MATHEMATICS 1st YEAR

UNIT # 13

INVERSE TRIGONOMETRIC FUNCTIONS

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Contents

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اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Inverse Trigonometric Functions

The functions $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, $\operatorname{cosec}^{-1}$, $\sec^{-1}$ and $\cot^{-1}$ are called inverse trigonometric functions.

* Inverse trigonometric functions are not single valued functions.

Principal Valued function

Since inverse trigonometric functions are not sigle valued functions, but they can be made sigle valued ∴ if we restrict their domains.

These functions are called principal valued functions denoted by Sin^{-1} Cos^{-1}, Tan^{-1}...., etc

Remember that, Sin^{-1}, Cos^{-1}, Tan^{-1},... are single valued function while $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$... are not single valued function.

The inverse sine function:-

The inverse sine function is denoted by Sin^{-1} and defined as;

$$y = \operatorname{Sin}^{-1}x \quad \text{iff} \quad x = \sin y$$

where $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ or $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

and $x \in [-1, 1]$ or $-1 \le x \le 1$

Example 1. Find the value of

(i) $\sin^{-1}\frac{\sqrt{3}}{2}$ (ii) $\sin^{-1}(-\frac{1}{2})$

Solution:- (i) $\sin^{-1}\frac{\sqrt{3}}{2}$

Let $\alpha = \sin^{-1}\frac{\sqrt{3}}{2} \longrightarrow$ (I)

$\sin\alpha = \frac{\sqrt{3}}{2} \rightarrow \alpha = \frac{\pi}{3}$ where $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

so (I) $\rightarrow$ $\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$

(ii) $\sin^{-1}(-\frac{1}{2})$

Let $\alpha = \sin^{-1}(-\frac{1}{2}) \longrightarrow$ (I)

$\sin\alpha = -\frac{1}{2} \rightarrow \alpha = -\frac{\pi}{6}$ where $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

so (I) $\rightarrow$ $\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$

The inverse cosine function:-

The inverse cosine function is denoted by Cos^{-1} and defined as;

$$y = \operatorname{Cos}^{-1}x \quad \text{iff} \quad x = \cos y$$

where $y \in [0, \pi]$ or $0 \le y \le \pi$

and $x \in [-1, 1]$ or $-1 \le x \le 1$

Example 2. Find the value of

(i) $\cos^{-1}1$ (ii) $\cos^{-1}(-\frac{1}{2})$

Solution:- (i) $\cos^{-1}1$

Let $\alpha = \cos^{-1}1 \longrightarrow$ (I) where $\alpha \in [0, \pi]$

$\cos\alpha = 1 \rightarrow \alpha = 0$

so (I) $\rightarrow$ $\cos^{-1}1 = 0$

(ii) $\cos^{-1}(-\frac{1}{2})$

Let $\alpha = \cos^{-1}(-\frac{1}{2}) \longrightarrow$ (I) where $\alpha \in [0, \pi]$

$\cos\alpha = -\frac{1}{2} \rightarrow \alpha = \frac{2\pi}{3}$

so (I) $\rightarrow$ $\cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$

The inverse tangent function:-

The inverse tangent function is denoted by Tan^{-1} and is defined as;

$$y = \operatorname{Tan}^{-1}x \quad \text{iff} \quad x = \tan y$$

where $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$ or $-\frac{\pi}{2} < y < \frac{\pi}{2}$

and $x \in (-\infty, \infty)$ or $-\infty < x < \infty$ or $x \in R$

Example 3. Find the value of

(i) $\tan^{-1}1$ (ii) $\tan^{-1}(-\sqrt{3})$

Solution:- (i) $\tan^{-1}(1)$

Let $\alpha = \tan^{-1}1 \longrightarrow$ (I)

$\tan\alpha = 1 \rightarrow \alpha = \frac{\pi}{4}$ where $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$

so (I) $\rightarrow$ $\tan^{-1}1 = \frac{\pi}{4}$

(ii) $\tan^{-1}(-\sqrt{3})$

Let $\alpha = \tan^{-1}(-\sqrt{3}) \longrightarrow$ (I) where $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\tan\alpha = -\sqrt{3} \rightarrow \alpha = -\frac{\pi}{3}$

so (I) $\rightarrow \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

Note:- It must be remember that $\sin^{-1}x \neq (\sin x)^{-1}$, $\cos^{-1}x \neq (\cos x)^{-1}$, $\tan^{-1}x \neq (\tan x)^{-1}$

The inverse cosecant function:-

The inverse cosecant function is denoted by $\operatorname{cosec}^{-1}$ and defined as $y = \operatorname{cosec}^{-1}x$ iff $x = \operatorname{cosec} y$ where $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ but $y \neq 0$ and $x \in [-1, 1]$ or $-1 \leq x \leq 1$

The inverse Secant function:-

The inverse secant function is denoted by $\sec^{-1}$ and defined as $y = \sec^{-1}x$ iff $x = \sec y$ where $y \in [0, \pi]$ but $y \neq \frac{\pi}{2}$ and $x \in [-1, 1]$ or $-1 \leq x \leq 1$

The inverse Cotangent function:-

The inverse cotangent function is denoted by $\cot^{-1}$ and defined as $y = \cot^{-1}x$ iff $x = \cot y$ where $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ but $y \neq 0$ and $x \in (-\infty, \infty)$ or $x \in R$

Domains and Ranges of Principal Trigonometric function and Inverse Trigonometric functions

$y = \sin x$

Domain; $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Range; $-1 \leq x \leq 1$

$y = \sin^{-1}x$

Domain; $-1 \leq x \leq 1$

Range; $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$y = \cos x$

Domain; $0 \leq x \leq \pi$ Range; $-1 \leq x \leq 1$

$y = \cos^{-1}x$

Domain; $-1 \leq x \leq 1$ Range; $0 \leq x \leq \pi$

$y = \tan x$

Domain; $-\frac{\pi}{2} < x < \frac{\pi}{2}$ Range; $(-\infty, \infty)$ or R

$y = \tan^{-1}x$

Domain; $(-\infty, \infty)$ or R Range; $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$y = \cot x$

Domain; $0 < x < \pi$ Range; $(-\infty, \infty)$ or R

$y = \cot^{-1}x$

Domain; $(-\infty, \infty)$ or R Range; $0 < x < \pi$

$y = \sec x$

Domain; $[0, \pi]$, $x \neq \frac{\pi}{2}$ Range; $y \leq -1$ or $y \geq 1$

$y = \sec^{-1}x$

Domain; $x \geq -1$ or $x \leq 1$ Range; $[0, \pi]$, $y \neq \frac{\pi}{2}$

$y = \csc x$

Domain; $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $x \neq 0$ Range; $y \leq -1$ or $y \geq 1$

$y = \csc^{-1}x$

Domain; $x \leq -1$ or $x \geq 1$ Range; $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $y \neq 0$

Example 4. Show that

$$\cos^{-1}\frac{12}{13} = \sin^{-1}\frac{5}{13}$$

Solution:-

Let $\alpha = \cos^{-1}\frac{12}{13} \longrightarrow$ (I)

$\rightarrow \cos\alpha = \frac{12}{13}$

$\because \sin\alpha = \sqrt{1-\cos^2\alpha}$

$= \sqrt{1 - (\frac{12}{13})^2}$

$= \sqrt{1 - \frac{144}{169}} = \sqrt{\frac{169-144}{169}}$

$\sin\alpha = \sqrt{\frac{25}{169}} \rightarrow \sin\alpha = \frac{5}{13}$

$\alpha = \sin^{-1}\frac{5}{13} \longrightarrow$ (II)

By (I) and (II)

$\cos^{-1}\frac{12}{13} = \sin^{-1}\frac{5}{13}$

Hence proved

Example 5. Find the value of

i) $\sin(\cos^{-1}\frac{\sqrt{3}}{2})$ ii) $\cos(\tan^{-1}0)$

iii) $\sec[\sin^{-1}(-\frac{1}{2})]$

Solution:- i) $\sin(\cos^{-1}\frac{\sqrt{3}}{2})$

Let $\alpha = \cos^{-1}\frac{\sqrt{3}}{2} \longrightarrow (I)$ where $\alpha \in [0, \pi]$

$\rightarrow \cos\alpha = \frac{\sqrt{3}}{2} \rightarrow \alpha = \frac{\pi}{6}$

so (I) $\cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

Now $\sin(\cos^{-1}\frac{\sqrt{3}}{2}) = \sin\frac{\pi}{6} = \frac{1}{2}$

ii) $\cos(\tan^{-1}0)$

Let $\alpha = \tan^{-1}0 \longrightarrow (I)$

$\rightarrow \tan\alpha = 0 \rightarrow \alpha = 0$ where $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$

so (I) $\rightarrow \tan^{-1}0 = 0$

Now $\cos(\tan^{-1}0) = \cos(0) = 1$

iii) $\sec[\sin^{-1}(-\frac{1}{2})]$

Let $\alpha = \sin^{-1}(-\frac{1}{2}) \longrightarrow (I)$

$\sin\alpha = -\frac{1}{2} \rightarrow \alpha = -\frac{\pi}{6}$ where $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

so (I) $\rightarrow \sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$

Now $\sec[\sin^{-1}(-\frac{1}{2})] = \sec(-\frac{\pi}{6})$

$= \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$

Example 6. Prove that the inverse trigonometric functions satisfy the following identities:

i) $\sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x$ and

$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$

Solution:- $\sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x$

Let $\alpha = \frac{\pi}{2} - \cos^{-1}x$

$\rightarrow \cos^{-1}x = \frac{\pi}{2} - \alpha \longrightarrow (i)$

$\rightarrow x = \cos(\frac{\pi}{2} - \alpha)$

$= \cos\frac{\pi}{2}\cos\alpha + \sin\frac{\pi}{2}\sin\alpha$

$= (0)\cos\alpha + (1)\sin\alpha$

$\rightarrow x = \sin\alpha \rightarrow \alpha = \sin^{-1}x \longrightarrow (ii)$

from (i) and (ii)

$\sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x$

Also,

$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$

Solution:-

Let $\alpha = \frac{\pi}{2} - \sin^{-1}x \longrightarrow (i)$

$\rightarrow \sin^{-1}x = \frac{\pi}{2} - \alpha$

$\rightarrow x = \sin(\frac{\pi}{2} - \alpha)$

$= \sin\frac{\pi}{2}\cos\alpha - \cos\frac{\pi}{2}\sin\alpha$

$x = (1)\cos\alpha - (0)\sin\alpha$

$x = \cos\alpha \rightarrow \alpha = \cos^{-1}x \longrightarrow (ii)$

from (i) and (ii)

$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$

ii) $\tan^{-1}x = \frac{\pi}{2} - \cot^{-1}x$ and

$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$

Solution:- $\tan^{-1}x = \frac{\pi}{2} - \cot^{-1}x$

Let $\alpha = \frac{\pi}{2} - \cot^{-1}x \longrightarrow (i)$

$\rightarrow \cot^{-1}x = \frac{\pi}{2} - \alpha$

$\rightarrow x = \cot(\frac{\pi}{2} - \alpha)$

$= \frac{\cos(\frac{\pi}{2} - \alpha)}{\sin(\frac{\pi}{2} - \alpha)} = \frac{\cos\frac{\pi}{2}\cos\alpha + \sin\frac{\pi}{2}\sin\alpha}{\sin\frac{\pi}{2}\cos\alpha - \cos\frac{\pi}{2}\sin\alpha}$

$= \frac{(0)\cos\alpha + (1)\sin\alpha}{(1)\cos\alpha - (0)\sin\alpha} = \frac{\sin\alpha}{\cos\alpha}$

$x = \tan\alpha \rightarrow \alpha = \tan^{-1}x \longrightarrow (ii)$

from (i) and (ii)

$\tan^{-1}x = \frac{\pi}{2} - \cot^{-1}x$

Also,

$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$

Solution:-

Let $\alpha = \frac{\pi}{2} - \tan^{-1}x \longrightarrow (i)$

$\rightarrow \tan^{-1}x = \frac{\pi}{2} - \alpha$

$\rightarrow x = \tan\left(\frac{\pi}{2} - \alpha\right)$

$= \frac{\sin(\frac{\pi}{2}-\alpha)}{\cos(\frac{\pi}{2}-\alpha)}$

$= \frac{\sin\frac{\pi}{2}\cos\alpha - \cos\frac{\pi}{2}\sin\alpha}{\cos\frac{\pi}{2}\cos\alpha + \sin\frac{\pi}{2}\sin\alpha}$

$= \frac{(1)\cos\alpha - (0)\sin\alpha}{(0)\cos\alpha + (1)\sin\alpha} = \frac{\cos\alpha}{\sin\alpha}$

$x = \cot\alpha \rightarrow \cot^{-1}x = \alpha \longrightarrow (ii)$

from (i) and (ii),

$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$

iii) $\sec^{-1}x = \frac{\pi}{2} - \csc^{-1}x$ and

$\csc^{-1}x = \frac{\pi}{2} - \sec^{-1}x$

Solution:- $\sec^{-1}x = \frac{\pi}{2} - \csc^{-1}x$

Let $\alpha = \frac{\pi}{2} - \csc^{-1}x \longrightarrow (i)$

$\rightarrow \csc^{-1}x = \frac{\pi}{2} - \alpha$

$\rightarrow x = \csc\left(\frac{\pi}{2} - \alpha\right)$

$= \frac{1}{\sin(\frac{\pi}{2}-\alpha)}$

$= \frac{1}{\sin\frac{\pi}{2}\cos\alpha - \cos\frac{\pi}{2}\sin\alpha}$

$x = \frac{1}{(1)\cos\alpha - (0)\sin\alpha} = \frac{1}{\cos\alpha}$

$\rightarrow x = \sec\alpha$

$\rightarrow \sec^{-1}x = \alpha \longrightarrow (ii)$

from (i) and (ii)

$\sec^{-1}x = \frac{\pi}{2} - \csc^{-1}x$

Also,

$\csc^{-1}x = \frac{\pi}{2} - \sec^{-1}x$

Solution:-

Let $\alpha = \frac{\pi}{2} - \sec^{-1}x \longrightarrow (i)$

$\rightarrow \sec^{-1}x = \frac{\pi}{2} - \alpha$

$\rightarrow x = \sec\left(\frac{\pi}{2} - \alpha\right)$

$= \frac{1}{\cos(\frac{\pi}{2}-\alpha)}$

$= \frac{1}{\cos\frac{\pi}{2}\cos\alpha + \sin\frac{\pi}{2}\sin\alpha}$

$x = \frac{1}{(0)\cos\alpha + (1)\sin\alpha} = \frac{1}{\sin\alpha}$

$\rightarrow x = \csc\alpha \rightarrow \alpha = \csc^{-1}x \longrightarrow (ii)$

from (i) and (ii)

$\csc^{-1}x = \frac{\pi}{2} - \sec^{-1}x$

Exercise 13.1

Q1. Evaluate without using tables/calculator:

i) $\sin^{-1}(1)$

Solution:- Let $\alpha = \sin^{-1}(1) \longrightarrow (I)$

$\rightarrow \sin\alpha = 1 \rightarrow \alpha = \frac{\pi}{2}$

so (I) $\rightarrow \sin^{-1}(1) = \frac{\pi}{2}$

(ii) $\sin^{-1}(-1)$

Solution:- Let $\alpha = \sin^{-1}(-1) \longrightarrow (i)$

$\rightarrow \sin\alpha = -1 \rightarrow \alpha = -\frac{\pi}{2}$

so (i) $\rightarrow \sin^{-1}(-1) = -\frac{\pi}{2}$

(iii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution:- Let $\alpha = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \rightarrow (i)$

$\rightarrow \cos\alpha = \frac{\sqrt{3}}{2} \rightarrow \alpha = \frac{\pi}{6}$

so (i) $\rightarrow \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

(iv) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution:- Let $\alpha = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) \rightarrow (i)$

$\rightarrow \tan\alpha = -\frac{1}{\sqrt{3}}$

$\Rightarrow \alpha = -\frac{\pi}{6}$ so (I)

$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

(v) $\cos^{-1}\left(\frac{1}{2}\right)$

Solution:- Let $\alpha = \cos^{-1}\left(\frac{1}{2}\right) \longrightarrow (i)$

$\Rightarrow \cos\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}$

so (i) $\Rightarrow \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

(vi) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Solution:- Let $\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \longrightarrow (i)$

$\Rightarrow \tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$

so (i) $\Rightarrow \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

(vii) $\cot^{-1}(-1)$

Solution:- Let $\alpha = \cot^{-1}(-1) \longrightarrow (i)$

where $\alpha \in (0, \pi)$

$\Rightarrow \cot\alpha = -1 = -\cot\frac{\pi}{4}$

$= \cot\left(\pi - \frac{\pi}{4}\right) = \cot\frac{3\pi}{4}$

since $\cot(\pi - \alpha) = -\cot\alpha$

$\Rightarrow \alpha = \frac{3\pi}{4}$ so (i) $\Rightarrow \cot^{-1}(-1) = \frac{3\pi}{4}$

(viii) $\operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

Solution:- Let $\alpha = \operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right) \longrightarrow (i)$

$\operatorname{cosec}\alpha = -\frac{2}{\sqrt{3}} \Rightarrow \alpha = -\frac{\pi}{3}$

where $\alpha \in [0, \pi]$

so (i) $\Rightarrow \operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{\pi}{3}$

but $\alpha \neq \frac{\pi}{2}$

(ix) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution:- Let $\alpha = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) \longrightarrow (i)$

$\Rightarrow \sin\alpha = -\frac{1}{\sqrt{2}} \Rightarrow \alpha = -\frac{\pi}{4}$

so (i) $\Rightarrow \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{2}$

Q2. Without using table/calculator show that:

i) $\tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13}$

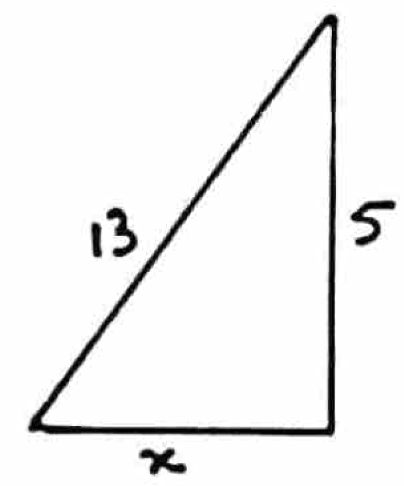
ii) $2\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25}$

iii) $\cos^{-1}\frac{4}{5} = \cot^{-1}\frac{4}{3}$

Solution:- (i) $\tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13}$

Let $\sin^{-1}\frac{5}{13} = \alpha \longrightarrow (i)$

$\Rightarrow \sin\alpha = \frac{5}{13}$



$\tan\alpha = \frac{5}{12}$

$\Rightarrow \alpha = \tan^{-1}\left(\frac{5}{12}\right) \longrightarrow (ii)$

from (i) and (ii)

$\tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13}$

Hence proved

$x^2 + (5)^2 = (13)^2$

$x^2 + 25 = 169$

$x^2 = 169 - 25$

$x^2 = 144$

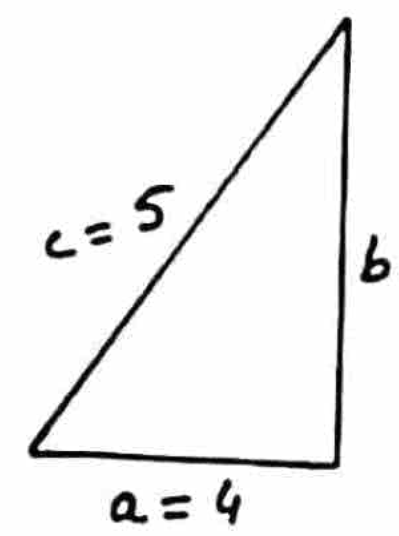
$\Rightarrow x = 12$

(ii) $2\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25}$

Solution:-

Let $\cos^{-1}\frac{4}{5} = \alpha \longrightarrow (i)$

$\Rightarrow \cos\alpha = \frac{4}{5}$



$\sin\alpha = \frac{3}{5}$

Now

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)$

$\sin 2\alpha = \frac{24}{25}$

$\Rightarrow 2\alpha = \sin^{-1}\left(\frac{24}{25}\right)$

from (i)

$2\left(\cos^{-1}\frac{4}{5}\right) = \sin^{-1}\left(\frac{24}{25}\right)$

$\Rightarrow 2\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25}$ Hence proved

By Pathagoras

$a^2 + b^2 = c^2$

$b^2 = c^2 - a^2$

$= (5)^2 - (4)^2$

$= 25 - 16$

$b^2 = 9$

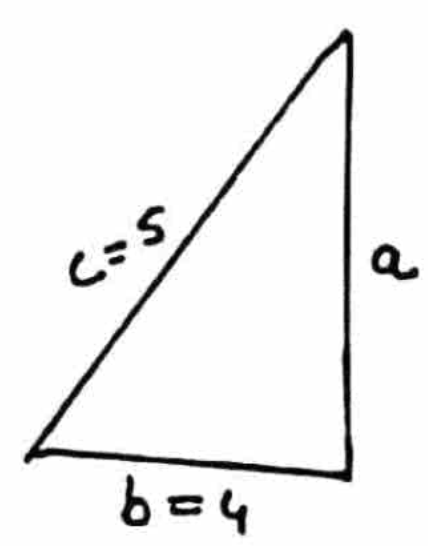
$\Rightarrow b = 3$

(iii) $\cos^{-1}\frac{4}{5} = \cot^{-1}\frac{4}{3}$

Solution:-

Let $\alpha = \cos^{-1}\frac{4}{5} \longrightarrow (i)$

$\Rightarrow \cos\alpha = \frac{4}{5}$



By Pathagoras

$a^2+b^2=c^2$

$\rightarrow a^2=c^2-b^2$

$=(5)^2-(4)^2$

$a^2=9$

$\rightarrow a=3$

$\tan\alpha=\frac{3}{4}$

$\rightarrow \cot\alpha=\frac{4}{3}$

$\rightarrow \alpha=\cot^{-1}\left(\frac{4}{3}\right)\longrightarrow$ (ii)

from (i) and (ii)

$\cos^{-1}\frac{4}{5}=\cot^{-1}\frac{4}{3}$ Hence proved

Q3. Find the values of each expression:

i) $\cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)$

Solution:- Let $\alpha=\sin^{-1}\frac{1}{\sqrt{2}}\longrightarrow$ (i)

$\rightarrow \sin\alpha=\frac{1}{\sqrt{2}} \rightarrow \alpha=\frac{\pi}{4}$

so (i) $\rightarrow \sin^{-1}\frac{1}{\sqrt{2}}=\frac{\pi}{4}$

Now $\cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)=\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

ii) $\sec\left(\cos^{-1}\frac{1}{2}\right)$

Solution:- Let $\alpha=\cos^{-1}\frac{1}{2}\longrightarrow$ (i)

$\rightarrow \cos\alpha=\frac{1}{2} \rightarrow \alpha=\frac{\pi}{3}$

so (i) $\rightarrow \cos^{-1}\frac{1}{2}=\frac{\pi}{3}$

Now $\sec\left(\cos^{-1}\frac{1}{2}\right)=\sec\left(\frac{\pi}{3}\right)=\frac{1}{\cos\frac{\pi}{3}}$

$=\frac{1}{1/2}=2$

iii) $\tan\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$

Solution:- Let $\alpha=\cos^{-1}\frac{\sqrt{3}}{2}\longrightarrow$ (i)

$\rightarrow \cos\alpha=\frac{\sqrt{3}}{2} \rightarrow \alpha=\frac{\pi}{6}$

so (i) $\rightarrow \cos^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{6}$ Now

$\tan\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)=\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}$

iv) $\csc\left(\tan^{-1}(-1)\right)$

Solution:- Let $\alpha=\tan^{-1}(-1)\longrightarrow$ (i)

$\rightarrow \tan\alpha=-1 \rightarrow \alpha=-\frac{\pi}{4}$

Now (i) $\rightarrow \tan^{-1}(-1)=-\frac{\pi}{4}$

So

$\csc\left(\tan^{-1}(-1)\right)=\csc\left(-\frac{\pi}{4}\right)$

$=-\csc\frac{\pi}{4}=-\frac{1}{\sin\frac{\pi}{4}}=-\frac{1}{1/\sqrt{2}}$

$=-\sqrt{2}$

v) $\sec\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

Solution:- Let $\alpha=\sin^{-1}\left(-\frac{1}{2}\right)\longrightarrow$ (i)

$\rightarrow \sin\alpha=-\frac{1}{2} \rightarrow \alpha=-\frac{\pi}{6}$

so (i) $\rightarrow \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

Now $\sec\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)=\sec\left(-\frac{\pi}{6}\right)$

$=\sec\frac{\pi}{6}=\frac{1}{\cos\frac{\pi}{6}}=\frac{1}{\sqrt{3}/2}$

$=\frac{2}{\sqrt{3}}$

vi) $\tan\left(\tan^{-1}(-1)\right)$

Solution:- Let $\alpha=\tan^{-1}(-1)\longrightarrow$ (i)

$\rightarrow \tan\alpha=-1 \rightarrow \alpha=-\frac{\pi}{4}$

so (i) $\rightarrow \tan^{-1}(-1)=-\frac{\pi}{4}$

Now $\tan\left(\tan^{-1}(-1)\right)=\tan\left(-\frac{\pi}{4}\right)$

$=-\tan\frac{\pi}{4}=-1$

vii) $\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$

Solution:- Let $\alpha=\sin^{-1}\left(\frac{1}{2}\right)\longrightarrow$ (i)

$\rightarrow \sin\alpha=\frac{1}{2} \rightarrow \alpha=\frac{\pi}{6}$

so (i) $\rightarrow \sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$

Now $\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right)=\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$

viii) $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

Solution:- Let $\alpha=\sin^{-1}\left(-\frac{1}{2}\right)\longrightarrow$ (i)

$\rightarrow \sin\alpha=-\frac{1}{2} \rightarrow \alpha=-\frac{\pi}{6}$

so (i) $\rightarrow \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

Now $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)=\tan\left(-\frac{\pi}{6}\right)$

$=-\tan\pi/6=-\frac{1}{\sqrt{3}}$

ix) $\sin\left(\tan^{-1}(-1)\right)$

Solution:- Let $\alpha=\tan^{-1}(-1)\longrightarrow$ (i)

$\rightarrow \tan\alpha=-1 \rightarrow \alpha=-\frac{\pi}{4}$

so (i) $\rightarrow \tan^{-1}(-1) = -\frac{\pi}{4}$

Now $\sin(\tan^{-1}(-1)) = \sin(-\frac{\pi}{4})$

$= -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

Addition and Subtraction Formulas

1) Prove that:

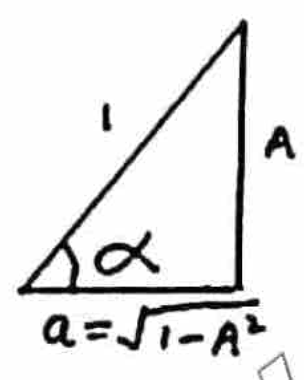
$\sin^{-1}A + \sin^{-1}B = \sin^{-1}(A\sqrt{1-B^2} + B\sqrt{1-A^2})$

Proof:-

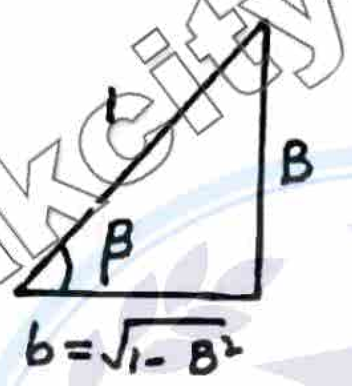
Let $\alpha = \sin^{-1}A$, $\beta = \sin^{-1}B$

$\rightarrow \sin\alpha = \frac{A}{1} = A$, $\sin\beta = \frac{B}{1} = B$

$\boxed{a^2 + A^2 = 1 \rightarrow a^2 = 1 - A^2 \quad \rightarrow a = \sqrt{1-A^2} \text{ By Pathagoras}}$



$\boxed{b^2 + B^2 = 1 \rightarrow b^2 = 1 - B^2 \quad \rightarrow b = \sqrt{1-B^2} \text{ By Pathagoras}}$



So $\cos\alpha = \frac{\sqrt{1-A^2}}{1} = \sqrt{1-A^2}$

$\cos\beta = \frac{\sqrt{1-B^2}}{1} = \sqrt{1-B^2}$

Now $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\rightarrow \alpha + \beta = \sin^{-1}(A\sqrt{1-B^2} + B\sqrt{1-A^2})$

$\rightarrow \sin^{-1}A + \sin^{-1}B = \sin^{-1}(A\sqrt{1-B^2} + B\sqrt{1-A^2})$

Hence proved

2) Prove that:

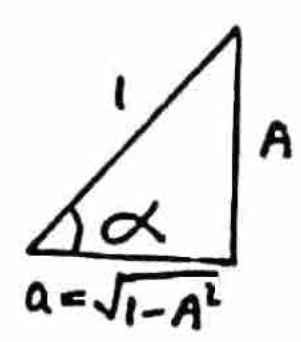
$\sin^{-1}A - \sin^{-1}B = \sin^{-1}(A\sqrt{1-B^2} - B\sqrt{1-A^2})$

Proof:-

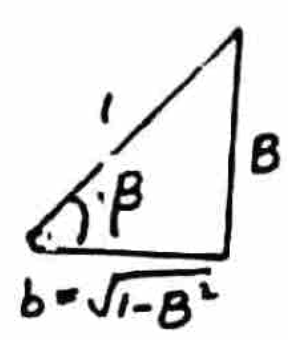
Let $\alpha = \sin^{-1}A$, $\beta = \sin^{-1}B$

$\rightarrow \sin\alpha = \frac{A}{1} = A$, $\sin\beta = \frac{B}{1} = B$

$\boxed{a^2 + A^2 = 1 \rightarrow a^2 = 1 - A^2 \quad \rightarrow a = \sqrt{1-A^2} \text{ By pathagoras}}$



$\boxed{b^2 + B^2 = 1 \rightarrow b^2 = 1 - B^2 \quad \rightarrow b = \sqrt{1-B^2} \text{ By Pathagoras}}$



so $\cos\alpha = \frac{\sqrt{1-A^2}}{1} = \sqrt{1-A^2}$

$\cos\beta = \frac{\sqrt{1-B^2}}{1} = \sqrt{1-B^2}$

Now

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\rightarrow \alpha - \beta = \sin^{-1}(A\sqrt{1-B^2} - B\sqrt{1-A^2})$

$\rightarrow \sin^{-1}A - \sin^{-1}B = \sin^{-1}(A\sqrt{1-B^2} - B\sqrt{1-A^2})$

Hence proved

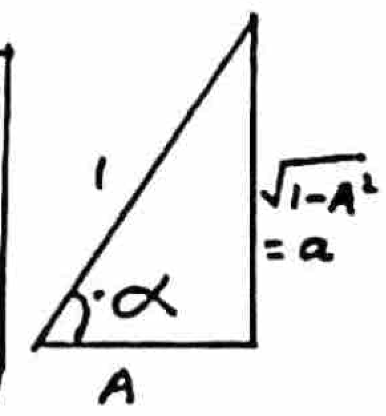
3) Prove that:

$\cos^{-1}A + \cos^{-1}B = \cos^{-1}(AB - \sqrt{(1-A^2)(1-B^2)})$

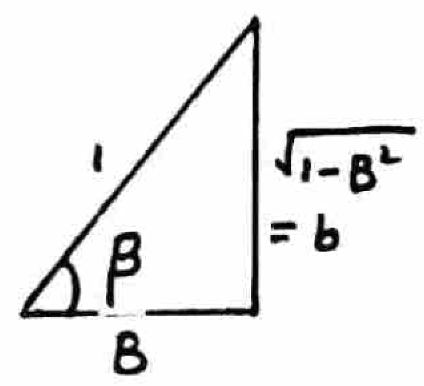
Proof:- Let $\alpha = \cos^{-1}A$, $\beta = \cos^{-1}B$

$\rightarrow \cos\alpha = \frac{A}{1} = A$, $\cos\beta = \frac{B}{1} = B$

$\boxed{a^2 + A^2 = 1 \text{ By Pathagoras} \quad \rightarrow a^2 = 1 - A^2 \quad \rightarrow a = \sqrt{1-A^2}}$



$\boxed{b^2 + B^2 = 1 \text{ By Pathagoras} \quad \rightarrow b^2 = 1 - B^2 \quad \rightarrow b = \sqrt{1-B^2}}$



So $\sin\alpha = \frac{\sqrt{1-A^2}}{1} = \sqrt{1-A^2}$

$\sin\beta = \frac{\sqrt{1-B^2}}{1} = \sqrt{1-B^2}$

$\because \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\rightarrow \alpha + \beta = \cos^{-1}(AB - \sqrt{1-A^2}\sqrt{1-B^2})$

$\rightarrow \cos^{-1}A + \cos^{-1}B = \cos^{-1}(AB - \sqrt{(1-A^2)(1-B^2)})$

Hence proved

4) Prove that:

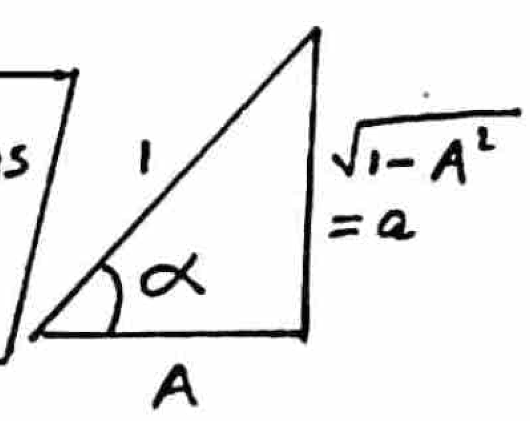
$\cos^{-1}A - \cos^{-1}B = \cos^{-1}(AB + \sqrt{(1-A^2)(1-B^2)})$

Proof:-

Let $\alpha = \cos^{-1}A$, $\beta = \cos^{-1}B$

$\rightarrow \cos\alpha = \frac{A}{1} = A$, $\cos\beta = \frac{B}{1} = B$

$\boxed{a^2 + A^2 = 1 \text{ By Pathagoras} \quad \rightarrow a^2 = 1 - A^2 \quad \rightarrow a = \sqrt{1-A^2}}$

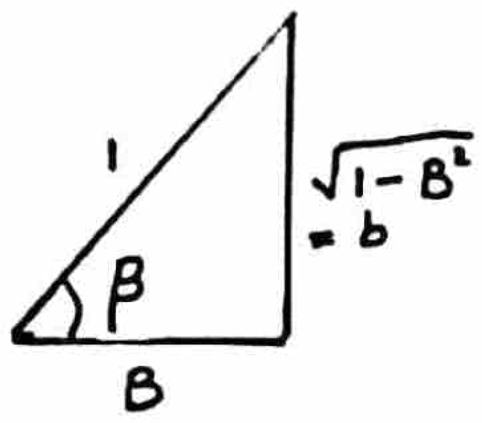


By Pathagoras

$b^2 + B^2 = 1$

$\rightarrow b^2 = 1 - B^2$

$\rightarrow b = \sqrt{1-B^2}$



so $\sin\alpha = \frac{\sqrt{1-A^2}}{1} = \sqrt{1-A^2}$

$\sin\beta = \frac{\sqrt{1-B^2}}{1} = \sqrt{1-B^2}$

$\because \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\rightarrow \alpha - \beta = \cos^{-1}(AB + \sqrt{1-A^2}\sqrt{1-B^2})$

$\cos^{-1}A - \cos^{-1}B = \cos^{-1}(AB + \sqrt{(1-A^2)(1-B^2)})$

Hence proved

5) Prove that:

$\tan^{-1}A + \tan^{-1}B = \tan^{-1}\frac{A+B}{1-AB}$

Proof:-

Let $\alpha = \tan^{-1}A$, $\beta = \tan^{-1}B$

$\rightarrow \tan\alpha = A$, $\tan\gamma = B$

$\because \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\rightarrow \alpha + \beta = \tan^{-1}\frac{A+B}{1-AB}$ Hence proved

6) Prove that:

$\tan^{-1}A - \tan^{-1}B = \tan^{-1}\frac{A-B}{1+AB}$

Proof:-

Let $\alpha = \tan^{-1}A$, $\beta = \tan^{-1}B$

$\rightarrow \tan\alpha = A$, $\tan\beta = B$

$\because \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$\rightarrow \alpha - \beta = \tan^{-1}\frac{A-B}{1+AB}$

$\tan^{-1}A - \tan^{-1}B = \tan^{-1}\frac{A-B}{1+AB}$

Hence proved

7) Prove that:

$2\tan^{-1}A = \tan^{-1}\frac{2A}{1-A^2}$

Proof:-

we know that

$\tan^{-1}A + \tan^{-1}B = \tan^{-1}\frac{A+B}{1-AB}$

Put $B = A$

$\tan^{-1}A + \tan^{-1}A = \tan^{-1}\left(\frac{A+A}{1-A\cdot A}\right)$

$\rightarrow 2\tan^{-1}A = \tan^{-1}\frac{2A}{1-A^2}$

Hence proved

Exercise 13.2

Prove the following:

Q1. $\sin^{-1}\frac{5}{13} + \sin^{-1}\frac{7}{25} = \cos^{-1}\frac{253}{325}$

Solution:- Let $\alpha = \sin^{-1}\frac{5}{13}$, $\beta = \sin^{-1}\frac{7}{25}$

$\rightarrow \sin\alpha = \frac{5}{13}$, $\sin\beta = \frac{7}{25}$

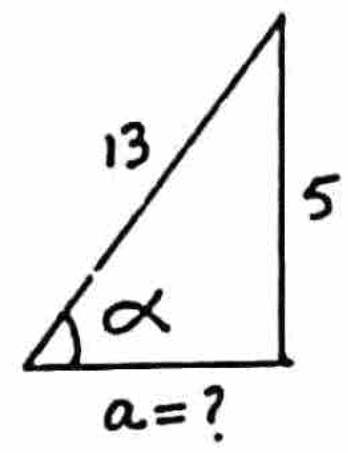
By Pathagoras

$a^2 + (5)^2 = (13)^2$

$\rightarrow a^2 = 169 - 25$

$a^2 = 144$

$\rightarrow a = 12$



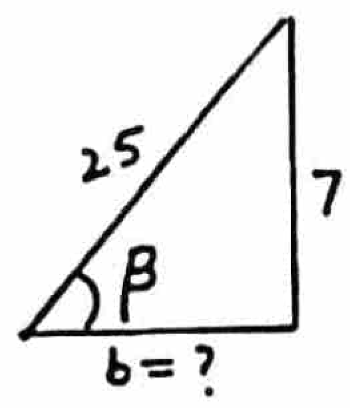
so $\cos\alpha = \frac{12}{13}$

$b^2 + (7)^2 = (25)^2$

$\rightarrow b^2 = 625 - 49$

$b^2 = 576$

$\rightarrow b = 24$



so $\cos\beta = \frac{24}{25}$

$\because \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\rightarrow \alpha + \beta = \cos^{-1}\left[\left(\frac{12}{13}\right)\left(\frac{24}{25}\right) - \left(\frac{5}{13}\right)\left(\frac{7}{25}\right)\right]$

$= \cos^{-1}\left(\frac{288}{325} - \frac{35}{325}\right)$

$\alpha + \beta = \cos^{-1}\frac{253}{325}$

$\rightarrow \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{7}{25} = \cos^{-1}\frac{253}{325}$

Hence proved

Q2. $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{9}{19}$

Solution:-

$\because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\frac{A+B}{1-AB}$

Put $A = \frac{1}{4}$, $B = \frac{1}{5}$ then

$\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{\frac{1}{4}+\frac{1}{5}}{1-(1/4)(1/5)}$

$$= \tan^{-1}\frac{\frac{5+4}{20}}{1-\frac{1}{20}}$$

$$= \tan^{-1}\left(\frac{9}{20}\times\frac{20}{19}\right)$$

$$\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{9}{19}$$

Hence proved

Q3. $2\tan^{-1}\frac{2}{3} = \sin^{-1}\frac{12}{13}$

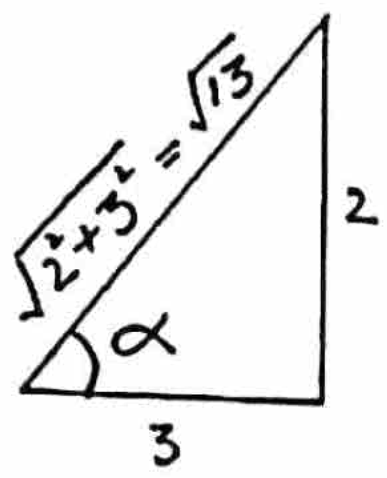
Solution:-

Let $\tan^{-1}\frac{2}{3} = \alpha \rightarrow \tan\alpha = \frac{2}{3}$

Now

$\sin\alpha = \frac{2}{\sqrt{13}}$ and

$\cos\alpha = \frac{3}{\sqrt{13}}$



$\because \sin 2\alpha = 2\sin\alpha\cos\alpha$

$$= 2\left(\frac{12}{\sqrt{13}}\right)\left(\frac{3}{\sqrt{13}}\right)$$

$$\sin 2\alpha = \frac{12}{13}$$

$\rightarrow 2\tan^{-1}\frac{2}{3} = \sin^{-1}\frac{12}{13}$

Hence proved

Q4. $\tan^{-1}\frac{120}{119} = 2\cos^{-1}\frac{12}{13}$

Solution:-

Let $\cos^{-1}\frac{12}{13} = \alpha \rightarrow \cos\alpha = \frac{12}{13}$

$\because \sin^2\alpha = 1-\cos^2\alpha$

$$\sin^2\alpha = 1-\frac{144}{169} = \frac{169-144}{169} = \frac{25}{169}$$

$\rightarrow \sin\alpha = \frac{5}{13}$ ($\because \sin\alpha$ is +ive in domain of $\cos\alpha$)

$$\because \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{5/13}{12/13} = \frac{5}{12}$$

Also, $$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(5/12\right)}{1-25/144}$$

$$= \frac{\frac{10}{12}}{\frac{144-25}{144}} = \frac{\frac{10}{12}}{\frac{119}{144}}$$

$$\tan 2\alpha = \frac{10}{12}\times\frac{144}{19}$$

$\rightarrow \tan 2\alpha = \frac{120}{119}$

$\rightarrow 2\alpha = \tan^{-1}\frac{120}{119}$

$\rightarrow 2\cos^{-1}\frac{12}{13} = \tan^{-1}\frac{120}{119}$ Hence proved

Q5. $\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3 = \frac{\pi}{4}$

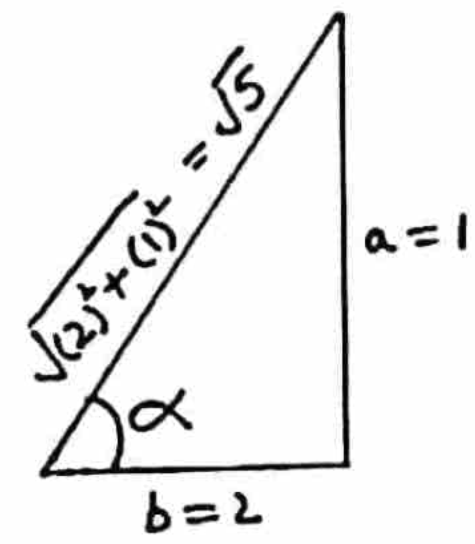
Solution:-

Let $\sin^{-1}\frac{1}{\sqrt{5}} = \alpha \rightarrow \sin\alpha = \frac{1}{\sqrt{5}}$

$\cot^{-1}3 = \beta \rightarrow \cot\beta = 3$

$\tan\alpha = \frac{1}{2}$ and

$\tan\beta = \frac{1}{3}$



$\because \tan(\alpha+\beta)$

$$= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$= \frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)} = \frac{\frac{3+2}{6}}{1-\frac{1}{6}}$$

$$\tan(\alpha+\beta) = \frac{5/6}{5/6} = 1$$

$\rightarrow \alpha+\beta = \tan^{-1}(1)$

$\rightarrow \sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3 = \frac{\pi}{4}$

Hence proved

Q6. $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \sin^{-1}\frac{77}{85}$

Solution:-

Let $\alpha = \sin^{-1}\frac{3}{5} \Rightarrow \sin\alpha = \frac{3}{5}$

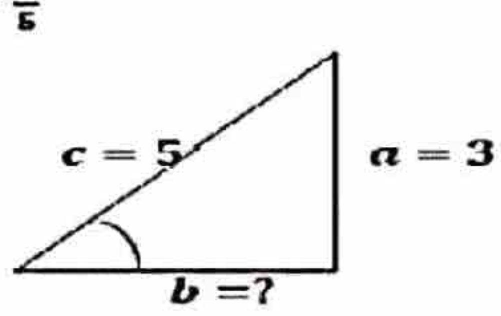
$c^2 = a^2 + b^2$

$(5)^2 = (3)^2 + b^2$

$25 - 9 = b^2 \Rightarrow b^2 = 16$

So $b = 4$

Now $\cos\alpha = \frac{4}{5}$



Let $\beta = \sin^{-1}\frac{8}{17} \Rightarrow \sin\beta = \frac{8}{17}$

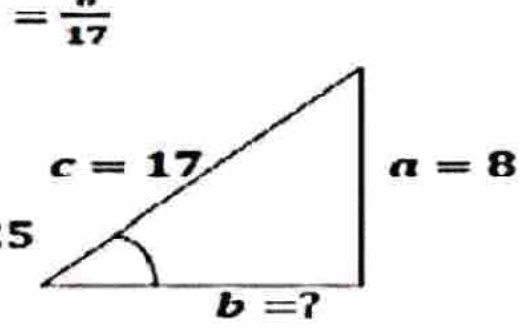
$c^2 = a^2 + b^2$

$(17)^2 = (8)^2 + b^2$

$289 - 64 = a^2 \Rightarrow a^2 = 225$

So $a = 15$

Now $\cos\beta = \frac{15}{17}$



Since $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \left(\frac{3}{5}\right)\left(\frac{15}{17}\right) + \left(\frac{4}{5}\right)\left(\frac{8}{17}\right) = \frac{45}{85} + \frac{32}{85}$$

$\sin(\alpha+\beta) = \frac{77}{85} \Rightarrow \alpha+\beta = \sin^{-1}\frac{77}{85}$

Hence $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \sin^{-1}\frac{77}{86}$

Q7. $\sin^{-1}\frac{77}{85} - \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{15}{17}$

Solution:-

Let $\alpha = \sin^{-1}\frac{77}{85} \rightarrow \sin\alpha = \frac{77}{85}$

$\beta = \sin^{-1}\frac{3}{5} \rightarrow \sin\beta = \frac{3}{5}$

$\because \cos^2\alpha = 1 - \sin^2\alpha$

$= 1 - \left(\frac{77}{85}\right)^2 = 1 - \frac{5929}{7225}$

$\cos^2\alpha = \frac{7225-5929}{7225}$

$\cos^2\alpha = \frac{1296}{7225}$

$\rightarrow \cos\alpha = \frac{36}{85}$ (cos. is +ive in domain of sine)

Also $\cos^2\beta = 1 - \sin^2\beta = 1 - \frac{9}{25}$

$\rightarrow \cos^2\beta = 1 - \left(\frac{3}{5}\right)^2$

$\cos^2\beta = \frac{25-9}{25} = \frac{16}{25}$

$\rightarrow \cos\beta = \frac{4}{5}$ ($\because$ cos is +ive in domain of sine)

$\because \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\rightarrow \alpha - \beta = \cos^{-1}\left(\left(\frac{36}{85}\right)\left(\frac{4}{5}\right) + \left(\frac{77}{85}\right)\left(\frac{3}{5}\right)\right)$

$= \cos^{-1}\left(\frac{144}{425} + \frac{231}{425}\right)$

$\alpha - \beta = \cos^{-1}\frac{375}{425} = \cos^{-1}\frac{15}{17}$

$\rightarrow \sin^{-1}\frac{77}{85} - \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{15}{17}$

Hence proved.

Q8. $\cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = \sin^{-1}\frac{3}{5}$

Solution:-

$\because 2\tan^{-1}A = \tan^{-1}\frac{2A}{1-A^2}$ so

$\cos^{-1}\frac{63}{65} + \tan^{-1}\frac{2(1/5)}{1-(1/5)^2} = \sin^{-1}\frac{3}{5}$

$\cos^{-1}\frac{63}{65} + \tan^{-1}\frac{\frac{2}{5}}{24/25} = \sin^{-1}\frac{3}{5}$

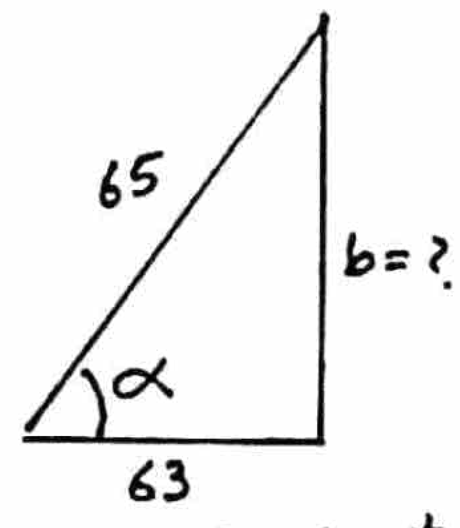
$\cos^{-1}\frac{63}{65} + \tan^{-1}\left(\frac{2}{5}\times\frac{25}{24}\right) = \sin^{-1}\frac{3}{5}$

$\cos^{-1}\frac{63}{65} + \tan^{-1}\frac{5}{12} = \sin^{-1}\frac{3}{5}$

Let $\cos^{-1}\frac{63}{65} = \alpha \rightarrow \cos\alpha = \frac{63}{65}$

$\tan^{-1}5/12 = \beta$

$\rightarrow \tan\beta = \frac{5}{12}$



$b^2 + (63)^2 = (65)^2$

$b^2 = 4225 - 3969$

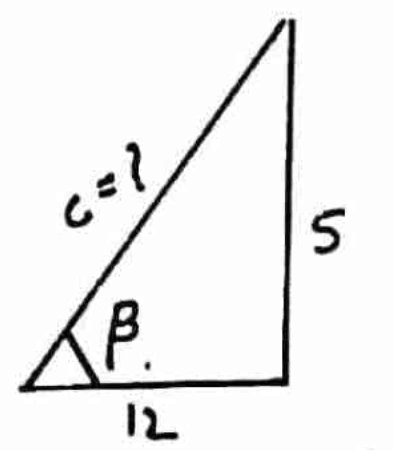
$b^2 = 256$

$\rightarrow b = 16$

so

$\sin\alpha = \frac{16}{65}$ and

$\cos\beta = \frac{12}{13}, \sin\beta = \frac{5}{13}$



$c^2 = (12)^2 + (5)^2$

$= 144 + 25$

$c^2 = 169$

$\rightarrow c = 13$

$\because \sin(\alpha+\beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(\frac{16}{65}\right)\left(\frac{12}{13}\right) + \left(\frac{63}{65}\right)\left(\frac{5}{13}\right)$

$\sin(\alpha+\beta) = \frac{192}{845} + \frac{315}{845}$

$= \frac{507}{845}$

$\sin(\alpha+\beta) = \frac{3}{5}$

$\rightarrow \alpha + \beta = \sin^{-1}\frac{3}{5}$

$\rightarrow \cos^{-1}\frac{63}{65} + \tan^{-1}\frac{5}{12} = \sin^{-1}\frac{3}{5}$

$\rightarrow \cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = \sin^{-1}\frac{3}{5}$

Hence proved

Q9. $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19} = \frac{\pi}{4}$

Solution:-

L.H.S $= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19}$

$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\cdot\frac{3}{5}}\right) - \tan^{-1}\frac{8}{19}$

$= \tan^{-1}\left(\frac{\frac{15+12}{20}}{1-9/20}\right) - \tan^{-1}\frac{8}{19}$

$= \tan^{-1}\frac{27/20}{11/20} - \tan^{-1}8/19$

$= \tan^{-1}\frac{27}{11} - \tan^{-1}\frac{8}{19}$

$$= \tan^{-1}\left(\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\cdot\frac{8}{19}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{513-88}{209}}{\frac{209+216}{209}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{425}{209}}{\frac{425}{209}}\right) = \tan^{-1}(1)$$

$$= \frac{\pi}{4} = R.H.S$$

Hence proved.

Q10. $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}$

Solution:-

$$L.H.S = \sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left(\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}+\frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right)+\sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left(\frac{4}{5}\sqrt{\frac{169-25}{169}}+\frac{5}{13}\sqrt{\frac{25-16}{25}}\right)+\sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left(\frac{4}{5}\left(\frac{12}{13}\right)+\frac{5}{13}\left(\frac{3}{5}\right)\right)+\sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left(\frac{48}{65}+\frac{15}{65}\right)+\sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\frac{63}{65}+\sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left(\frac{63}{65}\sqrt{1-\left(\frac{16}{65}\right)^2}+\frac{16}{65}\sqrt{1-\left(\frac{63}{65}\right)^2}\right)$$

$$= \sin^{-1}\left(\frac{63}{65}\sqrt{1-\frac{256}{4225}}+\frac{16}{65}\sqrt{1-\frac{3969}{4225}}\right)$$

$$= \sin^{-1}\left(\frac{63}{65}\sqrt{\frac{3969}{4225}}+\frac{16}{65}\sqrt{\frac{256}{4225}}\right)$$

$$= \sin^{-1}\left(\frac{63}{65}\left(\frac{63}{65}\right)+\frac{16}{65}\left(\frac{16}{65}\right)\right)$$

$$= \sin^{-1}\left(\frac{3969}{4225}+\frac{256}{4225}\right)$$

$$= \sin^{-1}\left(\frac{3969+256}{4225}\right)$$

$$= \sin^{-1}\left(\frac{4225}{4225}\right) = \sin^{-1}(1)$$

$$= \frac{\pi}{2} = R.H.S$$ Hence Proved

Q11. $\tan^{-1}\frac{1}{11}+\tan^{-1}\frac{5}{6}=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}$

Proof:- $L.H.S = \tan^{-1}\frac{1}{11}+\tan^{-1}\frac{5}{6}$

$$= \tan^{-1}\left(\frac{\frac{1}{11}+\frac{5}{6}}{1-\left(\frac{1}{11}\right)\left(\frac{5}{6}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{6+55}{66}}{1-\frac{5}{66}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{61}{66}}{\frac{61}{66}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$R.H.S = \tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}$$

$$= \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Hence proved $L.H.S = R.H.S$

Q12. $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$

Solution:-

$$L.H.S = 2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left(\frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}\right)+\tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{1-\frac{1}{9}}\right)+\tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right)+\tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right)+\tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\left(\frac{3}{4}\right)\left(\frac{1}{7}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{21+4}{28}}{1-\frac{3}{28}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right) = \tan^{-1}(1)$$

$$= \frac{\pi}{4} = R.H.S$$

Hence proved

Q13. Show that
$\cos(\sin^{-1}x) = \sqrt{1-x^2}$

Solution:-

Let $\sin^{-1}x = \alpha \rightarrow x = \sin\alpha$

$\because \cos^2\alpha = 1 - \sin^2\alpha$

$\cos^2\alpha = 1 - x^2$

$\rightarrow \cos\alpha = \sqrt{1-x^2}$ ($\cos\alpha$ is +ive in domain of $\sin\alpha$)

$\rightarrow \cos(\sin^{-1}x) = \sqrt{1-x^2}$

Hence proved

Q14. Show that
$\sin(2\cos^{-1}x) = 2x\sqrt{1-x^2}$

Solution:-

Let $\cos^{-1}x = \alpha \rightarrow x = \cos\alpha$

$\because \sin^2\alpha = 1 - \cos^2\alpha$

$\sin^2\alpha = 1 - x^2$

$\rightarrow \sin\alpha = \sqrt{1-x^2}$

$\because \sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2\sqrt{1-x^2}\cdot x$

$\rightarrow \sin 2(\cos^{-1}x) = 2x\sqrt{1-x^2}$

Hence proved

Q15. Show that
$\cos(2\sin^{-1}x) = 1 - 2x^2$

Solution:-

Let $\sin^{-1}x = \alpha$

$\rightarrow x = \sin\alpha$

$\because \cos 2\alpha = 1 - 2\sin^2\alpha$

$\cos 2\alpha = 1 - 2x^2$

$\rightarrow \cos(2\sin^{-1}x) = 1 - 2x^2$

Hence proved

Q16. Show that:
$\tan^{-1}(-x) = -\tan^{-1}x$

Solution:-

$\tan^{-1}(-x) = -\tan^{-1}x$

$\rightarrow \tan^{-1}(-x) + \tan^{-1}x = 0$

L.H.S $= \tan^{-1}(-x) + \tan^{-1}x$

$= \tan^{-1}\left(\frac{-x+x}{1-(-x)(x)}\right) = \tan^{-1}\left(\frac{0}{1+x^2}\right)$

$= \tan^{-1}(0) = 0 =$ R.H.S

Hence L.H.S = R.H.S

Q17. Show that
$\sin^{-1}(-x) = -\sin^{-1}x$

Solution:-

$Sin^{-1}(-x) = -Sin^{-1}x$

Or $Sin^{-1}(-x) + Sin^{-1}x = 0$

$L.H.S = Sin^{-1}(-x) + Sin^{-1}x$

$= Sin^{-1}\left[(-x)\sqrt{1-x^2} + x\sqrt{1-(-x)^2}\right]$

$= Sin^{-1}\left(-x\sqrt{1-x^2} + x\sqrt{1-x^2}\right)$

$= Sin^{-1}(0) = 0 = R.H.S$

Hence Proved

Q18. Show that
$\cos^{-1}(-x) = \pi - \cos^{-1}x$

Solution:-

$\cos^{-1}(-x) = \pi - \cos^{-1}x$

$\rightarrow \cos^{-1}(-x) + \cos^{-1}x = \pi$

L.H.S $= \cos^{-1}(-x) + \cos^{-1}x$

$= \cos^{-1}\left((-x)(x) - \sqrt{(1-(-x)^2)(1-x^2)}\right)$

$= \cos^{-1}\left(-x^2 - \sqrt{(1-x^2)(1-x^2)}\right)$

$= \cos^{-1}\left(-x^2 - \sqrt{(1-x^2)^2}\right)$

$= \cos^{-1}\left(-x^2 - (1-x^2)\right)$

$= \cos^{-1}(-x^2 - 1 + x^2)$

$= \cos^{-1}(-1) = \pi =$ R.H.S

Hence L.H.S = R.H.S

Q19. Show that $\tan(\sin^{-1}x) = \dfrac{x}{\sqrt{1-x^2}}$

Solution:- Let $\sin^{-1}x = \alpha \rightarrow x = \sin\alpha$

$\because \cos^2\alpha = 1 - \sin^2\alpha \rightarrow \cos^2\alpha = 1 - x^2$

$\because \tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} \rightarrow \tan\alpha = \dfrac{x}{\sqrt{1-x^2}}$

or $\tan(\sin^{-1}x) = \dfrac{x}{\sqrt{1-x^2}}$ Hence proved

Q20. Given that $x = \sin^{-1}\frac{1}{2}$, find the values of following trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\csc x$.

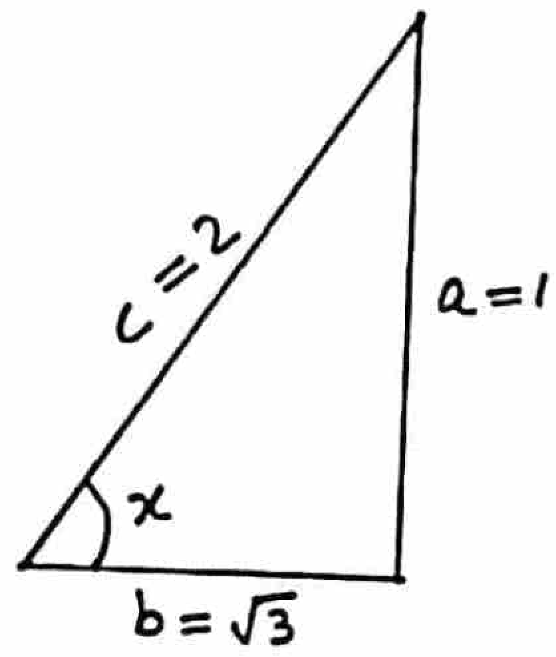
Solution:-

$x = \sin^{-1}\frac{1}{2}$

$\rightarrow \sin x = \frac{1}{2}$, $\csc x = \frac{2}{1} = 2$

$\cos x = \frac{\sqrt{3}}{2}$, $\sec x = \frac{2}{\sqrt{3}}$

$\tan x = \frac{1}{\sqrt{3}}$, $\cot x = \frac{\sqrt{3}}{1} = \sqrt{3}$



$\because a^2 + b^2 = c^2$

$b^2 = c^2 - a^2$

$= (2)^2 - (1)^2 = 4 - 1$

$b^2 = 3 \rightarrow b = \sqrt{3}$