$a_6 = 24+8 = 32$

$a_7 = 32+9 = 41$

ii) 1, 3, 12, 60, ... , a_6

Solution:-

$a_1 = 1$, $a_2 = 1(3) = 3$

$a_3 = 3(4) = 12$, $a_4 = 12(5) = 60$

$a_5 = 60(6) = 360$, $a_6 = (360)(7) = 2520$

iii) $1, \frac{3}{2}, \frac{5}{4}, \frac{7}{8}, ..., a_7$

Solution:-

$a_1 = 1$, $a_2 = \frac{1+2}{1\times2} = \frac{3}{2}$

$a_3 = \frac{3+2}{2\times2} = \frac{5}{4}$, $a_4 = \frac{5+2}{4\times2} = \frac{7}{8}$

$a_5 = \frac{7+2}{8\times2} = \frac{9}{16}$, $a_6 = \frac{9+2}{16\times2} = \frac{11}{32}$

$a_7 = \frac{11+2}{32\times2} = \frac{13}{64}$

iv) 1, 1, -3, 5, -7, ... , a_8

Solution:-

$a_1 = 1$, $a_2 = 1$

$a_3 = 1-4 = -3$, $a_4 = 1+4 = 5$

$a_5 = -3-4 = -7$, $a_6 = 5+4 = 9$

$a_7 = -7-4 = -11$, $a_8 = 9+4 = 13$

v) 1, -3, 5, -7, 9, -11, ... , a_8

Solution:-

$a_1 = 1$, $a_2 = -3$

$a_3 = 1+4 = 5$, $a_4 = -3-4 = -7$

$a_5 = 5+4 = 9$, $a_6 = -7-4 = -11$

$a_7 = 9+4 = 13$, $a_8 = -11-4 = -15$

Q3. Find the next two terms of the following sequences:

i) 7, 9, 12, 16....

Solution:-

$a_1 = 7$, $a_2 = 7+2 = 9$

$a_3 = 9+3 = 12$, $a_4 = 12+4 = 16$

$a_5 = 16+5 = 21$, $a_6 = 21+6 = 27$

ii) 1, 3, 7, 15, 31, . . —.

Solution:-

$a_1 = 1$, $a_2 = 1+2 = 3$

$a_3 = 3+4 = 7$, $a_4 = 7+8 = 15$

$a_5 = 15+16 = 31$, $a_6 = 31+32 = 63$

$a_7 = 63+64 = 127$

iii) $-1, 2, 12, 40, \ldots$

Solution:-

$a_1 = -1$ or $a_1 = 2^0 \times (-1) = -1$

$a_2 = 2^1 \times (1) = 2$, $a_3 = 2^2 \times (3) = 12$

$a_4 = 2^3 \times (5) = 8 \times 5 = 40$

$a_5 = 2^4 \times 7 = 16 \times 7 = 112$

$a_6 = 2^5 \times 9 = 32 \times 9 = 288$

iv) $1, -3, 5, -7, 9, -11, \ldots$

Solution:-

$a_1 = 1$, $a_2 = -3$

$a_3 = 1 + 4 = 5$, $a_4 = -3 - 4 = -7$

$a_5 = 5 + 4 = 9$, $a_6 = -7 - 4 = -11$

$a_7 = 9 + 4 = 13$, $a_8 = -11 - 4 = -15$

Arithmetic Progression (A.P)

A sequence $\{a_n\}$ is an Arithmetic sequence or Arithmetic progression (A.P) if $a_n - a_{n-1}$ is same for all $n \in N$ and $n > 1$.

* The difference $a_n - a_{n-1}$ i.e., the difference of two consecutive terms of an A.P., is called the common difference and is usually denoted by d.

Important Note:- "when the common difference (d) of any two consecutive terms of a sequence is same, then this sequence will be called an Arithmetic sequence."

Rule for the nth term of an A.P.

$$\boxed{a_n = a_1 + (n-1)\,d}$$

Proof:- We know that

$a_n - a_{n-1} = d$, $n > 1$

or $a_n = a_{n-1} + d \longrightarrow (i)$

Put $n = 2, 3, 4, \ldots$ in (i), we get

$a_2 = a_1 + d = a_1 + (2-1)d \longrightarrow (ii)$

$a_3 = a_2 + d = a_1 + d + d = a_1 + 2d$

$\Rightarrow a_3 = a_1 + (3-1)d \longrightarrow (iii)$

$a_4 = a_3 + d$

$\Rightarrow a_4 = a_1 + 2d + d = a_1 + 3d$

$\Rightarrow a_4 = a_1 + (4-1)d \longrightarrow (iv)$

from (ii), (iii) and (iv) we observe that

$a_n = a_1 + (n-1)d$

Hence proved

Example 1. Find the general term and the eleventh term of the A.P., whose first term and common difference are 2 and −3 respectively. Also write its first four terms.

Solution:- Here $a_1 = 2$, $d = -3$

$a_n = ?$ $a_{11} = ?$

$\because a_n = a_1 + (n-1)d$

$= 2 + (n-1)(-3)$

$= 2 - 3n + 3$

$a_n = 5 - 3n \longrightarrow (i)$

Put $n = 11$ in (i)

$\Rightarrow a_{11} = 5 - 3(11) = 5 - 33 = -28$

Put $n = 2, 3, 4$ in (i)

$a_2 = 5 - 3(2) = 5 - 6 = -1$

$a_3 = 5 - 3(3) = 5 - 9 = -4$

$a_4 = 5 - 3(4) = 5 - 12 = -7$

Hence first four terms are $2, -1, -4, -7$.

Example 2. If the 5th term of an A.P., is 13 and 17th terms is 49, find a_n and a_{13}.

Solution:- Here $a_5 = 13$, $a_{17} = 49$

$a_n = ?$, $a_{13} = ?$

$\because a_5 = a_1 + 4d$ $\because a_n = a_1 + (n-1)d$

$\rightarrow 13 = a_1 + 4d \longrightarrow (i)$

Also $a_{17} = a_1 + 16d$

$\rightarrow 49 = a_1 + 16d \longrightarrow (ii)$

By (ii) - (i) $\rightarrow$

$$\begin{array}{r} a_1 + 16d = 49 \\ \underline{-a_1 \pm 4d = -13} \\ 12d = 36 \end{array}$$

$\rightarrow d = 3$ put in (i)

$13 = a_1 + 4(3) \rightarrow 13 = a_1 + 12$

$\rightarrow a_1 = 13 - 12 \rightarrow a_1 = 1$

Now $a_n = a_1 + (n-1)d$

$\rightarrow a_n = 1 + (n-1)(3)$ $\because a_1 = 1$, $d = 3$

$\rightarrow a_n = 1 + 3n - 3$

$a_n = 3n - 2 \longrightarrow (iii)$

Put $n = 13$ in (iii)

$a_{13} = 3(13) - 2 = 39 - 2 = 37$

$\rightarrow a_{13} = 37$

Example 3. Find the number of terms in the A.P., if $a_1 = 3$, $d = 7$ and $a_n = 59$.

Solution:- Here $a_1 = 3$, $d = 7$

$a_n = 59$ and $n = ?$

$\because a_n = a_1 + (n-1)d$

$\rightarrow 59 = 3 + (n-1)(7)$

$59 - 3 = (n-1)7$

$\rightarrow \frac{56}{7} = n - 1$

or $n - 1 = 8$

$\rightarrow n = 8 + 1 \rightarrow n = 9$

Example 4. If $a_{n-2} = 3n - 11$, find the nth term of the sequence.

Solution:- Here $a_n = ?$ and

$a_{n-2} = 3n - 11 \longrightarrow (i)$

Put $n = 3, 4, 5$ in (i)

$a_{3-2} = a_1 = 3(3) - 11 = 9 - 11 = -2$

$a_{4-2} = a_2 = 3(4) - 11 = 12 - 11 = 1$

$a_{5-2} = a_3 = 3(5) - 11 = 15 - 11 = 4$

so the seq. is $-2, 1, 4, \ldots$

Here $a_1 = -2$, $d = 1 - (-2) = 1 + 2$

$\rightarrow d = 3$

$\because a_n = a_1 + (n-1)d$

$\rightarrow a_n = -2 + (n-1)(3)$

$= -2 + 3n - 3$

$a_n = 3n - 5$

Exercise 6.2

Q1. Write the first four terms of the following arithmetic sequences, if

i) $a_1 = 5$ and other three consective terms are 23, 26, 29

Solution:- Here $a_1 = 5$ and

23, 26, 29 $\rightarrow d = 26 - 23 = 3$

Now $a_2 = a_1 + d = 5 + 3 = 8$

$a_3 = a_1 + 2d = 5 + 2(3) = 5 + 6 = 11$

$a_4 = a_1 + 3d = 5 + 3(3) = 5 + 9 = 14$

∴ First four terms are 5, 8, 11, 14

ii) $a_5 = 17$ and $a_9 = 37$

Solution:-

$\because a_5 = a_1 + 4d$

$\rightarrow 17 = a_1 + 4d \longrightarrow (i)$

$\because a_9 = a_1 + 8d$

$\rightarrow 37 = a_1 + 8d \longrightarrow (ii)$

By (ii) - (i) $\rightarrow$

$$\begin{array}{r} a_1 + 8d = 37 \\ \underline{-a_1 \pm 4d = -17} \\ 4d = 20 \end{array}$$

→ d = 5 put in (i)

$a_1 + 4(5) = 17$

→ $a_1 + 20 = 17$ → $a_1 = 17 - 20 = -3$

Now $a_2 = a_1 + d = -3 + 5 = 2$

$a_3 = a_1 + 2d = -3 + 2(5) = -3 + 10 = 7$

$a_4 = a_1 + 3d = -3 + 3(5) = -3 + 15 = 12$

∴ First four terms are -3, 2, 7, 12

iii) $3a_7 = 7a_4$ and $a_{10} = 33$

Solution:-

∵ $3a_7 = 7a_4$

→ $3(a_1 + 6d) = 7(a_1 + 3d)$

→ $3a_1 + 18d = 7a_1 + 21d$

or $7a_1 + 21d = 3a_1 + 18d$

or $7a_1 - 3a_1 + 21d - 18d = 0$

$4a_1 + 3d = 0 \longrightarrow (i)$

Also $a_{10} = 33$

→ $a_1 + 9d = 33 \longrightarrow (ii)$

By 4(ii) − (i) → $4a_1 + 36d = 132$

$\underset{-}{4a_1} \underset{-}{+} 3d = \underset{-}{0}$

$33d = 132$

→ d = 4 put in (i)

$4a_1 + 3(4) = 0$ → $4a_1 + 12 = 0$

or $4a_1 = -12$ → $a_1 = -3$

Now $a_2 = a_1 + d = -3 + 4 = 1$

$a_3 = a_1 + 2d = -3 + 2(4) = -3 + 8$

→ $a_3 = 5$

$a_4 = a_1 + 3d = -3 + 3(5)$

$a_4 = -3 + 15 = 12$

∴ First four terms are -3, 1, 5, 12

Q2. If $a_{n-3} = 2n - 5$, find the nth term of the sequence.

Solution:-

∵ $a_{n-3} = 2n - 5 \longrightarrow (i)$, $a_n = ?$

Put n = 4, 5, 6 in (i)

$a_{4-3} = a_1 = 2(4) - 5 = 8 - 5 = 3$

$a_{5-3} = a_2 = 2(5) - 5 = 10 - 5 = 5$

$a_{6-3} = a_3 = 2(6) - 5 = 12 - 5 = 7$

Now sequence is 3, 5, 7, ...

Here $a_1 = 3$, $d = 5 - 3 = 2$

∵ $a_n = a_1 + (n-1)d$

$a_n = 3 + (n-1)2$

$a_n = 3 + 2n - 2$

→ $a_n = 2n + 1$

Q3. If the 5th term of an A.P. is 16 and the 20th term is 46, what is its 12th term?

Solution:- Here $a_5 = 16$, $a_{20} = 46$, $a_{12} = ?$

∵ $a_5 = 16$

→ $a_1 + 4d = 16 \longrightarrow (i)$

and $a_{20} = 46$

→ $a_1 + 19d = 46 \longrightarrow (ii)$

By (ii) − (i) → $a_1 + 19d = 46$

$\underset{-}{a_1} \underset{-}{+} 4d = \underset{-}{16}$

$15d = 30$

→ d = 2 put in (i)

$a_1 + 4(2) = 16$ → $a_1 + 8 = 16$

$a_1 = 16 - 8$ or $a_1 = 8$

Now $a_{12} = a_1 + 11d$

$a_{12} = 8 + 11(2) = 8 + 22$

→ $a_{12} = 30$

Q4. Find the 13th term of the sequence x, 1, 2−x, 3−2x, ...

Solution:- Here $a_{13} = ?$ and

x, 1, 2−x, 3−2x, ...

$a_1 = x$, $d = 1 - x$

Now $a_{13} = a_1 + 12d$

$\rightarrow a_{13} = x+12(1-x)$

$a_{13} = x+12-12x = 12-11x$

or $a_{13} = 12-11x$

Q5. Find the 18th term of A.P. if its 6th term is 19 and the 9th term is 31.

Solution:- Here $a_{18} = ?$

$a_6 = 19$, $a_9 = 31$

$a_6 = 19 \rightarrow a_1+5d=19 \longrightarrow (i)$

$a_9 = 31 \rightarrow a_1+8d=31 \longrightarrow (ii)$

By (ii)-(i) $\rightarrow$ $a_1+8d=31$

$\underset{-}{a_1} \underset{-}{+} 5d \underset{-}{=} 19$

$3d = 12$

$\rightarrow d=4$ put in (i)

$a_1+5(4)=19 \rightarrow a_1+20=19$

$a_1 = 19-20 \rightarrow a_1 = -1$

Now $a_{18} = a_1+17d = -1+17(4)$

$a_{18} = -1+68 = 67$

$\rightarrow a_{18} = 67$

Q6. Which term of the A.P. 5, 2, -1, ... is -85 ?

Solution:- Given that

5, 2, -1, ..., -85

Here $a_1 = 5$, $d = 2-5 = -3$, $a_n = -85$

$\because a_n = a_1+(n-1)d$

$-85 = 5+(n-1)(-3)$

$\rightarrow -85-5 = (n-1)(-3)$

$-90 = (n-1)(-3)$

$\rightarrow \frac{-90}{-3} = n-1 \rightarrow 30 = n-1$

$\rightarrow n = 30+1 \rightarrow n=31$

Hence 31st term is -85

Q7. Which term of the A.P. -2, 4, 10, ... is 148?

Solution:- Given that

-2, 4, 10, ..., 148

Here $a_1 = -2$, $d = 4-(-2) = 6$, $a_n = 148$

$\because a_n = a_1+(n-1)d$

$148 = -2+(n-1)(6)$

$148+2 = (n-1)6$

$\frac{150}{6} = n-1 \rightarrow n-1 = 25$

$\rightarrow n = 25+1 \rightarrow n=26$

Hence 26th term is 148

Q8. How many terms are there in the A.P. in which $a_1 = 11$, $a_n = 68$, $d = 3$?

Solution:- Given that

$a_n = 68$, $a_1 = 11$, $d = 3$

$\because a_n = a_1+(n-1)d$

$\rightarrow 68 = 11+(n-1)(3)$

$68-11 = (n-1)3$

$\rightarrow 57 = (n-1)3$

$\rightarrow \frac{57}{3} = n-1 \rightarrow n-1 = 19$

$n = 20$

Hence there are 20 terms in given A.P.

Q9. If the nth term of the A.P. is 3n-1, find the A.P.

Solution:- Given that

$a_n = 3n-1 \longrightarrow (i)$

Put $n = 1, 2, 3, 4$ in (i)

$a_1 = 3(1)-1 = 3-1 = 2$

$a_2 = 3(2)-1 = 6-1 = 5$

$a_3 = 3(3)-1 = 9-1 = 8$

$a_4 = 3(4)-1 = 12-1 = 11$

Hence the required sequence is

2, 5, 8, 11, ...

Q10. Determine whether (i) -19, (ii) 2 are the terms of A.P. 17, 13, 9, ... or not?

Solution:- Given that

17, 13, 9, ...

Here $a_1 = 17$, $d = 13-17 = -4$

i) Here $a_n = -19$

$\because a_n = a_1 + (n-1)d$

$-19 = 17 + (n-1)(-4)$

$-19-17 = (n-1)(-4)$

$\overset{9}{\cancel{-36}} = (n-1)(\cancel{-4})$

$\rightarrow n-1 = 9 \rightarrow n = 9+1$

or $n = 10$ (Possible $\because n \in Z^+$)

Thus -19 is present in given A.P.

ii) Here $a_n = 2$

$\because a_n = a_1 + (n-1)d$

$\rightarrow 2 = 17 + (n-1)(-4)$

$2-17 = (n-1)(-4)$

$\rightarrow -15 = (n-1)(-4)$

$\frac{-15}{-4} = n-1 \rightarrow n-1 = \frac{15}{4}$

or $n = \frac{15}{4}+1 \rightarrow n = \frac{15+4}{4}$

$\rightarrow n = \frac{19}{4}$ (Impossible $\because n \notin Z^+$)

Hence 2 is not present in given A.P.

Q11. If l, m, n are the pth, qth and rth terms of an A.P., show that

i) $l(q-r)+m(r-p)+n(p-q)=0$

ii) $p(m-n)+q(n-l)+r(l-m)=0$

Solution:-

i) $l(q-r)+m(r-p)+n(p-q)=0$

$\because l = a_p$ $\quad\quad \because a_n = a_1+(n-1)d$

$\rightarrow l = a_1 + (p-1)d$

$m = a_q$

$\rightarrow m = a_1 + (q-1)d$ $\quad$ and

$n = a_r$

$\rightarrow n = a_1 + (r-1)d$ $\quad$ Now

$L.H.S = l(q-r)+m(r-p)+n(p-q)$

$= [a_1+(p-1)d](q-r) + [a_1+(q-1)d](r-p)$

$+ [a_1+(r-1)d](p-q)$

$= (a_1+pd-d)(q-r) + (a_1+qd-d)(r-p)$

$+ (a_1+rd-d)(p-q)$

$= \cancel{a_1q} - \cancel{a_1r} + \cancel{pdq} - \cancel{pdr} - \cancel{dq} + \cancel{dr}$

$+ \cancel{a_1r} - \cancel{a_1p} + \cancel{qdr} - \cancel{qdp} - \cancel{dr} + \cancel{dp}$

$+ \cancel{a_1p} - \cancel{a_1q} + \cancel{rdp} - \cancel{rdq} - \cancel{dp} + \cancel{dq}$

$= 0 = R.H.S$

Hence proved

ii) $p(m-n)+q(n-l)+r(l-m)=0$

Solution:-

$\because l = a_p$ $\quad\quad \because a_n = a_1+(n-1)d$

$\rightarrow l = a_1+(p-1)d$

$l = a_1 + pd - d \longrightarrow (i)$

$m = a_q$

$\rightarrow m = a_1+(q-1)d$

$m = a_1 + qd - d \longrightarrow (ii)$

$n = a_r$

$\rightarrow n = a_1+(r-1)d$

$n = a_1 + rd - d \longrightarrow (iii)$

Now (i) − (ii)

$l = \cancel{a_1} + pd - \cancel{d}$

$\underset{-}{m} = \underset{-}{\cancel{a_1}} \underset{-}{+} qd \underset{+}{\cancel{-}} d$

$l-m = (p-q)d \longrightarrow (iv)$

(ii) − (iii)

$m = \cancel{a_1} + qd - \cancel{d}$

$\underset{-}{n} = \underset{-}{\cancel{a_1}} \underset{-}{+} rd \underset{+}{\cancel{-}} d$

$m-n = (q-r)d \longrightarrow (v)$

By $\frac{(iv)}{(v)} \rightarrow \frac{l-m}{m-n} = \frac{(p-q)d}{(q-r)d}$

$\rightarrow \frac{l-m}{m-n} = \frac{p-q}{q-r}$

or $(l-m)(q-r) = (m-n)(p-q)$

$\rightarrow lq - lr - mq + mr = mp - mq - np + nq$

$\rightarrow pm - pn - \cancel{qm} + qn - lq + lr + \cancel{mq} - mr = 0$

$\rightarrow p(m-n) + q(n-l) + r(l-m) = 0$

Hence proved

Q12. Find the nth term of the sequence, $\left(\frac{4}{3}\right)^2, \left(\frac{7}{3}\right)^2, \left(\frac{10}{3}\right)^2, \ldots$

Solution:- $\quad a_n = ?$

Take $4, 7, 10, \ldots$

Here $a_1 = 4$, $d = 7-4 = 3$

$\because a_n = a_1 + (n-1)d$

$a_n = 4 + (n-1)3$

$a_n = 4 + 3n - 3$

→ $a_n = 3n+1$

Thus $a_n = \left(\frac{3n+1}{2}\right)^2$

Q13. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in A.P., show that $b = \frac{2ac}{a+c}$

Solution:- Given

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P

so common difference will be same. i-e.,

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

$$\rightarrow \frac{1}{b} + \frac{1}{b} = \frac{1}{c} + \frac{1}{a}$$

$$\rightarrow \frac{1+1}{b} = \frac{a+c}{ac}$$

$$\rightarrow \frac{2}{b} = \frac{a+c}{ac}$$

$$\text{or } \frac{b}{2} = \frac{ac}{a+c}$$

$$\rightarrow b = \frac{2ac}{a+c}$$ Hence proved

Q14. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in A.P., show that the common difference is $\frac{a-c}{2ac}$.

Solution:- Given that

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P

so common difference $= d$ is

$$d = \frac{1}{b} - \frac{1}{a} \longrightarrow (i)$$

also $$d = \frac{1}{c} - \frac{1}{b} \longrightarrow (ii)$$

By (i) + (ii) → $d. + d = \cancel{\frac{1}{b}} - \frac{1}{a} + \frac{1}{c} - \cancel{\frac{1}{b}}$

$$\rightarrow 2d = \frac{1}{c} - \frac{1}{a}$$

$$2d = \frac{a-c}{ac}$$

$$\rightarrow d = \frac{a-c}{2ac}$$ Hence proved

Arithmetic Mean (A.M)

A number A is said to be arithmetic mean between two numbers a and b if a, A, b are in A.P

Thus $d = A - a$ and $d = b - A$

$$\rightarrow A - a = b - A$$

$$\rightarrow A + A = a + b$$

$$\rightarrow 2A = a + b \rightarrow A = \frac{a+b}{2}$$

Thus $A.M = \frac{a+b}{2}$

Note:- Middle term of three consecutive terms in A.P. is the A.M. between the extreme terms.

* The numbers $A_1, A_2, A_3, \ldots, A_n$ are said to be n A.M's between a and b if $a, A_1, A_2, A_3, \ldots, A_n, b$ are in A.P

* If $A_1, A_2, A_3, \ldots A_{n-1}, A_n$ are n A.M's between a and b and d be common difference then

$A_1 = a + d$, $A_2 = A_1 + d = a_1 + 2d$

$A_3 = A_2 + d = a_1 + 3d$, $A_4 = A_3 + d = a_1 + 4d$

$\ldots\ldots A_n = A_{n-1} + d = a_1 + nd$

Example 1. Find three A.M's between $\sqrt{2}$ and $3\sqrt{2}$

Solution:- Let A_1, A_2 and A_3 be three A.M's between $\sqrt{2}$ and $3\sqrt{2}$ then

$\sqrt{2}, A_1, A_2, A_3, 3\sqrt{2}$ are in A.P

Here $a_1 = \sqrt{2}$, $n = 5$, $a_5 = 3\sqrt{2}$

$$\because a_n = a_1 + (n-1)d$$

$$\rightarrow a_5 = a_1 + (5-1)d$$

$$a_5 = a_1 + 4d$$

so $3\sqrt{2} = \sqrt{2} + 4d$

or $3\sqrt{2} - \sqrt{2} = 4d \rightarrow 2\sqrt{2} = 4d$

or $d = \frac{2\sqrt{2}}{4}$ or $d = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Now $A_1 = a_1 + d = \sqrt{2} + \frac{1}{\sqrt{2}} = \frac{2+1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

$A_2 = A_1 + d = \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{3+1}{\sqrt{2}} = \frac{4}{\sqrt{2}} = \frac{2\times 2}{\sqrt{2}}$

$A_2 = 2\sqrt{2}$

$A_3 = A_2 + d \rightarrow A_3 = 2\sqrt{2} + \frac{1}{\sqrt{2}}$

$A_3 = \frac{2(2)+1}{\sqrt{2}} = \frac{4+1}{\sqrt{2}} = \frac{5}{\sqrt{2}}$

Thus three A.M's are $\frac{3}{\sqrt{2}}, 2\sqrt{2}, \frac{5}{\sqrt{2}}$ between $\sqrt{2}$ and $3\sqrt{2}$.

Example 2. Find n A.M's between a and b.

Solution:- Let $A_1, A_2, A_3, \ldots, A_n$ be n arithmetic means between a and b. then $a, A_1, A_2, A_3, \ldots, A_n, b$ are in A.P in which $a_1 = a$, $n = n+2$ and $a_{n+2} = b$,

using $a_n = a_1 + (n-1)d$

$\rightarrow a_{n+2} = a + (n+2-1)d$

$\rightarrow b = a + (n+1)d$

$b - a = (n+1)d$

$\rightarrow d = \frac{b-a}{n+1}$

Thus $A_1 = a_1 + d = a + \frac{b-a}{n+1}$

$A_1 = \frac{an+a+b-a}{n+1} = \frac{an+b}{n+1}$

$A_2 = a_1 + 2d = a + 2\left(\frac{b-a}{n+1}\right)$

$A_2 = \frac{a(n+1)+2b-2a}{n+1}$

$\rightarrow A_2 = \frac{an+a+2b-2a}{n+1}$

$\rightarrow A_2 = \frac{(n-2)a+2b}{n+1}$

$A_3 = a_1 + 3d$

$A_3 = a + 3\left(\frac{b-a}{n+1}\right)$

$A_3 = \frac{a(n+1)+3b-3a}{n+1}$

$A_3 = \frac{an+a+3b-3a}{n+1}$

$A_3 = \frac{(n-3)a+3b}{n+1}$

In similar way

$A_n = a_1 + nd$

$A_n = a + n\left(\frac{b-a}{n+1}\right)$

$A_n = \frac{a(n+1)+nb-na}{n+1}$

$A_n = \frac{an+a+nb-na}{n+1}$

$A_n = \frac{a+nb}{n+1}$

Exercise 6.3

Q1. Find A.M between i) $3\sqrt{5}$ and $5\sqrt{5}$

Solution:- Here $a = 3\sqrt{5}$, $b = 5\sqrt{5}$

then $A.M = \frac{a+b}{2} = \frac{3\sqrt{5}+5\sqrt{5}}{2}$

$A.M = \frac{8\sqrt{5}}{2} = 4\sqrt{5}$

ii) $x-3$ and $x+5$

Solution:- Here $a = x-3$, $b = x+5$

then $A.M = \frac{a+b}{2} = \frac{x-3+x+5}{2}$

$A.M = \frac{2x+2}{2} = \frac{2(x+1)}{2} = x+1$

iii) $1-x+x^2$ and $1+x+x^2$

Solution:- Here $a = 1-x+x^2$ and $b = 1+x+x^2$ then

$A.M = \frac{a+b}{2} = \frac{1-x+x^2+1+x+x^2}{2}$

$A.M = \frac{2+2x^2}{2} = \frac{2(1+x^2)}{2} = 1+x^2$

Q2. If 5, 8 are two A.Ms between a and b, find a and b.

Solution:- Here

a, 5, 8, b are in A.P

$\rightarrow 8-5 = 5-a$

$\rightarrow 3 = 5-a \rightarrow a = 5-3$

so $a = 2$

Also $b-8 = 8-5$

$\rightarrow b-8 = 3$

$b = 3+8$ so $b = 11$

Q3. Find 6 A.Ms between 2 and 5.

Solution:- suppose A_1, A_2, A_3, A_4, A_5 A_6 are 6 A.Ms between 2 and 5.

then $2, A_1, A_2, A_3, A_4, A_5, A_6, 5$ are in A.P

Here $a_1 = 2$ and $n = 8$

so $a_8 = 5$

$\rightarrow a_1 + 7d = 5$ $\quad \because a_n = a_1 + (n-1)d$

so $2 + 7d = 5$ $\quad (\because a_1 = 2)$

$\rightarrow 7d = 5 - 2 \rightarrow 7d = 3 \rightarrow d = \frac{3}{7}$

Thus $A_1 = a_1 + d = 2 + \frac{3}{7} = \frac{14+3}{7} = \frac{17}{7}$

$A_2 = a_1 + 2d = 2 + 2(\frac{3}{7}) = 2 + \frac{6}{7} = \frac{20}{7}$

$A_3 = a_1 + 3d = 2 + 3(\frac{3}{7}) = 2 + \frac{9}{7} = \frac{14+9}{7} = \frac{23}{7}$

$A_4 = a_1 + 4d = 2 + .4(\frac{3}{7}) = 2 + \frac{12}{7} = \frac{14+12}{7}$

$A_4 = \frac{26}{7}$

$A_5 = a_1 + 5d = 2 + 5(\frac{3}{7}) = \frac{14+15}{7}$

$A_5 = \frac{29}{7}$

$A_6 = a_1 + 6d = 2 + 6(\frac{3}{7}) = 2 + \frac{18}{7}$

$A_6 = \frac{14+18}{7} = \frac{32}{7}$

Hence 6 A.Ms between 2 and 5 are $\frac{3}{7}, \frac{17}{7}, \frac{20}{7}, \frac{23}{7}, \frac{26}{7}, \frac{29}{7}, \frac{32}{7}$

Q4. Find four A.Ms between $\sqrt{2}$ and $\frac{12}{\sqrt{2}}$

Solution:- suppose A_1, A_2, A_3, A_4 are four A.Ms between $\sqrt{2}$ and $\frac{12}{\sqrt{2}}$. then

$\sqrt{2}, A_1, A_2, A_3, A_4, \frac{12}{\sqrt{2}}$ are in A.P

Here $a_1 = \sqrt{2}$, $n = 6$, $a_6 = \frac{12}{\sqrt{2}}$

Now $a_1 + 5d = \frac{12}{\sqrt{2}}$ $\quad \because a_n = a_1 + (n-1)d$

$\rightarrow \sqrt{2} + 5d = \frac{12}{\sqrt{2}}$

$5d = \frac{12}{\sqrt{2}} - \sqrt{2} = \frac{12-2}{\sqrt{2}} = \frac{10}{\sqrt{2}}$

$\rightarrow \cancel{5}d = \frac{\cancel{5} \times 2}{\sqrt{2}} \rightarrow d = \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}}$

$\rightarrow d = \sqrt{2}$ Now

$A_1 = a_1 + d = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

$A_2 = a_1 + 2d = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$

$A_3 = a_1 + 3d = \sqrt{2} + 3\sqrt{2} = 4\sqrt{2}$

$A_4 = a_1 + 4d = \sqrt{2} + 4\sqrt{2} = 5\sqrt{2}$

Thus four A.Ms between $\sqrt{2}$ and $\frac{12}{\sqrt{2}}$ are $2\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}, 5\sqrt{2}$

Q5. Insert 7 A.Ms between 4 and 8.

Solution:- Let $A_1, A_2, A_3, A_4, A_5, A_6, A_7$ be 7 A.Ms between 4 and 8. then

$4, A_1, A_2, A_3, A_4, A_5, A_6, A_7, 8$ are in A.P

Here $a_1 = 4$, $n = 9$ and $a_9 = 8$

$\rightarrow a_1 + 8d = 8$ $\quad \because a_n = a_1 + (n-1)d$

$\rightarrow 4 + 8d = 8$

$8d = 8 - 4 \rightarrow 8d = 4$

or $d = \frac{1}{2}$ Now

$A_1 = a_1 + d = 4 + \frac{1}{2} = \frac{8+1}{2} = \frac{9}{2}$

$A_2 = a_1 + 2d = 4 + 2(\frac{1}{2}) = 4 + 1 = 5$

$A_3 = a_1 + 3d = 4 + 3(\frac{1}{2}) = 4 + \frac{3}{2} = \frac{8+3}{2}$

$\rightarrow A_3 = \frac{11}{2}$

$A_4 = a_1 + 4d = 4 + 4(\frac{1}{2}) = 4 + 2 = 6$

$A_5 = a_1 + 5d = 4 + 5(\frac{1}{2}) = 4 + \frac{5}{2} = \frac{8+5}{2}$

$\rightarrow A_5 = \frac{13}{2}$

$A_6 = a_1 + 6d = 4 + 6(\frac{1}{2}) = 4 + 3 = 7$

$A_7 = a_1 + 7d = 4 + 7(\frac{1}{2}) = 4 + \frac{7}{2} = \frac{8+7}{2} = \frac{15}{2}$

Hence 7 A.Ms are $\frac{9}{2}, 5, \frac{11}{2}, 6, \frac{13}{2}, 7, \frac{15}{2}$ between 4 and 8.

Q6. Find three A.Ms between 3 and 11.

Solution:- Let A_1, A_2, A_3 be three A.Ms between 3 and 11 then

$3, A_1, A_2, A_3, 11$ are in A.P

Here $a_1 = 3$, $n = 5$ so $a_5 = 11$

→ $a_1+4d=11$ ∵ $a_n=a_1+(n-1)d$

→ $3+4d=11$

→ $4d=11-3$ → $4d=8$ → $d=2$

Now $A_1=a_1+d=3+2=5$

$A_2=a_1+2d=3+2(2)=3+4=7$

$A_3=a_1+3d=3+3(2)=3+6=9$

Thus three A.Ms between 3 and 11 are 5, 7, 9.

Q7. Find n so that $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ may be the A.M. between a and b.

Solution:- If $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ be A.M between a and b then we have

$$\frac{a^n+b^n}{a^{n-1}+b^{n-1}}=\frac{a+b}{2}$$

→ $2(a^n+b^n)=(a+b)(a^{n-1}+b^{n-1})$

$2a^n+2b^n=aa^{n-1}+ab^{n-1}+ba^{n-1}+bb^{n-1}$

$2a^n+2b^n=a^n+ab^{n-1}+ba^{n-1}+b^n$

→ $2a^n-a^n+2b^n-b^n=ab^{n-1}+ba^{n-1}$

→ $a^n+b^n=a^{n-1}b+ab^{n-1}$

∵ $b^{n-1}\cdot b=b^n$

$a^{n-1}\cdot a=a^n$

→ $a^{n-1}a-a^{n-1}b=ab^{n-1}-b^{n-1}\cdot b$

→ $a^{n-1}\cancel{(a-b)}=b^{n-1}\cancel{(a-b)}$

→ $a^{n-1}=b^{n-1}$

or $\frac{a^{n-1}}{b^{n-1}}=\frac{b^{n-1}}{b^{n-1}}$ (÷ by b^{n-1})

or $\left(\frac{a}{b}\right)^{n-1}=1$

→ $\left(\frac{a}{b}\right)^{n-1}=\left(\frac{a}{b}\right)^0$

or $n-1=0$ → $n=1$

Q8. Show that the sum of n A.Ms between a and b is equal to n times their A.M.

Solution:-

Let $A_1, A_2, A_3, \ldots, A_n$ be n A.Ms between a and b. then $a, A_1, A_2, A_3, \ldots, A_n, b$ are in A.P

Here

$a_1=a$, $n=n+2$, $a_{n+2}=b$

∵ $a_n=a_1+(n-1)d$

→ $a_{n+2}=a_1+(n+2-1)d$

or $a_{n+2}=a+(n+1)d$

or $b=a+(n-1)d$

$b-a=(n-1)d$

→ $d=\frac{b-a}{n-1}$ Now

$A_1+A_2+A_3,\ldots+A_n=\frac{n}{2}[A_1+A_n]$

$=\frac{n}{2}[a_1+d+a_1+nd]$

$=\frac{n}{2}[2a_1+(n+1)d]$

$=\frac{n}{2}\left[2a+(n-1)\left(\frac{b-a}{n-1}\right)\right]$ (∵ $a_1=a$)

$=\frac{n}{2}[2a+b-a]$

$=n\left(\frac{a+b}{2}\right)$

$=n$ (A.M between a and b)

Hence $A_1+A_2+A_3+\ldots+A_n=n(A.M)$

Series

"The sum of terms of a sequence is called series".

We know that $a_1, (a_1+d), (a_1+2d), \ldots, a_n$ is an arithmetic sequence so $a_1+(a_1+d)+(a_1+2d)+\ldots+a_n$ is an arithmetic series.

Finit series:- If the number of terms in a series is finite, then the series is called a finite series.

Infinite series:- If the number of terms in a series is infinite, then the series is called infinite series.

Prove that $S_n=\frac{n}{2}(a_1+a_n)$ or $S_n=\frac{n}{2}[2a_1+(n-1)d]$

Proof:- Let $a_1, (a_1+d), (a_1+2d), \ldots$ $\ldots (a_n-2d), (a_n-d), a_n$ are n-terms of an arithmetic sequence. Let S_n be the sum of n-terms then

$$S_n = a_1+(a_1+d)+(a_1+2d)+\ldots+(a_n-2d) + (a_n-d)+a_n \longrightarrow (i)$$

Re-writting (i) in reverse order

$$S_n = a_n+(a_n-d)+(a_n-2d)+\ldots+(a_1+2d) +(a_1+d)+a_1 \longrightarrow (ii)$$

By (i) + (ii), we get

$$S_n+S_n = (a_1+a_n)+(a_1+d+a_n-d) + (a_1+2d+a_n-2d)+\ldots+(a_n-2d+a_1+2d) + (a_n-d+a_1+d)+(a_n+a_1)$$

$$\rightarrow 2S_n = (a_1+a_n)+(a_1+a_n)+(a_1+a_n)+\ldots \ldots + (a_1+a_n)+(a_1+a_n)+(a_1+a_n)$$

$$2S_n = (a_1+a_n)+(a_1+a_n)+(a_1+a_n)+\ldots n\text{ terms}$$

$$2S_n = n[a_1+a_n]$$

$$\rightarrow S_n = \frac{n}{2}[a_1+a_n]$$

Put $a_n = a_1+(n-1)d$

$$S_n = \frac{n}{2}[a_1+a_1+(n-1)d]$$

$$S_n = \frac{n}{2}(2a_1+(n-1)d)$$

Hence proved

Example1. Find the 19th term and the partial sum of 19 terms of the arithmetic series :

$$2+\frac{7}{2}+5+\frac{13}{2}+\ldots$$

Solution:-

Here $a_1=2$, $d=\frac{7}{2}-2=\frac{7-4}{2}=\frac{3}{2}$

$$\because a_{19} = a_1+18d$$
$$a_{19} = 2+18(3/2)$$
$$a_{19} = 2+9(3)$$
$$a_{19} = 2+27 = 29$$

we know that

$$S_n = \frac{n}{2}[a_1+a_n]$$
$$\rightarrow S_{19} = \frac{19}{2}(a_1+a_{19})$$
$$S_{19} = \frac{19}{2}(2+29) = \frac{19}{2}(31)$$
$$\rightarrow S_{19} = \frac{589}{2}$$

Example 2. Find the arithmetic series if its fifth term is 19 and $S_4 = a_9+1$

Solution:- Given

$a_5 = 19$. $S_4 = a_9+1$

$$\therefore a_5 = a_1+4d$$
$$\rightarrow a_1+4d = 19 \longrightarrow (i)$$
$$\because S_n = \frac{n}{2}[2a_1+(n-1)d]$$
$$\rightarrow S_4 = \frac{4}{2}[2a_1+(4-1)d]$$
$$S_4 = 2[2a_1+3d] = 4a_1+6d$$
$$\rightarrow S_4 = 4a_1+6d \longrightarrow (ii)$$

But given that

$$S_4 = a_9+1$$
$$S_4 = a_1+8d+1 \longrightarrow (iii)$$

By (ii) and (iii)

$$a_1+8d+1 = 4a_1+6d$$
$$\rightarrow 4a_1+6d-a_1-8d-1=0$$
$$3a_1-2d-1=0$$
$$\text{or } 6a_1-4d=2 \longrightarrow (iv)$$

By (i) + (iv) →

$7a_1 = 21 \rightarrow a_1 = 3$ put in (i)

$3+4d = 19 \rightarrow 4d = 19-3$

$4d = 16 \rightarrow d = 4$

Thus series is

$a_1 + (a_1+d) + (a_1+2d) + \cdots$

$3 + 7 + 11 + \cdots$

Example 3. How many terms of the series $-9-6-3+0+\ldots$ amount to 66?

Solution:- Here $a_1 = -9$

$d = -6-(-9) = -6+9 = 3$

$S_n = 66$

$\because S_n = \frac{n}{2}[2a_1+(n-1)d]$

$\rightarrow 66 = \frac{n}{2}(2(-9)+(n-1)(3))$

$\rightarrow 132 = n(-18+3n-3)$

$132 = n(3n-21)$

$132 = 3n^2-21n$

or $44 = n^2-7n$ ($\div$ by 3)

or $n^2-7n-44=0$

using Quadratic formula

$$n = \frac{-(-7) \pm \sqrt{(-7)^2-4(1)(-44)}}{2(1)}$$

$$n = \frac{7 \pm \sqrt{49+176}}{2}$$

$$n = \frac{7 \pm \sqrt{225}}{2}$$

$$n = \frac{7 \pm 15}{2}$$

$$n = \frac{7+15}{2}, \quad n = \frac{7-15}{2}$$

$$n = \frac{22}{2}, \quad n = -\frac{8}{2}$$

or $n = 11$, $n = -4$ (Not possible)

so sum of 11 terms of given series is 66.

Exercise 6.4

Q1. Find the sum of all the integral multiples of 3 between 4 and 97.

Solution:- Sum of all Integral multiple of 3 between 4 and 97 is

$6+9+12+15+\cdots+96$

Here $a_1 = 6$, $d = 9-6 = 3$, $a_n = 96$

$n = ?$

$\because a_n = a_1+(n-1)d$

$\rightarrow 96 = 6+(n-1)(3)$

$96 = 6+3n-3 = 3n-3$

or $96 = 3n+3 \rightarrow 96-3 = 3n$

$\rightarrow 3n = 93 \rightarrow n = 31$

Now $S_n = \frac{n}{2}(a_1+a_n)$

$\rightarrow S_{31} = \frac{31}{2}(6+96)$

$S_{31} = \frac{31}{2}(102)$

$S_{31} = (31)(51) = 1581$

Q2. Sum the series

i) $-3+(-1)+1+3+5+\cdots+a_{16}$

Solution:- Here $a_1 = -3$, $d = -1-(-3)$

$d = -1+3 = 2$

$n = 16$

so $S_n = \frac{n}{2}(2a_1+(n-1)d)$

$\rightarrow S_{16} = \frac{16}{2}(2(-3)+(16-1)(2))$

$S_{16} = 8(-6+(15)(2))$

$S_{16} = 8(-6+30) = 8(24) = 192$

ii) $\frac{3}{\sqrt{2}} + 2\sqrt{2} + \frac{5}{\sqrt{2}} + \cdots + a_{13}$

Solution:- Here $a_1 = \frac{3}{\sqrt{2}}$

$d = 2\sqrt{2} - \frac{3}{\sqrt{2}} = \frac{2(2)-3}{\sqrt{2}} = \frac{4-3}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$n = 13$

Now $S_n = \frac{n}{2}[2a_1+(n-1)d]$

→ $S_{13} = \frac{13}{2}\left(2\left(\frac{3}{\sqrt{2}}\right)+(13-1)\left(\frac{1}{\sqrt{2}}\right)\right)$

$S_{13} = \frac{13}{2}\left(\frac{6}{\sqrt{2}}+\frac{12}{\sqrt{2}}\right) = \frac{13}{2}\left(\frac{18}{\sqrt{2}}\right)$

$S_{13} = \frac{(13)(9)}{\sqrt{2}} = \frac{117}{\sqrt{2}}$

iii) $1.11 + 1.41 + 1.71 + \dots + a_{10}$

Solution:- Here $a_1 = 1.11$

$d = 1.41 - 1.11 = 0.30$, $n = 10$

$\because S_n = \frac{n}{2}(2a_1 + (n-1)d)$

$S_{10} = \frac{10}{2}(2(1.11) + (10-1)(0.30))$

$S_{10} = 5(2.22 + (9)(0.30))$

$S_{10} = 5(2.22 + 2.7) = 5(4.92)$

→ $S_{10} = 24.60$

iv) $-8 - 3\frac{1}{2} + 1 + \dots + a_{11}$

Solution:- Here $a_1 = -8$,

$d = -\frac{7}{2} - (-8) = -\frac{7}{2} + 8 = \frac{-7+16}{2} = \frac{9}{2}$

$n = 11$

$\because S_n = \frac{n}{2}(2a_1 + (n-1)d)$

→ $S_{11} = \frac{11}{2}\left(2(-8) + (11-1)\left(\frac{9}{2}\right)\right)$

$S_{11} = \frac{11}{2}\left(-16 + (10)\left(\frac{9}{2}\right)\right)$

$S_{11} = \frac{11}{2}(-16 + (5)(9))$

$S_{11} = \frac{11}{2}(-16 + 45) = \frac{11}{2}(29)$

$S_{11} = \frac{319}{2} = 159.5$

v) $(x-a) + (x+a) + (x+3a) + \dots$ to n terms

Solution:- Here $a_1 = x - a$

$d = x + a - (x - a) = x + a - x + a$

$d = 2a$, $n = n$

$\because S_n = \frac{n}{2}(2a_1 + (n-1)d)$

$S_n = \frac{n}{2}(2(x-a) + (n-1)2a)$

$S_n = \frac{2n}{2}(x - a + (n-1)a)$

$S_n = n(x - a + an - a)$

$S_n = n(x + an - 2a)$

→ $S_n = n(x + (n-2)a)$

vi) $\frac{1}{1-\sqrt{x}} + \frac{1}{1-x} + \frac{1}{1+\sqrt{x}} + \dots$ to n terms

Solution:- Here $a_1 = \frac{1}{1-\sqrt{x}}$

$d = \frac{1}{1-x} - \frac{1}{1-\sqrt{x}}$, $n = n$

$d = \frac{1}{(1-\sqrt{x})(1+\sqrt{x})} - \frac{1}{1-\sqrt{x}}$

$d = \frac{1-(1+\sqrt{x})}{(1-\sqrt{x})(1+\sqrt{x})} = \frac{1-1-\sqrt{x}}{1-x}$

$d = \frac{-\sqrt{x}}{1-x}$

$\because S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$= \frac{n}{2}\left[2\left(\frac{1}{1-\sqrt{x}}\right) + (n-1)\left(\frac{-\sqrt{x}}{1-x}\right)\right]$

$= \frac{n}{2}\left[\frac{2}{1-\sqrt{x}} - \frac{(n-1)\sqrt{x}}{1-x}\right]$

$= \frac{n}{2}\left[\frac{2}{1-\sqrt{x}} - \frac{(n-1)\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right]$

$= \frac{n}{2}\left[\frac{2(1+\sqrt{x}) - (n-1)\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right]$

$S_n = \frac{n}{2}\left[\frac{2 + 2\sqrt{x} - n\sqrt{x} + \sqrt{x}}{1-x}\right]$

$S_n = \frac{n}{2}\left[\frac{2 + 3\sqrt{x} - n\sqrt{x}}{1-x}\right]$

$S_n = \frac{n}{2}\left[\frac{2 + (3-n)\sqrt{x}}{1-x}\right]$

vii) $\frac{1}{1+\sqrt{x}} + \frac{1}{1-x} + \frac{1}{1-\sqrt{x}} + \dots +$ to n terms

Solution:- Here $a_1 = \frac{1}{1+\sqrt{x}}$

$d = \frac{1}{1-x} - \frac{1}{1+\sqrt{x}}$, $n = n$

$= \frac{1}{(1-\sqrt{x})(1+\sqrt{x})} - \frac{1}{1+\sqrt{x}}$

$d = \frac{1-(1-\sqrt{x})}{(1-\sqrt{x})(1+\sqrt{x})} = \frac{1-1+\sqrt{x}}{1-x}$

→ $d = \frac{\sqrt{x}}{1-x}$

$$\because S_n = \frac{n}{2}\left[2a_1+(n-1)d\right]$$

$$= \frac{n}{2}\left[2\left(\frac{1}{1+\sqrt{x}}\right)+(n-1)\cdot\frac{\sqrt{x}}{1-x}\right]$$

$$= \frac{n}{2}\left[\frac{2}{1+\sqrt{x}}+\frac{(n-1)\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right]$$

$$= \frac{n}{2}\left[\frac{2(1-\sqrt{x})+(n-1)\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right]$$

$$= \frac{n}{2}\left[\frac{2-2\sqrt{x}+n\sqrt{x}-\sqrt{x}}{1-x}\right]$$

$$S_n = \frac{n}{2}\left[\frac{2+n\sqrt{x}-3\sqrt{x}}{1-x}\right]$$

$$\rightarrow S_n = \frac{n}{2}\left[\frac{2+(n-3)\sqrt{x}}{1-x}\right]$$

Q3. How many terms of the series

i) $-7+(-5)+(-3)+\ldots$ amount to 65?

Solution:- Here $a_1=-7$,

$d=-5-(-7)=-5+7=2$, $n=?$

$S_n=65$

$$\because S_n = \frac{n}{2}\left[2a_1+(n-1)d\right]$$

$$\rightarrow 65 = \frac{n}{2}\left[2(-7)+(n-1)(2)\right]$$

$$65 = \frac{n}{2}\left(-14+2n-2\right)$$

$$65 = \frac{n}{2}\left(2n-16\right)$$

$$65 = \frac{2n}{2}\left(n-8\right)$$

or $65 = n(n-8)$

$\rightarrow n^2-8n=65$

or $n^2-8n-65=0$

$n^2-13n+5n-65=0$

$n(n-13)+5(n-13)=0$

$(n-13)(n+5)=0$

$\rightarrow n-13=0$ or $n+5=0$

$n=13$, $n=-5$ (Not possible)

Hence $n=13$

ii) $-7+(-4)+(-1)+\ldots$ amount to 114?

Solution:- Here $a_1=-7$, $d=-4-(-7)$

$d=-4+7=3$, $n=?$, $S_n=114$

$$\because S_n = \frac{n}{2}\left(2a_1+(n-1)d\right)$$

$$\rightarrow 114 = \frac{n}{2}\left(2(-7)+(n-1)(3)\right)$$

$$114 = \frac{n}{2}\left(-14+3n-3\right)$$

$$114 = \frac{n}{2}\left(3n-17\right)$$

$228 = 3n^2-17n$

or $3n^2-17n-228=0$

$\rightarrow 3n^2-36n+19n-228=0$

$3n(n-12)+19(n-12)=0$

$(n-12)(3n+19)=0$

$n-12=0$, $3n+19=0$

$n=12$, $n=-\frac{19}{3}$ (Not possible)

Hence $n=12$

Q4. Sum the series

i) $3+5-7+9+11-13+15+17-19+\ldots$ to $3n$ terms

Solution:-

By adding three terms, we get

$(3+5-7)+(9+11-13)+(15+17-19)+\ldots$ to n terms

$1+7+13+\ldots$ to n terms

Here $a_1=1$, $d=7-1=6$, $n=n$

$$\because S_n = \frac{n}{2}\left(2a_1+(n-1)d\right)$$

$$S_n = \frac{n}{2}\left(2(1)+(n-1)(6)\right)$$

$$S_n = \frac{n}{2}\left(2+6n-6\right)$$

$$= \frac{n}{2}\left(6n-4\right) = \frac{2n}{2}\left(3n-2\right)$$

$$S_n = n(3n-2)$$

ii) $1+4-7+10+13-16+19+22-25+\ldots$ to $3n$ terms

Solution:-

By adding three terms, we get

$(1+4-7)+(10+13-16)+(19+22-25)+\ldots$ to n terms

$-2+7+16+\ldots$ to n terms

Here $a_1=-2$, $\therefore d=7-(-2)=7+2=9$

$n=n$

$\because S_n=\frac{n}{2}(2a_1+(n-1)d)$

$S_n=\frac{n}{2}(2(-2)+(n-1)(9))$

$S_n=\frac{n}{2}(-4+9n-9)$

$\Rightarrow S_n=\frac{n}{2}(9n-13)$

Q5. Find the sum of 20 terms of the series whose rth term is 3r+1.

Solution:- Here $a_r=3r+1$, $S_{20}=?$

$a_r=3r+1 \longrightarrow (i)$

Put $r=1, 2, 3, 4$ in (i)

$a_1=3(1)+1=3+1=4$

$a_2=3(2)+1=6+1=7$

$a_3=3(3)+1=9+1=10$

$a_4=3(4)+1=12+1=13$

Hence series is

$4+7+10+13+\ldots$

Here $a_1=4$, $d=7-4=3$, $n=20$

$\because S_n=\frac{n}{2}(2a_1+(n-1)d)$

$\Rightarrow S_{20}=\frac{20}{2}(2(4)+(20-1)(3))$

$S_{20}=10(8+(19)(3))$

$\Rightarrow S_{20}=10(8+57)=10(65)=650$

Q6. If $S_n=n(2n-1)$, then find the series.

Solution:-

$S_n=n(2n-1) \longrightarrow (i)$

Put $n=1, 2, 3, \ldots$ in (i)

$S_1=a_1=1(2(1)-1)=1(2-1)=1$

$\Rightarrow a_1=1$

$S_2=a_1+a_2=2(2(2)-1)$

or $a_1+a_2=2(4-1)=2(3)=6$

$\Rightarrow a_1+a_2=6$

$1+a_2=6$

$\Rightarrow a_2=6-1$ or $a_2=5$ $\because a_1=1$

$S_3=a_1+a_2+a_3=3(2(3)-1)$

$\Rightarrow a_1+a_2+a_3=3(6-1)=3(5)=15$

or $a_1+a_2+a_3=15$

$\Rightarrow 1+6+a_3=15$

or $a_3=15-1-6=9 \Rightarrow a_3=9$

Thus the series is

$1+5+9+\ldots$

Q7. The ratio of the sums of n terms of two series in A.P is 3n+2:n+1. Find the ratio of their 8th terms.

Solution:- Let

$S_n=\frac{n}{2}(2a+(n-1)d)$ and

$S_n'=\frac{n}{2}(2a'+(n-1)d')$

According to given condition

$S_n : S_n' = 3n+2 : n+1$

$\Rightarrow \frac{S_n}{S_n'}=\frac{3n+2}{n+1}$

or $\frac{\frac{n}{2}[2a+(n-1)d]}{\frac{n}{2}[2a'+(n-1)d']}=\frac{3n+2}{n+1}$

$\Rightarrow \frac{\cancel{\frac{2n}{2}}[a+(\frac{n-1}{2})d]}{\cancel{\frac{2n}{2}}[a'+(\frac{n-1}{2})d']}=\frac{3n+2}{n+1}$

$\Rightarrow \frac{a+(\frac{n-1}{2})d}{a'+(\frac{n-1}{2})d'}=\frac{3n+2}{n+1}$

$\because$ 8th term $=a+7d$

so put $\frac{n-1}{2}=7 \Rightarrow n-1=14 \Rightarrow n=15$

so $\frac{a+7d}{a'+7d'}=\frac{3(15)+2}{15+1}=\frac{45+2}{16}=\frac{47}{16}$

which is ratio of 8th terms.

Q8. If S_2, S_3, S_5 are the sums of $2n, 3n, 5n$ terms of an A.P., show that $S_5 = 5(S_3 - S_2)$

Solution:- According to the given condition

$\because S_2 = \frac{2n}{2}[2a_1+(2n-1)d]$

$S_3 = \frac{3n}{2}[2a_1+(3n-1)d]$

$S_5 = \frac{5n}{2}[2a_1+(5n-1)d]$

Now

$S_3 - S_2 = \frac{3n}{2}[2a_1+(3n-1)d] - \frac{2n}{2}[2a_1+(2n-1)d]$

$= \frac{n}{2}[3(2a_1+(3n-1)d)-2(2a_1+(2n-1)d)]$

$= \frac{n}{2}[3(2a_1+3nd-d)-2(2a_1+2nd-d)]$

$= \frac{n}{2}[6a_1+9nd-3d-4a_1-8nd+2d]$

$= \frac{n}{2}[2a_1+5nd-d]$

$\rightarrow S_3 - S_2 = \frac{n}{2}[2a_1+(5n-1)d]$

or $5(S_3-S_2) = \frac{5n}{2}[2a_1+(5n-1)d]$

$5(S_3-S_2) = S_5$

or $S_5 = 5(S_3-S_2)$ Hence proved

Q9. Obtain the sum of all integers in the first 1000 integers which are neither divisible by 5 or by 2.

Solution:- First thousand (1000) integers which are neither divisible by 5 nor by 2 are

$1+3+7+9+11+13+17+19+21+23+27+29$
$+\dots+991+993+997+999$

Adding four, four numbers

$(1+3+7+9)+(11+13+17+19)+(21+23+27+29)$
$+\dots\dots+(991+993+997+999)$

$20+60+100+\dots+3980$

To find n, $a_n = a_1+(n-1)d$

$\rightarrow 3980 = 20+(n-1)40$

$\frac{3980-20}{40} = n-1$

$\rightarrow \frac{3960}{40} = n-1 \rightarrow n-1 = 99$

$n = 99+1 \rightarrow n = 100$

$\because S_n = \frac{n}{2}(a_1+a_n)$, $a_1 = 20$, $a_n = 3980$

$\rightarrow S_{100} = \frac{100}{2}(20+3980)$

$S_{100} = 50(4000) = 200000$

Q10. S_8 and S_9 are the sums of the first eight and nine terms of an A.P., find S_9 if $50S_9 = 63S_8$ and $a_1 = 2$

Solution:- Here $S_9 = ?$ $a_1 = 2$

$50S_9 = 63S_8$ (given)

$\rightarrow \overset{25}{\cancel{50}}\cdot\frac{9}{\cancel{2}}[2a_1+(9-1)d] = 63\cdot\frac{\overset{4}{\cancel{8}}}{\cancel{2}}[2a_1+(8-1)d]$

$\rightarrow 225(2(2)+8d) = 252(2(2)+7d)$

$225(4+8d) = 252(4+7d)$

or $900+1800d = 1008+1764d$

$\rightarrow 1800d-1764d = 1008-900$

$\rightarrow \cancel{36}d = \overset{3}{\cancel{108}} \rightarrow d = 3$

$\because S_n = \frac{n}{2}[2a_1+(n-1)d]$

$\rightarrow S_9 = \frac{9}{2}(2(2)+(9-1)(3))$

$S_9 = \frac{9}{2}(4+8(3))$

$S_9 = \frac{9}{2}(4+24) = \frac{9}{2}(28)$

$S_9 = 9(14)$

$\rightarrow S_9 = 126$

Q11. The sum of 9 terms of an A.P is 171 and its eighth term is 31. Find the series.

Solution:- Here $S_9 = 171$, $a_8 = 31$

$\because a_8 = a_1 + 7d = 31$

and $S_9 = \frac{9}{2}(2a_1 + 8d)$

$\rightarrow 171 = \frac{9}{2}(2a_1 + 8d)$

$171 = \frac{9(2)}{2}(a_1 + 4d)$

$171 = 9(a_1 + 4d)$

or $9a_1 + 36d = 171 \longrightarrow (i)$

Also $a_1 + 7d = 31 \longrightarrow (ii)$

'x' (ii) by 9, we get

$9a_1 + 63d = 279 \longrightarrow (iii)$

By (iii) − (i) $\rightarrow$ $9a_1 + 63d = 279$

$\underline{-9a_1 \pm 36d = -171}$

$27d = 108$

$\rightarrow d = 4$ put in (ii)

$a_1 + 7(4) = 31 \rightarrow a_1 + 28 = 31$

$a_1 = 31 - 28 \rightarrow a_1 = 3$

Now

$a_1 = 3$, $a_2 = a_1 + d = 3 + 4 = 7$

$a_3 = a_1 + 2d = 3 + 2(4) = 3 + 8 = 11$

so series $3 + 7 + 11 + \ldots\ldots$

Q12. The sum of S_9 and S_7 is 203 and $S_9 - S_7 = 49$. S_7 and S_9 being the sum of first 7 and 9 terms of an A.P. respectively. Determine the series.

Solution:- Given that

$S_9 + S_7 = 203 \longrightarrow (i)$

$S_9 - S_7 = 49 \longrightarrow (ii)$

By (i) + (ii) $\rightarrow 2S_9 = 252$

$\rightarrow S_9 = 126$ put in (i)

$126 + S_7 = 203$

$\rightarrow S_7 = 203 - 126$

$S_7 = 77$

$\because S_n = \frac{n}{2}(2a_1 + (n-1)d)$

$\rightarrow S_9 = \frac{9}{2}(2a_1 + (9-1)d)$

$S_9 = \frac{9}{2}(2a_1 + 8d) \longrightarrow (iii)$

and $S_7 = \frac{7}{2}(2a_1 + (7-1)d)$

$S_7 = \frac{7}{2}(2a_1 + 6d) \longrightarrow (iv)$

from (iii)

$126 = \frac{9}{2}(2a_1 + 8d)$ $\because S_9 = 126$

$126 = 9(a_1 + 4d)$

$\rightarrow 9a_1 + 36d = 126$

$a_1 + 4d = 14 \longrightarrow (v)$

from (iv)

$77 = \frac{7}{2}(2a_1 + 6d)$ $\because S_7 = 77$

$\rightarrow 77 = 7(a_1 + 3d)$

or $a_1 + 3d = 11 \longrightarrow (vi)$

By (v) − (vi) $\rightarrow$ $a_1 + 4d = 14$

$\underline{-a_1 \pm 3d = -11}$

$d = 3$ put in (v)

$a_1 + 4(3) = 14 \rightarrow a_1 = 14 - 12$

$a_1 = 2$ Now

$a_1 = 2$, $a_2 = a_1 + d = 2 + 3 = 5$

$a_3 = a_1 + 2d = 2 + 2(3) = 2 + 6 = 8$

so series $2 + 5 + 8 + \ldots\ldots$

Q13. S_7 and S_9 are the sums of the first 7 and 9 terms of an A.P. respectively. If $\frac{S_9}{S_7} = \frac{18}{11}$ and $a_7 = 20$, find the series.

Solution:- Given that

$\frac{S_9}{S_7} = \frac{18}{11}$

$\rightarrow \frac{\frac{9}{2}[2a_1 + 8d]}{\frac{7}{2}[2a_1 + 6d]} = \frac{18}{11}$

→ $\frac{9(a_1+4d)}{7(a_1+3d)} = \frac{18}{11}$

→ $\frac{9a_1+36d}{7a_1+21d} = \frac{18}{11}$

→ $99a_1+396d = 126a_1+378d$

$396d - 378d = 126a_1 - 99a_1$

or $18d = 27a_1$

→ $27a_1 - 18d = 0 \longrightarrow (i)$

Also $a_7 = a_1+6d = 20$ (given)

→ $a_1+6d = 20$

→ $3a_1+18d = 60 \longrightarrow (ii)$

By (i) + (ii) → $27a_1 - 18d = 0$

$3a_1 + 18d = 60$

$30a_1 = 60$

→ $a_1 = 2$ put in (ii)

$3(2)+18d = 60$

$18d = 60-6$

$18d = 54 \rightarrow d = 3$

Now $a_1 = 2$, $a_2 = a_1+d = 2+3 = 5$

$a_3 = a_1+2d = 2+2(3) = 2+6 = 8$

so series $2+5+8+\dots$

Q14. The sum of three numbers in an A.P. is 24 and their product is 440. Find the number.

Solution:- Suppose the numbers are a_1-d , a_1 , a_1+d in A.P then

$a_1-d+a_1+a_1+d = 24$

→ $3a_1 = 24 \rightarrow a_1 = 8$

and $(a_1-d)(a_1)(a_1+d) = 440$

→ $a_1(a_1^2-d^2) = 440$

$8(64-d^2) = 440$

$64-d^2 = 55$

→ $64-55 = d^2$

$d^2 = 9 \rightarrow d = \pm 3$

$d = 3$, $d = -3$

when $d=-3$ then

$a_1-d = 8-(-3) = 8+3 = 11$, $a_1 = 8$

$a_1+d = 8+(-3) = 8-3 = 5$

Hence ∴ 11,8,5 required numbers.

when $d = 3$ then

$a_1-d = 8-3 = 5$, $a_1 = 8$, $a_1+d = 8+3 = 11$

Hence 5, 8, 11 required numbers.

Q15. Find four numbers in A.P. whose sum is 32 and the sum of whose square is 276.

Solution:- Suppose four numbers in A.P. are a_1-3d , a_1-d, a_1+d, a_1+3d in A.P

I condition → $a_1-3d+a_1-d+a_1+d+a_1+3d = 32$

→ $4a_1 = 32 \rightarrow a_1 = 8$

II condition →

$(a_1-3d)^2+(a_1-d)^2+(a_1+d)^2+(a_1+3d)^2 = 276$

$a_1^2 - \cancel{6a_1d} + 9d^2 + a_1^2 - \cancel{2a_1d} + d^2 + a_1^2 + d^2 + \cancel{2a_1d} + a_1^2 + 9d^2 + \cancel{6a_1d} = 276$

→ $4a_1^2+20d^2 = 276$

→ $a_1^2+5d^2 = 69$ (÷ by 4)

or $(8)^2+5d^2 = 69$ ($\because a_1 = 8$)

$64+5d^2 = 69$

$5d^2 = 69-64 \rightarrow 5d^2 = 5$

$d^2 = 1 \rightarrow d = \pm 1$

when $d = 1$ then

$a_1-3d = 8-3(1) = 8-3 = 5$

$a_1-d = 8-1 = 7$

$a_1+d = 8+1 = 9$

$a_1+3d = 8+3(1) = 8+3 = 11$

Hence 5,7,9,11 required numbers

when $d = -1$ then

$a_1-3d = 8-3(-1) = 8+3 = 11$

$a_1-d = 8-(-1) = 8+1 = 9$

$a_1+3d = 8+3(-1) = 8-3 = 5$

$a_1+d = 8-1 = 7$

Hence 11,9,7,5 required numbers.

Q16. Find the five numbers in A.P. whose sum is 25 and the sum of whose square is 135.

Solution:- suppose five numbers in A.P. are $a_1-2d, a_1-d, a_1, a_1+d, a_1+2d$

I condition →

$a_1-2d+a_1-d+a_1+a_1+d+a_1+2d=25$

$\rightarrow 5a_1=25 \rightarrow a_1=5$

II condition →

$(a_1-2d)^2+(a_1-d)^2+a_1^2+(a_1+d)^2+(a_1+2d)^2=135$

$a_1^2-4\cancel{a_1}d+4d^2+a_1^2-2\cancel{a_1}d+d^2+a_1^2+a_1^2+2\cancel{a_1}d$

$+d^2+a_1^2+4d^2+\cancel{4a_1}d=135$

$\rightarrow 5a_1^2+10d^2=135$

$\rightarrow a_1^2+2d^2=27$

or $(5)^2+2d^2=27 \qquad \because a_1=5$

$\rightarrow 25+2d^2=27$

$\rightarrow 2d^2=27-25 \rightarrow 2d^2=2$

or $d^2=1$ or $d=\pm 1$

when $d=1$

$a_1-2d=5-2(1)=3$, $a_1-d=5-1=4$

$a_1=5$, $a_1+d=5+1=6$

$a_1+2d=5+2(1)=5+2=7$

Hence 3, 4, 5, 6, 7 required numbers.

when $d=-1$

$a_1-2d=5-2(-1)=5+2=7$

$a_1-d=5-(-1)=5+1=6$

$a_1=5$, $a_1+2d=5+2(-1)=5-2=3$

$a_1+d=5+(-1)=5-1=4$

Hence 7, 6, 5, 4, 3 req. numbers

Q17. The sum of the 6th and 8th terms of an A.P. is 40. and the product of the 4th and 7th term is 220. Find the A.P.

Solution:- Given that

$a_6+a_8=40 \longrightarrow (i)$ and

$(a_4)(a_7)=220 \longrightarrow (ii)$

$i) \rightarrow a_1+5d+a_1+7d=40$

$2a_1+12d=40$

$\rightarrow a_1+6d=20 \longrightarrow (iii)$

$ii) \rightarrow (a_1+3d)(a_1+6d)=220$

$\rightarrow (a_1+3d)\cancel{(20)}=\overset{11}{\cancel{220}}$ from (iii)

or $a_1+3d=11 \longrightarrow (iv)$

By $(iii)-(iv) \rightarrow a_1+6d=20$

$\underset{-}{}a_1 \pm 3d=\underset{-}{}11$

$3d=9$

or $d=3$ put in (iv)

$a_1+3(3)=11 \rightarrow a_1+9=11$

$a_1=11-9 \rightarrow a_1=2$ Now

$a_1=2$, $a_2=a_1+d=2+3=5$

$a_3=a_1+2d=2+2(3)=2+6=8$

Hence A.P is 2, 5, 8.

Q18. If a^2, b^2 and c^2 are in A.P., show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

Solution:- $\because a^2, b^2, c^2$ are in A.P

so $c^2-b^2=b^2-a^2 \longrightarrow (i)$

Now if $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P

then

$$\frac{1}{a+b}-\frac{1}{c+a}=\frac{1}{c+a}-\frac{1}{b+c}$$

$$\rightarrow \frac{c+a-(a+b)}{(a+b)(c\cancel{+a})}=\frac{b+c-(c+a)}{(\cancel{c+a})(b+c)}$$

$$\rightarrow \frac{c+a-a-b}{a+b}=\frac{b+c-c-a}{b+c}$$

$$\text{or} \quad \frac{c-b}{a+b}=\frac{b-a}{b+c}$$

$\rightarrow c^2-b^2=b^2-a^2$

$\rightarrow c^2-b^2=c^2-b^2 \longrightarrow$ use (i)

(common difference is same)

Hence $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

Word Problems on A.P

Example 1. Tickets for a certain show were printed bearing numbers from 1 to 100. Odd number tickets were sold by receving paisas equal to thrice of the number on the ticket while even number tickets were issued by receiving paisas equal to twice of the number on the ticket. How much amount was received by issuing agency?

Solution:- Let S_1 and S_2 be the amounts received for odd and even number tickets resp. then

$$S_1 = 3\left[1+3+5+\dots+99\right] \quad \because a_1=1,\ a_n=99,\ d=3-1=2,\ n=50$$

$$S_2 = 2\left[2+4+6+\dots+100\right] \quad a_1=2,\ d=4-2=2,\ a_n=100,\ n=50$$

Now

$$S_1 + S_2 = 3\times\frac{50}{2}(1+99)+2\times\frac{50}{2}(2+100)$$

$$= 7500 + 5100 = 12600$$

$$\because S_n = \frac{n}{2}(a_1+a_n)$$

Hence

total amount = 12600 paisas = Rs.126

Example 2. A man repays his loan of Rs. 1120 by Paying Rs. 15 in the first installment and then increases the payment by Rs. 10 every month. How long will it take to clear his loan?

Solution:-

Total loan = $S_n = 1120$

First installment = $a_1 = 15$

increasment = $d = 10$

period = $n = ?$

$$\because S_n = \frac{n}{2}\left(2a_1+(n-1)d\right)$$

$$1120 = \frac{n}{2}\left(2(15)+(n-1)(10)\right)$$

$$\Rightarrow 2240 = n(30+10n-10)$$

$$2240 = n(20+10n)$$

$$2240 = 10n^2+20n$$

$$\Rightarrow 224 = n^2+2n$$

$$\text{or } n^2+2n-224=0$$

$$n^2+16n-14n-224=0$$

$$n(n+16)-14(n+16)=0$$

$$(n+16)(n-14)=0$$

$$\text{or } n+16=0, \quad n-14=0$$

$$n=-16, \quad n=14$$

(ignore)

Thus 14 months is the period to clear his loan.

Example 3. A manufacturer of radio sets produced 625 units in the 4th year and 700 units in the 7th year. Assuming that production uniformly increases by a fixed number every year, find

i) The Production in the first year

ii) The total production in 8 years

iii) The production in the 11th year.

Solution:- Given that

Production of radio sets in 4th year = $a_4 = 625$

Production in 7th year = $a_7 = 700$

(i) $a_1 = ?$ (ii) $S_8 = ?$ (iii) $a_{11} = ?$

$$\because a_4 = a_1+3d, \quad a_7 = a_1+6d$$

$$\Rightarrow a_1+3d = 625 \longrightarrow (i), \quad a_1+6d = 700 \longrightarrow (ii)$$

By (ii) − (i) →

$$\begin{array}{r} a_1+6d = 700 \\ \underset{-}{a_1}+\underset{-}{3d} = \underset{-}{625} \\ \hline 3d = 75 \end{array} \Rightarrow d = 25$$

i) put $d=25$ in (i) $a_1+3(25)=625$

$$a_1 = 625-75 = 550$$

ii) $\because S_n = \frac{n}{2}\left(2a_1+(n-1)d\right)$

$$\Rightarrow S_8 = \frac{8}{2}\left(2(550)+(8-1)(25)\right)$$

$$S_8 = 4(1100+175) = 4(1275)$$

$$S_8 = 5100 \text{ units}$$

iii) $\because a_{11} = a_1 + 10d$

$\rightarrow a_{11} = 550 + 10(25) = 550 + 250$

$a_{11} = 800$ units

Exercise 6.5

Q1. A man deposits in a bank Rs. 10 in the first month; Rs. 15 in the second month; Rs. 20 in the third month and so on. Find how much he will have deposited in the bank by the 9th month.

Solution:- Given

$10 + 15 + 20 + \ldots\ldots + a_9$

$a_1 = 10,\ d = 15 - 10 = 5,\quad n = 9$

$\because S_n = \frac{n}{2}(2a_1 + (n-1)d)$

$S_9 = \frac{9}{2}(2(10) + (9-1)(5))$

$= \frac{9}{2}(20 + 8(5))$

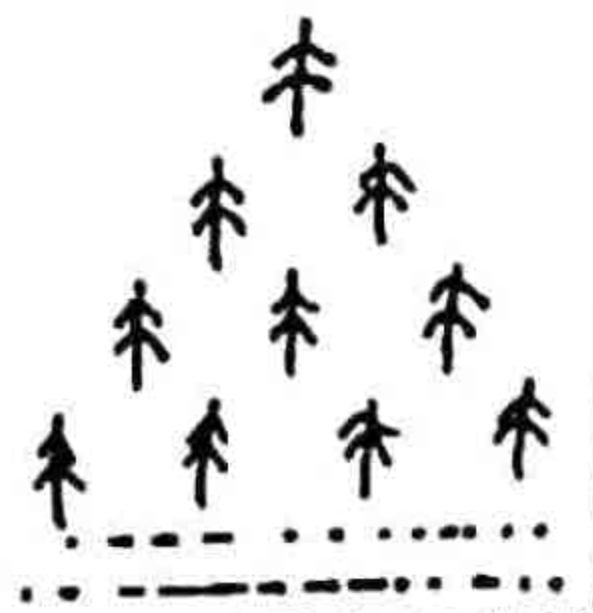
$= \frac{9}{2}(20 + 40) = \frac{9}{2}(60)$

$\rightarrow S_9 = 9(30) \rightarrow S_9 = 270$

Q2. 378 trees are planted in rows in the shape of an isosceles triangle, the number in successive rows decreasing by one from the base to the top. How many trees are there in the row which forms the base of the triangle?

Solution:- In first row we have 1 tree, in second 2 trees, in third 3 and so on, we have

$1 + 2 + 3 + \ldots\ldots + a_n = 378$



$S_n = 378$

Here

$S_n = 378,\ a_1 = 1,\ d = 2 - 1 = 1$

$n = ?$

$S_n = \frac{n}{2}(2a_1 + (n-1)d)$

$\rightarrow 378 = \frac{n}{2}(2(1) + (n-1)(1))$

$378 = \frac{n}{2}(2 + n - 1)$

$756 = n(n+1)$

$\rightarrow n^2 + n - 756 = 0$

$n^2 + 28n - 27n - 756 = 0$

$n(n+28) - 27(n+28) = 0$

$\rightarrow (n+28)(n-27) = 0$

$n + 28 = 0,\quad n - 27 = 0$

$n = -28,\quad n = 27$

(ignore)

Hence $n = 27 =$ total rows in isosceles

Q3. A man borrows Rs. 1100 and agree to repay with a total interest of Rs. 230 in 14 installments, each installment being less than the preceding by Rs. 10. What should be his first installment?

Solution:-

Total amount to repay $= 1100 + 230 = 1330$

so $S_n = 1330,\ n = 14,\ d = -10,\ a_1 = ?$

$\because S_n = \frac{n}{2}(2a_1 + (n-1)d)$

$1330 = \frac{14}{2}(2a_1 + (14-1)(-10))$

$1330 = 7(2a_1 + 13(-10))$

$1330 = 7(2a_1 - 130)$

$\rightarrow 190 = 2a_1 - 130\quad (\div \text{ by } 7)$

$2a_1 = 190 + 130$

$2a_1 = 320 \rightarrow a_1 = 160$

so first installment $= 160$

Q4. A clock strikes once when its hour hand is at one, twice when it is at two and so on. How many times does the clock strike in twelve hours?

Solution:-

According to statement

$1 + 2 + 3 + \ldots\ldots + a_{12} = ?$

$a_1 = 1,\ d = 2 - 1 = 1,\ n = 12,\ S_n = ?$

$S_n = \frac{n}{2}(2a_1 + (n-1)d)$

→ $S_{12} = \frac{12}{2}(2(1)+(12-1)) = 6(2+11)$

→ $S_{12} = 6(13) = 78$

Q5. A student save Rs. 12 at the end of the first week and goes on increasing his saving Rs. 4 weekly. After how many weeks will he be able to save Rs. 2100?

Solution:- In I^{st} week = 12

In 2^{nd} week = 12+4 = 16

In 3^{rd} week = 16+4 = 20

so series is

$12+16+20+\dots\dots+a_n = 2100$

$a_1 = 12,\ d = 16-12 = 4,\ n = ?$

$S_n = 2100$

$\because S_n = \frac{n}{2}(2a_1+(n-1)d)$

→ $2100 = \frac{n}{2}(2(12)+(n-1)4)$

$= \frac{n}{2}(24+4n-4)$

$2100 = \frac{n}{2}(20+4n)$

$2100 = \frac{2n}{2}(10+2n)$

→ $n(10+2n) = 2100$

→ $2n^2+10n-2100 = 0$

→ $n^2+5n-1050 = 0$ (÷ by 2)

$n^2+35n-30n-1050=0$

→ $n(n+35)-30(n+35)=0$

$(n+35)(n-30)=0$

→ $n+35=0,\quad n-30=0$

$n=-35,\quad n=30$

(Not possible)

He will have 2100 in 30 weeks.

Q6. An object falling rest, falls 9 meters during the first second, 27 meters during the next second, 45 meters during the third second and so on.

i) How far will it fall during the fifth second?

ii) How far will it fall up to the fifth second?

Solution:- Given

$9, 27, 45 \dots\dots$

i) $a_1 = 9,\ d = 27-9 = 18,\ n = 5$

$\because a_n = a_1+(n-1)d$

→ $a_5 = 9+(5-1)(18)$

→ $a_5 = 9+(4)(18) = 9+72$

→ $a_5 = 81$ meters

ii) $a_1 = 9,\ d = 18,\ n = 5,$

$\because S_n = \frac{n}{2}(2a_1+(n-1)d)$

→ $S_5 = \frac{5}{2}(2(9)+(5-1)(18))$

→ $S_5 = \frac{5}{2}(18+4(18))$

→ $S_5 = \frac{5}{2}(18+72) = \frac{5}{2}(90)$

→ $S_5 = 225$ meters

Q7. An investor earned Rs.6000 for year 1980 and Rs. 12000 for year 1990 on the same investment. If his earing have increased by the same amount each year, how much income he has recieved from the investment over the past eleven years?

Solution:- Here

$a_1 = 600,\ n = 11,\ a_n = 12000,$

$S_n = ?$

$\because S_n = \frac{n}{2}(a_1+a_n) = \frac{11}{2}(6000+12000)$

$S_{11} = \frac{11}{2}(18000) = 99000$

Q8. The sum of interior angles of polygons having sides 3, 4, 5, ... etc form an A.P. Find the sum of the interior angles for a 16 sided polygon.

Solution:-

Sum of angle 3-sided polygon = π

sum of angle 4-sided polygon = 2π

sum of angle 5-sided polygon = 3π

A.P. is $\pi, 2\pi, 3\pi, \ldots\ldots a_{14}$

$\because a_n = a_1 + (n-1)d \quad (a_1 = \pi,\ d = 2\pi - \pi = \pi,\ n = 14)$

$\rightarrow a_{14} = \pi + (14-1)(\pi)$

$a_{14} = \pi + 13\pi = 14\pi$

Q9. The prize money Rs. 60,000 will be distributed among the eight teams according to their positions determined in the match series. The award increases by the same amount for each higher positions. If the last place team is given Rs. 4000, how much will be awarded to the first place team?

Solution:- Given $S_n = 60,000$, $n = 8$, $a_8 = 4000$, $a_1 = ?$

$\because S_n = \frac{n}{2}(a_1 + a_{\cdot})$

$\rightarrow 60,000 = \frac{8}{2}(a_1 + 4000)$

$\rightarrow 60,000 = 4(a_1 + 4000)$

$15000 = a_1 + 4000$

$\rightarrow a_1 = 15000 - 4000 = 11000$

$\rightarrow a_1 = 11000$

Q10. An equilateral triangular base is filled by placing eight balls in the first row, 7 balls in the second row and so on with one ball in the last row. After this base layer, second layer is formed by placing 7 balls in the first row. Continuing this process, a pyramid of balls is formed with one ball on top. How many balls are their in the pyramid?

Let $S_1, S_2, \ldots\ldots, S_8$ denote sum of $1, 2, \ldots\ldots, 8$ layer so

$S_8 = 8 + 7 + 6 + \ldots\ldots + 1$

$= \frac{n}{2}(a_1 + a_n) = \frac{8}{2}(8+1) = 36$

$\therefore S_7 = 7 + 6 + 5 + \ldots\ldots + 1$

$= \frac{7}{2}(7+1) = \frac{7}{2}(8) = 28$

$S_6 = 6 + 5 + \ldots\ldots + 1$

$= \frac{6}{2}(6+1) = \frac{6}{2}(7) = 21$

$S_5 = 5 + 4 + 3 + 2 + 1 = 15$

$S_4 = 4 + 3 + 2 + 1 = 10$

$S_3 = 3 + 2 + 1 = 6$

$S_2 = 2 + 1 = 3$

$S_1 = 1 = 1$

Total balls $= 36 + 28 + 21 + 15 + 10 + 6 + 3 + 1$

$= 128$

Geometric Progression (G.P)

A sequence $\{a_n\}$ is a geometric sequence if $\frac{a_n}{a_{n-1}}$ is same for all terms when $n \in N$ and $n > 1$.

* The quotient $\frac{a_n}{a_{n-1}}$ is denoted by r and is called common ratio of the G.P.

* The common ratio $r = \frac{a_n}{a_{n-1}}$ is defined only if $a_{n-1} \neq 0$. i-e., No term of G.P. is zero.

Note:- When common ratio of any two consecutive terms of a sequence is called a geometric sequence or geometric progression.

Prove that $a_n = a_1 r^{n-1}$

Proof:-

we know that

$$r = \frac{a_n}{a_{n-1}}, \quad n>1 \longrightarrow (i)$$

Put $n = 2, 3, 4 \ldots$ in (i)

$\rightarrow r = \frac{a_2}{a_1} \rightarrow a_2 = a_1 r$

$r = \frac{a_3}{a_2} \rightarrow a_3 = a_2 r$

or $a_3 = (a_1 r) r \rightarrow a_3 = a_1 r^2$ $(\because a_2 = a_1 r)$

$r = \frac{a_4}{a_3} \rightarrow a_4 = a_3 r = (a_1 r^2) r$

$\rightarrow a_4 = a_1 r^3$ $(\because a_3 = a_1 r^2)$

In similar way, we have

$a_n = a_1 r^{n-1}$ Hence proved

Example 1. Find the 5th term of the G.P., 3, 6, 12, ...

Solution:-

Here $a_1 = 3$, $a_2 = 6$, $a_3 = 12$

$r = \frac{a_2}{a_1} = \frac{6}{3} = 2$, $a_5 = ?$

$\because a_5 = a_1 r^4$ $\quad \because a_n = a_1 r^{n-1}$

$a_5 = (3)(2)^4 = (3)(16) = 48$

Example 2. Find a_n if $a_4 = \frac{8}{27}$ and $a_7 = -\frac{64}{729}$ of a G.P.

Solution:- Here $a_n = ?$

$a_4 = \frac{8}{27}$ $\quad \because a_n = a_1 r^{n-1}$

$\rightarrow a_1 r^3 = \frac{8}{27} \longrightarrow (i)$

and $a_7 = -\frac{64}{729}$

$\rightarrow a_1 r^6 = \frac{-64}{729} \longrightarrow (ii)$

By $\frac{(ii)}{(i)} \rightarrow \frac{a_1 r^6}{a_1 r^3} = \frac{-64/729}{8/27}$

$\rightarrow r^3 = \frac{-64}{729} \times \frac{27}{8}$

$\rightarrow r^3 = -\frac{8}{27}$

or $(r)^3 = \left(-\frac{2}{3}\right)^3$

$\rightarrow r = -\frac{2}{3}$ put in (i)

$a_1 \left(-\frac{2}{3}\right)^3 = \frac{8}{27}$

$\rightarrow a_1 = \frac{8}{27} \times \frac{-27}{8}$

$a_1 = -1$

$\because a_n = a_1 r^{n-1}$

$\rightarrow a_n = (-1)\left(\frac{-2}{3}\right)^{n-1}$

$a_n = (-1)^1 (-1)^{n-1} \left(\frac{2}{3}\right)^{n-1}$

$a_n = (-1)^n \left(\frac{2}{3}\right)^{n-1}$ for $n \geq 1$

Example 3. If the numbers 1, 4 and 3 are added to there consecutive terms of G.P., the resulting numbers are in A.P. Find the numbers if their sum is 13.

Solution:- Let three consecutive terms of G.P. are

a, ar, ar^2

I condition $\rightarrow$

$a+1, ar+4, ar^2+3$ are in A.P.

Thus

$(ar+4)-(a+1) = (ar^2+3)-(ar+4)$

$\rightarrow ar+4-a-1 = ar^2+3-ar-4$

$ar-a+3 = ar^2-ar-1$

or $ar^2-ar-1-ar+a-3 = 0$

$ar^2-2ar+a = 4$

$\rightarrow a(r^2-2r+1) = 4 \longrightarrow (i)$

II Condition $\rightarrow a+ar+ar^2 = 13$

$\rightarrow a(1+r+r^2) = 13 \longrightarrow (ii)$

By $\frac{(ii)}{(i)} \rightarrow \frac{\cancel{a}(r^2+r+1)}{\cancel{a}(r^2-2r+1)} = \frac{13}{4}$

$\rightarrow 4(r^2+r+1) = 13(r^2-2r+1)$

$4r^2+4r+4 = 13r^2-26r+13$

or $13r^2-26r+13-4r^2-4r-4=0$

or $9r^2-30r+9=0$

$\rightarrow 3r^2 - 10r + 3 = 0 \quad (\div \text{by } 3)$

$3r^2 - 9r - r + 3 = 0$

$3r(r-3) - 1(r-3) = 0$

$(r-3)(3r-1) = 0$

$r - 3 = 0, \quad 3r - 1 = 0$

$r = 3 \quad , \quad r = \frac{1}{3}$

when $r = 3$ so (ii)

$a[(3)^2 + 3 + 1] = 13 \rightarrow a(13) = 13$

or $a = 1$ Now

$ar = (1)(3) = 3 \quad , \quad ar^2 = (1)(3)^2 = 9$

so req. numbers are 1, 3, 9

when $r = \frac{1}{3}$

$a\left[\left(\frac{1}{3}\right)^2 + \frac{1}{3} + 1\right] = 13 \rightarrow a\left(\frac{1}{9} + \frac{1}{3} + 1\right) = 13$

$a\left(\frac{1+3+9}{9}\right) = 13 \rightarrow a\left(\frac{\cancel{13}}{9}\right) = \cancel{13}$

$\rightarrow a = 9$ so

$ar = 9\left(\frac{1}{3}\right) = 3 \quad , \quad ar^2 = 9\left(\frac{1}{3}\right)^2 = 9\left(\frac{1}{9}\right)$

$ar^2 = 1$

Thus req. numbers are 9, 3, 1

Exercise 6.6

Q1. Find the 5th term of the G. P 3, 6, 12, ...

Solution:-

Her $a_1 = 3, \quad r = \frac{6}{3} = 2, \quad a_5 = ?$

$\because a_5 = a_1 r^4 \qquad \because a_n = a_1 r^{n-1}$

$a_5 = (3)(2)^4 = 3(16) = 48$

Q2. Find the 11th term of the sequence $1+i, 2i, -2+2i, \ldots$

Solution:-

Here $a_1 = 1+i \quad , \quad r = \frac{2}{1+i} \quad , \quad a_{11} = ?$

$\because a_{11} = a_1 r^{10} \qquad \because a_n = a_1 r^{n-1}$

$= (1+i)\left(\frac{2}{1+i}\right)^{10}$

$= (1+i)\frac{2^{10}}{(1+i)^{10}}$

$= \frac{(1+i)2^{10}}{[(1+i)^2]^5}$

$\because (1+i)^2$
$= (1)^2 + (i)^2 + 2(1)(i)$
$= 1 + i^2 + 2i$
$= 1 - 1 + 2i = 2i$
$\because i^2 = -1$

$= \frac{(1+i)1024}{(2i)^5}$

$= \frac{(1+i)1024}{32i^5}$

$= \frac{(1+i)(1024)}{32i^4 \cdot i} = \frac{(1+i)32}{(i^2)^2 \cdot i}$

$= \frac{(1+i)32}{(-1)^2 i} = \frac{(1+i)32}{i}$

$= \frac{(1+i)32}{i} \times \frac{i}{i} = \frac{32(i+i^2)}{i^2}$

$= \frac{32(i-1)}{-1} = -32(i-1)$

$(\because i^2 = -1)$

$a_{11} = 32(1-i)$

Q3. Find the 12th term of $1+i, 2i, -2+2i, \ldots$

Solution:-

Here $a_1 = 1+i, \quad r = \frac{2i}{1+i} \quad , \quad a_{12} = ?$

$\because a_{12} = a_1 r^{11} \qquad \because a_n = a_1 r^{n-1}$

$= (1+i)\left(\frac{2i}{1+i}\right)^{11}$

$= \frac{(1+i)(2i)^{11}}{(1+i)^{11}} = \frac{2^{11} \times i^{11}}{(1+i)^{10}}$

$= \frac{2048 \times i^{10} \cdot i}{[(1+i)^2]^5} = \frac{2048 \times i^{11}}{(2i)^5}$

$= \frac{\overset{64}{\cancel{2048}}\, i^{11}}{\cancel{32}\, i^5}$

$\because (1+i)^2$
$= (1)^2 + (i)^2 + 2(1)i$
$= 1 + i^2 + 2i$
$= 1 - 1 + 2i \quad \because i^2 = -1$
$= 2i$

$= 64\, i^{11-5}$

$= 64\, i^6$

$= 64(i^2)^3 = 64(-1)^3$

$= -64 \qquad \rightarrow a_{12} = -64$

Q4. Find the 11th term of the sequence, $1+i, 2, 2(1-i)$

Solution:- Here

$a_1 = 1+i, \quad r = \frac{2}{1+i} \quad , \quad a_{11} = ?$

∵ $a_{11} = a_1 r^{10}$ ∵ $a_n = a_1 r^{n-1}$

$$= (1+i)\left(\frac{2}{1+i}\right)^{10}$$

$$= \frac{(1+i)\;2^{10}}{(1+i)^{10}}$$

$$= \frac{(1+i)\;2^{10}}{\left[(1+i)^2\right]^5}$$

$$= \frac{(1+i)\;1024}{(2i)^5}$$

$$= \frac{(1+i)\;\overset{32}{\cancel{1024}}}{\cancel{32}\,i^5}$$

∵ $(1+i)^2$
$= (1)^2+(i)^2+2(1)(i)$
$= 1+i^2+2i$
$= 1-1+2i$
$= 2i$ ∵ $i^2=-1$

$$= \frac{(1+i)\,32}{i^4\cdot i} = \frac{(1+i)\,32}{(i^2)^2\cdot i}$$

$$= \frac{(1+i)\,32}{(-1)^2\cdot i} = \frac{(1+i)\,32}{i}$$

$$= \frac{(1+i)\,32}{i}\times\frac{i}{i} = \frac{(i+i^2)\,32}{i^2}$$

$$a_{11} = \frac{(i-1)\,32}{-1} = -32\,(i-1)$$

$a_{11} = 32(1-i)$ (∵ $i^2=-1$)

Q5. If an automobile depreciates in value 5% every year, at the end of 4 years. what is the value of the automobile purchased for Rs. 12,000?

Solution:- Since 5% depreciates

so

$r = 100\% - 5\% = 95\% = \frac{95}{100} = 0.95$

$a_1 = 12000$, $n=5$

∵ $a_5 = a_1 r^4$ $\boxed{∵ a_n = a_1 r^{n-1}}$

→ $a_5 = (12000)(0.95)^4$

$a_5 = (12000)(0.8145) = 9774$ Rs.

Q6. Which term of the sequence: $x^2-y^2,\ x+y,\ \frac{x+y}{x-y}, \ldots\ldots$ is $\frac{x+y}{(x-y)^9}$?

Solution:- Here

$a_1 = x^2-y^2$, $a_n = \frac{x+y}{(x-y)^9}$,

$r = \frac{x+y}{x^2-y^2} = \frac{x+y}{(x-y)(x+y)} = \frac{1}{x-y}$, $n=?$

∵ $a_n = a_1 r^{n-1}$

$$\rightarrow \frac{x+y}{(x-y)^9} = (x^2-y^2)\left(\frac{1}{x-y}\right)^{n-1}$$

$$\frac{\cancel{x+y}}{(x-y)^9} = (x-y)\cancel{(x+y)}\cdot\frac{1}{(x-y)^{n-1}}$$

$$\frac{1}{(x-y)^9} = \frac{1}{(x-y)^{n-2}} \rightarrow n-2=9$$

→ $n = 9+2$

→ $n = 11$

Q7. If a, b, c, d are in G.P., Prove that

i) $a-b,\ b-c,\ c-d$ are in G.P.,

Solution:-

∵ a, b, c, d are in G.P then

$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$ or

$\frac{b}{a} = \frac{c}{b}$, $\frac{c}{b} = \frac{d}{c}$, $\frac{d}{c} = \frac{b}{a}$

→ $b^2 = ac$, $c^2 = bd$, $ad = bc$ } → (i)

Now if $a-b,\ b-c,\ c-d$ are in G.P then

$$\frac{c-d}{b-c} = \frac{b-c}{a-b}$$

→ $(c-d)(a-b) = (b-c)(b-c)$

→ $(a-b)(c-d) = (b-c)^2$

L.H.S $= (a-b)(c-d)$

$= ac - ad - bc + bd$

$= b^2 - bc - bc + c^2$ use (i)

$= b^2 - 2bc + c^2$

$= (b-c)^2 =$ R.H.S

Hence $(a-b), (b-c), (c-d)$ are in G.P.

ii) $a^2-b^2,\ b^2-c^2,\ c^2-d^2$ are in G.P

Solution:-

If $a^2-b^2,\ b^2-c^2,\ c^2-d^2$ are in G.P.,

then $\frac{b^2-c^2}{a^2-b^2} = \frac{c^2-d^2}{b^2-c^2}$

$\rightarrow (b^2-c^2)^2 = (a^2-b^2)(c^2-d^2)$

R.H.S $= (a^2-b^2)(c^2-d^2)$

$= a^2c^2 - a^2d^2 - b^2c^2 + b^2d^2$

$= (ac)^2 - (ad)^2 - b^2c^2 + (bd)^2$

$= (b^2)^2 - b^2c^2 - b^2c^2 + (c^2)^2$ use(i)

$= (b^2)^2 - 2b^2c^2 + (c^2)^2$

$= (b^2-c^2)^2 =$ L.H.S

Hence $a^2-b^2, b^2-c^2, c^2-d^2$ are in G.P.,

iii) $a^2+b^2, b^2+c^2, c^2+d^2$ are in G.P.

Solution:-

If $a^2+b^2, b^2+c^2, c^2+d^2$ are in G.P

then $\dfrac{b^2+c^2}{a^2+b^2} = \dfrac{c^2+d^2}{b^2+c^2}$

$\rightarrow (b^2+c^2)^2 = (c^2+d^2)(a^2+b^2)$

R.H.S $= (a^2+b^2)(c^2+d^2)$

$= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$

$= (ac)^2 + (ad)^2 + b^2c^2 + (bd)^2$

$= (b^2)^2 + b^2c^2 + b^2c^2 + (c^2)^2$

$= (b^2)^2 + 2b^2c^2 + (c^2)^2$

$= (b^2+c^2)^2 =$ L.H.S

Hence $a^2+b^2, b^2+c^2, c^2+d^2$ are in G.P.

Q8. Show that the reciprocals of the terms of the geometric sequence $a_1, a_1r^2, a_1r^4, \ldots$ form another geometric sequence.

Solution:-

We have to prove that

$\dfrac{1}{a_1}, \dfrac{1}{a_1r^2}, \dfrac{1}{a_1r^4}, \ldots$ are in G.P

so $r = \dfrac{\frac{1}{a_1r^2}}{\frac{1}{a_1}} = \dfrac{1}{a_1r^2} \times \dfrac{a_1}{1}$

$\rightarrow r = \dfrac{1}{r^2}$

Also,

$r = \dfrac{\frac{1}{a_1r^4}}{\frac{1}{a_1r^2}}$

$= \dfrac{1}{a_1r^4} \times \dfrac{a_1r^2}{1}$

$r = \dfrac{1}{r^2}$

Hence ratio are same.

so $\dfrac{1}{a_1}, \dfrac{1}{a_1r^2}, \dfrac{1}{a_1r^4}$ are in G.P.

Q9. Find the nth term of the geometric sequence if;

$\dfrac{a_5}{a_3} = \dfrac{4}{9}$ and $a_2 = \dfrac{4}{9}$

Solution:- Given that

$\dfrac{a_5}{a_3} = \dfrac{4}{9} \rightarrow \dfrac{a_1r^4}{a_1r^2} = \dfrac{4}{9}$

$\rightarrow r^2 = \dfrac{4}{9} \rightarrow r = \pm\dfrac{2}{3}$

Also, $a_2 = \dfrac{4}{9} \rightarrow a_1r = \dfrac{4}{9} \longrightarrow (i)$

when $r = \frac{2}{3}$ so (i) $a_1\left(\frac{2}{3}\right) = \frac{4}{9} \rightarrow a_1 = \frac{2}{3}$

when $r = -2/3$ so (i) $\rightarrow a_1\left(-\frac{2}{3}\right) = \frac{4}{9} \rightarrow a_1 = -\frac{2}{3}$

Now if $a_1 = \frac{2}{3}$ and $r = \frac{2}{3}$ then

$a_n = a_1r^{n-1}$

$= \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{n-1}$

$a_n = \left(\frac{2}{3}\right)^n$

if $a_1 = -\frac{2}{3}$ and $r = -\frac{2}{3}$ then

$a_n = a_1r^{n-1}$

$= \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)^{n-1}$

$a_n = \left(-\frac{2}{3}\right)^n = (-1)^n\left(\frac{2}{3}\right)^n$

Q10. Find three, consecutive numbers in G.P whose sum is 26 and their product is 216.

Solution:- Let three consecutive numbers in G.P are a, ar, ar^2

I condition $\rightarrow a + ar + ar^2 = 26$

$\rightarrow a(1+r+r^2) = 26 \longrightarrow (i)$

II condition $\rightarrow (a)(ar)(ar^2) = 216$

$\rightarrow a^3r^3 = 216$

or $(ar)^3 = (6)^3$

$\rightarrow ar = 6 \longrightarrow (ii)$

By $\frac{(i)}{(ii)} \rightarrow \frac{a(r^2+r+1)}{ar} = \frac{26}{6}$

or $\frac{r^2+r+1}{r} = \frac{26}{6}$

$\rightarrow 26r = 6r^2 + 6r + 6$

or $6r^2 + 6r + 6 - 26r = 0$

$6r^2 - 20r + 6 = 0$

$(\div by 2)$ $3r^2 - 10r + 3 = 0$

$3r^2 - 9r - r + 3 = 0$

$3r(r-3) - 1(r-3) = 0$

$(r-3)(3r-1) = 0$

$r-3 = 0$, $3r-1 = 0$

$r = 3$, $r = 1/3$

so (ii) , so (ii)

$a(3) = 6$, $a(1/3) = 6$

$\rightarrow a = 2$, $a = 18$

$\rightarrow ar = (2)(3) = 6$, $ar = (18)(1/3) = 6$

$ar^2 = (2)(3)^2 = 18$, $ar^2 = (18)(\frac{1}{3})^2 = 2$

Thus required numbers are

2, 6, 18 or 18, 6, 2

Q11. If the sum of the four consecutive terms of a G.P is 80 and A.M of the second and fourth of them is 30. Find the terms.

Solution:- suppose four numbers in G.P are a, ar, ar^2, ar^3

I condition $\rightarrow$

$a + ar + ar^2 + ar^3 = 80 \longrightarrow (i)$

II condition $\rightarrow \frac{ar + ar^3}{2} = 30$

$\rightarrow ar + ar^3 = 60 \xrightarrow{(ii)}$ put in (i)

$a + ar^2 + 60 = 80$

$\rightarrow a + ar^2 = 80 - 60$

$a + ar^2 = 20 \longrightarrow (iii)$

'x' (iii) by r

$\rightarrow ar + ar^3 = 20r$

from (ii) $\rightarrow 60 = 20r$

or $r = 3$ put in (iii)

$a + a(3)^2 = 20$

$a + 9a = 20 \rightarrow 10a = 20$

or $a = 2$

so $a = 2$, $ar = (2)(3) = 6$

$ar^2 = (2)(3)^2 = (2)(9) = 18$

$ar^3 = (2)(3)^3 = (2)(27) = 54$

Hence required numbers are

2, 6, 18, 54

Q12. If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in G.P. show that the common ratio is $\pm\sqrt{\frac{a}{c}}$

Solution:-

$\because \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in G.P. so common ratio r is

$r = \frac{\frac{1}{b}}{1/a} = \frac{1}{b} \times \frac{a}{1} = \frac{a}{b} \longrightarrow (i)$

Also $r = \frac{1/c}{1/b} = \frac{1}{c} \times \frac{b}{1} = \frac{b}{c} \longrightarrow (ii)$

By $(i) \times (ii) \rightarrow r^2 = \frac{a}{b} \times \frac{b}{c}$

$\rightarrow r^2 = \frac{a}{c}$ or $r = \pm\sqrt{\frac{a}{c}}$

Hence proved

Q13. If the numbers 1, 4 and 3 are subtracted from three consecutive terms of an A.P., the resulting numbers are in G.P. Find the sum if their sum is 21.

Solution:- Suppose three nos. in A.P are $a_1 - d, a_1, a_1 + d$

II condition $\rightarrow$

$a_1 - d + a_1 + a_1 + d = 21$

or $3a_1 = 21 \rightarrow a_1 = 7$

I condition →

$a_1-d-1,\ a_1-4,\ a_1+d-3$ are in G.P

or $7-d-1,\ 7-4,\ 7+d-3$ are in G.P

→ $6-d,\ 3,\ 4+d$ are in G.P ∵ $a_1=7$

→ $\frac{4+d}{3}=\frac{3}{6-d}$ (common ratio)

or $(4+d)(6-d)=9$

$24-4d+6d-d^2=9$

→ $24+2d-d^2-9=0$

→ $-d^2+2d+15=0$

or $d^2-2d-15=0$

$d^2-5d+3d-15=0$

$d(d-5)+3(d-5)=0$

$(d-5)(d+3)=0$

→ $d-5=0,\quad d+3=0$

→ $d=5,\quad d=-3$

when $d=5,\quad a_1=7$

$a_1-d=7-5=2,\quad a_1=7,$

$a_1+d=7+5=12$

when $d=-3,\quad a_1=7$

$a_1-d=7-(-3)=7+3=10$

$a_1=7,\quad a_1+d=7-3=4$

Required nos. are 2,7,12 or 10,7,4

Q14. If three consecutive numbers in A.P. are increased by 1,4,15 respectively, the resulting numbers are in G.P. Find the original numbers if their sum is 6.

Solution:- Suppose three nos. in A.P are $a_1-d,\ a_1,\ a_1+d$

Condition II → $a_1-d+a_1+a_1+d=6$

→ $3a_1=6$ → $a_1=2$

condition I →

$a_1-d+1,\ a_1+4,\ a_1+d+15$ are in G.P

or $2-d+1,\ 2+4,\ 2+d+15$ are in G.P

→ $3-d,\ 6,\ 17+d$ in G.P (∵ $a_1=2$)

→ $\frac{17+d}{6}=\frac{6}{3-d}$ (common ratio)

→ $(17+d)(3-d)=36$

$51-17d+3d-d^2=36$

$-d^2-14d+51-36=0$

or $-d^2-14d+15=0$

→ $d^2+14d-15=0$

$d^2+15d-d-15=0$

$d(d+15)-1(d+15)=0$

$(d+15)(d-1)=0$

$d+15=0,\quad d-1=0$

$d=-15,\quad d=1$

when $d=-15,\ a_1=2$

$a_1+d=2+(-15)=-13,\quad a_1-d=2-(-15)=17$

when $d=1,\ a_1=2$

$a_1+d=2+1=3,\quad a_1-d=2-1=1$

Required numbers are -13, 2, 17 or 1, 2, 3

Geometric Means (G.M)

"A number G is said to be geometric mean (G.M) between two numbers a and b if a, G, b are in G.P."

In this case

$r=\frac{G}{a}$ and $r=\frac{b}{G}$

→ $\frac{G}{a}=\frac{b}{G}$ → $G^2=ab$

or $G=\pm\sqrt{ab}$

n Geometric means between two given numbers

The numbers $G_1, G_2, G_3, \ldots, G_n$ are called n Geometric means between a and b if $a, G_1, G_2, G_3, \ldots, G_n, b$ are in G.P

Now

$a, G_1, G_2, G_3, \ldots, G_n, b$ are in G.P

Here $a_1=a,\quad n=n+2,\quad a_{n+2}=b$

→ $a_{n+2}=a_1 r^{n+2-1}$ (∵ $a_n=a_1r^{n-1}$)

→ $b = ar^{n+1}$

or $\frac{b}{a} = r^{n+1}$ → $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Thus $r = \frac{G_1}{a}$ → $G_1 = ar = a\left(\frac{b}{a}\right)^{1/n+1}$

$r = \frac{G_2}{G_1}$ → $G_2 = G_1 r = ar\cdot r$

→ $G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/n+1}$

$G_3 = ar^3 = a\left(\frac{b}{a}\right)^{3/n+1}$

$\vdots$

$G_n = ar^n = a\left(\frac{b}{a}\right)^{n/n+1}$

Note:-

$G_1.G_2.G_3....G_n = a^n\left(\frac{b}{a}\right)^{\frac{(1+2+3+..+n)}{n+1}} = a^n\left(\frac{b}{a}\right)^{n/2}$

and $\sqrt[n]{(G_1.G_2.G_3.... G_n} = a\left(\frac{b}{a}\right)^{1/2} = \sqrt{ab}$

$= G =$ G.M between a and b

Example 1. Find the geometric mean between 4 and 16.

Solution:- Here $a = 4$, $b = 16$

$\because G = \pm\sqrt{ab} = \pm\sqrt{(4)(16)}$

$G = \pm\sqrt{64} = \pm 8$

Remember that

- ⋆ If the number of required G.M's is even, there is only one set of G.M's. i-e., we take value of r only positive.
- ⋆ If the number of required G.M's is odd, there is two sets of G.M's i-e., we take value of r both positive and negative.

Example 2. Insert three G.Ms between 2 and $\frac{1}{2}$.

Solution:-

Let G_1, G_2, G_3 be three G.M,s between 2 and $\frac{1}{2}$. so

$2, G_1, G_2, G_3, \frac{1}{2}$ are in G.P.

Here $a_1 = 2$, $n = 5$, $a_5 = \frac{1}{2}$

$\because a_n = a_1 r^{n-1}$

→ $a_5 = a_1 r^4$

$\frac{1}{2} = 2r^4$ → $r^4 = \frac{1}{4}$

→ $r^2 = \pm\frac{1}{2}$

so $r^2 = \frac{1}{2}$, $r^2 = -\frac{1}{2}$

→ $r = \pm\frac{1}{\sqrt{2}}$, $r^2 = \frac{i^2}{2}$ $\because i^2 = -1$

→ $r = \pm\frac{1}{\sqrt{2}}i$

when $r = \frac{1}{\sqrt{2}}$

$G_1 = a_1 r = (2)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$, $G_2 = a_1 r^2 = (2)\left(\frac{1}{\sqrt{2}}\right)^2$

→ $G_2 = (2)\left(\frac{1}{2}\right) = 1$

$G_3 = a_1 r^3$

$G_3 = (2)\left(\frac{1}{\sqrt{2}}\right)^3 = \cancel{2}\left(\frac{1}{\cancel{2}\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$

when $r = -\frac{1}{\sqrt{2}}$

$G_1 = (2)\left(-\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$, $G_2 = (2)\left(-\frac{1}{\sqrt{2}}\right)^2 = 1$

$G_3 = (2)\left(-\frac{1}{\sqrt{2}}\right)^3 = -\frac{1}{\sqrt{2}}$

when $r = \frac{1}{\sqrt{2}}i$

$G_1 = (2)\left(\frac{1}{\sqrt{2}}i\right) = \sqrt{2}i$, $G_2 = (2)\left(\frac{1}{\sqrt{2}}i\right)^2$

→ $G_2 = (\cancel{2})\left(\frac{1}{\cancel{2}}i^2\right)$

$G_3 = (2)\left(\frac{1}{\sqrt{2}}i\right)^3$ → $G_2 = -1$

$G_3 = (2)\left(\frac{1}{2\sqrt{2}}i^3\right) = \frac{1}{\sqrt{2}}i^2\cdot i = -\frac{1}{\sqrt{2}}i$

when $r = -\frac{1}{\sqrt{2}}i$

$G_1 = 2\left(-\frac{1}{\sqrt{2}}i\right) = -\sqrt{2}i$, $G_2 = (2)\left(-\frac{1}{\sqrt{2}}i\right)^2$

$G_2 = -1$

$G_3 = (2)\left(\frac{-1}{\sqrt{2}}i\right)^3 = -\frac{1}{\sqrt{2}}i$

Example 3. If a, b, c and d are in G.P. show that a+b, b+c, c+d are in G.P

Solution:-

$\because a, b, c, d$ are in G.P so

$r = \frac{b}{a}$, $r = \frac{c}{b}$, $r = \frac{d}{c}$

→ $\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$

Now $\frac{b}{a}=\frac{c}{b}$, $\frac{c}{b}=\frac{d}{c}$, $\frac{d}{c}=\frac{b}{a}$

$\rightarrow b^2=ac,\ c^2=bd,\ ad=bc \} \longrightarrow (i)$

Thus $a+b, b+c, c+d$ are in G.P

if $\frac{b+c}{a+b}=\frac{c+d}{b+c}$

$\rightarrow (b+c)^2=(a+b)(c+d)$

R.H.S $= (a+b)(c+d)$

$= ac+ad+bc+bd$

$= b^2+bc+bc+c^2$ use (i)

$= b^2+2bc+c^2$

$= (b+c)^2 =$ L.H.S

Hence $a+b, b+c, c+d$ are in G.P

Exercise 6.7

Q1. Find G.M between
i) -2 and 8 ii) $-2i$ and $8i$

Solution:- i) -2 and 8

Here $a=-2$, $b=8$

$\because G=\pm\sqrt{ab}=\pm\sqrt{(-2)(8)}$

$=\pm\sqrt{-16}=\pm\sqrt{16}\sqrt{-1}$

$G=\pm 4i \quad \because \sqrt{-1}=i$

ii) $-2i$ and $8i$

Here $a=-2i$, $b=8i$

$\because G=\pm\sqrt{ab}=\pm\sqrt{(-2i)(8i)}$

$=\pm\sqrt{-16i^2}=\pm\sqrt{(-16)(-1)}$

$G=\pm\sqrt{16}=\pm 4 \quad (\because i^2=-1)$

Q2. Insert two G.Ms between
i) 1 and 8 ii) 2 and 16

Solution:- i) 1 and 8

Let G_1, G_2 be two G.Ms between 1 and 8 then

$1, G_1, G_2, 8$ are in G.P

Here $a_1=1$, $n=4$, $a_4=8$

$\rightarrow a_4=a_1r^3 \quad \because a_n=a_1r^{n-1}$

$\rightarrow 8=(1)r^3$

$\rightarrow (r)^3=(2)^3$

or $r=2$

Thus $G_1=a_1r=(1)(2)=2$

$G_2=a_1r^2=(1)(2)^2=4$

ii) 2 and 16

Let G_1 and G_2 be two G.Ms between 2 and 16 then

$2, G_1, G_2, 16$ are in G.P.

Here $a_1=2$, $n=4$, $a_4=16$

so $a_4=a_1r^3 \quad \because a_n=a_1r^{n-1}$

$\rightarrow 16=(2)r^3$

$\rightarrow 8=r^3$ or $(r)^3=(2)^3$

$\rightarrow r=2$

Thus $G_1=a_1r=(2)(2)=4$

$G_2=a_1r^2=(2)(2)^2=8$

Q3. Insert three G.Ms between
i) 1 and 16 ii) 2 and 32

Solution:- i) 1 and 16

Let G_1, G_2, G_3 be three G.Ms between 1 and 16 then

$1, G_1, G_2, G_3, 16$ are in G.P

Here $a_1=1$, $n=5$, $a_5=16$

so $a_5=a_1r^4 \quad \because a_n=a_1r^{n-1}$

$\rightarrow 16=(1)r^4$

$\rightarrow (2)^4=r^4 \rightarrow r=2$

Now $G_1=a_1r=(1)(2)=2$

$G_2=a_1r^2=(1)(2)^2=4$

$G_3=a_1r^3=(1)(2)^3=8$

ii) 2 and 32

Let G_1, G_2, G_3 be three G.Ms between 2 and 32 then

$2, G_1, G_2, G_3, 32$ are in G.P.

Here $a_1=2$, $n=5$, $a_5=32$

$\rightarrow a_5=a_1r^4 \quad \because a_n=a_1r^{n-1}$

or $32=(2)r^4$

$\rightarrow r^4=16$ or $(r)^4=(2)^4$

so $r=2$

so $G_1 = a_1 r = (2)(2) = 4$

$G_2 = a_1 r^2 = (2)(2)^2 = 8$

$G_3 = a_1 r^3 = (2)(2)^3 = 16$

Q4. Insert four real geometric means between 3 and 96.

Solution:- Let G_1, G_2, G_3, G_4 are four G.Ms between 3 and 96. then $3, G_1, G_2, G_3, G_4, 96$ are in G.P

Here $a_1 = 3$, $n = 6$, $a_6 = 96$

$\rightarrow a_6 = a_1 r^5 \quad \because a_n = a_1 r^{n-1}$

or $96 = (3) r^5$

$\rightarrow 32 = r^5 \rightarrow (r)^5 = (2)^5$

or $r = 2$ so

$G_1 = a_1 r = (3)(2) = 6$

$G_2 = a_1 r^2 = (3)(2)^2 = (3)(4) = 12$

$G_3 = a_1 r^3 = (3)(2)^3 = (3)(8) = 24$

$G_4 = a_1 r^4 = (3)(2)^4 = (3)(16) = 48$

Thus 6, 12, 24, 48 are four G.Ms.

Q5. If both x and y are positive distinct real numbers, show that the geometric mean between x and y is less than their arithmetic mean.

Solution:- Given that $x > 0$ and $y > 0$

Here $a = x$, $b = y$ so

$G.M = \sqrt{xy}$ and $A.M = \frac{x+y}{2}$

$\left(\because G.M = \sqrt{ab} \;,\; A.M = \frac{a+b}{2} \right)$

Now $A.M - G.M = \frac{x+y}{2} - \sqrt{xy}$

$= \frac{x+y-2\sqrt{xy}}{2}$

$= \frac{(\sqrt{x})^2 + (y)^2 - 2\sqrt{xy}}{2}$

$A.M - G.M = \frac{(\sqrt{x} - \sqrt{y})^2}{2} > 0$

$\rightarrow A.M - G.M > 0$

or $A.M > G.M$

or $G.M < A.M$

Hence proved.

Q6. For what value of n, $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the positive geometric mean between a and b?

Solution:-

If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ be G.M between a and b then

$$\frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \sqrt{ab} = (ab)^{\frac{1}{2}}$$

$\rightarrow a^n + b^n = (a^{n-1} + b^{n-1})(a^{\frac{1}{2}} b^{\frac{1}{2}})$

$\rightarrow a^n + b^n = a^{n-1+\frac{1}{2}} b^{\frac{1}{2}} + b^{n-1+\frac{1}{2}} a^{\frac{1}{2}}$

$\rightarrow a^n + b^n = a^{n-\frac{1}{2}} b^{\frac{1}{2}} + b^{n-\frac{1}{2}} a^{\frac{1}{2}}$

$\rightarrow a^n - a^{n-\frac{1}{2}} b^{\frac{1}{2}} = a^{\frac{1}{2}} b^{n-\frac{1}{2}} - b^n$

$\rightarrow a^{n-\frac{1}{2}+\frac{1}{2}} - a^{n-\frac{1}{2}} b^{\frac{1}{2}} = a^{\frac{1}{2}} b^{n-\frac{1}{2}} - b^{n-\frac{1}{2}+\frac{1}{2}}$

$\rightarrow a^{n-\frac{1}{2}} \cancel{(a^{\frac{1}{2}} - b^{\frac{1}{2}})} = b^{n-\frac{1}{2}} \cancel{(a^{\frac{1}{2}} - b^{\frac{1}{2}})}$

$\rightarrow a^{n-\frac{1}{2}} = b^{n-\frac{1}{2}}$

or $\frac{a^{n-\frac{1}{2}}}{b^{n-\frac{1}{2}}} = 1 \qquad (\div \text{ by } b^{n-\frac{1}{2}})$

or $\left(\frac{a}{b}\right)^{n-\frac{1}{2}} = \left(\frac{a}{b}\right)^0 \rightarrow n - \frac{1}{2} = 0$

or $n = \frac{1}{2}$

Q7. The A.M of two positive integral numbers exceeds their (positive) G.M by 2 and their sum is 20, find the numbers.

Solution:- Suppose two numbers are a and b then

I condition $\rightarrow \frac{a+b}{2} = \sqrt{ab} + 2$

or $a + b = 2\sqrt{ab} + 4 \longrightarrow (i)$

$\text{II condition} \rightarrow a+b=20 \rightarrow a=20-b \xrightarrow{} (ii)$

Put $a=20-b$ in (i) so

$$20-b+b = 2\sqrt{(20-b)b} + 4$$

$$20 = 2\sqrt{20b-b^2} + 4$$

$$\rightarrow 20-4 = 2\sqrt{20b-b^2}$$

$$16 = 2\sqrt{20b-b^2} \rightarrow \sqrt{20b-b^2} = 8$$

Now squaring both sides

$$\left(\sqrt{20b-b^2}\right)^2 = (8)^2$$

$$\rightarrow 20b - b^2 = 64$$

$$\rightarrow b^2 - 20b + 64 = 0$$

$$b^2 - 16b - 4b + 64 = 0$$

$$\rightarrow b(b-16) - 4(b-16) = 0$$

$$(b-16)(b-4) = 0$$

$$\therefore b-16 = 0, \quad b-4 = 0$$

$$\rightarrow b = 16, \quad b = 4$$

when $b=16$ then $a=20-16=4$

when $b=4$ then $a=20-4=16$

Hence two numbers are 4, 16 or 16, 4

Q8. The A.M between two numbers is 5 and their (positive) G.M is 4. Find the numbers.

Solution:-

Suppose two numbers are a and b then

$$\text{I condition} \rightarrow \frac{a+b}{2} = 5$$

$$\rightarrow a+b = 10$$

$$\text{or } a = 10-b \xrightarrow{} (i)$$

$$\text{II condition} \rightarrow \sqrt{ab} = 4$$

$$\rightarrow ab = 16$$

$$\rightarrow (10-b)b = 16 \quad \text{from (i)}$$

$$10b - b^2 = 16$$

$$\text{or } b^2 - 10b + 16 = 0$$

$$b^2 - 8b - 2b + 16 = 0$$

$$\rightarrow b(b-8) - 2(b-8) = 0$$

$$\rightarrow (b-8)(b-2) = 0$$

$$\rightarrow b-8 = 0, \quad b-2 = 0$$

$$b = 8, \quad b = 2$$

when $b=8$ then $a=10-8=2$

when $b=2$ then $a=10-2=8$

Hence two numbers are 8, 2 or 2, 8.

Sum of n terms of a Geometric Series

$$S_n = \frac{a_1(r^n-1)}{r-1}, |r|>1$$

$$\text{and } S_n = \frac{a_1(1-r^n)}{1-r}, |r|<1$$

Proof:- We know that if the sequence $\{a_n\}$ is a geometric sequence then

$$S_n = a_1 + a_1r + a_1r^2 + \ldots\ldots + a_1r^{n-1}$$

'x' both sides by $(1-r)$

$$(1-r)S_n = (1-r)(a_1 + a_1r + a_1r^2 + \ldots + a_1r^{n-1})$$

$$= a_1 + a_1r + a_1r^2 + \ldots + a_1r^{n-1} - a_1r - a_1r^2 - a_1r^3 \ldots\ldots a_1r^{n-1} - a_1r^n$$

$$(1-r)S_n = a_1 - a_1r^n$$

$$(1-r)S_n = a_1(1-r^n)$$

$$\rightarrow S_n = \frac{a_1(1-r^n)}{1-r} \quad \text{if } |r|<1$$

$$\text{and } S_n = \frac{a_1(r^n-1)}{r-1} \quad \text{if } |r|>1$$

Example 1. Find the sum of n terms of the geometric series if

$$a_n = (-3)\left(\frac{2}{5}\right)^n$$

Solution:- Here $S_n = ?$

$$a_n = (-3)\left(\frac{2}{5}\right)^n$$

$$\rightarrow a_n = (-3)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)^{n-1}$$

$$a_n = \left(-\frac{6}{5}\right)\left(\frac{2}{5}\right)^{n-1}$$

By comparing with $a_n = a_1 r^{n-1}$

we have $a_1 = -\frac{6}{5}$, $r = \frac{2}{5}$

Here $r = \frac{2}{5} < 1$ so

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_n = \frac{(-\frac{6}{5})(1-(\frac{2}{5})^n)}{1-\frac{2}{5}} = \frac{(-\frac{6}{5})(1-(\frac{2}{5})^n)}{\frac{5-2}{5}}$$

$$S_n = \frac{(-\frac{6}{5})(1-(\frac{2}{5})^n)}{3/5} = -2\left(1-(\tfrac{2}{5})^n\right)$$

→ $S_n = -2\left[1-(\frac{2}{5})^n\right]$

Example 2. The growth of a certain plant is 5% of its length monthly. when will the plant be of 4.41 cm if its initial length is 4cm?

Solution:-

Here $a_1 = 4cm$

Since 5% increases , so

$r = 100\% + 5\% = 105\% = \frac{105}{100} = 1.05$

Now length:-

After one month $(a_2) = a_1 r = 4(1.05) = 4.20$

After two months $(a_3) = a_1 r^2 = 4(1.05)^2 = 4.41$

Hence after two months length will be 4.41 cm.

The Infinite Geometric Series

$$S_\infty = \frac{a_1}{1-r} \quad \text{if } |r|<1$$

Proof:- we know that

$a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} + \cdots \infty$ is called an infinite geometric series. It is denoted by S_∞.

Also we know that

$$S_n = \frac{a_1(1-r^n)}{1-r} \quad \text{if } |r|<1$$

But we do not know how to add infinitely many terms of the series. so for this purpose applying limit as $n \to \infty$ i-e.,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{a_1(1-r^n)}{1-r}\right]$$

$\because |r|<1$, it follows that r^n gets smaller and smaller as n gets larger and larger so when $n \to \infty$ then $r^n \to 0$

Now

$$S_\infty = \frac{a_1(1-0)}{1-r}$$

→ $S_\infty = \frac{a_1}{1-r}$

Note:- S_∞ is only possible if $|r|<1$

if $|r|>1$

then r^n does not tend to zero when $n \to \infty$

→ S_n does not tend to a limit and the series does not converge in this case so the series is divergent.

if $r=1$

then the series becomes as

$a_1 + a_1 + a_1 + \cdots$ and

$S_n = n a_1$. In this case S_n does not tend to a limit when $n \to \infty$ and the series does not converge.

if $r=-1$

then the series becomes as

$a_1 - a_1 + a_1 - a_1 + a_1 - a_1 + \cdots$

and $S_n = \frac{a_1 - (-1)^n a_1}{2}$

i-e., $S_n = a_1$ if n is +ive odd number.

$S_n = 0$ if n is +ive even number.

Thus S_n does not tend to a definit number when $n \to \infty$

In such a case we say that the series is oscillatory.

Example 3. Find the sum of the infinite G.P. $2, \sqrt{2}, 1, \ldots$

Solution:-

Here $S_\infty = ?$, $a_1 = 2$

$$r = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} < 1$$

$$\because S_\infty = \frac{a_1}{1-r} = \frac{2}{1-\frac{1}{\sqrt{2}}}$$

$$S_\infty = \frac{2}{\frac{\sqrt{2}-1}{\sqrt{2}}} = \frac{2\sqrt{2}}{\sqrt{2}-1}$$

$$S_\infty . = \frac{2\sqrt{2}}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$S_\infty = \frac{2\sqrt{2}(\sqrt{2}+1)}{(\sqrt{2})^2-(1)^2} = \frac{2(2)+2\sqrt{2}}{2-1}$$

$$\rightarrow S_\infty = \frac{4+2\sqrt{2}}{1} = 4+2\sqrt{2}$$

Example 4. Convert the recurring decimal $2.\dot{2}\dot{3}$ into an equivalent common fraction (vulgur fraction)

Solution:-

$$2.\dot{2}\dot{3} = 2.232323\ldots\ldots$$

$$= 2 + 0.232323\ldots\ldots$$

$$2.\dot{2}\dot{3} = 2 + [0.23 + 0.0023 + 0.000023 + \ldots\ldots]$$

Here $a_1 = 0.23$, $r = \frac{0.0023}{0.23} = 0.01$

$$= 2 + \frac{a_1}{1-r}$$

$$= 2 + \frac{0.23}{1-0.01} = 2 + \frac{0.23}{0.99}$$

$$= 2 + \frac{23}{99} = \frac{198+23}{99} = \frac{221}{99}$$

Example 5. The sum of an infinite geometric series is half the sum of the squares of its terms. If the sum of its first two terms is $\frac{9}{2}$, find the series.

Solution:-

Let the series be

$$a_1 + a_1 r + a_1 r^2 + \ldots\ldots$$

so $S_\infty = \frac{a_1}{1-r}$

The sum of squares of its terms is

$$a_1^2 + a_1^2 r^2 + a_1^2 r^4 + \ldots\ldots$$

so $S_\infty = \frac{a_1^2}{1-r^2}$

I condition →

$$\frac{a_1}{1-r} = \frac{1}{2}\left(\frac{a_1^2}{1-r^2}\right)$$

$$\rightarrow \frac{a_1}{1-r} = \frac{1}{2}\left(\frac{a_1^2}{(1-r)(1+r)}\right)$$

$$\rightarrow 1 = \frac{a_1}{2(1+r)}$$

$$\rightarrow a_1 = 2(1+r) \longrightarrow (i)$$

II condition →

$$a_1 + a_1 r = \frac{9}{2} \rightarrow a_1(1+r) = \frac{9}{2}$$

$$\rightarrow 2(1+r)(1+r) = \frac{9}{2} \quad \text{from (i)}$$

$$\rightarrow (1+r)^2 = \frac{9}{4}$$

or $1+r = \pm\frac{3}{2}$

$1+r = \frac{3}{2}$, $1+r = -\frac{3}{2}$

$r = \frac{3}{2}-1$, $r = -\frac{3}{2}-1$

$r = \frac{1}{2}$, $r = -\frac{5}{2}$

$$\rightarrow |r| = \left|-\frac{5}{2}\right| = \frac{5}{2} > 1$$

(Not possible)

so we take $r = \frac{1}{2}$ only

$$(i) \rightarrow a_1 = 2\left(1+\frac{1}{2}\right) = 2\left(\frac{3}{2}\right)$$

$$\rightarrow a_1 = 3$$

Now $a_1 = 3$, $a_2 = a_1 r = (3)\left(\frac{1}{2}\right) = \frac{3}{2}$

$$a_3 = a_1 r^2 = (3)\left(\frac{1}{2}\right)^2 = (3)\left(\frac{1}{4}\right) = \frac{3}{4}$$

$$a_4 = a_1 r^3 = (3)\left(\frac{1}{2}\right)^3 = (3)\left(\frac{1}{8}\right) = \frac{3}{8}$$

Hence required series is

$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots\cdots$

$3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots\cdots$

Example 6. If $a = 1 - x + x^2 - x^3 + \cdots$ $|x|<1$

$b = 1 + x + x^2 + x^3 + \cdots\cdots$ $|x|<1$

show that $2ab = a+b$

Solution:-

$a = 1 - x + x^2 - x^3 + \cdots\cdots$ $|x|<1$

Here $a_1 = 1$

$r = \frac{-x}{1} = -x$

$S_\infty = a$

$\because S_\infty = \frac{a_1}{1-r}$

$a = \frac{1}{1-(-x)} = \frac{1}{1+x}$

$\rightarrow a = \frac{1}{1+x}$ or $1 + x = \frac{1}{a} \rightarrow (i)$

Also, $b = 1 + x + x^2 + x^3 + \cdots\cdots$ $|x|<1$

Here $a_1 = 1$

$r = \frac{x}{1} = x$

$S_\infty = b$

$\because S_\infty = \frac{a_1}{1-r}$

$\rightarrow b = \frac{1}{1-x}$

or $1 - x = \frac{1}{b} \longrightarrow (ii)$

By $(i) + (ii) \rightarrow$

$\frac{1}{a} + \frac{1}{b} = 1 + x + 1 - x$

$\rightarrow \frac{b+a}{ab} = 2$

$\rightarrow a + b = 2ab$

Thus $2ab = a+b$ Hence proved.

Exercise 6.8

Q1. Find the sum of first 15 terms of the geometric sequence $1, \frac{1}{3}, \frac{1}{9}, \cdots\cdots$

Solution:-

Here $a_1 = 1$, $r = \frac{1/3}{1} = \frac{1}{3} < 1$

$S_{15} = ?$

$\because S_n = \frac{a_1(1-r^n)}{1-r}$ if $r<1$

$\rightarrow S_{15} = \frac{(1)(1-(\frac{1}{3})^{15})}{1 - 1/3}$

$= \frac{1 - \frac{1}{3^{15}}}{\frac{3-1}{3}} = \frac{\frac{3^{15}-1}{3^{15}}}{\frac{2}{3}}$

$S_{15} = \frac{3}{2}\left[\frac{14348907 - 1}{3^{15}}\right]$

$S_{15} = \frac{14348906}{2 \cdot 3^{14}}$

$S_{15} = \frac{14348906}{2(4782969)} = \frac{7174453}{4782969}$

Q2. Sum to n terms, the series

i) $.2 + .22 + .222 + \cdots$

ii) $3 + 33 + 333 + \cdots\cdots$

Solution:- i) $0.2 + 0.22 + 0.222 + \cdots$ n terms

$= 2[0.1 + 0.11 + 0.111 + \cdots\cdots \text{ n terms}]$

$= \frac{2}{9}(0.9 + 0.99 + 0.999 + \cdots\cdots \text{ n terms})$

$= \frac{2}{9}\left(\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \cdots\cdots \text{ n terms}\right)$

$= \frac{2}{9}\left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{100}\right) + \left(1 - \frac{1}{1000}\right) + \cdots\cdots \text{ n terms}\right]$

$= \frac{2}{9}\left[(1+1+1+\cdots\cdots \text{ n terms}) - \left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots\cdots \text{ n terms}\right)\right]$

$= \frac{2}{9}\left[n - \frac{\frac{1}{10}\left(1 - \left(\frac{1}{10}\right)^n\right)}{1 - \frac{1}{10}}\right]$

$\because a_1 = \frac{1}{10}$

$r = \frac{1}{10} < 1$

$= \frac{2}{9}\left(n - \frac{\frac{1}{\cancel{10}}\left(1 - \frac{1}{10^n}\right)}{\frac{9}{\cancel{10}}}\right)$

$= \frac{2}{9}\left(n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right)$

ii) $3 + 33 + 333 + \cdots\cdots$

$= 3(1 + 11 + 111 + \cdots\cdots \text{ n terms})$

$= \frac{3}{9}(9 + 99 + 999 + \cdots\cdots \text{ n terms})$

$= \frac{1}{3}[(10-1) + (100-1) + (1000-1) + \cdots\cdots \text{ n terms}]$

$$= \frac{1}{3}\left[(10+100+1000+\dots n\text{ terms}) - (1+1+1+\dots\dots n\text{ terms})\right]$$

$a_1 = 10, \quad r = 10 > 1$

$$= \frac{1}{3}\left[\frac{(10)((10)^n - 1)}{10-1} - n\right]$$

$$= \frac{1}{3}\left(\frac{10(10^n - 1)}{9} - n\right)$$

$$= \frac{1}{3}\left(\frac{10}{9}(10^n - 1) - n\right)$$

Q3. Sum to n terms the series

i) $1 + (a+b) + (a^2+ab+b^2) + (a^3+a^2b+ab^2+b^3) + \dots$

Solution:-

$1 + (a+b) + (a^2+ab+b^2) + (a^3+a^2b+ab^2+b^3) + \dots$

'x' and '÷' by $(a-b)$

$$= \frac{1}{a-b}\left[(a-b) + (a^2-b^2) + (a^3-b^3) + \dots n\text{ terms}\right]$$

$$= \frac{1}{a-b}\left[(a+a^2+a^3+\dots n\text{ terms}) - (b+b^2+b^3+\dots n\text{ term}\right]$$

$\because a_1 = a, \ r = a \qquad \because a_1 = b, \ r = b$

$$= \frac{1}{a-b}\left[\frac{a(1-a^n)}{1-a} - \frac{b(1-b^n)}{1-b}\right]$$

$$= \frac{1}{a-b}\left[\frac{a(1-b)(1-a^n) - b(1-a)(1-b^n)}{(1-a)(1-b)}\right]$$

$$= \frac{a(1-b)(1-a^n) - b(1-a)(1-b^n)}{(a-b)(1-a)(1-b)}$$

ii) $r + (1+k)r^2 + (1+k+k^2)r^3 + \dots$

Solution:-

$r + (1+k)r^2 + (1+k+k^2)r^3 + \dots n$ terms

'x' by '÷' By $1-k$

$$= \frac{1}{1-k}\left[(1-k)r + (1-k^2)r^2 + (1-k^3)r^3 + \dots n\text{ terms}\right]$$

$$= \frac{1}{1-k}\left[r - rk + r^2 - r^2k^2 + r^3 - r^3k^3 + \dots n\text{ terms}\right]$$

$$= \frac{1}{1-k}\left[(r+r^2+r^3+\dots n\text{ terms}) - (kr+k^2r^2+k^3r^3+\dots n\text{ terms})\right]$$

$$= \frac{1}{1-k}\left[\frac{r(1-r^n)}{1-r} - \frac{kr(1-(kr)^n)}{1-kr}\right]$$

$$= \frac{r}{1-k}\left[\frac{1-r^n}{1-r} - \frac{k(1-k^n r^n)}{1-kr}\right]$$

Q4. Sum the series $2 + (1-i) + \frac{1}{i} + \dots$ to 8 terms

Solution:-

$2 + (1-i) + \frac{1}{i} + \dots\dots$ to 8 terms.

Here $a_1 = 2, \ r = \frac{1-i}{2}, \ n = 8 < 1$

$$\because S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_8 = \frac{2\left[1 - \left(\frac{1-i}{2}\right)^8\right]}{1 - \frac{1-i}{2}}$$

$$S_8 = \frac{2\left[2^8 - (1-i)^8\right]}{2^8\left(\frac{2-1+i}{2}\right)}$$

$$S_8 = \frac{2^8 - (1-i)^8}{2^6(1+i)}$$

$$S_8 = \frac{256 - [(1-i)^2]^4}{64(1+i)}$$

$$S_8 = \frac{256 - [1+i^2-2i]^4}{64(1+i)}$$

$$S_8 = \frac{256 - (1-1-2i)^4}{64(1+i)} \qquad \because i^2 = -1$$

$$S_8 = \frac{256 - (-2i)^4}{64(1+i)}$$

$$\rightarrow S_8 = \frac{256 - 16i^4}{64(1+i)}$$

$$\rightarrow S_8 = \frac{256 - 16}{64(1+i)} \qquad \because i^4 = (i^2)^2 = (-1)^2 = 1$$

$$\rightarrow S_8 = \frac{240}{64(1+i)} = \frac{15}{4(1+i)}$$

$$\rightarrow S_8 = \frac{15}{4(1+i)} \times \frac{1-i}{1-i}$$

$$S_8 = \frac{15(1-i)}{4(1-i^2)} = \frac{15(1-i)}{4(1-(-1))}$$

$$S_8 = \frac{15(1-i)}{4(2)} = \frac{15(1-i)}{8}$$

Thus $S_8 = \frac{15(1-i)}{8}$

Q5. Find the sums of the following infinite geometric series:

i) $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\ldots$

Solution:-

Here $a_1=\frac{1}{5}$, $r=\frac{\frac{1}{25}}{\frac{1}{5}}=\frac{1}{25}\times\frac{5}{1}=\frac{1}{5}$

$\because S_\infty=\frac{a_1}{1-r}=\frac{1/5}{1-\frac{1}{5}}=\frac{1/5}{4/5}$

$S_\infty=\frac{1}{5}\times\frac{5}{4}=\frac{1}{4}$

ii) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$

Solution:-

Here $a_1=\frac{1}{2}$, $r=\frac{\frac{1}{4}}{1/2}=\frac{1}{4}\times\frac{2}{1}=\frac{1}{2}$

$\because S_\infty=\frac{a_1}{1-r}=\frac{1/2}{1-\frac{1}{2}}=\frac{\frac{1}{2}}{\frac{1}{2}}=1$

iii) $\frac{9}{4}+\frac{3}{2}+1+\frac{2}{3}+\ldots$

Solution:-

$a_1=\frac{9}{4}$, $r=\frac{\frac{3}{2}}{\frac{9}{4}}=\frac{3}{2}\times\frac{4}{9}=\frac{2}{3}$

$\because S_\infty=\frac{a_1}{1-r}=\frac{9/4}{1-\frac{2}{3}}=\frac{9/4}{\frac{3-2}{3}}$

$S_\infty=\frac{\frac{9}{4}}{1/3}=\frac{9}{4}\times\frac{3}{1}=\frac{27}{4}$

iv) $2+1+0.5+\ldots$

Solution:-

Here $a_1=2$, $r=\frac{1}{2}$

$\because S_\infty=\frac{a_1}{1-r}=\frac{2}{1-\frac{1}{2}}=\frac{2}{\frac{1}{2}}=2\times\frac{2}{1}$

$\Rightarrow S_\infty=4$

v) $4+2\sqrt{2}+2+\sqrt{2}+1+\ldots$

Solution:-

Here $a_1=4$, $r=\frac{2\sqrt{2}}{4}=\frac{\sqrt{2}}{2}=\frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}\sqrt{2}}$

$\Rightarrow r=\frac{1}{\sqrt{2}}$

$\because S_\infty=\frac{a_1}{1-r}=\frac{4}{1-\frac{1}{\sqrt{2}}}$

$S_\infty=\frac{4}{\frac{\sqrt{2}-1}{\sqrt{2}}}=\frac{4\sqrt{2}}{\sqrt{2}-1}$

$=\frac{4\sqrt{2}}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}$

$=\frac{4\sqrt{2}(\sqrt{2}+1)}{(\sqrt{2})^2-(1)^2}=\frac{4((2)+\sqrt{2})}{2-1}$

$S_\infty=\frac{4(2+\sqrt{2})}{1}=4(2+\sqrt{2})$

vi) $0.1+0.05+0.025+\ldots$

Solution:-

$a_1=0.1$, $r=\frac{0.05}{0.1}=0.5$

$\because S_\infty=\frac{a_1}{1-r}=\frac{0.1}{1-0.5}=\frac{0.1}{0.5}$

$\Rightarrow S_\infty=0.2$

Q6. Find vulgar fractions equivalent to the following recurring decimals.

i) $1.\dot{3}\dot{4}$

Solution:-

$1.\dot{3}\dot{4}=1.343434\ldots$

$=1+0.343434\ldots$

$=1+(0.34+0.0034+\ldots)$

$a_1=0.34$, $r=\frac{0.0034}{0.34}=0.01$

$=1+\frac{a_1}{1-r}$

$=1+\frac{0.34}{1-0.01}=1+\frac{0.34}{0.99}$

$=1+\frac{34}{99}=\frac{99+34}{99}=\frac{133}{99}$

ii) $0.\dot{7}$

Solution:-

$0.\dot{7}=0.7777\ldots$

$=0.7+0.07+0.007+\ldots$

$a_1=0.7$, $r=\frac{0.07}{0.7}=0.1$

$\because S_\infty=\frac{a_1}{1-r}=\frac{0.7}{1-0.1}=\frac{0.7}{0.9}$

$S_\infty=\frac{\frac{7}{10}}{\frac{9}{10}}=\frac{7}{9}$

iii) $0.\dot{2}5\dot{9}$

Solution:-

$$0.\dot{2}5\dot{9} = 0.259259259.....$$

$$= 0.259 + 0.000259 + ...$$

$$a_1 = 0.259, \quad r = \frac{0.000259}{0.259} = 0.001$$

$$S_\infty = \frac{a_1}{1-r} = \frac{0.259}{1-0.001}$$

$$= \frac{0.259}{0.999} = \frac{\frac{259}{1000}}{\frac{999}{1000}} = \frac{259}{999}$$

iv) $1.\dot{5}\dot{3}$

Solution:-

$$1.\dot{5}\dot{3} = 1.535353.....$$

$$= 1 + 0.535353.....$$

$$= 1 + (0.53 + 0.0053 + ...)$$

$$a_1 = 0.53, \quad r = \frac{0.0053}{0.53} = 0.01$$

$$= 1 + \frac{a_1}{1-r} = 1 + \frac{0.53}{1-0.01}$$

$$= 1 + \frac{0.53}{0.99} = 1 + \frac{\frac{53}{100}}{\frac{99}{100}}$$

$$= 1 + \frac{53}{99} = \frac{99+53}{99} = \frac{152}{99}$$

v) $0.\dot{1}5\dot{9}$

Solution:-

$$0.\dot{1}5\dot{9} = 0.159159159.....$$

$$= 0.159 + 0.000159 + ...$$

$$a_1 = 0.159, \quad r = \frac{0.000159}{0.159} = 0.01$$

$$0.\dot{1}5\dot{9} = \frac{0.159}{1-0.001} \qquad \because S_\infty = \frac{a_1}{1-r}$$

$$= \frac{0.159}{0.999} = \frac{\frac{159}{1000}}{\frac{999}{1000}} = \frac{159}{999}$$

vi) $1.1\dot{4}\dot{7}$

Solution:-

$$1.1\dot{4}\dot{7} = 1.1474747.....$$

$$= 1.1 + 0.0474747...$$

$$= 1.1 + 0.047 + 0.00047 + ...$$

$$a_1 = 0.047, \quad r = \frac{0.00047}{0.047} = 0.01$$

$$= 1.1 + \frac{a_1}{1-r}$$

$$= 1.1 + \frac{0.047}{1-0.01} = 1.1 + \frac{0.047}{0.99}$$

$$= 1.1 + \frac{\frac{47}{1000}}{\frac{99}{100}} = 1.1 + \frac{47}{1000} \times \frac{100}{99}$$

$$= 1.1 + \frac{47}{990} = \frac{11}{10} + \frac{47}{990}$$

$$= \frac{1089+47}{990} = \frac{1136}{990}$$

Vulgar Fraction:- "A fraction whose numerator and denomenator both are integers is called vulgar fraction or common fraction"

Proper fraction:- "If the numerator of the fraction is less than the denominator, the fraction is called proper fraction"

Q7. Find the sum to infinity of the series; $r + (1+k)r^2 + (1+k+k^2)r^3 + ...$ r and k being proper fractions.

Solution:-

$$r + (1+k)r^2 + (1+k+k^2)r^3 + ...$$

'x' and '÷' by $1-k$

$$= \frac{1}{1-k}\left[(1-k)r + (1-k^2)r^2 + (1-k^3)r^3 + ... n \text{ terms}\right]$$

$$= \frac{1}{1-k}\left[r - rk + r^2 - k^2r^2 + r^3 - k^3r^3 + ... n \text{ terms}\right]$$

$$= \frac{1}{1-k}\left[(r + r^2 + r^3 + ... n \text{ terms}) - (kr + k^2r^2 + k^3r^3 + ... n \text{ terms})\right]$$

$$= \frac{1}{1-k}\left(\frac{r}{1-r} - \frac{kr}{1-kr}\right] \qquad \because S_\infty = \frac{a_1}{1-r}$$

$$= \frac{1}{1-k}\left[\frac{r(1-kr) - kr(1-r)}{(1-r)(1-kr)}\right]$$

$$= \frac{1}{1-k}\left(\frac{r - kr^2 - kr + kr^2}{(1-r)(1-kr)}\right)$$

$$= \frac{1}{1-k}\left(\frac{r - kr}{(1-r)(1-kr)}\right)$$

$$= \frac{1}{1-K}\left[\frac{r(1-K)}{(1-r)(1-Kr)}\right] = \frac{r}{(1-r)(1-Kr)}$$

Q8. If $y = \frac{x}{2} + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \dots$ and if $0 < x < 2$, then prove that

$$x = \frac{2y}{1+y}$$

Solution:-

$$y = \frac{x}{2} + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \dots$$

$$a_1 = \frac{x}{2}, \quad r = \frac{\frac{1}{4}x^2}{\frac{1}{2}x} = \frac{1}{4}x^2 \times \frac{2}{x}$$

$$\rightarrow r = \frac{x}{2}$$

$$\because S_\infty = \frac{a_1}{1-r}$$

$$\rightarrow y = \frac{\frac{x}{2}}{1-\frac{x}{2}} = \frac{\frac{x}{2}}{\frac{2-x}{2}}$$

$$\rightarrow y = \frac{x}{2-x}$$

$$\rightarrow y(2-x) = x$$

$$2y - xy = x \rightarrow 2y = x + xy$$

$$\text{or } 2y = x(1+y)$$

$$\text{or } x = \frac{2y}{1+y}$$

Hence proved

Q9. If $y = \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \dots$ and if $0 < x < \frac{3}{2}$, then show that

$$x = \frac{3y}{2(1+y)}$$

Solution:-

$$y = \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \dots$$

$$a_1 = \frac{2}{3}x, \quad r = \frac{\frac{4}{9}x^2}{\frac{2}{3}x} = \frac{4}{9}x^2 \times \frac{3}{2x}$$

$$\rightarrow r = \frac{2x}{3}$$

$$\because S_\infty = \frac{a_1}{1-r}$$

$$\rightarrow y = \frac{\frac{2}{3}x}{1-\frac{2x}{3}} = \frac{\frac{2x}{3}}{\frac{3-2x}{3}}$$

$$\text{or } y = \frac{2x}{3-2x}$$

$$y(3-2x) = 2x$$

$$\rightarrow 3y - 2xy = 2x$$

$$\text{or } 3y = 2x + 2xy$$

$$3y = 2x(1+y)$$

$$\rightarrow \frac{3y}{(1+y)} = 2x$$

$$\text{or } x = \frac{3y}{2(1+y)}$$

Hence proved

Q10. A ball is dropped from a height of 27 meters and it rebounds two-third of the distance it falls. If it continues to fall in the same way what distance will be ball travel before coming to rest?

Solution:-

according to the given condition

Height, 1st rebound, 2nd rebound,....

$27, \quad (27)\times\frac{2}{3} = 18, \quad (18)\times\left(\frac{2}{3}\right) = 12, \dots\dots$

or 27 ,18 , 12 , ... so the distance

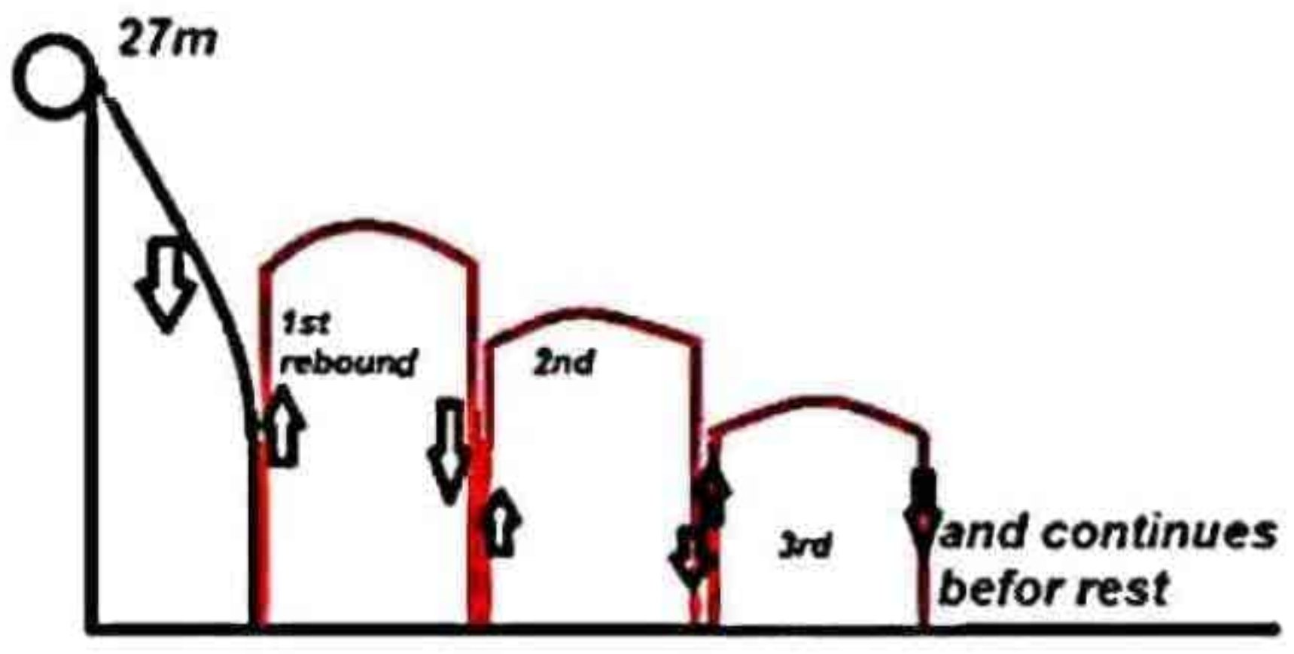
$$S_\infty = 27 + 2(18) + 2(12) + \dots$$

$$= 27 + 2(18 + 12 + \dots) \quad \left(a_1 = 18, r = \frac{12}{18} = \frac{2}{3}\right)$$

$$= 27 + 2\left(\frac{a_1}{1-r}\right)$$

$$= 27 + 2\left(\frac{18}{1-\frac{2}{3}}\right)$$

$$= 27 + 2\left(\frac{18}{\frac{3-2}{3}}\right)$$



$$= 27 + 2\left(\frac{18}{\frac{1}{3}}\right) = 27 + 2\times18\times\frac{3}{1}$$

$$d = 27 + 108 = 135\text{ m}$$

Q11. What distance will a ball travel before coming to rest if it is dropped from a height of 75 meters and after each fall it rebounds $\frac{2}{5}$ of the distance it fell?

Solution:-

According to given condition

Height, 1st rebound, 2nd rebound, ...

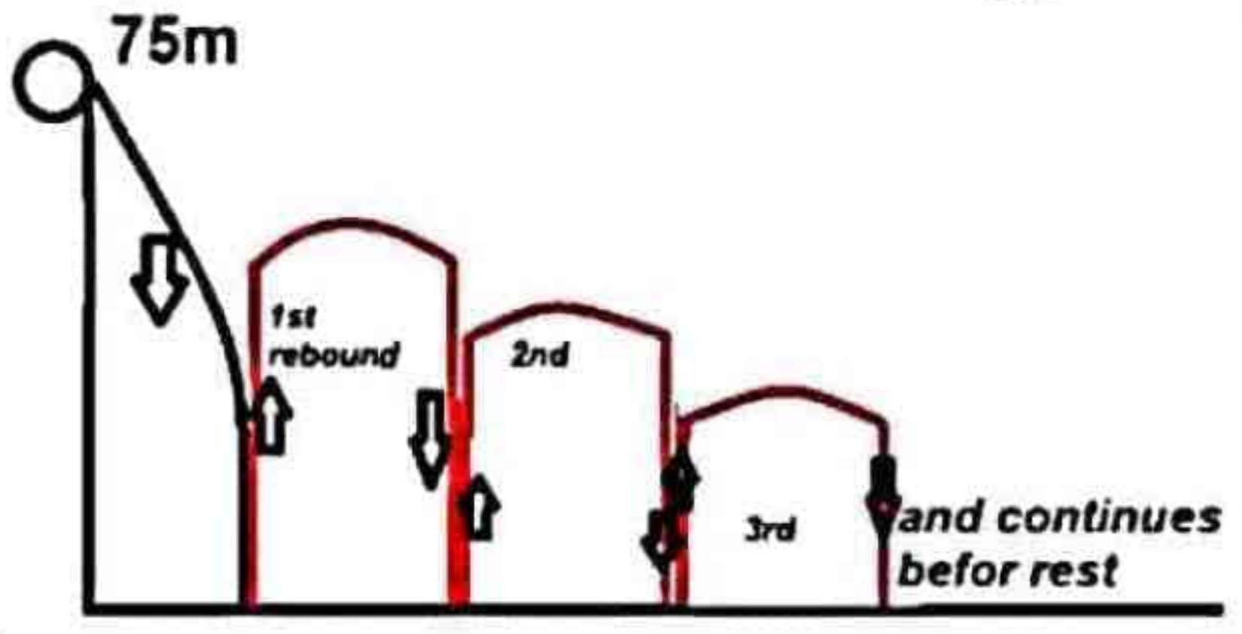
$75, \quad 75\times\frac{2}{5} = 30 \quad 30\times\left(\frac{2}{5}\right) = 12, \dots$

or 75, 30 , 12,..... so the distance

$S_\infty = 75+2(30)+2(12)+.......$

$=75+2(30+12+\cdots)$ $\left(\begin{array}{l} a_1=30 \\ r=\frac{12}{30}=\frac{2}{5}\end{array}\right)$

$=75+2\left(\frac{a_1}{1-r}\right)$



$=75+2\left(\frac{30}{1-\frac{2}{5}}\right)$

$=75+2\left(\frac{30}{\frac{5-2}{5}}\right)$

$=75+2\left(\frac{30}{\frac{3}{5}}\right) = 75+2\times30\times\frac{5}{3}$

$d = 75+100 = 175$ meters

Q12. If $y=1+2x+4x^2+8x^3+\cdots$

i) show that $x=\frac{y-1}{2y}$

ii) Find the interval in which the series is convergent.

Solution:-

$y=1+2x+4x^2+8x^3+\cdots$

$a_1=1,\quad r=\frac{2x}{1}=2x$

$\because S_\infty = \frac{a_1}{1-r}$

$\rightarrow y=\frac{1}{1-2x}$

or $y(1-2x)=1$

$y-2xy=1$

$\rightarrow y-1=2xy$

$\rightarrow \frac{y-1}{2y}=x$

or $x=\frac{y-1}{2y}$ Hence proved

For interval Series will be convergent if

$|r|<1 \rightarrow |2x|<1$

or $|x|<\frac{1}{2}$

$\rightarrow -\frac{1}{2}<x<\frac{1}{2}$

Q13. If $y=1+\frac{x}{2}+\frac{x^2}{4}+\cdots$

i) Show that $x=2\left(\frac{y-1}{y}\right)$

ii) Find the interval in which the series is convergent.

Solution:-

$y=1+\frac{x}{2}+\frac{x^2}{4}+\cdots$

$a_1=1,\quad r=\frac{\frac{x}{2}}{1}=\frac{x}{2}$

$\because S_\infty = \frac{a_1}{1-r}$

$\rightarrow y=\frac{1}{1-\frac{x}{2}} = \frac{1}{\frac{2-x}{2}}$

$y=\frac{2}{2-x} \rightarrow y(2-x)=2$

or $2y-xy=2$

$\rightarrow 2y-2=xy$

$2(y-1)=xy$

$\rightarrow \frac{2(y-1)}{y}=x$

or $x=\frac{2(y-1)}{y}$ Hence proved

For interval series will be convergent if

$|r|<1 \rightarrow \left|\frac{x}{2}\right|<1$

$\rightarrow |x|<2 \rightarrow -2<x<2$

Q14. The sum of an infinite geometric series is 9 and the sum of the squeres of its terms is $\frac{81}{5}$. Find the series?

Solution:-

suppose infinite series

$a_1+a_1r+a_1r^2+\cdots\cdots$

I condition $\rightarrow S_\infty = \frac{a_1}{1-r}=9$

$\rightarrow a_1 = 9(1-r) \longrightarrow (i)$

II condition $\rightarrow$

$a_1^2+a_1^2r^2+a_1^2r^4+\cdots = \frac{81}{5}$

$\rightarrow \frac{a_1^2}{1-r^2}=\frac{81}{5}$

$\to 5a_1^2 = 81(1-r^2)$

$5[9(1-r)]^2 = 81(1-r)(1+r)$ from (i)

$\to 5(\cancel{81})(1-r)^{\cancel{2}} = \cancel{81}(\cancel{1-r})(1+r)$

or $5(1-r) = 1+r$

$5-5r = 1+r$

$5-1 = r+5r \to 4 = 6r$

$\to r = \frac{4}{6} \to r = \frac{2}{3}$ put in (i)

$a_1 = 9(1-\frac{2}{3}) = 9(\frac{3-2}{3}) = 3(1) = 3$

$\to a_1 = 3$ so infinite series

is $3 + 3(\frac{2}{3}) + 3(\frac{2}{3})^2 + \cdots$

or $3 + 2 + \frac{4}{3} + \cdots\cdots$

Word Problems on G.P.

Example 1. A man deposits in a bank Rs. 20 in the first year; Rs. 40 in the second year; Rs. 80 in the third year and so on. Find the amount he will have deposited in the bank by the seventh year:

Solution:-

Given

$20 + 40 + 80 + \cdots$.

Here $a_1 = 20$, $r = \frac{40}{20} = 2 > 1$

$\because S_n = \frac{a_1(r^n-1)}{r-1}$ if $r>1$

$\to S_7 = \frac{(20)(2^7-1)}{2-1} = \frac{20(128-1)}{1}$

$S_7 = 20(127) = 2540$

Example 2. A person invests Rs. 2000/- at 4% interest compounded annually. What is total amount will he get after 5 years?

Solution:- Here $a_1 = 2000$

$r = 1 + 4\% = 1 + \frac{4}{100} = 1 + 0.04$

$\to r = 1.04$

so amount payable after 5 years $= a_6 = a_1 r^5$

$= 2000(1.04)^5$

$= 2000(1.26532)$

$= 2530.64$

Example 3. The population of a big town is 972405 at present and four years before it was 800,000. Find its rate of increase if it increased geometrically.

Solution:-

Here $a_1 = 800,000$

Growth rate $= r\% = \frac{r}{100}$

$a_5 = 972405$

$\because a_5 = a_1 r^4$

$972405 = (800,000)(1+\frac{r}{100})^4$

$\to (1+\frac{r}{100})^4 = (\frac{21}{20})^4$

$\to 1 + \frac{r}{100} = \frac{21}{20}$

$\frac{r}{100} = \frac{21}{20} - 1 = \frac{21-20}{20}$

$\frac{r}{100} = \frac{1}{20} \to r = \frac{100}{20} = 5$

Hence growth rate is 5% anually.

Exercise 6.9

Q1. A man deposits in a bank Rs. 8 in the first year, Rs. 24 in the second year, Rs. 72 in the third year and so on. Find the amount he will have deposited in the bank by fifth year.

Solution:-

Given $8 + 24 + 72 + \cdots\cdots a_5$

$a_1 = 8$, $r = \frac{24}{8} = 3 > 1$, $n = 5$

$\because S_n = \frac{a_1(r^n-1)}{r-1}$, $|r| > 1$

$\rightarrow S_5 = \frac{8(3^5-1)}{3-1} = \frac{8(243-1)}{2}$

$\rightarrow S_5 = 4(242) = 968$

Q2. A man borrows Rs.32760 without interest and agrees to repay the loan in installments, each installment being twice the preceding one. Find the amount of the last installment, if the amount of first installment is Rs. 8

Solution:- Here

$Sn = 32760 \ , \ r = 2 \ , \ a_1 = 8 \ , \ a_n = ?$

$\because Sn = \frac{a_1(r^n-1)}{r-1} \ , \ r>1$

$\rightarrow 32760 = \frac{8(2^n-1)}{2-1}$

$32760 = 8(2^n-1)$

$\rightarrow 2^n - 1 = \frac{32760}{8}$

$2^n - 1 = 4095$

$2^n = 4095+1$

$\rightarrow 2^n = 4096$

$2^n = 2^{12} \rightarrow n = 12$

Now $a_n = a_1 r^{n-1} = 8(2)^{12-1}$

$= 8(12)^{11} = 8(2048)$

$a_n = 16384$

Q3. The population of a certain village is 62500. What will be its population after 3 years if it increases geometrically at the rate of 4% annually?

Solution:-

$a_1 = 62500 \ , \ n = 4$

$r = 1 + 4\% = 1 + \frac{4}{100} = 1 + 0.04$

$r = 1.04$

$\because a_n = a_1 r^{n-1} \rightarrow a_4 = a_1 r^{4-1}$

$a_4 = a_1 r^3$

$= (62500)(1.04)^3$

$a_4 = 62500(1.1249) = 70304$

Q4. The enrollment of a famous school doubled after every eight years from 1970 to 1994. If the enrollment was 6000 in 1994, what was it enrollment in 1970?

Solution:-

According to given condition

1970, 1978, 1986, 1994

Here $a_4 = 6000 \ , \ a_1 = ?$

$r = 2 \ , \ n = 4$

$\because a_n = a_1 r^{n-1}$

$a_4 = a_1 r^3 = a_1 (2)^3$

$\rightarrow 6000 = a_1(8) \rightarrow a_1 = \frac{6000}{8}$

$\rightarrow a_1 = 750$

Q5. A singular cholera bacteria produces two complete bacteria in $\frac{1}{2}$ hours. If we start with a colony of A bacteria, how many bacteria will we have in n hours?

Solution:-

$\frac{1}{2} + \frac{1}{2} = 1 \text{hour} = 4A$

$1 + \frac{1}{2} = \frac{3}{4} \text{hour} = 8A$

$\frac{3}{2} + \frac{1}{2} = 2 \text{ hour} = 16A$

$2 + \frac{1}{2} = \frac{5}{2} \text{ hour} = 32A$

$\frac{5}{2} + \frac{1}{2} = 3 \text{ hour} = 64A$

So in 1, 2, 3 hours 4A, 16A, 64A, ...n

$\because a_n = a_1 r^{n-1} \qquad a_1 = 4A, r = 4$

$a_n = 4A(4)^{n-1}$

$a_n = A(4)^n = A(2)^{2n}$ bacteria

Q6. Joining the mid points of the sides of an equilateral triangle, an equilateral triangle having half the parimeter of the orignal triangle is obtained. We form a sequence of nested equilateral triangles in the same manner described above with the orignal triangle having perimeter $\frac{3}{2}$ what will be the total perimeter of all the triangles formed in this way?

Solution:-

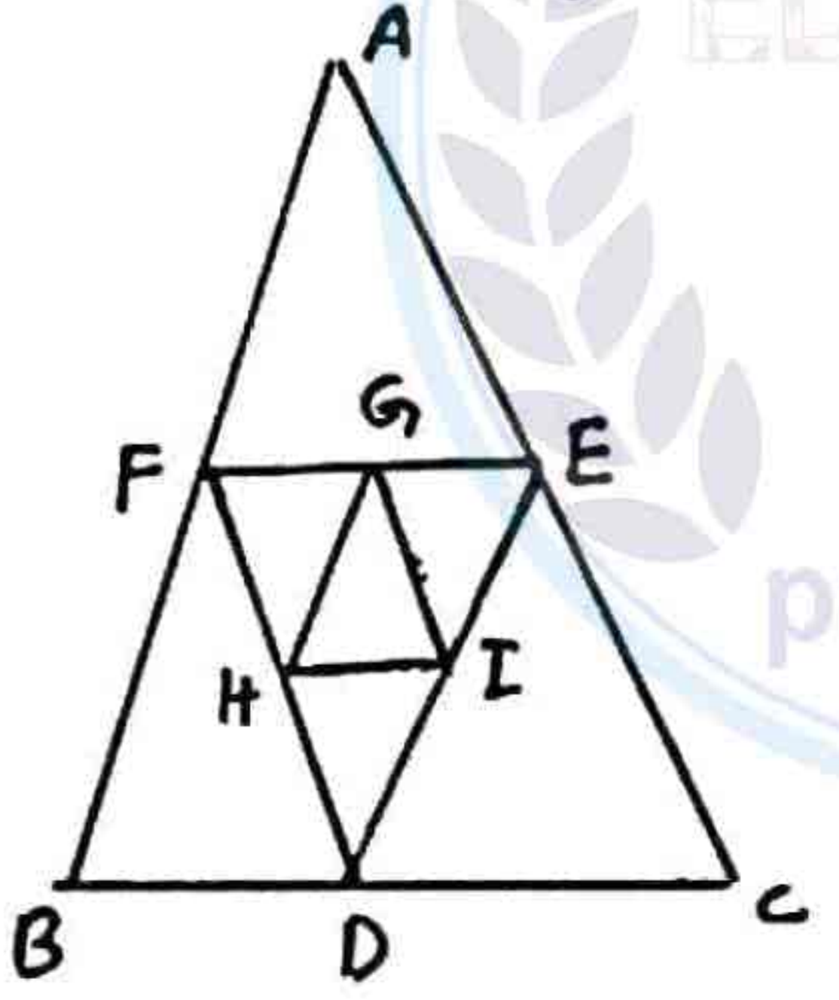
According to the given condition

perimeter of $\Delta ABC = \frac{3}{2}$

Perimeter of $\Delta DEF = \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}$

Perimeter of $\Delta GHI = \frac{1}{2}$(permeter of ΔDEF)

$= \frac{1}{2}\left(\frac{3}{4}\right) = \frac{3}{8}$



so series is

$= \frac{2}{3} + \frac{3}{4} + \frac{3}{8} + \dots$

$a_1 = \frac{3}{2}$, $r = \frac{\frac{3}{4}}{\frac{2}{3}}$

$r = \frac{3}{4} \times \frac{2}{3}$

$r = \frac{1}{2}$

$$\because S_\infty = \frac{a_1}{1-r} = \frac{\frac{3}{2}}{1-\frac{1}{2}} = \frac{\frac{3}{2}}{\frac{1}{2}} = 3$$

Harmonic Progression (H.P)

A sequence of numbers is called a Harmonic sequence or Harmonic Progression if the reciprocals of its terms are in arithmetic Progression (A.P).

e.g., $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \dots\dots$ is in H.P

where $3, 6, 9, \dots$ is in A.P

* The reciprocal of zero is not defined so zero cannot be the term of a harmonic sequence.
* The general form of H.P is

$\frac{1}{a_1}, \frac{1}{a_1+d}, \frac{1}{a_1+2d}, \dots\dots$ where

$a_1, a_1+d, a_1+2d, \dots$ in A.P

Example 1. Find the nth and 8th terms of H.P; $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \dots$

Solution:- Given that

$\because \frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \dots$ is in H.P

$a_n = ?$, $a_8 = ?$

Now $2, 5, 8, \dots$ is in A.P

Here $a_1 = 2$, $d = 5-2 = 3$

$\because a_n = a_1 + (n-1)d$

$\rightarrow a_n = 2 + (n-1)3$

$= 2 + 3n - 3$

$a_n = 3n - 1$

For $n=8$, $a_8 = 3(8)-1 = 24-1 = 23$

Hence in H.P; $a_n = \frac{1}{3n-1}$, $a_8 = \frac{1}{23}$

Example 2. If the 4th term and 7th term of an H.P are $\frac{2}{13}$ and $\frac{2}{25}$ respectively, find the sequence.

Solution:- Given that

$a_4 = \frac{2}{13}$, $a_7 = \frac{2}{25}$ $\Big\} \rightarrow$ In H.P

so $a_4 = \frac{13}{2}$, $a_7 = \frac{25}{2}$ $\Big\} \rightarrow$ In A.P

Now $a_1 + 3d = \frac{13}{2} \rightarrow (i)$ $\because a_n = a_1 + (n-1)d$

and $a_1 + 6d = \frac{25}{2} \rightarrow (ii)$

By (ii) − (i) $\rightarrow$

$$\begin{array}{r} a_1 + 6d = \frac{25}{2} \\ \underset{-}{a_1} \underset{-}{+} 3d = \underset{-}{\frac{13}{2}} \\ \hline 3d = \frac{25}{2} - \frac{13}{2} \end{array}$$

$\rightarrow 3d = \frac{25-13}{2}$

$3d = \frac{12}{2}$

$\rightarrow 3d = 6$ or $d = 2$ put in (i)

$a_1 + 3(2) = \frac{13}{2} \rightarrow a_1 + 6 = \frac{13}{2}$

$\rightarrow a_1 = \frac{13}{2} - 6 \rightarrow a_1 = \frac{13-12}{2}$

or $a_1 = \frac{1}{2}$ Now

$a_1, a_1+d, a_1+2d, \dots$ in A.P

$\frac{1}{2}, (\frac{1}{2}+2), (\frac{1}{2}+2(2)), \dots$ in A.P

$\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \dots$ in A.P

so $\frac{2}{1}, \frac{2}{5}, \frac{2}{9}, \dots$ in H.P

which is required seq.

Harmonic Mean:-

A number H is said to be Harmonic mean (H.M) between two numbers a and b if

a, H, b are in H.P

$\rightarrow \frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P

In this case

$d = \frac{1}{H} - \frac{1}{a}$ and $d = \frac{1}{b} - \frac{1}{H}$

$\rightarrow \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$

$\rightarrow \frac{1}{H} + \frac{1}{H} = \frac{1}{b} + \frac{1}{a}$

$\rightarrow \frac{1+1}{H} = \frac{a+b}{ab}$

$\rightarrow \frac{2}{H} = \frac{a+b}{ab}$

or $\frac{H}{2} = \frac{ab}{a+b}$

or $\boxed{H = \frac{2ab}{a+b}}$

n Harmonic Means between two numbers

$H_1, H_2, H_3, \dots H_n$ are n Harmonic means between a and b if

$a, H_1, H_2, H_3, \dots, H_n, b$ are in H.P.

If we want to find n H.M between a and b, we first find n A.Ms $A_1, A_2, A_3, \dots A_n$ between $\frac{1}{a}$ and $\frac{1}{b}$, then take reciprocals to get n H.Ms between a and b, that is, $\frac{1}{A_1}, \frac{1}{A_2}, \dots, \frac{1}{A_n}$ will be required n H.Ms between a and b.

Example 3. Find three harmonic means between $\frac{1}{5}$ and $\frac{1}{17}$.

Solution:-

Let H_1, H_2, H_3 be H.Ms between $\frac{1}{5}$ and $\frac{1}{17}$ then

$\frac{1}{5}, H_1, H_2, H_3, \frac{1}{17}$ are in H.P

$\rightarrow 5, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, 17$ are in A.P

$a_1 = 5$, $a_5 = 17$, $n = 5$

$\because a_5 = a_1 + 4d = 17$

$\rightarrow 5 + 4d = 17$

$4d = 17 - 5 \rightarrow 4d = 12$

or $d = 3$ Now

$\frac{1}{H_1} = a_1 + d = 5 + 3 = 8 \rightarrow H_1 = \frac{1}{8}$

$\frac{1}{H_2} = \frac{1}{H_1} + d = 8 + 3 = 11 \rightarrow H_2 = \frac{1}{11}$

$\frac{1}{H_3} = \frac{1}{H_2} + d = 11 + 3 = 14 \rightarrow H_3 = \frac{1}{14}$

Thus $\frac{1}{8}, \frac{1}{11}, \frac{1}{14}$ are req. H.Ms.

Example 4. Find n H.Ms between a and b.

Solution:-

Let $H_1, H_2, H_3, \dots, H_n$ be n H.Ms between a and b. then

$a, H_1, H_2, H_3, \dots, H_n, b$ are in H.P

$\rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \dots, \frac{1}{H_n}, \frac{1}{b}$ are in A.P

$\because a_n = a_1 + (n-1)d$

$\rightarrow \frac{1}{b} = \frac{1}{a} + (n-1)d$

$a_1 = \frac{1}{a}$
$a_n = \frac{1}{b}$
$n = n+2$

$\frac{1}{b}-\frac{1}{a} = (n+2-1)d$

$\rightarrow \frac{a-b}{ab} = (n+1)d$

$\rightarrow d = \frac{a-b}{ab(n+1)}$ Now

$\frac{1}{H_1} = \frac{1}{a}+d = \frac{1}{a}+\frac{a-b}{ab(n+1)}$

$= \frac{b(n+1)+a-b}{ab(n+1)} = \frac{bn+b+a-b}{ab(n+1)}$

$\rightarrow \frac{1}{H_1} = \frac{bn+a}{ab(n+1)} \rightarrow H_1 = \frac{ab(n+1)}{bn+a}$

$\frac{1}{H_2} = \frac{1}{H_1}+d = \frac{bn+a}{ab(n+1)} + \frac{a-b}{ab(n+1)}$

$= \frac{bn+a+a-b}{ab(n+1)}$

$\frac{1}{H_2} = \frac{b(n-1)+2a}{ab(n+1)}$

or $H_2 = \frac{ab(n+1)}{b(n-1)+2a}$

Similarly $H_3 = \frac{ab(n+1)}{b(n-2)+3a}$

$\vdots$

$H_n = \frac{ab(n+1)}{[n-(n-1)]b+na}$

$H_n = \frac{ab(n+1)}{(n-n+1)b+na}$

$H_n = \frac{ab(n+1)}{b+na}$

Hence n H.Ms between a and b are $\frac{ab(n+1)}{nb+a}, \frac{ab(n+1)}{(n-1)b+2a}, \frac{ab(n+1)}{(n-2)b+2a}$ .... $\frac{ab(n+1)}{b+na}$

Relations between A.M, G.M and H.M

Prove that A, G, H are in G.P.

or $G^2 = A \times H$

or $\frac{G}{A} = \frac{H}{G}$

Proof:- We know that

$A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $H = \frac{2ab}{a+b}$

Now $G^2 = (\sqrt{ab})^2 = ab \longrightarrow (i)$

$A \times H = \frac{a+b}{2} \times \frac{2ab}{a+b}$

$A \times H = ab \longrightarrow (ii)$

By (i) and (ii)

$\boxed{G^2 = A \times H}$ or $G \times G = A \times H$

$\rightarrow \boxed{\frac{G}{A} = \frac{H}{G}}$ it is clear that

$\rightarrow$ A, G, H are in G.P

Prove that A > G > H

if a, b are any two distinct +ive numbers

Proof:-

we know that for two +ive distinct numbers a and b

$A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $H = \frac{2ab}{a+b}$

Let $A > G \rightarrow \frac{a+b}{2} > \sqrt{ab}$

squaring both sides

$\left(\frac{a+b}{2}\right)^2 > (\sqrt{ab})^2$

$\frac{(a+b)^2}{4} > ab$

$(a+b)^2 > 4ab$

$a^2+b^2+2ab-4ab > 0$

$a^2+b^2-2ab > 0$

$\rightarrow (a-b)^2 > 0$ which is always true

so $A > G \longrightarrow (i)$

Now let $G > H$

$\rightarrow \sqrt{ab} > \frac{2ab}{a+b}$ squaring both sides

$\rightarrow ab > \frac{4a^2b^2}{(a+b)^2}$

→ $(a+b)^2 > \frac{4a^2b^2}{ab}$

→ $a^2+b^2+2ab > 4ab$

$a^2+b^2+2ab-4ab>0$

$a^2+b^2-2ab>0$

→ $(a-b)^2>0$ which is always true

so $G > H \longrightarrow$ (ii)

From (i) and (ii)

$A > G > H$ Hence proved

Prove that $A < G < H$

if a, b are any two distinct -ive numbers.

Proof:-

Let $a=-m$ and $b=-n$

where $m, n \in R^+$

$\because A = \frac{a+b}{2}$ → $A = \frac{-m+(-n)}{2}$

$G = -\sqrt{ab}$ → $G = -\sqrt{(-m)(-n)}$

$G = -\sqrt{mn}$

$H = \frac{2ab}{a+b} = \frac{2(-m)(-n)}{-m+(-n)} = \frac{2mn}{-m-n}$

Let $A < G$

→ $\frac{-m-n}{2} < -\sqrt{mn}$

$-\left(\frac{m+n}{2}\right) < -\sqrt{mn}$ $\left(\because -2<-1 \rightarrow 2>1\right)$

→ $\frac{m+n}{2} > \sqrt{mn}$

$\left(\frac{m+n}{2}\right)^2 > mn$ (squaring)

$m^2+n^2+2mn > 4mn$

$m^2+n^2+2mn-4mn > 0$

$m^2+n^2-2mn > 0$

→ $(m-n)^2 > 0$ always true

so $A < G$ —— (i)

Now let $G < H$

→ $-\sqrt{mn} < \frac{2mn}{-m-n}$

→ $-\sqrt{mn} < -\left(\frac{2mn}{m+n}\right)$

→ $\sqrt{mn} > \frac{2mn}{m+n}$ $\left(\because -2<-1,\ 2>1\right)$

$mn > \frac{4m^2n^2}{(m+n)^2}$ (squaring)

$(m+n)^2 > \frac{4m^2n^2}{mn}$

$m^2+n^2+2mn > 4mn$

→ $m^2+n^2+2mn-4mn > 0$

$m^2+n^2-2mn > 0$

→ $(m-n)^2 > 0$ (always true)

so $G < H \longrightarrow$ (ii)

from (i) and (ii)

$A < G < H$ Hence proved.

Prove that $A > H$ (Page # 223)

for any two distinct +ive numbers a and b

Solution:-

$A > H$ if

$\frac{a+b}{2} > \frac{2ab}{a+b}$

→ $(a+b)^2 > 4ab$

$a^2+b^2+2ab-4ab>0$

$a^2+b^2-2ab>0$

$(a-b)^2>0$ always true

Hence $A > H$

Exercise 6.10

Q1. Find the 9th term of the harmonic sequence.

i) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ ii) $-\frac{1}{5}, -\frac{1}{3}, -1, \ldots$

Solution:- i) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots, a_9 = ?$

$\because \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ in H.P

$\rightarrow 3, 5, 7, \ldots$ in A.P

$a_1 = 3, \quad d = 5 - 3 = 2$

$\because a_n = a_1 + (n-1)d$

$\rightarrow a_9 = a_1 + 8d$

$= 3 + 8(2)$

$a_9 = 3 + 16$

$\rightarrow a_9 = 16$ in A.P

and $a_9 = \frac{1}{16}$ in H.P

ii) $-\frac{1}{5}, -\frac{1}{3}, -1, \ldots, a_9 = ?$

$\because -\frac{1}{5}, -\frac{1}{3}, -1, \ldots$ in H.P

$\rightarrow -5, -3, -1, \ldots$ in A.P

$a_1 = -5, \quad d = -3-(-5) = -3+5 = 2$

$\because a_9 = a_1 + 8d$

$= -5 + 8(2) = -5 + 16$

$a_9 = 11$ in A.P

$\rightarrow a_9 = \frac{1}{11}$ in H.P

Q2. Find the 12th terms of the following harmonic sequences.

i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$ ii) $\frac{1}{3}, \frac{2}{9}, \frac{1}{6}, \ldots$

Solution:- i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots \quad a_{12} = ?$

$\because \frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$ in H.P

$\rightarrow 2, 5, 8, \ldots$ in A.P

$a_1 = 2, \quad d = 5 - 2 = 3$

$\because a_n = a_1 + (n-1)d$

$a_{12} = 2 + (12-1)(3)$

$a_{12} = 2 + (11)(3) = 2 + 33 = 35$

$a_{12} = 35$ in A.P

$\rightarrow a_{12} = \frac{1}{35}$ in H.P

ii) $\frac{1}{3}, \frac{2}{9}, \frac{1}{6}, \ldots \quad a_{12} = ?$

$\because \frac{1}{3}, \frac{2}{9}, \frac{1}{6}, \ldots$ in H.P

$\rightarrow 3, \frac{9}{2}, 6, \ldots$ in A.P

$a_1 = 3, \quad d = \frac{9}{2} - 3 = \frac{9-6}{2} = \frac{3}{2}$ $\quad \because a_n = a_1 + (n-1)d$

$\because a_{12} = a_1 + 11d$

$= 3 + 11\left(\frac{3}{2}\right) = 3 + \frac{33}{2}$

$a_{12} = \frac{6+33}{2} = \frac{39}{2}$

$\rightarrow a_{12} = \frac{39}{2}$ in A.P

so $a_{12} = \frac{2}{39}$ in H.P

Q3. Insert five harmonic means between the following given numbers.

i) $-\frac{2}{5}$ and $\frac{2}{13}$ ii) $\frac{1}{4}$ and $\frac{1}{24}$

Solution:- i) $-\frac{2}{5}$ and $\frac{2}{13}$

Let H_1, H_2, H_3, H_4, H_5 be H.Ms between $-\frac{2}{5}$ and $\frac{2}{13}$ then

$-\frac{2}{5}, H_1, H_2, H_3, H_4, H_5, \frac{2}{13}$ are in H.P

$\rightarrow -\frac{5}{2}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{1}{H_5}, \frac{13}{2}$ in A.P

$a_1 = -\frac{5}{2}, \quad n = 7, \quad a_7 = \frac{13}{2}$

$\rightarrow a_1 + 6d = \frac{13}{2}$

$\rightarrow -\frac{5}{2} + 6d = \frac{13}{2} \rightarrow 6d = \frac{13}{2} + \frac{5}{2}$

$\rightarrow \cancel{6}d = \frac{\cancel{18}^{3}}{2} \rightarrow d = \frac{3}{2}$

Now $\frac{1}{H_1} = a_1 + d = -\frac{5}{2} + \frac{3}{2} = -\frac{2}{2} = -1 \rightarrow H_1 = -1$

$\frac{1}{H_2} = \frac{1}{H_1} + d = -1 + \frac{3}{2} = \frac{-2+3}{2} = \frac{1}{2}$

$\rightarrow H_2 = 2$

$\frac{1}{H_3} = \frac{1}{H_2} + d$

$\frac{1}{H_3} = \frac{1}{2} + \frac{3}{2} = \frac{1+3}{2} = \frac{4}{2} = 2$

$\rightarrow H_3 = \frac{1}{2}$

$\frac{1}{H_4} = \frac{1}{H_3} + d = 2 + \frac{3}{2} = \frac{7}{2}$

$\rightarrow H_4 = \frac{2}{7}$

$\frac{1}{H_5} = \frac{1}{H_4} + d = \frac{7}{2} + \frac{3}{2} = \frac{10}{2} = 5$

$\rightarrow H_5 = \frac{1}{5}$

Req. 5 H.Ms are $-1, 2, \frac{1}{2}, \frac{2}{7}, \frac{1}{5}$

ii) $\frac{1}{4}$ and $\frac{1}{24}$

Solution:-

Let H_1, H_2, H_3, H_4, H_5 be 5 H.Ms between $\frac{1}{4}$ and $\frac{1}{24}$.

$\frac{1}{4}, H_1, H_2, H_3, H_4, H_5, \frac{1}{24}$ in H.P

$\rightarrow 4, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{1}{H_5}, 24$ in A.P

$a_1 = 4, \quad n = 7, \quad a_7 = 24$

$a_7 = a_1 + 6d = 24$

$\rightarrow 4 + 6d = 24$

$6d = 24 - 4$

$\rightarrow 6d = 20 \rightarrow d = \frac{20}{6} = \frac{10}{3}$

Now $\frac{1}{H_1} = a_1 + d = 4 + \frac{10}{3} = \frac{12+10}{3} = \frac{22}{3}$

$\rightarrow H_1 = \frac{3}{22}$

$\frac{1}{H_2} = \frac{1}{H_1} + d = \frac{22}{3} + \frac{10}{3} = \frac{32}{3}$

$\rightarrow H_2 = \frac{3}{32}$

$\frac{1}{H_3} = \frac{1}{H_2} + d = \frac{32}{3} + \frac{10}{3} = \frac{42}{3}$

$\rightarrow H_3 = \frac{3}{42}$

$\frac{1}{H_4} = \frac{1}{H_3} + d = \frac{42}{3} + \frac{10}{3} = \frac{52}{3}$

$\rightarrow H_4 = \frac{3}{52}$

$\frac{1}{H_5} = \frac{1}{H_4} + d = \frac{52}{3} + \frac{10}{3} = \frac{62}{3}$

$\rightarrow H_5 = \frac{3}{62}$

Req. 5 H.Ms are $\frac{3}{22}, \frac{3}{32}, \frac{3}{42}, \frac{3}{52}, \frac{3}{62}$

Q4. Insert four harmonic means between the following given numbers.

i) $\frac{1}{3}$ and $\frac{1}{23}$ ii) $\frac{7}{3}$ and $\frac{7}{11}$

iii) 4 and 20

Solution:- i) $\frac{1}{3}$ and $\frac{1}{23}$

Let H_1, H_2, H_3, H_4 be 4 H.Ms between $\frac{1}{3}$ and $\frac{1}{23}$ then

$\frac{1}{3}, H_1, H_2, H_3, H_4, \frac{1}{23}$ in H.P

$\rightarrow 3, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, 23$ in A.P

$a_1 = 3, \quad n = 6, \quad a_6 = 23$

$\rightarrow a_1 + 5d = 23$

$3 + 5d = 23$

$5d = 23 - 3$

$5d = 20 \rightarrow d = 4$

Now $\frac{1}{H_1} = a_1 + d = 3 + 4 = 7 \rightarrow H_1 = \frac{1}{7}$

$\frac{1}{H_2} = \frac{1}{H_1} + d = 7 + 4 = 11 \rightarrow H_2 = \frac{1}{11}$

$\frac{1}{H_3} = \frac{1}{H_2} + d = 11 + 4 = 15 \rightarrow H_3 = \frac{1}{15}$

$\frac{1}{H_4} = \frac{1}{H_3} + d = 15 + 4 = 19 \rightarrow H_4 = \frac{1}{19}$

Thus $\frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \frac{1}{19}$ are req. 4 H.Ms.

ii) $\frac{7}{3}$ and $\frac{7}{11}$

Solution:-

Let H_1, H_2, H_3, H_4 are 4 H.Ms between $\frac{7}{3}$ and $\frac{7}{11}$ then

$\frac{7}{3}, H_1, H_2, H_3, H_4, \frac{7}{11}$ are in H.P

$\rightarrow \frac{3}{7}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{11}{7}$ are in A.P

$a_1 = \frac{3}{7}, \quad n = 6, \quad a_6 = \frac{11}{7}$

$a_6 = a_1 + 5d = \frac{11}{7}$

$\rightarrow \frac{3}{7} + 5d = \frac{11}{7}$

$5d = \frac{11}{7} - \frac{3}{7} = \frac{8}{7}$

$\rightarrow d = \frac{8}{35}$

Now $\frac{1}{H_1} = a_1 + d = \frac{3}{7} + \frac{8}{35} = \frac{3(5)+8}{35}$

$\frac{1}{H_1} = \frac{15+8}{35} = \frac{23}{35} \rightarrow H_1 = \frac{35}{23}$

$\frac{1}{H_2} = \frac{1}{H_1} + d = \frac{23}{35} + \frac{8}{35} = \frac{31}{35}$

$\rightarrow H_2 = \frac{35}{31}$

$\frac{1}{H_3} = \frac{1}{H_2} + d = \frac{31}{35} + \frac{8}{35} = \frac{39}{35}$

$\rightarrow H_3 = \frac{35}{39}$

$\frac{1}{H_4} = \frac{1}{H_3} + d = \frac{39}{35} + \frac{8}{35} = \frac{47}{35}$

$\rightarrow H_4 = \frac{35}{47}$

Thus four H.Ms are $\frac{35}{23}, \frac{35}{31}, \frac{35}{39}$ and $\frac{35}{47}$.

iii) 4 and 20

Solution:-

Let H_1, H_2, H_3 and H_4 be four H.Ms between 4 and 20, then

$4, H_1, H_2, H_3, H_4, 20$ in H.P

$\rightarrow \frac{1}{4}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{1}{20}$ in A.P

$a_1 = \frac{1}{4}, \quad n = 6, \quad a_6 = \frac{1}{20}$

$a_6 = a_1 + 5d = \frac{1}{20}$

$\rightarrow \frac{1}{4} + 5d = \frac{1}{20}$

$5d = \frac{1}{20} - \frac{1}{4} = \frac{1-5}{20} = \frac{-4}{20}$

$\rightarrow 5d = -\frac{1}{5} \rightarrow d = \frac{-1}{25}$

Now $\frac{1}{H_1} = a_1 + d = \frac{1}{4} + \left(-\frac{1}{25}\right) = \frac{1}{4} - \frac{1}{25}$

$\frac{1}{H_1} = \frac{25-4}{100} = \frac{21}{100} \rightarrow H_1 = \frac{100}{21}$

$\frac{1}{H_2} = \frac{1}{H_1} + d = \frac{21}{100} + \left(-\frac{1}{25}\right)$

$= \frac{21}{100} - \frac{1}{25} = \frac{21-4}{100}$

$\frac{1}{H_2} = \frac{17}{100} \rightarrow H_2 = \frac{100}{17}$

$\frac{1}{H_3} = \frac{1}{H_2} + d = \frac{17}{100} + \left(-\frac{1}{25}\right)$

$\frac{1}{H_3} = \frac{17}{100} - \frac{1}{25} = \frac{17-4}{100}$

$\frac{1}{H_3} = \frac{13}{100} \rightarrow H_3 = \frac{100}{13}$

$\frac{1}{H_4} = \frac{1}{H_3} + d = \frac{13}{100} + \left(-\frac{1}{25}\right)$

$= \frac{13}{100} - \frac{1}{25} = \frac{13-4}{100}$

$\frac{1}{H_4} = \frac{9}{100} \rightarrow H_4 = \frac{100}{9}$

Hence four H.Ms between 4 and 20 are $\frac{100}{21}, \frac{100}{17}, \frac{100}{13}, \frac{100}{9}$.

Q5. If the 7th and 10th terms of an H.P are $\frac{1}{3}$ and $\frac{5}{21}$ respectively, find its 14th term.

Solution:- Given that

$a_7 = \frac{1}{3}, \quad a_{10} = \frac{5}{21} \Big\} \rightarrow$ in H.P

$a_7 = 3, \quad a_{10} = \frac{21}{5} \Big\} \rightarrow$ in A.P

$\rightarrow a_1 + 6d = 3 \longrightarrow$ (i)

$a_1 + 9d = \frac{21}{5} \longrightarrow$ (ii)

By (ii) − (i) $\rightarrow a_1 + 9d = \frac{21}{5}$

$\underset{-}{a_1} \underset{-}{+} 6d = \underset{-}{} 3$

$3d = \frac{21}{5} - 3$

$\rightarrow 3d = \frac{21-15}{5} \rightarrow 3d = \frac{6}{5}$

$\rightarrow d = \frac{2}{5}$ put in (i)

$a_1 + 6\left(\frac{2}{5}\right) = 3 \rightarrow a_1 + \frac{12}{5} = 3$

$\rightarrow a_1 = 3 - \frac{12}{5} \rightarrow a_1 = \frac{15-12}{5}$

$a_1 = \frac{3}{5}$

Now $a_{14} = a_1 + 13d = \frac{3}{5} + 13\left(\frac{2}{5}\right)$

$a_{14} = \frac{3}{5} + \frac{26}{5} = \frac{29}{5}$

$\rightarrow a_{14} = \frac{29}{5}$ in A.P

$\rightarrow a_{14} = \frac{5}{29}$ in H.P

Q6. The first term of an H.P is $-\frac{1}{3}$ and the fifth term is $\frac{1}{5}$. Find its 9th term.

Solution:- Given that

$a_1 = -\frac{1}{3}$, $a_5 = \frac{1}{5}$ $\} \rightarrow$ In H.P

$\rightarrow a_1 = -3$, $a_5 = 5$ $\} \rightarrow$ In A.P

$\rightarrow a_1 + 4d = 5$ ...(i) $\because a_n = a_1 + (n-1)d$

$-3 + 4d = 5$

$4d = 5+3$

$\rightarrow 4d = 8 \rightarrow d = 2$ put in (i)

$a_1 + 4(2) = 5 \rightarrow a_1 + 8 = 5$

$\rightarrow a_1 = 5-8 \rightarrow a_1 = -3$

Now $a_9 = a_1 + 8d$

$a_9 = -3 + 8(2) = -3 + 16$

$a_9 = 13$ in A.P

$\rightarrow a_9 = \frac{1}{13}$ in H.P

Q7. If 5 is the harmonic mean between 2 and b, find b.

Solution:-

Here H.M = 5 , a = 2, b = b

$\because H.M = \frac{2ab}{a+b}$

$\rightarrow 5 = \frac{2(2)b}{2+b}$

or $5(2+b) = 4b$

or $10 + 5b = 4b$

$5b - 4b = -10$

$\rightarrow b = -10$

Q8. If the numbers $\frac{1}{K}$, $\frac{1}{2K+1}$ and $\frac{1}{4K-1}$ are in harmonic sequence, find K.

Solution:-

$\because \frac{1}{K}, \frac{1}{2K+1}, \frac{1}{4K-1}$ are in H.P

$\rightarrow K, 2K+1, 4K-1$ are in A.P

$\rightarrow 4K - 1 - (2K+1) = 2K+1-K$

$\rightarrow 4K-1-2K-1 = 2K+1-K$

$2K - 2 = K+1$

$\rightarrow 2K - K = 1+2$

or $K = 3$

Q9. Find n, so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be H.M. between a and b.

Solution:-

$\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ be H.M between a and b

$\rightarrow \frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{2ab}{a+b}$

$\rightarrow (a^{n+1}+b^{n+1})(a+b) = (a^n+b^n)(2ab)$

$a^{n+2} + a^{n+1}b + b^{n+1}a + b^{n+2} = 2a^{n+1}b + 2ab^{n+1}$

$a^{n+2} + b^{n+2} = 2a^{n+1}b + 2ab^{n+1} - a^{n+1}b - b^{n+1}a$

$a^{n+2} + b^{n+2} = a^{n+1}b + ab^{n+1}$

$\rightarrow a^{n+2} - a^{n+1}b = ab^{n+1} - b^{n+2}$

$\rightarrow a^{n+1}\cdot a - a^{n+1}b = ab^{n+1} - b^{n+1}\cdot b$

$\rightarrow a^{n+1}\cancel{(a-b)} = b^{n+1}\cancel{(a-b)}$

or $a^{n+1} = b^{n+1}$

$\rightarrow \frac{a^{n+1}}{b^{n+1}} = 1$ $(\div$ by $b^{n+1})$

or $\left(\frac{a}{b}\right)^{n+1} = \left(\frac{a}{b}\right)^0$

$\rightarrow n+1 = 0 \rightarrow n = -1$

Q10. If a^2, b^2 and c^2 are in A.P. show that $a+b$, $c+a$ and $b+c$ are in H.P.

Solution:-

$\because a^2, b^2, c^2$ in A.P

$\rightarrow b^2 - a^2 = c^2 - b^2$

$\rightarrow (b-a)(b+a) = (c-b)(c+b)$

or $-(a-b)(a+b) = -(b-c)(b+c)$

$\rightarrow (a-b)(a+b) = (b-c)(b+c)$

or $\frac{a-b}{b+c} = \frac{b-c}{a+b} \longrightarrow (i)$

Now we prove that

a+b, c+a, b+c are in H.P

$\rightarrow \frac{1}{a+b}, \frac{1}{c+a}, \frac{1}{b+c}$ are in A.P

$\rightarrow \frac{1}{c+a} - \frac{1}{a+b} = \frac{1}{b+c} - \frac{1}{c+a}$

$\rightarrow \frac{a+b-c-a}{(\cancel{c+a})(a+b)} = \frac{c+a-b-c}{(\cancel{c+a})(b+c)}$

$\rightarrow \frac{b-c}{a+b} = \frac{a-b}{b+c}$

See eq (i)

Hence a+b, c+a, b+c are in H.P.

Q11. The sum of the first and fifth terms of the harmonic sequence is $\frac{4}{7}$, if the first term is $\frac{1}{2}$, find the sequence.

Solution:- Given that

$a_1 + a_5 = \frac{4}{7}$, $a_1 = \frac{1}{2}$ in H.P

$\rightarrow \frac{1}{2} + a_5 = \frac{4}{7}$

$a_5 = \frac{4}{7} - \frac{1}{2} = \frac{8-7}{14} = \frac{1}{14}$

$\rightarrow a_5 = \frac{1}{14}$, $a_1 = \frac{1}{2}$ in H.P

so $a_5 = 14$, $a_1 = 2$ in A.P

$\rightarrow a_1 + 4d = 14$ $\because a_n = a_1 + (n-1)d$

$2 + 4d = 14$

$4d = 14 - 2 \rightarrow 4d = 12 \rightarrow d = 3$

Now $a_1 = 2$, $a_2 = a_1 + d = 2 + 3 = 5$

$a_3 = a_1 + 2d = 2 + 2(3) = 8$

so 2, 5, 8, ... in A.P

$\rightarrow \frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots\ldots$ in H.P

which is req. series.

Q12. If A, G and H are the arithmetic, geometric and harmonic means between a and b respectively, show that $G^2 = AH$.

Solution:-

We know that

$A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $H = \frac{2ab}{a+b}$

Now $AH = \left(\frac{a+b}{2}\right)\left(\frac{2ab}{a+b}\right)$

$AH = ab \longrightarrow (i)$

Also $G = \sqrt{ab}$

$\rightarrow G^2 = ab \longrightarrow (ii)$

By (i) and (ii) $\rightarrow G^2 = AH$

Hence proved.

Q13. Find A, G, H and show that $G^2 = AH$ if

i) $a = -2$, $b = -6$

Solution:-

$\because A = \frac{a+b}{2} = \frac{-2-6}{2} = -\frac{8}{2} = -4$

$G = \pm\sqrt{ab} = \pm\sqrt{(-2)(-6)} = \pm\sqrt{12}$

$H = \frac{2ab}{a+b} = \frac{2(-2)(-6)}{-2-6} = \frac{24}{-8} = -3$

Now $G^2 = (\pm\sqrt{12})^2 = 12 \longrightarrow (i)$

$AH = (-4)(-3) = 12 \longrightarrow (ii)$

By (i) and (ii) Hence $G^2 = AH$

ii) $a = 2i$, $b = 4i$

Solution:-

$\because A = \frac{a+b}{2} = \frac{2i+4i}{2} = \frac{6i}{2} = 3i$

$G = \pm\sqrt{ab} = \pm\sqrt{(2i)(4i)} = \pm\sqrt{8i^2} = \pm\sqrt{-8}$

$H = \frac{2ab}{a+b} = \frac{2(2i)(4i)}{2i+4i} = \frac{16i^2}{6i} = \frac{-8}{3i}$

Now $G^2 = (\pm\sqrt{-8})^2 = -8 \longrightarrow (i)$

$AH = (3i)\left(-\frac{8}{3i}\right) = -8 \longrightarrow (ii)$

By (i) and (ii) $\rightarrow G^2 = AH$

iii) $a = 9,\ b = 4$

Solution:-

$$A = \frac{a+b}{2} = \frac{9+4}{2} = \frac{13}{2}$$

$$G = \pm\sqrt{ab} = \pm\sqrt{(9)(4)} = \pm\sqrt{36} = \pm 6$$

$$H = \frac{2ab}{a+b} = \frac{2(9)(4)}{9+4} = \frac{72}{13}$$

Now $G^2 = (\pm 6)^2 = 36 \longrightarrow (i)$

$$AH = \left(\frac{13}{2}\right)\left(\frac{72}{13}\right) = \frac{72}{2} = 36 \longrightarrow (ii)$$

from (i) and (ii) → $G^2 = AH$

Q14. Find A, G, H and verify that $A > G > H$ $(G>0)$ if

i) $a = 2,\ b = 8$

Solution:-

$$\because A = \frac{a+b}{2} = \frac{2+8}{2} = 5$$

$$G = \sqrt{ab} = \sqrt{(2)(8)} = \sqrt{16} = 4\ (G>0)$$

$$H = \frac{2ab}{a+b} = \frac{2(2)(8)}{2+8} = \frac{32}{10} = \frac{16}{5}$$

clearly $5 > 4 > \frac{16}{5}$

Hence $A > G > H$

ii) $a = \frac{2}{5},\ b = \frac{8}{5}$

Solution:-

$$\because A = \frac{a+b}{2} = \frac{\frac{2}{5}+\frac{8}{5}}{2} = \frac{1}{2}\left(\frac{2+8}{5}\right)$$

$$A = \frac{1}{2}\left(\frac{10}{5}\right) = \frac{10}{10} = 1$$

$$G = \sqrt{ab} = \sqrt{\left(\frac{2}{5}\right)\left(\frac{8}{5}\right)} = \sqrt{\frac{16}{25}}$$

$$G = \frac{4}{5}\ (G>0)$$

$$H = \frac{2ab}{a+b} = \frac{2\left(\frac{2}{5}\right)\left(\frac{8}{5}\right)}{\frac{2}{5}+\frac{8}{5}}$$

$$= \frac{\frac{32}{25}}{\frac{10}{5}} = \frac{32}{25} \times \frac{5}{10}$$

$$H = \frac{16}{25}$$

clearly $1 > \frac{4}{5} > \frac{16}{25}$

→ $A > G > H$

Q15. Find A, G, H and verify that $A < G < H$ $(G<0)$, if

i) $a = -2,\ b = -8$

Solution:-

$$\because A = \frac{a+b}{2} = \frac{-2-8}{2} = -\frac{10}{2} = -5$$

$$G = -\sqrt{ab} = -\sqrt{(-2)(-8)} = -\sqrt{16} = -4\ (G<0)$$

$$H = \frac{2ab}{a+b} = \frac{2(-2)(-8)}{-2-8} = \frac{-32}{-10} = \frac{32}{10}$$

clearly $-5 < -4 < \frac{32}{10}$

→ $A < G < H$

ii) $a = -\frac{2}{5},\ b = -\frac{8}{5}$

Solution:-

$$\because A = \frac{a+b}{2} = \frac{-\frac{2}{5}+\left(-\frac{8}{5}\right)}{2}$$

$$= \frac{1}{2}\left(\frac{-2-8}{5}\right) = \frac{1}{2}\left(\frac{-10}{5}\right) = \frac{-10}{10} = -1$$

$$G = -\sqrt{ab} = -\sqrt{\left(-\frac{2}{5}\right)\left(-\frac{8}{5}\right)} = -\sqrt{\frac{16}{25}}$$

$$G = -\frac{4}{5}\ (G<0)$$

$$H = \frac{2ab}{a+b} = \frac{2\left(-\frac{2}{5}\right)\left(-\frac{8}{5}\right)}{-\frac{2}{5}-\frac{8}{5}}$$

$$= \frac{\frac{32}{25}}{\frac{-10}{5}} = \frac{32}{25} \times \left(\frac{-5}{10}\right) = \frac{-16}{25}$$

clearly $-1 < -\frac{4}{5} < -\frac{16}{25}$

→ $A < G < H$

Q16. If the H.M and A.M between two numbers are 4 and $\frac{9}{2}$ respectively, find the numbers.

Solution:- Let a and b be numbers

H.M = 4 , A.M = $\frac{9}{2}$

$\rightarrow \frac{2ab}{a+b} = 4$ —(i) $\frac{a+b}{2} = \frac{9}{2}$

$\rightarrow a+b = 9 \longrightarrow (ii)$

i) $\rightarrow 4 = \frac{2ab}{9}$ $\because a+b=9$ from (ii)

$\rightarrow 36 = 2ab$

$ab = 18 \longrightarrow$ (iii)

from (ii) $\rightarrow a+b=9$

$\rightarrow a = 9-b$ put in (iii)

$b(9-b) = 18$

$\rightarrow 9b - b^2 = 18 \rightarrow b^2 - 9b + 18 = 0$

$b^2 - 6b - 3b + 18 = 0$

$b(b-6) - 3(b-6) = 0$

$(b-6)(b-3) = 0$

$\rightarrow b-6=0, \quad b-3=0$

$b = 6, \quad b = 3$

when $b=6$ so (iii) $a(6) = 18$

$\rightarrow a = 3$

when $b=3$ so (iii) $a(3) = 18$

$\rightarrow a = 6$

so numbers are 3, 6 or 6, 3

Q17. If the (positive) G.M and H.M between two numbers are 4 and $\frac{16}{5}$, find the numbers.

Solution:- Let a and b be numbers so

$H.M = \frac{16}{5}$, $G.M = 4$

$\rightarrow \frac{2ab}{a+b} = \frac{16}{5} \longrightarrow$ (i) , $\sqrt{ab} = 4$

$\rightarrow ab = 16 \longrightarrow$ (ii)

i) $\rightarrow \frac{2(16)}{a+b} = \frac{16}{5}$ ($\because ab = 16$ from (ii))

$\rightarrow \frac{2}{a+b} = \frac{1}{5}$

or $a+b = 10$

$\rightarrow a = 10-b$ put in (ii)

$(10-b)b = 16$

$10b - b^2 = 16$

$\rightarrow b^2 - 10b + 16 = 0$

$b^2 - 8b - 2b + 16 = 0$

$b(b-8) - 2(b-8) = 0$

$\rightarrow (b-8)(b-2) = 0$

$b-8 = 0, \quad b-2 = 0$

$b = 8, \quad b = 2$

when $b = 8$, so (ii) $\rightarrow a(8) = 16$

$\rightarrow a = 2$

when $b = 2$, so (ii) $\rightarrow a(2) = 16$

$\rightarrow a = 8$

Hence numbers are 2, 8 or 8, 2

Q18. If the numbers $\frac{1}{2}$, $\frac{4}{21}$ and $\frac{1}{36}$ are subtracted from the three consecutive terms of a G.P, the resulting numbers are in H.P. Find the numbers if their product is $\frac{1}{27}$.

Solution:- Let three consecutive terms of G.P., are a_1, a_1r, a_1r^2

I condition $\rightarrow$ $a_1 - \frac{1}{2}, a_1r - \frac{4}{21}, a_1r^2 - \frac{1}{36}$ are in H.P

II condition $\rightarrow$ $(a_1)(a_1r)(a_1r^2) = \frac{1}{27} \rightarrow a_1^3 r^3 = \frac{1}{27}$

$\rightarrow (a_1r)^3 = \frac{1}{(3)^3}$

so $a_1 r = \frac{1}{3} \rightarrow r = \frac{1}{3a_1}$

Now we have H.P. is

$a_1 - \frac{1}{2}, \; a_1\left(\frac{1}{3a_1}\right) - \frac{4}{21}, \; a_1\left(\frac{1}{3a_1}\right)^2 - \frac{1}{36}$ in H.P

$a_1 - \frac{1}{2}, \; \frac{1}{3} - \frac{4}{21}, \; \frac{1}{9a_1} - \frac{1}{36}$ in H.P

$\frac{2a_1-1}{2}, \; \frac{7-4}{21}, \; \frac{4-a_1}{36a_1}$ in H.P

$\frac{2a_1-1}{2}, \; \frac{1}{7}, \; \frac{4-a_1}{36a_1}$ in H.P

$\rightarrow \frac{2}{2a_1-1}, \; \frac{7}{1}, \; \frac{36a_1}{4-a_1}$ in A.P

so $7 - \frac{2}{2a_1-1} = \frac{36a_1}{4-a_1} - 7$

$$\rightarrow 7+7 = \frac{36a_1}{4-a_1} + \frac{2}{2a_1-1}$$

$$\rightarrow 14 = \frac{36a_1(2a_1-1)+2(4-a_1)}{(4-a_1)(2a_1-1)}$$

$$\rightarrow 14 = \frac{72a_1^2-36a_1+8-2a_1}{8a_1-4-2a_1^2+a_1}$$

$$14 = \frac{72a_1^2-38a_1+8}{-2a_1^2+9a_1-4}$$

$$\rightarrow 14(-2a_1^2+9a_1-4) = 72a_1^2-38a_1+8$$

$$-28a_1^2+126a_1-56 = 72a_1^2-38a_1+8$$

$$\rightarrow 72a_1^2-38a_1+8+28a_1^2-126a_1+56=0$$

$$100a_1^2-164a_1+64=0$$

$$25a_1^2-41a_1+16=0 \quad (\div \text{ by } 4)$$

$$\rightarrow 25a_1^2-25a_1-16a_1+16=0$$

$$25a_1(a_1-1)-16(a_1-1)=0$$

$$(a_1-1)(25a_1-16)=0$$

$$a_1-1=0 \ , \quad 25a_1-16=0$$

$$\rightarrow a_1 = 1 \ , \quad a_1 = \frac{16}{25}$$

If $a_1 = 1$ then $r = \frac{1}{3(1)} = \frac{1}{3}$

If $a_1 = \frac{16}{25}$ then $r = \frac{1}{3(\frac{16}{25})} = \frac{25}{48}$

If $a_1 = 1$, $r = \frac{1}{3}$ then numbers

a_1 , a_1r , a_1r^2

1 , $(1)(\frac{1}{3})$, $(1)(\frac{1}{3})^2$

or 1 , $\frac{1}{3}$, $\frac{1}{9}$

If $a_1 = \frac{16}{25}$, $r = \frac{25}{48}$ then

numbers are a_1 , a_1r, a_1r^2

$\frac{16}{25}$, $\frac{16}{25}(\frac{25}{48})$, $(\frac{16}{25})(\frac{25}{48})^2$

or $\frac{16}{25}$, $\frac{1}{3}$, $\frac{25}{144}$

Sigma Notation (or Summation Notation)

The Greek letter Σ (sigma) is used to sum a sequence of numbers. We write the sum in sigma notation as,

$$a_1 + a_2 + a_3 + \ldots\ldots + a_n = \sum_{K=1}^{n} a_K$$

Here Σ indicates the sum and K is called index of sumation. The summation begins from $K=1$ and ends with $K=n$. $K=1$ is called lower limit while $K=n$ is called upper limit.

Remember that

i) $\sum_{K=1}^{n} 1 = n(1) = n$

ii) $\sum_{K=1}^{n} c = nc$ (c is constant)

iii) $\sum_{K=1}^{n} (a_K + b_K) = \sum_{K=1}^{n} a_K + \sum_{K=1}^{n} b_K$

iv) $\sum_{K=1}^{n} (a_K - b_K) = \sum_{K=1}^{n} a_K - \sum_{K=1}^{n} b_K$

v) $\sum_{K=1}^{n} c a_K = c \sum_{K=1}^{n} a_K$

we may write $\sum_{K=1}^{n} a_K = \sum_{1}^{K} a_K$

and $\sum_{K=1}^{n} [K^m - (K-1)^m] = n^m$

To find the Formulae for the sums

i) $\sum_{K=1}^{n} K$ ii) $\sum_{K=1}^{n} K^2$ iii) $\sum_{K=1}^{n} K^3$

i) $\sum_{K=1}^{n} K$

we know that

$$(K-1)^2 = K^2 - 2K + 1$$

$$\rightarrow 2K-1 = K^2 - (K-1)^2$$

$$\text{or} \quad K^2 - (K-1)^2 = 2K-1$$

Taking summation on both sides

$$\sum_{K=1}^{n}\left[K^2-(K-1)^2\right]=\sum_{K=1}^{n}(2K-1)$$

$$\sum_{K=1}^{n}\left[K^2-(K-1)^2\right]=\sum_{K=1}^{n}2K-\sum_{K=1}^{n}1$$

$$\sum_{K=1}^{n}\left[K^2-(K-1)^2\right]=2\sum_{K=1}^{n}K-\sum_{K=1}^{n}1$$

$$\rightarrow 2\sum_{K=1}^{n}K=\sum_{K=1}^{n}\left[K^2-(K-1)^2\right]+\sum_{K=1}^{n}1$$

$$\because \sum_{K=1}^{n}\left[K^m-(K-1)^m\right]=n^m$$

and $\sum_{K=1}^{n}1=n$

$$\rightarrow 2\sum_{K=1}^{n}K=n^2+n$$

$$\rightarrow \boxed{\sum_{K=1}^{n}K=\frac{n(n+1)}{2}}$$

ii) $\sum_{K=1}^{n}K^2$

We know that

$$(K-1)^3=K^3-3K^2+3K-1$$

$$\rightarrow 3K^2-3K+1=K^3-(K-1)^3$$

or $K^3-(K-1)^3=3K^2-3K+1$

Taking summation on both sides

$$\sum_{K=1}^{n}\left[K^3-(K-1)^3\right]=\sum_{K=1}^{n}(3K^2-3K+1)$$

$$=3\sum_{K=1}^{n}K^2-3\sum_{K=1}^{n}K+\sum_{K=1}^{n}1$$

$$\rightarrow 3\sum_{K=1}^{n}K^2=\sum_{K=1}^{n}\left[K^3-(K-1)^3\right]+3\sum_{K=1}^{n}K-\sum_{K=1}^{n}1$$

$$\because \sum_{K=1}^{n}\left[K^m-(K-1)^m\right]=n^m$$

$$\rightarrow 3\sum_{K=1}^{n}K^2=n^3+\frac{3n(n+1)}{2}-n$$

$$=\frac{2n^3+3n(n+1)-2n}{2}$$

$$=\frac{n[2n^2+3n+3-2]}{2}$$

$$3\sum_{K=1}^{n}K^2=\frac{n[2n^2+3n+1]}{2}$$

$$\rightarrow \sum_{K=1}^{n}K^2=\frac{n[2n^2+2n+n+1]}{6}$$

$$=\frac{n(2n(n+1)+1(n+1))}{6}$$

$$\sum_{K=1}^{n}K^2=\frac{n(2n+1)(n+1)}{6}$$

$$\rightarrow \boxed{\sum_{K=1}^{n}K^2=\frac{n(n+1)(2n+1)}{6}}$$

iii) $\sum_{K=1}^{n}K^3$

we know that

$$(K-1)^4=K^4-4K^3+6K^2-4K+1$$

$$\because (a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4$$

$$\rightarrow 4K^3-6K^2+4K-1=K^4-(K-1)^4$$

or $K^4-(K-1)^4=4K^3-6K^2+4K-1$

Taking summation on both sides

$$\sum_{K=1}^{n}\left[K^4-(K-1)^4\right]=\sum_{K=1}^{n}(4K^3-6K^2+4K-1)$$

$$=4\sum_{K=1}^{n}K^3-6\sum_{K=1}^{n}K^2+4\sum_{K=1}^{n}K-\sum_{K=1}^{n}1$$

$$\rightarrow 4\sum_{K=1}^{n}K^3=\sum_{K=1}^{n}\left[K^4-(K-1)^4\right]+6\sum_{K=1}^{n}K^2+4\sum_{K=1}^{n}K-\sum_{K=1}^{n}1$$

$$=n^4+\frac{6n(n+1)(2n+1)}{6}-\frac{4n(n+1)}{2}+n$$

$$4\sum_{K=1}^{n}K^3=n^4+n(n+1)(2n+1)-2n(n+1)+n$$

$$\sum_{K=1}^{n}K^3=\frac{n^4+n(n+1)(2n+1)-2n(n+1)+n}{4}$$

$$=\frac{n}{4}\left[n^3+2n^2+n+2n+1-2n-2+1\right]$$

$$=\frac{n}{4}\left[n^3+2n^2+n\right]$$

$$\sum_{K=1}^{n}K^3=\frac{n^2}{4}(n^2+2n+1)=\frac{n^2}{4}(n+1)^2$$

$$\rightarrow \boxed{\sum_{K=1}^{n}K^3=\left[\frac{n(n+1)}{2}\right]^2}$$

Example 1. Find the sum of the series $1^3+3^3+5^3+\ldots\ldots$ to n terms

Solution:-

$T_K = (1+(K-1)2)^3$ $\quad\because a_n = a_1+(n-1)d$, $a_1=1$, $d=3-1=2$

$\rightarrow T_K = (1+2K-2)^3$

$= (2K-1)^3$

$= (2K)^3-(1)^3-3(2K)^2(1)+3(2K)(1)$

$= 8K^3-1-12K^2+6K$

$T_K = 8K^3-12K^2+6K-1$

Now

$S_n = \sum_{K=1}^{n} T_K = \sum_{K=1}^{n}(8K^3-12K^2+6K-1)$

$= 8\sum_{K=1}^{n}K^3-12\sum_{K=1}^{n}K^2+6\sum_{K=1}^{n}K-\sum_{K=1}^{n}1$

$= 8\left[\frac{n(n+1)}{2}\right]^2-12\left[\frac{n(n+1)(2n+1)}{6}\right]$

$+6\left[\frac{n(n+1)}{2}\right]-n$

$= \frac{8n^2(n+1)^2}{4}-2n(n+1)(2n+1)+3n(n+1)-n$

$= 2n^2(n^2+1+2n)-2n(2n^2+n+2n+1)+3n^2+3n-n$

$= 2n^2(n^2+1+2n)-2n(2n^2+3n+1)+3n^2+2n$

$= 2n[n^3+n+2n^2-2n^2-3n-1]+n(3n+2)$

$= 2n[n^3-2n-1]+n(3n+2)$

$= n[2n^3-4n-1+3n+2]$

$= n[2n^3-n] = n^2(2n^2-1)$

Example 2. Find the sum of n terms of series whose nth terms is $n^3+\frac{3}{2}n^2+\frac{1}{2}n+1$

Solution:- Given that

$T_n = n^3+\frac{3}{2}n^2+\frac{1}{2}n+1$

$\rightarrow T_K = K^3+\frac{3}{2}K^2+\frac{1}{2}K+1$

and $S_n = \sum_{K=1}^{n} T_K$

$= \sum_{K=1}^{n}(K^3+\frac{3}{2}K^2+\frac{1}{2}K+1)$

$S_n = \sum_{K=1}^{n}K^3+\frac{3}{2}\sum_{K=1}^{n}K^2+\frac{1}{2}\sum_{K=1}^{n}K+\sum_{K=1}^{n}1$

$= \frac{n^2(n+1)^2}{4}+\frac{3}{2}\frac{n(n+1)(2n+1)}{6}+\frac{1}{2}\frac{n(n+1)}{2}+n$

$= \frac{n^2(n+1)^2}{4}+\frac{n(n+1)(2n+1)}{4}+\frac{n(n+1)}{4}+n$

$= \frac{n}{4}[n(n+1)^2+2n^2+n+2n+1)+n+1+4]$

$= \frac{n}{4}[n(n^2+1+2n)+2n^2+4n+5]$

$= \frac{n}{4}[n^3+n+2n^2+2n^2+4n+5]$

$= \frac{n}{4}[n^3+4n^2+5n+5]$

Exercise 6.11

Sum the following series upto n terms.

Q1. $1\times1+2\times4+3\times7+\ldots\ldots$

Solution:-

$T_K = (\text{Kth term of } 1,2,3,\ldots)(\text{Kth term of } 1,4,7,\ldots)$

$T_K = [1+(K-1)(1)]\times[1+(K-1)(3)]$

$= [1+K-1]\times[1+3K-3]$

$T_K = K(3K-2) = 3K^2-2K$

$\rightarrow S_n = \sum_{K=1}^{n}T_K = \sum_{K=1}^{n}(3K^2-2K)$

$= 3\sum_{K=1}^{n}K^2-2\sum_{K=1}^{n}K$

$= 3\frac{n(n+1)(2n+1)}{6}-\frac{2n(n+1)}{2}$

$= \frac{n(n+1)(2n+1)}{2}-\frac{2n(n+1)}{2}$

$= \frac{n(n+1)}{2}[2n+1-2]$

$S_n = \frac{n(n+1)}{2}(2n-1) = \frac{n(n+1)(2n-1)}{2}$

Q2. $1\times3+3\times6+5\times9+\ldots$

Solution:-

$T_K = (\text{Kth term of } 1,3,5,\ldots)(\text{Kth term of } 3,6,9,\ldots)$

$T_K = (1+(K-1)(2))(3+(K-1)(3))$

$$T_K = (1+2K-2)(3+3K-3)$$
$$= (2K-1)(3K)$$
$$T_K = 6K^2 - 3K$$
$$\text{Now } S_n = \sum_{K=1}^{n} T_K = \sum_{K=1}^{n} (6K^2 - 3K)$$
$$= 6\sum_{K=1}^{n} K^2 - 3\sum_{K=1}^{n} K$$
$$= 6\frac{n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2}$$
$$= n(n+1)(2n+1) - \frac{3}{2}n(n+1)$$
$$= n(n+1)\left[2n+1 - \frac{3}{2}\right]$$
$$= n(n+1)\left[\frac{4n+2-3}{2}\right]$$
$$S_n = \frac{n(n+1)(4n-1)}{2}$$

Q3. $1\times4+2\times7+3\times10+\cdots\cdots$

Solution:-

$$T_K = \left(\begin{matrix}\text{Kth term of}\\ 1, 2, 3, \ldots\end{matrix}\right)\left(\begin{matrix}\text{Kth term of}\\ 4, 7, 10, \ldots\end{matrix}\right)$$
$$= (1+(K-1)(1))(4+(K-1)3)$$
$$= (1+K-1)(4+3K-3)$$
$$= K(3K+1)$$
$$T_K = 3K^2+K$$
$$\rightarrow S_n = \sum_{K=1}^{n} T_K = \sum_{K=1}^{n}(3K^2+K)$$
$$= 3\sum_{K=1}^{n} K^2 + \sum_{K=1}^{n} K$$
$$= \frac{3n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)(2n+1)}{2} + \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{2}\left[2n+1+1\right]$$
$$= \frac{n(n+1)}{2}(2n+2) = \frac{2n(n+1)(n+1)}{2}$$
$$S_n = n(n+1)^2$$

Q4. $3\times5+5\times9+7\times13+\cdots$

Solution:-

$$T_K = \left(\begin{matrix}\text{Kth term of}\\ 3, 5, 7, \ldots\end{matrix}\right)\left(\begin{matrix}\text{Kth term of}\\ 5, 9, 13, \ldots\end{matrix}\right)$$
$$= (3+(K-1)(2))(5+(K-1)4)$$
$$= (3+2K-2)(5+4K-4)$$
$$= (2K+1)(4K+1)$$
$$= (8K^2+2K+4K+1)$$
$$T_K = 8K^2+6K+1$$
$$\rightarrow S_n = \sum_{K=1}^{n} T_K = \sum_{K=1}^{n}(8K^2+6K+1)$$
$$= 8\sum_{K=1}^{n} K^2 + 6\sum_{K=1}^{n} K + \sum_{K=1}^{n} 1$$
$$= 8\frac{n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2} + n$$
$$= \frac{4n(n+1)(2n+1)}{3} + 3n(n+1) + n$$
$$= \frac{n}{3}\left[4(2n^2+n+2n+1)+9n+9+3\right]$$
$$= \frac{n}{3}(8n^2+12n+4+9n+9+3)$$
$$= \frac{n}{3}(8n^2+21n+16)$$

Q5. $1^2+3^2+5^2+\cdots\cdots$

Solution:-

$$T_K = \left(\begin{matrix}\text{Kth term of}\\ 1, 3, 5, \ldots\end{matrix}\right)^2$$
$$T_K = (1+(K-1)(2))^2$$
$$= (1+2K-2)^2 = (2K-1)^2$$
$$T_K = 4K^2+1-4K$$
$$T_K = 4K^2-4K+1$$
$$\rightarrow S_n = \sum_{K=1}^{n} T_K = \sum_{K=1}^{n}(4K^2-4K+1)$$
$$= 4\sum_{K=1}^{n} K^2 - 4\sum_{K=1}^{n} K + \sum_{K=1}^{n} 1$$
$$= 4.\frac{n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$$
$$= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n$$
$$= n\left[\frac{2(n^2+n+2n+1)}{3} - 2n - 2 + 1\right]$$

$$= n\left[\frac{4n^2+\cancel{6n}+2-\cancel{6n}-6+3}{3}\right]$$

$$= \frac{n}{3}(4n^2-1)$$

Q6. $2^2+5^2+8^2+\cdots\cdots$

Solution:-

$$T_K = \left(\begin{matrix}\text{Kth term of}\\ 2,5,8,\ldots\end{matrix}\right)^2$$

$$= \left(2+(K-1)(3)\right)^2$$

$$= (2+3K-3)^2 = (3K-1)^2$$

$$T_K = 9K^2-6K+1$$

$$\rightarrow S_n = \sum_{K=1}^{n}(9K^2-6K+1)$$

$$= 9\sum_{K=1}^{n}K^2 - 6\sum_{K=1}^{n}K + \sum_{K=1}^{n}1$$

$$= 9\cdot\frac{n(n+1)(2n+1)}{6} - \frac{6n(n+1)}{2} + n$$

$$= \frac{3n(n+1)(2n+1)}{2} - 3n(n+1)+n$$

$$= \frac{n}{2}\left[3(2n^2+n+2n+1)-6(n+1)+2\right]$$

$$= \frac{n}{2}(6n^2+9n+3-6n-6+2)$$

$$S_n = \frac{n}{2}(6n^2+3n-1)$$

Q7. $2\times1^2+4\times2^2+6\times3^2+\cdots\cdots$

Solution:-

$$T_K = \left(\begin{matrix}\text{Kth term}\\ \text{of } 2,4,6,\ldots\end{matrix}\right)\left(\begin{matrix}\text{Kth term of}\\ 1,2,3,\ldots\end{matrix}\right)^2$$

$$= \left(2+(K-1)(2)\right)\left(1+(K-1)(1)\right)^2$$

$$= (2+2K-2)(1+K-1)^2$$

$$T_K = (2K)(K)^2 = 2K^3$$

$$\rightarrow S_n = \sum_{K=1}^{n}(2K^3) = 2\sum_{K=1}^{n}K^3$$

$$= 2\cdot\left[\frac{n(n+1)}{2}\right]^2 = 2\frac{n^2(n+1)^2}{4}$$

$$S_n = \frac{n^2(n+1)^2}{2}$$

Q8. $3\times2^2+5\times3^2+7\times4^2+\cdots\cdots$

Solution:-

$$T_K = \left(\begin{matrix}\text{Kth term}\\ \text{of } 3,5,7,\ldots\end{matrix}\right)\left(\begin{matrix}\text{Kth term of}\\ 2,3,4,\ldots\end{matrix}\right)^2$$

$$T_K = \left(3+(K-1)2\right)\left(2+(K-1)(1)\right)^2$$

$$= (3+2K-2)(2+K-1)^2$$

$$= (2K+1)(K+1)^2$$

$$= (2K+1)(K^2+1+2K)$$

$$= 2K^3+2K+4K^2+K^2+1+2K$$

$$T_K = 2K^3+5K^2+4K+1$$

$$\rightarrow S_n = \sum_{K=1}^{n}T_K = \sum_{K=1}^{n}(2K^3+5K^2+4K+1)$$

$$= 2\sum_{K=1}^{n}K^3+5\sum_{K=1}^{n}K^2+4\sum_{K=1}^{n}K+\sum_{K=1}^{n}1$$

$$= 2\,\frac{n^2(n+1)^2}{4}+\frac{5n(n+1)(2n+1)}{6}+\frac{4n(n+1)}{2}+n$$

$$= \frac{n^2(n+1)^2}{2}+\frac{5n(n+1)(2n+1)}{6}+2n(n+1)+n$$

$$= n\left[\frac{n(n^2+1+2n)}{2}+\frac{5(2n^2+n+2n+1)}{6}+2n+2+1\right]$$

$$= n\left[\frac{3n(n^2+1+2n)+5(2n^2+3n+1)+12n+12+6}{6}\right]$$

$$= \frac{n}{6}\left[3n^3+3n+6n^2+10n^2+15n+5+12n+18\right]$$

$$= \frac{n}{6}(3n^3+16n^2+30n+23)$$

Q9. $2\times4\times7+3\times6\times10+4\times8\times13+\cdots\cdots$

Solution:-

$$T_K = \left(\begin{matrix}\text{Kth term of}\\ 2,3,4,\ldots\end{matrix}\right)\left(\begin{matrix}\text{Kth term}\\ \text{of } 4,6,8,\ldots\end{matrix}\right)\left(\begin{matrix}\text{Kth term of}\\ 7,10,13,\ldots\end{matrix}\right)$$

$$= \left(2+(K-1)(1)\right)\left(4+(K-1)(2)\right)\left(7+(K-1)3\right)$$

$$= (2+K-1)(4+2K-2)(7+3K-3)$$

$$= (K+1)(2K+2)(3K+4)$$

$$= (2K^2+2K+2K+2)(3K+4)$$

$$= (2K^2+4K+2)(3K+4)$$

$$= 6K^3+8K^2+12K^2+16K+6K+8$$

$$= 6K^3+20K^2+22K+8$$

$$T_K = 2(3K^3+10K^2+11K+4)$$

$$\rightarrow S_n = \sum_{K=1}^{n}T_K = 2\cdot\sum_{K=1}^{n}(3K^3+10K^2+11K+4)$$

$$= 2\left[3\sum_{K=1}^{n}K^3+10\sum_{K=1}^{n}K^2+11\sum_{K=1}^{n}K+\sum_{K=1}^{n}4\right]$$

$$= 2\left[3\cdot\frac{n^2(n+1)^2}{4}+\frac{10n(n+1)(2n+1)}{6}+\frac{11n(n+1)}{2}+4n\right]$$

$$= 2\left[3\frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{3} + \frac{11n(n+1)}{2} + 4n\right]$$

$$= 2n\left[\frac{9n(n^2+1+2n)+20(2n^2+n+2n+1)+66n+66+48}{12}\right]$$

$$= \frac{n}{6}\left[9n^3+9n+18n^2+40n^2+60n+20+66n+66+48\right]$$

$$= \frac{n}{6}(9n^3+58n^2+135n+134)$$

Q10. $1\times4\times6+4\times7\times10+7\times10\times14+\dots$

Solution:-

$$T_K = \begin{pmatrix}\text{Kth term of}\\ 1,4,7,\dots\end{pmatrix}\begin{pmatrix}\text{Kth term}\\ \text{of } 4,7,10,\dots\end{pmatrix}\begin{pmatrix}\text{Kth term of}\\ 6,10,14,\dots\end{pmatrix}$$

$$= (1+(K-1)3)(4+(K-1)3)(6+(K-1)4)$$

$$= (1+3K-3)(4+3K-3)(6+4K-4)$$

$$= (3K-2)(3K+1)(4K+2)$$

$$= (9K^2+3K-6K-2)\cdot2(2K+1)$$

$$= 2(9K^2-3K-2)(2K+1)$$

$$= 2(18K^3+9K^2-6K^2-3K-4K-2)$$

$$T_n = 2(18K^3+3K^2-7K-2)$$

$$\rightarrow S_n = \sum_{K=1}^{n} T_n = 2\sum_{K=1}^{n}(18K^3+3K^2-7K-2)$$

$$= 2\left[18\sum_{K=1}^{n}K^3+3\sum_{K=1}^{n}K^2-7\sum_{K=1}^{n}K-\sum_{K=1}^{n}2\right]$$

$$= 2\left[18\cdot\frac{n^2(n+1)^2}{4}+\frac{3n(n+1)(2n+1)}{6}-\frac{7n(n+1)}{2}-2n\right]$$

$$= 2n\left[\frac{9n(n^2+1+2n)}{2}+\frac{(n+1)(2n+1)}{2}-\frac{7(n+1)}{2}-2\right]$$

$$= 2n\left[\frac{9n^3+9n+18n^2+2n^2+n+2n+1-7n-7-4}{2}\right]$$

$$S_n = n(9n^3+20n^2+5n-10)$$

Q11. $1+(1+2)+(1+2+3)+\dots$

Solution:-

$$T_n = 1+2+3+\dots+n$$

$$T_n = \frac{n(n+1)}{2} \qquad \because 1+2+3+\dots+n=\frac{n(n+1)}{2}$$

$$\rightarrow T_K = \frac{K(K+1)}{2}$$

$$\rightarrow S_n = \sum_{K=1}^{n}\frac{K(K+1)}{2}$$

$$= \frac{1}{2}\sum_{K=1}^{n}(K^2+K)$$

$$= \frac{1}{2}\left[\sum_{K=1}^{n}K^2+\sum_{K=1}^{n}K\right]$$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right]$$

$$= \frac{1}{2}\,\frac{n(n+1)}{2}\left[\frac{2n+1}{3}+1\right]$$

$$= \frac{n(n+1)}{4}\left[\frac{2n+1+3}{3}\right]$$

$$= \frac{n(n+1)}{4}\left(\frac{2n+4}{3}\right) = \frac{n(n+1)(2n+4)}{12}$$

$$S_n = \frac{n(n+1)(n+2)}{6}$$

Q12. $1^2+(1^2+2^2)+(1^2+2^2+3^3)+\dots$

Solution:-

$$T_n = 1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$T_K = \frac{K(K+1)(2K+1)}{6}$$

$$= \frac{(K^2+K)(2K+1)}{6} = \frac{2K^3+K^2+2K^2+K}{6}$$

$$T_K = \frac{2K^3+3K^2+K}{6}$$

$$\rightarrow S_n = \frac{1}{6}\sum_{K=1}^{n}(2K^3+3K^2+K)$$

$$= \frac{1}{6}\left[2\sum_{K=1}^{n}K^3+3\sum_{K=1}^{n}K^2+\sum_{K=1}^{n}K\right]$$

$$= \frac{1}{6}\left[2\left(\frac{n(n+1)}{2}\right)^2+\frac{3n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right]$$

$$= \frac{1}{6}\left[\frac{\cancel{2}n^2(n+1)^2}{\cancel{4}2}+\frac{n(n+1)(2n+1)}{2}+\frac{n(n+1)}{2}\right]$$

$$= \frac{1}{6}\cdot\frac{n(n+1)}{2}\left[n(n+1)+2n+1+1\right]$$

$$= \frac{n(n+1)}{12}\left[n^2+n+2n+2\right]$$

$$= \frac{n(n+1)}{12}(n^2+3n+2)$$

Q13. $2+(2+5)+(2+5+8)+\dots$

Solution:-

$$T_n = 2+5+8+\dots n \text{ terms}$$

$$a_1 = 2,\ d = 3,\ n = n$$

$$\left(\because S_n = \frac{n}{2}\left(2a_1+(n-1)d\right)\right)$$

so $T_n = \frac{n}{2}\left[2(2)+(n-1)3\right]$

$= \frac{n}{2}\left[4+3n-3\right]$

$T_n = \frac{n}{2}(3n+1)$

$\rightarrow T_n = \frac{3n^2+n}{2}$ $\rightarrow T_K = \frac{1}{2}(3K^2+K)$

so $S_n = \sum_{K=1}^{n} T_K = \sum_{K=1}^{n}\left[\frac{1}{2}(3K^2+K)\right]$

$S_n = \frac{1}{2}\left[3\sum_{K=1}^{n} K^2 + \sum_{K=1}^{n} K\right]$

$= \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$

$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{2} + \frac{n(n+1)}{2}\right]$

$= \frac{1}{2}\cdot\frac{n(n+1)}{2}\left[2n+1+1\right]$

$= \frac{n(n+1)}{4}(2n+2) = \frac{n(n+1)(n+1)}{2}$

$S_n = \frac{n(n+1)^2}{2}$

Q14. Sum the series.

i) $1^2-2^2+3^2-4^2+\dots+(2n-1)^2-(2n)^2$

Solution:-

$\because T_n = (2n-1)^2-(2n)^2$

$= 4n^2+1-4n-4n^2$

$T_n = -4n+1$

$\rightarrow T_K = -4K+1$

Now $S_n = \sum_{K=1}^{n} T_K = \sum_{K=1}^{n}(-4K+1)$

$= -4\sum_{K=1}^{n} K + \sum_{K=1}^{n} 1$

$= -4\frac{n(n+1)}{2} + n$

$= -2n(n+1)+n = -2n^2-2n+n$

$S_n = -2n^2-n = -n(2n+1)$

ii) $1^2-3^2+5^2-7^2+\dots+(4n-3)^2-(4n-1)^2$

Solution:-

$T_n = (4n-3)^2-(4n-1)^2$

$= 16n^2+9-24n-(16n^2+1-8n)$

$T_n = 16n^2+9-24n-16n^2-1+8n$

$\rightarrow T_n = -16n+1$

or $T_K = -16K+1$

$S_n = \sum_{K=1}^{n} T_K = \sum_{K=1}^{n}(-16K+8)$

$= -16\sum_{K=1}^{n} K + \sum_{K=1}^{n} 8$

$= -16\frac{n(n+1)}{2} + 8n$

$= -8n(n+1)+8n = -8n^2-8n+8n$

$S_n = -8n^2$

iii) $\frac{1^2}{1} + \frac{1^2+2^2}{2} + \frac{1^2+2^2+3^2}{3} + \dots$ to n terms

Solution:-

$T_n = \frac{1^2+2^2+3^2+\dots+n^2}{n}$

$= \frac{\frac{n(n+1)(2n+1)}{6}}{n}$

$T_n = \frac{n(n+1)(2n+1)}{6n}$

$= \frac{(n+1)(2n+1)}{6} = \frac{2n^2+n+2n+1}{6}$

$T_n = \frac{2n^2+3n+1}{6}$

$\rightarrow T_K = \frac{1}{6}\left[2K^2+3K+1\right]$

$S_n = \sum_{K=1}^{n} T_K = \frac{1}{6}\left[2\sum_{K=1}^{n} K^2 + 3\sum_{K=1}^{n} K + \sum_{K=1}^{n} 1\right]$

$= \frac{1}{6}\left[2\frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + n\right]$

$= \frac{n}{6}\left[\frac{2(n+1)(2n+1)+9n+9+6}{6}\right]$

$= \frac{n}{36}\left(2(2n^2+n+2n+1)+9n+15\right)$

$= \frac{n}{36}\left(4n^2+6n+2+9n+15\right)$

$= \frac{n}{36}\left(4n^2+15n+17\right)$

Q15. Find the sum to n terms of the series whose nth terms are given

i) $3n^2+n+1$

ii) n^2+4n+1

Solution:- i) $3n^2+n+1$

$T_n = 3n^2+n+1$

$\rightarrow T_K = 3K^2+K+1$

$$S_n = \sum_{K=1}^{n} T_K = \sum_{K=1}^{n} (3K^2+K+1)$$

$$= 3\sum_{K=1}^{n} K^2 + \sum_{K=1}^{n} K + \sum_{K=1}^{n} 1$$

$$= 3\,\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n$$

$$= n\left[\frac{(n+1)(2n+1)}{2} + \frac{n+1}{2} + 1\right]$$

$$= \frac{n}{2}\left[2n^2+n+2n+1+n+1+2\right]$$

$$= \frac{n}{2}(2n^2+4n+4) = \frac{2n}{2}(n^2+n+2)$$

$$S_n = n(n^2+n+2)$$

ii) n^2+4n+1

$$T_n = n^2+4n+1$$

$$\rightarrow T_K = K^2+4K+1$$

$$S_n = \sum_{K=1}^{n} T_K = \sum_{K=1}^{n}(K^2+4K+1)$$

$$= \sum_{K=1}^{n} K^2 + 4\sum_{K=1}^{n} K + \sum_{K=1}^{n} 1$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} + n$$

$$= n\left[\frac{2n^2+n+2n+1}{6} + 2n+2+1\right]$$

$$= \frac{n}{6}\left[2n^2+3n+1+12n+12+6\right]$$

$$= \frac{n}{6}(2n^2+15n+19)$$

Q16. Given nth terms of the series, find the sum to 2n terms. i) $3n^2+2n+1$

ii) n^3+2n+3

Solution:- i) $3n^2+2n+1$

$\because T_n = 3n^2+2n+1$, $S_{2n} = ?$

$$T_K = 3K^2+2K+1$$

$$\because S_n = \sum_{K=1}^{n} T_K = \sum_{K=1}^{n}(3K^2+2K+1)$$

$$S_n = 3\sum_{K=1}^{n} K^2 + 2\sum_{K=1}^{n} K + \sum_{K=1}^{n} 1$$

$$= 3\,\frac{n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} + n$$

$$= \frac{n(n+1)(2n+1)}{2} + n(n+1) + n$$

$$= n\left[\frac{2n^2+n+2n+1}{2} + n+1+1\right]$$

$$= \frac{n}{2}(2n^2+3n+1+2n+2+2)$$

$$= \frac{n}{2}(2n^2+5n+5)$$

$$\rightarrow S_n = \frac{n}{2}(2n^2+5n+5)$$

so $S_{2n} = \frac{2n}{2}\left[2(2n)^2+5(2n)+5\right]$

$$\rightarrow S_{2n} = n\left[8n^2+10n+5\right]$$

ii) n^3+2n+3

$T_n = n^3+2n+3$, $S_{2n} = ?$

$$T_K = K^3+2K+3$$

$$\because S_n = \sum_{K=1}^{n} T_K = \sum_{K=1}^{n}(K^3+2K+3)$$

$$= \sum_{K=1}^{n} K^3 + 2\sum_{K=1}^{n} K + \sum_{K=1}^{n} 3$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + \frac{2n(n+1)}{2} + 3n$$

$$= \frac{n^2(n+1)^2}{4} + n(n+1) + 3n$$

$$= \frac{n}{4}\left[n(n^2+1+2n) + 4(n+1) + 12\right]$$

$$= \frac{n}{4}\left[n^3+n+2n^2+4n+4+12\right]$$

$$= \frac{n}{4}\left[n^3+2n^2+5n+16\right]$$

$$\rightarrow S_n = \frac{n}{4}\left[n^3+2n^2+5n+16\right]$$

Now

$$S_{2n} = \frac{2n}{4}\left[(2n)^3+2(2n)^2+5(2n)+16\right]$$

$$S_{2n} = \frac{n}{2}(8n^3+8n^2+10n+16)$$

$$= \frac{2n}{2}(4n^3+4n^2+5n+8)$$

$$S_{2n} = n(4n^3+4n^2+5n+8)$$