MATHEMATICS 1st YEAR

UNIT # 12

APPLICATIONS OF TRIGONOMETRY

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Contents

Sherazi Mathematics

اچھی باتیں

1۔ جو کسی کا برا نہیں چاہتے ان کے ساتھ کوئی برا نہیں کر سکتا یہ میرے رب کا وعدہ ہے۔

2۔ برے سلوک کا بہترین جواب اچھا سلوک اور جہالت کا جواب "خاموشی" ہے۔

3۔ کوئی مانے یا نہ مانے لیکن زندگی میں دو ہی اپنے ہوتے ہیں ایک خود اور ایک خدا۔

4۔ جو دو گے وہی لوٹ کے آئے گا عزت ہو یا دھوکہ۔

5۔ جس سے اس کے والدین خوشی سے راضی نہیں اس سے اللہ بھی راضی نہیں۔

Solution of the triangle

A triangle has six important elements; three angles and three sides.

If any three out of these six elements, out of which atleast one side, are given, the remaining three elements can be determined.

This process of finding the unknown elements is called the "Solution of the triangle.

In $\triangle ABC$, angles are denoted by α, β and γ and three sides opposite to them are denoted by a, b and c.

Tables of Trigonometric Ratios

In the solution of triangles, we may have to solve problems involving angles other than $0°$, $30°$, $45°$, $60°$ and $90°$. In such cases, we shall have to consult natural sin/cos/tan tables.

Note:- Tables/calculator will also be used for finding the measures of the angles when value of trigonometric ratios are given. e.g., to find θ when $\sin\theta = x$

Important points about Tables

* In four-figure tables, the interval is 6 minutes and difference corresponding to 1, 2, 3, 4, 5 minutes are given in the difference columns.
* In given tables we have total 16-columns. First column on extreme left is of angles in degrees from $0°$ to $90°$
* Next ten columns are of angles in minute with difference of 6 minute.
* Last five columns are columns of difference of 1 minute.

Example 1. Find the value of

i) $\sin 38°24'$ ii) $\sin 38°28'$ iii) $\tan 65°30'$

Solution:-

i) $\sin 38°24' = 0.6211$

ii) $\sin 38°28' = 0.6220$

iii) $\tan 65°30' = 2.1943$

Note:- As $\sin\theta$, $\sec\theta$ and $\tan\theta$ go on increasing as θ increases from $0°$ to $90°$. so the number in the columns of differences for $\sin\theta$, $\sec\theta$ and $\tan\theta$ are added.

Since $\cos\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$ decrease as θ increases from $0°$ to $90°$. so for $\cos\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$ the numbers in the column of the differences are subtracted

Example 2. If $\sin x = (0.5100)$, find x

Solution:-

$\because \sin x = 0.5100$

$\rightarrow x = \sin^{-1}(0.5100)$

$x = 30°40'$

Exercise 12.1

Q1. Find the values of:

Solution:-

i) $\sin 53°40' = 0.8056$

ii) $\cos 36°20' = 0.8056$

iii) $\tan 19°30' = 0.3541$

iv) $\cot 33°50' = 1.4920$

v) $\cos 42°38' = 0.7357$

vi) $\tan 25°34' = 0.4784$

vii) $\sin 18°31' = 0.3176$

viii) $\cos 52^\circ 13' = 0.6127$

ix) $\cot 89^\circ 9' = 0.0149$

Q2. Find θ, if:

Solution:-

i) $\sin\theta = 0.5791$

$\rightarrow \theta = \sin^{-1}(0.5791) = 35^\circ 23'$

ii) $\cos\theta = 0.9316$

$\rightarrow \theta = \cos^{-1}(0.9316) = 21^\circ 91'$

iii) $\cos\theta = 0.5257$

$\rightarrow \theta = \cos^{-1}(0.5257) = 58^\circ 17'$

iv) $\tan\theta = 1.705$

$\rightarrow \theta = \tan^{-1}(1.705) = 59^\circ 36'$

v) $\tan\theta = 21.943$

$\rightarrow \theta = \tan^{-1}(21.943) = 87^\circ 23'$

vi) $\sin\theta = 0.5186$

$\rightarrow \theta = \sin^{-1}(0.5186) = 31^\circ 14'$

Solution of Right Triangles

To solve a right triangle, we have to find:

i) the measures of two acute angles

and ii) the lengths of the three sides

Note:- A trigonometric ratio of an acute angle of a right triangle involves 3 quantities "lengths of two sides and measure of an angle". Thus if two out of these three quantities are known, we can find the third quantity.

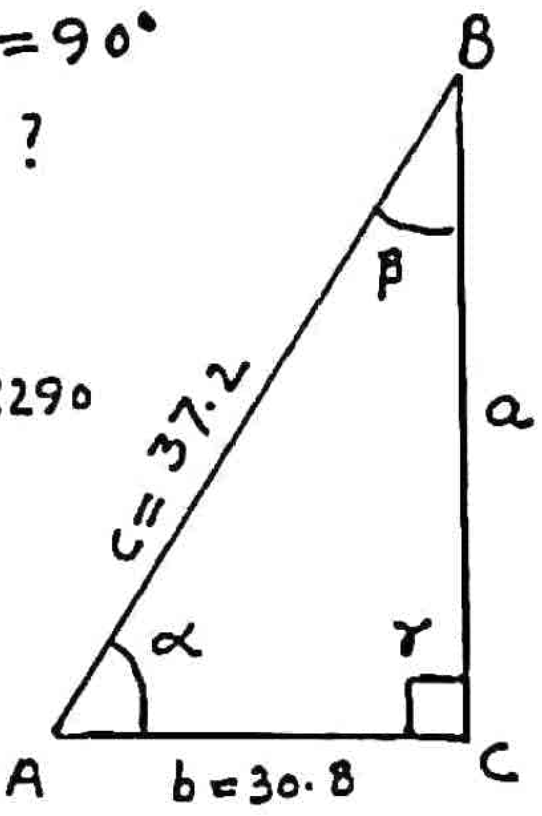
Case I. When Measures of Two Sides are Given

Example 1. Solve the right triangle ABC, in which $b = 30.8$, $c = 37.2$ and $\gamma = 90^\circ$.

Solution:-

$b = 30.8$, $c = 37.2$, $\gamma = 90^\circ$

$a = ?$, $\beta = ?$ $\alpha = ?$



From fig

$\cos\alpha = \frac{b}{c} = \frac{30.8}{37.2} = 0.8290$

$\rightarrow \alpha = \cos^{-1}(0.8290)$

$\alpha = 34^\circ 6'$

$\because \alpha + \beta + \gamma = 180^\circ$

$\rightarrow \beta = 180^\circ - \alpha - \gamma$

$\beta = 180^\circ - 34^\circ 6' - 90^\circ$

$\rightarrow \beta = 55^\circ 54'$

$\because \sin\alpha = \frac{a}{c} \rightarrow a = c\sin\alpha$

$\rightarrow a = 37.2 \sin 34^\circ 6'$

$a = (37.2)(0.5606) = 20.855$

Hence $a = 20.855$, $\alpha = 34^\circ 6'$, $\beta = 55^\circ 54'$

Case II. When Measures of One Side and One Angle are Given

Example 2. Solve the right triangle, in which $\alpha = 58^\circ 13'$, $b = 125.7$ and $\gamma = 90^\circ$

Solution:-

$\alpha = 58^\circ 13'$, $b = 125.7$, $\gamma = 90^\circ$

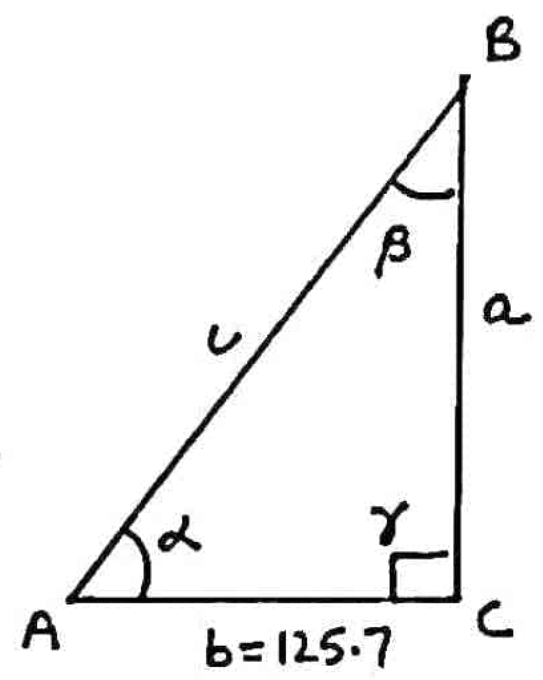
$\beta = ?$, $a = ?$, $c = ?$

$\because \alpha + \beta + \gamma = 180^\circ$

$\rightarrow \beta = 180^\circ - \alpha - \gamma = 180^\circ - 58^\circ 13' - 90^\circ$

$\rightarrow \beta = 31^\circ 47'$

from fig.,



$\tan\alpha = \frac{a}{b}$

$\rightarrow a = b\tan\alpha$

$= (125.7)\tan 58^\circ 13'$

$= (125.7)(1.6139)$

$a = 202.865$

and $\sin\alpha = \frac{a}{c} \rightarrow c = \frac{a}{\sin\alpha} = \frac{202.865}{\sin 58^\circ 13'}$

$\rightarrow c = \frac{202.865}{0.8501} = 238.7$

Hence $\beta = 31^\circ 47'$, $a = 202.865$

$c = 238.7$

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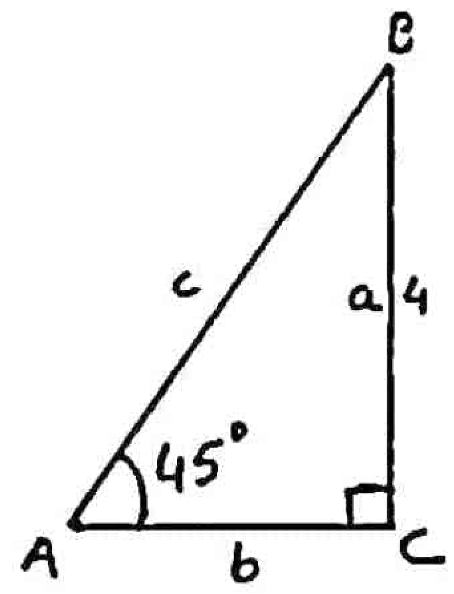
Exercise 12.2

Q1. Find the unknown angles and sides of the following triangles:

Solution:- (i) Here

$a=4,\ \alpha=45^\circ,\ \gamma=90^\circ$

$b=?\ \ c=?\ ,\ \ \beta=?$



$\because \alpha+\beta+\gamma=180^\circ$

$\rightarrow \beta=180^\circ-\alpha-\gamma$

$=180^\circ-45^\circ-90^\circ$

$\rightarrow \beta=45^\circ$

$\because \tan\alpha=\frac{a}{b} \rightarrow b=\frac{a}{\tan\alpha}=\frac{4}{\tan 45^\circ}$

$\rightarrow b=\frac{4}{1} \rightarrow b=4$

$\because \cos\alpha=\frac{b}{c} \rightarrow c=\frac{b}{\cos\alpha}=\frac{4}{\cos 45^\circ}$

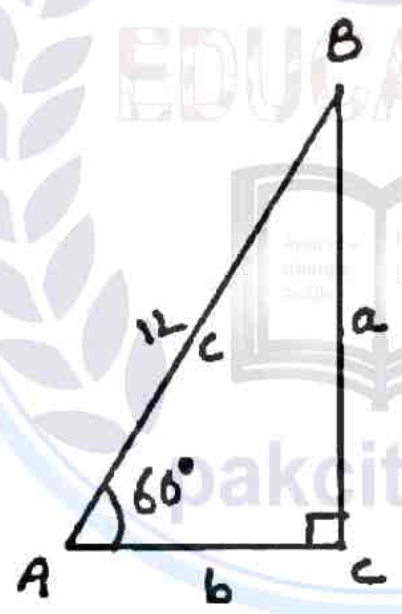
$\rightarrow c=\frac{4}{1/\sqrt{2}}=4\sqrt{2}$

Hence $b=4,\ \ c=4\sqrt{2},\ \ \beta=45^\circ$

(ii) Here,

$\alpha=60^\circ,\ \gamma=90^\circ,\ c=12$

$\beta=?\ ,\ a=?\ ,\ b=?$



$\because \alpha+\beta+\gamma=180^\circ$

$\rightarrow \beta=180^\circ-\alpha-\gamma$

$=180^\circ-60^\circ-90^\circ$

$\rightarrow \beta=30^\circ$

$\because \cos\alpha=\frac{b}{c} \rightarrow b=c\cos\alpha$

$\rightarrow b=(12)\cos 60^\circ=(12)(\frac{1}{2})=6$

$\because \sin\alpha=\frac{a}{c} \rightarrow a=c\sin\alpha$

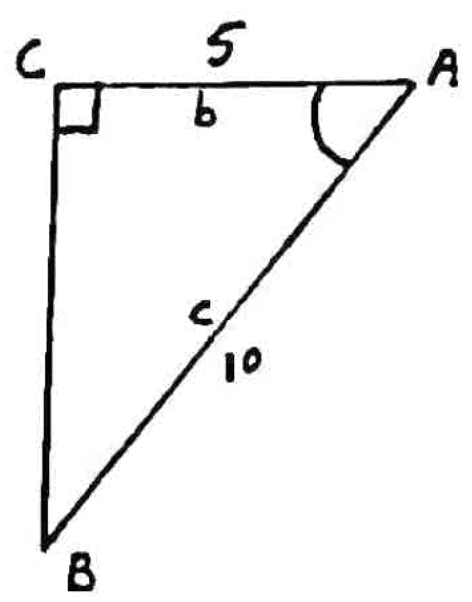
$\rightarrow a=12\sin 60^\circ=12(\frac{\sqrt{3}}{2})=6\sqrt{3}$

Hence $a=6\sqrt{3}\ ,\ b=6,\ \beta=30^\circ$

(iii) Here,

$b=5,\ c=10\ ,\ \gamma=90^\circ$

$\alpha=?\ ,\ \beta=?\ ,\ a=?$



$\because \cos\alpha=\frac{b}{c}=\frac{5}{10}=\frac{1}{2}$

$\rightarrow \alpha=\cos^{-1}(\frac{1}{2})$

$\alpha=60^\circ$

$\because \sin\alpha=\frac{a}{c} \rightarrow a=c\sin\alpha$

$\rightarrow a=10\sin 60^\circ=10(\frac{\sqrt{3}}{2})=5\sqrt{3}$

$\because \alpha+\beta+\gamma=180^\circ$

$\rightarrow \beta=180^\circ-\alpha-\gamma$

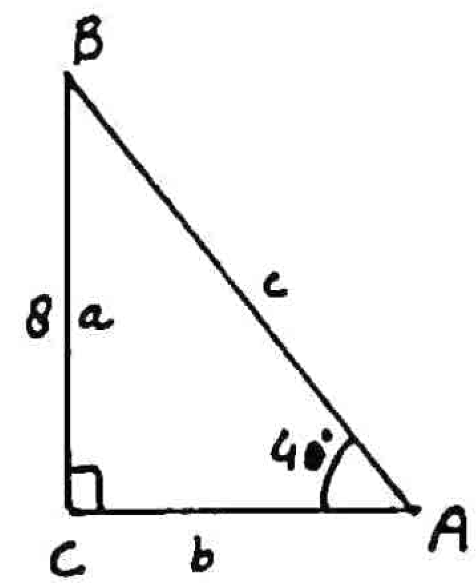
$\beta=180^\circ-60^\circ-90^\circ=30^\circ$

Hence $\alpha=60^\circ,\ \beta=30^\circ,\ a=5\sqrt{3}$

(iv) Here

$a=8,\ \alpha=40^\circ,\ \gamma=90^\circ$

$b=?\ ,\ c=?\ ,\ \beta=?$



$\because \alpha+\beta+\gamma=180^\circ$

$\rightarrow \beta=180^\circ-\alpha-\gamma$

$\beta=180^\circ-40^\circ-90^\circ$

$\rightarrow \beta=50^\circ$

$\because \sin\alpha=\frac{a}{c} \rightarrow c=\frac{a}{\sin\alpha}=\frac{8}{\sin 40^\circ}$

$\rightarrow c=\frac{8}{0.643}=12.44$

$\because \cos\alpha=\frac{b}{c} \rightarrow b=c\cos\alpha$

$\rightarrow b=12.44\cos 40^\circ=(12.44)(0.766)$

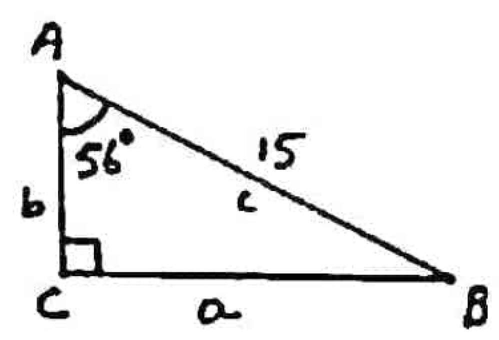
$\rightarrow b=9.529$

Hence $b=9.529,\ c=12.44\ ,\ \beta=50^\circ$

(v) Here

$c=15,\ \alpha=56^\circ,\ \gamma=90^\circ$

$a=?\ ,\ \beta=?\ ,\ b=?$



$\because \alpha+\beta+\gamma=180^\circ$

$\rightarrow \beta=180^\circ-\alpha-\gamma$

$=180^\circ-56^\circ-90^\circ$

$\rightarrow \beta=34^\circ$

$\because \sin\alpha=\frac{a}{c} \rightarrow a=c\sin\alpha$

$\rightarrow a=(15)\sin 56^\circ=(15)(0.829)=12.435$

$\because \cos\alpha=\frac{b}{c} \rightarrow b=c\cos\alpha$

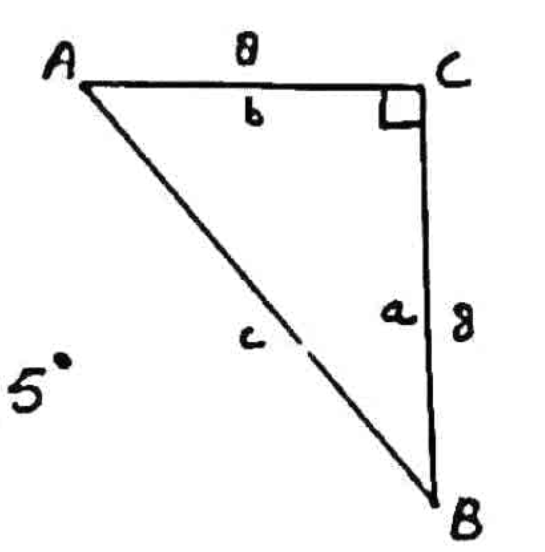
$\rightarrow b=15\cos 56^\circ=(15)(0.559)=8.4$

Hence, $b=8.4,\ a=12.435,\ \beta=34^\circ$

(vi) Here,

$a=8,\ \ b=8\ ,\ \gamma=90^\circ$

$c=?\ ,\beta=?\ ,\alpha=?$



$\because \tan\alpha=\frac{b}{a}=\frac{8}{8}=1$

$\rightarrow \alpha=\tan^{-1}(1)=45^\circ$

$\because \sin\alpha = \frac{a}{c} \rightarrow c = \frac{a}{\sin\alpha} = \frac{8}{\sin 45^\circ}$

$\rightarrow c = \frac{8}{1/\sqrt{2}} = 8\sqrt{2}$

$\because \alpha+\beta+\gamma = 180^\circ \rightarrow \beta = 180^\circ - \alpha - \gamma$

$\rightarrow \beta = 180^\circ - 45^\circ - 90^\circ = 45^\circ$

Hence, $c = 8\sqrt{2}$, $\alpha = 45^\circ$, $\gamma = 45^\circ$

Solve the right triangle ABC, in which $\gamma = 90^\circ$

Q2. $\alpha = 37^\circ 20'$, $a = 243$

Solution:- Here,

$\alpha = 37^\circ 20'$, $a = 243$, $\gamma = 90^\circ$

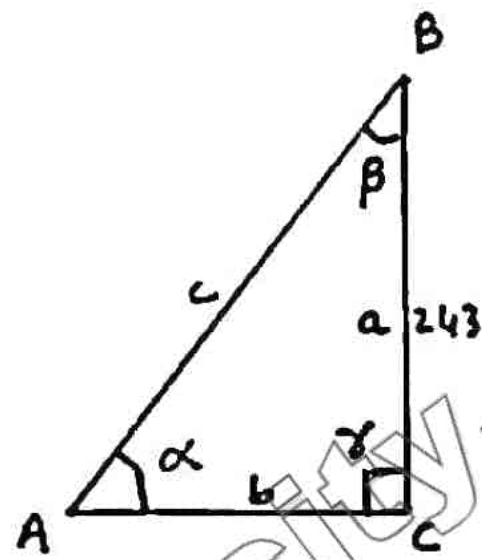
$\beta = ?$, $b = ?$, $c = ?$

$\because \alpha+\beta+\gamma = 180^\circ$

$\rightarrow \beta = 180^\circ - \alpha - \gamma$

$= 180^\circ - 37^\circ 20' - 90^\circ$

$\rightarrow \beta = 52^\circ 40'$



$\because \sin\alpha = \frac{a}{c} \rightarrow c = \frac{a}{\sin\alpha} = \frac{243}{\sin 37^\circ 20'}$

$\rightarrow c = \frac{243}{0.606} = 401$

$\because \frac{b}{c} = \cos\alpha \rightarrow b = c\cos\alpha$

$\rightarrow b = 401\cos 37^\circ 20' = (401)(0.795)$

$\rightarrow b = 318.795$

Hence, $c = 401$, $b = 318.795$, $\beta = 52^\circ 40'$

Q3. $\alpha = 62^\circ 40'$, $b = 796$

Solution:- Here

$\alpha = 62^\circ 40'$, $b = 796$, $\gamma = 90^\circ$

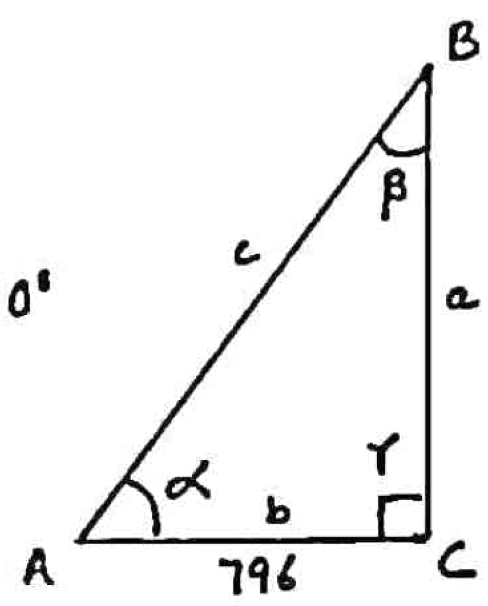
$\beta = ?$, $a = ?$, $c = ?$

$\because \alpha+\beta+\gamma = 180^\circ$

$\rightarrow \beta = 180^\circ - \alpha - \gamma$

$= 180^\circ - 62^\circ 40' - 90^\circ$

$\beta = 27^\circ 20'$



$\because \cos\alpha = \frac{b}{c} \rightarrow c = \frac{b}{\cos\alpha}$

$\rightarrow c = \frac{796}{\cos 62^\circ 40'} = \frac{796}{0.459} = 1734$

$\because \sin\alpha = \frac{a}{c} \rightarrow a = c\sin\alpha$

$\rightarrow a = (1734)\sin 62^\circ 40'$

$\rightarrow a = (1734)(0.888) = 1540$

Hence, $a = 1540$, $c = 1734$, $\beta = 27^\circ 20'$

Q4. $a = 3.28$, $b = 5.74$

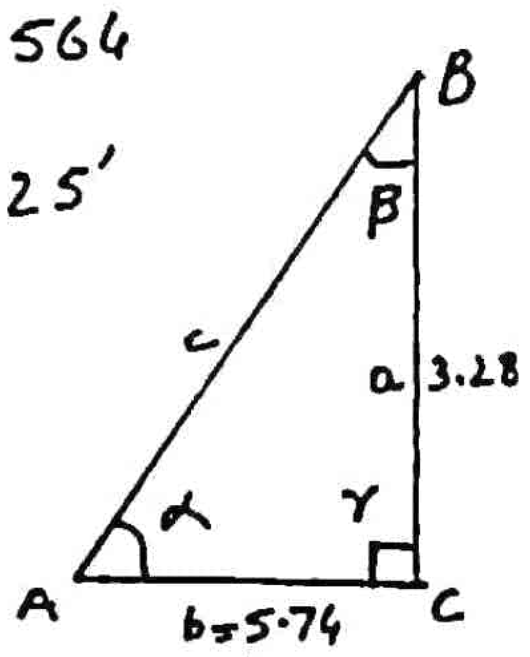
Solution:- Here

$a = 3.28$, $b = 5.74$, $\gamma = 90^\circ$

$c = ?$, $\beta = ?$, $\alpha = ?$

$\because \tan\alpha = \frac{a}{b} = \frac{3.28}{5.74} = 0.564$

$\rightarrow \alpha = \tan^{-1}(0.564) = 29^\circ 25'$



$\because \sin\alpha = \frac{a}{c} \rightarrow c = \frac{a}{\sin\alpha}$

$\rightarrow c = \frac{3.28}{\sin 29^\circ 25'} = \frac{3.28}{0.49}$

$\rightarrow c = 6.69$

$\because \alpha+\beta+\gamma = 180^\circ$

$\rightarrow \beta = 180^\circ - \alpha - \gamma = 180^\circ - 29^\circ 25' - 90^\circ$

$\rightarrow \beta = 60^\circ 35'$

Hence $c = 6.69$, $\beta = 60^\circ 35'$, $\alpha = 29^\circ 25'$

Q5. $b = 68.4$, $c = 96.2$

Solution:- Here,

$b = 68.4$, $c = 96.2$, $\gamma = 90^\circ$

$a = ?$, $\alpha = ?$, $\beta = ?$

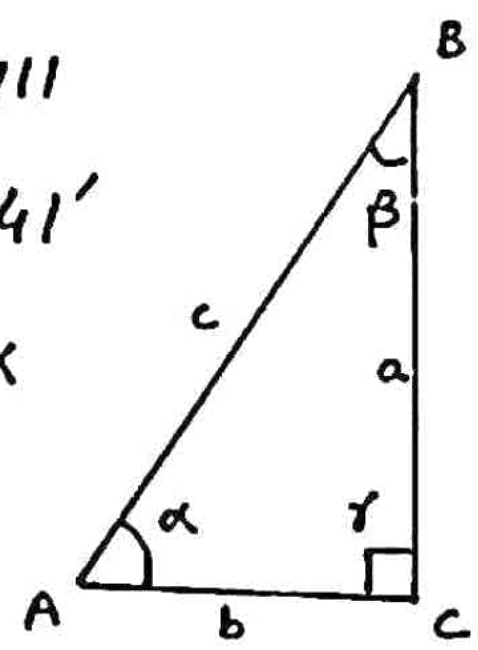
$\because \cos\alpha = \frac{b}{c} = \frac{68.4}{96.2} = 0.711$

$\rightarrow \alpha = \cos^{-1}(0.711) = 44^\circ 41'$

$\because \sin\alpha = \frac{a}{c} \rightarrow a = c\sin\alpha$

$\rightarrow a = (96.2)\sin 44^\circ 41'$

$a = (96.2)(0.703) = 67.6$



$\because \alpha+\beta+\gamma = 180^\circ$

$\rightarrow \beta = 180^\circ - \alpha - \gamma = 180^\circ - 44^\circ 41' - 90^\circ$

$\rightarrow \beta = 45^\circ 19'$

Hence $a = 67.6$, $\alpha = 44^\circ 41'$, $\beta = 45^\circ 19'$

Q6. $a = 5429$, $c = 6294$

Solution:- Here

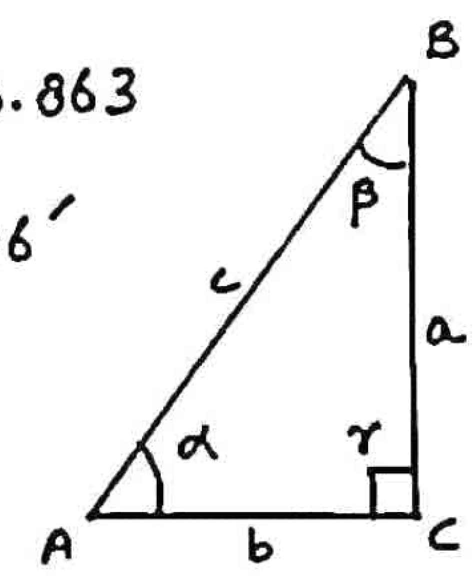
$a = 5429$, $c = 6294$, $\gamma = 90^\circ$

$b = ?$, $\alpha = ?$, $\beta = ?$

$\because \sin\alpha = \frac{a}{c} = \frac{5429}{6294} = 0.863$

$\rightarrow \alpha = \sin^{-1}(0.863) = 59^\circ 36'$

$\because \cos\alpha = \frac{b}{c}$



$\rightarrow b = c\cos\alpha = 6294\cos 59°36'$

$\rightarrow b = (6294)(0.506) = 3184$

$\because \alpha + \beta + \gamma = 180°$

$\rightarrow \beta = 180° - \alpha - \gamma = 180° - 59°36' - 90°$

$\rightarrow \beta = 30°24'$

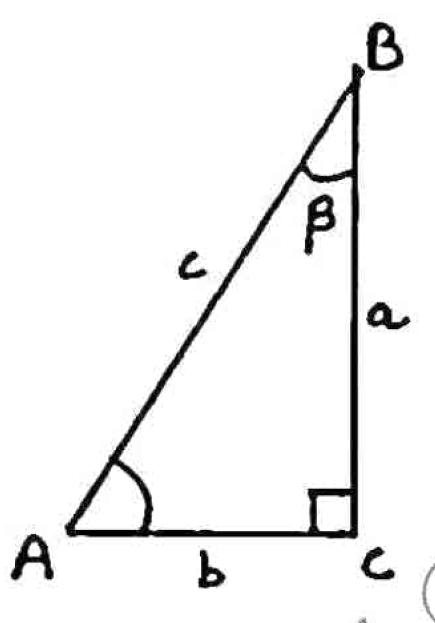
Hence $b = 3184$, $\alpha = 59°36'$, $\beta = 30°24'$

Q7. $\beta = 50°10'$, $c = 0.832$

Solution:- Here

$\beta = 50°10'$, $c = 0.832$, $\gamma = 90°$

$a = ?$, $b = ?$, $\alpha = ?$



$\because \alpha + \beta + \gamma = 180°$

$\rightarrow \alpha = 180° - \beta - \gamma$

$= 180° - 50°10' - 90°$

$\alpha = 39°50'$

$\because \cos\alpha = \frac{b}{c} \rightarrow b = c\cos\alpha$

$\rightarrow b = (0.832)\cos 39°50'$

$b = (0.832)(0.767) = 0.638$

$\because \sin\alpha = \frac{a}{c} \rightarrow a = c\sin\alpha$

$\rightarrow a = (0.832)\sin 39°50'$

$a = (0.832)(0.64) = 0.533$

Hence, $a = 0.533$, $b = 0.638$, $\alpha = 39°50'$

Hights and Distances

To find heights and distances, the following procedure is adopted:

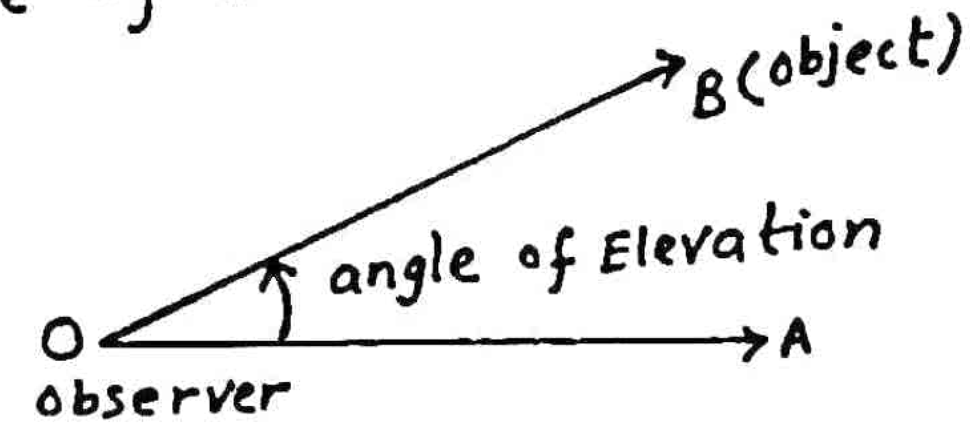
1) Construct a clear labelled diagram, showing the known measurements.
2) Establish the relationships between the quantities in the diagram to form equations containing trigonometric ratios.
3) Use tables/calculator to find the solution.

Angles of Elevation and Depression

Angle of Elevation:-

when an object is at higher level from the observers' eye then the angle made by observer's eye is called angle of elevation.

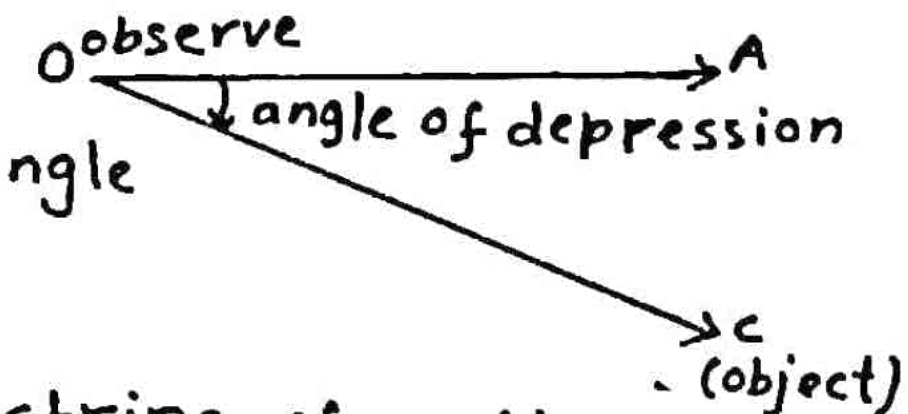
In fig., $m\angle AOB$ is angle of elevation.



Angle of Depression:-

when an object is at lower level from the observer's eye then the angle made by observer's eye is called angle of depression.

In fig., $m\angle AOC$ is angle of depression.



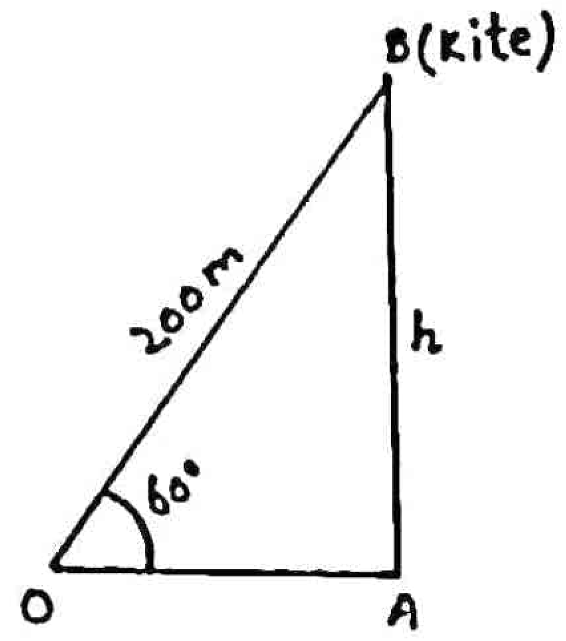
Example 1. A string of a flying kite is 200 meters long, and its angle of elevation is 60°. Find the height of the kite above the ground taking the string to be streches.

Solution:- Let

O = position of observer

B = position of kite

AB = height of kite = h



In fig OAB,

$\sin 60° = \frac{h}{200}$

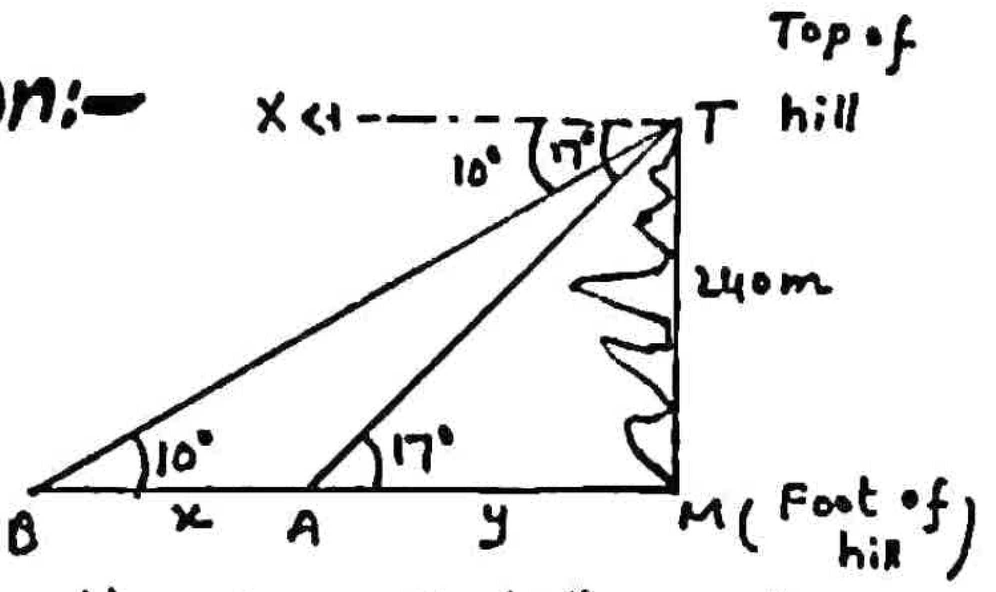
$\rightarrow h = 200\sin 60°$

$h = (200)(0.866) = 173.2m$

Thus height of kite is 173.2m

Example 2. A surveyor stands on the top of 240m high hill by the side of a lake. He observes two boats at the angles of depression of measures 17° and 0°. If the boats are the same straight line with the foot of the hill just below the observer. Find the distance between the two boats, if they are on the same side of the hill.

Solution:-



Let T be the top of hill and M be the foot of hill. Let A and B be the positions of two boats. so

$m\angle TBM = m\angle XTB = 10^\circ$

$m\angle TAM = m\angle XTA = 17^\circ$

$TM = 240m$

Let $BA = x$, $AM = y$

In $\triangle TAM$, $\tan 17^\circ = \frac{TM}{AM}$

$\rightarrow \tan 17^\circ = \frac{240}{y} \rightarrow y = \frac{240}{\tan 17^\circ}$

$\rightarrow y = \frac{240}{0.3057} = 785m$

In $\triangle TBM$, $\tan 10^\circ = \frac{TM}{BM}$

$\rightarrow \tan 10^\circ = \frac{240}{x+y}$

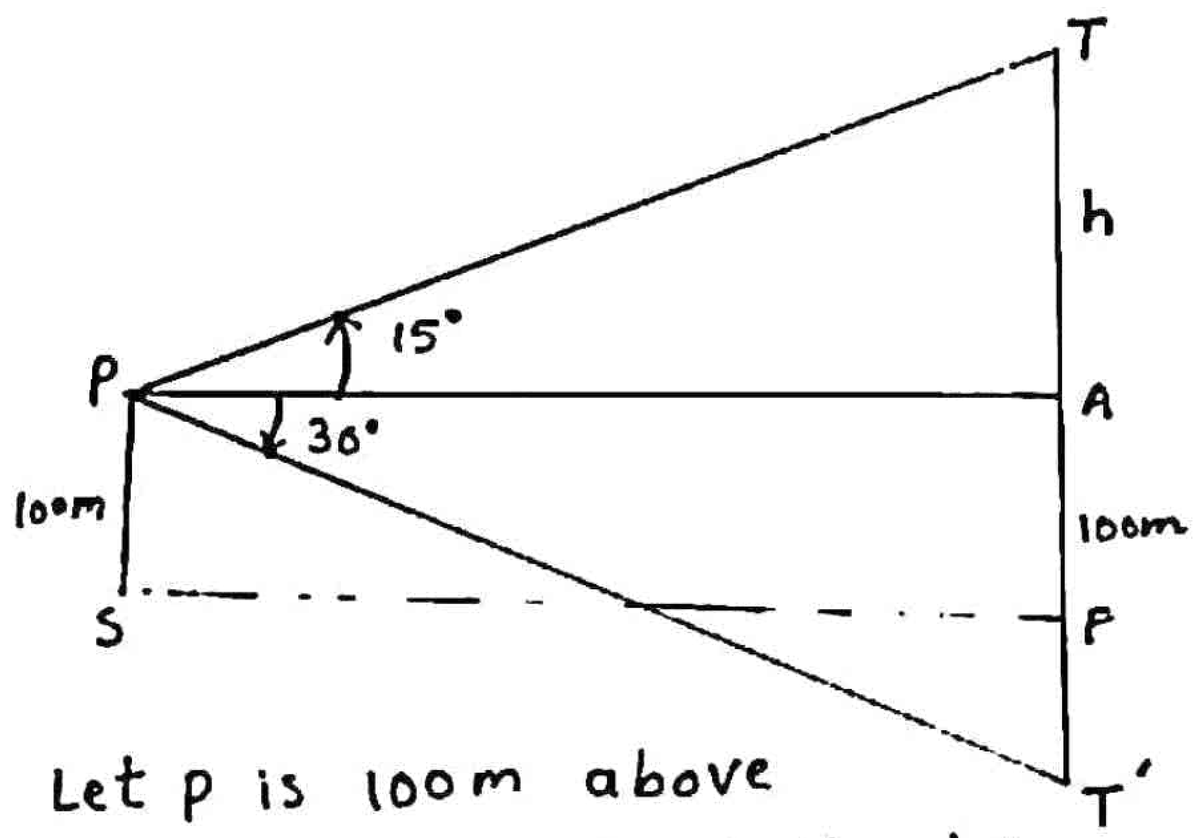
$\rightarrow x+y = \frac{240}{\tan 10^\circ} = \frac{240}{0.1763}$

$\rightarrow x + 785 = 1361.10$

$\rightarrow x = 1361.10 - 785 = 576.10m$

Example3. From a point 100m above the surface of a lake, the angle of elevation of lake of a cliff is found to be 15° and the angle of depression of the image of the lake is 30°. Find the height of the peak.

Solution:-



Let P is 100m above from surface S of the lake. Let T be top and F be the foot of cliff, also T' be image of top(peak) of the cliff. Now

In fig

$m\angle TPA = 15^\circ$, $m\angle T'PA = 30^\circ$

$PS = AF = 100m$, Let $TA = h$

$PA = x$

In $\triangle PTA$, $\tan 15^\circ = \frac{TA}{PA} = \frac{h}{x}$

In $\triangle PT'A$, $\tan 30^\circ = \frac{T'A}{PA}$

$\because T'A = T'F + AF$ $\quad \because T'F = TF = h+100$

$\rightarrow T'A = TF + AF$

$T'A = h+100+100 = h+200$

so $\tan 30^\circ = \frac{h+200}{x}$

$\tan 30^\circ = \frac{h}{x} + \frac{200}{x}$

$\rightarrow \tan 30^\circ = \tan 15^\circ + \frac{200}{x}$

$\rightarrow \frac{200}{x} = \tan 30^\circ - \tan 15^\circ$

$= 0.5774 - 0.2679$

$\frac{200}{x} = 0.3094$

$\rightarrow x = \frac{200}{0.3094} = 646.41$

$\because \frac{h}{x} = \tan 15^\circ \rightarrow h = x \tan 15^\circ$

$\rightarrow h = (646.41)(\tan 15^\circ)$

$h = 173.20$

so height of peak $= h + 100 = 173.20 + 100$

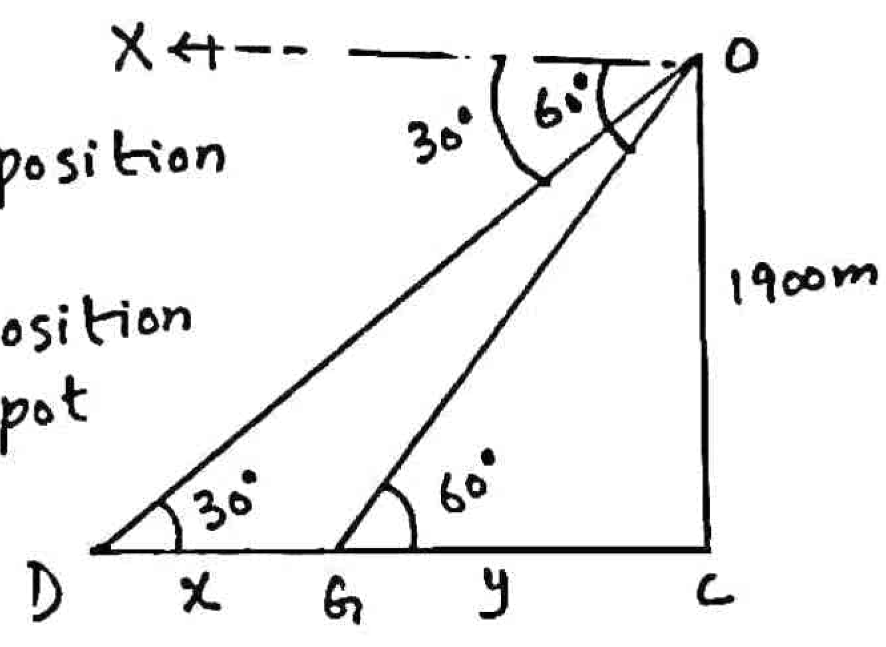
$= 273.20m$

Engineering and Hights and Distances

Example4. An O.P., sitting on a cliff 1900meters high, finds himself in the vertical plane with an anti-air-craft gun and an amunition depot of the enemy. Observes that the angles of depression of the gun and the depot are 60° and 30° resp. He passes this information on the headquaters. calculate the distance between the gun and depot.

Solution:-

Let O be the position of O.P., G and D be position of gun and depot resp.



In fig., $OC = 1900m$

$DG = x$, $GC = y$

$m\angle XOD = m\angle ODC = 30^\circ$

$m\angle XOG = m\angle OGC = 60^\circ$

In $\triangle OGC$, $\tan 60^\circ = \frac{OC}{GC} = \frac{1900}{y}$

$\rightarrow y = \frac{1900}{\tan 60^\circ} = 1096.97m$

In $\triangle ODC$, $\tan 30^\circ = \frac{OC}{DC}$

$\rightarrow \tan 30^\circ = \frac{1900}{x+y}$

$\rightarrow x+y = \frac{1900}{\tan 30^\circ} = 3290.89$

$\rightarrow x = 3290.89 - 1096.97$

$x = 2193.93m$

Thus distance b/w gun and depot is 2194m.

Exercise 12.3

Q1. A vertical pole is 8m high and the length of its shadow is 6m. What is the angle of elevation of the sun at that moment?

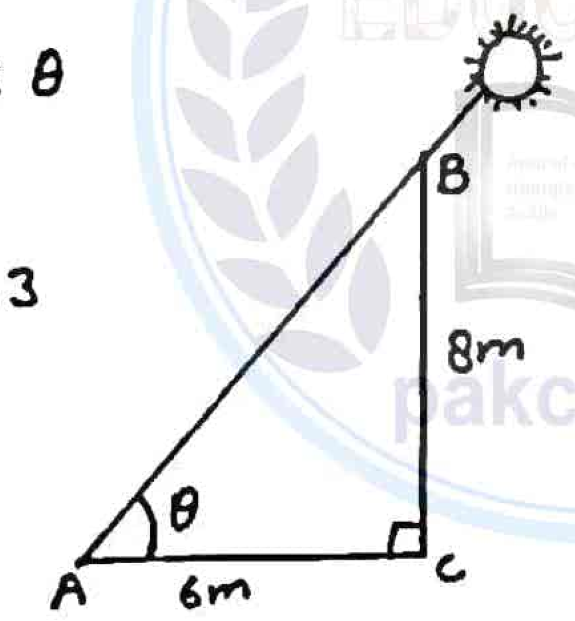
Solution:-

Let required angle is θ then

$\tan\theta = \frac{\overline{BC}}{\overline{AC}} = \frac{8}{6} = 1.33$

$\rightarrow \theta = \tan^{-1}(1.33)$

$\rightarrow \theta = 53^\circ 7' 48''$



Q2. A man 18dm tall observes that the angle of elevation of the top of a tree at a distance of 12m from him is 32°. What is height of the tree?

Solution:- Let $\overline{AE}$ be height of man and h be the height of tree.

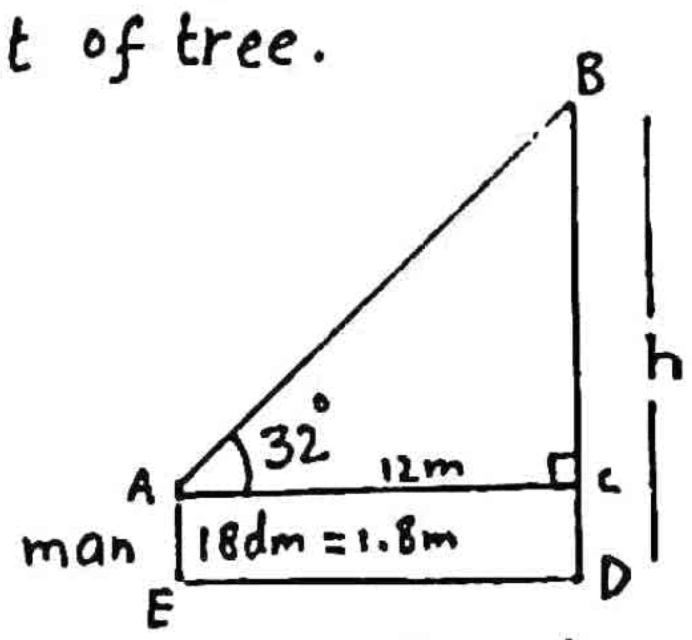
then $\tan\theta = \frac{\overline{BC}}{\overline{AC}}$

$\rightarrow \tan 32^\circ = \frac{\overline{BC}}{12}$

$\overline{BC} = 12(0.624)$

$\overline{BC} = 7.5m$

$\because h = \overline{BC} + \overline{CD}$



$\overline{AE} = \overline{CD} = 18dm = 1.8m$

$\rightarrow h = 7.5 + 1.8 = 9.3m$

Q3. At the top of a cliff 80m high, the angle of depression of a boat is 12°. How far is the boat from the cliff?

Solution:- Let x be required distance

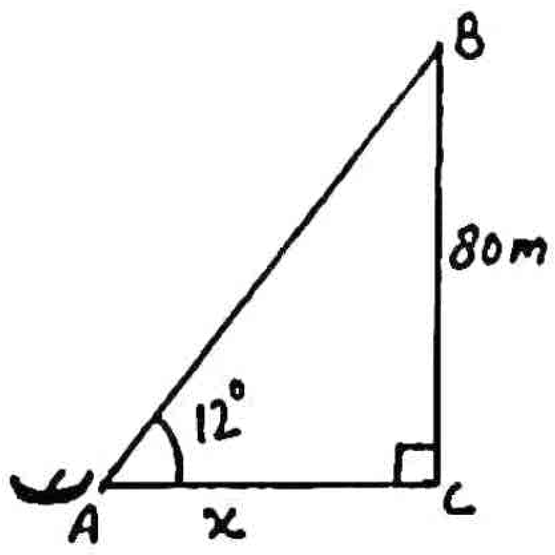
then $\tan\theta = \frac{\overline{BC}}{\overline{AC}}$

$\rightarrow \tan 12^\circ = \frac{80}{x}$

$\rightarrow x = \frac{80}{\tan 12^\circ}$

$= \frac{80}{0.2125}$

$x = 376.37m$



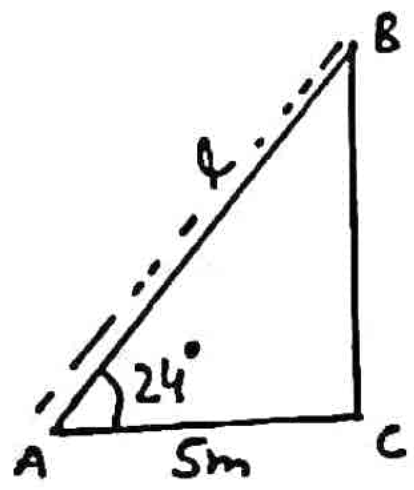
Q4. A ladder leaning against a vertical wall makes an angle of 24° with the wall, its foot is 5m from the wall. Find its length.

Solution:- Let l be the length of ladder then

$\cos\theta = \frac{\overline{BC}}{\overline{AC}}$

$\rightarrow \cos 24^\circ = \frac{5}{l}$

$\rightarrow l = \frac{5}{0.9135} = 5.47m$



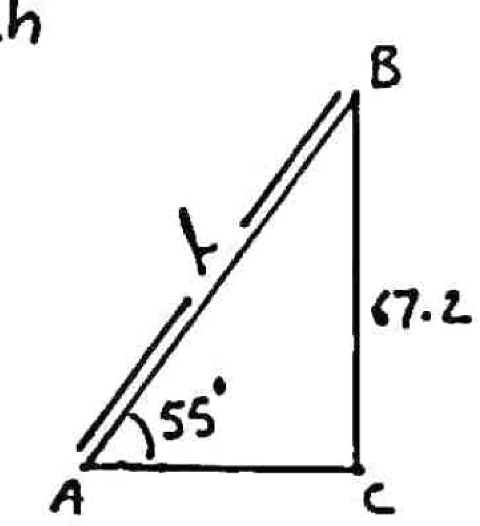
Q5. A kite flying at a height of 67.2m is attached to a fully stretched string inclined at an angle of 55° to the horizontal. Find the length of the string.

Solution:- Let l be length of string then

$\sin\theta = \frac{\overline{BC}}{\overline{AB}} = \frac{6.72}{l}$

$\rightarrow \sin 55^\circ = \frac{6.72}{l}$

$\rightarrow l = \frac{6.72}{\sin 55^\circ} = 82.03m$

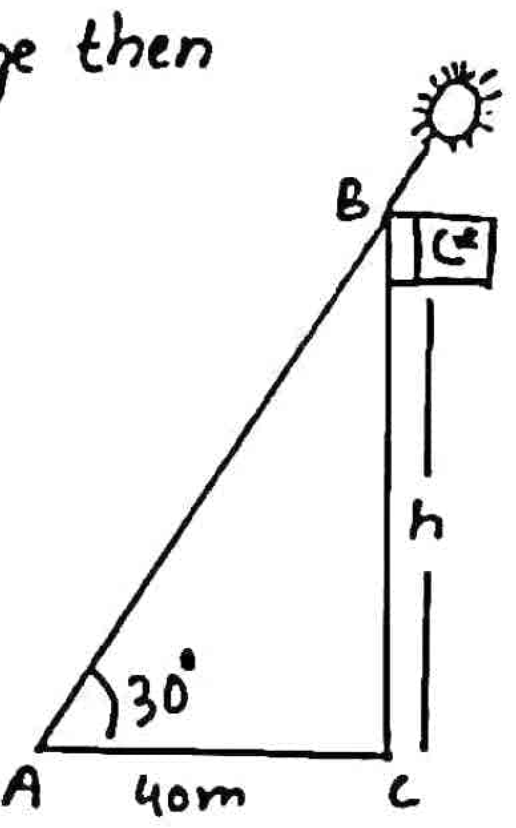


Q6. When the angle between the ground and the sun is 30°, flag pole casts a shadow of 40m long. Find the hight of the top of the flag.

Solution:-

Let h be height of flage then

$\tan\theta = \frac{\overline{BC}}{\overline{AC}} = \frac{h}{40}$



$\rightarrow \tan 30° = \frac{h}{40}$

$\rightarrow h = 40\tan 30°$

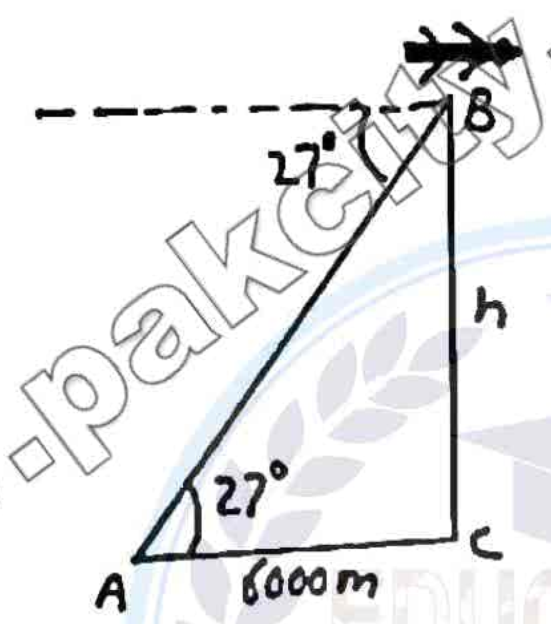
$= 40(0.577)$

$\rightarrow h = 23.1m$

Q7. A plane flying directly above a post 6000m. away from an anti-aircraft gum observes the gun at an angle of depression of 27°. Find the height of the plane.

Solution:- Let h be the height of plane then

$\tan\theta = \frac{\overline{BC}}{\overline{AC}}$



$\rightarrow \tan 27° = \frac{h}{6000}$

$\rightarrow h = 6000\tan 27°$

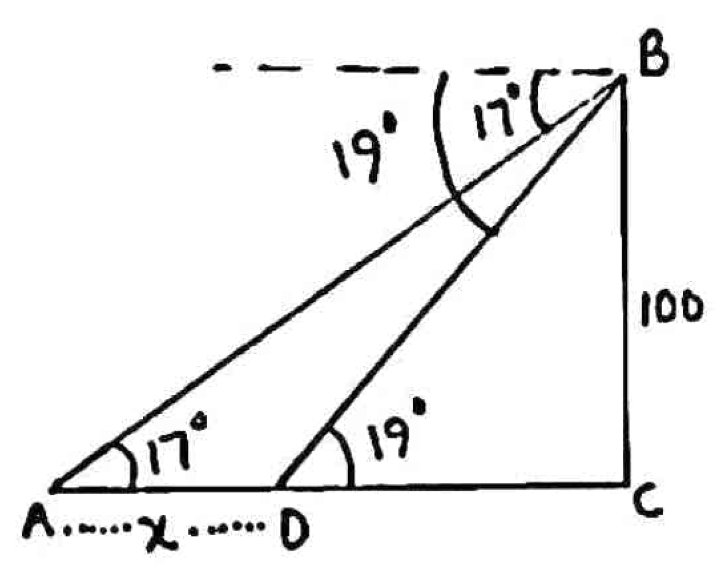
$= 6000(0.5095)$

$h = 3057.15$

Q8. A man on the top of a 100m high listed-house is in line with two ships on the same side of it, whose angle of depression from the man are 17° and 19° respecting. Find the distance between the ships.

Solution:- Let distance between two ships is x then

$\tan\theta = \frac{\overline{BC}}{\overline{AC}}$



$\rightarrow \tan 17° = \frac{100}{\overline{AC}}$

$\rightarrow \overline{AC} = \frac{100}{\tan 17°}$

$\rightarrow \overline{AC} = \frac{100}{0.3057} = 327.08$

$\tan 19° = \frac{\overline{BC}}{\overline{CD}} = \frac{100}{\overline{CD}}$

$\rightarrow \overline{CD} = \frac{100}{\tan 19°} = \frac{100}{0.3443} = 290.42$

$x = \overline{AD} = \overline{AC} - \overline{CD}$

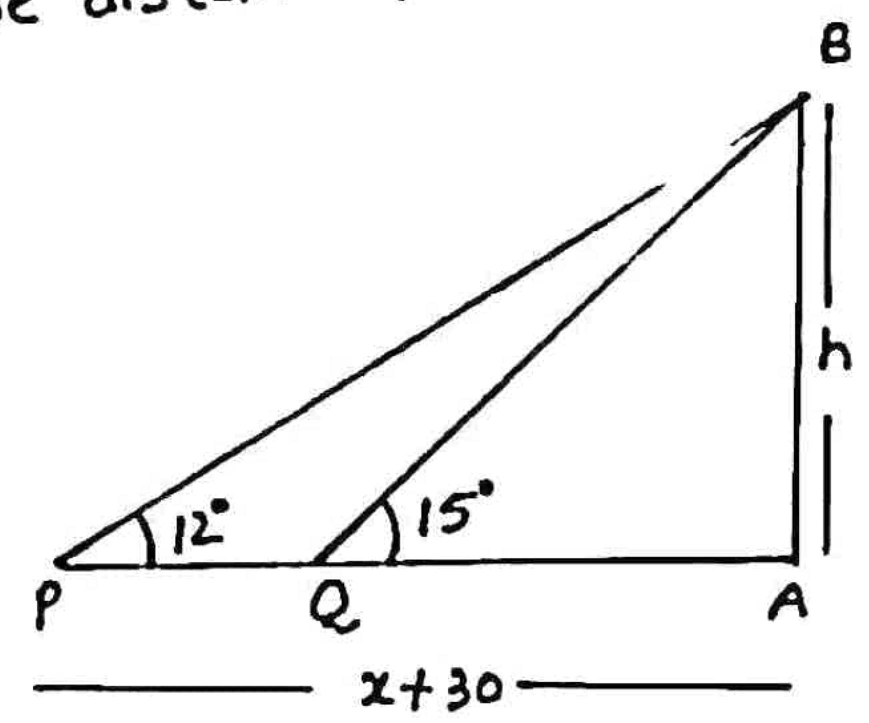
$= 327.08 - 290.42$

$\rightarrow x = 36.7m$

Q9. P and Q are two points in line with a tree. If the distance between P and Q be 30m and the angles of elevation of the top of the tree at P and Q be 12° and 15° resp. Find the height of the tree.

Solution:- Let h be the height of tree and x be distance from p to A then

$\tan\theta = \frac{\overline{AB}}{\overline{AP}}$



$\tan 12° = \frac{h}{x+30}$

$\rightarrow x+30 = \frac{h}{\tan 12°}$

$\rightarrow x = \frac{h}{\tan 12°} - 30 \longrightarrow (i)$

Now $\tan 15° = \frac{\overline{AB}}{\overline{AQ}} = \frac{h}{x}$

$\rightarrow x = \frac{h}{\tan 15°} = \frac{h}{0.2679}$

$\rightarrow x = \frac{h}{0.2679} \longrightarrow (ii)$

Comparing (i) and (ii)

$\frac{h}{0.2679} = \frac{h}{0.2125} - 30$

$\rightarrow \frac{h}{0.2125} - \frac{h}{0.2679} = 30$

$\rightarrow h\left(\frac{1}{0.2125} - \frac{1}{0.2679}\right) = 30$

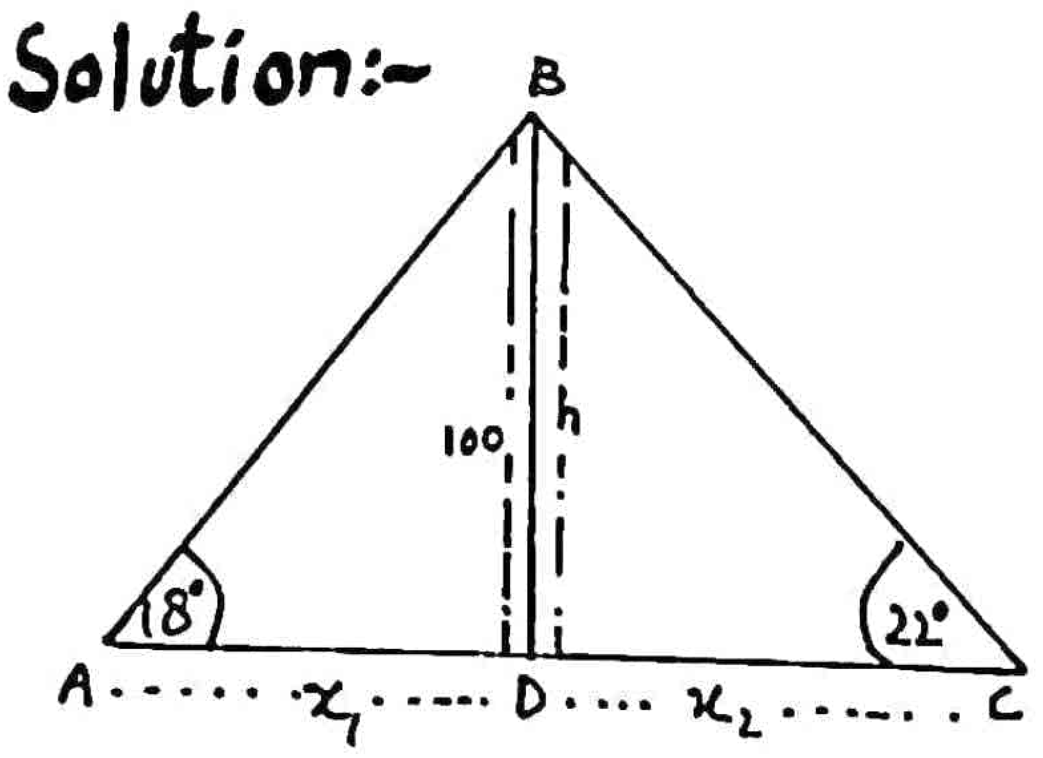
$\rightarrow h(4.7058 - 3.7327) = 30$

$(0.9730)h = 30 \rightarrow h = \frac{30}{0.9730}$

$\rightarrow h = 30.9m$

Q10. Two men are on the opposite sides of a 100m high tower. If the measure of the angles of elevation of the top of the tower are 18° and 22° resp. Find the distance between them.

Solution:-



Let distance b/w A and D is x_1 and distance b/w D and C is x_2 also h is height.

$\tan\theta = \frac{\overline{BD}}{\overline{AD}} \rightarrow \tan 18° = \frac{100}{x_1}$

$\rightarrow x_1 = \frac{100}{\tan 18°} \rightarrow x_1 = 307.76$

Also $\tan 22° = \frac{\overline{BD}}{\overline{DC}} = \frac{100}{x_2}$

$\rightarrow x_2 = \frac{100}{\tan 22°} = \frac{100}{0.4040}$

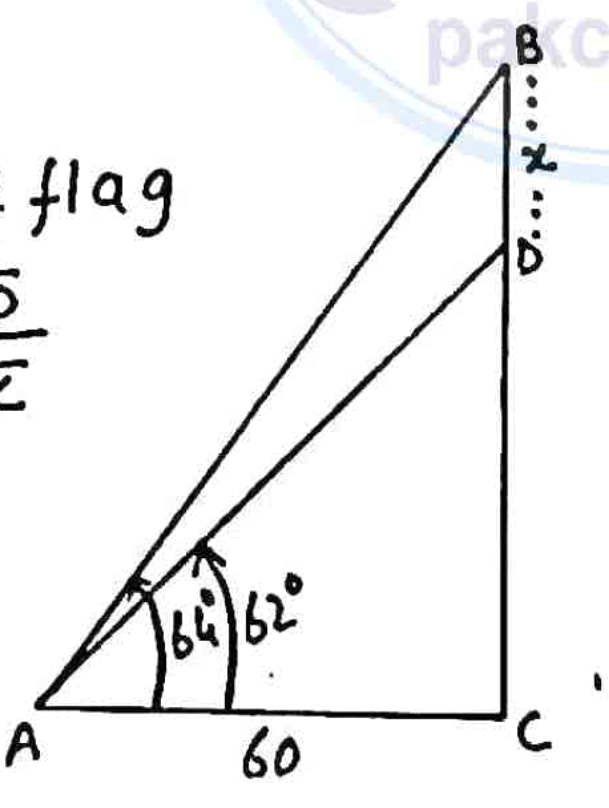
$\rightarrow x_2 = 247.5$

Req. distance $= x_1 + x_2 = 307.76 + 247.5$

$= 555.26m$

Q11. A man standing 60m away from a tower notices that the angles of the elevation of the top and the bottom of a flag staff on the top of the tower are 64° and 62° respectively. Find the length of the flag staff.

Solution:-



Let x be the length of flag staff then $\tan\theta = \frac{\overline{CD}}{\overline{AC}}$

$\rightarrow \tan 62° = \frac{\overline{CD}}{60}$

$\rightarrow \overline{CD} = 60 \tan 62°$

$= 60(1.88)$

$\rightarrow \overline{CD} = 112.84$

Now $\tan 64° = \frac{\overline{BC}}{\overline{AC}} = \frac{\overline{BC}}{60}$

$\rightarrow \overline{BC} = 60 \tan 64°$

$= 60(2.0540)$

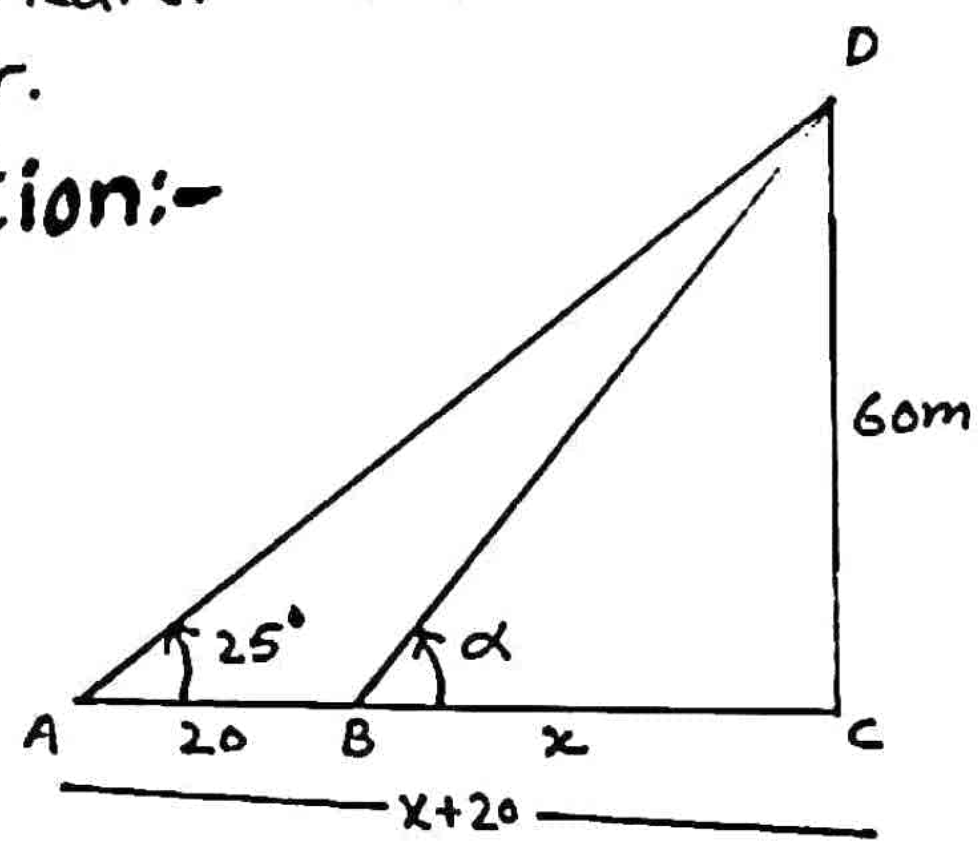
$\overline{BC} = 123.01$

So $x = \overline{BC} - \overline{CD} = 123.01 - 112.84$

$\rightarrow x = 10.17m$

Q12. The angle of elevation of the top of a 60m high tower from a point A, on the same level as the foot of the tower is 25°. Find the angle of the top of the tower from a point B, 20m nearer to A from the foot of the tower.

Solution:-



Let α be the required angle. Let x be the distance from B to C.

$\because \tan\theta = \frac{\overline{CD}}{\overline{AC}} \rightarrow \tan 25° = \frac{60}{x+20}$

$\rightarrow x+20 = \frac{60}{\tan 25°} = \frac{60}{0.4663}$

$\rightarrow x+20 = 128.67 \rightarrow x = 128.67 - 20$

$\rightarrow x = 108.67$

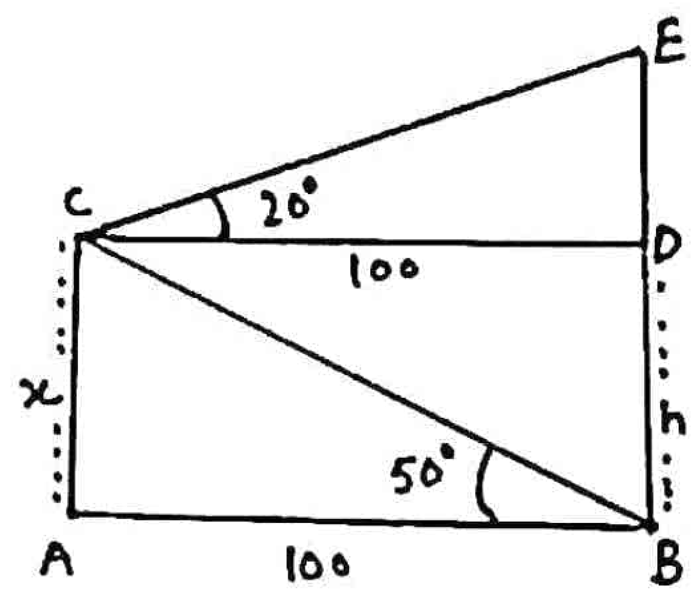
Now $\tan\alpha = \frac{\overline{CD}}{\overline{BC}} = \frac{60}{108.67}$

$\rightarrow \alpha = \tan^{-1}(0.5521) = 28°54'16''$

Q13. Two buildings A and B are 100m apart. The angle of elevation from the top of the building A to the top of the building B is 20°. the angle of elevation from the base of the building B to the top of the building A is 50°. find the height of the building B.

Solution:-

Let h be the height of building B and x be the height of A then



$\tan\theta = \frac{\overline{DE}}{\overline{CD}} \rightarrow \tan 20° = \frac{\overline{DE}}{100}$

$\rightarrow \overline{DE} = 100(0.3639)$

$\rightarrow \overline{DE} = 36.39$

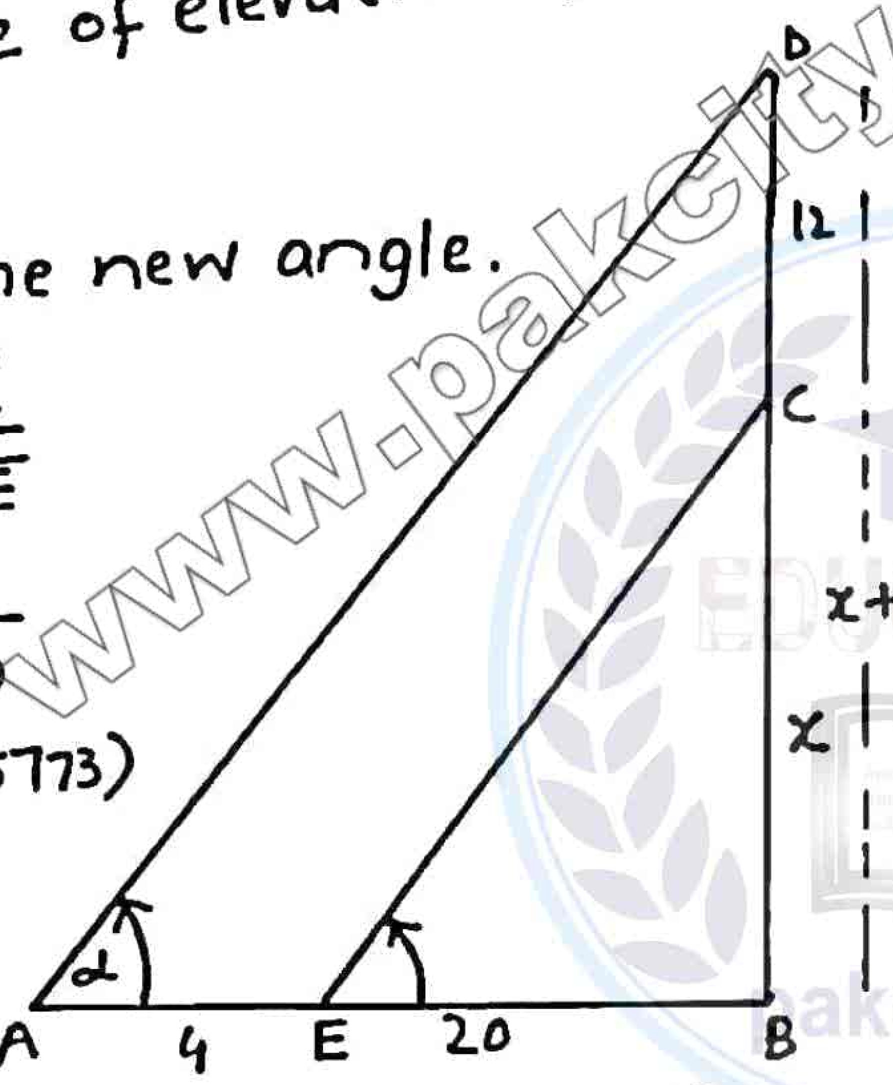
Now $\tan 50° = \frac{\overline{AC}}{100}$ $\quad \because \overline{AB} = 100$

$\rightarrow \overline{AC} = 100(1.1917) = 119.17$

$\because h = \overline{BD} + \overline{DE} = \overline{AC} + \overline{DE} = 119.17 + 36.39$

$h = 155.56m$

Q14. A window washer is working in a hotal building. An observer at a distance of 20m from the building finds the angle of elevation of the worker to be of 30°. The work climbs up 12m and the observer moves 4m farther away from the building. Find the new angle of elevation of the worker.



Solution:-

Let α be the new angle.

$\tan\theta = \frac{\overline{BC}}{\overline{BE}}$

$\tan 30° = \frac{x}{20}$

$\rightarrow x = 20(0.5773)$

$x = 11.54$

$\tan\alpha = \frac{\overline{BD}}{\overline{AB}} = \frac{x+12}{24} = \frac{11.54+12}{24}$

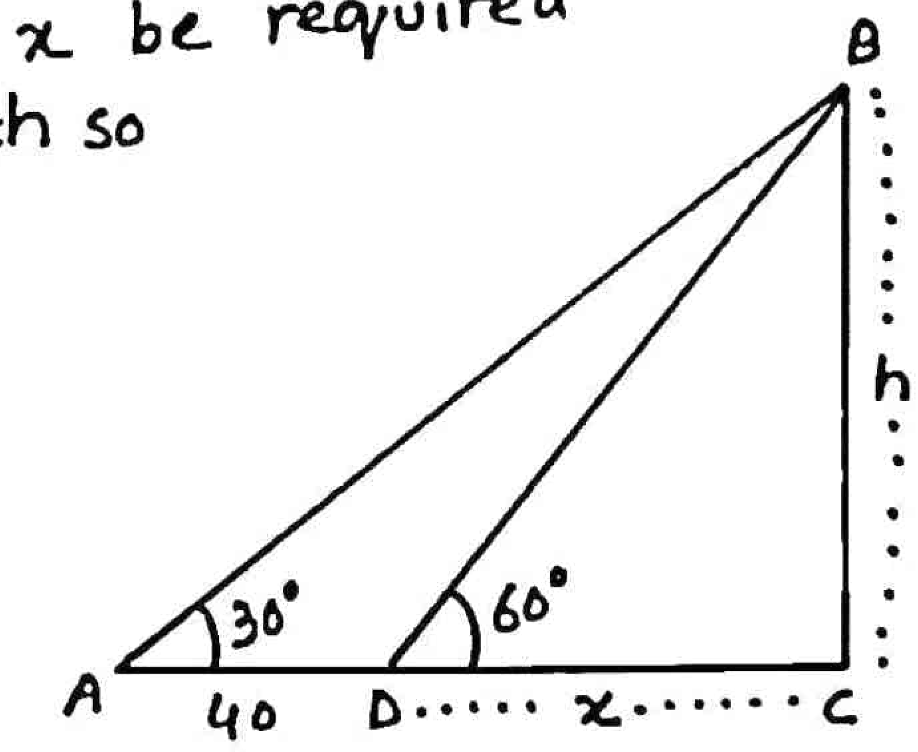
$\rightarrow \tan\alpha = \frac{23.54}{24} = 0.98$

$\rightarrow \alpha = \tan^{-1}(0.98) = 44°25'$

Q15. A man standing on the bank of canal observes that the measure of the angle of elevation of a tree on the other side of the canal, is 60°. on retreating 40 meters from the bank, he finds the measure of the angle of elevation of the tree as 30°. Find the height of the tree and the width of the canal.

Solution:-

Let h be required height and x be required width so



$\tan\theta = \frac{\overline{BC}}{\overline{CD}} \rightarrow \tan 60° = \frac{h}{x}$

$\rightarrow 1.7320 = \frac{h}{x}$

$\rightarrow h = 1.7320x \longrightarrow (i)$

Now $\tan 30° = \frac{\overline{BC}}{\overline{AC}}$

$\rightarrow 0.5773 = \frac{h}{x+40}$

$\rightarrow (x+40)\cdot(0.5773) = h$

$\rightarrow h = 0.5773x + 23.0940 \longrightarrow (ii)$

Comparing (i) and (ii)

$1.7320x = 0.5773x + 23.0940$

$\rightarrow 1.1547x = 23.0940$

$\rightarrow x = \frac{23.0940}{1.1547} = 20m$

$\rightarrow$ width $= x = 20m$ put in (i)

$\rightarrow h = 1.7320(20) = 34.64$

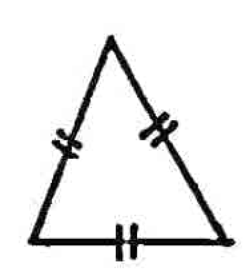
height $= 34.64m$

Oblique Triangles

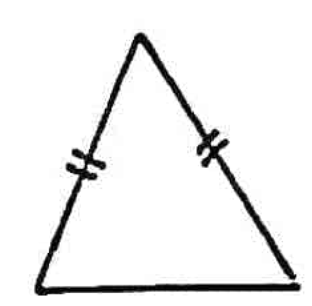
A triangle which is not right angled is called oblique triangle..

Examples:-

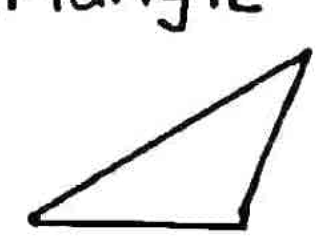
Equilateral triangle



Isosceles triangle



scalene triangle



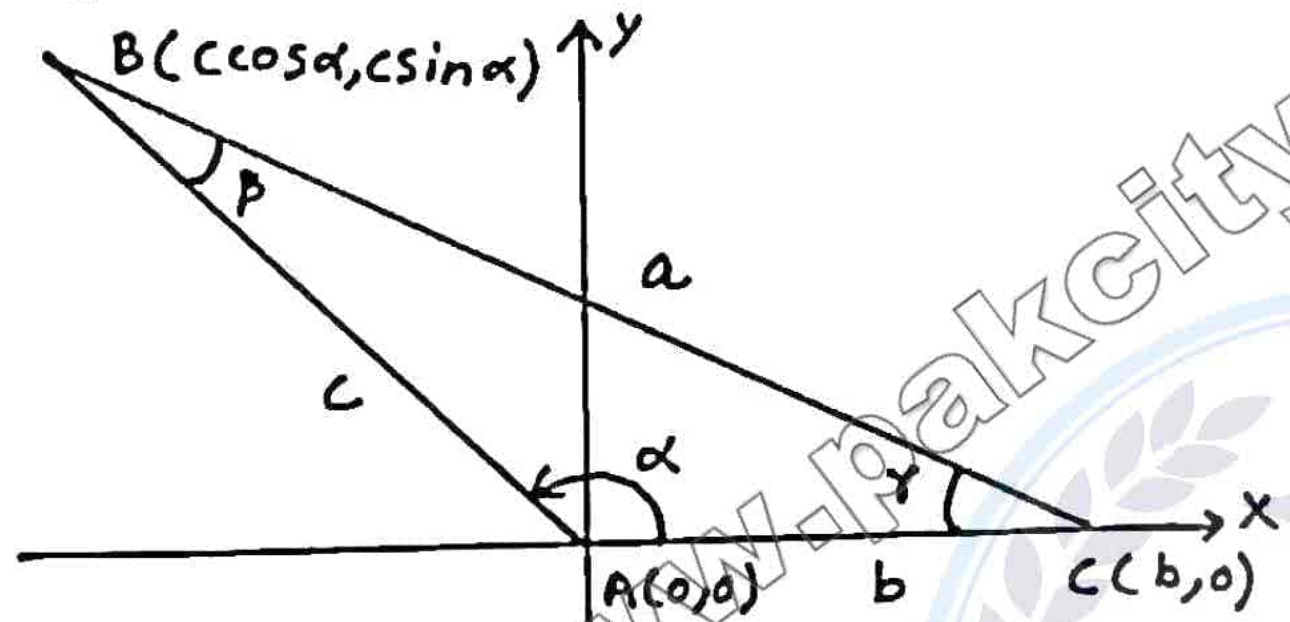
To solve oblique triangles we use
i) Law of Sine ii) Law of cosine
iii) Law of tangents iv) Half angle formulas

The Law of Cosine

In any triangle ABC, with usual notations, prove that:

i) $a^2 = b^2 + c^2 - 2bc\cos\alpha$
ii) $b^2 = c^2 + a^2 - 2ca\cos\beta$
iii) $c^2 = a^2 + b^2 - 2ab\cos\gamma$

Proof:- In any triangle ABC, coordinates of points are $A(0,0)$ $B(c\cos\alpha, c\sin\alpha)$ and $C(b,0)$



By distance formula

$$|BC|^2 = (c\cos\alpha - b)^2 + (c\sin\alpha - 0)^2$$

$$\Rightarrow a^2 = c^2\cos^2\alpha + b^2 - 2bc\cos\alpha + c^2\sin^2\alpha$$

$$= c^2(\cos^2\alpha + \sin^2\alpha) + b^2 - 2bc\cos\alpha$$

$$\Rightarrow a^2 = c^2(1) + b^2 - 2bc\cos\alpha$$

$$\Rightarrow \boxed{a^2 = b^2 + c^2 - 2bc\cos\alpha}$$

Similarly,

$$\boxed{\begin{aligned} b^2 &= c^2 + a^2 - 2ca\cos\beta \\ c^2 &= a^2 + b^2 - 2ab\cos\gamma \end{aligned}}$$

They can also expressed as:

$$\boxed{\begin{aligned} \cos\alpha &= \frac{b^2 + c^2 - a^2}{2bc} \\ \cos\beta &= \frac{c^2 + a^2 - b^2}{2ca} \\ \cos\gamma &= \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}}$$

Note:- If $\triangle ABC$ is right, then Law of cosines reduces to pythagorous Theorem i-e.,

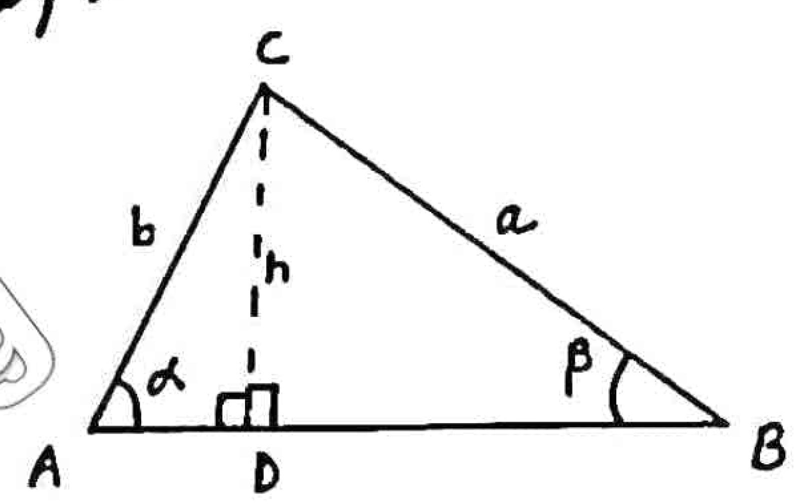
If $\alpha = 90^\circ$ then $b^2 + c^2 = a^2$
or If $\beta = 90^\circ$ then $c^2 + a^2 = b^2$
If $\gamma = 90^\circ$ then $a^2 + b^2 = c^2$

The Law of Sines

In any triangle ABC, with usual notations, prove that:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Proof:-



In any triangle draw a ⊥ar C to $\overline{AB}$ at D then, In right triangle CAD

$$\sin\alpha = \frac{\overline{CD}}{\overline{AC}} \rightarrow \sin\alpha = \frac{h}{b}$$

$$\rightarrow h = b\sin\alpha \longrightarrow (i)$$

In right triangle CBD

$$\sin\beta = \frac{\overline{CD}}{\overline{BC}} \rightarrow \sin\beta = \frac{h}{a}$$

$$\rightarrow h = a\sin\beta \longrightarrow (ii)$$

from (i) $a\sin\beta = b\sin\alpha$

$$\rightarrow \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \longrightarrow (iii)$$

Similarly if we draw a ⊥ar from A to $\overline{BC}$ then $\frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \longrightarrow (iv)$

Combining (iii) and (iv) we get

$$\boxed{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}}$$

Hence proved

The Law of Tangents

In any triangle ABC, with usual notations, prove that

i) $\dfrac{a-b}{a+b} = \dfrac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}}$

ii) $\dfrac{b-c}{b+c} = \dfrac{\tan\frac{\beta-\gamma}{2}}{\tan\frac{\beta+\gamma}{2}}$

iii) $\dfrac{c-a}{c+a} = \dfrac{\tan\frac{\gamma-\alpha}{2}}{\tan\frac{\gamma+\alpha}{2}}$

Proof:-

we have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = D(\text{say})$$

then $a = D\sin\alpha$, $b = D\sin\beta$, $c = D\sin\gamma$

$$\frac{a-b}{a+b} = \frac{D\sin\alpha - D\sin\beta}{D\sin\alpha + D\sin\beta} = \frac{\cancel{D}(\sin\alpha-\sin\beta)}{\cancel{D}(\sin\alpha+\sin\beta)}$$

$$= \frac{\cancel{2}\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{\cancel{2}\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}$$

$$= \cot\frac{\alpha+\beta}{2}\tan\frac{\alpha-\beta}{2}$$

$$\rightarrow \frac{a-b}{a+b} = \frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}} \quad \text{if } a>b$$

similarly

$$\frac{b-c}{b+c} = \frac{\tan\frac{\beta-\gamma}{2}}{\tan\frac{\beta+\gamma}{2}} \quad \text{if } b>c \text{ and}$$

$$\frac{c-a}{c+a} = \frac{\tan\frac{\gamma-\alpha}{2}}{\tan\frac{\gamma+\alpha}{2}} \quad \text{if } c>a$$

Half Angle formulas

a) The Sine Half the angle in terms of the sides

In any triangle ABC, prove that

i) $\sin\frac{\alpha}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$

ii) $\sin\beta/_2 = \sqrt{\dfrac{(s-c)(s-a)}{ca}}$

iii) $\sin\gamma/_2 = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$

where $2s = a+b+c$

Proof:-

$\because$ Law of cosines

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

Also $\cos\alpha = 1-2\sin^2\frac{\alpha}{2}$

$$\rightarrow 2\sin^2\frac{\alpha}{2} = 1-\cos\alpha$$

$$= 1 - \frac{b^2+c^2-a^2}{2bc}$$

$$2\sin^2\frac{\alpha}{2} = \frac{2bc-b^2-c^2+a^2}{2bc}$$

$$\rightarrow \sin^2\frac{\alpha}{2} = \frac{a^2-(b^2+c^2-2bc)}{4bc}$$

$$= \frac{a^2-(b-c)^2}{4bc}$$

$$\sin^2\frac{\alpha}{2} = \frac{(a+b-c)(a-(b-c))}{4bc}$$

$$= \frac{(a+b-c)(a-b+c)}{4bc}$$

$$\sin^2\frac{\alpha}{2} = \frac{(a+b-c)(a+c-b)}{4bc}$$

$\because s = \frac{a+b+c}{2} \rightarrow a+b+c = 2s$

$\rightarrow a+b = 2s-c$

$\rightarrow a+b-c = 2s-c-c$

$\rightarrow a+b-c = 2s-2c = 2(s-c)$

Also $a+b+c = 2s$

$\rightarrow a+c = 2s-b$

$\rightarrow a+c-b = 2s-b-b$

$a+c-b = 2s-2b = 2(s-b)$

So

$$\sin^2\frac{\alpha}{2} = \frac{2(s-c)\cdot 2(s-b)}{4bc}$$

$$= \frac{4(s-c)(s-b)}{4bc}$$

$$\sin^2\frac{\alpha}{2} = \frac{(s-c)(s-b)}{bc}$$

$$\rightarrow \sin\frac{\alpha}{2} = \sqrt{\frac{(s-c)(s-b)}{bc}}$$

$\because \alpha$ is measure of angle of $\triangle ABC$

$\therefore \frac{\alpha}{2} < 90^\circ \rightarrow \sin\frac{\alpha}{2} = +\text{ive}$

Similarly,

$\sin\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$, $\sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

b) The cosine of Half the angle in term of the sides

In any ΔABC, with usual notations, prove that

i) $\cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$

ii) $\cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}$

iii) $\cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}$

where $2s = a+b+c$

Proof:-

$\because$ Law of cosines

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

Also $\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1$

$$\Rightarrow 2\cos^2\frac{\alpha}{2} = 1+\cos\alpha$$

$$= 1+\frac{b^2+c^2-a^2}{2bc}$$

$$2\cos^2\frac{\alpha}{2} = \frac{2bc+b^2+c^2-a^2}{2bc}$$

$$2\cos^2\frac{\alpha}{2} = \frac{(b+c)^2-a^2}{2bc}$$

$$\Rightarrow \cos^2\frac{\alpha}{2} = \frac{(b+c)^2-a^2}{4bc}$$

$$\cos^2\frac{\alpha}{2} = \frac{(b+c+a)(b+c-a)}{4bc}$$

$$\cos^2\frac{\alpha}{2} = \frac{(a+b+c)(b+c-a)}{4bc}$$

$\because a+b+c = 2s$

$\Rightarrow b+c = 2s-a$

$b+c-a = 2s-a-a = 2s-2a$

$= 2(s-a)$

So $\cos^2\frac{\alpha}{2} = \frac{(2s)\cdot 2(s-a)}{4bc}$

$$\cos^2\frac{\alpha}{2} = \frac{4s(s-a)}{4bc}$$

$$\Rightarrow \cos^2\frac{\alpha}{2} = \frac{s(s-a)}{bc}$$

$$\Rightarrow \cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$\because \alpha$ is measure of angle of ΔABC

$\therefore \frac{\alpha}{2}$ is acute $\Rightarrow \cos\frac{\alpha}{2} = +ive$

Similarly,

$\cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ca}}$, $\cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}$

c) The Tangent of Half the angle in terms of the sides

In any triangle ABC, with usual notation, prove that:

i) $\tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

ii) $\tan\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

iii) $\tan\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

where $2s = a+b+c$

Proof:-

$\because \sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ and

$\cos\frac{\alpha}{2} = \sqrt{\frac{s(s-\alpha)}{bc}}$

$$\Rightarrow \tan\frac{\alpha}{2} = \frac{\sin\alpha/2}{\cos\alpha/2} = \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$\Rightarrow \tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Similarly,

$\tan\beta/2 = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$,

$\tan\gamma/2 = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Solution of Oblique Triangle

We know that, A triangle can be constructed if: i) one side and two angles are given, or
ii) two sides and their included angle are given, or
iii) three sides are given.

In the same way,
An Oblique triangle can be solve if
i) one side and two angles are given,
or ii) two sides and their included angle are given, or
iii) three sides are given.

Case I.

When measures of one side and two angles are given

Example 1. Solve the triangle ABC, given that $\alpha = 35°17'$, $\beta = 45°13'$, $b = 421$

Solution:- $\because \alpha + \beta + \gamma = 180°$

$\rightarrow \gamma = 180° - \alpha - \beta = 180° - 35°17' - 45°13'$

$\rightarrow \gamma = 99°30'$

By Law of sine; $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$

$\rightarrow a = \dfrac{b\sin\alpha}{\sin\beta} = \dfrac{(421)\sin 35°17'}{\sin 45°13'}$

$a = \dfrac{421(0.5776)}{0.7098} = 343$

Again, $\dfrac{c}{\sin\gamma} = \dfrac{b}{\sin\beta}$

$\rightarrow c = \dfrac{b\sin\gamma}{\sin\beta} = \dfrac{(421)\sin 90°30'}{\sin 45°13'}$

$\rightarrow c = \dfrac{421(0.9863)}{0.7098} = 585$

Hence, $\gamma = 99°30'$, $a = 343$, $c = 585$

Exercise 12.4

Solve the triangle ABC, if

Q1. $\beta = 60°$; $\gamma = 15°$, $b = \sqrt{6}$

Solution:- $\because \alpha + \beta + \gamma = 180°$

$\rightarrow \alpha = 180° - \beta - \gamma = 180° - 60° - 15°$

$\alpha = 105°$

Now Law of sine; $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$

$\rightarrow a = \dfrac{b\sin\alpha}{\sin\beta} = \dfrac{\sqrt{6}\sin 105°}{\sin 60°}$

$a = \dfrac{(\sqrt{6})(0.965)}{0.866} = 2.73 = 1.73 + 1$

$\rightarrow a = \sqrt{3} + 1 \quad \because \sqrt{3} = 1.73$

Again, $\dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$

$\rightarrow c = \dfrac{b\sin\gamma}{\sin\beta} = \dfrac{\sqrt{6}\sin 15°}{\sin 60°}$

$c = \dfrac{\sqrt{6}(0.26)}{0.87} = 0.73$

$\rightarrow c = 0.73 = 1.73 - 1 = \sqrt{3} - 1$

Hence, $\alpha = 105°$, $a = \sqrt{3} + 1$, $c = \sqrt{3} - 1$

Q2. $\beta = 52°$, $\gamma = 89°35'$, $a = 89.35$

Solution:- $\because \alpha + \beta + \gamma = 180°$

$\rightarrow \alpha = 180° - \beta - \gamma$

$\alpha = 180° - 52° - 89°35' = 38°25'$

By Law of sine, $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$

$\rightarrow b = \dfrac{a\sin\beta}{\sin\alpha} = \dfrac{89.35\sin 52°}{\sin 38°25'}$

$\rightarrow b = \dfrac{(89.35)(0.788)}{0.621}$

$b = 113.37$

Again, $\dfrac{a}{\sin\alpha} = \dfrac{c}{\sin\gamma}$

$\rightarrow c = \dfrac{a\sin\gamma}{\sin\alpha} = \dfrac{(89.35)\sin 89°35'}{\sin 38°25'}$

$c = \dfrac{(89.35)(0.999)}{0.621} = 143.78$

Hence $\alpha = 38°25'$, $b = 113.37$, $c = 143.78$

Q3. $b = 125$, $\gamma = 53^\circ$, $\alpha = 47^\circ$

Solution:-

$\because \alpha + \beta + \gamma = 180^\circ$

$\rightarrow \beta = 180^\circ - \alpha - \gamma = 180^\circ - 47^\circ - 53^\circ$

$\rightarrow \beta = 80^\circ$

By Law of sine; $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$

$\rightarrow a = \dfrac{b\sin\alpha}{\sin\beta} = \dfrac{125\sin 47^\circ}{\sin 80^\circ}$

$a = \dfrac{(125)(0.73)}{0.98} = 93.1$

Again, $\dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$

$\rightarrow c = \dfrac{b\sin\gamma}{\sin\beta} = \dfrac{125\sin 53^\circ}{\sin 80^\circ}$

$c = \dfrac{(125)(0.798)}{0.98} = 101.7$

Hence $\beta = 80^\circ$, $a = 93.1$, $c = 101.7$

Q4. $c = 16.1$, $\alpha = 42^\circ 45'$, $\gamma = 74^\circ 32'$

Solution:- $\because \alpha + \beta + \gamma = 180^\circ$

$\rightarrow \beta = 180^\circ - \alpha - \gamma = 180^\circ - 42^\circ 45' - 74^\circ 32'$

$\rightarrow \beta = 62^\circ 43'$

By Law of sine; $\dfrac{a}{\sin\alpha} = \dfrac{c}{\sin\gamma}$

$\rightarrow a = \dfrac{c\sin\alpha}{\sin\gamma} = \dfrac{(16.1)\sin 42^\circ 45'}{\sin 74^\circ 32'}$

$a = \dfrac{(16.1)(0.678)}{0.963} = 11.33$

Again, $\dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$

$\rightarrow b = \dfrac{c\sin\beta}{\sin\gamma}$

$= \dfrac{(16.1)\sin 62^\circ 43'}{\sin 74^\circ 32'}$

$b = \dfrac{(16.1)(0.888)}{0.963} = 14.84$

Hence,
$\beta = 62^\circ 43'$, $a = 11.33$, $b = 14.84$

Q5. $a = 53$, $\beta = 88^\circ 36'$, $\gamma = 31^\circ 54'$

Solution:-

$\because \alpha + \beta + \gamma = 180^\circ$

$\rightarrow \alpha = 180^\circ - \beta - \gamma = 180^\circ - 88^\circ 36' - 31^\circ 54'$

$\alpha = 59^\circ 30'$

Now by Law of sine; $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$

$\rightarrow b = \dfrac{a\sin\beta}{\sin\alpha} = \dfrac{(53)\sin 88^\circ 36'}{\sin 59^\circ 30'}$

$\rightarrow b = \dfrac{(53)(0.099)}{0.96} = 61.01$

Again, $\dfrac{a}{\sin\alpha} = \dfrac{c}{\sin\gamma}$

$\rightarrow \therefore c = \dfrac{a\sin\gamma}{\sin\alpha} = \dfrac{(53)\sin 31^\circ 54'}{\sin 59^\circ 30'}$

$c = \dfrac{(53)(0.528)}{0.861} = 32.5$

Thus $\alpha = 59^\circ 30'$, $b = 61.01$, $c = 32.5$

Case II

When measures of two sides and their included angle are given

In this case, we can use

i) First law of cosine and then law of sines, OR

ii) First law of tangents and then law of sines.

Example 1. Solve the triangle ABC, by using the cosine and sine laws given that $b = 3$, $c = 5$ and $\alpha = 120^\circ$

Solution:-

By cosines laws,

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

$\rightarrow a^2 = (3)^2 + (5)^2 - 2(3)(5)\cos 120^\circ$

$\rightarrow a^2 = 9 + 25 - 30(-0.5) = 34 + 15$

$a^2 = 49 \rightarrow a = 7$

Now by Law of sine;

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$\rightarrow \sin\beta = \frac{b\sin\alpha}{a} = \frac{3}{7}\sin 120°$

$\sin\beta = \frac{3}{7}(0.866) = (0.428)(0.866)$

$\rightarrow \beta = \sin^{-1}(0.3711) = 21°47'$

$\because \alpha+\beta+\gamma = 180°$

$\rightarrow \gamma = 180° - \alpha - \beta$

$\rightarrow \gamma = 180° - 120° - 21°47' - 38°12'5''$

Hence $\beta = 21°47'$, $\gamma = 38°12'5''$, $a = 7$

Example 2. Solve the triangle ABC in which: $a = 36.21$, $c = 30.14$, $\beta = 78°10'$

Solution:- $\because a > c \rightarrow \alpha > \gamma$

$\because \alpha+\beta+\gamma = 180°$

$\rightarrow \alpha+\gamma = 180° - \beta$

$= 180° - 78°10'$

$\alpha+\gamma = 101°50' \longrightarrow$ (i)

$\rightarrow \frac{\alpha+\gamma}{2} = \frac{101°50'}{2} = 50°55'$

By Law of tangents

$$\frac{\tan\frac{\alpha-\gamma}{2}}{\tan\frac{\alpha+\gamma}{2}} = \frac{a-c}{a+c}$$

$\rightarrow \tan\frac{\alpha-\gamma}{2} = \frac{a-c}{a+c}\tan\frac{\alpha+\gamma}{2}$

$\tan\frac{\alpha-\gamma}{2} = \frac{36.21-30.14}{36.21+30.14}\tan 50°55'$

$= \frac{6.07}{66.35}(1.2312) = 0.1126$

$\frac{\alpha-\gamma}{2} = \tan^{-1}(0.1126)$

$\frac{\alpha-\gamma}{2} = 6°26'$

$\rightarrow \alpha-\gamma = 12°52' \longrightarrow$ (ii)

By (i) + (ii) $\rightarrow 2\alpha = 114°42'$

$\rightarrow \alpha = 57°21'$ put in (i)

(i) $\rightarrow 57°21' + \gamma = 101°50'$

$\gamma = 101°50' - 57°21' = 44°29'$

By Law of sine

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha}$$

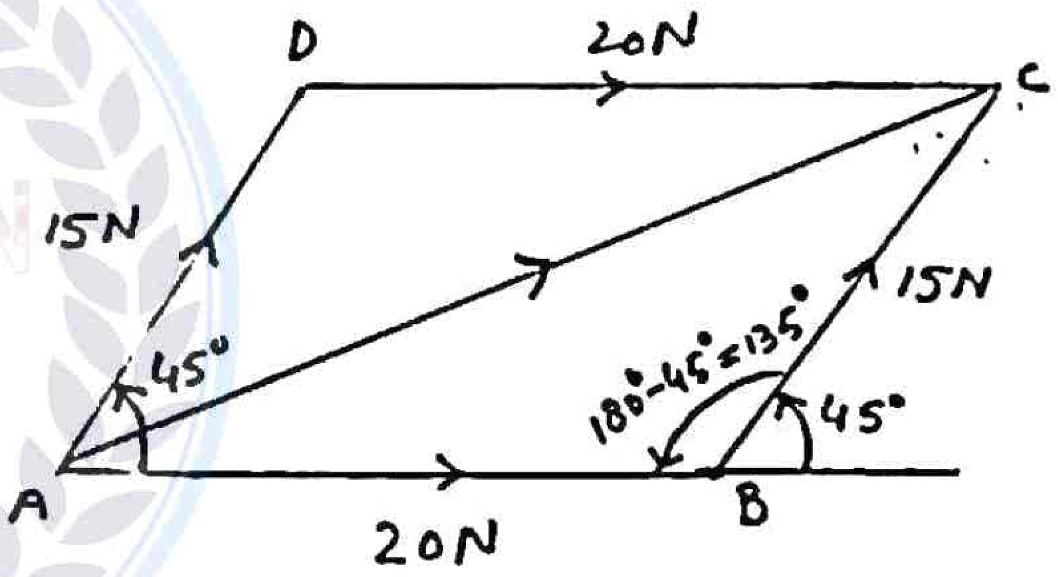
$\rightarrow b = \frac{a\sin\beta}{\sin\alpha} = \frac{(36.21)\sin 78°10'}{\sin 57°21'}$

$b = \frac{(36.21)(0.978)}{0.842} = 42.09$

Hence $b = 42.09$, $\gamma = 44°29'$, $\alpha = 57°21'$

Example 3. Two forces of 20 Newtons and 15 Newtons, inclined at an angle of 45° are applied at a point on a body. If these forces are represented by two adjacent sides of a parallelogram then, their resultant is represented by its diagonal. Find the resultant force and also the angle which the resultant makes with the force of 20 Newton.

Solution:-



Let ABCD be a $\parallel^m$, such that

$|\overrightarrow{AB}| = 20N$, $|\overrightarrow{AD}| = 15N$

also $m\angle BAD = 45°$

$m\angle ABC = 180° - m\angle BAD = 180° - 45°$

$m\angle ABC = 135°$

using Law of cosines,

$(|\overrightarrow{AC}|)^2 = (|\overrightarrow{AB}|)^2 + (|\overrightarrow{BC}|)^2 - 2|\overrightarrow{AB}|\times|\overrightarrow{BC}|\cos 135°$

$= (20)^2 + (15)^2 - 2(20)(15)(-0.707)$

$= 400 + 225 + 424.2$

$|\overrightarrow{AC}|^2 = 1049.2$

$\rightarrow \overrightarrow{AC} = 32.4\,N$

By Law of sines,

$$\frac{\overrightarrow{BC}}{\sin m\angle BAC} = \frac{\overrightarrow{AC}}{\sin m\angle ABC}$$

$$\rightarrow \sin m\angle BAC = \frac{\overrightarrow{BC}\sin m\angle ABC}{\overrightarrow{AC}}$$

$$= \frac{15\sin 135^\circ}{33.4} = \frac{15(0.707)}{33.4}$$

$\sin m\angle BAC = 0.3274$

$\rightarrow m\angle BAC = \sin^{-1}(0.3274)$

$m\angle BAC = 19^\circ 6'$

Hence $\overrightarrow{AC} = 32.4N$, $m\angle BAC = 19^\circ 6'$

Exercise 12.5

Solve the triangle ABC in which:

Q1. $b = 95$, $c = 34$, $\alpha = 52^\circ$

Solution:-

$\because$ Law of cosines,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$= (95)^2 + (34)^2 - 2(95)(34)\cos 52^\circ$$

$$= 9025 + 1156 - 6460(0.61566)$$

$$a^2 = 6203.8269$$

$$\rightarrow a = 78.764$$

Now Law of sine,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\rightarrow \sin\beta = \frac{b}{a}\sin\alpha$$

$$= \frac{95}{78.764}\sin 52^\circ$$

$$\sin\beta = \frac{95\times 0.788}{78.764} = 0.95$$

$$\rightarrow \beta = \sin^{-1}(0.95) = 71^\circ 53'$$

$$\because \alpha + \beta + \gamma = 180^\circ$$

$$\rightarrow \gamma = 180^\circ - \alpha - \beta$$

$$\gamma = 180^\circ - 52^\circ - 71^\circ 53'$$

$$\rightarrow \gamma = 56^\circ 7'$$

Hence

$a = 78.764$, $\beta = 71^\circ 53'$, $\gamma = 56^\circ 7'$

Q2. $b = 12.5$, $c = 23$, $\alpha = 38^\circ 20'$

Solution:-

$\because$ Law of cosines,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$= (12.5)^2 + (23)^2 - 2(12.5)(23)\cos 38^\circ 20'$$

$$a^2 = 156.25 + 529 - 575(0.784)$$

$$a^2 = 234.211 \rightarrow a = 15.304$$

By Law of sine,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\rightarrow \sin\beta = \frac{b}{a}\sin\alpha$$

$$\sin\beta = \frac{12.5}{15.304}\sin 38^\circ 20'$$

$$= (0.816)(0.62)$$

$$\sin\beta = 0.5061$$

$$\rightarrow \beta = \sin^{-1}(0.5061) = 30^\circ 24'$$

$$\because \alpha + \beta + \gamma = 180^\circ$$

$$\rightarrow \gamma = 180^\circ - \alpha - \beta$$

$$= 180^\circ - 38^\circ 20' - 30^\circ 24'$$

$$\rightarrow \gamma = 111^\circ 15'$$

Hence $a = 15.304$, $\beta = 30^\circ 24'$, $\gamma = 111^\circ 15'$

Q3. $a = \sqrt{3} - 1$, $b = \sqrt{3} + 1$, $\gamma = 60^\circ$

We take a=0.7320 , b=2.7320

Solution:- $\because$ Law of cosines,

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$= (0.7320)^2 + (2.7320)^2 - 2(0.7320)(2.7320)\cos 60^\circ$$

$$= 0.53582 + 7.4638 - 3.999(0.5)$$

$$= 6.00012$$

$$\rightarrow c^2 = 6 \rightarrow c = \sqrt{6}$$

By Law of Sine,

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}$$

$$\rightarrow \sin\alpha = \frac{a\sin\gamma}{c}$$

$\sin\alpha = \dfrac{0.7320 \sin 60^\circ}{\sqrt{6}} = \dfrac{(0.7320)(0.8660)}{2.4494}$

$\sin\alpha = \dfrac{0.6339}{2.4494} = 0.25884$

$\alpha = \sin^{-1}(0.25884)$

$\alpha = 15^\circ$

$\because \alpha+\beta+\gamma = 180^\circ$

$\rightarrow \beta = 180^\circ - \alpha - \gamma$

$\beta = 180^\circ - 15^\circ - 60^\circ = 105^\circ$

Hence $\alpha = 15^\circ$, $\beta = 105$, $c = \sqrt{6}$

Q4. $a = 3$, $c = 6$, $\beta = 36^\circ 20'$

Solution:- $\because$ Law of cosine

$b^2 = a^2 + c^2 - 2ac\cos\beta$

$= (3)^2 + (6)^2 - 2(3)(6)\cos 36^\circ 20'$

$= 9 + 36 - 36(0.8055)$

$b^2 = 45 - 29.0010 = 15.99$

$\rightarrow b = 3.998$ or $b = 4$

Now Law of sine,

$\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} \rightarrow \sin\alpha = \dfrac{a\sin\beta}{b}$

$\sin\alpha = \dfrac{3\sin 36^\circ 20'}{4} = \dfrac{3(0.5924)}{4} = 0.4443$

$\alpha = \sin^{-1}(0.4443)$

$\alpha = 26^\circ 23' 4''$

$\because \alpha+\beta+\gamma = 180^\circ$

$\rightarrow \gamma = 180^\circ - \alpha - \beta$

$= 180^\circ - 26^\circ 23' 4'' - 36^\circ 20'$

$\gamma = 117^\circ 16' 56''$

Hence $\alpha = 26^\circ 23' 4''$, $\gamma = 117^\circ 16' 56''$, $b = 4$

Q5. $a = 7$, $b = 3$, $\gamma = 38^\circ 13'$

Solution:-

$\because$ Law of cosines,

$c^2 = a^2 + b^2 - 2ab\cos\gamma$

$c^2 = (7)^2 + (3)^2 - 2(7)(3)\cos 38^\circ 13'$

$\rightarrow c^2 = 49 + 9 - 42(0.7856)$

$c^2 = 58 - 32.9984 = 25.0016 \approx 25$

$\rightarrow c = 5$

Now By Law of Sine,

$\dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma} \rightarrow \sin\beta = \dfrac{b\sin\gamma}{c}$

$\sin\beta = \dfrac{3\sin 38^\circ 13'}{5} = \dfrac{3(0.6186)}{5} = 0.3716$

$\beta = \sin^{-1}(0.3716)$

$\rightarrow \beta = 21^\circ 47'$

$\because \alpha+\beta+\gamma = 180^\circ$

$\rightarrow \alpha = 180^\circ - \beta - \gamma$

$\alpha = 180^\circ - 21^\circ 47' - 38^\circ 13' = 120^\circ$

Hence $\alpha = 120^\circ$, $\beta = 21^\circ 47'$, $c = 5$

Solve the following triangles, using first law of tangents and then law of sines:

Q6. $a = 36.21$, $b = 42.09$, $\gamma = 44^\circ 29'$

Solution:- $\because b > a \rightarrow \beta > \alpha$

$\because \alpha+\beta+\gamma = 180^\circ$

$\rightarrow \alpha+\beta = 180^\circ - \gamma = 180^\circ - 44^\circ 29'$

$\alpha+\beta = 135^\circ 31' \longrightarrow (i)$

$\rightarrow \dfrac{\alpha+\beta}{2} = \dfrac{135^\circ 31'}{2} = 67^\circ 45' 31''$

$\because$ Law of tangents

$\dfrac{\tan\frac{\beta-\alpha}{2}}{\tan\frac{\beta+\alpha}{2}} = \dfrac{b-a}{b+a}$

$\rightarrow \dfrac{\tan\frac{\beta-\alpha}{2}}{\tan 67^\circ 45' 31''} = \dfrac{42.09-36.21}{42.09+36.21}$

$\tan\frac{\beta-\alpha}{2} = \dfrac{5.88}{78.3}(2.4453)$

$\tan\frac{\beta-\alpha}{2} = 0.1836$

$\rightarrow \frac{\beta-\alpha}{2} = \tan^{-1}(0.1836) = 10.4036$

$\rightarrow \beta - \alpha = 20^\circ 45' 26'' \longrightarrow (ii)$

By (i) + (ii) $\rightarrow$ $2\beta = 156°19'26''$

$\rightarrow \beta = 78°9'43''$ put in (i)

(i) $\rightarrow$ $\alpha + 78°9'43'' = 135°31'$

$\rightarrow \alpha = 135°31' - 78°9'43''$

$\alpha = 57°21'17''$

Now Law of sines,

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$\rightarrow c = \frac{b\sin\gamma}{\sin\beta} = \frac{(42.09)\sin 44°29'}{\sin 78°9'43''}$$

$$c = \frac{(42.09)(0.7007)}{0.9787} = 30.13$$

$\rightarrow c = 30.13$

Hence $c = 30.13$, $\alpha = 57°21'17''$, $\beta = 78°9'43''$

Q7. $a = 93$, $c = 101$, $\beta = 80°$

Solution:- $\because \alpha + \beta + \gamma = 180°$

$\rightarrow \alpha + \gamma = 180° - \beta = 180° - 80°$

$\rightarrow \alpha + \gamma = 100° \longrightarrow$ (i)

$$\frac{\alpha+\gamma}{2} = \frac{100°}{2} = 50°$$

By Law of tangents,

$$\frac{\tan\frac{\gamma-\alpha}{2}}{\tan\frac{\gamma+\alpha}{2}} = \frac{c-a}{c+a}$$

$$\rightarrow \tan\frac{\gamma-\alpha}{2} = \frac{101-93}{101+93}\tan 50°$$

$$\tan\frac{\gamma-\alpha}{2} = \frac{8}{194}(1.19) = 0.0476$$

$$\rightarrow \frac{\gamma-\alpha}{2} = \tan^{-1}(0.0476)$$

$\frac{\gamma-\alpha}{2} = 2.72 \rightarrow \gamma - \alpha = 5.45$

$\rightarrow \gamma - \alpha = 5°27' \longrightarrow$ (ii)

By (i) + (ii) $\rightarrow$ $2\gamma = 105°27'$

$\rightarrow \gamma = \frac{105°27'}{2} = 52°43'$ put in (i)

(i) $\rightarrow$ $\alpha + 52°43' = 100°$

$\alpha = 100° - 52°43' = 47°16'$

By Law of sine,

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$\rightarrow b = \frac{c\sin\beta}{\sin\gamma} = \frac{(101)\sin 80°}{\sin 52°43'}$$

$$b = \frac{(101)(0.98)}{0.795} = 124.5$$

Hence $\alpha = 47°16'$, $\gamma = 52°43'$, $b = 124.5$

Q8. $b = 14.8$, $c = 16.1$, $\alpha = 42°45'$

Solution:- $\because \alpha + \beta + \gamma = 180°$

$\rightarrow \beta + \gamma = 180° - \alpha$

$\rightarrow \beta + \gamma = 180° - 47°16'$

$\rightarrow \beta + \gamma = 137°15' \longrightarrow$ (i)

$$\frac{\beta+\gamma}{2} = 68°37'15''$$

By law of tangents,

$$\frac{\tan\frac{\beta-\gamma}{2}}{\tan\frac{\beta+\gamma}{2}} = \frac{b-c}{b+c}$$

$$\rightarrow \tan\frac{\beta-\gamma}{2} = \frac{14.8-16.1}{14.8+16.1}\tan 68°37'15''$$

$$\tan\frac{\beta-\gamma}{2} = \frac{-1.3}{30.9}(2.55) = -0.1072$$

$$\rightarrow \frac{\beta-\gamma}{2} = \tan^{-1}(-0.1072) = -6°7'7''$$

$\beta - \gamma = -12°14'14'' \longrightarrow$ (ii)

By (i) + (ii) $\rightarrow$ $2\beta = 125°$

$\rightarrow \beta = 62°30'22''$ put in (i)

$62°30'22'' + \gamma = 137°15'$

$\rightarrow \gamma = 137°15' - 62°30'22''$

$\gamma = 74°44'37''$

By Law of sines,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$\rightarrow a = \frac{b\sin\alpha}{\sin\beta} = \frac{14.8\sin 42°45'}{\sin 62°30'}$$

$$\rightarrow a = \frac{(14.8)(0.678)}{0.887} = 11.31$$

Hence $a = 11.31$, $\gamma = 74°44'37''$

$\beta = 62°30'22''$

Q9. $a = 319$, $b = 168$, $\gamma = 110^\circ 22'$

Solution:- $\because \alpha+\beta+\gamma = 180^\circ$

$\rightarrow \alpha+\beta = 180^\circ - \gamma = 180^\circ - 110^\circ 22'$

$\rightarrow \alpha+\beta = 69^\circ 38' \longrightarrow (i)$

$\rightarrow \frac{\alpha+\beta}{2} = 34^\circ 49'$

By Law of tangents

$$\frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}} = \frac{a-b}{a+b}$$

$\rightarrow \tan\frac{\alpha-\beta}{2} = \frac{319-168}{319+168}\tan 34^\circ 49'$

$\tan\frac{\alpha-\beta}{2} = \frac{151}{487}(0.695) = 0.215$

$\rightarrow \frac{\alpha-\beta}{2} = \tan^{-1}(0.215)$

$\frac{\alpha-\beta}{2} = 12^\circ 8' 1''$

$\rightarrow \alpha-\beta = 24^\circ 16' 3'' \longrightarrow (ii)$

By (i)+(ii) $\rightarrow 2\alpha = 93^\circ 54' 3''$

$\rightarrow \alpha = 46^\circ 57' 1''$ put in (i)

$46^\circ 57' 1'' + \beta = 69^\circ 38'$

$\beta = 69^\circ 38' - 46^\circ 57' 1'' = 22^\circ 40' 58''$

By Law of sines,

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}$$

$\rightarrow c = \frac{a\sin\gamma}{\sin\alpha} = \frac{(319)\sin 110^\circ 22'}{\sin 46^\circ 57' 1''}$

$\rightarrow c = \frac{(319)(0.937)}{0.73} = 409.12$

Hence $c = 409.12$, $\beta = 22^\circ 40' 58''$, $\alpha = 46^\circ 57' 1''$

Q10. $b = 61$, $c = 32$, $\alpha = 59^\circ 30'$

Solution:- $\because \alpha+\beta+\gamma = 180^\circ$

$\rightarrow \beta+\gamma = 180^\circ - \alpha$

$= 180^\circ - 59^\circ 30'$

$\beta+\gamma = 120^\circ 30' \longrightarrow (i)$

$\rightarrow \frac{\beta+\gamma}{2} = 60^\circ 15'$

By law of tangents,

$$\frac{\tan\frac{\beta-\gamma}{2}}{\tan\frac{\beta+\gamma}{2}} = \frac{b-c}{b+c}$$

$\rightarrow \tan\frac{\beta-\gamma}{2} = \frac{61-32}{61+32}\tan 60^\circ 15'$

$\tan\frac{\beta-\gamma}{2} = \frac{29}{93}(1.749) = 0.544$

$\rightarrow \frac{\beta-\gamma}{2} = \tan^{-1}(0.544) = 28^\circ 32' 46''$

$\rightarrow \beta-\gamma = 57^\circ 53' 2'' \longrightarrow (ii)$

By (i)+(ii) $\rightarrow 2\beta = 117^\circ 35' 32''$

$\rightarrow \beta = 88^\circ 47' 46''$ put in (i)

$88^\circ 47' 46'' + \gamma = 120^\circ 30'$

$\gamma = 120^\circ 30' - 88^\circ 47' 46''$

$\gamma = 31^\circ 42' 13''$

By law of sines,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$\rightarrow a = \frac{b\sin\alpha}{\sin\beta} = \frac{61\sin 59^\circ 30'}{\sin 88^\circ 47' 46''}$

$a = \frac{61(0.8616)}{0.9997} = 52.573$

Hence, $a = 52.573$, $\gamma = 31^\circ 42' 13''$, $\beta = 88^\circ 47' 46''$

Q11. Measures of two sides of a triangle are in the ratio 3:2 and they include an angle of measure 57°. Find the remaining two angles.

Solution:- $b = 2$, $c = 3$, $\alpha = 57^\circ$, $\beta = ?$, $\gamma = ?$

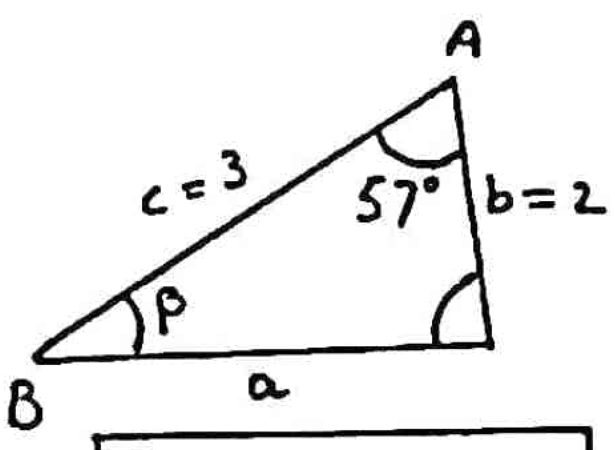
$\because \alpha+\beta+\gamma = 180^\circ$

$\rightarrow \beta+\gamma = 180^\circ - \alpha$

$\beta+\gamma = 180^\circ - 57^\circ$

$\beta+\gamma = 123^\circ \longrightarrow (i)$

$\rightarrow \frac{\beta+\gamma}{2} = 61^\circ 30'$



Note

$c : b = 3 : 2$

$\frac{c}{b} = \frac{3}{2}$

$\frac{c-b}{c+b} = \frac{3-2}{3+2}$

By law of tangents,

$$\frac{c-b}{c+b} = \frac{\tan\frac{\gamma-\beta}{2}}{\tan\frac{\gamma+\beta}{2}}$$

$$\frac{3-2}{3+2} = \frac{\tan\frac{\gamma-\beta}{2}}{\tan 61^\circ 30'}$$

$$\rightarrow \frac{1}{5} = \frac{\tan\frac{\gamma-\beta}{2}}{1.8417}$$

$$\rightarrow \tan\frac{\gamma-\beta}{2} = \frac{1.8417}{5} = 0.3683$$

$$\rightarrow \frac{\gamma-\beta}{2} = \tan^{-1}(0.3683)$$

$$\frac{\gamma-\beta}{2} = 20^\circ 13' 17''$$

$$\rightarrow \gamma-\beta = 40^\circ 26' 34'' \longrightarrow (ii)$$

By (i)+(ii) $\rightarrow 2\gamma = 163^\circ 26' 34''$

$\rightarrow \gamma = 81^\circ 43' 17''$ put in (i)

$$\beta + 81^\circ 43' 17'' = 123^\circ$$

$$\rightarrow \beta = 123^\circ - 81^\circ 43' 17'' = 41^\circ 16' 43''$$

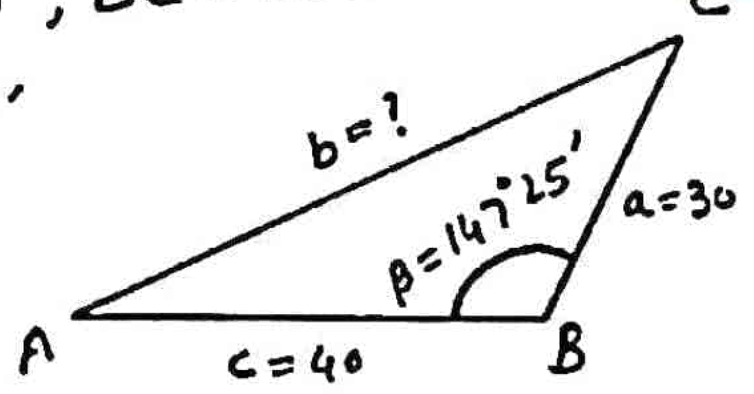
Hence $\beta = 41^\circ 16' 43''$, $\gamma = 81^\circ 43' 17''$

Q12. Two forces of 40N and 30N are represented by $\overrightarrow{AB}$ and $\overrightarrow{BC}$ which are inclined at an angle of $147^\circ 25'$. Find $\overrightarrow{AC}$, the resultant of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

Solution:-

Here $\overline{AB} = 40N$, $\overline{BC} = 30N$

and $\beta = 147^\circ 25'$



By Law of cosines,

$$|\overline{AC}|^2 = |\overline{AB}|^2 + |\overline{BC}|^2 - 2|\overline{AB}||\overline{BC}|\cos 147^\circ 25'$$

$$= (40)^2 + (30)^2 - 2(40)(30)(-0.843)$$

$$= 1600 + 900 - 2400(-0.843)$$

$$|\overline{AC}|^2 = 2500 + 2022.262$$

$$\rightarrow |\overline{AC}|^2 = 4522.262$$

$$\rightarrow \overline{AC} = 67.25N$$

Case III

When measures of three sides are given

In this case, we can use

i) the law of cosine;

or ii) the half angle formulas:

Example 1. Solve the triangle ABC, by using the law of cosine when

$a = 7,\ b = 3,\ c = 5$

Solution:-

$$\because \cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{(3)^2+(5)^2-(7)^2}{2(3)(5)} = \frac{9+25-49}{30}$$

$$\cos\alpha = -\frac{15}{30} = -\frac{1}{2}$$

$$\rightarrow \alpha = \cos^{-1}(-\tfrac{1}{2}) = 120^\circ$$

$$\because \cos\beta = \frac{c^2+a^2-b^2}{2ca} = \frac{(5)^2+(7)^2-(3)^2}{2(5)(7)}$$

$$\cos\beta = \frac{25+49-9}{70} = \frac{65}{70} = 0.9286$$

$$\rightarrow \beta = \cos^{-1}(0.9286) = 21^\circ 47'$$

$$\because \gamma = 180^\circ - \alpha - \beta$$

$$= 180^\circ - 120^\circ - 21^\circ 47' = 38^\circ 13'$$

Example 2. Solve the triangle ABC, by half angle formula, when

$a = 283,\ b = 317,\ c = 428$

Solution:-

$$\because s = \frac{a+b+c}{2} = \frac{283+317+428}{2}$$

$$s = \frac{1028}{2} = 514$$

$$s-a = 514 - 283 = 231$$

$$s-b = 514 - 317 = 197$$

$$s-c = 514 - 428 = 86$$

Now

$$\tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{197 \times 86}{514 \times 231}}$$

$$\tan\frac{\alpha}{2} = \sqrt{0.1426} = 0.377$$

$\Rightarrow \frac{\alpha}{2} = \tan^{-1}(0.3777) = 20°42'$

$\Rightarrow \alpha = 41°24'$

$\because \tan\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \sqrt{\frac{86 \times 231}{514 \times 197}}$

$\tan\frac{\beta}{2} = \sqrt{0.196} = 0.4429$

$\Rightarrow \frac{\beta}{2} = \tan^{-1}(0.4429) = 23°53'$

$\Rightarrow \beta = 47°46'$

$\because \gamma = 180° - \alpha - \beta = 180° - 41°24' - 47°46'$

$\gamma = 90°50'$

Exercise 12.6

Solve the following triangles, in which

Q1. $a=7$, $b=7$, $c=9$

Solution:-

$\because \cos\alpha = \frac{b^2+c^2-a^2}{2bc} = \frac{(7)^2+(9)^2-(7)^2}{2(7)(9)}$

$\cos\alpha = \frac{81}{126} = 0.643$

$\Rightarrow \alpha = \cos^{-1}(0.643) = 50°$

$\cos\beta = \frac{a^2+c^2-b^2}{2ac} = \frac{(7)^2+(9)^2-(7)^2}{2(7)(9)}$

$\cos\beta = \frac{81}{126} = 0.643$

$\Rightarrow \beta = \cos^{-1}(0.643) = 50°$

$\because \gamma = 180° - \alpha - \beta$

$\gamma = 180° - 50° - 50° = 80°$

Hence, $\alpha = 50°$, $\beta = 50°$, $\gamma = 80°$

Q2. $a = 32$, $b = 40$, $c = 66$

Solution:-

$\because \cos\alpha = \frac{b^2+c^2-a^2}{2bc}$

$= \frac{(40)^2+(66)^2-(32)^2}{2(40)(66)}$

$\Rightarrow \cos\alpha = \frac{1600+4356-1024}{5280} = \frac{4932}{5280}$

$\Rightarrow \cos\alpha = 0.934$

$\Rightarrow \alpha = \cos^{-1}(0.934) = 20°55'58''$

$\cos\beta = \frac{a^2+c^2-b^2}{2ac} = \frac{(32)^2+(66)^2-(40)^2}{2(32)(66)}$

$\cos\beta = \frac{1024+4356-1600}{2(32)(66)} = \frac{3780}{4224}$

$\cos\beta = 0.895$

$\Rightarrow \beta = \cos^{-1}(0.895) = 26°29'29''$

$\because \gamma = 180° - \alpha - \beta$

$= 180° - 20°55'58'' - 26°29'29''$

$\gamma = 132°34'32''$

Hence $\alpha = 20°55'58''$, $\beta = 26°29'29''$

$\gamma = 132°34'32''$

Q3. $a = 28.3$, $b = 31.7$, $c = 42.8$

Solution:-

$\because \cos\alpha = \frac{b^2+c^2-a^2}{2bc} = \frac{(31.7)^2+(42.8)^2-(28.3)^2}{2(31.7)(42.8)}$

$= \frac{1004.89+1831.84-800.89}{2713.5}$

$\cos\alpha = 0.75$

$\Rightarrow \alpha = \cos^{-1}(0.75) = 41°24'34''$

$\because \cos\beta = \frac{c^2+a^2-b^2}{2ac} = \frac{(42.8)^2+(28.3)^2-(31.7)^2}{2(42.8)(28.3)}$

$= \frac{1831.84+800.89-1004.89}{2422.48}$

$\cos\beta = 0.671$

$\Rightarrow \beta = \cos^{-1}(0.671) = 47°47'10''$

$\because \gamma = 180° - \alpha - \beta$

$= 180° - 41°24'34'' - 47°47'10''$

$\gamma = 90°48'15''$

Hence

$\alpha = 41°24'34''$, $\beta = 47°47'10''$

$\gamma = 90°48'15''$

Q4. $a = 31.9$, $b = 56.31$, $c = 42.8$

Solution:-

$\because \cos\alpha = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{(56.31)^2+(42.8)^2-(31.9)^2}{2(56.31)(42.8)}$

$= \dfrac{3170.81+1621.67-1017.61}{4}$

$\cos\alpha = \dfrac{3447.88}{4535.21} = 0.8323$

$\Rightarrow \alpha = \cos^{-1}(0.8323) = 33^\circ 39' 51''$

$\because \cos\beta = \dfrac{c^2+a^2-b^2}{2ca} = \dfrac{(42.8)^2+(31.9)^2-(56.31)^2}{2(42.8)(31.9)}$

$= \dfrac{1621.67+1017.61-3170.81}{2569.2}$

$\cos\beta = -0.207$

$\Rightarrow \beta = \cos^{-1}(-0.207) = 101^\circ 56' 47''$

$\because \gamma = 180^\circ - \alpha - \beta$

$= 180^\circ - 33^\circ 39' 51'' - 101^\circ 56' 47''$

$\gamma = 44^\circ 23' 21''$

Hence $\alpha = 33^\circ 39' 51''$, $\beta = 101^\circ 56' 47''$

$\gamma = 44^\circ 23' 21''$

Q5. $a = 4584$, $b = 5140$, $c = 3624$

Solution:-

$\because \cos\alpha = \dfrac{b^2+c^2-a^2}{2bc}$

$= \dfrac{(5140)^2+(3624)^2-(4584)^2}{2(5140)(3624)}$

$= \dfrac{26419600+13133376-21013056}{37254720}$

$\cos\alpha = 0.4977$

$\Rightarrow \alpha = \cos^{-1}(0.4977) = 60^\circ 9' 7''$

$\because \cos\beta = \dfrac{c^2+a^2-b^2}{2ca}$

$= \dfrac{(3624)^2+(4584)^2-(5140)^2}{2(3624)(4584)}$

$= \dfrac{13133376+21013056-26419600}{33224832}$

$\Rightarrow \cos\beta = 0.233$

$\Rightarrow \beta = \cos^{-1}(0.233) = 76^\circ 30' 33''$

$\because \gamma = 180^\circ - \alpha - \beta$

$\Rightarrow \gamma = 180^\circ - 60^\circ 9' 7'' - 76^\circ 30' 33''$

$\gamma = 43^\circ 20' 22''$

Hence $\alpha = 60^\circ 9' 7''$, $\beta = 76^\circ 30' 33''$

and $\gamma = 43^\circ 20' 22''$

Q6. Find the smallest angle of the triangle ABC, when $a = 37.34$, $b = 3.24$, $c = 35.06$

Solution:- Here

$b < a$ and $b < c$ so,

smallest angle is β.

$\because \cos\beta = \dfrac{a^2+c^2-b^2}{2ac}$

$= \dfrac{(37.34)^2+(35.06)^2-(3.24)^2}{2(37.34)(35.06)}$

$= \dfrac{1394.28+1229.20-10.56}{2618.28}$

$\cos\beta = \dfrac{2612.98}{2618.28} = 0.998$

$\Rightarrow \beta = \cos^{-1}(0.998) = 3^\circ 37' 27''$

Q7. Find the measure of greatest angle, if sides of the triangle are 16, 20, 33.

Solution:-

Let $a = 16$, $b = 20$, $c = 33$.

Here $c > a$ and $c > b$ so

greatest angle is γ.

$\because \cos\gamma = \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{(16)^2+(20)^2-(33)^2}{2(16)(20)}$

$= \dfrac{256+400-1089}{640} = \dfrac{-433}{640}$

$\cos\gamma = -0.68 \Rightarrow \gamma = \cos^{-1}(-0.68)$

$\Rightarrow \gamma = 132^\circ 50' 37''$

Q8. The sides of a triangle are x^2+x+1, $2x+1$ and x^2-1. Prove that the greatest angle of the triangle is 120°.

Solution:-

Let $a = x^2+x+1$, $b = 2x+1$
$c = x^2-1$

clearly $a > b$ and $a > c$ so greatest angle is α.

$$\because \cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{(2x+1)^2+(x^2-1)^2-(x^2+x+1)^2}{2(2x+1)(x^2-1)}$$

$$= \frac{4x^2+4x+1+x^4-2x^2+1-(x^4+x^2+1+2x^3+2x^2+2x)}{2(2x^3-2x+x^2-1)}$$

$$= \frac{4x^2+4x+1+x^4-2x^2+1-x^4-x^2-1-2x^3-2x^2-2x}{2(2x^3-2x+x^2-1)}$$

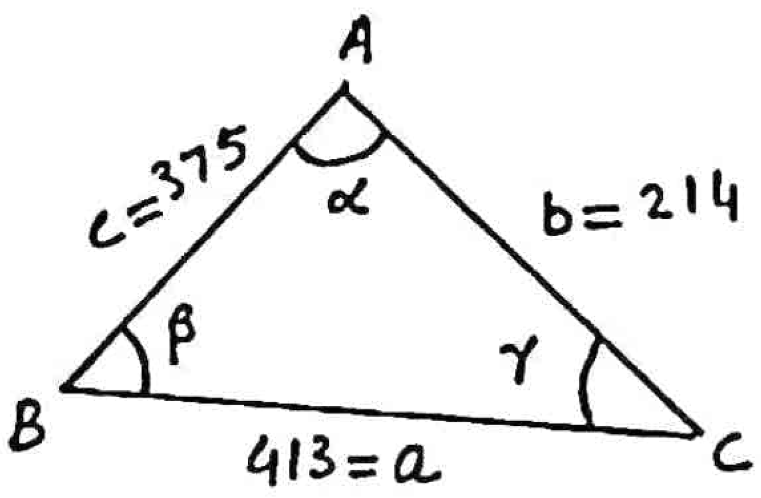
$$\cos\alpha = \frac{-2x^3-x^2+2x+1}{2(2x^3+x^2-2x-1)} = \frac{-(2x^3+x^2-2x-1)}{2(2x^3+x^2-2x-1)}$$

$$\cos\alpha = -\frac{1}{2} \rightarrow \alpha = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\rightarrow \alpha = 120^\circ$$

Q9. The measures of side of a triangular plot are 413, 214 and 375 meters. Find the measures of the corner angles of the plot.

Solution:-



Let $a = 413$, $b = 214$, $c = 375$

$$\because \cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{(214)^2+(375)^2-(413)^2}{2(214)(375)}$$

$$= \frac{45796+140625-170569}{160500}$$

$$\cos\alpha = \frac{15852}{160500} = 0.987$$

$$\rightarrow \alpha = \cos^{-1}(0.987)$$

$$\alpha = 84^\circ 19' 54''$$

$$\because \cos\beta = \frac{c^2+a^2-b^2}{2ca}$$

$$= \frac{(375)^2+(413)^2-(214)^2}{2(375)(413)}$$

$$\cos\beta = \frac{265398}{309750} = 0.856$$

$$\rightarrow \beta = \cos^{-1}(0.856) = 31^\circ 2' 21''$$

$$\because \gamma = 180^\circ - \alpha - \beta$$

$$= 180^\circ - 84^\circ 19' 54'' - 31^\circ 2' 21''$$

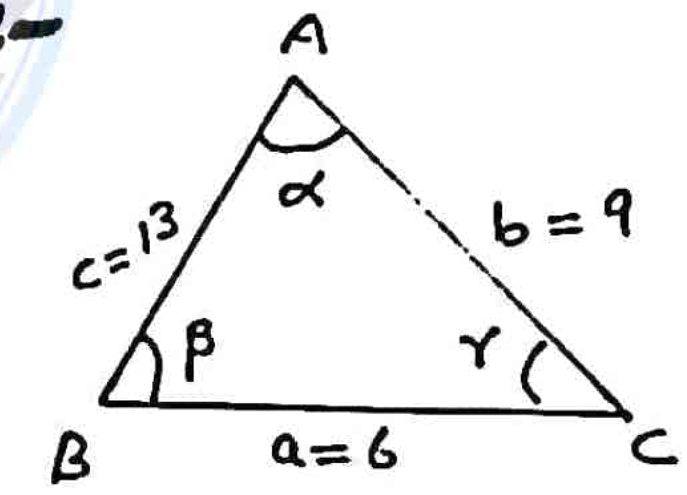
$$\gamma = 64^\circ 37' 44''$$

Hence

$\alpha = 84^\circ 19' 54''$, $\beta = 31^\circ 2' 21''$, $\gamma = 64^\circ 37' 44''$

Q10. Three villages A, B and C are connected by straight roads 6 Km, 9 Km and 13 Km. What angles these roads make with each other?

Solution:-



Let $a = 6$, $b = 9$, $c = 13$

$$\because \cos\alpha = \frac{b^2+c^2-a^2}{2bc} = \frac{(9)^2+(13)^2-(6)^2}{2(9)(13)}$$

$$\cos\alpha = \frac{81+169-36}{234} = \frac{214}{234} = 0.9145$$

$$\alpha = \cos^{-1}(0.9145) = 23^\circ 51' 39''$$

$$\because \cos\beta = \frac{c^2+a^2-b^2}{2ca} = \frac{(13)^2+(6)^2-(9)^2}{2(13)(6)}$$

$$\cos\beta = \frac{169+36-81}{156} = \frac{124}{156} = 0.7949$$

$$\rightarrow \beta = \cos^{-1}(0.7949) = 37^\circ 21' 24''$$

$$\because \gamma = 180^\circ - \alpha - \beta$$

$\rightarrow \gamma = 180^\circ - 23^\circ 51' 39'' - 37^\circ 21' 24''$

$\rightarrow \gamma = 118^\circ 46' 56''$

Hence $\alpha = 23^\circ 51' 39''$, $\beta = 37^\circ 21' 24''$
$\gamma = 118^\circ 46' 56''$

Area of Triangle

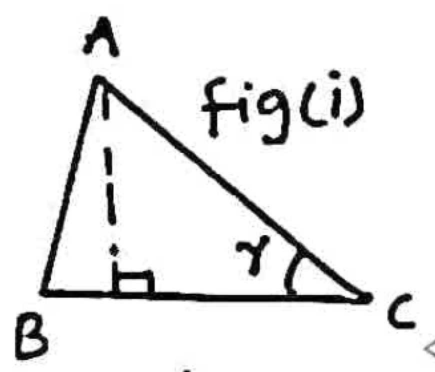
Case I

Area of Triangle in Terms of the Measures of two sides and their included angle

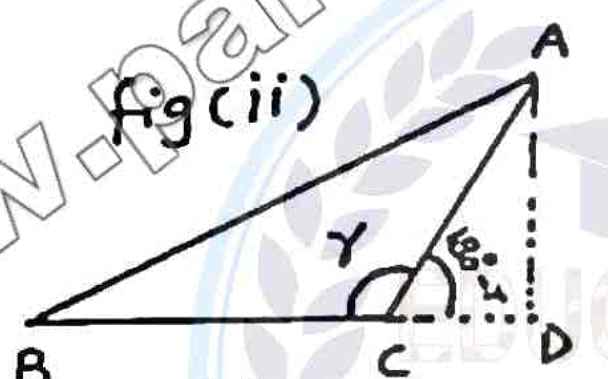
"with usual notations, prove that

Area of triangle ABC
$= \frac{1}{2} bc \sin\alpha = \frac{1}{2} ca \sin\beta$
$= \frac{1}{2} ab \sin\gamma$"

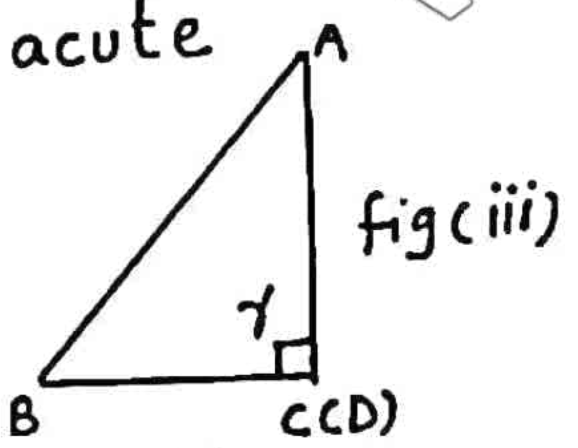
Proof:- consider three different kinds of ΔABC with $m\angle C = \gamma$ as



acute



obtuse



right

In fig (i) $\frac{AD}{AC} = \sin\gamma \longrightarrow (i)$

In fig (ii) $\frac{AD}{AC} = \sin(180^\circ - \gamma) = \sin\gamma \longrightarrow (ii)$

In fig (iii) $\frac{AD}{AC} = \sin 90^\circ = \sin\gamma \longrightarrow (iii)$

from (i), (ii) and (iii) it is clear that $\frac{AD}{AC} = \sin\gamma$

$\rightarrow AD = AC \sin\gamma$

$AD = b \sin\gamma$

Now

Area of $\Delta ABC = \frac{1}{2}(\text{base})(\text{altitude})$

$= \frac{1}{2}(BC)(AD)$

$\Delta = \frac{1}{2} ab \sin\gamma$

Similarly,

$\Delta = \frac{1}{2} bc \sin\alpha = \frac{1}{2} ca \sin\beta$

Case II

Area of triangle in terms of One side and two angles

"In any triangle ΔABC, with usual notations, prove that

Area of triangle $= \frac{a^2 \sin\beta \sin\gamma}{2\sin\alpha}$

$= \frac{b^2 \sin\gamma \sin\alpha}{2\sin\beta} = \frac{c^2 \sin\alpha \sin\beta}{2\sin\gamma}$

Proof:- we know that

$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$

$\rightarrow a = \frac{c\sin\alpha}{\sin\gamma}$ and

$b = \frac{c\sin\beta}{\sin\gamma}$

Now $\Delta = \frac{1}{2} ab \sin\gamma$

$= \frac{1}{2} \frac{c\sin\alpha}{\sin\gamma} \cdot \frac{c\sin\beta}{\sin\gamma} \sin\gamma$

$\Delta = \frac{1}{2} \frac{c^2 \sin\alpha \sin\beta}{\sin\gamma}$

Similarly,

$\Delta = \frac{1}{2} \frac{a^2 \sin\beta \sin\gamma}{\sin\alpha}$, $\Delta = \frac{1}{2} \frac{b^2 \sin\alpha \sin\gamma}{\sin\beta}$

Case III

Area of Triangle in terms of the Measures of its sides

Hero's formula

In a ΔABC, with usual notation, prove that:

Area of triangle $= \sqrt{s(s-a)(s-b)(s-c)}$

where $s = \frac{a+b+c}{2}$

Proof:- we know that

$$\Delta = \frac{1}{2} bc\sin\alpha$$

$\because \sin 2(\frac{\alpha}{2}) = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$ so

$$\Delta = \frac{1}{2} bc(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2})$$

$$= bc\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$$

$$\Delta = bc\sqrt{\frac{s(s-a)(s-b)(s-c)}{b^2c^2}}$$

$$\Delta = bc\frac{\sqrt{s(s-a)(s-b)(s-c)}}{bc}$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

which is called Hero's formula

Example 1. Find the area of triangle ABC, in which $b = 21.6$, $c = 30.2$, and $\alpha = 52^\circ 40'$

Solution:-

$$\because \Delta ABC = \frac{1}{2} bc\sin\alpha$$

$$= \frac{1}{2}(21.6)(30.2)\sin 52^\circ 40'$$

$$= \frac{1}{2}(21.6)(30.2)(0.7951)$$

$$\Delta ABC = 259.3 \text{ sq. units}$$

Example 2. Find the area of the triangle ABC, when $\alpha = 35^\circ 17'$, $\gamma = 45^\circ 13'$ and $b = 42.1$

Solution:-

$$\because \alpha + \beta + \gamma = 180^\circ$$

$$\rightarrow \beta = 180^\circ - \alpha - \gamma$$

$$= 180^\circ - 35^\circ 17' - 45^\circ 13'$$

$$\beta = 99^\circ 30'$$

$$\because \Delta = \frac{b^2 \sin\gamma \sin\alpha}{2\sin\beta}$$

$$= \frac{(42.1)^2 \sin 45^\circ 13' \sin 35^\circ 17'}{2\sin 99^\circ 30'}$$

$$= \frac{(1772.4)(0.7097)(0.5776)}{2(0.9863)}$$

$$\therefore \Delta = 368.3 \text{ sq. units}$$

Example 3. Find the area of the triangle ABC, in which $a = 275.4$, $b = 303.7$, $c = 342.5$

Solution:-

$$\because s = \frac{a+b+c}{2} = \frac{275.4+303.7+342.5}{2}$$

$$s = \frac{921.6}{2} = 460.8$$

Now

$$s-a = 460.8 - 275.4 = 185.4$$

$$s-b = 460.8 - 303.7 = 157.1$$

$$s-c = 460.8 - 342.5 = 118.3$$

$$\therefore \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{(460.8)(185.4)(157.1)(118.3)}$$

$$\Delta = 39847 \text{ sq. units}$$

Exercise 12.7

Q1. Find the area of the triangle ABC, given two sides and their included angle

i) $a = 200$, $b = 120$, $\gamma = 150^\circ$

Solution:-

$$\because \Delta = \frac{1}{2} ab\sin\gamma$$

$$= \frac{1}{2}(200)(120)\sin 150^\circ$$

$$\Delta = 12000(0.5) = 6000 \text{ sq. units}$$

ii) $b = 37$, $c = 45$, $\alpha = 30^\circ 50'$

Solution:-

$$\because \Delta = \frac{1}{2} bc\sin\alpha$$

$$= \frac{1}{2}(37)(45)\sin 30^\circ 50'$$

$$\Delta = 426.69$$

iii) $a = 4.33$, $b = 9.25$, $\gamma = 56^\circ 44'$

Solution:-

$$\because \Delta = \frac{1}{2} ab\sin\gamma$$

$$= \frac{1}{2}(4.33)(9.25)\sin 56^\circ 44'$$

$$= \frac{1}{2}(4.33)(9.25)(0.8361)$$

$$\Delta = 16.74 \text{ sq. units}$$

Q2. Find the area of the triangle ABC, given one side and two angles:

i) $b = 25.4$, $\gamma = 36°41'$, $\alpha = 45°17'$

Solution:-

$\because \alpha + \beta + \gamma = 180°$

$\Rightarrow \beta = 180° - \alpha - \gamma$

$= 180° - 45°17' - 36°41'$

$\beta = 98°2'$

$\because \Delta ABC = \dfrac{b^2 \sin\alpha \sin\gamma}{2\sin\beta}$

$= \dfrac{(25.4)^2 \sin 45°17' \sin 36°41'}{2\sin 98°2'}$

$= \dfrac{(645.16)(0.711)(0.597)}{2(0.99)} = \dfrac{273.85}{1.98}$

$\Delta ABC = 138.29$ sq. units

ii) $c = 32$, $\alpha = 47°24'$, $\beta = 70°16'$

Solution:-

$\because \alpha + \beta + \gamma = 180°$

$\Rightarrow \gamma = 180° - \beta - \alpha$

$= 180° - 70°16' - 47°24'$

$\gamma = 62°20'$

$\because$ Area of $\Delta ABC = \dfrac{c^2 \sin\alpha \sin\beta}{2\sin\gamma}$

$= \dfrac{(32)^2 \sin 47°24' \sin 70°16'}{2\sin 62°20'}$

$= \dfrac{(1024)(0.7361)(0.9413)}{2(0.8857)}$

$\Delta ABC = \dfrac{709.52}{1.7714} = 400.54$ sq. units

iii) $a = 4.8$, $\alpha = 83°42'$, $\gamma = 37°12'$

Solution:-

$\because \alpha + \beta + \gamma = 180°$

$\Rightarrow \beta = 180° - \alpha - \gamma$

$= 180° - 83°42' - 37°12'$

$\beta = 59°6'$

$\because$ Area of $\Delta ABC = \dfrac{a^2 \sin\beta \sin\gamma}{2\sin\alpha}$

$= \dfrac{(4.8)^2 \sin 59°6' \sin 37°12'}{2\sin 83°42'}$

$= \dfrac{(23.04)(0.8581)(0.6046)}{2(0.99)}$

$= \dfrac{11.953}{1.986} = 6.02$ sq. units

Q3. Find the area of the triangle ABC, given the sides:

i) $a = 18$, $b = 24$, $c = 30$

Solution:-

$\because s = \dfrac{a+b+c}{2} = \dfrac{18+24+30}{2} = 36$

$s - a = 36 - 18 = 18$

$s - b = 36 - 24 = 12$

$s - c = 36 - 30 = 6$

$\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{36(18)(12)(6)}$

$\Delta = \sqrt{46656} = 216$ sq. units

ii) $a = 524$, $b = 276$, $c = 315$

Solution:-

$\because s = \dfrac{a+b+c}{2} = \dfrac{524+276+315}{2}$

$s = \dfrac{1165}{2} = 557.25$

$s - a = 557.25 - 524 = 33.5$

$s - b = 557.25 - 276 = 281.5$

$s - c = 557.25 - 315 = 242.5$

$\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{(557.25)(33.5)(281.5)(242.5)}$

$= \sqrt{127491.09}$

$\Delta = 35705.89$ sq. units

iii) $a = 32.65$, $b = 42.81$, $c = 64.92$

Solution:-

$\because s = \dfrac{a+b+c}{2} = \dfrac{32.65+42.81+64.92}{2}$

$s = \dfrac{140.38}{2} = 70.19$

Now

$s-a = 70.19 - 32.65 = 37.54$

$s-b = 70.19 - 42.81 = 27.38$

$s-c = 70.19 - 64.92 = 5.27$

$\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{(70.19)(37.54)(27.38)(5.27)}$

$\Delta = \sqrt{380201.28} = 616.6$ sq. units

Q4. The area of triangle is 2437. if $a = 79$, and $c = 97$, then find angle β.

Solution:- Here $\Delta = 2437$, $a = 79$

$c = 97$, $\beta = ?$

$\because \Delta = \frac{1}{2} ac \sin\beta$

$\rightarrow 2437 = \frac{1}{2}(79)(97)\sin\beta$

$\rightarrow \sin\beta = \frac{2 \times 2437}{79 \times 97} = \frac{4874}{7663}$

$\rightarrow \beta = \sin^{-1}(0.636) = 39°30'$

Q5. The area of triangle is 121.34. If $\alpha = 32°15'$, $\beta = 65°37'$, then find c and angle γ.

Solution:- Here $\Delta = 121.34$, $\alpha = 32°15'$

$\beta = 65°37'$, $c = ?$, $\gamma = ?$

$\because \alpha + \beta + \gamma = 180°$

$\rightarrow \gamma = 180° - \alpha - \beta$

$= 180° - 32°15' - 65°37'$

$\gamma = 82°8'$

$\because \Delta = \frac{c^2 \sin\alpha \sin\beta}{2\sin\gamma}$

$121.34 = \frac{c^2 \sin 32°15' \sin 65°37'}{2\sin 82°8'}$

$121.34 = \frac{c^2 (0.5336)(0.9108)}{2(0.9906)}$

$\rightarrow c^2 = \frac{(121.34)(2)(0.9906)}{(0.5336)(0.9108)}$

$c^2 = 494.64$

$\rightarrow c = 22.24$

Q6. One side of a triangular garden is 30m. If its two corner angles are $22°\frac{1}{2}$ and $112°\frac{1}{2}$. Find the cost of planting the grass at the rate of Rs. 5 per squar meter.

Solution:-

Let $a = 30m$, $\beta = 22°30'$

$\gamma = 112°30'$

$\because \alpha + \beta + \gamma = 180°$

$\rightarrow \alpha = 180° - \beta - \gamma$

$\alpha = 180° - 22°30' - 112°30'$

$\alpha = 45°$

$\because \Delta = \frac{a^2 \sin\beta \sin\gamma}{2\sin\alpha}$

$= \frac{(30)^2 \sin 22°30' \sin 112°30'}{2\sin 45°}$

$\Delta = \frac{900(0.382)(0.9238)}{2(0.707)}$

$\Delta = 225$ sq. units

$\because$ cost per sq. meter = Rs. 5

So tatal cost is $= 5 \times 225$

$= 1125$ Rs.

Circles connected with Triangle

we have three kinds of circles related to a triangle:

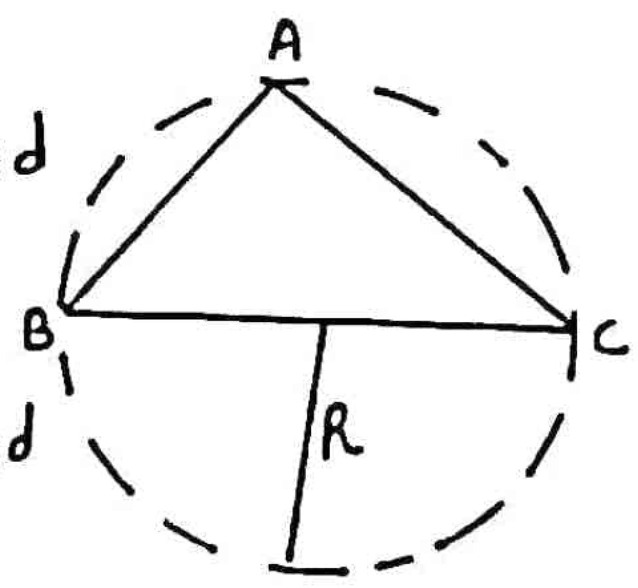
i) Circum-circle ii) In-Circle

iii) Ex-Circle

i) Circum-Circle:-

The circle passing through the vertices of a triangle is called a Circum-circle.

* Its centre is called circum-centre and its radius is called circum-radius and is denoted R.



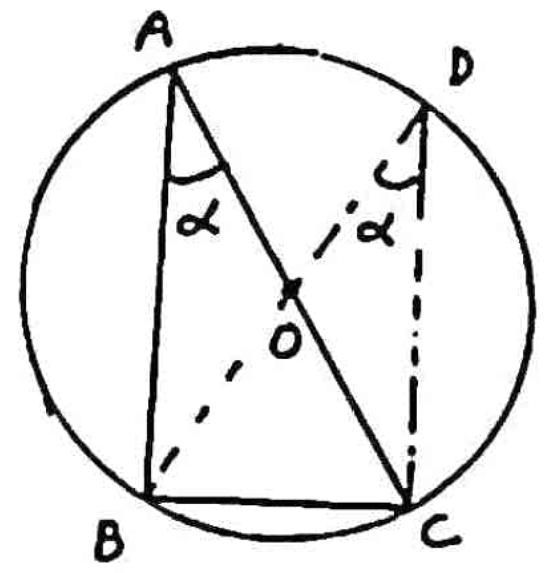
Prove that:

$$R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma},$$

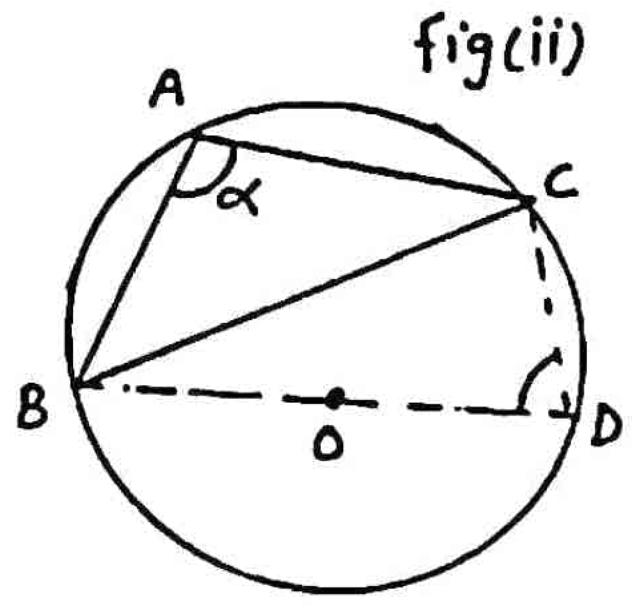
with usual notations.

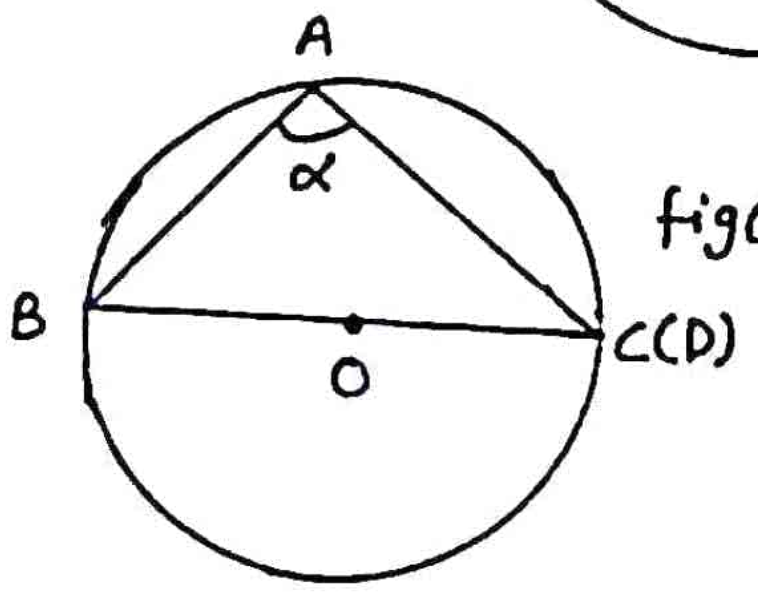
Proof:-

fig (i)



fig(ii)





fig(iii)

In fig (i), in right triangle $\triangle BCD$

$$\frac{m\overline{BC}}{m\overline{BD}} = \sin\alpha \quad (\alpha \cong m\angle BDC) \longrightarrow (I)$$

In fig (ii) $m\angle BDC + m\angle BAC = 180°$

(∵ sum of opposite angle of cyclical quadrilateral = 180°)

$\Rightarrow m\angle BDC = 180° - m\angle A = 180° - \alpha$

In right triangle BCD

$$\frac{m\overline{BC}}{m\overline{BD}} = \sin m\angle BDC$$

$$\Rightarrow \frac{m\overline{BC}}{m\overline{BD}} = \sin(180° - \alpha) = \sin\alpha \quad \text{———} (II)$$

In fig (iii), clearly

$$\frac{m\overline{BC}}{m\overline{BD}} = 1 = \sin 90° = \sin\alpha \quad \text{———} (III)$$

From (I), (II), (III)

$$\frac{m\overline{BC}}{m\overline{BD}} = \sin\alpha \Rightarrow \sin\alpha = \frac{a}{2R}$$

where $\overline{BC} = a$, $\overline{BD} = 2R$

$$\Rightarrow R = \frac{a}{2\sin\alpha}$$

Similarly, $R = \frac{b}{2\sin\beta}$ and

$$R = \frac{c}{2\sin\gamma}$$

a) Deduction of Law of Sines:

we know that

$$R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$$

$$\Rightarrow \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

$\therefore \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$, which is the law of sines.

b) Prove that: $R = \frac{abc}{4\Delta}$

Proof:-

we know that: $R = \frac{a}{2\sin\alpha}$

$$\Rightarrow R = \frac{a}{2 \cdot 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

$$\left(\because \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)$$

$$= \frac{a}{4\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

$$= \frac{a}{4\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}}$$

$$R = \frac{a}{4\sqrt{\frac{s(s-a)(s-b)(s-c)}{b^2c^2}}}$$

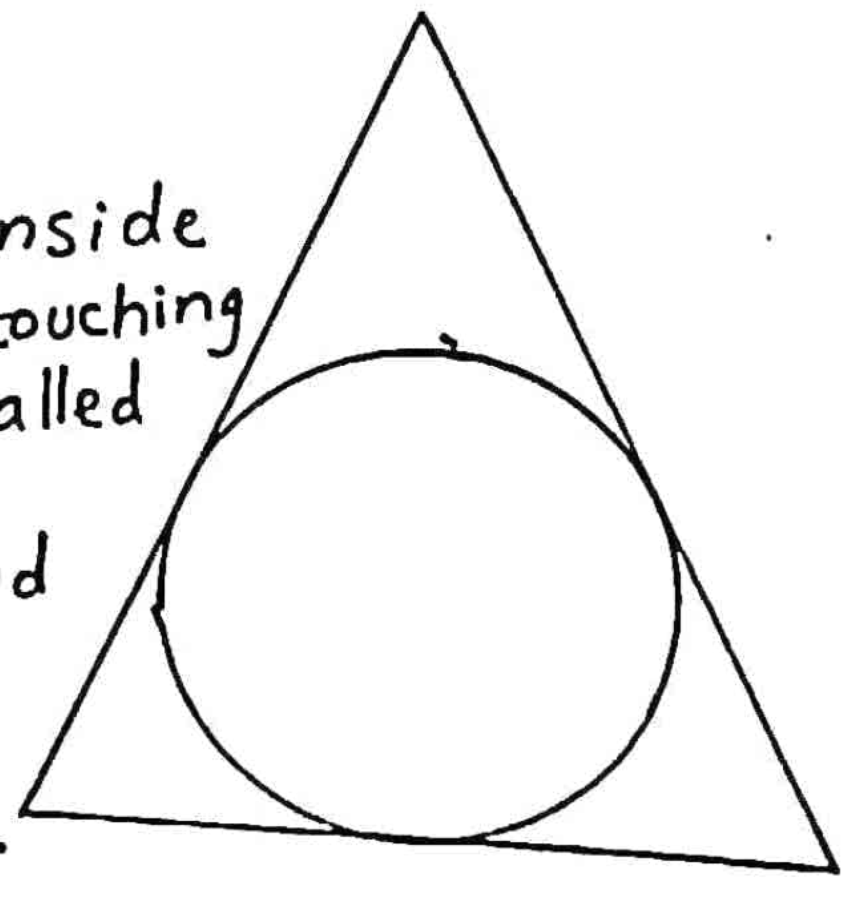
$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

$$\Rightarrow R = \frac{abc}{4\Delta} \quad \text{Hence proved}$$

Incircle:-

A circle drawn inside the triangle and touching its three sides is called an in circle.

* Its centre is called incentre and its radius is called inradius. denoted by r.



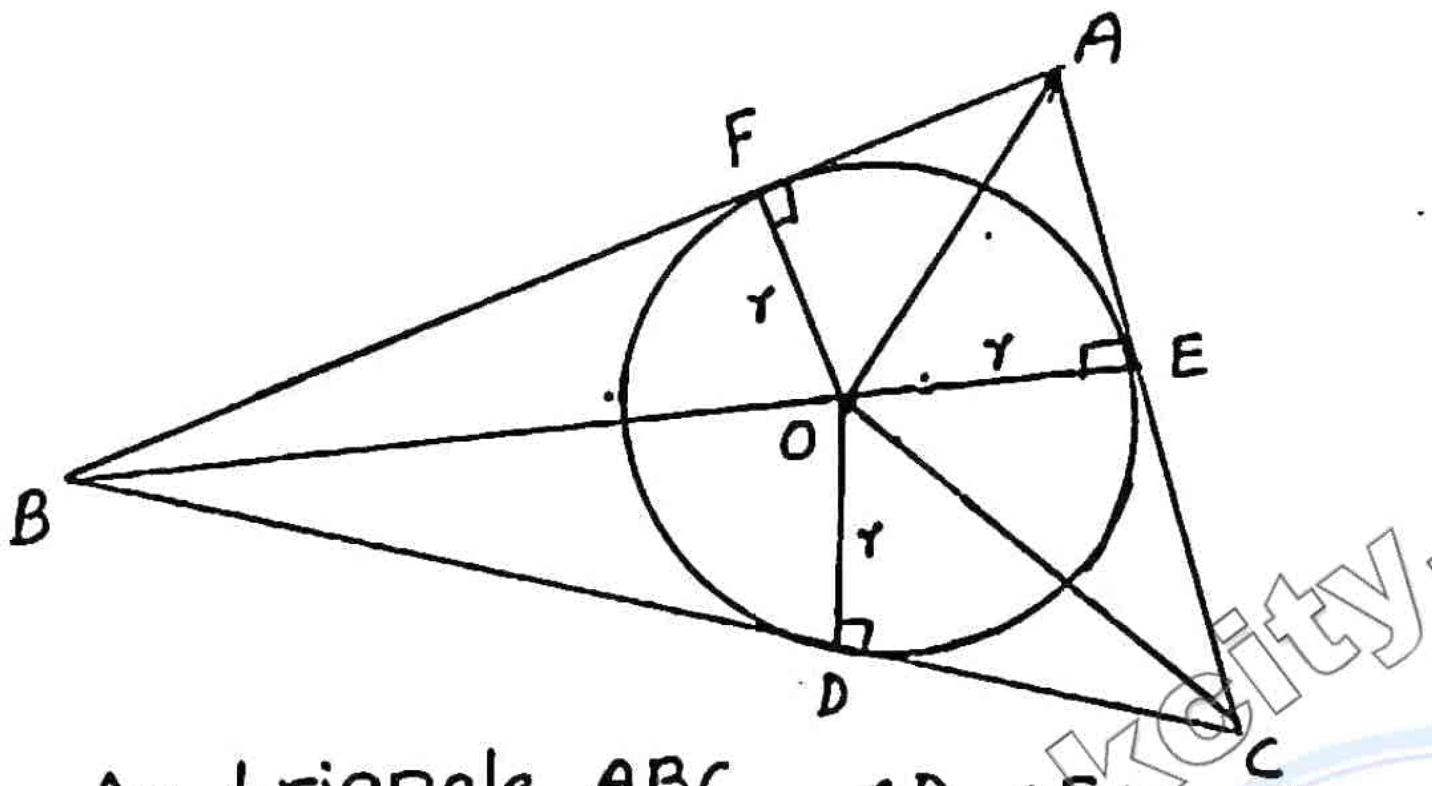
Important note:-

- Circum-center is the point of intersection of the perpendicular bisectors of sides of a triangle.
- In-centre is the point of intersection of the angle bisectors of a triangle.

Prove that:

$\gamma = \dfrac{\Delta}{S}$ with usual notation

Proof:-



In triangle ABC, OD, OE and OF are ⊥ar to BC, AC and AB resp. then ∴ from fig

area of ΔABC = area of ΔOBC + area of ΔOCA + area of ΔOAB

$\Rightarrow \Delta = \frac{1}{2} BC \times OD + \frac{1}{2} CA \times OE + \frac{1}{2} AB \times OF$

$= \frac{1}{2} ar + \frac{1}{2} br + \frac{1}{2} cr$

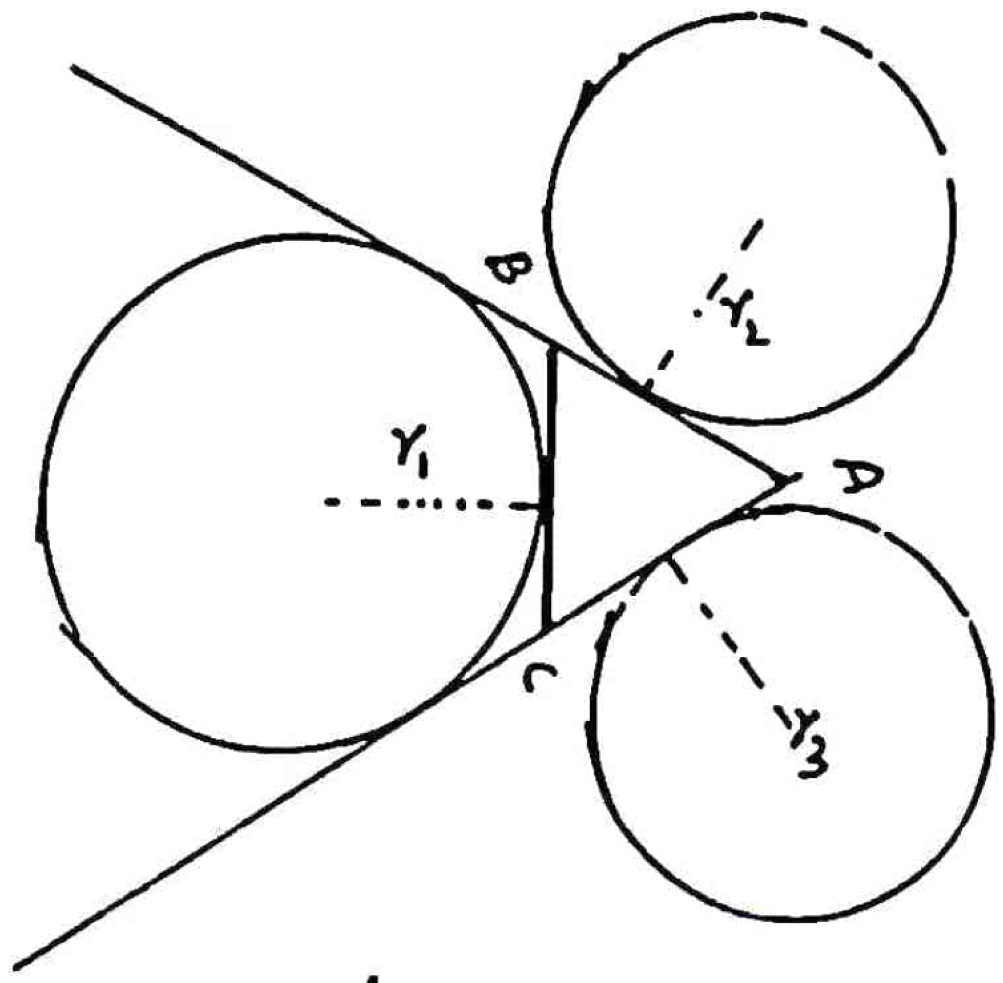
$= \frac{1}{2} r(a+b+c)$

$= \frac{1}{2} r(2S) \quad \because a+b+c = 2S$

$\Delta = rS \Rightarrow r = \dfrac{\Delta}{S}$

Escribed Circles

A circle which touches one side of a triangle externally and other two produced sides internally is called escribed-circle (e-circle).



Important note:-

- The radius of escribed circle is called escribed radius or e-radius.
- The centre of escribed circle is called escribed centre or e-center.
- Escribed circles are of three kindes

i) Circle drawn opposite to vertex A has radius r_1.

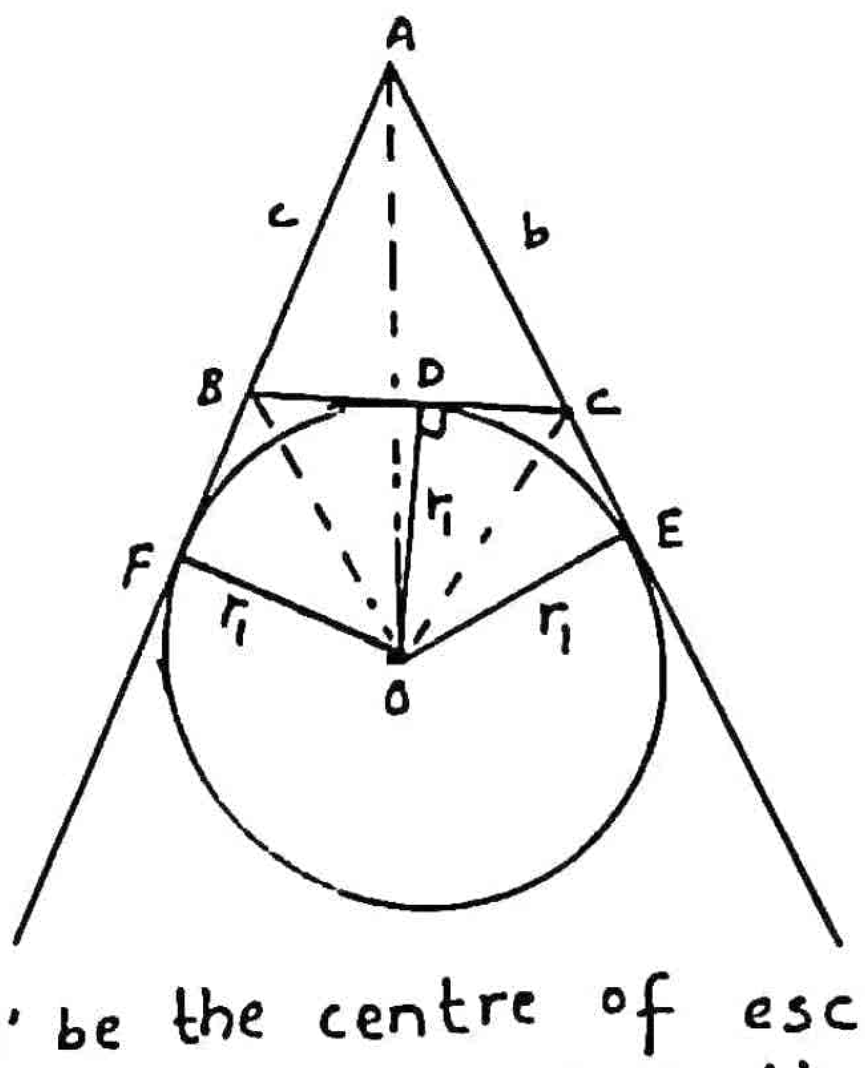
ii) circle draw opposite to vertex B has radius r_2.

iii) circle draw opposite to vertex C has radius r_3.

Prove that:

$r_1 = \dfrac{\Delta}{s-a}$, $r_2 = \dfrac{\Delta}{s-b}$ and $r_3 = \dfrac{\Delta}{s-c}$

Proof:-



Let 'O' be the centre of escribed circle. Draw ⊥ar D, E, F then

$\Delta ABC = \Delta OAB + \Delta AOC - \Delta OBC$

$= \frac{1}{2}(AB)(OF) + \frac{1}{2}(AC)(OE) - \frac{1}{2}(BC)(OD)$

$$\Delta = \frac{1}{2}cr_1 + \frac{1}{2}br_1 - \frac{1}{2}ar_1$$
$$= \frac{1}{2}r_1(c+b-a)$$
$$\Delta = \frac{1}{2}r_1(b+c-a)$$
$$= \frac{1}{2}r_1(2s-a-a) \qquad \because a+b+c=2s \Rightarrow b+c=2s-a$$
$$= \frac{1}{2}r_1(2s-2a)$$
$$= \frac{1}{2}2r_1(s-a)$$
$$\Delta = r_1(s-a)$$
$$\Rightarrow r_1 = \frac{\Delta}{s-a} \quad \text{Hence proved}$$

Similarly,
$$r_2 = \frac{\Delta}{s-b}, \quad r_3 = \frac{\Delta}{s-c}$$

Example 1. Show that
$$r = (s-a)\tan\frac{\alpha}{2} = (s-b)\tan\frac{\beta}{2} = (s-c)\tan\frac{\gamma}{2}$$

Solution:-
$$R.H.S = (s-a)\tan\frac{\alpha}{2}$$
$$= (s-a)\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$
$$= \sqrt{\cancel{s-a}}\sqrt{s-a} \times \frac{\sqrt{(s-b)(s-c)}}{\sqrt{s}\sqrt{\cancel{s-a}}}$$
$$= \frac{\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}} \times \frac{\sqrt{s}}{\sqrt{s}}$$
$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\Delta}{s} = r = L.H.S$$

Also,
$$R.H.S = (s-b)\tan\frac{\beta}{2}$$
$$= (s-b)\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$
$$= \sqrt{\cancel{s-b}}\sqrt{s-b} \times \frac{\sqrt{(s-a)(s-c)}}{\sqrt{s}\sqrt{\cancel{s-b}}}$$
$$= \frac{\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}} \times \frac{\sqrt{s}}{\sqrt{s}}$$
$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\Delta}{s} = r = L.H.S$$

Also,
$$R.H.S = (s-c)\tan\frac{\gamma}{2}$$
$$= (s-c)\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$
$$= \sqrt{s-c}\sqrt{\cancel{s-c}} \times \frac{\sqrt{(s-a)(s-b)}}{\sqrt{s}\sqrt{\cancel{s-c}}}$$
$$= \frac{\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}} \times \frac{\sqrt{s}}{\sqrt{s}}$$
$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\Delta}{s} = r = L.H.S$$

Hence proved.

Example 2. Show that
$$r_1 = 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

Solution:-
$$R.H.S = 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$
$$= \cancel{4}\frac{abc}{\cancel{4}\Delta}\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-b)}{ca}}\sqrt{\frac{s(s-c)}{ab}}$$
$$= \frac{abc}{\Delta}\sqrt{\frac{s^2(s-b)^2(s-c)^2}{a^2b^2c^2}}$$
$$= \frac{\cancel{abc}}{\Delta}\cdot\frac{s(s-b)(s-c)}{\cancel{abc}}$$
$$= \frac{s(s-a)(s-b)(s-c)}{\Delta(s-a)} = \frac{\Delta^2}{\Delta(s-a)} = \frac{\Delta}{s-a}$$
$$= r_1 = L.H.S \quad \text{Hence proved}$$

Example 3. Prove that
$$\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{\Delta^2}$$

Solution:-
$$L.H.S = \frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2}$$
$$= \frac{s^2}{\Delta^2} + \frac{(s-a)^2}{\Delta^2} + \frac{(s-b)^2}{\Delta^2} + \frac{(s-c)^2}{\Delta^2}$$
$$= \frac{s^2+(s-a)^2+(s-b)^2+(s-c)^2}{\Delta^2}$$
$$= \frac{s^2+s^2+a^2-2as+s^2+b^2-2bs+s^2+c^2-2cs}{\Delta^2}$$

$$= \frac{4s^2 - 2s(a+b+c) + a^2 + b^2 + c^2}{\Delta^2}$$

$$= \frac{4s^2 - 2s(2s) + a^2 + b^2 + c^2}{\Delta^2}$$

$$= \frac{4s^2 - 4s^2 + a^2 + b^2 + c^2}{\Delta^2}$$

$$= \frac{a^2+b^2+c^2}{\Delta^2} = R.H.S$$

Hence proved

Example 4. If the measures of the sides of a triangle ABC are 17, 10, 21 Find R, r, r_1, r_2 and r_3.

Solution:- Let $a = 17$, $b = 10$, $c = 21$

$$\because s = \frac{a+b+c}{2} = \frac{17+10+21}{2} = \frac{48}{2} = 24$$

$s-a = 24-17 = 7$, $s-b = 24-10 = 14$

$s-c = 24-21 = 3$

Now

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{(24)(7)(14)(3)} = \sqrt{7056} = 84$$

Now

$$R = \frac{abc}{4\Delta} = \frac{(17)(10)(21)}{4(84)} = \frac{85}{8}$$

$$r = \frac{\Delta}{s} = \frac{84}{24} = \frac{7}{2}$$

$$r_1 = \frac{\Delta}{s-a} = \frac{84}{7} = 12$$

$$r_2 = \frac{\Delta}{s-b} = \frac{84}{14} = 6$$

$$r_3 = \frac{\Delta}{s-c} = \frac{84}{3} = 28$$

Exercise 12.8

Q1. Show that

i) $r = 4R \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sin\frac{\gamma}{2}$

Solution:-

$$R.H.S = 4R \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sin\frac{\gamma}{2}$$

$$= 4\frac{abc}{4\Delta} \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{(s-a)(s-c)}{ac}} \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= \frac{abc}{\Delta} \sqrt{\frac{(s-a)^2(s-b)^2(s-c)^2}{a^2b^2c^2}}$$

$$= \frac{\cancel{abc}}{\Delta} \cdot \frac{(s-a)(s-b)(s-c)}{\cancel{abc}}$$

$$= \frac{s(s-a)(s-b)(s-c)}{s\Delta} = \frac{\Delta^2}{s\Delta}$$

$$= \frac{\Delta}{s} = r = L.H.S$$

Hence proved

ii) $s = 4R \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$

Solution:-

$$R.H.S = 4R \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$$

$$= 4\frac{abc}{4\Delta} \sqrt{\frac{s(s-a)}{bc}} \sqrt{\frac{s(s-b)}{ac}} \sqrt{\frac{s(s-c)}{ab}}$$

$$= \frac{abc}{\Delta} \sqrt{\frac{s^2 \cdot s(s-a)(s-b)(s-c)}{a^2b^2c^2}}$$

$$= \frac{\cancel{abc}}{\Delta} \cdot s\, \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\cancel{abc}}$$

$$= \frac{s\Delta}{\Delta} = s = L.H.S$$

Hence proved $\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Q2. Show that

$$r = a\sin\frac{\beta}{2} \sin\frac{\gamma}{2} \sec\frac{\alpha}{2} = b\sin\frac{\gamma}{2} \sin\frac{\alpha}{2} \sec\frac{\beta}{2}$$

$$= c \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sec\frac{\gamma}{2}$$

Solution:-

$$R.H.S = a\sin\frac{\beta}{2} \sin\frac{\gamma}{2} \sec\frac{\alpha}{2}$$

$$= a\sqrt{\frac{(s-a)(s-c)}{ac}} \sqrt{\frac{(s-a)(s-b)}{ab}} \sqrt{\frac{bc}{s(s-a)}}$$

$$= a\sqrt{\frac{(s-a)^2(s-b)(s-c)bc}{s(s-a)a^2bc}}$$

$$= \frac{a}{a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s \times s}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\Delta}{s} = r = L.H.S$$

Hence proved

Also,

$$R.H.S = b\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sec\frac{\beta}{2}$$

$$= b\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{ac}{s(s-b)}}$$

$$= b\sqrt{\frac{(s-a)(s-b)^2(s-c)ac}{ab^2c\,s(s-b)}}$$

$$= \frac{b}{b}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s\times s}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$$

$$= \frac{\Delta}{s} = r = L.H.S$$

Hence proved

Also,

$$R.H.S = c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$= c\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{ab}{s(s-c)}}$$

$$= c\sqrt{\frac{(s-a)(s-b)(s-c)^2}{abc^2\,s(s-c)}}$$

$$= \frac{c}{c}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s\times s}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\Delta}{s}$$

$$= r = L.H.S$$

Hence proved

Q3. Show that:

i) $r_1 = 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

Solution:-

$$R.H.S = 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$= 4\frac{abc}{4\Delta}\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{s^2(s-b)^2(s-c)^2}{a^2b^2c^2}}$$

$$= \frac{\cancel{abc}}{\Delta}\,\frac{s(s-b)(s-c)}{\cancel{abc}} = \frac{s(s-b)(s-c)}{\Delta}$$

$$= \frac{s(s-a)(s-b)(s-c)}{(s-a)\cdot\Delta} = \frac{\Delta^2}{(s-a)\cdot\Delta}$$

$$= \frac{\Delta}{s-a} = r_1 = L.H.S$$

Hence proved

ii) $r_2 = 4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$

Solution:-

$$R.H.S = 4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$= 4\frac{abc}{4\Delta}\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{s^2(s-a)^2(s-c)^2}{a^2b^2c^2}}$$

$$= \frac{\cancel{abc}}{\Delta}\cdot\frac{s(s-a)(s-c)}{\cancel{abc}} = \frac{s(s-a)(s-c)}{\Delta}$$

$$= \frac{s(s-a)(s-b)(s-c)}{\Delta(s-b)} = \frac{\Delta^2}{\Delta(s-b)}$$

$$= \frac{\Delta}{s-b} = r_2 = L.H.S$$

Hence proved

iii) $r_3 = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}$

Solution:-

$$R.H.S = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}$$

$$= 4\frac{abc}{4\Delta}\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{s^2(s-a)^2(s-b)^2}{a^2b^2c^2}}$$

$$= \frac{\cancel{abc}}{\Delta}\cdot\frac{s(s-a)(s-b)}{\cancel{abc}} = \frac{s(s-a)(s-b)}{\Delta}$$

$$= \frac{s(s-a)(s-b)(s-c)}{\Delta\cdot(s-c)} = \frac{\Delta^2}{\Delta(s-c)}$$

$$= \frac{\Delta}{s-c} = r_3 = L.H.S$$

Hence proved

Q4. Show that

i) $r_1 = s\tan\frac{\alpha}{2}$

Solution:-

$R.H.S = s\tan\frac{\alpha}{2}$

$= s\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

$= s\sqrt{\frac{s(s-a)(s-b)(s-c)}{s(s-a).s(s-a)}}$

$= \frac{s\sqrt{s(s-a)(s-b)(s-c)}}{s(s-a)}$

$= \frac{\Delta}{s-a} = r_1 = L.H.S$

Hence proved

ii) $r_2 = s\tan\frac{\beta}{2}$

Solution:-

$R.H.S = s\tan\frac{\beta}{2}$

$= s\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

$= s\sqrt{\frac{s(s-a)(s-b)(s-c)}{s(s-b).s(s-b)}}$

$= \frac{s\sqrt{s(s-a)(s-b)(s-c)}}{s(s-b)}$

$= \frac{\Delta}{s-b} = r_2 = L.H.S$

Hence proved.

iii) $r_3 = s\tan\frac{\gamma}{2}$

Solution:-

$R.H.S = s\tan\frac{\gamma}{2}$

$= s\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

$= s\sqrt{\frac{s(s-a)(s-b)(s-c)}{s(s-c).s(s-c)}}$

$= \frac{s\sqrt{s(s-a)(s-b)(s-c)}}{s(s-c)}$

$= \frac{\Delta}{s-c} = r_3 = L.H.S$

Hence proved

Q5. Prove that:

i) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

Solution:-

$L.H.S = r_1r_2 + r_2r_3 + r_3r_1$

$= \left(\frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-b}\right) + \left(\frac{\Delta}{s-b}\right)\left(\frac{\Delta}{s-c}\right) + \left(\frac{\Delta}{s-c}\right)\left(\frac{\Delta}{s-a}\right)$

$= \frac{\Delta^2}{(s-a)(s-b)} + \frac{\Delta^2}{(s-b)(s-c)} + \frac{\Delta^2}{(s-c)(s-a)}$

$= \Delta^2\left[\frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)}\right]$

$= \Delta^2\left[\frac{s-c+s-a+s-b}{(s-a)(s-b)(s-c)}\right]$

$= \Delta^2\left[\frac{3s-(a+b+c)}{(s-a)(s-b)(s-c)}\right]$

$= \Delta^2\left[\frac{3s-2s}{(s-a)(s-b)(s-c)}\right]$ $\because a+b+c = 2s$

$= \Delta^2 . \frac{s \times s}{s(s-a)(s-b)(s-c)}$

$= \frac{\Delta^2 . s^2}{\Delta^2} = s^2 = R.H.S$

Hence proved

ii) $r r_1 r_2 r_3 = \Delta^2$

Solution:-

$L.H.S = r r_1 r_2 r_3$

$= \left(\frac{\Delta}{s}\right)\left(\frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-b}\right)\left(\frac{\Delta}{s-c}\right)$

$= \frac{\Delta^4}{s(s-a)(s-b)(s-c)}$

$= \frac{\Delta^4}{\Delta^2} = \Delta^2 = R.H.S$

Hence proved

iii) $r_1 + r_2 + r_3 - r = 4R$

Solution:-

$L.H.S = r_1 + r_2 + r_3 - r$

$= \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} - \frac{\Delta}{s}$

$$= \Delta\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}\right)$$

$$= \Delta\left(\frac{s-b+s-a}{(s-a)(s-b)}+\frac{s-(s-c)}{s(s-c)}\right)$$

$$= \Delta\left(\frac{2s-(a+b)}{(s-a)(s-b)}+\frac{s-s+c}{s(s-c)}\right)$$

$$= \Delta\left(\frac{2s-(2s-c)}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right)$$

$\because a+b+c=2s$

$\Rightarrow a+b=2s-c$

$$= \Delta\left(\frac{2s-2s+c}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right)$$

$$= \Delta\left(\frac{c}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right)$$

$$= c\Delta\left(\frac{1}{(s-a)(s-b)}+\frac{1}{s(s-c)}\right)$$

$$= c\Delta\left(\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right)$$

$$= c\Delta\left(\frac{s^2-sc+s^2-sb-as+ab}{\Delta^2}\right)$$

$$= c\cdot\frac{2s^2-s(a+b+c)+ab}{\Delta}$$

$$= c\cdot\frac{2s^2-s(2s)+ab}{\Delta}$$

$$= c\cdot\frac{2s^2-2s^2+ab}{\Delta}$$

$$= \frac{abc}{\Delta} = \frac{abc}{4\Delta}\times 4$$

$\because \frac{abc}{4\Delta}=R$

$$= 4R = R.H.S$$

Hence proved

iv) $r_1r_2r_3 = rs^2$

Solution:-

$$L.H.S = r_1r_2r_3$$

$$= \left(\frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-b}\right)\left(\frac{\Delta}{s-c}\right)$$

$$= \frac{\Delta^3}{(s-a)(s-b)(s-c)}$$

$$= \frac{s\Delta^3}{s(s-a)(s-b)(s-c)}$$

$$= \frac{s\Delta^3}{\Delta^2} = \Delta s$$

$\because \frac{\Delta}{s}=r$

$\Rightarrow \Delta = rs$

$$= s(rs)$$

$$= rs^2 = R.H.S$$

Hence proved

Q6. Find R, r, r_1, r_2 and r_3, if measures of the sides of triangle ABC are

i) $a=13$, $b=14$, $c=15$

ii) $a=34$, $b=20$, $c=42$

Solution:- i) $a=13$, $b=14$, $c=15$

$\because R=\frac{abc}{4\Delta}$, $r=\frac{\Delta}{s}$, $r_1=\frac{\Delta}{s-a}$, $r_2=\frac{\Delta}{s-b}$

$r_3=\frac{\Delta}{s-c}$

$\because s=\frac{a+b+c}{2}=\frac{13+14+15}{2}=\frac{42}{2}=21$

$s-a = 21-13=8$, $s-b=21-14=7$

$s-c = 21-15=6$

$\therefore \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = \sqrt{(21)(8)(7)(6)} = \sqrt{7056}=84$

$R=\frac{abc}{4\Delta}=\frac{(13)(14)(15)}{4(84)}=\frac{2730}{336}=8.125$

$r=\frac{\Delta}{s}=\frac{84}{21}=4$, $r_1=\frac{\Delta}{s-a}=\frac{84}{8}=10.5$

$r_2=\frac{\Delta}{s-b}=\frac{84}{7}=12$, $r_3=\frac{\Delta}{s-c}=\frac{84}{6}=14$

ii) $a=34$, $b=20$, $c=42$

$\because R=\frac{abc}{4\Delta}$, $r=\frac{\Delta}{s}$, $r_1=\frac{\Delta}{s-a}$

$r_2=\frac{\Delta}{s-b}$, $r_3=\frac{\Delta}{s-c}$

$\because s=\frac{a+b+c}{2}=\frac{34+20+42}{2}=\frac{96}{2}=48$

$s-a=48-34=14$, $s-b=48-20=28$

$s-c=48-42=6$

$\therefore \Delta=\sqrt{s(s-a)(s-b)(s-c)}$

$\Delta=\sqrt{48(14)(28)(6)}=\sqrt{112896}=336$

$R=\frac{abc}{4\Delta}=\frac{(34)(20)(42)}{4(336)}=21.25$

$r = \frac{\Delta}{s} = \frac{336}{48} = 7$

$r_1 = \frac{\Delta}{s-a} = \frac{336}{14} = 24$

$r_2 = \frac{\Delta}{s-b} = \frac{336}{28} = 12$

$r_3 = \frac{\Delta}{s-c} = \frac{336}{6} = 56$

Q7. Prove that in an equilateral triangle,

i) $r : R : r_1 = 1:2:3$

ii) $r : R : r_1 : r_2 : r_3 = 1:2:3:3:3$

Solution:-

In equilateral triangle

$a = b = c$ so

$$s = \frac{a+b+c}{2} = \frac{a+a+a}{2} = \frac{3a}{2}$$

$$\therefore \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\Delta = \sqrt{s(s-a)(s-a)(s-a)} = \sqrt{s(s-a)^3}$$

$$= \sqrt{\frac{3a}{2}\left(\frac{3a}{2}-a\right)^3} = \sqrt{\frac{3a}{2}\left(\frac{3a-2a}{2}\right)^3}$$

$$= \sqrt{\frac{3a}{2}\times\left(\frac{a}{2}\right)^3} = \sqrt{\frac{3a}{2}\times\frac{a^3}{8}}$$

$$\Delta = \sqrt{\frac{3a^4}{16}} = \frac{\sqrt{3}a^2}{4}$$

$$\because r = \frac{\Delta}{s} = \frac{\frac{\sqrt{3}a^2}{4}}{\frac{3a}{2}} = \frac{\sqrt{3}a^2}{4} \times \frac{2}{3a}$$

$$r = \frac{\sqrt{3}a}{2} \times \frac{1}{\sqrt{3}\sqrt{3}} = \frac{a}{2\sqrt{3}}$$

$$r_1 = \frac{\Delta}{s-a} = \frac{\frac{\sqrt{3}a^2}{4}}{\frac{3a}{2}-a} = \frac{\frac{\sqrt{3}a^2}{4}}{\frac{3a-2a}{2}}$$

$$r_1 = \frac{\frac{\sqrt{3}a^2}{4}}{\frac{a}{2}} = \frac{\sqrt{3}a^2}{4} \times \frac{2}{a} = \frac{\sqrt{3}a}{2}$$

$$\rightarrow r_1 = r_2 = r_3 = \frac{\sqrt{3}a}{2} \quad \left(\because a=b=c \text{ so } r_1 = r_2 = r_3\right)$$

$$R = \frac{abc}{4\Delta} = \frac{a \cdot a \cdot a}{4\frac{\sqrt{3}a^2}{4}} = \frac{a}{\sqrt{3}}$$

i) $r : R : r_1 = 1:2:3$

L.H.S $= r : R : r_1$

$$= \frac{a}{2\sqrt{3}} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}a}{2}$$

'x' by $\frac{2\sqrt{3}}{a}$

$$= \frac{a}{2\sqrt{3}} \times \frac{2\sqrt{3}}{a} : \frac{a}{\sqrt{3}} \times \frac{2\sqrt{3}}{a} : \frac{\sqrt{3}a}{2} \times \frac{2\sqrt{3}}{a}$$

$= 1:2:3 =$ R.H.S

Hence proved

(ii) $r : R : r_1 : r_2 : r_3 = 1:2:3:3:3$

L.H.S $= r : R : r_1 : r_2 : r_3$

$$= \frac{a}{2\sqrt{3}} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}a}{2} : \frac{\sqrt{3}a}{2} : \frac{\sqrt{3}a}{2}$$

'x' by $\frac{2\sqrt{3}}{a}$

$$= \frac{a}{2\sqrt{3}} \times \frac{2\sqrt{3}}{a} : \frac{a}{\sqrt{3}} \times \frac{2\sqrt{3}}{a} : \frac{\sqrt{3}a}{2} \times \frac{2\sqrt{3}}{a} : \frac{\sqrt{3}a}{2} \times \frac{2\sqrt{3}}{a} : \frac{\sqrt{3}a}{2} \times \frac{2\sqrt{3}}{a}$$

$= 1:2:3:3:3$

$=$ R.H.S

Hence proved

> **Remember,** "Any triangle whose all sides are equal is called equilateral triangle"

Q8. Prove that:

i) $\Delta = r^2 \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}$

Solution:-

R.H.S $= r^2 \cot\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}$

$$= \left(\frac{\Delta}{s}\right)^2 \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} \cdot \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \cdot \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{\Delta^2}{s^2} \sqrt{\frac{s^3(s-a)(s-b)(s-c)}{(s-a)^2(s-b)^2(s-c)^2}}$$

$$= \frac{\Delta^2}{s^2} \sqrt{\frac{s^3}{(s-a)(s-b)(s-c)}}$$

$$= \frac{\Delta^2}{s^2} \sqrt{\frac{s}{s} \times \frac{s^3}{(s-a)(s-b)(s-c)}}$$

$$= \frac{\Delta^2}{s^2} \sqrt{\frac{s^4}{s(s-a)(s-b)(s-c)}}$$

$$= \frac{\Delta^2}{s^2} \cdot \frac{s^2}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$= \frac{\Delta^2}{\Delta} = \Delta =$ L.H.S

Hence proved

ii) $r = s\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

Solution:-

R.H.S $= s\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

$= s\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

$= s\sqrt{\frac{(s-a)^2(s-b)^2(s-c)^2}{s^3(s-a)(s-b)(s-c)}}$

$= s\sqrt{\frac{(s-a)(s-b)(s-c)}{s^3}}$

$= s\sqrt{\frac{(s-a)(s-b)(s-c)}{s^3}\times\frac{s}{s}}$

$= s\sqrt{\frac{s(s-a)(s-b)(s-c)}{s^4}}$

$= \frac{s\sqrt{s(s-a)(s-b)(s-c)}}{s^2} = \frac{\Delta}{s}$

$= r =$ L.H.S

Hence proved

iii) $\Delta = 4Rr\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

Solution:-

R.H.S $= 4Rr\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

$= 4\frac{abc}{4\Delta}\cdot\frac{\Delta}{s}\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$

$= \frac{abc}{s}\sqrt{\frac{s^3(s-a)(s-b)(s-c)}{a^2b^2c^2}}$

$= \frac{\cancel{abc}}{s}\frac{\sqrt{s^2s(s-a)(s-b)(s-c)}}{\cancel{abc}}$

$= \frac{s}{s}\sqrt{s(s-a)(s-b)(s-c)}$

$= \Delta =$ L.H.S

Hence proved

Q9. Show that

i) $\frac{1}{2rR} = \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$

ii) $\frac{1}{r} = \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$

Solution:- i) $\frac{1}{2rR} = \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$

L.H.S $= \frac{1}{2rR}$

$= \frac{1}{2\left(\frac{\Delta}{r}\right)\left(\frac{abc}{4\Delta}\right)} = \frac{1}{\frac{abc}{2s}}$

$= \frac{2s}{abc} = \frac{a+b+c}{abc}$

$= \frac{\cancel{a}}{\cancel{a}bc}+\frac{\cancel{b}}{a\cancel{b}c}+\frac{\cancel{c}}{ab\cancel{c}}$ $(\because 2s = a+b+c)$

$= \frac{1}{bc}+\frac{1}{ac}+\frac{1}{ab} = \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$

$=$ R.H.S

Hence proved

ii) $\frac{1}{r} = \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$

R.H.S $= \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$

$= \frac{1}{\frac{\Delta}{s-a}}+\frac{1}{\frac{\Delta}{s-b}}+\frac{1}{\frac{\Delta}{s-c}}$

$= \frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}$

$= \frac{s-a+s-b+s-c}{\Delta}$ $\quad\because 2s = a+b+c$

$= \frac{3s-(a+b+c)}{\Delta} = \frac{3s-2s}{\Delta}$

$= \frac{s}{\Delta} = \frac{1}{\frac{\Delta}{s}} = \frac{1}{r} =$ L.H.S

Hence proved

Q10. Prove that:

$r = \frac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}} = \frac{b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}}$

$= \frac{c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$

Solution:-

R.H.S $= \frac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$

$= \frac{a\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}}{\sqrt{\frac{s(s-a)}{bc}}}$

$$= a\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{bc}{s(s-a)}}$$

$$= a\sqrt{\frac{(s-a)^2(s-b)(s-c).\cancel{bc}}{a^2\cancel{bc}\,s(s-a)}}$$

$$= \frac{a}{a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s\times s}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\Delta}{s}$$

$$= r = L.H.S$$

Hence proved

Also,

$$R.H.S = \frac{b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}}$$

$$= \frac{b\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-b)}{ab}}}{\sqrt{\frac{s(s-b)}{ac}}}$$

$$= b\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{ac}{s(s-b)}}$$

$$= b\sqrt{\frac{(s-a)(s-b)^2(s-c).\cancel{ac}}{\cancel{a}b^2\cancel{c}\,s(s-b)}}$$

$$= \frac{b}{b}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s\times s}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\Delta}{s}$$

$$= r = L.H.S$$

Hence proved

Also,

$$R.H.S = \frac{c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$$

$$= \frac{c\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-b)}{ab}}}{\sqrt{\frac{s(s-c)}{ab}}}$$

$$= c\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{ab}{s(s-c)}}$$

$$= c\sqrt{\frac{(s-a)(s-b)(s-c)^2.ab}{abc^2.\,s(s-c)}}$$

$$= \frac{c}{c}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s\times s}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\Delta}{s}$$

$$= r = L.H.S$$

Hence proved

Q11. Prove that:

$$abc(\sin\alpha+\sin\beta+\sin\gamma) = 4\Delta s$$

Solution:-

$$L.H.S = abc(\sin\alpha+\sin\beta+\sin\gamma)$$

$$= abc\left(\frac{a}{2R}+\frac{b}{2R}+\frac{c}{2R}\right) \quad \because R=\frac{a}{2\sin\alpha} \rightarrow \sin\alpha=\frac{a}{2R}$$

$$= abc\left(\frac{a+b+c}{2R}\right) \quad \because R=\frac{b}{2\sin\beta} \rightarrow \sin\beta=\frac{b}{2R}$$

$$= abc\left(\frac{2s}{2\left(\frac{abc}{4\Delta}\right)}\right) \quad \because R=\frac{c}{2\sin\gamma} \rightarrow \sin\gamma=\frac{c}{2R}$$

$$= \frac{s}{\frac{1}{4\Delta}} = 4\Delta s = R.H.S$$

Hence proved.

Q12. Prove that:

i) $(r_1+r_2)\tan\frac{\gamma}{2} = c$

ii) $(r_3-r)\cot\frac{\gamma}{2} = c$

Solution:- i) $(r_1+r_2)\tan\frac{\gamma}{2} = c$

$$L.H.S = (r_1+r_2)\tan\frac{\gamma}{2}$$

$$= \left(\frac{\Delta}{s-a}+\frac{\Delta}{s-b}\right)\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \Delta\left(\frac{1}{s-a}+\frac{1}{s-b}\right)\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \Delta\left(\frac{s-b+s-a}{(s-a)(s-b)}\right)\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \Delta\left(\frac{2s-(a+b)}{(s-a)(s-b)}\right)\frac{\sqrt{(s-a)(s-b)}}{\sqrt{s(s-c)}}$$

$$= \Delta.\frac{a+b+c-a-b}{\left(\sqrt{(s-a)(s-b)}\right)^{\cancel{2}}}\frac{\cancel{\sqrt{(s-a)(s-b)}}}{\sqrt{s(s-c)}}$$

$$= \Delta.\frac{c}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \frac{\Delta.c}{\Delta} = c = R.H.S$$

Hence proved.

ii) $(r_3 - r)\cot\frac{\gamma}{2} = c$

$L.H.S = (r_3 - r)\cot\frac{\gamma}{2}$

$$= \left(\frac{\Delta}{s-c} - \frac{\Delta}{s}\right).\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \Delta\left(\frac{1}{s-c} - \frac{1}{s}\right).\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \Delta\left(\frac{s-(s-c)}{s(s-c)}\right).\frac{\sqrt{s(s-c)}}{\sqrt{(s-a)(s-b)}}$$

$$= \Delta.\frac{s-s+c}{\left(\sqrt{s(s-c)}\right)^{\cancel{2}}}.\frac{\cancel{\sqrt{s(s-c)}}}{\sqrt{(s-a)(s-b)}}$$

$$= \frac{\Delta.\ c}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \frac{\Delta.c}{\Delta} = c = R.H.S$$

Hence proved